EQUIVARIANT COHERENT SHEAVES ON THE EXOTIC NILPOTENT CONE

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In loving memory of my grandfather, Nadaraja Rajamanikkam

ABSTRACT. Let $G = Sp_{2n}(\mathbb{C})$, and $\mathfrak{N}$ be Kato's exotic nilpotent cone. Following techniques used by Bezrukavnikov in 2003 to establish a bijection between Λ^+, the dominant weights for an arbitrary simple algebraic group H, and $\mathbf{O}$, the set of pairs consisting of a nilpotent orbit and a finite-dimensional irreducible representation of the isotropy group of the orbit, we prove an analogous statement for the exotic nilpotent cone. First we prove that dominant line bundles on the exotic Springer resolution $\tilde{\mathfrak{N}}$ have vanishing higher cohomology, and compute their global sections using techniques of Broer. This allows us to show that the direct images of these dominant line bundles constitute a quasi-exceptional set generating the category $D^b(\mathrm{Coh}^G(\mathfrak{N}))$, and deduce that the resulting t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ coincides with the perverse coherent t-structure. The desired result now follows from the bijection between costandard objects and simple objects in the heart of the t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$.

CONTENTS

1. INTRODUCTION

Let H be a simple algebraic group; denote by $\mathfrak{h}$ its Lie algebra and $\mathcal{N} \subset \mathfrak{h}$ its nilpotent cone. Let $\mathbf{\Lambda}^+$ denote the set of dominant weights for H, and let $\mathbf{O}$ denote the set of pairs $(\mathcal{O}, L)$, where $\mathcal{O}$ is an H-orbit on $\mathcal{N}$, and L is a finite-dimensional irreducible representation of the isotropy group G^x of the orbit $\mathcal{O}$, where $x \in \mathcal{O}$. Using geometric methods, in [10], Bezrukavnikov shows that there is a canonical bijection between $\mathbf{\Lambda}^+$ and $\mathbf{O}$ (a result that was previously conjectured by Lusztig and Vogan).

Now let $G = Sp_{2n}(\mathbb{C})$, and let $\mathfrak{N}$ be Kato's exotic nilpotent cone, defined as follows:

$$\mathfrak{N} := \mathbb{C}^{2n} \times \{x \in \mathrm{End}(\mathbb{C}^{2n}) \mid x \text{ nilpotent}, \langle xv, v \rangle = 0 \ \forall \ v \in \mathbb{C}^{2n}\}.$$

The main purpose is to establish an exotic analogue of Bezrukavnikov's bijection, i.e., a bijection between Λ^+, the dominant weights for G, and $\mathbb{O}$, the set of pairs $(\mathcal{O}, L)$, where $\mathcal{O}$ is a G-orbit on $\mathfrak{N}$, and L is a finite-dimensional irreducible representation of the isotropy group $G^{(v,x)}$ of the orbit $\mathcal{O}$, where $(v, x) \in \mathcal{O}$.

The exotic nilpotent cone was originally introduced by Kato in [18] to study multi-parameter affine Hecke algebras, via the equivariant K-theory of the exotic Steinberg variety (following techiques used by Kazhdan, Lusztig and Ginzburg in the case of one-parameter affine Hecke algebras). The G-orbits on $\mathfrak{N}$ are proven to be in bijection with $\mathcal{Q}_n$, the set of bi-partitions of n (and thus also with irreducible representations of the type C Weyl group). In [19], Kato explicitly realizes this bijection via an exotic Springer correspondence; this correspondence is somewhat cleaner than the type C Springer correspondence since there are no non-trivial local systems on G-orbits in $\mathfrak{N}$.

In [3], Achar and Henderson make precise conjectures describing the intersection cohomology of orbit closures in the exotic nilpotent cone. These have now been proven independently by Shoji and Sorlin (see Theorem 5.7 in [26]); and by Kato (see Theorem A in [20], Theorem A.1.8 in [21] and Remark 5.8 in [26]). The work of Achar, Henderson and Sommers in [5] studies special pieces for $\mathfrak{N}$, which turn out to have the same number of $\mathbb{F}_q$ points as Lusztig's special pieces for the ordinary nilpotent cone. These results all demonstrate a strong connection between the exotic nilpotent cone and the ordinary nilpotent cone of type C; the present work draws another parallel between the geometry of the exotic nilpotent cone $\mathfrak{N}$ and the geometry of the ordinary nilpotent cone $\mathcal{N}$ of type C.

Below we describe the main results of this paper in more detail.

Section 2: After recalling some of the basic properties of the exotic nilpotent cone, and some results of Achar, Henderson and Sommers on resolutions of special orbit closures, here we study the cohomology of dominant line bundles on the exotic Springer resolution $\widetilde{\mathfrak{N}}$. We closely follow the methods developed by Broer in [12], [13] that prove analogous results in the case of the ordinary nilpotent cone $\mathcal{N}$. Defining $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) := p^* \mathcal{O}_{G/B}(\lambda)$ where $p : \widetilde{\mathfrak{N}} \to G/B$ is the projection, we first prove that $H^i(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)) = 0$ using a theorem of Grauert-Riemenschneider. By using the additivity of the Euler characteristic along with Borel-Weil-Bott, we then compute the structure of $H^0(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda))$ as a G-module.

Section 3: First we recall the theory of quasi-exceptional sets (following Section 2 of [10]). Given a triangulated category $\mathcal{C}$, two ordered sets of objects $\nabla = \{\nabla_i | i \in I\}, \Delta = \{\Delta_i | i \in I\}$ are said to constitute a dualizable quasi-exceptional

set if they satisfy certain conditions. From a quasi-exceptional set generating a category $\mathcal{C}$, one obtains a t-structure on $\mathcal{C}$ whose heart is a quasi-hereditary category with simple objects also indexed by I. Letting $\mathcal{C} = D^b(\mathrm{Coh}^G(\mathfrak{N}))$, define $\nabla_\lambda := R\pi_* \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[\frac{\dim(\mathfrak{N})}{2}]$ where $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is the resolution of singularities. Then the main result is that $\nabla = \{\nabla_\lambda | \lambda \in \Lambda^+\}$ and $\Delta = \{\nabla_{w_0 \cdot \lambda} | \lambda \in \Lambda^+\}$ constitute a quasi-exceptional set generating $D^b(\mathrm{Coh}^G(\mathfrak{N}))$. The results of Section 2 are needed to prove this claim.

Section 4: After recalling some of the theory of perverse coherent sheaves (developed by Deligne and Bezrukavnikov), in this section we compare the t-structure constructed on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ with the perverse coherent t-structure corresponding to the middle perversity. We first prove that ∇_λ is a perverse coherent sheaf for all $\lambda \in \Lambda$ (this essentially follows from the fact that $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is semi-small). We then deduce that the quasi-exceptional t-structure coincides with the perverse coherent t-structure. Since the simple perverse coherent sheaves are indexed by $\mathbb{O}$, the bijection between simple objects and co-standard objects in the heart of the t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ gives the required bijection $\Theta : \mathbb{O} \leftrightarrow \Lambda^+$.

Section 5: We comment on some open questions relating to this construction. First, one may ask for an explicit combinatorial description of the construction Θ. One may also try to extend some results regarding canonical bases in equivariant K-theory, proved by Bezrukavnikov in [9], to the exotic setting.

We remark that there are a few differences between the proof of the bijection $\mathbf{O} \leftrightarrow \Lambda^+$ for $\mathcal{N}$, and the bijection $\mathbb{O} \leftrightarrow \Lambda^+$ for $\mathfrak{N}$. First, while the canonical line bundle on the Springer resolution $\widetilde{\mathcal{N}}$ is trivial, the canonical line bundle on the exotic Springer resolution $\widetilde{\mathfrak{N}}$ is the anti-dominant line bundle $\mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$. As a result, for the proofs in Section 3, we use the twisted Weyl group action on Λ, $w \cdot \lambda = w(\lambda + \theta) - \theta$ with $\theta = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$ instead of the ordinary action.

Second, in the case of the ordinary nilpotent cone $\mathcal{N}$, the proof that the image of the functor $R\pi_* : D^b(\mathrm{Coh}^H(\widetilde{\mathcal{N}})) \to D^b(\mathrm{Coh}^H(\mathcal{N}))$ generates $D^b(\mathrm{Coh}^H(\mathcal{N}))$ (see Lemma 7 in [10]) uses the Jacobson-Morozov resolution of orbit closures in $\mathcal{N}$. In the case of $\mathfrak{N}$, the resolutions of an arbitrary orbit closure $\overline{\mathbb{O}_{\mu,\nu}}$ are (to the best of our knowledge) not known. However, resolutions of special orbit closures have been recently developed in [5], and these are sufficient to prove the corresponding result in the case of the exotic nilpotent cone. It seems possible that there is an easier proof of this result (Proposition 3.15) without using the results of [5].

2. Geometry of the exotic nilpotent cone

2.1. **Recollections.** Here we recall the definition and basic properties of Kato's exotic nilpotent cone, following [18] and Section 6 of [3].

Definition 2.1. Let $\mathbb{V}$ be a $2n$-dimensional $\mathbb{C}$-vector space with a symplectic form $\langle \cdot, \cdot \rangle$, let $G = Sp(\mathbb{V})$ the associated symplectic group, and $\mathfrak{g} = \mathfrak{sp}(\mathbb{V})$ the associated symplectic Lie algebra. Define $\mathfrak{s} \subset \mathfrak{gl}(\mathbb{V})$ as below (note $\mathfrak{s} \oplus \mathfrak{sp}(\mathbb{V}) = \mathfrak{gl}(\mathbb{V})$):

$$\mathfrak{s} = \{x \in \mathrm{End}(\mathbb{V}) | \langle xv, v \rangle = 0 \ \forall \ v \in \mathbb{V}\}.$$

If $\mathcal{N}$ denotes the set of nilpotent endomorphims in $\mathfrak{gl}(\mathbb{V})$, we define the **exotic nilpotent cone** to be $\mathfrak{N} = \mathbb{V} \times (\mathfrak{s} \cap \mathcal{N})$.

Proposition 2.2. *The orbits of G on $\mathfrak{N}$ are in bijection with the poset $\mathcal{Q}_n$ of bipartitions of n. Under this correspondence, given $(\mu, \nu) \in \mathcal{Q}_n$, the orbit $\mathbb{O}_{\mu,\nu}$*

consists of $(v, x) \in \mathfrak{N}$ such that the Jordan type of x acting on the subspace $E^x v \subset \mathbb{V}$ is $\mu \cup \mu$, and the Jordan type of x acting on the quotient $V/E^x v$ is $\nu \cup \nu$. Here E^x denotes the centralizer of x in $End(\mathbb{V})$, and for a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\lambda \cup \lambda$ denotes the partition $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \cdots)$. The closure ordering for G-orbits on $\mathfrak{N}$ corresponds to the natural ordering on $\mathcal{Q}_n$ (i.e., $(\mu, \nu) \geq (\mu', \nu')$ if $\sum_{1 \leq i \leq j}(\mu_i + \nu_i) \geq \sum_{1 \leq i \leq j}(\mu'_i + \nu'_i)$ and $\mu_{j+1} + \sum_{1 \leq i \leq j}(\mu_i + \nu_i) \geq \mu'_{j+1} + \sum_{1 \leq i \leq j}(\mu'_i + \nu'_i)$).

