

THE CAPELLI IDENTITY FOR GRASSMANN MANIFOLDS

SIDDHARTHA SAHI

ABSTRACT. The column space of a real $n \times k$ matrix x of rank k is a k -plane. Thus we get a map from the space X of such matrices to the Grassmannian \mathbb{G} of k -planes in \mathbb{R}^n , and hence a GL_n -equivariant isomorphism

$$C^\infty(\mathbb{G}) \approx C^\infty(X)^{GL_k}.$$

We consider the $O_n \times GL_k$ -invariant differential operator C on X given by

$$C = \det(x^t x) \det(\partial^t \partial), \quad \text{where } x = (x_{ij}), \quad \partial = \left(\frac{\partial}{\partial x_{ij}} \right).$$

By the above isomorphism, C defines an O_n -invariant operator on \mathbb{G} .

Since \mathbb{G} is a symmetric space for O_n , the irreducible O_n -submodules of $C^\infty(\mathbb{G})$ have multiplicity 1; thus, O_n -invariant operators act by scalars on these submodules. Our main result determines these scalars for a general class of such operators including C . This answers a question raised by Howe and Lee and also gives new Capelli-type identities for the orthogonal Lie algebra.

1. INTRODUCTION

1.1. The classical Capelli identity. Let $\{z_{ij} : 1 \leq i, j \leq n\}$ be n^2 variables and let $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$ be the corresponding partial derivatives. Consider the $n \times n$ matrices $z = (z_{ij})$ and $\partial = (\partial_{ij})$. The classical Capelli identity [3] is the following identity of differential operators

$$(1.1) \quad \det(z) \det(\partial) = \det(z^t \partial + (n-i) \delta_{ij}).$$

Here δ_{ij} is the Kronecker δ function, and the (noncommuting) determinant on the right is defined as follows:

$$\det(a_{ij}) = \sum_{w \in S_n} \text{sgn}(w) a_{w(1),1} \cdots a_{w(n),n}.$$

This identity plays a key role in the work of Herman Weyl [20] and others on classical invariant theory. It has also appeared more recently in the work of Atiyah, Bott, and Patodi [1] on the index theorem. We refer the reader to [7] and [2] for a discussion of the background of this identity. In the early 90s, the identity was generalized considerably by B. Kostant and the author ([13], [14], [19]) and by Howe and Umeda [10] to the context of Jordan algebras and multiplicity free actions. Since then there has been considerable interest in generalizations of Capelli-type identities, we refer the reader to [4], [11], [12], [15], [16], [17], [18] and the references therein.

The representation-theoretic meaning of (1.1) was explained in [7]. The polynomial algebra $\mathbb{C}[z_{ij}]$ is the affine coordinate ring of the space $M_{n,n}$ of $n \times n$ matrices.

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The group $GL_n \times GL_n$ acts on $M_{n,n}$ by left and right multiplication, and hence we get a representation π on $\mathbb{C}[z_{ij}]$. The operator $\det(z) \det(\partial)$ on the left of (1.1) is a $GL_n \times GL_n$ invariant differential operator on $\mathbb{C}[z_{ij}]$. It follows from (GL_n, GL_n) duality [7] that such an operator is necessarily of the form $\pi(\Omega)$ for some element Ω in the center of the enveloping algebra of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$. Note, however, that the element Ω is not unique, and the expression on the right of (1.1) merely identifies one possible choice of Ω . Indeed, the entries of the matrix $(z^t \partial)_{ij} = \sum_p z_{pi} \partial_{pj}$ come from the *right* action of GL_n alone, and one can find other similar expressions in terms of the matrix entries of the left action $(z \partial^t)_{ij} = \sum_p z_{ip} \partial_{jp}$, or the diagonal action $(z \partial^t - z^t \partial)_{ij}$, etc.

There is, however, an associated eigenvalue problem that does admit a unique answer. Again, by (GL_n, GL_n) duality, or in this case by the Peter-Weyl theorem, one knows that $\mathcal{P}_{n,n}$ decomposes as a direct sum of $GL_n \times GL_n$ -modules $\mathcal{P}_{n,n} = \bigoplus_\lambda (V^\lambda \otimes V^\lambda)$ where λ ranges over partitions of length $\leq n$ and V^λ is the GL_n representation with highest weight λ . By Schur's lemma $\det(z) \det(\partial)$ acts by a scalar $p(\lambda)$ on $V^\lambda \otimes V^\lambda$ and one may ask for an explicit formula for this scalar. This turns out to have a rather pretty answer:

$$(1.2) \quad p(\lambda) = \prod_{i=1}^n (\lambda_i + n - i).$$

As explained in [13], the eigenvalue formula (1.2) implies (1.1) via the Harish-Chandra homomorphism. Moreover, this point of view leads to a much larger class of identities in the context of Jordan algebras ([13], [14], [19]).

