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CONJUGACY CLASSES OF INVOLUTIONS AND KAZHDAN–LUSZTIG CELLS

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Abstract. According to an old result of Schützenberger, the involutions in a given two-sided cell of the symmetric group $\mathfrak{S}_n$ are all conjugate. In this paper, we study possible generalizations of this property to other types of Coxeter groups. We show that Schützenberger's result is a special case of a general result on "smooth" two-sided cells. Furthermore, we consider Kottwitz's conjecture concerning the intersections of conjugacy classes of involutions with the left cells in a finite Coxeter group. Our methods lead to a proof of this conjecture for classical types which, combined with further recent work, settles this conjecture in general.

1. Introduction

Let (W, S) be a Coxeter system, and let $\varphi : S \to \Gamma_{>0}$ be a *weight function*, that is, a map with values in a totally ordered abelian group Γ such that $\varphi(s) = \varphi(t)$ whenever s and t are conjugate in W. Associated with this datum, G. Lusztig has defined [Lu1] a partition of W into left, right or two-sided cells. (If φ is constant, then this was defined earlier by D. Kazhdan and G. Lusztig [KaLu]). There seems to be almost no connection between cells and conjugacy classes of elements of order greater than 2. However, several papers have investigated links between conjugacy classes of involutions and cells [Ko], [LuVo], [Lu6], [Ma].

To the best of our knowledge, the oldest result in this direction is the following. In the case where $W = \mathfrak{S}_n$, the two-sided cells are described by the Robinson-Schensted correspondence [KaLu], [Lu2]. It then follows from a result of M.-P. Schützenberger [Sch] (see also C. Hohlweg [Ho]) that, if $W = \mathfrak{S}_n$, then all the (Duflo) involutions contained in the same two-sided cell are conjugate. Of course, as it can be seen already in the Coxeter group of type B_2 (with φ constant), the same kind of result cannot be generalized as such. However, again if W is of type B_2 but if we now take φ to be non-constant, then again the same result holds. In this paper we shall investigate possible generalizations of M.-P. Schützenberger's result. For any subset $X \subseteq W$, we denote by $\mathscr{C}_2(X)$ the union of all conjugacy classes of involutions in W which have non-empty intersection with X.

Conjecture. *If C and C' are two left cells contained in the same two-sided cell, then $\mathscr{C}_2(C) = \mathscr{C}_2(C')$.*

As it can already be checked in type A_3, the obvious generalization of this conjecture to elements of any order is false. In this paper, we investigate this conjecture whenever W is finite. For simplification, all along this paper, we will

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say that *"Lusztig's P Conjectures hold"* if *"Lusztig's Conjectures P1, P2,..., P15 in* [Lu5, Chapter 14] *hold"*. Our aim is to prove the following result.

Theorem. *Assume that W is finite. Let $\mathfrak{C}$ be a two-sided cell of W and let C and C' be two left cells contained in $\mathfrak{C}$. Then:*

(a) *If φ is constant, then $\mathscr{C}_2(C) = \mathscr{C}_2(C')$.*
(b) *Assume that Lusztig's P Conjectures for (W, S, φ) hold. If $\mathfrak{C}$ is a* ***smooth*** *two-sided cell, then all the involutions in $\mathfrak{C}$ are conjugate. In particular, $\mathscr{C}_2(C) = \mathscr{C}_2(C')$ is a single conjugacy class.*
(c) *Assume that φ is constant and W is of classical type B_n or D_n. Then $|\mathscr{C} \cap C| = |\mathscr{C} \cap C'|$ where $\mathscr{C}$ is any conjugacy class of involutions in W.*

Here, a two-sided cell $\mathfrak{C}$ is called *smooth* if the family of irreducible characters associated with $\mathfrak{C}$ contains only one element. This definition is inspired by the theory of rational Cherednik algebras and Calogero–Moser cells (as developed in [BoRo1] or [BoRo2]). Note that smooth two-sided cells actually occur quite often; for example, all the two-sided cells of $\mathfrak{S}_n$ are smooth [KaLu], [Lu2], as well as all the two-sided cells when W is of type B_n and φ corresponds to the asymptotic case as in [BoIa], [Bon1]. See also Table 1 (p. 162) for more numerical data.

Part (a) of the Theorem will be shown in Proposition 5.1; for part (b) see Corollary 4.3. An essential ingredient in our proof is the fact that, if $\mathscr{C}$ is a conjugacy class of involutions in W, then $\sum_{w \in \mathscr{C}} T_w$ is central in the Hecke algebra $\mathscr{H}$; see Section 2. (Here, $(T_w)_{w \in W}$ is the standard basis of $\mathscr{H}$, as explained below.) Whenever $\mathscr{C}$ is a conjugacy class of reflections, this result is due to L. Iancu (unpublished).

Part (c) of the above result would follow from results of Lusztig [LuB, Chap. 12] and a general conjecture due to R.E. Kottwitz [Ko] concerning the intersections of conjugacy classes of involutions with left cells. Here, we prove (c) directly by the methods developed in Section 5 and then use this to actually show that Kottwitz's conjecture holds for W of classical type; see Sections 7, 8. As far as type D_n is concerned, this also relies on results in [Ge7]. Combined with further recent work (see the remarks following Conjecture 5.7), this completes the proof of Kottwitz's Conjecture in general.

Finally, we point out that our conjecture also makes sense for arbitrary Coxeter groups. It can be checked easily that it holds in the infinite dihedral case; it also follows from work of J. Guilhot [Gu] that, if W is affine and if $\mathfrak{C}_0$ is the lowest two-sided cell, then the above conjecture holds for left cells contained in $\mathfrak{C}_0$.

2. Hecke algebras, involutions, cells

Let (W, S) be a *finite* Coxeter system, let $\ell : W \to \mathbb{N}$ denote the length function, let Γ be a totally ordered abelian group and let $\varphi : S \to \Gamma_{>0}$ be a *weight function*, that is, a map such that $\varphi(s) = \varphi(t)$ whenever s and t are conjugate in W. We denote by A the group ring $\mathbb{R}[\Gamma]$, denoted exponentially: in other words, $A = \bigoplus_{\gamma \in \Gamma} \mathbb{R} v^\gamma$, with $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$. If $a = \sum_{\gamma \in \Gamma} a_\gamma v^\gamma \in A$, then we denote by $\deg(a)$ its degree, namely the maximal $\gamma \in \Gamma$ such that $\alpha_\gamma \neq 0$ (note that $\deg(0) = -\infty$).