Proof. We refer the reader to Theorem 6.1 of [3] and [18] for the statement regarding orbits; see also Corollary 2.9 in [3] and Theorem 1 in [28]. For the statement regarding orbit closures, see Theorem 6.3 in [3]. □

After fixing a Cartan subgroup $T \subset G$ and a Borel subgroup $B \subset G$, let $(\mathbb{V} \oplus \mathfrak{s})^+ \subset \mathfrak{N}$ denote the sum of the strictly positive weight spaces in the G-module $\mathbb{V} \oplus \mathfrak{s}$; note that it is a B-module.

Definition 2.3. Let $\widetilde{\mathfrak{N}} = G \times_B (\mathbb{V} \oplus \mathfrak{s})^+$, and let $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ be the map given by $\pi(g, (v, s)) = (gv, gsg^{-1})$. Then π is a resolution of singularities; accordingly we call $\widetilde{\mathfrak{N}}$ the exotic Springer resolution.

2.2. **Resolutions of special orbit closures and some subvarieties of $\widetilde{\mathfrak{N}}$.** In this section, for each orbit $\mathbb{O}_{\mu,\nu}$ (with $(\mu, \nu) \in \mathcal{Q}_n$), using resolutions of "special" orbit closures constructed in [5], we will construct a subvariety $\widehat{\mathbb{O}_{\mu,\nu}} \subset \widetilde{\mathfrak{N}}$ with a map $\pi_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \mathbb{O}_{\mu,\nu}$ whose fibres are acyclic. The need for this construction is to prove Lemma 3.14 (an exotic analogue of Lemma 7 from [10]). We start by recalling the Achar-Henderson-Sommer construction of resolutions for "special" orbit closures from [5]; for more details see Sections 2 and 5 of [5].

Remark 2.4. We will use C-distinguished partitions in the following; alternatively one may use B-distinguished partitions (see [5]) instead.

Proposition-Definition 1. *Let $\mathcal{Q}_n^C \subset \mathcal{Q}_n$ denote the subposet consisting of bi-partitions (μ, ν) satisfying $\mu_i \geq \nu_i - 1, \nu_i \geq \mu_{i+1} - 1$. If $\mathcal{P}_{2n}$ is the poset of partitions of $2n$ with the natural order, let $\mathcal{P}_{2n}^C \subset \mathcal{P}_{2n}$ denote the subposet consisting of partitions where each odd part occurs with even multiplicity. Define the map $\Phi^C : \mathcal{Q}_n \to \mathcal{P}_{2n}^C$ by sending the bi-partition (μ, ν) to the partition obtained from the composition $(2\mu_1, 2\nu_1, 2\mu_2, 2\nu_2, \cdots)$ by replacing successive terms $(2s, 2t)$ with $(s + t, s + t)$ if $s < t$. Define also the map $\widehat{\Phi}^C : \mathcal{P}_{2n}^C \to \mathcal{Q}_n^C$: given $\lambda \in \mathcal{P}_{2n}^C$, obtain the composition λ' by first halving any even parts, and replacing any string of odd parts $(2k+1, \cdots, 2k+1)$ by $(k, k+1, \cdots, k, k+1)$; then let $\widehat{\Phi}^C(\lambda') = (\mu, \nu)$ where $\mu = (\lambda'_1, \lambda'_3, \cdots), \nu = (\lambda'_2, \lambda'_4, \cdots)$. Then the maps $\widehat{\Phi}^C$ and $\Phi^C|_{\mathcal{Q}_n^C}$ give an isomorphism of posets $\mathcal{Q}_n^C \simeq \mathcal{P}_{2n}^C$. For $(\mu, \nu) \in \mathcal{Q}_n$, denote $(\mu, \nu)^C = \widehat{\Phi}^C(\Phi^C(\mu, \nu))$.*

Definition 2.5. Let $(\mu, \nu) \in \mathcal{Q}_n, \lambda = \Phi^C(\mu, \nu)$. A λ-filtration (which we also refer to as a (μ, ν)-filtration) of $\mathbb{V}$ is a sequence of subspaces $(\mathbb{V}_{\geq a})_{a \in \mathbb{Z}}$, satisfying the following:

- $\mathbb{V}_{\geq a} \subseteq \mathbb{V}_{\geq a-1}$,
- $\mathbb{V}_{\geq 1-a}^\perp = \mathbb{V}_{\geq a}$,
- $\dim \mathbb{V}_{\geq a} = \sum_{i \geq 1} \max(\lceil \frac{\lambda_i - a}{2} \rceil, 0) := \lambda_a$ for $a \geq 1$.

For $(v, x) \in \mathfrak{N}$, say that a λ-filtration is "C-adapted" to (v, x) if $v \in \mathbb{V}_{\geq 1}$ and $x(\mathbb{V}_{\geq a}) \subseteq \mathbb{V}_{\geq a+2}$ for all a.

Proposition 2.6. *If $(v, x) \in \mathbb{O}_{\mu,\nu}$, then there is a unique (μ, ν)-filtration $(\mathbb{V}_{\geq a})_{a \in \mathbb{Z}}$ that is C-adapted to (v, x).*

Proof. See Theorem 5.5 of [5]. $\hfill\square$

Definition 2.7. Fix a specific $(v, x) \in \mathbb{O}_{\mu,\nu}$, and the corresponding (μ, ν)-filtration $(\mathbb{V}_{\geq a})_{a \in \mathbb{Z}}$. Let $P \subset G$ be the parabolic subgroup stabilizing this isotropic flag. Define:

$$\mathfrak{s}_{\geq 2} = \{x \in \mathfrak{s} \, | \, x(V_{\geq a}) \subseteq V_{\geq a+2}\}.$$

With this choice, it follows that $\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}$ is a P-submodule of $\mathbb{V} \oplus \mathfrak{s}$; let $\widetilde{\mathbb{O}_{(\mu,\nu)^C}} = G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2})$.

Proposition 2.8. *The image of the natural map $G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to \mathfrak{N}$ is $\overline{\mathbb{O}_{(\mu,\nu)^C}}$, and the map $\pi_{(\mu,\nu)^C} : \widetilde{\mathbb{O}_{(\mu,\nu)^C}} \to \overline{\mathbb{O}_{(\mu,\nu)^C}}$ is a resolution of singularities. Further, $\pi_{(\mu,\nu)^C}$ is an isomorphism restricted to $\pi_{(\mu,\nu)^C}^{-1}(\mathbb{O}_{\mu,\nu})$.*

Proof. See Theorem 5.7 in [5]. $\hfill\square$

Note that the definition above of $\widetilde{\mathbb{O}_{(\mu,\nu)^C}}$, and the definition in Section 2.1 of $\widetilde{\mathfrak{N}}$ are not entirely canonical (they depend on choices of (v, x), and a Cartan T, respectively). This is slightly inconvenient as we now want to relate the two varieties; the easiest way to fix the problem is via the equivalent, canonical descriptions below.

Lemma 2.9. *Let $B' \subset P$ be a Borel subgroup.*

$$\widetilde{\mathfrak{N}} \simeq \{(0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_n \subset \cdots \subset \mathbb{V}_{2n} = \mathbb{V}), (v, s) \in \mathfrak{N} \mid \dim \mathbb{V}_i = i,$$
$$\mathbb{V}_i = \mathbb{V}_{2n-i}^{\perp}, v \in \mathbb{V}_n, s\mathbb{V}_{i+1} \subseteq \mathbb{V}_i\},$$

$$\widetilde{\mathbb{O}_{(\mu,\nu)^C}} \simeq \{(\mathbb{V}_{\geq i}), (v, x) \in \mathfrak{N} \mid (\mathbb{V}_{\geq i}) \text{ is } (\mu, \nu)\text{-adapted to } (v, x)\},$$

$$\widetilde{\mathfrak{N}} \supset G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \simeq \{((0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_n \subset \cdots \subset \mathbb{V}_{2n} = \mathbb{V}), (v, s)) \in \widetilde{\mathfrak{N}} \mid$$
$$(\mathbb{V}_{\lambda_a}) \text{ is } (\mu, \nu)\text{-adapted to } (v, x)\}.$$

Definition 2.10. Let $\widehat{\mathbb{O}_{\mu,\nu}} = G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \cap \pi^{-1}(\overline{\mathbb{O}_{\mu,\nu}})$ be a subvariety of $\widetilde{\mathfrak{N}}$.

Corollary 2.11. *Consider the natural map $\theta_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \overline{\mathbb{O}_{\mu,\nu}}$. The fibres of $\theta_{\mu,\nu}$ over a point in $\mathbb{O}_{\mu,\nu}$ are flag varieties for the Levi subgroup of P, and consequently have vanishing cohomology in degrees greater than 0.*

Proof. By construction, the fibres of $\theta_{\mu,\nu}$ are the same as the fibres of the composite map $G \times_{B'} (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) \to \overline{\mathbb{O}_{\mu,\nu}}$. The second map has singleton fibres over $\mathbb{O}_{\mu,\nu}$ by Proposition 2.8, and the first map has fibres $P/B' \simeq L/L \cap B'$ where L is the Levi subgroup of P. The cohomology vanishing for the structure sheaf follows from Borel-Weil-Bott. $\hfill\square$

Example 2.12. Let $n = 6$, $\mu = (1^3)$, $\nu = (3)$. Then replacing successive terms $(2s, 2t)$ of the sequence $(2, 6, 2, 0, 2, 0)$ with $(s+t, s+t)$ if $s < t$, we have $\Phi^C(\mu, \nu) = \lambda = (4, 4, 2, 1, 1) \in \mathcal{P}_{12}^C$. Then $\lambda' = (2, 2, 1, 0, 1, 0)$, so $(\mu, \nu)^C = \widetilde{\Phi}^C(\lambda) = ((2, 1^2), (2))$. Using Definition 2.5, $\mathbb{V}_{\geq 4} = 0$, $\dim \mathbb{V}_{\geq 3} = \dim \mathbb{V}_{\geq 2} = 2$, $\dim \mathbb{V}_{\geq 1} = 5$, $\dim \mathbb{V}_{\geq 0} = 7$, $\dim \mathbb{V}_{\geq -1} = \dim \mathbb{V}_{\geq -2} = 10$, $\mathbb{V}_{\geq -3} = \mathbb{V}$. Thus,

$$\widetilde{\mathbb{O}_{(\mu,\nu)^C}} = G \times_P (\mathbb{V}_{\geq 1} \oplus \mathfrak{s}_{\geq 2}) = \{(0 \subset \mathbb{V}_{\geq 3} \subset \mathbb{V}_{\geq 1} \subset \mathbb{V}_{\geq 0} \subset \mathbb{V}_{\geq -2} \subset \mathbb{V}), (v, x) |$$
$$v \in \mathbb{V}_{\geq 1}, x\mathbb{V} \subseteq \mathbb{V}_{\geq -2}, x\mathbb{V}_{\geq -2} \subseteq \mathbb{V}_{\geq 1}, x\mathbb{V}_{\geq 0} \subseteq \mathbb{V}_{\geq 3}, x\mathbb{V}_{\geq 3} = 0\}$$

