A GEOMETRIC CONSTRUCTION OF TYPES
FOR THE SMOOTH REPRESENTATIONS
OF PGL(2) OVER A LOCAL FIELD

PAUL BROUSSOUS

Dedicated to Guy Henniart on his 60th birthday

Abstract. We show that almost all (Bushnell and Kutzko) types of PGL(2, F), F a non-Archimedean locally compact field of odd residue characteristic, naturally appear in the cohomology of finite graphs.

1. Introduction

Let F be a non-Archimedean locally compact field and G the group PGL(2, F). We assume that the residue characteristic of F is not 2. In previous works ([2], [3]) we defined a tower of directed graphs (X_n)_{n \geq 0} lying G-equivariantly over the Bruhat-Tits tree X of G. We proved the two following facts:

Theorem 1 ([3], Theorem (3.2.4), page 502). Let (\pi, V) be a non-spherical generic smooth irreducible representation. Then (\pi, V) is a quotient of the cohomology space with compact support H^1_c(\hat{X}_n(\pi), \mathbb{C}) where n(\pi) is the conductor of \pi.

Theorem 2 ([3], Theorem (5.3.2), page 512). If (\pi, V) is a supercuspidal smooth irreducible representation of G, then we have \dim \text{Hom}_G[H^1_c(\hat{X}_n(\pi), \mathbb{C}), V] = 1.

In this paper we make the G-module structure of H^1_c(\hat{X}_n, \mathbb{C}) more explicit for all n \geq 0, and draw some interesting consequences.

Let us fix an edge [s_0, s_1] of X and denote by K_0 and K_1 the stabilizers in G of s_0 and [s_0, s_1], respectively. Then K_0 and K_1 form a set of representatives of the two conjugacy classes of maximal compact subgroups in G. If n is even, we have a G-equivariant mapping p_n : \hat{X}_n \rightarrow X which respects the graph structures. We denote by \Sigma_n the subgraph p_n^{-1}([s_0, s_1]). If n is odd, then after passing to the first barycentric subdivisions, we have a G-equivariant mapping p_n : \hat{X}_n \rightarrow X which respects the graph structures. We denote by \Sigma_n the subgraph p_n^{-1}(S(s_0, 1/2)), where S(s_0, 1/2) denotes the set of points x in X such that d(x, s_0) \leq 1/2 (here d is the natural distance on the standard geometric realization of X, normalized in such a way that d(s_0, s_1) = 1).

Then for all n, \Sigma_n is a finite graph, equipped with an action of K_1 if n is even, and K_0 if n is odd. So the cohomology spaces H^1_c(\Sigma_n, \mathbb{C}) provide finite dimensional smooth representations of K_1 or K_0, according to the parity of n.
For each \( n \geq 0 \), we define a finite set \( \mathcal{P}_n \) of pairs \((\mathcal{K}, \lambda)\) formed of a maximal compact subgroup \( \mathcal{K} \in \{ \mathcal{K}_0, \mathcal{K}_1 \} \) and of an irreducible smooth representation of \( \mathcal{K} \). By definition we have \((\mathcal{K}, \lambda) \in \mathcal{P}_n \) if and only if there exists \( k \in \{0, 1, \ldots, n\} \) such that \((\mathcal{K}, \lambda)\) is an irreducible constituent of the representation \( H^1(\Sigma_k, \mathbb{C}) \). For \((\mathcal{K}, \lambda) \in \mathcal{P}_n \) and \( k \leq n \), we denote by \( m^k_\lambda \) the multiplicity of \( \lambda \) in \( H^1_c(\Sigma_k, \mathbb{C}) \) and we set \( m_n,\lambda = m_\lambda = m^0_\lambda + \cdots + m^n_\lambda \). Note that \( m_\lambda \) depends on \((\mathcal{K}, \lambda)\) and \( n \).

The main results of this article are the following.

Theorem A. For all \( n \geq 0 \), we have the direct sum decomposition

\[
H^1_c(\tilde{X}_n, \mathbb{C}) = \text{St}_G \oplus \bigoplus_{(\mathcal{K}, \lambda) \in \mathcal{P}_n} (c\text{-ind}^G_{\mathcal{K}} \lambda)^{m_\lambda}.
\]

(Here \( \text{St}_G \) denotes the Steinberg representation of \( G \)).

Theorem B. For all \( n \geq 0 \), any element of \( \mathcal{P}_n \) is:

a) either a type in the sense of Bushnell and Kutzko’s type theory [6], which is not a type for the unramified principal series,

b) or a pair of the form \((\mathcal{K}_0, \chi \circ \det \otimes \text{St}_{\mathcal{K}_0})\), where \( \chi \) is a smooth character of \( F^\times \) of order 2, trivial on the group of 1-units in \( F^\times \), and \( \text{St}_{\mathcal{K}_0} \) is the representation inflated from the Steinberg representation of \( \text{PGL}(2) \) of the residue field of \( F \),

c) or the pair \((\mathcal{K}_1, 1_{\mathcal{K}_1})\), where 1 denotes a trivial character.

It is worth noting that if the pairs of cases a) and b) are not types, they are typical in the sense of [9].

Corollary C. Let \( n \geq 0 \). If \((\mathcal{K}, \lambda) \in \mathcal{P}_n \) is a cuspidal type, then \( m_{n,\lambda} = 1 \).

Indeed this follows from Theorems 2 and A using Frobenius reciprocity for compact induction.

By Theorem 1, any Bernstein component of \( G \), different from the unramified principal series component, must have a type in \( \mathcal{P}_n \) for \( n \) large enough. Hence the graphs \( \tilde{X}_n, n \geq 0 \), provide a geometric construction of types for almost all Bernstein components of \( G \).

We conjecture that if \((\mathcal{K}, \lambda) \in \mathcal{P}_n \) is a type of \( G \), then \( m_\lambda = 1 \).

Finally, let us observe that this construction gives a new proof that the irreducible supercuspidal representations of \( G \) are obtained by compact induction. Our proof differs from Kutzko’s original proof ([10], also see [4]) only at the exhaustion steps. Indeed our “supercuspidal” types are the same as Kutzko’s, but we prove that any irreducible supercuspidal representation contains such a type by using an argument based on [2] and [3], that is mainly on the existence of the new vector.

The article is organized as follows. The proof of Theorem A relies first on combinatorial properties of the graphs \( \tilde{X}_n \) that are stated and proved in §2. Using these combinatorial properties and some homological arguments, we show in §3 how to relate the cohomology of \( \tilde{X}_n \) to that of \( \tilde{X}_{n-1} \). The irreducible components of \( H^1(\Sigma_n) \) are determined in §4 when \( n \) is even, and in §5 and §6 when \( n \) is odd. A synthesis of the arguments of paragraphs 2 to 6, leading to a proof of Theorems A and B, is given in §7.

We shall assume that the reader is familiar with the language of Bushnell and Kutzko’s type theory [6] and with the language of strata ([5], [4]).
2. Notation

We shall denote by
F a non-Archimedean non-discrete locally compact field,
\( \sigma \) its valuation ring,
p the maximal ideal of \( \sigma \),
\( \varpi \) the choice of a uniformizer of \( \sigma \),
k = \( \sigma / p \) the residue field of \( F \),
p the characteristic of \( k \),
q = \( p^f \) the cardinal of \( k \),
G the group \( \text{PGL}(2, F) \).
t the image of the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \) in \( G \).

The results of this article are obtained by the following.

Hypothesis. The characteristic of \( k \) is not 2

We shall often define an element, a subset, or a subgroup of \( G \) by giving a (set of) representative(s) in \( \text{GL}(2, F) \).

We write \( T \) for the diagonal torus of \( G \) and \( B \supset T \) for the upper standard \( Borel \) subgroup. We denote by \( T_0 \) the maximal compact subgroup of \( T \), i.e., the set of matrices with coefficients in \( \sigma \times \sigma \), and by \( T_n \) the subgroup of matrices with coefficients in \( 1 + p^n \), \( n > 0 \).

Let \( k, l \) be integers satisfying \( k + l \geq 0 \). Then
\[ A(k, l) = \begin{pmatrix} \sigma & p^l \\ p^k & \sigma \end{pmatrix} \]
is an \( \sigma \)-order of \( M(2, F) \). We denote by \( \Gamma_0(k, l) \) the image in \( G \) of its group of units.