We denote by $\mathscr{H} = \mathscr{H}(W, S, \varphi)$ the Hecke algebra with parameter φ. As a module, $\mathscr{H} = \bigoplus_{w \in W} A\, T_w$ and the multiplication is completely determined by

the following two rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ T_s^2 = 1 + (v^{\varphi(s)} - v^{-\varphi(s)})T_s & \text{if } s \in S. \end{cases}$$

The Bruhat–Chevalley order on W will be denoted by $\leqslant$.

Remark 2.1. If $\mathscr{P}$ is an assertion, then we define $\delta_{\mathscr{P}}$ by $\delta_{\mathscr{P}} = 1$ (resp. 0) if $\mathscr{P}$ is true (resp. false). For instance, $\delta_{i=j}$ replaces the usual Kronecker symbol $\delta_{i,j}$. With this notation, we have

$$\begin{aligned} T_s T_w &= T_{sw} + \delta_{sw<w}(v^{\varphi(s)} - v^{-\varphi(s)})T_w, \\ T_w T_s &= T_{ws} + \delta_{ws<w}(v^{\varphi(s)} - v^{-\varphi(s)})T_w \end{aligned}$$

for all $s \in S$ and $w \in W$.

Lemma 2.2. *Let $\mathscr{C}$ be a union of conjugacy classes of involutions in W. Then*

$$\boldsymbol{T}_{\mathscr{C}} := \sum_{w \in \mathscr{C}} T_w \quad \textit{is central in } \mathscr{H}.$$

Proof. Since $(T_s)_{s \in S}$ generates the A-algebra $\mathscr{H}$, it is sufficient to show that $T_s \boldsymbol{T}_{\mathscr{C}} = \boldsymbol{T}_{\mathscr{C}} T_s$ for all $s \in S$. But, by Remark 2.1, we have

$$\begin{aligned} T_s \boldsymbol{T}_{\mathscr{C}} &= \sum_{w \in \mathscr{C}} T_{sw} + \sum_{w \in \mathscr{C}} \delta_{sw<w}(v^{\varphi(s)} - v^{-\varphi(s)})T_w, \\ \boldsymbol{T}_{\mathscr{C}} T_s &= \sum_{w \in \mathscr{C}} T_{ws} + \sum_{w \in \mathscr{C}} \delta_{ws<w}(v^{\varphi(s)} - v^{-\varphi(s)})T_w. \end{aligned}$$

Now, as $\mathscr{C}$ is a union of conjugacy classes, we have $s\mathscr{C} = \mathscr{C}s$. Moreover, as elements of $\mathscr{C}$ are involutions, we have $sw < w$ if and only if $ws < w$ (for any $w \in \mathscr{C}$). The result follows. □

If $\mathscr{C}$ is a conjugacy class of reflections, the above result is mentioned in [GeJa, Exp. 3.3.8]; in this case, it is due to L. Iancu (unpublished).

Remark 2.3. Let $f\colon W \to \mathbb{R}$ be any class function on W. Let $\mathscr{C}$ be a union of conjugacy classes of involutions in W. Then we also have that

$$\boldsymbol{T}_{\mathscr{C}}^f := \sum_{w \in \mathscr{C}} f(w) T_w \quad \text{is central in } \mathscr{H}.$$

(Indeed, it is sufficient to prove this in the case where $\mathscr{C}$ is a single conjugacy class, in which case we have $\boldsymbol{T}_{\mathscr{C}}^f = f(t)\boldsymbol{T}_{\mathscr{C}}$ where $t \in \mathscr{C}$ is fixed.) In particular, applying this to the sign character ε of W, we obtain

$$\boldsymbol{T}_{\mathscr{C}}^{\varepsilon} = \sum_{w \in \mathscr{C}} (-1)^{\ell(w)} T_w \quad \text{is central in } \mathscr{H}.$$

For any $a = \sum_{\gamma \in \mathbb{R}} a_\gamma v^\gamma$, we set $\overline{a} = \sum_{\gamma \in \Gamma} a_\gamma v^{-\gamma}$. This can be extended to an antilinear automorphism $\mathscr{H} \to \mathscr{H}$, $h \mapsto \bar{h}$, by the formula

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \overline{a}_w T_{w^{-1}}^{-1}.$$

We set $A_{<0} = \bigoplus_{\gamma<0} Av^\gamma$ and $\mathscr{H}_{<0} = \bigoplus_{w\in W} A_{<0}T_w$. By [Lu5, Theorem 5.2(a)], there exists a unique A-basis $(c_w)_{w\in W}$ of $\mathscr{H}$, called the *Kazhdan–Lusztig basis*, such that

$$\begin{cases} \overline{c}_w = c_w, \\ c_w \equiv T_w \mod \mathscr{H}_{<0}. \end{cases}$$

We now define $\leqslant_L$ (resp. $\leqslant_R$, resp. $\leqslant_{LR}$) as the coarsest preorder such that, for all $w \in W$, $\bigoplus_{y\leqslant_L w} Ac_y$ (resp. $\bigoplus_{y\leqslant_R w} Ac_y$, resp. $\bigoplus_{y\leqslant_{LR} w} Ac_y$) is a left (resp. right, resp. two-sided) ideal of $\mathscr{H}$. We define $\sim_L$ (resp. $\sim_R$, resp. $\sim_{LR}$) as the equivalence relation associated with $\leqslant_L$ (resp. $\leqslant_R$, resp. $\leqslant_{LR}$): its equivalence classes are called the *left* (resp. *right*, resp. *two-sided*) *cells*. We denote by $\mathrm{Cell}_L(W)$ (resp. $\mathrm{Cell}_R(W)$, resp. $\mathrm{Cell}_{LR}(W)$) the set of left (resp. right, resp. two-sided) cells of W.

In order to define the corresponding *cell modules* it will be convenient, as in the later chapters of [Lu5], to work with a slightly modified version of the basis $(c_w)_{w\in W}$. Let $h \mapsto h^\dagger$ denote the unique A-algebra automorphism of $\mathscr{H}$ such that

$$T_s^\dagger = -T_s^{-1} \qquad \text{for all } s \in W.$$

(See [Lu5, 3.5].) Then, clearly, $(c_w^\dagger)_{w\in W}$ also is an A-basis of $\mathscr{H}$.