1.2. The Capelli identity for Grassmann manifolds. In [9] Howe and Lee pose a similar eigenvalue problem in the context of Grassmannians. In order to describe this problem we need some notation. Fix integers k, l with $k < l$ and write $n = k + l$. Let $\mathbb{G} = \mathbb{G}_{n,k}$ denote the Grassmannian of k planes in \mathbb{C}^n , and let $\mathcal{R} = \bigoplus_{m=0}^\infty \mathcal{R}_m$ be its homogeneous coordinate ring. The ring \mathcal{R} admits an explicit description via the Plücker imbedding that we now recall briefly, referring the reader to [5, Ch. 9] for details. Let $\mathcal{P} = \mathbb{C}[z_{11}, \dots, z_{nk}]$ be the coordinate ring of the space $M_{n,k}$ of $n \times k$ matrices, regarded as a $GL_n \times GL_k$ -module. Then \mathcal{R} is the subalgebra of \mathcal{P} generated by the determinants of $k \times k$ minors. Alternatively we have

$$\mathcal{R} = \mathcal{P}^{SL_k}, \mathcal{R}_m = \mathcal{P}^{(GL_k, \det^m)}$$

with the usual notation for invariants and equivariants for the right action of GL_k on $M_{n,k}$. This description also shows that each \mathcal{R}_m is naturally a module under the left action of GL_n . By (e.g. [7]) \mathcal{R}_m has a multiplicity-free O_n -decomposition,

$$\mathcal{R}_m = \bigoplus_\mu \mathcal{R}_m^\mu,$$

and the occurring summands are indexed by partitions of length $\leq k$, whose parts are less than m and have the same parity as m . In other words, μ ranges over integer sequences $\mu = (\mu_1, \dots, \mu_k)$ satisfying

$$m \geq \mu_1 \geq \dots \geq \mu_k \geq 0, \mu_i \equiv m \pmod{2} \text{ for all } i.$$

Consider now the $n \times k$ matrices $z = (z_{ij})$ and $\partial = \left(\frac{\partial}{\partial z_{ij}} \right)$ and define

$$\gamma(z) = \det(z^t z), \quad L = \det(\partial^t \partial).$$

The differential operator L and the multiplication operator γ are O_n -invariant and transform under GL_k by the characters \det^{-2} and \det^2 , respectively. Thus they define maps

$$(1.3) \quad \gamma : \mathcal{R}_m^\mu \rightarrow \mathcal{R}_{m+2}^\mu, L : \mathcal{R}_m^\mu \rightarrow \mathcal{R}_{m-2}^\mu.$$

It follows that for any two integers $d, d' \geq 0$, the operator

$$(1.4) \quad C_{d,d'} = \gamma^d \circ L^{d+d'} \circ \gamma^{d'}$$

is an O_n -invariant operator on \mathcal{R}_m^μ . By Schur's lemma $C_{d,d'}$ acts by a scalar on \mathcal{R}_m^μ , and Howe and Lee [9] ask for an explicit formula for these scalar eigenvalues. To be precise they consider only the operator $L\gamma = C_{0,1}$, but the general case is not much more difficult as we explain below. For $k = 1$ the problem reduces to the classical theory of harmonic polynomials, and Howe and Lee [9] solve the problem for $k = 2$ by an explicit calculation. Their calculation is elementary but fairly intricate and it is unclear how to extend it for $k \geq 3$.

In this paper we solve the eigenvalue problem for all k using completely different methods. Let us define the following k -tuple of rational numbers

$$(1.5) \quad \rho = (\rho_1, \dots, \rho_k) \in \mathbb{Q}^k, \quad \rho_i = \frac{n}{2} - i.$$

Also, let s and $\tau = (\tau_1, \dots, \tau_k)$ be $k + 1$ indeterminates and define polynomials

$$(1.6) \quad q_{1,0}(s, \tau) = \prod_{i=1}^k (s^2 - \tau_i^2), \quad q_{d,d'}(s, \tau) = \prod_{j=0}^{d+d'-1} q_{1,0}(s + 2d' - 2j, \tau).$$

Our main result is as follows:

Theorem 1.1. $C_{d,d'}$ acts on \mathcal{R}_m^μ by the scalar $q_{d,d'}(m + \rho_1, \mu + \rho)$.