There are two conjugacy classes of maximal compact subgroups of \( G \). The first one has representative \( K = \Gamma_0(0, 0) \). A representative \( I \) of the second one is the semidirect product of the cyclic group generated by \( \Pi = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \) with the Iwahori subgroup \( I = \Gamma_0(1, 0) \).

The group \( K \) is filtered by the normal compact open subgroups
\[ K_n = \begin{pmatrix} 1 + p^n & p^n \\ p^n & 1 + p^n \end{pmatrix}, \quad n \geq 1. \]
The group \( I \) is filtered by the normal compact subgroups \( I_n, n \geq 1 \), defined by
\[ I_{2k+2} = \begin{pmatrix} 1 + p^{k+1} & p^{k+1} \\ p^{k+2} & 1 + p^{k+1} \end{pmatrix}, \quad I_{2k+1} = \begin{pmatrix} 1 + p^{k+1} & p^k \\ p^{k+1} & 1 + p^{k+1} \end{pmatrix}, \quad k \geq 0. \]
The subgroups \( I_n, n \geq 1 \), are normalized by \( \Pi \).

We denote by \( X \) the Bruhat-Tits building of \( G \). This is a uniform tree with valency \( q+1 \). As a \( G \)-set and as a simplicial complex \( X \) identifies with the following complex. Its vertices are the homothety classes \( \left[ L \right] \) of full \( \sigma \)-lattices \( L \) in the vector space \( V = F^2 \). Two vertices \( \left[ L \right] \) and \( \left[ M \right] \) define an edge if and only if there exists a basis \( (e_1, e_2) \) of \( V \) such that, up to homothety, we have \( L = \sigma e_1 \oplus \sigma e_2 \) and \( M = \sigma e_1 \oplus pe_2 \).

The vertices of the standard apartment (i.e., the apartment stabilized by \( T \)) are the \( s_k = [\sigma \oplus p^k], k \in \mathbb{Z} \). The element \( t_{\varpi} \) acts as \( t_{\varpi} s_k = s_{k+1}, k \in \mathbb{Z} \). The maximal compact subgroup \( K \) is the stabilizer of \( s_0 \) and \( I \) (resp. \( I \)) is the global stabilizer (resp. pointwise stabilizer) of the edge \( [s_0, s_1] \). If \( l \geq k \), the pointwise stabilizer of the segment \( [s_k, s_l] \) is \( \Gamma_0(l, -k) \).
3. Combinatorics of $\tilde{X}_n$

We recall the construction of the directed graphs $\tilde{X}_n$, $n \geq 1$.

For any integer $k \geq 1$, an oriented $k$-path in $X$ is an injective sequence of vertices $(s_i)_{i=0,\ldots,k}$ in $X$ such that, for $k = 0, \ldots, k-1$, $\{s_i, s_{i+1}\}$ is an edge in $X$. We shall allow the index $i$ to run over any interval of integers of length $k + 1$. Let us fix an integer $n \geq 1$. The directed graph $\tilde{X}_n$ is constructed as follows. Its edge set (resp. vertex set) is the set of oriented (resp. vertex) $k$-paths in $X$. If $a = \{s_0, s_1, \ldots, s_{n+1}\}$ is an edge of $\tilde{X}_n$, its head (resp. its tail) is $a^+ = \{s_1, s_2, \ldots, s_{n+1}\}$ (resp. $a^- = \{s_0, s_1, \ldots, s_n\}$). The graphs we obtain in this way are actually simplicial complex. The group $G$ acts on $\tilde{X}_n$ in an obvious way; the action preserves the structure of directed graph.

When $n = 2m$ is even, we have a natural simplicial projection $p = p_n : \tilde{X}_n \to X$ given on vertices by $p(s_m, \ldots, s_0, s_0, \ldots, s_m) = s_0$. It is $G$-equivariant. Let $c = \{s_0, t_0\}$ be an edge of $X$. We are going to describe the finite simplicial complex $p^{-1}(e)$. An edge in $\tilde{X}_n$ above the edge $e$ corresponds to an oriented $(2m+1)$-path of one of the following forms:

i) $(s_m, s_{m+1}, \ldots, s_0, t_0, \ldots, t_{m-1}, t_m)$,

ii) $(s_m, t_{m-1}, \ldots, s_0, t_0, \ldots, s_{m-1}, s_m)$.

Let $C_{2m-1}(e)$ be the set of $(2m-1)$-paths $c = (s_{m+1}, \ldots, s_0, t_0, \ldots, t_{m-1})$. We say that $c \in C_{2m-1}(e)$ lies above $e$. Fix $c \in C_{2m-1}(e)$ and consider the simplicial subcomplex $\tilde{X}_{2m}[e, c]$ of $\tilde{X}_{2m}$ whose edges correspond to the $(2m+1)$-paths of the form

$$(a, s_{m+1}, \ldots, s_0, t_0, \ldots, t_{m-1}, b).$$

So $a$ (resp. $b$) can be any neighbour of $s_{m+1}$ (resp. $t_{m-1}$) different from $s_{m+2}$ (resp. $t_{m-1}$), with the convention that $s_1 = t_0$ and $t_1 = s_0$. The simplicial complex $\tilde{X}_{2m}[e, c]$ is connected. It is indeed isomorphic to the complete bipartite graph with sets of vertices:

$$\{a \mid a \text{ neighbour of } s_{m+1}, a \neq s_{m+2}\} \text{ and } \{b \mid b \text{ neighbour of } t_{m-1}, b \neq t_{m-2}\}.$$

**Lemma 3.1.** Let $e$ and $e'$ be two edges of $X$ and $c \in C_{2m-1}(e)$, $c' \in C_{2m-1}(e')$. Then $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] \neq \emptyset$ if and only if we are in one of the following cases:

i) $e = e'$ and $c = c'$ (so that $\tilde{X}_{2m}[e, c] = \tilde{X}_{2m}[e', c']$);

ii) $e \cap e'$ is reduced to one vertex of $X$ and $c \cup c'$ is an oriented $2n$-path in $X$. In that case $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$ is reduced to the vertex of $\tilde{X}_{2m}$ corresponding to the $2n$-path $c \cup c'$.

**Proof.** If $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] \neq \emptyset$, then $e \cap e' = p(\tilde{X}_{2m}[e, c]) \cap p(\tilde{X}_{2m}[e', c']) \neq \emptyset$. Assume first that $e = e'$. Then $c = c'$, for if $c \neq c'$, then $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] = \emptyset$; indeed, if $\tilde{s}$ is a vertex of $\tilde{X}_{2m}[e, c]$, then it determines $c$ uniquely. Now assume that $e \cap e'$ is a vertex. Let $\tilde{s} \in \tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$. Then $\tilde{s}$ contains $e$ and $e'$ as subsequences, with $c \neq c'$. So by a length argument $s = c \cup c'$. Conversely if $c \cup c'$ is an oriented $2n$-path, then $c \cup c'$ is a vertex of $\tilde{X}$ lying in $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$. □

**Corollary 3.2.** For any edge $e$ of $X$, the connected components of $p^{-1}(e)$ are the $\tilde{X}_{2m}[e, c]$, where $c$ runs over $C_{2m-1}(e)$.

Define a 1-dimensional simplicial complex $Y_{2m-1}$ in the following way. Its vertices are the connected components $\tilde{X}_{2m}[e, c]$, where $e$ runs over the edges of $X$ and $c$
over \( C_{2m-1}(e) \), and two vertices \( \tilde{X}_{2m}[e,c] \) and \( \tilde{X}_{2m}[e',c'] \) are linked by an edge if they intersect. Note that \( Y_{2m-1} \) is naturally a \( G \)-simplicial complex.

**Corollary 3.3.** As a \( G \)-simplicial complex, \( Y_{2m-1} \) is canonically isomorphic to the complex \( \tilde{X}_{2m-1} \).