Remark 2.4. By [Lu5, Theorem 5.2(b)], we have

$$c_w \equiv T_w \mod \Big(\bigoplus_{y<w} A_{<0}T_y\Big) \qquad \text{for all } w \in W.$$

Since $c_w = \overline{c}_w$, we also have

$$c_w^\dagger \equiv (-1)^{\ell(w)}T_w \mod \Big(\bigoplus_{y<w} A_{>0}T_y\Big) \qquad \text{for all } w \in W.$$

Now, for every left cell C, we can construct a left $\mathscr{H}$-module V_C, called a *left cell module*, as follows. For $x, y \in W$, let us write

$$c_xc_y = \sum_{z\in W} h_{x,y,z}c_z, \qquad \text{where} \qquad h_{x,y,z} \in A.$$

Then, as an A-module, V_C is free with a basis $\{e_x \mid x \in C\}$. The action of $\mathscr{H}$ on V_C is given by the formula (see [Lu5, 21.1])

$$c_x^\dagger.e_y = \sum_{z\in C} h_{x,y,z}e_z, \qquad \text{where } x \in W \text{ and } y \in C.$$

We can perform similar constructions for right and two-sided ideals, giving rise to right $\mathscr{H}$-modules and $(\mathscr{H},\mathscr{H})$-bimodules, respectively.

Now, let K denote the fraction field of A and, if M is an A-module, let $KM = K\otimes_A M$. Then it is well-known (see, for example, [GePf, 9.3.5]) that the K-algebra $K\mathscr{H}$ is split and semisimple, so, by Tits' deformation Theorem, there is a bijection

$$\begin{array}{ccc} \mathrm{Irr}(W) & \xrightarrow{\sim} & \mathrm{Irr}(K\mathscr{H}) \\ \chi & \longmapsto & \chi_\varphi. \end{array}$$

Here, χ can be retrieved from χ_φ through the specialization $v^\gamma \mapsto 1$.

Definition 2.5 ([KaLu], [Lu1])**.** We define a partition of $\mathrm{Irr}(W)$, depending on φ, as follows. For a two-sided cell $\mathfrak{C}$, we denote by $\mathrm{Irr}_{\mathfrak{C}}(W)$ the set of irreducible

characters χ of W such that χ_φ is an irreducible constituent of KV_C, where C is a left cell contained in $\mathfrak{C}$. Then:

$$\mathrm{Irr}(W) = \coprod_{\mathfrak{C}\in \mathrm{Cell}_{LR}(W)} \mathrm{Irr}_{\mathfrak{C}}(W).$$

Note that, for each two-sided cell $\mathfrak{C}$, we have

$$|\mathfrak{C}| = \sum_{\chi\in \mathrm{Irr}_{\mathfrak{C}}(W)} \chi(1)^2.$$

If C is a left cell, we denote by $[C]$ the character of W obtained by specialization through $v^\gamma \mapsto 1$ from the character of $K\mathscr{H}$ afforded by V_C. An indication of the connection between left cells and involutions is given by the following result.

Proposition 2.6 ([Ge5]). *Let C be a left cell in W. Then the number of involutions in C is equal to the number of irreducible constituents of $[C]$ (counting multiplicities).*

We denote by $\mathscr{G}_\varphi(W)$ the following graph: its vertices are the irreducible characters of W, and two irreducible characters χ and χ' are joined by an edge if there exists a left cell C such that χ and χ' are irreducible components of $[C]$. In order to relate the graph $\mathscr{G}_\varphi(W)$ to the partition of $\mathrm{Irr}(W)$ in Definition 2.5 we need the following result.

Proposition 2.7 ([LuB, Theorem 12.15] and [Ge5, Corollary 3.9]). *Let C and C' be two left cells. Then $\langle [C], [C']\rangle_W = |C' \cap C^{-1}|$.*

As two-sided cells are unions of left cells, the sets $\mathrm{Irr}_{\mathfrak{C}}(W)$ are unions of connected components of the graph $\mathscr{G}_\varphi(W)$. It is conjectured that the converse holds:

Corollary 2.8. *Assume that Lusztig's P Conjectures for (W, S, φ) hold. Then the sets $\mathrm{Irr}_{\mathfrak{C}}(W)$ are the connected components of the graph $\mathscr{G}_\varphi(W)$.*

Proof. Indeed, if Lusztig's P Conjectures for (W, S, φ) hold, then $\sim_{LR}$ is the equivalence relation generated by $\sim_L$ and $\sim_R$; see [Lu5, §14.2, Conjecture P9]. So the result follows from Proposition 2.7. □

We shall also need the following result whose proof relies on some case–by–case arguments and explicit computations.

Proposition 2.9 (Cf. [Lu5, Chap. 22]). *Assume that Lusztig's P Conjectures hold for (W, S, φ). Let $\chi \in \mathrm{Irr}(W)$. Then there exists a left cell C of W such that $\langle [C], \chi\rangle_W = 1$.*

Proof. By the explicit results in [Lu5, §22] (see also [Ge2, §7] and the references there for the non-crystallographic types), every $\chi \in \mathrm{Irr}(W)$ appears with multiplicity 1 in some "contructible" character, as defined in [Lu5, 22.1]. (For Weyl groups and the equal parameter case, this statement already appeared in [LuB, 5.30].) On the other hand, since Lusztig's P Conjectures for (W, S, φ) are assumed to hold, we can apply [Lu5, Lemma 22.2], which shows that every constructible character is of the form $[C]$ for some left cell C. □

3. Leading coefficients

Lusztig has associated with any $\chi \in \mathrm{Irr}(W)$ two invariants $\mathbf{a}_\chi \in \Gamma_{\geq 0}$ and $f_\chi \in \mathbb{R}_{>0}$; see [LuB, Chap. 4], [Lu4], [Lu5, §20]. Let us briefly recall how this is done. It is known that $\chi_\varphi(T_w) \in A$ for all $w \in W$; see [GePf, 9.3.5]. Thus, we can define

$$\mathbf{a}_\chi := \min\{\gamma \in \Gamma_{\geq 0} \mid v^\gamma\, \chi_\varphi(T_w) \in A_{\geq 0} \text{ for all } w \in W\}.$$