Proposition 2.8 now implies that the fibres of the map $\pi_{\mu,\nu^C} : \widetilde{\mathbb{O}_{(\mu,\nu)^C}} \to \mathbb{O}_{(\mu,\nu)^C}$ over $\mathbb{O}_{\mu,\nu}$ are singletons. To check this, it suffices to compute the fibre of π_{μ,ν^C} over the following point $(v,x) \in \mathbb{O}_{\mu,\nu}$:

$$
v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 & & & & & & & & & \\ & 0 & 1 & & & & & & & & \\ & & 0 & 1 & & & & & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & 1 & \\ & & & & & & & & & 0 & 1 \\ & & & & & & & & & & 0 & 1 \\ & & & & & & & & & & & 0 \end{pmatrix}.
$$

We have dim $(\mathrm{Im}\ (x^2) \oplus \mathbb{C}v) = 5$; since $\mathrm{Im}\ (x^2) \oplus \mathbb{C}v \subseteq \mathbb{V}_{\geq 1}$ it follows $\mathbb{V}_{\geq 1} = \mathrm{Im}\ (x^2) \oplus \mathbb{C}v, \mathbb{V}_{\geq 0} = (\mathrm{Im}\ (x^2) \oplus \mathbb{C}v)^\perp$. Since $\mathrm{Im}(x^3) \subset \mathbb{V}_{\geq 3}$ and both vector spaces have dimension 2, $\mathbb{V}_{\geq 3} = \mathrm{Im}(x^3), \mathbb{V}_{\geq -2} = \mathrm{Im}\ (x^3)^\perp$. Hence $\pi_{(\mu,\nu)^C}^{-1}(v,x)$ is a single flag, as expected.

2.3. Vanishing higher cohomology of dominant line bundles on $\widetilde{\mathfrak{N}}$.

Definition 2.13. Denote by $p : \widetilde{\mathfrak{N}} \to G/B$ the natural projection. Let $\Lambda^+ \subset \Lambda$ denote respectively the dominant weights, and weight lattice of G. For $\lambda \in \Lambda^+$, let $\mathbb{C}_\lambda$ be the 1-dimensional representation of B where the torus T acts by λ. Given a B-representation V, denote $\mathcal{L}_{G/B}(V)$ for the sheaf of sections of the vector bundle $G \times_B V$; in particular, define $\mathcal{O}_{G/B}(\lambda) = \mathcal{L}_{G/B}(\mathbb{C}_\lambda^*)$. Following Borel-Weil, $H^0(G/B, \mathcal{O}_{G/B}(\lambda)) = V_\lambda^*$ is the dual of the finite-dimensional irreducible G-module with highest weight λ. Denote $\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda) = p^*\mathcal{O}_{G/B}(\lambda)$.

Lemma 2.14.
$$V_\lambda^* \simeq V_\lambda.$$

Proof. On page 116 of [17], it is proven that V_λ^* is isomorphic to $V(-\sigma\lambda)$ (here $\sigma \in W$ is the unique element of W which sends Δ to $-\Delta$; here Δ is the set of roots). In type C it is easy to check that one can find $\sigma \in W$, such that $\sigma\lambda = -\lambda$ for all $\lambda \in \Lambda$. The conclusion follows. $\qquad\square$

Theorem 2.15. *For* $\lambda \in \Lambda^+, H^i(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) = 0$ *for* $i > 0$.

Recall the following theorem of Grauert-Riemenschneider in Kempf's version (see [22], Theorem 4, and also [24], Theorem 3.4).

Proposition 2.16. *Let* U *be an algebraic variety, and* ω_U *the canonical line bundle. If there is a proper generically finite morphism* $U \to X$, *where* X *is an affine variety, then* $H^i(U, \omega_U) = 0$ *for* $i > 0$.

Lemma 2.17. *Let* $U = G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})$. *Then* $H^i(U, \omega_U) = 0$ *for* $i > 0$.

Proof. In accordance with 2.16, it suffices to find an affine variety X with a proper, generically finite map $U \to X$. Let $X' = (\mathbb{V} \oplus \mathfrak{s})^+ \oplus V_{\lambda'}$, and define $\pi : U \to X'$

by $\pi(g, (x, y)) = (gx, gy)$ where $g \in G$, $x \in (\mathbb{V} \oplus \mathfrak{s})^+$, $y \in \mathbb{C}_{\lambda'} \subset V_{\lambda'}$. The map π is proper as we can factorize it as follows (note G/B is projective):

$$G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}) \hookrightarrow G \times_B ((\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'}) \simeq G/B$$
$$\times ((\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'}) \to (\mathbb{V} \oplus \mathfrak{s}) \oplus V_{\lambda'}$$

Let $X = \operatorname{im}(\pi)$; we claim X has the required property. Note that the map $U \to X$ generically has singleton fibres over a point $(s, 0) \in X$, with $s \in (\mathbb{V} \oplus \mathfrak{s})^+$. Since it is also proper, its fibres are generically finite, as required. $\square$

Denote the natural projection $p_U : G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}) \to G/B$, where $\lambda' = \lambda + \epsilon_1 + \cdots + \epsilon_n$.

Lemma 2.18. *We have an isomorphism $\omega_U \simeq p_U^* \mathcal{O}_{G/B}(\lambda)$.*

Proof. Since $p_U : U \to G/B$ has fibre $(\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}$, we have a short exact sequence of vector bundles on U (noting $\Omega_{G/B} \simeq \mathcal{L}_{G/B}(\mathfrak{u})$):

$$0 \to p_U^* \mathcal{L}_{G/B}(\mathfrak{u}) \to \Omega_U \to p_U^* \mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})^*) \to 0.$$

Since $\omega_U = \wedge^{top}(\Omega_U)$, it now follows that $\omega_U \simeq p_U^* \mathcal{L}_{G/B}(V)$ where:

$$V = \wedge^{top}(\mathfrak{u} \oplus ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'})^*) = \wedge^{top}(\mathfrak{u}) \otimes \wedge^{top}((\mathbb{V} \oplus \mathfrak{s})^{+*}) \otimes \mathbb{C}_{\lambda'}^*.$$

Noting that, as B-modules, $\mathfrak{u}$ has weights $\epsilon_i - \epsilon_j$ for $i < j$, $\epsilon_i + \epsilon_j$, and $2\epsilon_i$ and $(\mathbb{V} \oplus \mathfrak{s})^+$ has weights $\epsilon_i - \epsilon_j$ for $i < j$, $\epsilon_i + \epsilon_j$, and ϵ_i; we can conclude that $V = \mathbb{C}_{\epsilon_1 + \cdots + \epsilon_n} \otimes \mathbb{C}_{\lambda'}^* = \mathbb{C}_\lambda^*$, as required. $\square$

Proof of Theorem. Using the above two lemmas, we compute as follows (for $i > 0$):

$$0 = H^i(U, \omega_U) = H^i(G \times_B ((\mathbb{V} \oplus \mathfrak{s})^+ \oplus \mathbb{C}_{\lambda'}), p_U^* \mathcal{L}_{G/B}(\mathbb{C}_\lambda^*))$$

$$= H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda^* \otimes S((\mathbb{V} \oplus \mathfrak{s})^{+*} \oplus \mathbb{C}_{\lambda'}^*)))$$

$$= \bigoplus_{j,n \geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda^* \otimes S^j(\mathbb{V} \oplus \mathfrak{s})^{+*} \otimes S^n(\mathbb{C}_{\lambda'}^*)))$$

$$= \bigoplus_{j,n \geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_{\lambda+n\lambda'}^* \otimes S^j(\mathbb{V} \oplus \mathfrak{s})^{+*}))$$

$$= \bigoplus_{n \geq 0} H^i(G \times_B (\mathbb{V} \oplus \mathfrak{s})^+, p^* \mathcal{L}_{G/B}(\mathbb{C}_{\lambda+n\lambda'}^*)).$$

Taking the $n = 0$ summand in the above gives the desired result. $\square$

Corollary 2.19. *We have $H^i(\widetilde{\mathfrak{N}}, \omega_{\widetilde{\mathfrak{N}}}) = 0$ for $i > 0$.*

Proof. First we calculate that $\omega_{\widetilde{\mathfrak{N}}} = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$. Following the method used in the proof of Lemma 2.18, we have the short exact sequence:

$$0 \to p^* \mathcal{L}_{G/B}(\mathfrak{u}) \to \Omega_{\widetilde{\mathfrak{N}}} \to p^* \mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^{+*}) \to 0.$$

So it follows that $\omega_{\widetilde{\mathfrak{N}}} = \wedge^{top}(\Omega_{\widetilde{\mathfrak{N}}}) = p^* \mathcal{L}_{G/B}(V)$, where $V = \wedge^{top}(\mathfrak{u}) \otimes \wedge^{top}((\mathbb{V} \oplus \mathfrak{s})^{+*}) = \mathbb{C}_{\epsilon_1 + \cdots + \epsilon_n}$ using above calculations. Thus $\omega_{\widetilde{\mathfrak{N}}} \simeq \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$, as required.

Now note that the proof of Theorem 2.14 actually holds under the weaker assumption that $\lambda' = \lambda + \epsilon_1 + \cdots + \epsilon_n$ is dominant; since $\lambda = -\epsilon_1 - \cdots - \epsilon_n$ satisfies this hypothesis, the result follows. $\square$

2.4. **Global sections of dominant line bundles on $\widetilde{\mathfrak{N}}$.** In this section, we will use the results of the previous section to compute the global sections of dominant line bundles on $\widetilde{\mathfrak{N}}$. Given a coherent sheaf $\mathcal{E}$ on G/B, since $H^i(G/B, \mathcal{E})$ acquires the structure of a G-module, we adopt a convenient abuse of notation whereby $H^i(G/B, \mathcal{E})$ denotes the corresponding element of $K(\text{Rep}(G))$; denote χ_λ to be the image in $K(\text{Rep}(G))$ of the irreducible module with highest weight λ. We will also need the following notation.