Specializing to $k = 2, d = 0, d' = 1$, we see that $q_{0,1}(m + \rho_1, \mu + \rho)$ equals

$$(m - \mu_1 + 2)(m + \mu_1 + n)(m - \mu_2 + 3)(m + \mu_2 + n - 1),$$

which agrees with Theorem 6.1 of [9], using formulas in [9, p. 356] for $\lambda = \mu$.

We now briefly sketch the proof of Theorem 1.1. In the next section, we reduce the problem to the special case of the operator $C = C_{1,0} = \gamma L$, and prove a simple lemma about the kernel of C . We also explain how to imbed the problem into the more general real analytic setting of the symmetric space $Y = SO_n(\mathbb{R}) / [SO_k(\mathbb{R}) \times SO_l(\mathbb{R})]$. The operator C gives rise to a family of invariant operators C_s on Y . The structure of invariant differential operators on a symmetric space is given by the Harish-Chandra homomorphism. In Sections 3 and 4 we recall this theory and specialize it to Y . In Section 5 we prove the eigenvalues of C_s are given by a polynomial function with suitable symmetry properties. Finally, in Section 6 we use this polynomiality result, together with the knowledge of the kernel of C , to prove Theorem 2.1 up to an overall constant, which we then show to be 1 by an auxiliary computation.

In the appendix, we combine the results of this paper with those of [11] and [15] to obtain identities of invariant differential operators on the symmetric space $SO(p + q) / SO(p) \times SO(q)$. The main theorem also leads to a new inversion formula for the Radon transform on Grassmannians, which we plan to discuss in a subsequent paper.

2. REDUCTION OF THE PROBLEM

We first explain how deduce Theorem 1.1 from a special case. Define

$$C = C_{1,0} = \gamma L.$$

Theorem 2.1. *C acts on \mathcal{R}_m^μ by the scalar $q_{1,0}(m + \rho_1, \mu + \rho)$.*

Proof of Theorem 1.1. We abbreviate $q(m) = q_{1,0}(m + \rho_1, \mu + \rho)$. If v in \mathcal{R}_m^μ , then $\gamma^{d'}v \in \mathcal{R}_{m+2d'}^\mu$, and by Theorem 2.1 we get

$$L(\gamma^{d'}v) = q(m + 2d')\gamma^{d'-1}v.$$

Iterating this we get

$$L^{d+d'}\gamma^{d'}v = \left[\prod_{i=0}^{d+d'-1} q(m + 2d' - 2i) \right] \gamma^{-d}v.$$

We multiply both sides by γ^d to get the result. □

We now sketch the proof of Theorem 2.1. The following simple result on the kernel of C plays a key role.

Lemma 2.2. *If $\mu_1 = m$, then C acts by 0 on \mathcal{R}_m^μ .*

Proof. As noted above, $L : \mathcal{R}_m^\mu \rightarrow \mathcal{R}_{m-2}^\mu$. If $\mu_1 = m$, then $\mathcal{R}_{m-2}^\mu = 0$, thus L acts by 0 on \mathcal{R}_m^μ and so does $C = \gamma L$. □

In the next few sections we establish some general results about the eigenvalues of operators such as C , which enable us to deduce Theorem 2.1 from the above lemma. The key result is Proposition 5.3 below, which says that these eigenvalues are given by a polynomial function in the parameters m, μ with certain symmetry properties. In order to prove this proposition, it is important to extend the eigenvalue problem to a larger real analytic setting that we now explain.

The (holomorphic) polynomials in \mathcal{P} and \mathcal{R} are determined by their restrictions to the space $M_{n,k}(\mathbb{R})$ of real $n \times k$ matrices, and even to the open subset

$$X = M'_{n,k}(\mathbb{R})$$

of rank k matrices. We will write x_{ij} for the (real) coordinate functions on $M_{n,k}(\mathbb{R})$ and X , and regard \mathcal{P} and \mathcal{R} as spaces of polynomials in x_{ij} . Similarly, we write $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ and consider the restrictions of the various differential operators,

$$x = (x_{ij}), \partial = (\partial_{ij}), \gamma(x) = \det(x^t x), L = \det(\partial^t \partial), C = \gamma L.$$

Let $H = GL_k^+(\mathbb{R}) = \{g \in GL_k(\mathbb{R}) \mid \det g > 0\}$ and define

$$Y = X/H.$$

The left action of the group $SO_n(\mathbb{R})$ on Y is transitive and allows one to identify Y with the compact symmetric space $SO_n(\mathbb{R}) / [SO_k(\mathbb{R}) \times SO_l(\mathbb{R})]$. The general structure of invariant differential operators on such a space is well understood, and we briefly recall the main ideas below.

3. DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES

In this section we briefly recall some well-known results on compact symmetric spaces, referring the reader to [6] for details and proofs.

Let K be a connected compact group with involution σ , let K^σ denote the σ -fixed subgroup and let $(K^\sigma)_0$ denote its identity component. A subgroup M of K is said to be a symmetric subgroup if $(K^\sigma)_0 \subset M \subset K^\sigma$, and in this case we say that K/M is a symmetric space. In the case of interest to us $M = (K^\sigma)_0$ so we make this simplifying assumption in the subsequent discussion. Let $\mathfrak{k}, \mathfrak{m}$ denote the complexified Lie algebras of K and M . The involution σ on K defines an involution, still denoted σ , on \mathfrak{k} and we obtain the corresponding Cartan decomposition

$$\mathfrak{k} = \mathfrak{m} + \mathfrak{s}$$

into ± 1 eigenspaces of σ . We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and let \mathfrak{t} denote the centralizer of \mathfrak{a} in \mathfrak{m} . Then $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$ is a Cartan subalgebra of \mathfrak{k} . We let $\Sigma \subset \mathfrak{h}^*$ denote the root system of \mathfrak{h} in \mathfrak{k} , with Weyl group $W = W(\Sigma)$, and fix a positive subsystem Σ^+ . The irreducible representations of K are classified by their highest weight $\lambda \in \Lambda$, where $\Lambda \subset \mathfrak{h}^*$ denotes the lattice cone of dominant integral elements.

The representations of K with an M -fixed vector are those for which the highest weight μ belongs to the subcone $\Lambda_0 \subset \Lambda$ consisting of *even* integral elements that vanish on \mathfrak{t} . Thus, if \mathcal{H} denotes the space of K -finite functions on K/M , then we have a multiplicity free decomposition into irreducible K -modules

$$\mathcal{H} = \bigoplus_{\mu \in \Lambda_0} \mathcal{H}^\mu.$$

It is convenient to work with the restricted root system $\Sigma_0 = \Sigma|_{\mathfrak{a}} \setminus \{0\} \subset \mathfrak{a}^*$, the restricted Weyl group W_0 and the positive subsystem $\Sigma_0^+ = \Sigma^+|_{\mathfrak{a}} \cap \Sigma_0$. We regard Λ_0 as a lattice contained in \mathfrak{a}^* and we also define $\rho \in \mathfrak{a}^*$ by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_0^+} m_\alpha \alpha,$$

where m_α denotes the multiplicity of the restricted root α .

Let $\mathbf{D}(K/M)$ denote the algebra of K -invariant differential operators on K/M . Then one has an isomorphism, the Harish-Chandra isomorphism

$$D_0 \mapsto \Gamma_{D_0} : \mathbf{D}(K/M) \rightarrow \mathcal{I}(\mathfrak{a}),$$

where $\mathcal{I}(\mathfrak{a}) \approx \mathcal{S}(\mathfrak{a})^{W_0}$ denotes the algebra of W_0 -invariant elements in the symmetric algebra $\mathcal{S}(\mathfrak{a})$. We regard $\mathcal{S}(\mathfrak{a})$ and $\mathcal{I}(\mathfrak{a})$ as spaces of the polynomial functions on \mathfrak{a}^* , and we write $\mathcal{I}_d(\mathfrak{a}) \subset \mathcal{I}(\mathfrak{a})$ for the space of invariant polynomials of degree $\leq d$. The key property of the Harish-Chandra homomorphism is as follows:

Proposition 3.1. *If $D_0 \in \mathbf{D}(K/M)$ has order $\leq d$, then $\Gamma_{D_0} \in \mathcal{I}_d(\mathfrak{a})$. Moreover, D_0 acts on \mathcal{H}^μ by the scalar $\Gamma_{D_0}(\mu + \rho)$.*