Consequently, there are unique numbers $c_{w,\chi} \in \mathbb{R}$ ($w \in W$) such that

$$v^{\mathbf{a}_\chi}\, \chi_\varphi(T_w) \equiv (-1)^{\ell(w)}\, c_{w,\chi} \bmod A_{>0}.$$

These numbers are Lusztig's "leading coefficients of character values"; see [LuB], [Lu4]. Since $\chi_\varphi(T_w) = \chi_\varphi(T_{w^{-1}})$ for all $w \in W$ (see [GePf, 8.2.6]), we certainly have

$$c_{w,\chi} = c_{w^{-1},\chi} \qquad \text{for all } w \in W.$$

Given χ, there is at least one $w \in W$ such that $c_{w,\chi} \neq 0$ (by the definition of $\mathbf{a}_\chi$). Hence, the sum of all $c_{w,\chi}^2$ ($w \in W$) will be strictly positive, and so we can write that sum as $f_\chi\, \chi(1)$ where $f_\chi \in \mathbb{R}$ is strictly positive. We have the following orthogonality relations (see [GePf, Exc. 9.8]):

$$\sum_{w \in W} c_{w,\chi}\, c_{w,\chi'} = \begin{cases} f_\chi \chi(1) & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases}$$

The coefficients $c_{w,\chi}$ and the numbers f_χ are related to the left and two-sided cells of W. We shall now state a few results which make this relation more precise.

Proposition 3.1 ([LuB, 5.8] and [Ge5, 3.8])**.** *Let C be a left cell and $\chi, \chi' \in \mathrm{Irr}(W)$. Then*

$$\sum_{w \in C} c_{w,\chi}\, c_{w,\chi'} = \begin{cases} f_\chi\, \langle [C], \chi \rangle_W & \textit{if } \chi = \chi', \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 3.2. *Let $\chi \in \mathrm{Irr}(W)$ and $w \in W$. If $c_{w,\chi} \neq 0$, then $\langle [C], \chi \rangle_W \neq 0$, where C is the left cell containing w. In particular, $\chi \in \mathrm{Irr}_{\mathfrak{C}}(W)$, where $\mathfrak{C}$ is the two-sided cell such that $w \in \mathfrak{C}$.*

Proof. If $c_{w,\chi} \neq 0$ and $w \in C$, then the left hand side of the formula in Proposition 3.1 (where $\chi' = \chi$) is non-zero. Hence, so is the right hand side, that is, $\langle [C], \chi \rangle_W \neq 0$. □

Example 3.3. Let $W' \subseteq W$ be a standard parabolic subgroup, ε' the sign character of W' and $w_0' \in W'$ the longest element in W'. Let $\chi \in \mathrm{Irr}(W)$ be such that

$$\mathbf{a}_\chi = \varphi(w_0') \qquad \text{and} \qquad \big\langle \mathrm{Ind}_{W'}^W(\varepsilon'), \chi \big\rangle_W \neq 0.$$

Then $\chi \in \mathrm{Irr}_{\mathfrak{C}}(W)$, where $\mathfrak{C}$ is the two-sided cell which contains w_0'. (Indeed, by [GeJa, Cor. 2.8.6], we have

$$c_{w_0',\chi} = \pm \big\langle \mathrm{Ind}_{W'}^W(\varepsilon'), \chi \big\rangle_W \neq 0,$$

and it remains to use Corollary 3.2.)

Definition 3.4. We define the set of "*distinguished elements*" in W by

$$\mathscr{D} := \{w \in W \mid n_w \neq 0\}, \qquad \text{where} \qquad n_w := \sum_{\chi \in \mathrm{Irr}(W)} f_\chi^{-1}\, c_{w,\chi}.$$

(Note that $\mathscr{D}$ depends on φ.) If Lusztig's P Conjectures for (W, S, φ) hold, then [Ge4, Lemma 3.7] shows that this definition coincides with that in [Lu5, 14.1]. In particular, by Conjectures P5 and P6, we have $n_d = \pm 1$ and $d^2 = 1$ for all $d \in \mathscr{D}$; furthermore, by P13, every left cell contains a unique element of $\mathscr{D}$.

Proposition 3.5. *Assume that Lusztig's P Conjectures for (W, S, φ) hold. Let C be a left cell and $\mathscr{D} \cap C = \{d\}$. Then*

$$c_{d,\chi} = n_d \langle [C], \chi \rangle_W \qquad \textit{and} \qquad \sum_{\chi \in \mathrm{Irr}(W)} f_\chi^{-1} \langle [C], \chi \rangle_W = 1.$$

Proof. The first identity is contained in [Lu5, 20.6, 21.4]. Then the second identity immediately follows from the above formula for n_d. □

Definition 3.6. A two-sided cell $\mathfrak{C}$ is said to be ***smooth*** if $|\mathrm{Irr}_{\mathfrak{C}}(W)| = 1$. The set of smooth two-sided cells will be denoted by $\mathrm{Cell}_{LR}^{\mathrm{smooth}}(W)$.

The next result gives a characterization of smooth two-sided cells whenever Lusztig's P Conjectures hold:

Lemma 3.7. *Assume that Lusztig's P Conjectures for (W, S, φ) hold. Let $\mathfrak{C}$ be a two-sided cell. We denote $\mathfrak{C}_{(2)} = \{w \in \mathfrak{C} \mid w^2 = 1\}$. Then the following are equivalent:*

(1) *$\mathfrak{C}$ is "smooth", that is, $|\mathrm{Irr}_{\mathfrak{C}}(W)| = 1$.*
(2) *There exists a left cell $C \subseteq \mathfrak{C}$ such that $[C] \in \mathrm{Irr}(W)$.*
(3) *$f_\chi = 1$ for some $\chi \in \mathrm{Irr}_{\mathfrak{C}}(W)$.*
(4) *For any left cell $C \subseteq \mathfrak{C}$, we have $[C] \in \mathrm{Irr}(W)$.*
(5) *$|\mathfrak{C}| = |\mathfrak{C}_{(2)}|^2$.*
(6) *$\mathfrak{C}_{(2)} \subseteq \mathscr{D}$, that is, all involutions in $\mathfrak{C}$ are "distinguished".*

Note also that the condition "$[C] \in \mathrm{Irr}(W)$" can be replaced by "$|C \cap C^{-1}| = 1$"; see Proposition 2.7.