Definition 2.20. Let $p, p' : \Lambda \to \mathbb{Z}$ denote the Kostant partition function in types B and C, respectively, defined as follows:

$$\frac{1}{\prod_{i<j}(1 - e^{\epsilon_i - \epsilon_j})(1 - e^{\epsilon_i + \epsilon_j}) \prod_i (1 - e^{\epsilon_i})} = \sum_{\mu \in \Lambda} p(\mu) e^\mu,$$

$$\frac{1}{\prod_{i<j}(1 - e^{\epsilon_i - \epsilon_j})(1 - e^{\epsilon_i + \epsilon_j}) \prod_i (1 - e^{2\epsilon_i})} = \sum_{\mu \in \Lambda} p'(\mu) e^\mu.$$

Denote $\rho \in \Lambda^+$ to be the half-sum of the positive roots. Also, for $\mu \in \Lambda$, denote $\text{conv}(\mu)$ to be the intersection of Λ with the convex hull of the set $\{w\mu \mid w \in W\}$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ (here W denotes the Weyl group of G). Denote also $\text{conv}^0(\lambda)$ to be the complement of $\{w\lambda \mid w \in W\}$ in $\text{conv}(\lambda)$.

Theorem 2.21. *For $\lambda, \mu \in \Lambda^+$,*

$$\dim \text{Hom}_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = \sum_{w \in W} \text{sgn}(w) p'(w(\mu + \rho) - (\lambda + \rho)).$$

Proof. Using Theorem 2.15 and the additivity of the Euler characteristic, we can compute that:

$$H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) = \sum_{i \geq 0} (-1)^i H^i(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))$$

$$= \sum_{i \geq 0} (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda^* \otimes S((\mathbb{V} \oplus \mathfrak{s})^{+*})))$$

$$= \sum_{\mu \in \Lambda} p'(\mu) \sum_{i \geq 0} (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_{\lambda + \mu}^*)).$$

Above we have used the filtration of $S((\mathbb{V} \oplus \mathfrak{s})^{+*})$ by 1-dimensional B-modules, where $\mathbb{C}_\mu^*$ occurs with multiplicity $p'(\mu)$. We continue using the Borel-Weil-Bott Theorem, which states $\sum_{i \geq 0} H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_\lambda^*)) = \text{sgn}(w)\chi_{w(\lambda + \rho) - \rho}$, where $w \in W$ is the unique Weyl group element such that $w(\lambda + \rho) - \rho \in \Lambda^+$; this implies that $\sum_{i \geq 0} (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathbb{C}_{\lambda + \mu}^*)) = \text{sgn}(w)\chi_\mu$ for $\mu \in \Lambda^+ \Leftrightarrow \mu = w^{-1}(\mu + \rho) - (\lambda + \rho)$. The result now follows:

$$H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = \sum_{w \in W, \mu \in \Lambda^+} \text{sgn}(w) p'(w^{-1}(\mu + \rho) - (\lambda + \rho))\chi_\mu$$

$$= \sum_{\mu \in \Lambda^+} \chi_\mu \sum_{w \in W} \text{sgn}(w) p'(w(\mu + \rho) - (\lambda + \rho)). \qquad \square$$

Recall also the Weyl character formulae (see e.g. [17]):

Proposition 2.22. *For $\mu \in \Lambda^+$, let m_μ^λ denote the multiplicity of the weight λ in V_μ. Then*

$$m_\mu^\lambda = \sum_{w \in W} \operatorname{sgn}(w) p(w(\mu + \rho) - (\lambda + \rho)).$$

Proposition 2.23. $\operatorname{Hom}_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = 0$ *unless* $\lambda \in \operatorname{conv}(\mu)$.

Proof. From Definition 2.20, first note the following identity:

$$\prod_i (1 + e^{\epsilon_i})(\sum_{\mu \in \Lambda} p(\mu) e^\mu) = \sum_{\mu \in \Lambda} p'(\mu) e^\mu$$

$$p'(\mu) = \sum_{S \subseteq \{1, \cdots, n\}} p(\mu - \sum_{i \in S} \epsilon_i)$$

$$\dim \operatorname{Hom}_G(V_\mu, H^0(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda))) = \sum_{w \in W} \operatorname{sgn}(w) p'(w(\mu + \rho) - (\lambda + \rho))$$

$$= \sum_{w \in W, S \subseteq \{1, \cdots, n\}} \operatorname{sgn}(w) p(w(\mu + \rho) - (\lambda + \sum_{i \in S} \epsilon_i + \rho))$$

$$= \sum_{S \subseteq \{1, \cdots, n\}} m_\mu^{\lambda + \sum_{i \in S} \epsilon_i}.$$

It is well known that $m_\mu^\lambda = 0$ unless $\lambda \in \operatorname{conv}(\mu)$ (since $m_\mu^\lambda = 0$ unless $\lambda \preccurlyeq \mu$, and $m_\mu^\lambda = m_\mu^{w\lambda}$ for $w \in W$). Since $\lambda \in \Lambda^+$, it then follows that for the above quantity to be nonzero, $\lambda \in \operatorname{conv}(\mu)$. $\qquad\square$

3. A QUASI-EXCEPTIONAL SET GENERATING $D^b(\operatorname{Coh}^G(\mathfrak{N}))$

3.1. Recollections. For the reader's convenience, in this section we give a summary of the results in Section 1 of [10] (see also [25]) regarding quasi-exceptional sets and quasi-hereditary categories that will be relevant to us in the next section.

Let $\mathcal{C}$ be a triangulated category; for $C_1, C_2 \in \mathcal{C}$, denote $\operatorname{Hom}^\bullet(C_1, C_2) = \bigoplus_n \operatorname{Hom}(C_1, C_2[n])$ (and $\operatorname{Hom}^{<0}(C_1, C_2) = \bigoplus_n \operatorname{Hom}(C_1, C_2[n])$). Given two sets $\mathcal{S}_1, \mathcal{S}_2$ of objects in $\mathcal{C}$, define $\mathcal{S}_1 \star \mathcal{S}_2$ to be the set of objects X for which there is a distinguished triangle $S_1 \to X \to S_2 \to S_1[1]$; the axioms of a triangulated category imply that this operation is associative. Given a set of objects $\mathcal{S}$ in $\mathcal{C}$, define $\langle \mathcal{S} \rangle = \mathcal{S} \cup \mathcal{S} \star \mathcal{S} \cup \mathcal{S} \star \mathcal{S} \star \mathcal{S} \cup \cdots$. Then $\langle \bigcup_i \mathcal{S}[i] \rangle$ is the smallest strictly full triangulated subcategory containing $\mathcal{S}$.

Definition 3.1. Let I be a totally ordered set. A dualizable quasi-exceptional set in $\mathcal{C}$ consists of a subset $\nabla = \{\nabla_i | i \in I\}$, and its dual $\Delta = \{\Delta_i | i \in I\}$, satisfying the following properties:

- $\operatorname{Hom}^\bullet(\nabla_i, \nabla_j) = 0$ if $i < j$.
- $\operatorname{Hom}^{<0}(\nabla_i, \nabla_i) = 0$ and $\operatorname{Hom}(\nabla_i, \nabla_i) = \mathbb{C}$ for all $i \in I$.
- $\operatorname{Hom}(\Delta_j, \nabla_i) = 0$ if $j > i$.
- $\Delta_i \simeq \nabla_i (\operatorname{mod} \mathcal{C}/\mathcal{C}_{<i})$, where $\mathcal{C}_{<i}$ is the smallest strictly full triangulated subcategory containing the objects $\{\nabla_j | j < i\}$, and $\mathcal{C}/\mathcal{C}_{<i}$ is the Verdier quotient category.

Proposition 3.2. *Suppose that $\{\nabla_i | i \in I\}$ generate $\mathcal{C}$. Then $\mathcal{C}$ has a unique t-structure $(\mathcal{C}^{\geq 0}, \mathcal{C}^{\leq 0})$ such that $\nabla_i \in \mathcal{C}^{\geq 0}, \Delta_i \in \mathcal{C}^{\leq 0}$. It is given by $\mathcal{C}^{\geq 0} = \langle \{\nabla_i[d], i \in I, d \leq 0\} \rangle, \mathcal{C}^{\leq 0} = \langle \{\Delta_i[d], i \in I, d \geq 0\} \rangle$.*

Proof. See Proposition 1 in [4].   □

We define a quasi-hereditary category in preparation for the next result, which shows that the heart of the above *t*-structure is quasi-hereditary.

Definition 3.3. An abelian category $\mathcal{A}$ has simple objects $\{S_i\}$ indexed by an index set I is quasi-hereditary if it satisfies the following properties:

- For each simple S_i, there is an object A_i with a non-zero morphism $\alpha : A_i \to S_i$, known as its "standard cover", such that:
  - $\ker(\alpha_i) \in \mathcal{A}_{<i}$
  - $\mathrm{Hom}(A_i, S_j) = \mathrm{Ext}^1(A_i, S_j) = 0$ for $j < i$.
- For each simple S_i, there is an object B_i with a non-zero morphism $\beta_i : S_i \to B_i$, known as its "costandard hull", such that
  - $\mathrm{coker}(\beta_i) \in \mathcal{A}_{<i}$.
  - $\mathrm{Hom}(S_j, B_i) = \mathrm{Ext}^1(S_j, B_i) = 0$ for $j < i$.

Remark 3.4. The usual definition of "quasi-hereditary" categories is that $\mathrm{Hom}(A_i, S_j) = \mathrm{Ext}^1(A_i, S_j) = 0$ for $j \leq i$, and $\mathrm{Hom}(S_j, B_i) = \mathrm{Ext}^1(S_j, B_i) = 0$ for $j \leq i$. However, we will need to use the weaker definition above.

Denote by $\tau_{\geq 0}, \tau_{\leq 0}$ the truncation functors associated to the above *t*-structure. Let $\Delta_i' = \tau_{\geq 0}(\Delta_i), \nabla_i' = \tau_{\leq 0}(\nabla_i)$ be objects in the heart of the *t*-structure.

Proposition 3.5. *There exists a morphism $\theta_i : \Delta_i' \to \nabla_i'$, such that $S_i := im(\theta_i)$ is a simple object in $C^{\geq 0} \cap C^{\leq 0}$. Each simple object in $C^{\geq 0} \cap C^{\leq 0}$ is isomorphic to S_i for some i. The map $\Delta_i' \to S_i$ is a standard cover of S_i, and the inclusion $S_i \to \nabla_i'$ is a costandard hull for S_i. Thus, $C^{\geq 0} \cap C^{\leq 0}$ is a quasi-hereditary category.*

3.2. **Main Results.** Following the techniques in [10], we may now prove analogous results about $\mathfrak{N}$ by applying the results in Section 1. However, we will need to use the following twisted action of W on Λ instead of the usual one:

Definition 3.6. Let $\theta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)$. Given $w \in W, \lambda \in \Lambda$, define the twisted action $w \cdot \lambda = w(\lambda + \theta) - \theta$.