4. DIFFERENTIAL OPERATORS ON GRASSMANNIANS

We specialize the results of the previous section to the case

$$K = SO_n(\mathbb{R}), \quad M = SO_k(\mathbb{R}) \times SO_l(\mathbb{R}), \quad Y = K/M,$$

assuming as before that $n = k + l$ and $k < l$. In this case the restricted root system is of type B_k and we choose the usual positive subsystem

$$\Sigma_0 = \{\pm e_i \pm e_j, \pm e_j\}, \quad \Sigma_0^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\},$$

where e_1, \dots, e_k are the unit vectors in $\mathfrak{a}^* \approx \mathbb{C}^k$. The root multiplicities are 1 for $e_i \pm e_j$, and $(l - k)$ for e_j . Thus we get

$$\rho = \frac{1}{2} \sum_i [2(k - i) + (l - k)] e_i = \sum_i \left[\frac{n}{2} - i \right] e_i,$$

which we note agrees with formula (1.5). The lattice cone Λ_0 consists of integer k -tuples $\mu = (\mu_1, \dots, \mu_k)$ satisfying

$$\mu_1 \geq \dots \geq \mu_k \geq 0, \mu_i \equiv \mu_j \pmod{2} \text{ for all } i, j.$$

The Weyl group W_0 acts on \mathbb{C}^k by sign changes and permutations of the coordinates. Thus,

$$W_0 \approx S_k \times (\mathbb{Z}/2)^k$$

and $\mathcal{I} = \mathcal{I}(\mathfrak{a})$ is the ring of polynomials in k variables, invariant under permutations and sign changes. Now Proposition 3.1 specializes as follows:

Proposition 4.1. *Let $D_0 \in \mathbf{D}(Y)$ be an $SO_n(\mathbb{R})$ -invariant differential operator of order $\leq d$, then D_0 acts on the space \mathcal{H}^μ by the scalar $\Gamma_{D_0}(\mu + \rho)$, where $\Gamma_{D_0}(\tau)$ is a polynomial of degree $\leq d$ in k variables $\tau = (\tau_1, \dots, \tau_k)$ which is invariant under permutations and sign changes.*

5. TWISTED DIFFERENTIAL OPERATORS

We continue with the notation $K = SO_n(\mathbb{R})$, $M = SO_k(\mathbb{R}) \times SO_l(\mathbb{R})$ and $Y = K/M$ as in the previous section, and also recall from Section 2 that

$$X = M'_{n,k}(\mathbb{R}), H = GL_k^+(\mathbb{R}), Y = X/H.$$

Let D be a $K \times H$ invariant differential operator of order d on X . Then D descends to a K -invariant differential operator D_0 on $Y = K/M$, whose Harish-Chandra image Γ_{D_0} belongs to the space $\mathcal{I}_d = \{p \in \mathcal{I} \mid \deg(p) \leq d\}$. Now the function $\gamma = \det(x^t x)$ is strictly positive on X , hence for any real (or complex) number s , we can consider the twisted differential operator $\gamma^{-s} \circ D \circ \gamma^s$, which descends to a K -invariant differential operator D_s on Y .

Theorem 5.1. *If D is a $K \times H$ invariant differential operator of order $\leq d$ on X , then the map $p_D : \mathbb{R} \rightarrow \mathcal{I}_d$ defined by $p_D(s) = \Gamma_{D_s}$ is a polynomial map in s of degree $\leq d$. More precisely, there exist elements $c_j = c_{j,D} \in \mathcal{I}_j$ such that $p_D(s) = \sum_{j=0}^d c_j s^{n-j}$.*

For the proof we need the following simple result:

Lemma 5.2. *Let V be a finite dimensional vector space and let $p : \mathbb{R} \rightarrow V$ be a smooth function such that for all t , $p(s + t) - p(s)$ is a polynomial of degree $\leq d - 1$ in s . Then p is a polynomial function of degree $\leq d$.*

Proof. Taking the d th order partial derivative of $p(s + t) - p(s)$ with respect to s , we deduce that for all s, t ,

$$p^{(d)}(s + t) - p^{(d)}(s) = 0.$$

Substituting $s = 0$, we conclude that $p^{(d)}(t) = p^{(d)}(0)$ is a constant for all t . It follows that $p^{(d+1)}(t) = 0$, whence p must be a polynomial of degree $\leq d$. \square