Proof. First we show the equivalence of (1), (2), (3), (4).

"(1) ⇒ (2)" Let $\mathrm{Irr}_{\mathfrak{C}}(W) = \{\chi\}$. Let C be a left cell as in Proposition 2.9. Since $\langle [C], \chi \rangle_W \neq 0$, we have $C \subseteq \mathfrak{C}$; see Definition 2.5. Furthermore, by Corollary 2.8, *every* irreducible constituent of $[C]$ belongs to $\mathfrak{C}$. Hence, χ is the only constituent of $[C]$. Since it occurs with multiplicity 1, we have $[C] = \chi \in \mathrm{Irr}(W)$.

"(2) ⇒ (3)" If $\chi := [C] \in \mathrm{Irr}(W)$, then the identity in Proposition 3.5 reduces to $1 = f_\chi^{-1}$, and so $f_\chi = 1$.

"(3) ⇒ (4)" Let C be a left cell as in Proposition 2.9. Then, as above, we have $C \subseteq \mathfrak{C}$. The identity in Proposition 3.5 now shows that

$$1 = 1 + \sum_{\chi \neq \psi \in \mathrm{Irr}(W)} f_\psi^{-1} \langle [C], \psi \rangle_W.$$

Hence, we have $\langle [C], \psi \rangle_W = 0$ for all $\psi \neq \chi$ and so $[C] = \chi \in \mathrm{Irr}(W)$. Now let C' be another left cell contained in $\mathfrak{C}$. By Corollary 2.8, there exists a sequence $C = C_0, C_1, \ldots, C_n = C'$ of left cells contained in $\mathfrak{C}$ such that $\langle [C_i], [C_{i+1}] \rangle_W \neq 0$ for all i. We shall prove by induction on i that $[C_i] = [C]$. This is clear if $i = 0$, so assume that $[C_i] = [C]$ and let us show that $[C_{i+1}] = [C]$. By assumption, we have

$\langle [C_i], [C_{i+1}]\rangle_W \neq 0$, which means that $\langle [C_i], \psi\rangle_W \leq \langle [C_{i+1}], \psi\rangle_W$ for all $\psi \in \mathrm{Irr}(W)$. Applying the identity in Proposition 3.5 to both C_i and C_{i+1}, we obtain

$$1 = \sum_{\psi\in\mathrm{Irr}(W)} f_\psi^{-1}\langle [C_i], \psi\rangle_W \leq \sum_{\psi\in\mathrm{Irr}(W)} f_\psi^{-1}\langle [C_{i+1}], \psi\rangle_W = 1.$$

Hence, we must have $\langle [C_i], \psi\rangle_W = \langle [C_{i+1}], \psi\rangle_W$ for all $\psi \in \mathrm{Irr}(W)$, and so $[C_{i+1}] = [C_i] \in \mathrm{Irr}(W)$, as required. Thus, (4) holds.

"(4) $\Rightarrow$ (1)" By Corollary 2.8, we necessarily have $\chi := [C] = [C']$ for all left cells $C, C' \subseteq \mathfrak{C}$, and then $\mathrm{Irr}_{\mathfrak{C}}(W) = \{\chi\}$.

Now we show the remaining equivalences.

"(1) $\Leftrightarrow$ (5)" Let $|\mathrm{Irr}_{\mathfrak{C}}(W)| = n \geq 1$ and write $\mathrm{Irr}_{\mathfrak{C}}(W) = \{\chi_1, \ldots, \chi_n\}$. Then, as noted in Definition 2.5, we have

$$|\mathfrak{C}| = \chi_1(1)^2 + \cdots + \chi_n(1)^2.$$

On the other hand, it easily follows from Proposition 2.6 that $|\mathfrak{C}_{(2)}| = \chi_1(1) + \cdots + \chi_n(1)$; see [Ge5, Cor. 3.12], Hence, we have

$$|\mathfrak{C}_{(2)}|^2 = \big(\chi_1(1) + \cdots + \chi_n(1)\big)^2,$$

which implies that $|\mathfrak{C}| = |\mathfrak{C}_{(2)}|^2$ if and only if $n = 1$.

"(4) $\Leftrightarrow$ (6)" Recall that, by Lusztig's Conjecture P13, every left cell contains a unique element of $\mathscr{D}$; furthermore, by P6, we have $d^2 = 1$ for all $d \in \mathscr{D}$. So the equivalence immediately follows from Proposition 2.6. $\square$

TABLE 1. Number of smooth cells (equal parameters)

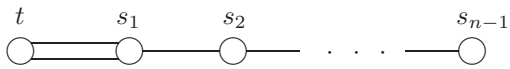
Type of W	$\lvert\mathrm{Cell}_{LR}(W)\rvert$	$\lvert\mathrm{Cell}_{LR}^{\mathrm{smooth}}(W)\rvert$
$I_2(m)$	3	2
B_3	6	4
B_4	10	5
B_5	16	6
B_6	26	10
B_7	40	12
B_8	60	15
D_4	11	10
D_5	14	12
D_6	27	22
D_7	35	25
D_8	60	40
E_6	17	14
E_7	35	24
E_8	46	23
F_4	11	8
H_3	7	4
H_4	13	6

Example 3.8. Assume that we are in the equal parameter case where φ is constant. In this case, it is known that Lusztig's P Conjectures for (W, S, φ) hold; see [Lu2], [Lu5, Chap. 15] (for Weyl groups) and [Du] (for the remaining types).

Note that "smooth" two-sided cells actually occur quite often in this case. For example, assume that (W, S) is of type A_{n-1} where $W = \mathfrak{S}_n$ is the symmetric group. Then we are automatically in the equal parameter case and we have $f_\chi = 1$ for all $\chi \in \mathrm{Irr}(W)$; see, for example, [LuB, 5.16] and [GePf, 9.4.5]. Hence, every two-sided cell in W is smooth in this case.

For further information, we give in Table 1 the number of smooth two-sided cells (equal parameter case) whenever $|S| \leq 8$ and (W, S) is not of type A. To compute this table it suffices, by Lemma 3.7, to find all $\chi \in \mathrm{Irr}(W)$ such that $f_\chi = 1$, and this information is easily available from the tables in [LuB], [GePf, Appendix].