Definition 3.7. Let $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ denote the derived bounded category of G-equivariant coherent sheaves on $\mathfrak{N}$. For $\lambda \in \Lambda$, define $\mathbf{AJ}_\lambda = R\pi_* \mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[d]$, where $d = \frac{\dim \mathfrak{N}}{2}$. Given $S \subset \Lambda$, denote by $\mathfrak{D}_S$ to be the smallest strictly full triangulated subcategory of $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ containing the objects $\mathbf{AJ}_\lambda$ for $\lambda \in S$.

Lemma 3.8 (Grothendieck-Serre Duality)**.** *The (equivariant) dualizing sheaf on $\widetilde{\mathfrak{N}}$ is isomorphic to $\omega_{\widetilde{\mathfrak{N}}}[2d]$; the functor $\mathcal{S} : D^b(\mathrm{Coh}^G(\mathfrak{N})) \to D^b(\mathrm{Coh}^G(\mathfrak{N}))^{op}$ of Grothendieck-Serre duality is an anti-autoequivalence satisfying $\mathcal{S}(\mathbf{AJ}_\lambda) = \mathbf{AJ}_{-\lambda-2\theta}$.*

Proof. Since $H^i(\widetilde{\mathfrak{N}}, \omega_{\widetilde{\mathfrak{N}}}) = 0$ for $i > 0$ using 2.19, the equivariant dualizing sheaf on $\widetilde{\mathfrak{N}}$ is isomorphic to $\omega_{\widetilde{\mathfrak{N}}}[2d]$. Since $\omega_{\widetilde{\mathfrak{N}}} \simeq \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\epsilon_1 - \cdots - \epsilon_n)$ (again from 2.19), we have

$$R\underline{\mathrm{Hom}}(\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)[d], \omega_{\widetilde{\mathfrak{N}}}[2d]) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\lambda - \epsilon_1 - \cdots - \epsilon_n)[d].$$

Recalling that by definition, $\mathcal{S}(\mathcal{F}) = R\underline{\mathrm{Hom}}(\mathcal{F}, \mathbb{D}C)$, where $\mathbb{D}C$ is the equivariant dualizing sheaf; and that Grothendieck-Serre duality commutes with $R\pi_*$ since π is proper, the result follows.   □

Definition 3.9. For $\mu \in \Lambda$, let $\widetilde{\mathrm{conv}}(\mu)$ denote the intersection of the convex hull of the set $\{w \cdot \mu | w \in W\}$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with Λ, and denote $\widetilde{\mathrm{conv}}^0(\mu)$ to be the complement of $\{w \cdot \mu | w \in W\}$ in $\widetilde{\mathrm{conv}}(\mu)$.

Proposition 3.10. *For $w \in W$, $\mathbf{AJ}_{w \cdot \lambda} \simeq \mathbf{AJ}_\lambda (\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)})$.*

Proof. Let α be a simple root for G (i.e., $\alpha \in \{\epsilon_1 - \epsilon_2, \cdots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$); let $\widetilde{\alpha} = \alpha$ if $\alpha \neq 2\epsilon_n$, and $\widetilde{\alpha} = \epsilon_n$ if $\alpha = 2\epsilon_n$. By induction on the length of the word $w \in W$, it suffices to prove that $\mathbf{AJ}_\lambda \simeq \mathbf{AJ}_{s_\alpha \cdot \lambda}(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)})$, for λ satisfying $\langle \lambda, \check{\alpha} \rangle = n > 0$. Let P_α be the minimal parabolic in G corresponding to the root α; denote by p_α the projection $G/B \to G/P_\alpha$. Since G is simply connected, we may choose $\lambda' \in \Lambda$ such that $\langle \lambda', \check{\alpha} \rangle = n - 1$. Define $\mathcal{V}_{\lambda'} = p_{\alpha,*} p_\alpha^* \mathcal{O}_{G/B}(\lambda')$ to be the G-equivariant vector bundle on G/B with a filtration whose subquotients are $\mathcal{O}_{G/B}(\lambda'), \cdots, \mathcal{O}_{G/B}(\lambda' - (n-1)\alpha)$. Let $(\mathbb{V} \oplus \mathfrak{s})_\alpha^+$ be the P_α-submodule of $(\mathbb{V} \oplus \mathfrak{s})^+$ obtained by taking the sum of all weight spaces excluding the weight space corresponding to $\widetilde{\alpha}$. Let $\widetilde{\mathfrak{N}}_\alpha = G \times_B (\mathbb{V} \oplus \mathfrak{s})_\alpha^+$, and let $\iota_\alpha : \widetilde{\mathfrak{N}}_\alpha \hookrightarrow \widetilde{\mathfrak{N}}$ be the embedding of this divisor. Let $\pi_\alpha : \widetilde{\mathfrak{N}}_\alpha = G \times_B (\mathbb{V} \oplus \mathfrak{s})_\alpha^+ \to G \times_{P_\alpha} (\mathbb{V} \oplus \mathfrak{s})_\alpha^+$ denote the projection, whose fibres are isomorphic to $\mathbb{P}^1$. Since $\langle \lambda - \lambda' - \alpha, \check{\alpha} \rangle = -1$, the restriction of $p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$ is isomorphic to the sum of several copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ when restricted to any fibre of π_α. Thus we have:

$$\pi_{\alpha*}(\mathcal{O}_{\widetilde{\mathfrak{N}_\alpha}} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0,$$
$$\pi_*(\mathcal{O}_{\widetilde{\mathfrak{N}_\alpha}} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0.$$

Under the embedding of the divisor ι_α, $\mathcal{O}(\widetilde{\mathfrak{N}}_\alpha) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha})$ (see Lemma 3.11 for a proof of this statement), giving us the exact sequence:

$$0 \to \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha}) \to \mathcal{O}_{\widetilde{\mathfrak{N}}} \to \iota_{\alpha*} \mathcal{O}_{\widetilde{\mathfrak{N}}_\alpha} \to 0.$$

Since $\pi_*(\mathcal{O}_{\widetilde{\mathfrak{N}_\alpha}} \otimes p^*(\mathcal{V}_{\lambda'})(\lambda - \lambda' - \alpha)) = 0$, tensoring this exact sequence with $p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha)$ and taking direct image under π, we have:

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha + \widetilde{\alpha}) \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha).$$

Case 1. Suppose $\alpha \neq 2\epsilon_n$. Then $\alpha = \widetilde{\alpha}$, so

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda') \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \alpha).$$

Since $\mathcal{V}_{\lambda'}(\lambda - \lambda')$ has a filtration whose quotients are $\mathcal{O}_{G/B}(\lambda), \cdots, \mathcal{O}_{G/B}(\lambda - (n-1)\alpha)$, we find that $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \in [\mathbf{AJ}_{\lambda-(n-1)\alpha}] * \cdots * [\mathbf{AJ}_\lambda]$. Since $\mathbf{AJ}_{\lambda-(n-1)\alpha}, \cdots, \mathbf{AJ}_{\lambda-\alpha} \in \mathfrak{D}_{\widetilde{conv}^0(\lambda)}$, this implies

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \simeq \mathbf{AJ}_\lambda(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)}).$$

Similarly, we find that $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda')[d] \in [\mathbf{AJ}_{\lambda-n\alpha}] * \cdots * [\mathbf{AJ}_{\lambda-\alpha}]$, and hence $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda-\lambda'-\alpha)[d] \simeq \mathbf{AJ}_{\lambda-n\alpha}(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)})$. Comparing, the required identity follows (note $s_\alpha \cdot \lambda = \lambda - n\alpha$ in this case): $\mathbf{AJ}_{s_\alpha \cdot \lambda} \simeq \mathbf{AJ}_\lambda(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)})$.

Case 2. Suppose $\alpha = 2\epsilon_n$; then $\widetilde{\alpha} = \epsilon_n$, so

$$\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n) \simeq \pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n).$$

Note that $s_{2\epsilon_n} \cdot (\lambda - \epsilon_n) = \lambda - 2n\epsilon_n$. Similarly to the above, since $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n)[d] \in [\mathbf{AJ}_{\lambda-(2n-1)\epsilon_n}] * \cdots * [\mathbf{AJ}_{\lambda-\epsilon_n}]$, we find $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - \epsilon_n)[d] \simeq \mathbf{AJ}_{\lambda-\epsilon_n}(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda-\epsilon_n)})$. Since $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n)[d] \in [\mathbf{AJ}_{\lambda-2n\epsilon_n}] * \cdots *$

$[\mathbf{AJ}_{\lambda-2\epsilon_n}]$, we find $\pi_* p^* \mathcal{V}_{\lambda'}(\lambda - \lambda' - 2\epsilon_n)[d] \simeq \mathbf{AJ}_{\lambda-2n\epsilon_n} (\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda-\epsilon_n)})$. Comparing, we see that $\mathbf{AJ}_{\lambda-\epsilon_n} \simeq \mathbf{AJ}_{s_{2\epsilon_n}\cdot(\lambda-\epsilon_n)}(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda-\epsilon_n)})$. Replacing $\lambda - \epsilon_n$ by λ, we get the result. $\qquad\square$

Lemma 3.11. *We have* $\mathcal{O}(\widetilde{\mathfrak{N}}_\alpha) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\widetilde{\alpha})$.

Proof. By definition, $\widetilde{\mathfrak{N}}$ is the total space of the vector bundle $\mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^+)$. Consider the pull-back line bundle $\mathcal{E} = p^* \mathcal{L}_{G/B}((\mathbb{V} \oplus \mathfrak{s})^+)$ on $\widetilde{\mathfrak{N}}$; denote by τ the canonical section of $\Gamma(\widetilde{\mathfrak{N}}, \mathcal{E})$. The surjection of B-modules $(\mathbb{V}\oplus\mathfrak{s})^+ \to (\mathbb{V}\oplus\mathfrak{s})^+/(\mathbb{V}\oplus\mathfrak{s})_\alpha^+ \simeq \mathbb{C}_{\widetilde{\alpha}}$ gives a map $\mathcal{L}_{G/B}(\mathbb{V} \oplus \mathfrak{s})^+ \to \mathcal{O}_{G/B}(-\tilde{\alpha})$, and hence a map $\phi_\alpha : \mathcal{E} \twoheadrightarrow \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha})$. Let $\tau_\alpha \in \Gamma(\widetilde{\mathfrak{N}}, \mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha}))$ denote the image of τ. Then by construction $\widetilde{\mathfrak{N}}_\alpha$ is the zero set of the section τ_α. It follows that $\mathcal{O}_{\widetilde{\mathfrak{N}}}(-\tilde{\alpha}) = \mathcal{O}_{\widetilde{\mathfrak{N}}}(d\widetilde{\mathfrak{N}}_\alpha)$ for some $d \in \mathbb{Z}^+$; but since $\mathrm{Pic}(\widetilde{\mathfrak{N}}) = \Lambda$ we find $d = 1$, as required. $\qquad\square$

Proposition 3.12. (1) *For* $\lambda \in \Lambda^+$, $\mathrm{Hom}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = \mathbb{C}$ *and* $\mathrm{Hom}^{<0}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = 0$.