Proof of Theorem 5.1. We proceed by induction on d . The result is clear for $d = 0$, since an operator of order 0 is multiplication by a function, which by virtue of $K \times H$ -invariance reduces to a constant, and hence belongs to \mathcal{I}_0 . We now assume that $d > 0$ and that the result holds for all invariant operators of order $\leq d - 1$. Fix $t \in \mathbb{R}$ and define

$$D' = \gamma^{-t} \circ D \circ \gamma^t - D = \gamma^{-t} [D, \gamma^t].$$

Then D' is invariant by the first expression and is of order $\leq d - 1$ by the last expression. Therefore, by induction $p_{D'}(s)$ is a polynomial function of degree $\leq d - 1$ taking values in \mathcal{I}_{d-1} . However, $D'_s = D_{s+t} - D_s$ and so

$$p_{D'}(s) = \Gamma_{D_{s+t}} - \Gamma_{D_s} = p_D(s + t) - p_D(s).$$

Thus, we conclude that for all t , $p_D(s + t) - p_D(s)$ is a polynomial of degree $\leq d - 1$ in s . From the definition, it is easy to see that $p_D(s)$ is a smooth function of s . It follows from the previous lemma that p_D is a polynomial of degree $\leq d$ in s .

Thus, we have an expression $p_D(s) = \sum_{j=0}^d c_j s^{d-j}$ with $c_j \in \mathcal{I}_d$ and it remains to show that $c_j \in \mathcal{I}_j$ for all $j \leq d - 1$. We proceed once again by induction on d . The case $d = 0$ follows as in the previous paragraph. Let us consider the expansion of $h(s) = p_D(s + 1) - p_D(s)$:

$$h(s) = \sum_{j=0}^{d-1} c'_j s^{d-1-j} = \sum_{j=0}^d c_j (s + 1)^{d-j} - \sum_{j=0}^d c_j s^{d-j}.$$

It follows that $c'_j = (d - j)c_j +$ a combination of c_k for $k < j$. Inverting this we deduce that for $j \leq d - 1$, we have

$$c_j = \frac{1}{d - j} c'_j + \text{a combination of } c'_k \text{ for } k < j.$$

Now as explained in the previous paragraph, the polynomial h is of the form $p_{D''}$ for the invariant operator $D'' = \gamma^{-1} \circ D \circ \gamma - D$ of degree $d - 1$. Therefore, by induction $c'_j \in \mathcal{I}_j$ for all $j \leq d - 1$, and by the above expression for c_j we conclude $c_j \in \mathcal{I}_j$ as well. □

Proposition 5.3. *If D is a $K \times H$ invariant differential operator of order $\leq d$ on X , then there is a polynomial $p_D(s, \tau)$ of total degree $\leq d$ such that D_s acts on \mathcal{H}^μ by the scalar $p_D(s, \mu + \rho)$. Moreover, p_D is invariant under permutations $\tau_i \leftrightarrow \tau_j$ and sign changes $\tau_i \rightarrow -\tau_i$.*

Proof. This follows by combining Theorem 5.1 and Proposition 4.1. □

6. PROOF OF THEOREM 2.1

We apply the result of the previous section to the invariant differential operator $C = C_{1,0} = \det(x^t x) \det(\partial^t \partial)$, which has degree $2k$. By the previous proposition there exists a polynomial $p_C(s, \tau)$ of degree $\leq 2k$ such that C_s acts on \mathcal{H}^μ by the character $p_C(s, \mu + \rho)$.

Proposition 6.1. *C acts on \mathcal{R}_m^μ by the character $p_C\left(\frac{m}{2}, \mu + \rho\right)$.*

Proof. Let $f \in \mathcal{R}_m^\mu$, then $\gamma^{-m/2} f$ is H -invariant, hence $\gamma^{-m/2} f \in \mathcal{H}^\mu$. Evidently, $\gamma^{-s} C = C_s \gamma^{-s}$, thus by the previous corollary we get

$$\gamma^{-m/2} C f = C_{m/2} [\gamma^{-m/2} f] = p_C\left(\frac{m}{2}, \mu + \rho\right) [\gamma^{-m/2} f].$$

Multiplying both sides by $\gamma^{m/2}$ we get the desired result. □

In view of the above result, let us define

$$(6.1) \quad q(s, \tau) = p_C \left(\frac{s - \rho_1}{2}, \tau \right).$$

Then by the previous proposition C acts on \mathcal{R}_m^μ by the scalar $q(m + \rho_1, \mu + \rho)$, and Theorem 2.1 reduces to showing the following result.

Theorem 6.2. *We have $q = q_{1,0}$.*

We will prove this in two stages.