**Lemma 3.4.** For all \( n \geq 0 \), the simplicial complex \( X_n \) is connected.

**Proof.** When \( n \) is even this is \( [2, \text{Lemma (4.1)}] \). For odd \( n \), the proof is similar. \( \square \)

Assume that \( m \geq 1 \). We say that an edge of \( \tilde{X}_{2m-1} \) lies above a vertex \( s_0 \) of \( X \) if as an oriented \( 2m \)-path it has the form \((s_{-m}, \ldots, s_0, \ldots, s_m)\). For any vertex \( s_0 \) of \( X \) we write \( \tilde{X}_{2m-1}[s_0] \) for the subsimplicial complex of \( \tilde{X}_{2m-1} \) formed of the edges lying above \( s_0 \).

**Lemma 3.5.** When \( m = 1 \) the simplicial complexes \( \tilde{X}_{2m-1}[s_0] = \tilde{X}_1[s_0] \) are connected.

**Proof.** We may identify the neighbour vertices of \( s_0 \) in \( X \) with the points of the projective line \( \mathbb{P}^1(M) \cong \mathbb{P}^1(k) \), where \( s_0 = [M] \) and \( M = M/pM \). The vertices of \( \tilde{X}_1[s_0] \) are the oriented 1-paths \((s_0, x)\), \((y, s_0)\), \( x, y \in \mathbb{P}^1(M) \). Two oriented 1-paths of the form \((x, s_0)\) and \((s_0, y)\) are linked by the edge \((x, s_0, y)\). Let \((x, s_0), (y, s_0)\) be two oriented 1-paths with \( x \neq y \). Since \( |\mathbb{P}^1(k)| \geq 3 \), there exists \( z \in \mathbb{P}^1(M) \) distinct from \( x \) and \( y \). Then \((x, s_0)\) is linked to \((s_0, z)\) via the path \((x, s_0, z)\) and \((s_0, z)\) is linked to \((y, s_0)\) via the path \((y, s_0, z)\). For vertices of the form \((s_0, x)\), \((s_0, y)\) the proof is similar. \( \square \)

We now assume that \( m > 1 \). We write \( C_{2m-2}(s_0) \) for the set \((2m-2)\)-paths of the form \((s_{-m+1}, \ldots, s_0, \ldots, s_{m-1})\). For any \( c \in C_{2m-2}(s_0) \), we consider the subsimplicial complex \( \tilde{X}_{2m-1}[s_0, c] \) of \( \tilde{X}_{2m-1} \) whose edges corresponds to the \( 2m \)-paths of the form \((a, s_{-m+1}, \ldots, s_0, s_{n-1}, b)\). We have results similar to Lemma 3.1 and Corollaries 3.2 and 3.3.

**Lemma 3.6.** i) For any vertex \( s_0 \) of \( X \) and for \( c \in C_{2m-2}(s_0) \), \( \tilde{X}_{2m-1}[s_0, c] \) is connected. It is indeed isomorphic to a complete bipartite graph constructed on two sets of \( |k| \) elements.

ii) Let \( s \) and \( s' \) be vertices of \( X \), \( c \in C_{2m-2}(s) \) and \( c' \in C_{2m-2}(s') \). Then \( \tilde{X}_{2m-1}[s,c] \cap \tilde{X}_{2m-1}[s',c'] = \emptyset \) if and only if \( s = s' \) and \( c = c' \), or \( \{s, s'\} \) is an edge in \( X \) and \( c \cup c' \) is an oriented \( 2n-1 \)-path. In this last case \( \tilde{X}_{2m-1}[s,c] \cap \tilde{X}_{2m-1}[s',c'] = \{s\} \), where the vertex \( s \) of \( \tilde{X}_{2m-1} \) corresponds to the \((2n-1)\)-path \( c \cup c' \).

iii) For any vertex \( s \) of \( X \), the connected components of \( \tilde{X}_{2m-1}[s] \) are \( \tilde{X}_{2m-1}[s,c] \), \( c \) running over \( C_{2m-2}(s) \).

We can consider the 1-dimensional simplicial complex \( Z_{2m-2} \) whose vertices are the connected components \( \tilde{X}_{2m-1}[s,c] \), \( s \) running over the vertices of \( X \) and \( c \) over \( C_{2m-2}(s) \), and where two connected components define an edge if and only if they intersect. Note that \( Z_{2m-2} \) is naturally a \( G \)-simplicial complex.

**Corollary 3.7.** As a \( G \)-simplicial complex \( Z_{2m-2} \) is isomorphic to \( X_{2n-2} \).

4. The cohomology of \( \tilde{X}_n \): First reductions

If \( \Sigma \) is a locally finite 1-dimensional simplicial complex, we write \( \Sigma^0 \) (resp. \( \Sigma^{(1)}, \Sigma^1 \)) for its set of vertices (resp. non-oriented edges, oriented edges). We
let \( C_0(\Sigma) \) (resp. \( C_1(\Sigma) \)) denote the \( \mathbb{C} \)-vector space with basis \( \Sigma^0 \) (resp. \( \Sigma^1 \)). We define the space \( C_0^0(\Sigma, \mathbb{C}) = C_0(\Sigma) \) (resp. \( C_1^0(\Sigma, \mathbb{C}) = C_1(\Sigma) \)) of oriented simplicial 0-cochains (resp. 1-cochains) with compact support by:

\[
C_0^0(\Sigma) = \text{space of all linear forms } f : C_0(\Sigma) \to \mathbb{C} \text{ such that } f(s) = 0 \text{ except for a finite number of vertices } s;
\]

\[
C_1^0(\Sigma) = \text{space of all linear forms } \omega : C_1(\Sigma) \to \mathbb{C} \text{ such that } \omega([a, b]) = 0 \text{ except for a finite number of oriented edges } [a, b] \text{ and } \omega([a, b]) = -\omega([b, a]).
\]

We set \( C_i^k(\Sigma) = 0 \) for \( k \in \mathbb{Z}\setminus\{0, 1\} \) and define a coboundary map \( d : C_0^0(\Sigma) \to C_1^0(\Sigma) \) by \( df([a, b]) = f(b) - f(a) \). The cohomology of the cochain complex \((C_\bullet^0(\Sigma), d)\) computes the cohomology with compact support \( H_i^0(\Sigma, \mathbb{C}) = H_i^0(\Sigma) \) (of the standard geometric realization of) \( \Sigma \). If \( \Sigma \) is acted upon by a group \( H \) whose action is simplicial, then \((C_\bullet^0(\Sigma), d)\) is in a straightforward way a complex of \( H \)-modules, and its cohomology computes \( H_i^1(\Sigma) \) as a \( H \)-module. When \( \Sigma \) is finite we drop the subscripts \( c \).

Since the stabilizer of a finite number of vertices of \( X \) is open in \( G \), we see that for \( n \geq 1 \), the \( G \)-modules \( C_0^0(\tilde{X}_n), C_1^0(\tilde{X}_n) \) and therefore \( H_i^1(\tilde{X}_n) \) are smooth.

In the sequel we fix \( m \geq 1 \) and we abbreviate \( \tilde{X}_{2m} = \tilde{X} \). The disjoint union \( \tilde{X}^1 = \bigcup_{e \in X^{(1)}} \tilde{X}_e \), where \( \tilde{X}_e = p^{-1}(e) \), induces an isomorphism:

\[
\begin{align*}
C_i^0(\tilde{X}) &\cong \bigoplus_{e \in X^{(1)}} C_i^0(\tilde{X}_e), \\
\omega &\mapsto (\omega|_{C_i^0(\tilde{X}_e)})_{e \in X^{(1)}}.
\end{align*}
\]

Similarly, the non-disjoint union \( \tilde{X}_e^0 = \bigcup_{e \in X^{(1)}} \tilde{X}_e^0 \) induces an injection:

\[
\begin{align*}
J : C_i^0(\tilde{X}) &\hookrightarrow \bigoplus_{e \in X^{(1)}} C_i^0(\tilde{X}_e), \\
f &\mapsto (f|_{C_i^0(\tilde{X}_e)})_{e \in X^{(1)}}.
\end{align*}
\]

We have the following commutative diagram of \( G \)-modules:

\[
\begin{array}{cccccc}
H_0^0(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} H_0^0(\tilde{X}_e) & \overset{\varphi}{\longrightarrow} & \text{coker } j \\
0 & \longrightarrow & C_0^0(\tilde{X}) & \overset{j}{\longrightarrow} & \bigoplus_{e \in X^{(1)}} C_0^0(\tilde{X}_e) & \overset{\text{coker } j}{\longrightarrow} & 0 \\
0 & \longrightarrow & C_1^0(\tilde{X}) & \overset{d}{\longrightarrow} & \bigoplus_{e \in X^{(1)}} C_1^0(\tilde{X}_e) & \longrightarrow & 0 \\
0 & \longrightarrow & H_1^1(\tilde{X}) & \longrightarrow & \bigoplus_{e \in X^{(1)}} H_1^1(\tilde{X}_e) & \longrightarrow & 0
\end{array}
\]

Here, for \( e \in X^{(1)} \), \( d_e \) denote the coboundary map \( C_0^0(\tilde{X}_e) \to C_1^0(\tilde{X}_e) \). Since \( \tilde{X} \) is connected (Lemma 4.1) and non-compact, we have \( H_0^0(\tilde{X}) = 0 \). So the snake lemma gives the kernel-cokernel exact sequence:

\[
0 \to \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \to \text{coker } j \to H_1^1(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1(\tilde{X}_e) \to 0,
\]

that is,

\[
0 \to \text{coker } j/\varphi(\bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e)) \to H_1^1(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1(\tilde{X}_e) \to 0.
\]
Abbreviate \( Y = Y_{2m-1} \).