    (2) *For* $\lambda \in \Lambda^+$, *we have* $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$ *if* $\lambda \notin \widetilde{\mathrm{conv}}(\mu)$.
    (3) *For* $\lambda, \mu \in \Lambda^+$, $\lambda \neq \mu$, *we have* $\mathrm{Hom}^\bullet(\mathbf{AJ}_{w_0 \cdot \mu}, \mathbf{AJ}_\lambda) = 0$.

Proof. (1) By Theorem 2.15, $\mathbf{AJ}_\lambda$ is concentrated in a single degree, so

$$\mathrm{Hom}^{<0}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = 0.$$

Since $\mathfrak{N}$ has an open, smooth G-orbit, and map $\phi : \mathbf{AJ}_\lambda \to \mathbf{AJ}_\lambda$ is determined by its restriction to the open orbit, where it must be a scalar, it follows that $\mathrm{Hom}(\mathbf{AJ}_\lambda, \mathbf{AJ}_\lambda) = \mathbb{C}$. (2) Fix λ. Assuming that for all $\mu' \in \Lambda^+ \cap \widetilde{\mathrm{conv}}^0(\mu)$, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\mu'}, \mathbf{AJ}_\lambda) = 0$; it suffices to prove that $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$. First assume $\mu \in \Lambda^+$ also. Noting that $V_\mu \otimes \mathcal{O}_{G/B} \in \mathrm{Coh}^G(G/B)$ has a filtration where the subquotient $\mathcal{O}_{G/B}(\lambda)$ occurs with multiplicity m_μ^λ, if the weights of V_μ are $\nu_1, \cdots, \nu_k$ it follows that:

$$V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d] = \pi_* p^*(V_\mu \otimes \mathcal{O}_{G/B})[d] \in [\mathbf{AJ}_{\nu_1}^{m_\mu^{\nu_1}}] * [\mathbf{AJ}_{\nu_2}^{m_\mu^{\nu_2}}] * \cdots * [\mathbf{AJ}_{\nu_k}^{m_\mu^{\nu_k}}].$$

Given $\mathcal{A}, \mathcal{B} \in D^b(\mathrm{Coh}^G(\mathfrak{N}))$, we view $\mathrm{Hom}^\bullet(\mathcal{A}, \mathcal{B})$ as an element of $D(\mathrm{Vect}_\mathbb{C})$. It now follows that:

$$\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d], \mathbf{AJ}_\lambda) \in [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_1}^{m_\mu^{\nu_1}}, \mathbf{AJ}_\lambda)]$$
$$* [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_2}^{m_\mu^{\nu_2}}, \mathbf{AJ}_\lambda)] * \cdots * [\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_k}^{m_\mu^{\nu_k}}, \mathbf{AJ}_\lambda)]$$

All $\nu_i \in \mathrm{conv}(\mu)$; by the induction hypothesis, if $\nu_i \in \mathrm{conv}^0(\mu) \subseteq \widetilde{\mathrm{conv}}^0(\mu)$, then $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}, \mathbf{AJ}_\lambda) = 0$; thus, for $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}^{m_\mu^{\nu_i}}, \mathbf{AJ}_\lambda) \neq 0$, we require $\nu_i = w\mu$ for some $w \in W$ (note $m_\mu^{w\mu} = 1$ in this case). Either $w\mu \in \widetilde{\mathrm{conv}}^0(\mu)$, or $w\mu = w \cdot \mu$. In the first case again we have $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}, \mathbf{AJ}_\lambda) = 0$ by the induction hypothesis. In the second case, by Proposition 3.10, $\mathbf{AJ}_{w \cdot \mu} \simeq \mathbf{AJ}_\mu(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\mu)})$, and by the induction hypothesis, $\mathrm{Hom}^\bullet(\mathcal{A}_\mu, \mathbf{AJ}_\lambda) = 0$ for any $\mathcal{A}_\mu \in \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\mu)}$, we may conclude $\mathrm{Hom}^\bullet(\mathbf{AJ}_{w \cdot \mu}, \mathbf{AJ}_\lambda) = \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)$. To summarize, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{\nu_i}^{m_\mu^{\nu_i}}, \mathbf{AJ}_\lambda)$ is either 0 or $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda)$. So rewriting the above:

$$\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_{\mathfrak{N}}[d], \mathbf{AJ}_\lambda) \in \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) * \cdots * \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda).$$

However, using Theorems 2.15 and Proposition 2.23, note that (since $\lambda \in \widetilde{\mathrm{conv}}(\mu) \to \lambda \in \mathrm{conv}(\mu)$):

$$\mathrm{Hom}^\bullet(V_\mu \otimes \mathcal{O}_\mathfrak{N}[d], \mathbf{AJ}_\lambda) = \mathrm{Hom}_G(V_\mu, R^\bullet\Gamma(\mathbf{AJ}_\lambda[-d])) = 0$$
$$\Rightarrow 0 \in \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) * \cdots * \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda).$$

Using Lemma 5, on page 12 of [10], we now conclude that $\mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$. Finally, as stated above, $\mathrm{Hom}^\bullet(\mathbf{AJ}_{w \cdot \mu}, \mathbf{AJ}_\lambda) = \mathrm{Hom}^\bullet(\mathbf{AJ}_\mu, \mathbf{AJ}_\lambda) = 0$; since any weight $\mu' \in \Lambda$ is of the form $w \cdot \mu$ for some $\mu \in \Lambda^+$, this removes the initial assumption that $\mu \in \Lambda^+$ and completing the proof of the statement. (3) If $\lambda \notin \widetilde{\mathrm{conv}}(\mu)$, then $\lambda \notin \widetilde{\mathrm{conv}}(w_0 \cdot \mu)$, and the result follows from the previous part. So suppose that $\lambda \in \widetilde{\mathrm{conv}}(\mu)$, i.e., that $\lambda + \theta \in \mathrm{conv}(\mu + \theta)$; then $\mu \notin \widetilde{\mathrm{conv}}(-\lambda - 2\theta)$. Recalling Lemma 3.8, from the previous part we deduce the required statement (noting that $w_0 \cdot \mu = w_0(\mu + \theta) - \theta = -\mu - 2\theta$ since $w_0\lambda = -\lambda$):

$$\mathrm{Hom}^\bullet(\mathbf{AJ}_{w_0 \cdot \mu}, \mathbf{AJ}_\lambda) = \mathrm{Hom}^\bullet(\mathcal{S}(\mathbf{AJ}_\lambda), \mathcal{S}(\mathbf{AJ}_{w_0 \cdot \mu})) = \mathrm{Hom}^\bullet(\mathbf{AJ}_{-\lambda - 2\theta}, \mathbf{AJ}_\mu) = 0.$$

$\square$

Lemma 3.13. $D^b(\mathrm{Coh}^G(\widetilde{\mathfrak{N}}))$ *is generated as a triangulated category by the objects* $\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)$ *for* $\lambda \in \Lambda$.

Proof. It is well known that $\mathrm{Coh}^G(G/B) \simeq \mathrm{Rep}(B)$, and is generated as a category by $\mathcal{O}_{G/B}(\lambda)$. Since $p : \widetilde{\mathfrak{N}} \to G/B$ gives $\widetilde{\mathfrak{N}}$ the structure of a G-equivariant vector bundle over G/B, using Lemma 6 in [10] (see also the last paragraph in page 266 of [15]), the required result follows. $\square$

Lemma 3.14. *The image of the functor* $R\pi_* : D^b(\mathrm{Coh}^G(\widetilde{\mathfrak{N}})) \to D^b(\mathrm{Coh}^G(\mathfrak{N}))$ *generates* $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ *as a triangulated category.*

Proof. Let $\mathfrak{D}$ denote the triangulated subcategory of $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ generated by the image of $R\pi_*$. It suffices to show that any $\mathcal{F} \in \mathrm{Coh}^G(\mathfrak{N})$ lies in $\mathfrak{D}$. Given this statement, it will then also follow that $\mathcal{F}[i] \in \mathfrak{D}$; since $\{\mathcal{F}[i] | \mathcal{F} \in \mathrm{Coh}^G(\mathfrak{N}), i \in \mathbb{Z}\}$ generates $D^b(\mathrm{Coh}^G(\mathfrak{N}))$, it would then follow that $\mathfrak{D} = D^b(\mathrm{Coh}^G(\mathfrak{N}))$. We proceed by induction on the dimension of the support to show that an arbitrary $\mathcal{F} \in D^b(\mathrm{Coh}^G(\mathfrak{N}))$ lies in $\mathfrak{D}$. It suffices to construct an $\tilde{\mathcal{F}} \in D^b(\mathrm{Coh}^G(\widetilde{\mathfrak{N}}))$, and a morphism $\phi : \mathcal{F} \to R\pi_* \tilde{\mathcal{F}}$ such that the cone $\mathcal{G}$ of ϕ has smaller support than $\mathcal{F}$ (since $\mathcal{G} \in \mathfrak{D}$ by induction, and $R\pi_* \tilde{\mathcal{F}} \in \mathfrak{D}$, it would follow that $\mathcal{F} \in \mathfrak{D}$). Suppose that $\mathbb{O}_{\mu,\nu}$ is open in the support of $\mathcal{F}$, for some $(\mu, \nu) \in \mathcal{Q}_n$. Denote the inclusion $\iota_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \hookrightarrow \widetilde{\mathfrak{N}}$. Let $\mathcal{F}' = (\pi \circ \iota_{\mu,\nu})^* \mathcal{F} \in \mathrm{Coh}^G(\widehat{\mathbb{O}_{\mu,\nu}})$. We claim that $\tilde{\mathcal{F}} = \iota_{\mu,\nu *} \mathcal{F}'$ has the required property. First construct the map ϕ as the composition of the map $\mathcal{F} \to (\pi \circ \iota_{\mu,\nu})_* \mathcal{F}'$ (coming the adjointness of $(\pi \circ \iota_{\mu,\nu})^*$ and $(\pi \circ \iota_{\mu,\nu})_*$), with the map $(\pi \circ \iota_{\mu,\nu})_* \mathcal{F}' \to R\pi_*(\iota_{\mu,\nu *} \mathcal{F}') = R\pi_* \tilde{\mathcal{F}}$. Since the fibres of the map $\pi \circ \iota_{\mu,\nu} : \widehat{\mathbb{O}_{\mu,\nu}} \to \mathfrak{N}$ over the orbit $\mathbb{O}_{\mu,\nu}$ are acyclic (see Corollary 2.11), the projection formula gives that ϕ is an isomorphism when restricted to $\mathbb{O}_{\mu,\nu}$. Thus the cone of ϕ is supported on $\overline{\mathbb{O}_{\mu,\nu}} \setminus \mathbb{O}_{\mu,\nu}$, and has smaller dimension. $\square$

Proposition 3.15. *The category* $D^b(Coh^G(\mathfrak{N}))$ *is generated as a triangulated category by the objects* $\{\mathbf{AJ}_\lambda | \lambda \in \Lambda^+\}$.