Proposition 6.3. *There is a constant a such that $q = aq_{1,0}$.*

Proof. Dividing q by $s - \tau_1$ we obtain the expression

$$(6.2) \quad q(s, \tau) = (s - \tau_1) h(s, \tau) + r(\tau).$$

Let $\mu \in \Lambda_0$ be arbitrary, then Lemma 2.2 implies that C acts on \mathcal{R}_m^μ by 0 for $m = \mu_1$. Thus we get

$$q(\mu_1 + \rho_1, \mu + \rho) = 0.$$

Substituting $s = \mu_1 + \rho_1$ and $\tau = \mu + \rho$ in formula (6.2) we deduce that, for all $\mu \in \Lambda_0$,

$$r(\mu + \rho) = 0.$$

Now it is easy to see by induction on k that the set $\Lambda_0 + \rho$ is Zariski dense in \mathbb{C}^k . It follows that $r(\tau)$ is identically 0, and thus $(s - t_1)$ divides q . Since p_C and hence q are invariant under permutations $\tau_i \leftrightarrow \tau_j$ and sign changes $\tau_i \rightarrow -\tau_i$, it follows that $(s \pm \tau_i)$ divides q for all i . Thus $q_{1,0} = \prod_{i=1}^k (s^2 - \tau_i^2)$ divides q , but since $\deg q \leq 2k$ we must have $q(s, \tau) = aq_{1,0}$ for some constant a . □

Now Theorem 2.1 reduces to proving $a = 1$. For the proof we need a simple result on polynomials. In order to formulate and prove this result it is convenient to proceed in somewhat greater generality, thus we temporarily introduce the following notation:

$$x = (x_1, \dots, x_n), \quad \partial_i = \partial/\partial x_i, \quad \partial = (\partial_1, \dots, \partial_n).$$

Let $g(x)$ be a polynomial, let s be an additional variable and define $g^s(x) = [g(x)]^s$, considered as an analytic function on a suitable open set in (x, s) . Let $f(x)$ be a homogenous polynomial of degree d and regard $f(\partial)$ as a constant coefficient differential operator of order d . We apply $f(\partial)$ to g^s and then multiply the result by g^{d-s} to obtain the expression

$$g^{d-s} (f(\partial) g^s).$$

It is easy to see that this expression is a polynomial in (x, s) . We now have the following result where we use the symbol \sim to denote equality modulo lower degree terms in s .

Lemma 6.4. *In the setting of the previous paragraph, we have*

$$g^{d-s} (f(\partial) g^s) \sim s^d f(\nabla g),$$

where $\nabla g = (\partial_1 g, \dots, \partial_n g)$ denotes the gradient of g .

Proof. It suffices to prove the result when f is a monomial. Thus we need to show that

$$g^{d-s} [\partial_{i_1} \cdots \partial_{i_d} g^s] \sim s^d (\partial_{i_1} g) \cdots (\partial_{i_d} g),$$

which follows easily by the chain rule and induction on d . □

We now return to the general discussion.

Proposition 6.4. *The constant a in Proposition 6.3 is 1.*

Proof. Let $\mathbf{1}$ denote the constant function in \mathcal{H}^0 . By the previous proposition we have

$$(6.3) \quad C_s \mathbf{1} = q(2s + \rho_1, \rho) \mathbf{1} = aq_{1,0}(2s + \rho_1, \rho) \mathbf{1} \sim (a4^k) s^{2k} \mathbf{1}.$$

On the other hand, recalling the definition of C_s , we get

$$(6.4) \quad \gamma^{2k-1} [C_s \mathbf{1}] = \gamma^{2k-s} [\det(\partial^t \partial)(\gamma^s)] \sim s^{2k} \det(y^t y),$$

after applying the previous lemma and writing $y = \nabla \gamma$.

Now multiplying (6.3) by γ^{2k-1} and comparing with (6.4) we get

$$(6.5) \quad s^{2k} \det(y^t y) \sim s^{2k} (a4^k) \gamma^{2k-1}.$$

Let I_k denote the $k \times k$ identity matrix, and let J denote the $n \times k$ matrix $\begin{bmatrix} I_k \\ 0 \end{bmatrix}$, then we have $J^t J = I_k, \gamma(J) = 1$. Moreover, an easy calculation shows that

$$y(J) = \nabla \gamma(J) = 2J.$$

Thus evaluating both sides of formula (6.5) at $x = J$ we get

$$\det(4I_k) s^{2k} \sim (a4^k) s^{2k}.$$

Since $\det(4I_k) = 4^k$, this proves $a = 1$ as desired. □

As noted above, this proves Theorem 6.2 and hence Theorem 2.1.