**Lemma 4.1.** We have a canonical isomorphism of \( G \)-modules
\[
\text{coker} j / \varphi \left( \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \right) \simeq H^1_c(Y).
\]

**Proof.** From Corollary 3.2 we have
\[
\bigoplus_{e \in X^{(1)}} C^0(\tilde{X}_e) = \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C_{2m-1}(e)} C^0(\tilde{X}_{2m}[e, c]).
\]
So the map \( j \) is given by \( f \mapsto \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C_{2m-1}(e)} f_{e, c} \), where \( f_{e, c} = f|_{C_0(\tilde{X}_{2m}[e, c])} \).

Consider the \( G \)-equivariant morphism of vector spaces
\[
\psi : \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C_{2m-1}(e)} C^0(\tilde{X}_{2m}[e, c]) \to C^1(Y)
\]
given as follows. If \( \sigma \) is an oriented edge of \( Y \), then there exist uniquely determined edges \( e_o, e_o' \) of \( X \), \( c_o \in C_{2m-1}(e_o), c_o' \in C_{2m-1}(e_o') \), such that \( \sigma \) corresponds to the intersection \( \tilde{X}_{2m}[e_o, c_o] \cap \tilde{X}_{2m}[e_o', c_o'] = \{ s_o \} \), \( s_o \in \tilde{X}^0 \). We then set
\[
\psi([f_{e, c}, e, c])(\sigma) = f_{e_o, c}(s_o) - f_{e_o', c}(s_o).
\]
Then \( \psi \) is surjective and its kernel is precisely \( j(C^0_c(\tilde{X})) \). So we may identify \( \text{coker} j \) with \( C^1_c(Y) \). From Corollary 3.2, we have
\[
\bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) = \bigoplus_{e \in X^{(1)}} \bigoplus_{c \in C_{2m-1}(e)} H^0(\tilde{X}_{2m}[e, c])
\]
so that we may identify \( \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \) with \( C^0_c(\tilde{Y}) \). Under our identifications the map \( \varphi : \bigoplus_{e \in X^{(1)}} H^0(\tilde{X}_e) \to \text{coker} j \) corresponds to the coboundary map \( d : C^0_c(Y) \to C^1_c(Y) \), and we are done since all our identifications are \( G \)-equivariant. \( \square \)

**Proposition 4.2.** For \( m \geq 1 \), we have an isomorphism of \( G \)-modules:
\[
H^1_c(\tilde{X}_{2m}) \simeq H^1_c(\tilde{X}_{2m-1}) \oplus c\text{-ind}_{K_{e_o}}^G H^1(\tilde{X}_{e_o})
\]
for any edge \( e_o \) of \( X \) and where \( K_{e_o} \) denotes the stabilizer of \( e_o \) in \( G \).

**Proof.** From the short exact sequence (4.3) and Lemma 4.1 we have the exact sequence of \( G \)-modules:
\[
0 \to H^1_c(Y) \to H^1_c(\tilde{X}) \to \bigoplus_{e \in X^{(1)}} H^1(\tilde{X}_e) \to 0.
\]
Since \( G \) acts transitively on the edges of \( X \), \( \bigoplus_{e \in X^{(1)}} H^1(\tilde{X}_e) \) identifies with the compactly induced representation \( c\text{-ind}_{K_{e_o}}^G H^1(\tilde{X}_{e_o}) \). Moreover, this induced representation is projective in the category of smooth complex representations of \( G \). This is classical and follows from Frobenius reciprocity for compact induction together with the fact that the category of smooth \( K_{e_o} \)-modules is semisimple. So the sequence (4.4) splits. \( \square \)

We assume that \( m \geq 1 \) and we abbreviate \( \tilde{X} = \tilde{X}_{2m-1} \). The disjoint union \( \tilde{X} = \bigsqcup_{s \in X^0} \tilde{X}_{2m-1}[s] \) induces an isomorphism:
\[
C^1_c(\tilde{X}) \cong \bigoplus_{s \in X^0} C^1(\tilde{X}_{2m-1}[s]),
\]
where
\[
\omega \overset{\cong}{\mapsto} \langle \omega|_{C_1(\tilde{X}_{2m-1}[s])} \rangle_{s \in X^0}.
\]
Similarly, the non-disjoint union $\tilde{X}^0 = \bigcup_{s \in X^0} \tilde{X}_s^0$ induces an injection:

$$ j : \ C^0_c(\tilde{X}) \to \bigoplus_{s \in X^0} C^0(\tilde{X}_{2m-1}[s]), $$

We have the following commutative diagram of $G$-modules:

\[
\begin{array}{ccccccccc}
H^0_c(\tilde{X}) & \longrightarrow & \bigoplus_{s \in X^0} H^0(\tilde{X}_{2m-1}[s]) & \overset{\varphi}{\longrightarrow} & \text{coker} j \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0_c(\tilde{X}) & \longrightarrow & \bigoplus_{s \in X^0} C^0(\tilde{X}_{2m-1}[s]) & \longrightarrow & \text{coker} j & \longrightarrow & 0 \\
\downarrow^d & & \downarrow & & \downarrow \oplus d_s \\
0 & \longrightarrow & C^1_c(\tilde{X}) & \longrightarrow & \bigoplus_{s \in X^0} C^1(\tilde{X}_{2m-1}[s]) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1_c(\tilde{X}) & \longrightarrow & \bigoplus_{s \in X^0} H^1(\tilde{X}_{2m-1}[s]) & \longrightarrow & 0 & & & & \\
\end{array}
\]

Here, for $s \in X^0$, $d_s$ denote the coboundary map $C^0(\tilde{X}_{2m-1}[s]) \to C^1(\tilde{X}_{2m-1}[s])$. By Lemma 3.4, $\tilde{X}$ is connected. So we have $H^0_c(\tilde{X}) = 0$ since $\tilde{X}$ is non-compact. The *snake lemma* gives the kernel-cokernel exact sequence:

$$ 0 \to \text{coker} j / \varphi(\bigoplus_{s \in X^0} H^0(\tilde{X}_{2m-1}[s])) \to H^1_c(\tilde{X}) \to \bigoplus_{s \in X^0} H^1(\tilde{X}_{2m-1}[s]) \to 0. $$

**Lemma 4.3.** We have a canonical isomorphism of $G$-modules

$$ \text{coker} j / \varphi(\bigoplus_{s \in X^0} H^0(\tilde{X}_{2m-1}[s])) \simeq H^1_c(\tilde{X}_{2m-2}). $$

**Proof.** It is similar to the proof of Lemma 4.1 and relies on Lemma 3.6 and Corollary 3.7. \qed

**Proposition 4.4.** For $m \geq 1$, we have an isomorphism of $G$-modules:

$$ H^1_c(\tilde{X}_{2m-1}) \simeq H^1_c(\tilde{X}_{2m-2}) \oplus \text{c-ind}^{G}_{K_{s_o}} H^1(\tilde{X}_{s_o}) $$

for any vertex $s_o$ and where $K_{s_o}$ denotes the stabilizer of $s_o$ in $G$.