Example 3.9. Let (W, S) be of type B_n and write $S = \{t, s_1, s_2, \ldots, s_{n-1}\}$ in such a way that the Dynkin diagram of (W, S) is given as follows.



We set $\varphi(t) = b$ and $\varphi(s_1) = \cdots = \varphi(s_{n-1}) = a$. Then it follows from the computation of constructible characters in [Lu5, Proposition 22.25] that:

(a) $\qquad f_\chi = 1$ *for all* $\chi \in \mathrm{Irr}(W) \qquad \Longleftrightarrow \qquad b \notin \{a, 2a, \ldots, (n-1)a\}$.

Hence, if Lusztig's P Conjectures for (W, S, φ) hold, then Lemma 3.7 shows that all two-sided cells of W are smooth if and only if $b \notin \{a, 2a, \ldots, (n-1)a\}$. Without assuming that Lusztig's P Conjectures for (W, S, φ) hold, the only known results are the following:

(b) *All the two-sided cells in* W *are smooth if* $a = 2b$ *or* $3a = 2b$ *or* $b > (n-1)a$.

If $a = 2b$ or $3a = 2b$, then (b) follows essentially from [Lu5, §16] (see [BGIL, Theorem 3.14] for some explanation). If $b > (n-1)a$, then (b) follows from [BoIa, Theorem 7.7] and [Bon1, Theorem 3.5 and Corollary 5.2].

4. A basic identity

Hypothesis. *Throughout this section we assume that Lusztig's P Conjectures hold for* (W, S, φ).

The main result of this section is the following basic identity, which links cells and involutions through the leading coefficients of character values.

Lemma 4.1 (The $(\mathfrak{C}, C, \mathscr{C})$-identity)**.** *Let* $\mathfrak{C}$ *be a two-sided cell and* C *a left cell contained in* $\mathfrak{C}$. *Let* $\mathscr{C}$ *be a union of conjugacy classes of involutions in* W. *Then*

$$\langle [C], \chi \rangle_W \sum_{w \in \mathscr{C} \cap \mathfrak{C}} c_{w,\chi} = \chi(1) \sum_{w \in \mathscr{C} \cap C} c_{w,\chi} \qquad \textit{for all } \chi \in \mathrm{Irr}(W).$$

Proof. Let $\mathrm{Z}(\mathscr{H})$ be the centre of $\mathscr{H}$. We denote by $\omega_\chi : \mathrm{Z}(\mathscr{H}) \to A$ the *central character* associated with χ_φ: if $z \in \mathrm{Z}(\mathscr{H})$, then $\omega_\chi(z) = \chi_\varphi(z)/\chi(1)$. Now consider the central element

$$\boldsymbol{T}^\varepsilon_{\mathscr{C}} = \sum_{w \in \mathscr{C}} (-1)^{\ell(w)} T_w \qquad \text{(see Remark 2.3).}$$

The desired identity will be obtained by evaluating χ_φ on $\boldsymbol{T}^\varepsilon_{\mathscr{C}} T_d$, where d is the unique element of $\mathscr{D}$ contained in C (see Lusztig's Conjecture P13). First note that, if $\chi \notin \mathrm{Irr}_{\mathfrak{C}}(W)$, then both sides of the identity are zero; see Corollary 3.2.

We can now assume that $\chi \in \mathrm{Irr}_{\mathfrak{C}}(W)$. Since $\boldsymbol{T}^{\varepsilon}_{\mathscr{C}} \in \mathrm{Z}(\mathscr{H})$, we have $\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}) = \chi(1)\omega_\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}})$ and $\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d) = \omega_\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}})\chi(T_d)$. Furthermore,

$$v^{\mathbf{a}_\chi}\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}) = \sum_{w\in\mathscr{C}} v^{\mathbf{a}_\chi}(-1)^{\ell(w)}\chi(T_w) \equiv \Big(\sum_{w\in\mathscr{C}} c_{w,\chi}\Big) \bmod A_{>0}.$$

It follows that

$$v^{2\mathbf{a}_\chi}\chi(1)\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d) \equiv \big(v^{\mathbf{a}_\chi}\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}})\big)\big(v^{\mathbf{a}_\chi}\chi(T_d)\big) \equiv (-1)^{\ell(d)}\Big(\sum_{w\in\mathscr{C}} c_{w,\chi}c_{d,\chi}\Big) \bmod A_{>0}.$$

Now, by Proposition 3.5, we have $c_{d,\chi} = n_d\langle[C],\chi\rangle_W$. Thus, we obtain

$$v^{2\mathbf{a}_\chi}\chi(1)\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d) \equiv (-1)^{\ell(d)}n_d\langle[C],\chi\rangle_W\Big(\sum_{w\in\mathscr{C}} c_{w,\chi}\Big) \bmod A_{>0}.$$

The summation on the right hand side can be taken over all $w \in \mathscr{C}\cap\mathfrak{C}$ (instead of $w\in\mathscr{C}$) since $c_{w,\chi} = 0$ unless $w\in\mathfrak{C}$; see Corollary 3.2. Next we re-write $\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d$ using the Kazhdan–Lusztig basis. For any $w\in W$, we have $T_w \equiv (-1)^{\ell(w)}c^{\dagger}_w \bmod \mathscr{H}_{>0}$; see Remark 2.4. This yields

$$\begin{aligned}\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d &= \sum_{w\in\mathscr{C}}(-1)^{\ell(w)}T_wT_d \in \sum_{w\in W}(c^{\dagger}_w + \mathscr{H}_{>0})((-1)^{\ell(d)}c^{\dagger}_d + \mathscr{H}_{>0})\\ &\subseteq \Big(\sum_{w\in\mathscr{C}}(-1)^{\ell(d)}c^{\dagger}_wc^{\dagger}_d\Big) + \mathscr{H}_{>0}\mathscr{H}_{\geq 0} + \mathscr{H}_{\geq 0}\mathscr{H}_{>0}.\end{aligned}$$

We certainly have $v^{\mathbf{a}_\chi}\chi(h)\in A_{\geq 0}$ for any $h\in\mathscr{H}_{\geq 0}$ and $v^{\mathbf{a}_\chi}\chi(h)\in A_{>0}$ for any $h\in\mathscr{H}_{>0}$. Hence, we obtain

$$v^{2\mathbf{a}_\chi}\chi(\boldsymbol{T}^{\varepsilon}_{\mathscr{C}}T_d) \equiv (-1)^{\ell(d)}\Big(\sum_{w\in\mathscr{C}} v^{2\mathbf{a}_\chi}\chi\big(c^{\dagger}_wc^{\dagger}_d\big)\Big) \bmod A_{>0}.$$