Proof. By applying the previous two lemmas, we find that (after a shift), $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ is generated by the objects $\{\mathbf{AJ}_\lambda | \lambda \in \Lambda\}$. The result will follow

once we establish that $\mathfrak{D}_{\widetilde{\mathrm{conv}}(\lambda)\cap\Lambda^+} = \mathfrak{D}_{\widetilde{\mathrm{conv}}(\lambda)}$ for each $\lambda \in \Lambda^+$. Suppose that this is true for all $\lambda' \in \Lambda^+ \cap \widetilde{\mathrm{conv}}^0(\lambda)$, then it follows that $\mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)\cap\Lambda^+} = \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)}$. But since $\mathbf{AJ}_{w\cdot\lambda} \simeq \mathbf{AJ}_\lambda(\mathrm{mod}\ \mathfrak{D}_{\widetilde{\mathrm{conv}}^0(\lambda)\cap\Lambda^+})$ by Proposition 3.10, it follows that $\mathbf{AJ}_{w\cdot\lambda} \in \mathfrak{D}_{\widetilde{\mathrm{conv}}(\lambda)\cap\Lambda^+}$; the result now follows. $\qquad\square$

Definition 3.16. Denote $\Delta_\lambda = \nabla_{w_0\cdot\lambda}$.

Theorem 3.17. *Fix any total order $\preceq$ on Λ^+, such that $\lambda \in \widetilde{\mathrm{conv}}(\mu) \Rightarrow \lambda \preceq \mu$. Then the sets $\nabla = \{\nabla_\lambda | \lambda \in \Lambda^+\}$, and $\Delta = \{\nabla_{w_0\cdot\lambda} | \lambda \in \Lambda^+\}$, respectively, are a quasi-exceptional set generating $D^b(\mathrm{Coh}^G(\mathfrak{N}))$, and a dual quasi-exceptional set.*

Proof. By definition (see section 3.1), we must check the following:
- To check that ∇ is quasi-exceptional, note that $\mathrm{Hom}^\bullet(\nabla_\lambda, \nabla_{\lambda'}) = 0$ if $\lambda \preceq \lambda'$ (and are distinct), since in this case $\lambda' \notin \widetilde{\mathrm{conv}}(\lambda)$ and we may apply Proposition 3.12; this proposition also implies that $\mathrm{Hom}^{<0}(\nabla_\lambda, \nabla_\lambda) = 0$ and $\mathrm{End}(\nabla_\lambda) = \mathbb{C}$.
- To check that Δ is a dual quasi-exceptional set, note $\mathrm{Hom}^\bullet(\Delta_\lambda, \nabla_{\lambda'}) = 0$ for $\lambda' \preceq \lambda$ using Proposition 3.12 (in fact, it is true provided $\lambda \neq \lambda'$) and that $\Delta_\lambda \simeq \nabla_\lambda(\mathrm{mod}\ \mathfrak{D}_{\preceq\lambda})$ using Proposition 3.10.
- Using Proposition 3.15, one notes that ∇_λ generates $D^b(\mathrm{Coh}^G(\mathfrak{N}))$. $\qquad\square$

Definition 3.18. Let $\mathfrak{D}^{q,\geq 0}, \mathfrak{D}^{q,\leq 0}$ denote the positive and negative subcategories corresponding to the t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ given by the quasi-exceptional set $\{\nabla_\lambda | \lambda \in \Lambda^+\}$.

4. THE PERVERSE COHERENT t-STRUCTURE

4.1. **Recollections.** For the reader's convenience, this section is a brief summary of some of the results in [8] (see also [6]) that we will need in the following section. Let X be an algebraic variety with an action of an algebraic group G; we will recount how to construct the "perverse coherent" t-structure on $D^b(\mathrm{Coh}^G(X))$.

Definition 4.1. Let X^{top} denote the set of generic points of closed G-invariant subschemes in X. For $x \in X^{top}$, let $d(x)$ denote the Krull dimension of the subscheme $\overline{x}$. A perversity function p is a function $X^{top} \to \mathbb{Z}$; associated to p define the dual perversity function p' by $p'(x) = -\dim(x) - p(x)$. The function p is "monotone" (resp. "strictly monotone") if $p(x') \geq p(x)$ (resp $p(x') > p(x)$) $\forall\ x' \in \overline{x}$, and (strictly) "co-monotone" if p' is (strictly) monotone.

Definition 4.2. Define $D^{p,\leq 0}, D^{p\geq 0} \subset D^b(\mathrm{Coh}^G(X))$ via:

$$\mathcal{F} \in D^{p,\geq 0} \text{ iff } \forall x \in X^{top}, i_x^!(\mathcal{F}) \in D^{\geq p(x)}(\mathcal{O}_x - \mathrm{mod}),$$

$$\mathcal{F} \in D^{p,\leq 0} \text{ iff } \forall x \in X^{top}, i_x^*(\mathcal{F}) \in D^{\leq p(x)}(\mathcal{O}_x - \mathrm{mod}).$$

Theorem 4.3. *If the perversity function p is monotone and co-monotone, then $(D^{p,\leq 0}, D^{p\geq 0})$ defines a t-structure on $D^b(\mathrm{Coh}^G(X))$, which we call the perverse coherent t-structure.*

Proof. See Theorem 1 in [8]. $\qquad\square$

Proposition 4.4. *Suppose that the perversity function is strictly monotone and co-monotone and G acts on X with finitely many orbits. Then the irreducible objects in the heart $\mathcal{P} = D^{p,\leq 0} \cap D^{p\geq 0}$ of the perverse t-structure are parametrized*

by a G-orbit $\mathcal{O}$ on X, and an irreducible G-equivariant vector bundle $\mathcal{L}$ on $\mathcal{O}$; the corresponding object is denoted $j_{!*}\mathcal{L}[p(\mathcal{O})]$. Further, $\mathcal{P}$ is Artinian.

Proof. See Corollaries 4 and 5 in [8]. $\qquad\square$

4.2. **Comparing the quasi-exceptional and perverse coherent t-structures on $D^b(\mathbf{Coh}^G(\mathfrak{N}))$.** Since all orbits in the exotic nilpotent cone have even dimension, we may consider the perverse coherent t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ arising from the middle perversity (i.e., for the orbit $\mathbb{O}_{\mu,\nu} \subset \mathfrak{N}, p(\mathbb{O}_{\mu,\nu}) = -\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}$); let $\mathfrak{D}^{p,\geq 0}, \mathfrak{D}^{p,\leq 0}$ denote the negative and positive subcategories as defined above, and let $\mathcal{P} = \mathfrak{D}^{p,\geq 0} \cap \mathfrak{D}^{p,\leq 0}$ denote the core of this t-structure. The goal of this section is to prove that the perverse coherent t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ coincides with the t-structure arising from the quasi-exceptional set ∇.

Proposition 4.5. $\mathbf{AJ}_\lambda \in \mathcal{P}$ for all $\lambda \in \Lambda$.

Proof. Using Lemma 3.8, $\mathcal{S}(\mathbf{AJ}_\lambda) = \mathbf{AJ}_{-\lambda-\epsilon_1-\cdots-\epsilon_n}$, and the perverse coherent t-structure is self-dual with respect to Grothendieck-Serre duality, it suffices to prove just the first of two conditions defining the perverse coherent t-structure (here $\iota_{\mathrm{gen},\mu,\nu} : (\mathbb{O}_{\mu,\nu})_{\mathrm{gen}} \hookrightarrow \mathfrak{N}$ denotes the inclusion of the generic point of the orbit $\mathbb{O}_{\mu,\nu}$):

$$\iota_{\mathrm{gen},\mu,\nu}^*(\mathbf{AJ}_\lambda) \in \mathfrak{D}^{\leq p(\mathbb{O}_{\mu,\nu})}(((\mathbb{O}_{\mu,\nu})_{\mathrm{gen}}) - \mathrm{mod}),$$

$$\iota_{\mathrm{gen},\mu,\nu}^*(R\pi_*\mathcal{O}_{\widetilde{\mathfrak{N}}}(\lambda)) \in \mathfrak{D}^{\leq p(\mathbb{O}_{\mu,\nu})+d}(((\mathbb{O}_{\mu,\nu})_{\mathrm{gen}}) - \mathrm{mod}).$$

Note first that $\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}$ is a semi-small resolution of singularities, i.e., $\dim(\pi^{-1}(x)) \leq \frac{1}{2}\mathrm{codim}(\mathbb{O}_{\mu,\nu}) = p(\mathbb{O}_{\mu,\nu}) + d$. Note also that from [16], Chapter 3, Corollary 11.2, if $f : X \to Y$ is a projective morphism of Noetherian schemes, such that $\dim(f^{-1}(y)) \leq r$ for all $y \in Y$, then for $i > r$, $R^i f_*\mathcal{F} = 0$ for all $\mathcal{F} \in \mathrm{Coh}(X)$. Since $\iota_{\mathrm{gen},\mu,\nu}^*$ is an exact functor, the required result follows. $\qquad\square$

Proposition 4.6. *The perverse coherent t-structure on $D^b(\mathrm{Coh}^G(\mathfrak{N}))$ coincides with the quasi-exceptional t-structure corresponding to the set $\{\nabla_\lambda | \lambda \in \Lambda\}$.*

Proof. Above we have proven that $\nabla_\lambda \in \mathfrak{D}^{p,\geq 0}$, so it follows that $\nabla_\lambda[d] \in \mathfrak{D}^{p,\geq 0}$ for $d \leq 0$. Since $\mathfrak{D}^{q,\geq 0}$ is generated by the objects $\nabla_\lambda[d]$, it follows that $\mathfrak{D}^{q,\geq 0} \subseteq \mathfrak{D}^{p,\geq 0}$. Since the above proof also gives us $\Delta_\lambda \in \mathfrak{D}^{p,\leq 0}$, it follows that $\Delta_\lambda[d] \in \mathfrak{D}^{p,\leq 0}$ for $d \geq 0$; hence $\mathfrak{D}^{q,\leq 0} \subseteq \mathfrak{D}^{p,\leq 0}$. It now follows from the axioms of a t-structure that we have equality in the above inclusions, i.e., two t-structures coincide. $\qquad\square$

Definition 4.7. Let $\mathbb{O}$ denote the set of pairs $(\mathbb{O}_{\mu,\nu}, V)$ of an orbit and an irreducible representation V of the isotropy group $G^{v_{\mu,\nu}}$, where $v_{\mu,\nu} \in \mathbb{O}_{\mu,\nu}$.