**Proof.** Similar to the proof of Proposition 4.2. \qed

Recall [3] that $\tilde{X}_0$ is different from $X$. This is a directed graph whose set of vertices is isomorphic to $X^0$ as a $G$-set and whose set of edges is isomorphic to the $G$-set of oriented edges of $X$.

5. Determination of the Inducing Representations – I

Let $m \geq 0$ be a fixed integer and $e_0 = [s_0, s_1]$ the standard edge. The aim of this section is to determine the $\mathcal{K}_{e_0}$-module $H^1(\tilde{X}_{2m}[e_0])$. Here we have $\mathcal{K}_{e_0} = \bar{I}$, the normalizer in $G$ of the standard Iwahori subgroup. We have the semidirect products

$$ \bar{I} = \langle \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} \rangle \rtimes I = E \times I $$

GEOMETRIC CONSTRUCTION OF TYPES 515
for any totally ramified subfield extension $E/F \subset M(2, F)$ such that $E^\times$ normalizes $I$.

We first assume that $m \geq 1$. By Corollary 3.2, we have the disjoint union

$$\tilde{X}_{2m}[e_0] = \bigsqcup_{c \in C_{2m-1}(e_0)} \tilde{X}_{2m}[e_0, c].$$

The group $\tilde{I}$ acts transitively on $C_{2m-1}(e_0)$. This comes form the standard fact that $I$, the pointwise stabilizer of $e_0$, acts transitively on the apartments of $X$ containing $e_0$.

Let $c_0 \in C_{2m-1}(e_0)$ be the path

$$s_{-m+1}, \ldots, s_0, s_1, \ldots, s_m.$$

The global stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ in $\tilde{I}$ is the pointwise stabilizer of $c_0$ in $\tilde{I}$, that is,

$$\Gamma_0(m, m-1) = \left( \begin{array}{cc} \phi^\times & p^{m-1} \\ p^m & \phi^\times \end{array} \right) = T^0 I_{2m-1}.$$

It follows that

$$(5.1) \quad H^1(\tilde{X}_{2m}[e_0]) = \text{ind}^I_{T^0 I_{2m-1}} H^1(\tilde{X}_{2m}[e_0, c_0]).$$

On the other hand, an easy calculation shows that the pointwise stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ is $T^1 I_{2m}$, where $T^1$ is the congruence subgroup of $T$ given by

$$T^1 = \left( \begin{array}{cc} 1 + p & 0 \\ 0 & 1 + p \end{array} \right).$$

So the $T^0 I_{2m-1}$-module $H^1(\tilde{X}_{2m}[e_0, c_0])$ may be viewed as a representation of the finite group $T^0 I_{2m-1}/T^1 I_{2m}$, that is, a semidirect product of the cyclic group $k^\times$ with the abelian group $I_{2m-1}/I_{2m} \simeq k \oplus k$.

Set $\Gamma = \tilde{X}_{2m}[e_0, c_0]$. This is a finite directed graph. Let $\Sigma_{-m}$ (resp. $\Sigma_{m+1}$) denote the set of vertices of $X$ that are neighbours of $s_{-m+1}$ and different from $s_{-m+2}$ (resp. neighbours of $s_m$ and different from $s_{m-1}$). Then the vertex set of $\Gamma$ is

$$\Gamma^0 = \{(a, s_{-m+1}, \ldots, s_0, \ldots, s_m); \ a \in \Sigma_{-m}\} \bigsqcup \{(s_{-m+1}, \ldots, s_0, \ldots, s_m, b); \ b \in \Sigma_{m+1}\}$$

$$\simeq \Sigma_{-m} \bigsqcup \Sigma_{m+1}$$

and its edge set is

$$\Gamma^1 = \{(a, s_{-m+1}, \ldots, s_0, \ldots, s_m, b); \ a \in \Sigma_{-m}, \ b \in \Sigma_{m+1}\} \simeq \Sigma_{-m} \times \Sigma_{m+1}.$$

In particular, $\Gamma$ is a bipartite graph based on two sets of $q$ elements. In particular, its Euler character is given by

$$\chi(\Gamma) = 1 - \dim_\mathbb{C} H^1(\Gamma) = 2q - q^2,$$

so that

$$(5.2) \quad \dim_\mathbb{C} H^1(\Gamma) = q^2 - 2q + 1 = (q - 1)^2.$$

Let $\mathbb{C}[\Gamma^1]$ be the space of complex function on $\Gamma^1$ and let $\mathcal{H}(\Gamma)$ be the space of harmonic 1-cochains on $\Gamma$:

$$\mathcal{H}(\Gamma) = \{ f \in \mathbb{C}[\Gamma^1]; \ \sum_{a \in \Gamma^1, \ s \in a} [a : s] f(a) = 0, \ \text{all} \ s \in \Gamma^0 \}.$$
Here $[a : s]$ denote an incidence number. In our case:

$$ (5.3) \quad f \in \mathcal{H}(\Gamma) \iff \begin{cases} \sum_{a \in \Sigma_{-m}} f(a, s_{-m+1}, \ldots, s_m, b) = 0, & \text{for all } b, \\ \sum_{b \in \Sigma_{m+1}} f(a, s_{-m+1}, \ldots, s_m, b) = 0, & \text{for all } a. \end{cases} $$

This is a standard result (see e.g. [3] Lemma (1.3.2)) that, as a $T^0I_{2m-1}/T^1I_{2m}$-module, $H^1(\Gamma)$ is isomorphic to the contragredient module of $\mathcal{H}(\Gamma)$.

An easy computation shows that we may identify $\Gamma^1$ with $k \times k$ in such a way that:

1) an element of $I_{2m-1} = \left( \begin{array}{cc} 1 + p^m & p^{m-1} \\ p^m & 1 + p^m \end{array} \right)$ acts as

$$ (1 + \left( \begin{array}{cc} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{array} \right) ) \cdot (x, y) = (x + \bar{b}, y + \bar{c}) $$

for $a, b, c, d \in a, x, y \in k$.

2) an element of $T^0$ acts as

$$ \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \cdot (x, y) = (\bar{a}d^{-1}x, \bar{a}^{-1}y) $$

and the condition (5.3) gives us:

$$ f \in \mathcal{H}(\Gamma) \iff \begin{cases} \sum_{x \in k} f(x, y) = 0, & \text{for all } y \in k, \\ \sum_{y \in k} f(x, y) = 0, & \text{for all } x \in k. \end{cases} $$

A basis of $\mathbb{C}[\Gamma]$ is formed of the functions $\chi_1 \otimes \chi_2(x, y) = \chi_1(x)\chi_2(y)$, where, for $i = 1, 2$, $\chi_i$ runs over the characters of $(k, +)$. It is clear that the $(q-1)^2$-dimensional subspace of $\mathbb{C}[\Gamma]$ generated by the $\chi_1 \otimes \chi_2$, $\chi_1 \neq 1$, $\chi_2 \neq 1$, is contained in $\mathcal{H}(\Gamma)$. So using (5.2), we obtain:

$$ (5.4) \quad \mathcal{H}(\Gamma) = \text{Span}\{\chi_1 \otimes \chi_2 ; \chi_i \in \widehat{k}, \chi_i \neq 1, i = 1, 2\}. $$

It follows from (5.4) that, as an $I_{2m-1}/I_{2m}$-module, the space $\mathcal{H}(\Gamma)$ is the direct sum of 1-dimensional representations corresponding to the characters $\alpha = \alpha(\chi_1, \chi_2), \chi_i \neq 1, i = 1, 2$, given by

$$ \alpha(1 + \left( \begin{array}{cc} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{array} \right) ) = \chi_1(b)\chi_2(c). $$

In particular, $\mathcal{H}(\Gamma)$ is isomorphic to its contragredient and therefore isomorphic to $H^1(\Gamma)$ as an $I_{2m-1}/I_{2m}$-module. In the language of strata (the reader may refer to [4] §4), for $\chi_i \neq 1, i = 1, 2$, the character $\alpha(\chi_1, \chi_2)$ corresponds to a stratum of the form $[I, 2m, 2m - 1, \beta]$, where $I$ is the standard Iwahori order and $\beta \in M(2, F)$ is an element of the form $\Pi^{2m-1} \left( \begin{array}{cc} u & 0 \\ 0 & v \end{array} \right)$, $u, v \in \alpha^\times$. In the terminology of [4] §4, page 98] this stratum is a ramified simple stratum.