We now look at the term $\chi(c^{\dagger}_wc^{\dagger}_d)$ (for $w\in\mathscr{C}$) in more detail. Following Lusztig's notation in [Lu5, §13], we set for x, $y\in W$,

$$c_xc_y = \sum_{z\in W} h_{x,y,z}c_z, \qquad \text{where } h_{x,y,z}\in A.$$

Furthermore, if $z\in W$, we define $\mathbf{a}(z) = \max\{\deg(h_{x,y,z}) \mid x,y\in W\}$. Let us now consider

$$v^{2\mathbf{a}_\chi}\chi(c^{\dagger}_wc^{\dagger}_d) = \sum_{x\in W}\big(v^{\mathbf{a}_\chi}h_{w,d,x}\big)\big(v^{\mathbf{a}_\chi}\chi(c^{\dagger}_x)\big).$$

Let $x\in W$ be such that $h_{w,d,x}\neq 0$ and $\chi(c^{\dagger}_x)\neq 0$. Since Lusztig's Conjecture P4 holds, the first condition implies that $\mathbf{a}(d)\leq\mathbf{a}(x)$. By [Ge4, Lemma 3.5], the second condition implies that $\mathbf{a}(x)\leq\mathbf{a}_\chi$. (Note that the function $\tilde{a}(w)$ in [Ge4, 3.5] agrees with $\mathbf{a}(w)$ by [Ge4, Prop. 3.6 and Rem. 4.2].) On the other hand, since $\chi\in\mathrm{Irr}_{\mathfrak{C}}(W)$, we have $\mathbf{a}_\chi = \mathbf{a}(d)$; see [Lu5, Proposition 20.6]. Thus, we must have $\mathbf{a}(d) = \mathbf{a}(x) = \mathbf{a}_\chi$. Furthermore, since $h_{w,d,x}\neq 0$ and $\mathbf{a}(d) = \mathbf{a}(x)$, we can now even conclude that $x\in C$, by Lusztig's Conjecture P9. Thus, we obtain

$$v^{2\mathbf{a}_\chi}\chi(c^{\dagger}_wc^{\dagger}_d) = \sum_{x\in C}\big(v^{\mathbf{a}(x)}h_{w,d,x}\big)\big(v^{\mathbf{a}_\chi}\chi(c^{\dagger}_x)\big).$$

Now, by Remark 2.4, we have $v^{\mathbf{a}_\chi}\chi_\varphi(c_w^\dagger) \equiv c_{w,\chi} \bmod A_{>0}$. Hence, by taking constant terms in the above identity, we obtain

$$v^{2\mathbf{a}_\chi}\chi(c_w^\dagger c_d^\dagger) \equiv \Big(\sum_{x\in C}\gamma_{w,d,x^{-1}}\, c_{x,\chi}\Big) \bmod A_{>0};$$

here, we denote by $\gamma_{w,d,x^{-1}}$ the constant term of $v^{\mathbf{a}(x)}h_{w,d,x}$, as in [Lu5, 13.6]. By Lusztig's Conjectures P2, P5 and P7, we have

$$\gamma_{w,d,x^{-1}} = \begin{cases} n_d & \text{if } x = w, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that $v^{2\mathbf{a}_\chi}\chi(c_w^\dagger c_d^\dagger) \equiv \delta_{w\in C} n_d\, c_{w,\chi} \bmod A_{>0}$, and so

$$\begin{aligned} v^{2\mathbf{a}_\chi}\chi(\boldsymbol{T}_{\mathscr{C}}^{\varepsilon}T_d) &\equiv (-1)^{\ell(d)}\sum_{w\in\mathscr{C}} v^{2\mathbf{a}_\chi}\chi\big(c_w^\dagger c_d^\dagger\big) \\ &\equiv (-1)^{\ell(d)} n_d \sum_{w\in\mathscr{C}} \delta_{w\in C} n_d\, c_{w,\chi} \\ &\equiv (-1)^{\ell(d)} n_d \Big(\sum_{w\in\mathscr{C}\cap C} c_{w,\chi}\Big) \bmod A_{>0}. \end{aligned}$$

Comparing this with our earlier expression for $v^{2\mathbf{a}_\chi}\chi(1)\chi(\boldsymbol{T}_{\mathscr{C}}^{\varepsilon}T_d) \bmod A_{>0}$ yields the desired identity. □

Example 4.2. Let $\mathfrak{C}$ be a two-sided cell which is "smooth", that is, we have

$$\mathrm{Irr}_{\mathfrak{C}}(W) = \{\chi\} \qquad \text{for some } \chi \in \mathrm{Irr}(W).$$

Let $d \in \mathfrak{C} \cap \mathscr{D}$ and $\mathscr{C}$ a union of conjugacy classes of involutions in W. Let C be the left cell containing d. Then we claim that the $(\mathfrak{C}, C, \mathscr{C})$-identity in Lemma 4.1 reduces to

$$\sum_{w\in\mathscr{C}\cap\mathfrak{C}} c_{w,\chi} = \begin{cases} \chi(1)n_d & \text{if } d \in \mathscr{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, by Lemma 3.7 and Corollary 3.2, we have $[C] \in \mathrm{Irr}_{\mathfrak{C}}(W)$, and so $\chi = [C]$. This yields the left hand side. On the other hand, by Proposition 3.5, we have $c_{d,\chi} = n_d\langle[C],\chi\rangle_W = 1$. Furthermore, by Lemma 3.7, all the involutions in C are contained in $\mathscr{D}$. Hence, $\mathscr{C}\cap C = \varnothing$ unless $d \in \mathscr{C}$, in which case $\mathscr{C}\cap C = \{d\}$. Thus, the right hand side of the $(\mathfrak{C}, C, \mathscr{C})$-identity reduces to the expression above.