7. APPENDIX

We apply the results of this paper to obtain identities in the algebra $\mathbf{D}(Y)$ of K -invariant differential operators on the symmetric space $Y = K/M$, where

$$K = SO_n(\mathbb{R}), \quad M = SO_k(\mathbb{R}) \times SO_{n-k}(\mathbb{R}).$$

As noted in Section 5 above we may also write $Y = X/H$, where

$$X = M'_{n,k}(\mathbb{R}), \quad H = GL_k^+(\mathbb{R}).$$

Thus, if D is a $K \times H$ invariant differential operator on X , then for any real (or complex) number s , the twisted differential operator $\gamma^{-s} \circ D \circ \gamma^s$ descends to an operator D_s in $\mathbf{D}(Y)$. This applies in particular to the element $C_{d,d'} = \gamma^d \circ L^{d+d'} \circ \gamma^{d'}$ in (1.4), and we write

$$C_{d,d',s} = \gamma^{-s} \circ C_{d,d'} \circ \gamma^s = \gamma^{d-s} \circ L^{d+d'} \circ \gamma^{d'+s}.$$

On the other hand, if z belongs to the center $\mathcal{Z}(\mathfrak{k})$ of the enveloping algebra $\mathcal{U}(\mathfrak{k})$ of $\mathfrak{k} = \mathfrak{so}(n)$, then z gives rise to an operator $\mathbf{D}(Y)$ as well. A number of authors have given explicit construction of elements in $\mathcal{Z}(\mathfrak{k})$. In particular, we refer the reader to [11, Sections 6 and 7] for a certain element $D^\dagger(\lambda) = D_F(\lambda) \in \mathcal{Z}(\mathfrak{k})$ defined in terms of a noncommuting determinant, and shown in [11, p. 487] to coincide with another element constructed in terms of the Sklyanin determinant in [15].

Theorem 7.1. *The following identity holds in $\mathbf{D}(Y)$,*

$$C_{d,d',s} = a_{d,d'}(s) \prod_{j=0}^{d+d'-1} D^\dagger(2(s+d'-j) + n/2 - 1),$$

where $a_{d,d'}(s)$ is a certain explicit constant.

Proof. Writing $C = C_{1,0} = \gamma \circ L$ and $C_s = \gamma^{1-s} \circ L \circ \gamma^s$ we have a factorization:

$$C_{d,d',s} = C_{s-d+1} \cdots C_{s+d'} = \prod_{j=0}^{d+d'-1} C_{s+d'-j}.$$

Thus it suffices to show that for some explicit constant $a(s)$ one has

$$(7.1) \quad C_s = a(s) D^\dagger(2s + n/2 - 1).$$

As before, let \mathcal{H} denote the space of K -finite functions on K/M , and let $\mathcal{H} = \bigoplus_{\mu \in \Lambda_0} \mathcal{H}^\mu$ denote its (multiplicity-free) decomposition into irreducible K -modules. It suffices to show that the two sides of (7.1) agree on each \mathcal{H}^μ . Let $\rho_i = n/2 - i$ as before, then by (6.1), C_s acts on \mathcal{H}^μ by the scalar

$$(7.2) \quad p_C(s, \mu + \rho) = q(2s + \rho_1, \mu + \rho) = \prod_{i=1}^k \left[(2s + \rho_1)^2 - (\mu_i + \rho_i)^2 \right].$$

Now the action of $D^\dagger(\lambda)$ on \mathcal{H}^μ is given via the Harish-Chandra homomorphism,

$$\bar{\gamma} : \mathcal{Z}(\mathfrak{k}) \rightarrow \mathcal{S}(\mathfrak{h}),$$

where \mathfrak{h} is a maximal toral subalgebra in \mathfrak{k} , followed by evaluation at

$$(\mu_1 + \rho_1, \dots, \mu_k + \rho_k, \rho_{k+1}, \dots, \rho_p),$$

where $p = \lfloor n/2 \rfloor$. By Theorem 7.2 of [11] we see that $D^\dagger(\lambda)$ acts on \mathcal{H}^μ by

$$\begin{aligned} & \prod_{i=k+1}^p (\lambda^2 - \rho_i^2) \prod_{i=1}^k \left[\lambda^2 - (\mu_i + \rho_i)^2 \right] && \text{if } n \text{ is even,} \\ (\lambda - 1/2) \prod_{i=k+1}^p (\lambda^2 - \rho_i^2) \prod_{i=1}^k \left[\lambda^2 - (\mu_i + \rho_i)^2 \right] && \text{if } n \text{ is odd.} \end{aligned}$$

Comparing this with (7.2) we obtain (7.1), which implies the result. □

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY

E-mail address: sahi@math.rutgers.edu

URL: <http://www.math.rutgers.edu/~sahi>