We now have enough material to prove the following result.

**Proposition 5.1.** Let $\lambda$ be an irreducible constituent of

$$ H^1(\tilde{X}_{2m}[e_0]) = \text{ind}_{T^0I_{2m-1}}^{I_{2m}} H^1(\tilde{X}_{2m}[e_0, c_0]). $$

Then the compactly induced representation $c\text{-}\text{ind}_I^G \lambda$ is irreducible, whence supercuspidal.

Proof. It is a standard result that an irreducible compactly induced representation is supercuspidal (see e.g. [8] page 194).

The proof of the irreducibility is also standard by an argument due to Kutzko. But we repeat it for convenience. By Frobenius reciprocity, the restriction of $\lambda$ to $I_{2m-1}$ contains a character $\lambda(\chi_1, \chi_2)$ corresponding to a (ramified) simple stratum. Since $\lambda$ is irreducible and since $\tilde{I}$ normalizes $I_{2m-1}$, the restriction $\lambda|_{I_{2m-1}}$ is a direct sum $\alpha_1 \oplus \cdots \oplus \alpha_r$ of $I$-conjugates of $\alpha(\chi_1, \chi_2)$. They all correspond to simple strata. Let $g \in G$ be an element intertwining $\lambda$ with itself. Then by restriction it intertwines a character $\alpha_i$ with a character $\alpha_j$ for some $j = 1, \ldots, r$. By [4, Lemma (16.1), page 111], such an element $G$ must belong to $\tilde{I}$. It follows that the $G$-intertwining of $\lambda$ is equal to $\tilde{I}$ and that the representation $c\text{-}\text{ind}_I^G \lambda$ is irreducible according to Mackey’s irreducibility criterion ([8, Proposition (1.5), page 195]). 

We finally consider the case $m = 0$. The directed graph $\tilde{X}_0$ has $X^0$ as vertex set. An edge $\{t, s\}$ in $X$ gives rise to two edges $[s, t]$ and $[t, s]$ in $\tilde{X}_0$. Since the action of $G$ on $\tilde{X}_0$ preserves the structure of the digraph, the $G$-module $H^1_c(\tilde{X}_0)$ may be computed using the complex

$$0 \rightarrow C^0_c(\tilde{X}_0) \rightarrow C^1_c(\tilde{X}_0),$$

where $C^1_c(\tilde{X}_0)$ is the space of (unoriented) 1-cochains, that is the space of maps from $\tilde{X}_0^{(1)}$ (unoriented edges) to $C$ with finite support. The coboundary map is given here by $df(a) = f(a^+) - f(a^-)$. Consider the $G$-equivariant injection $j : C^1_c(X) \rightarrow C^1_c(\tilde{X}_0)$ given by $j(\omega) : [s, t] \mapsto \omega([s, t])$. We have the commutative diagram of $G$-modules:

$$
\begin{array}{ccc}
0 & \rightarrow & C^0_c(X) \\
\downarrow & & \downarrow \\
0 & \rightarrow & C^0_c(\tilde{X}_0) \\
\downarrow & & \downarrow \\
0 & \rightarrow & C^1_c(X) \\
\downarrow & & j \\
0 & \rightarrow & C^1_c(\tilde{X}_0) \\
\downarrow & & \downarrow \\
0 & \rightarrow & C^1_c(\tilde{X}_0)/\text{Im} j \\
\downarrow & & 0 \\
0 & \rightarrow & H^1_c(X) \\
\downarrow & & H^1_c(\tilde{X}_0) \\
\downarrow & & c\text{-}\text{ind}_I^G 1_{\tilde{I}} \\
0 & \rightarrow & 0.
\end{array}
$$

The quotient $C^1_c(\tilde{X}_0)/\text{Im} j$ identifies with the subspace of $C^1_c(\tilde{X}_0)$ formed by those functions $f$ satisfying $f([s, t]) = f([t, s])$ for all edges $\{s, t\}$ of $X$. This subspace is nothing more than the compactly induced representation $c - \text{Ind}_I^G 1_{\tilde{I}}$. The cokernel exact sequence gives us:

$$0 \rightarrow H^1_c(X) \rightarrow H^1_c(\tilde{X}_0) \rightarrow c\text{-}\text{ind}_I^G 1_{\tilde{I}} \rightarrow 0.$$

Now we use the following two facts to obtain Proposition 5.2:

- the representation $c\text{-}\text{ind}_I^G 1_{\tilde{I}}$ is a projective object of the category of smooth representations of $G$,
- the $G$-module $H^1_c(X)$ is isomorphic to the Steinberg representation $\text{St}_G$ of $G$ ([7]).

**Proposition 5.2.** The $G$-module $H^1_c(\tilde{X}_0)$ is isomorphic to $\text{St}_G \oplus c\text{-}\text{ind}_I^G 1_{\tilde{I}}$. 
6. The Inducing Representations—II

We now determine the $\mathcal{K}_{s_0}$-module $H^1(\tilde{X}_{2m+1}[s_0])$. The arguments are very often similar to those of the previous section and we will not give all details. Since the case $m = 0$ requires slightly different techniques we postpone it to the end of the section and assume first that $m > 0$.

Recall that the stabilizer $\mathcal{K}_{s_0}$ of $s_0$ in $G$ is the image $K$ of $GL(2, \mathfrak{o})$ in $G$.

Let $c_0 \in C_{2m}(s_0)$ be the path $(s_{m-1}, s_m, \ldots, s_0, \ldots, s_{m+1})$. Its pointwise stabilizer is $\Gamma_0(m, m) = T^0K_m$. So as a $K$-module, $H^1(\tilde{X}_{2m+1}[s_0])$ is isomorphic to the induced representation $\text{Ind}_{T^{0}K_m}^{K}H^1(\tilde{X}_{2m+1}[s_0], c_0)$. Moreover, the pointwise stabilizer of $\tilde{X}_{2m+1}[s_0, c_0]$ is $T^1K_{m+1}$ and $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ may be viewed as a representation of $T^0K_m/T^1K_{m+1}$.

As in the previous section, one may consider the bipartite graph $\Omega$ whose both vertex sets identify with $\mathbb{k}$, equipped with an action of $K_m$ on $\Omega$ given by

$$[I_2 + \bar{x}^m \begin{pmatrix} a & b \\ c & d \end{pmatrix}] \cdot (x, y) = (x + \bar{b}, y + \bar{c}),$$

the action of $T^0$ being given by

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot (x, y) = (\bar{a}\bar{d}^{-1}x, \bar{d}\bar{a}^{-1}y).$$

Then the contragredient of the $T^0K_m/T^1K_{m+1}$-module $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ is isomorphic to the space $\mathcal{H}(\Omega)$ of harmonic cochains on $\Omega$. As in the previous section the latter space is generated by the functions $\chi_1 \otimes \chi_2$, where $\chi_i$, $i = 1, 2$, runs over the non-trivial characters of $(k, +)$. The line $\mathbb{C}\chi_1 \otimes \chi_2$ is acted upon by $K_m$ via the character $\alpha(\chi_1, \chi_2)$ given by

$$\alpha(\chi_1, \chi_2)(I_2 + \bar{x}^m \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \chi_1(b)\chi_2(c).$$

It follows that $\mathcal{H}(\Omega)$ is isomorphic to its contragredient and that $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ is the direct sum of the characters $\alpha(\chi_1, \chi_2)$, $\chi_i \neq 1$, $i = 1, 2$.

For $\chi_i \neq 1$, $i = 1, 2$, the character $\alpha(\chi_1, \chi_2)$ corresponds to a stratum of the form $[\mathbb{M}(2, \mathfrak{o}), m, m-1, \beta]$, where $\beta \in \mathbb{M}(2, F)$ is given by $\bar{x}^m \begin{pmatrix} v & u \\ 0 & 0 \end{pmatrix}$, $u, v \in \mathfrak{o}^x$. This stratum is either simple and non-scalar or split fundamental according to whether $uv \mod p$ is a square in $\mathbb{k}^x$ or not (here we have used the fact that $\text{Char}(\mathbb{k}) \neq 2$).