Proposition 4.8. *The irreducibles in $\mathcal{P}$ are indexed by $\mathbb{O}$. The bijection between costandard objects and irreducible objects in $\mathcal{P}$ gives a bijection $\Theta : \lambda \to (\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}}, L_\lambda)$ between Λ^+ and $\mathbb{O}$.*

Proof. The irreducible objects in the heart $\mathcal{P}$ of the perverse coherent t-structure are parametrized by an irreducible G-equivariant vector bundle $\mathcal{L}$ on an orbit $\mathbb{O}_{\mu,\nu}$, with the corresponding perverse coherent sheaf being given by $j_{!*}\mathcal{L}[-\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}]$. If $x \in \mathbb{O}_{\mu,\nu}$, then $\mathbb{O}_{\mu,\nu} \simeq G/G^x$; and the irreducible G-equivariant vector bundles are given by $G \times_{G^x} V$, where V is an irreducible representation of the isotropy group G^x. Let $IC_{\mathbb{O}_{\mu,\nu},V} := j_{!*}(G \times_{G^x} V)[-\frac{\dim(\mathbb{O}_{\mu,\nu})}{2}]$. Thus, using Proposition 3.5, we obtain the desired conclusion. $\qquad\square$

We deduce from the above that this bijection has the following properties:

Corollary 4.9. *The above bijection has the following properties, any of which uniquely characterize it.*

- $\mathrm{Hom}(IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda},\nabla_\lambda) \neq 0.$
- *There exists a morphism* $IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda} \to \nabla_\lambda$ *whose cone lies in* $\mathfrak{D}_{\widetilde{\mathrm{conv}^0(\lambda)}}.$
- $\mathrm{Hom}(\nabla_{w_0\cdot\lambda}, IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda}) \neq 0.$
- *There exists a morphism* $\nabla_{w_0\cdot\lambda} \to IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda}$ *whose cone lies in* $\mathfrak{D}_{\widetilde{\mathrm{conv}^0(\lambda)}}.$
- $IC_{\mathbb{O}_{\lambda^{(1)},\lambda^{(2)}},L_\lambda} \in \mathfrak{D}_{\widetilde{\mathrm{conv}^0(\lambda)}}.$

5. FURTHER DIRECTIONS

5.1. An explicit combinatorial bijection.
Thus far we have constructed an abstract bijection between $\mathbb{O}$ and Λ^+ using geometric techniques. One may ask whether it is possible to give a combinatorial description of this bijection. In the case of the bijection established in [10] for a simple algebraic group H between $\mathbf{O}$ and $\mathbf{\Lambda}^+$, an explicit combinatorial description has been given by Achar in [2] (see also [1]). While partial progress has been made in [1] to give a combinatorial description of the Bezrukavnikov's bijection in type C, outside of type A a complete explicit description of the bijection is, to the extent of our knowledge, not known.

One may give an explicit description of $\mathbb{O}$ as follows using work of Sun, [27]. Given $(\mu,\nu) \in \mathcal{Q}_n$, pick $(v,x) \in \mathbb{O}_{\mu,\nu}$. Following [27], the Levi component $K^{(v,x)}$ of $G^{(v,x)}$ is given as follows. Following Notation 3 on page 5 of [27], denote $\mu = (j_h^{n_{l_h}}), \nu = (k_h^{n_{l_h}})$ (such that for each h, either $j_h > j_{h+1}, k_h > k_{h+1}$, or both), and define $J = \{h | j_h > j_{h+1}, k_{h-1} > k_h\}$. Let $n'_{l_h} = n_{l_h}$ if $h \notin J$ and $n'_{l_h} = n_{l_h} - 1$ if $h \in J$. Then Theorem 3.10 in [27] states that:

$$K^{(v,x)} = \prod_{h \in J} Sp_{2n'_{l_h}}(\mathbb{C}).$$

Thus we can describe $\mathbb{O}$ as $\{(\mu,\nu) \in \mathcal{Q}_n, \underline{\lambda} | \underline{\lambda} = \{\underline{\lambda}_i\}_{i \in J}, \underline{\lambda}_i = (\underline{\lambda}_i^t)_{1 \leq t \leq n'_{l_h}}, \underline{\lambda}_i^t \geq \underline{\lambda}_i^{t+1}, \underline{\lambda}_i^t \in \mathbb{Z}_{\geq 0}\}$. It would be interesting to give a combinatorial description of the constructed bijection $\Theta : \mathbb{O} \leftrightarrow \Lambda^+$, following the techniques used by Achar in [1] and [2].

5.2. Canonical bases in equivariant K-theory.
Consider the setup in the case of the ordinary nilpotent cone, $\mathcal{N}$ for a simple algebraic group H. In [7], an exotic t-structure is constructed on the category $D^b(\mathrm{Coh}^G(\tilde{\mathcal{N}}))$ (and also on $D^b(\mathrm{Coh}^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}))$). Denote by $\boldsymbol{\pi}, \boldsymbol{\Lambda}, \boldsymbol{\Theta}$, etc., the analogues of the π, Λ, Θ, etc., in the above setting (with the nilpotent cone $\mathcal{N}$ of a simple algebraic group H, instead of the exotic nilpotent cone $\mathfrak{N}$). It is proved in Section 2.4 of [7] that:

Proposition 5.1. *The functor* $R\boldsymbol{\pi}_*$ *is t-exact with respect to the exotic t-structure on* $D^b(\mathrm{Coh}^G(\tilde{\mathcal{N}}))$, *and the perverse coherent t-structure on* $D^b(\mathrm{Coh}^G(\mathcal{N}))$. *The irreducible objects in the exotic t-structure on* $\mathcal{N}$ *are indexed by* $\boldsymbol{\Lambda}$; *denote them by* $\{E_{\boldsymbol{\lambda}} | \boldsymbol{\lambda} \in \boldsymbol{\Lambda}\}$. *Then* $R\boldsymbol{\pi}_*(E_{\boldsymbol{\lambda}}) = 0$ *if* $\boldsymbol{\lambda} \notin -\boldsymbol{\Lambda}^+$, *and* $R\boldsymbol{\pi}_*(E_{\boldsymbol{\lambda}}) = IC_{\boldsymbol{\Theta}(\boldsymbol{w_0 \lambda})}$ *if* $\boldsymbol{\lambda} \in -\boldsymbol{\Lambda}^+$.

Recall that the Steinberg variety is defined to be $\mathbf{St} := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. It is well known that the equivariant K-theory of the Steinberg variety gives us the affine Hecke algebra $\mathbf{H}$ associated to the simple algebraic group H (i.e., $K^{G \times \mathbb{C}^\times}(\mathbf{St}) = \mathbf{H}$; see Theorem 7.2.5 in [15]). The two projection maps $\boldsymbol{pr}_1, \boldsymbol{pr}_2 : \mathbf{St} = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \to \mathcal{N}$ give maps $\boldsymbol{pr}_{1*}, \boldsymbol{pr}_{2*} : K^{G \times \mathbb{C}^\times}(\mathbf{St}) \to K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}})$, while the map $\boldsymbol{\pi} : \tilde{\mathcal{N}} \to \mathcal{N}$ induces the map $\boldsymbol{\pi}_* : K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) \to K^{G \times \mathbb{C}^\times}(\mathcal{N})$. The projection $\boldsymbol{p} : \mathbf{St} \to \mathcal{N}$ now induces a map $\boldsymbol{p}_* : K^{G \times \mathbb{C}^\times}(\mathbf{St}) \to K^{G \times \mathbb{C}^\times}(\mathcal{N})$. In $K^{G \times \mathbb{C}^\times}(\mathcal{N})$, it was conjectured by Ostrik ([23]), that one has a canonical basis consisting of IC coherent sheaves indexed by $\mathbf{O}$, while in $K^{G \times \mathbb{C}^\times}(\mathbf{St}) = \mathbf{H}$ one has the Kazhdan-Lusztig bases $\{C_w | w \in W_{\mathrm{aff}}\}$, where W_{aff} is the extended affine Weyl group. This was proven by Bezrukavnikov in [9] and [10]. The following statement is proven in Corollary 1 of [9]:

Proposition 5.2. *Let $W_{\mathrm{aff}}^{\mathrm{fin}}$ denote the set of minimal length representatives of 2-sided cosets of the finite Weyl group W_{fin} in W_{aff}. Then if $w \notin W_{\mathrm{aff}}^{\mathrm{fin}}$, $p_*(C_w) = 0$, while if $w \in W_{\mathrm{aff}}^{\mathrm{fin}}$, then $p_*(C_w)$ is the class of an irreducible IC coherent sheaf in $K^{G \times \mathbb{C}^\times}(\mathcal{N})$.*

One may ask whether the above story carries over to the set-up of the exotic nilcone, $\mathfrak{N}$. First, we expect that an analogue of Proposition 5.1 holds in the exotic setting.

Define the exotic Steinberg variety to be $\mathrm{St} := \widetilde{\mathfrak{N}} \times_{\mathfrak{N}} \widetilde{\mathfrak{N}}$. Denote by $\mathfrak{N}_2$ the Hilbert nullcone of the representation $\mathbb{V}^{\oplus 2} \oplus \mathfrak{s}$, and let $\widetilde{\mathfrak{N}_2} := G \times_B (\mathbb{V}^{\oplus 2} \oplus \mathfrak{s})^+$; then we have a resolution map $\pi_2 : \widetilde{\mathfrak{N}_2} \to \mathfrak{N}_2$. Kato proves in Theorem 2.8 of [18] that $K^{G \times (\mathbb{C}^\times)^2}(\widetilde{\mathfrak{N}_2} \times_{\mathfrak{N}_2} \widetilde{\mathfrak{N}_2}) = \tilde{\mathbb{H}}$, a multi-parameter affine Hecke algebra associated to G (see Definition 2.1 in [18]). Using the techniques employed in Section 2 of [18], we can give a similar description of $K^{G \times \mathbb{C}^\times}(\mathrm{St})$. From Section 4, we have a basis of $K^{G \times \mathbb{C}^\times}(\mathfrak{N})$, indexed by $\mathbb{O}$, consisting of IC coherent sheaves. The natural map $p : \mathrm{St} \to \mathfrak{N}$ now induces a map $p_* : K^{G \times \mathbb{C}^\times}(\mathrm{St}) \to K^{G \times \mathbb{C}^\times}(\mathfrak{N})$. It would be interesting to construct a Kazhdan-Lusztig type basis of $\mathbb{H}$, satisfying the properties similar to that specified in Proposition 5.2.

Another question is to see whether Achar's results (see [4]), a version of the result proven in [10] but in positive characteristic, continue to hold for the exotic nilpotent cone.

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