It is clear that $T^0$ leaves the set of characters corresponding to simple strata (resp. split fundamental strata) stable. So we may write

$$H^1(\tilde{X}_{2m+1}[s_0, c_0]) = H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{simple}} \oplus H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{split}},$$

where $H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{simple}}$ (resp. $H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{split}}$) is the $T^0K_m$-submodule which decomposes as a $K_m/K_{m+1}$-module as a direct sum of (characters corresponding to) simple non-scalar strata (resp. split fundamental strata).

We have a result similar to Proposition 5.1 whose proof uses the same arguments.

**Proposition 6.1.** Let $\lambda$ be an irreducible constituent of

$$\text{Ind}_{T^0K_m}^{K}H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{simple}} \subset H^1(\tilde{X}_{2m+1}[s_0]).$$

Then the compactly induced representation $c\text{-ind}_{K}^{G}\lambda$ is irreducible, whence supercuspidal.
The study of $\text{Ind}^K_{\mathbf{T}^K} H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\text{split}}$ is the aim of the next section.

We are now going to determine the $K$-module structure of $H^1(\tilde{X}_1[s_0])$. Set $\mathbf{G} = \text{PGL}(2, \mathbb{K}) \simeq K/K^1$ and write $\mathbf{B}$ and $\mathbf{T}$ for the upper Borel subgroup and diagonal torus of $\mathbf{G}$, respectively. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. As a $K$-set the set of neighbour vertices of $s_0$ is isomorphic to $\mathbb{P}^1(\mathbb{K}) = \mathbf{G}/\mathbf{B}$.

The graph $\Omega = \tilde{X}_1[s_0]$ has for the vertex set the set of paths of the form $(s, s_0)$ or $(s_0, s)$ where $s$ runs over the neighbour vertices of $s_0$ in $X$. So the space $C^0(\Omega)$ of 0-cochains identifies with the space $\mathcal{F}(\mathbb{P}^1(\mathbb{K}) \coprod \mathbb{P}^1(\mathbb{K}))$ of complex-valued functions on the disjoint union $\mathbb{P}^1(\mathbb{K}) \coprod \mathbb{P}^1(\mathbb{K})$, thus has a $\mathbf{G}$-module $C^0(\Omega)$ that is isomorphic to $\mathbf{1}_G \oplus \text{St}_G \oplus \mathbf{1}_G \oplus \text{St}_G$, where $\mathbf{1}$ denotes a trivial representation and $\text{St}$ a Steinberg representation.

The $\mathbf{G}$-set $\Omega^1$ is the set of paths of the form $(s, s_0, t)$, where $s$ and $t$ are two different neighbour vertices of $s_0$. This $\mathbf{G}$-set is isomorphic to the quotient $\mathbf{G}/\mathbf{T}$.

The space $C^{(1)}(\Omega)$ of unoriented 1-cochains identifies as a $\mathbf{G}$-module with the space $\mathcal{F}(\mathbf{G}/\mathbf{T})$.

Fix a non-trivial character $\psi$ of $\mathbf{U}$. It is well known that the induced representation $\text{Ind}_\mathbf{U}^\mathbf{G} \psi$ is multiplicity free. Its irreducible constituent form is by definition the generic (irreducible) representations of $G$. Moreover, an irreducible representation is generic if and only if it is not a character.

We have a natural $G$-equivariant map $\Phi : \mathcal{F}(\mathbf{G}/\mathbf{T}) \rightarrow \text{Ind}_\mathbf{U}^\mathbf{G} \psi$, given by

$$\Phi(f)(g) = \sum_{u \in \mathbf{U}} f(gu) \bar{\psi}(u), \quad f \in \mathcal{F}(\mathbf{G}/\mathbf{T}), \quad g \in \mathbf{G}.$$ 

If a function $f$ lies in the kernel of $\Phi$, then we have $\sum_{u \in \mathbf{U}} f(gu) \theta(u) = 0$, for all $g \in \mathbf{G}$ and all non-trivial character $\theta$ of $\mathbf{U}$. Indeed, it suffices to use the fact that the action of $\mathbf{T}$ on $\mathbf{U}$ by conjugation acts transitively on the non-trivial characters of $\mathbf{U}$ and the right invariance of $f$ under the action of $\mathbf{T}$. So the kernel of $\Phi$ consists of the function $f$ such that $u \mapsto f(gu)$ is a constant function on $U$, for all $g \in G$.

In other words, $\text{Ker} \Phi = \mathcal{F}(G/B) \simeq \mathbf{1}_G \oplus \text{St}_G$. By a dimension argument, we see that $\Phi$ is surjective. It follows that

$$C^{(1)}(\Omega) \simeq \text{Ind}_\mathbf{U}^\mathbf{G} \psi \oplus \mathbf{1}_G \oplus \text{St}_G.$$ 

We have the cochain complex of $G$-modules:

$$0 \rightarrow C^0(\Omega) \rightarrow C^{(1)}(\Omega) \rightarrow 0.$$ 

Since $\Omega$ is connected the kernel of the coboundary operator is the trivial module $\mathbb{C}$. Hence in the Grothendieck groups of $G$-modules, we have $dC^0(\Omega) \simeq 2\mathbf{1}_G + 2\text{St}_G - \mathbf{1}_G = \mathbf{1}_G + 2\text{St}_G$. Therefore,

$$H^1(\Omega) = C^1(\Omega)/dC^0(\Omega) \simeq \text{Ind}_\mathbf{U}^\mathbf{G} \psi + \mathbf{1}_G + \text{St}_G - \mathbf{1}_G - 2\text{St}_G = \text{Ind}_\mathbf{U}^\mathbf{G} \psi - \text{St}_G.$$ 

Since $q = |\mathbb{K}|$ is odd, there exists a unique non-trivial character of $\mathbb{K}^\times/(\mathbb{K}^\times)^2$, that we denote by $\chi_0$. The irreducible constituents of the Gelfand-Graev representation $\text{Ind}_\mathbf{U}^\mathbf{G} \psi$ are the following:

- the irreducible cuspidal representations of $\mathbf{G}$,
- the principal series $\text{Ind}_\mathbf{B}^\mathbf{G} \chi \otimes \chi^{-1}$, where $\chi : \mathbb{K}^\times \rightarrow \mathbb{C}^\times$ is a character such that $\chi^2 \neq 1$ (i.e. $\chi \notin \{1, \chi_0\}$),
- the steinberg representation $\text{St}_G$,
- (when $q$ is odd) the twisted representation $\text{St}_G \otimes \chi_0$. 

If \( \sigma \) is a cuspidal representation of \( G = K/K^1 \), then the induced representation \( c\text{-ind}_K^G \sigma \) is irreducible and supercuspidal ([11 (11.5), page 81]). Such a representation of \( G \) is called a level 0 supercuspidal representation.

A principal series of \( G = K/K^1 \) may be written as \( \text{Ind}_I^K \rho \), where \( \rho \) is a character of \( I/I^1 \). The pair \( (I, \rho) \) is actually a type in the sense of Bushnell and Kutzko’s type theory. For technical reasons we postpone definitions and references to the next section. Since the representation \( \text{Ind}_I^K \rho \) is irreducible, it is a type of the same constituent as \( (I, \rho) \).

To sum up, we have proved the following.

**Proposition 6.2.** An irreducible constituent \( \lambda \) of \( H^1(\hat{X}_1[s_0]) \) is of one of the following forms:

(i) the inflation of a cuspidal representation of \( G \); in that case \( c\text{-ind}_K^G \lambda \) is a level 0 irreducible supercuspidal representation of \( G \),

(ii) the inflation to \( K \) of the representation \( \text{St}_G \otimes \chi_0 \),

(iii) a type of the form \( \text{Ind}_I^K \rho \), where the \( \rho \) is inflated from a character of \( I/I^1 \approx (k^\times \times k^\times)/k^\times \) of the form \( \chi \otimes \chi^{-1}, \chi^2 \neq 1 \).

Note that in (iii), the pair \( (K, \text{Ind}_I^K \rho) \) is a principal series type.

7. THE INDUCING REPRESENTATIONS – III

We keep the notation as in the previous section. To determine the structure of \( \text{Ind}_{T^0 K_0}^K H^1(\hat{X}_{2m+1}[s_0, c_0])^{\text{split}} \), we first recall crucial facts on split strata and types for principal series representations. The basic reference for type theory is [6].

Let \( \chi \) be a character of \( T \), that we view as a character of \( T^0 \) by restriction. Assume that the conductor of \( \chi \) is \( n > 0 : T^n \subset \text{Ker} \chi \) and \( n \) is minimal for this property. Set

\[
J_\chi = \begin{pmatrix} o^\times & o \\ p^n & o^\times \end{pmatrix} = \Gamma_0(p^n).
\]

If \( U \) and \( \bar{U} \) denote the groups of upper and lower unipotent matrices respectively, then \( J_\chi \) has an Iwahori decomposition,

\[
J_\chi = (J_\chi \cap \bar{U}) \cdot (J_\chi \cap T) \cdot (J_\chi \cap U)
\]

and one may define a character \( \rho_\chi \) of \( J_\chi \) by

\[
\rho_\chi(\bar{u}t^0u) = \chi(t^0), \quad \bar{u} \in J_\chi \cap \bar{U}, \ u \in J_\chi \cap U, \ t^0 \in T^0.
\]

Let \( \mathcal{R}_{[T, \chi]} \) be the Bernstein component of the category of smooth representations of \( G \) whose objects are the representations \( \mathcal{V} \) satisfying the following property: Any irreducible subquotient of \( \mathcal{V} \) occurs in a parabolically induced representation \( \text{Ind}_B^G(\chi \otimes \chi_0) \), where \( B \) is a Borel subgroup with Levi component \( T \) and \( \chi_0 \) an unramified character of \( T \). We then have:

**Theorem 7.1** (A. Roche). The pair \( (J_\chi, \rho_\chi) \) is a type for \( \mathcal{R}_{[T, \chi]} \).

This is indeed Theorem (7.7) of [11]. Note that our \( J_\chi \) is not exactly the same as Roche’s, but a conjugate under an element of \( T \) (see [11 Example (3.5)])

**Proposition 7.2.** With the notation as before, assume that \( \chi_{|T^0} \) is not of the form \( \alpha \circ \text{Det} \), where \( \alpha \) is a character of \( o^\times \) (necessarily of order 2). Then the induced representation \( \text{Ind}_J^K \rho_\chi \) is irreducible. In particular, it is a type for \( \mathcal{R}_{[T, \chi]} \).
Proof. Let $W$ be the extended affine Weyl group of $G$ w.r.t. $T$ and set $W_\chi = \{ w \in W : w\chi = \chi \}$. Then by Theorem (4.14) of [11], the $G$-intertwining of $\rho_\chi$ is $J_\chi W_\chi J_\chi$. The hypothesis on $\chi$ forces $W_\chi = T/T^0$. So $(J_\chi W_\chi J_\chi) \cap K = J_\chi T^0 J_\chi = J_\chi$, and we may apply Mackey’s criterion of irreducibility. \qed

For $n > 0$ and $q \in \{0, ..., n\}$, define compact open subgroups of $G$ as follows:

$q\mathfrak{h}_1 = \left( \begin{array}{ccc} 1 + p^n & p^q \\ p^{n+1} & 1 + p^n \end{array} \right)$ and $q\mathfrak{h}_2 = \left( \begin{array}{ccc} 1 + p^{n+1} & p^q \\ p^{n+1} & 1 + p^{n+1} \end{array} \right)$.

These groups are particular cases of groups considered in [11 §(2.3)]. The quotients $q\mathfrak{h}_1/q\mathfrak{h}_2$, $q = 0, ..., n$, are abelian, and for $\alpha \in k^\times$, one may define a character $\psi_\alpha$ of $q\mathfrak{h}_1/q\mathfrak{h}_2$ by the formula

$$\psi_\alpha(I_2 + \left( \begin{array}{ccc} \omega^n a \\ \omega^{n+1} + c \\ \omega^n d \end{array} \right)) = \psi(a - d),$$

where $\psi$ is a fixed non-trivial character of $(k, +)$. We shall need the following result.

Lemma 7.3. If a smooth representation of $K$ contains $(\psi_\alpha)|_{n\mathfrak{h}_1}$ by restriction, then it contains the character $(\psi_\alpha)|_{n\mathfrak{h}_1}$.

Proof. Since the characteristic of $k$ is not 2, then $\alpha \neq -\alpha$ and $(\psi_\alpha)|_{n\mathfrak{h}_1}$ is the restriction to $n\mathfrak{h}_1$ of a split fundamental stratum of $K_n/K_{n+1}$. Our lemma is then a particular case of [11 Lemma (2.4.5)]. \qed

Proposition 7.4. Let $\lambda$ be an irreducible constituent of $\text{Ind}_{T^0 K_m}^K H^1(\check{X}_{2m+1} [s_0, c_0])_{\text{spin}}$. Then with the notation as above, $\lambda$ is of the form $\text{Ind}_{J_\chi}^K \rho_\chi$, for some principal series type $(J_\chi, \rho_\chi)$ with $\chi$ of conductor $m + 1$.

Proof. We know that such a $\lambda$ contains a split fundamental stratum of the form $[M(2, \sigma), m, m - 1, b]$, where $b = \omega^{-m} \left( \begin{array}{cc} 0 & u \\ v & 0 \end{array} \right)$, $u, v \in \sigma^\times$, and $uv$ is a square modulo $p$. If $\alpha \in \sigma$ is such that $\alpha^2 \equiv uv \mod p$, then the stratum is equivalent to a $K$-conjugate of $[M(2, \sigma), m, m - 1, b']$, where $b' = \omega^{-m} \left( \begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right)$. So we deduce that $\lambda$ contains this latter stratum by restriction. Now consider the group $n\mathfrak{h}_1$ for $n = m$. The representation $\lambda$ contains the character $(\psi_\alpha)|_{n\mathfrak{h}_1}$ by restriction. By applying Lemma 7.3 we obtain that it contains the character $(\psi_\alpha)|_{n\mathfrak{h}_1}$. This character clearly extends to $T^0_n \mathfrak{h}_1 = \Gamma_0(m + 1, 0)$ and the quotient $T^0_n \mathfrak{h}_1/\mathfrak{h}_1$ is abelian. It follows that $\lambda$ contains an extension of $\psi_\alpha$ to $\Gamma_0(m + 1, 0)$. Such an extension is of the form $(J_\chi, \rho_\chi)$, for some character $\chi$ of $T$ of conductor $m + 1$. The fact that $\lambda$ is induced from $(J_\chi, \rho_\chi)$ follows from Proposition 7.2. \qed

8. Synthesis

We now prove Theorems A and B of the Introduction.

By Propositions 4.2 and 4.4 we have isomorphisms of $G$-modules:

\begin{align*}
(8.1) & \quad H^1_c(\check{X}_{2m}) \simeq H^1_c(\check{X}_{2m-1}) \oplus c\text{-}\text{ind}_{K_0}^G H^1(\Sigma_{2m}), \quad m \geq 1, \\
(8.2) & \quad H^1_c(\check{X}_{2m+1}) \simeq H^1_c(\check{X}_{2m}) \oplus c\text{-}\text{ind}_{K_0}^G H^1(\Sigma_{2m+1}), \quad m \geq 0.
\end{align*}

Recall that with the notation of the introduction, we have:

- $\Sigma_{2m} = \check{X}_{2m} [s_0]$, $\Sigma_{2m+1} = \check{X}_{2m+1} [s_0]$,
- $K_0 = K_{s_0}$, $K_1 = K_{c_0}$. 

Moreover, by Proposition 5.2, we have
\begin{equation}
H^1_c(\tilde{X}_0) \simeq \text{St}_G \oplus c\text{-ind}_{G}^{G_1} H^1(\Sigma_0)
\end{equation}
so that (1) holds for \( m = 0 \). Hence, Theorem A follows from (8.1) and (8.2) by a straightforward inductive argument.

Theorem B follows from the description of the irreducible components of \( H^1(\Sigma_n) \) given in Proposition 5.1 (\( n \) even and \( n > 0 \)), Proposition 5.2 (\( n = 0 \)), and Propositions 6.1 and 7.4 (\( n \) odd).

References


Département de Mathématiques, Université de Poitiers, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France

E-mail address: paul.broussous@math.univ-poitiers.fr