Corollary 4.3. *Recall our assumption that Lusztig's P Conjectures for (W, S, φ) hold. Let $\mathfrak{C}$ be a smooth two-sided cell. Then all the involutions in $\mathfrak{C}$ are conjugate in W.*

Proof. Let $\mathfrak{C} = C_1 \amalg \ldots \amalg C_n$ be the partition of $\mathfrak{C}$ into left cells. By Lusztig's Conjecture P13, for each i there is a unique $d_i \in \mathscr{D}\cap C_i$. On the other hand, by Lemma 3.7, all involutions in $\mathfrak{C}$ are contained in $\mathscr{D}$. It follows that $\{d_1, \ldots, d_n\}$ is precisely the set of involutions in $\mathfrak{C}$. Now let $\mathscr{C}$ be the conjugacy class containing d_1. Then the identity in Example 4.2 reads:

$$\sum_{w\in\mathscr{C}\cap\mathfrak{C}} c_{w,\chi} = \chi(1)n_{d_1} \neq 0 \qquad (\text{since } d_1 \in \mathscr{C}).$$

Similarly, for any $i \geq 2$, we have

$$\sum_{w \in \mathscr{C} \cap \mathfrak{C}} c_{w,\chi} = \begin{cases} \chi(1) n_{d_i} & \text{if } d_i \in \mathscr{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the left hand side is non-zero, we conclude that $d_i \in \mathscr{C}$, as claimed. □

Example 4.4. Let $W = \mathfrak{S}_n$ be of type A_{n-1} with generators given by the basic transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Then, as already mentioned in Example 3.8, all the two-sided cells in W are smooth, and so we now recover a known result of Schützenberger [Sch] in this case. An elementary proof that Lusztig's P Conjectures for (W, S, φ) hold is given in [Ge3] (see also [GeJa, §2.8]). We can now also explicitly determine the conjugacy class of involutions associated with a two-sided cell. Indeed, it is well-known that the irreducible characters of $W = \mathfrak{S}_n$ have a natural labelling by the partitions of n; we write this in the form

$$\mathrm{Irr}(\mathfrak{S}_n) = \{\chi^\alpha \mid \alpha \vdash n\}.$$

For example, $\chi^{(n)}$ is the trivial character and $\chi^{(1^n)}$ is the sign character. For $\alpha \vdash n$, let $\mathfrak{C}_\alpha$ be the unique two-sided cell such that $\chi^\alpha \in \mathrm{Irr}_{\mathfrak{C}_\alpha}(\mathfrak{S}_n)$. Since every two-sided cell is smooth, the sets $\{\mathfrak{C}_\alpha \mid \alpha \vdash n\}$ are precisely the two-sided cells of $\mathfrak{S}_n$. Given $\alpha \vdash n$, let $\mathscr{C}_\alpha$ be the unique conjugacy class of involutions such that $\mathscr{C}_\alpha \cap \mathfrak{C}_\alpha \neq \varnothing$. Let α^* denote the transpose partition and w_{α^*} be the longest element in the Young subgroup $\mathfrak{S}_{\alpha^*} \subseteq \mathfrak{S}_n$. Then it is well-known that

$$\big\langle \mathrm{Ind}_{\mathfrak{S}_{\alpha^*}}^{\mathfrak{S}_n}(\varepsilon_{\alpha^*}), \chi^\alpha \big\rangle_{\mathfrak{S}_n} = 1, \qquad \text{where} \qquad \varepsilon_{\alpha^*} = \text{ sign character of } \mathfrak{S}_{\alpha^*}.$$

Using the formula for $\mathbf{a}_{\chi^\alpha}$ in [LuB, 4.4], one also sees that $\mathbf{a}_{\chi^\alpha} = \ell(w_{\alpha^*})$. Hence, by Example 3.3, we have $w_{\alpha^*} \in \mathfrak{C}_\alpha$, and so

$$\mathscr{C}_\alpha = \text{ conjugacy class containing } w_{\alpha^*}.$$

The discussion of this example will be continued in Example 5.10.

Example 4.5. Assume that (W, S) if of type B_n, as in Example 3.9. Let $b > (n-1)a$. Then the fact that all the involutions contained in a two-sided cell are conjugate can be proved directly from the combinatorial description given in [BoIa, Theorem 7.7] and [Bon1, Theorem 3.5 and Corollary 5.2], by using Schützenberger's result for the symmetric group [Sch]. Also, for more general values of a, b, a conjectural description of left, right and two-sided cells is provided by [BGIL, Conjectures A$^+$ and B]; it would be interesting to see if the conjecture we have stated in the introduction is compatible with this conjectural combinatorial construction.

Remark 4.6. If W is of type F_4 or $I_2(m)$ and φ is a general weight function, then Lusztig's P Conjectures for (W, S, φ) are known to hold; see [Ge4, §5]. In these cases, using the explicit knowledge of the cells and the classes of involutions (see [Ge1] for type F_4 and [Lu5, §8] for type $I_2(m)$), one can directly check that if C and C' are two left cells contained in the same two-sided cell, then $\mathscr{C}_2(C) = \mathscr{C}_2(C')$. This provides some support for the general conjecture stated in the introduction.

5. The equal parameter case

Hypothesis. *From now until the end of this paper, we assume that we are in equal parameter case where $\Gamma = \mathbb{Z}$ and $\varphi(s) = 1$ for all $s \in S$.*

Under this hypothesis, as already mentioned in Example 3.8, it is known that Lusztig's P Conjectures for (W, S, φ) hold. One further distinctive feature of this case is the existence of *special* characters. For $\chi \in \mathrm{Irr}(W)$, let $\mathbf{b}_\chi$ denote the smallest integer $i \geq 0$ such that χ occurs in the ith symmetric power of the standard reflection representation of W. Then, following Lusztig [LuB, 4.1], χ is called *special* if $\mathbf{a}_\chi = \mathbf{b}_\chi$. Let

$$\mathscr{S}(W) := \{\chi \in \mathrm{Irr}(W) \ \mid \ \mathbf{a}_\chi = \mathbf{b}_\chi\}$$

be the set of special characters of W. It is known that

$$(\diamondsuit_1) \qquad |\mathscr{S}(W) \cap \mathrm{Irr}_{\mathfrak{C}}(W)| = 1 \qquad \text{for every two-sided cell } \mathfrak{C} \text{ of } W.$$

This is seen as follows. Consider the partition of $\mathrm{Irr}(W)$ in terms of "families", as defined in [LuB, 4.2]. (The same definition also works for groups of non-crystallographic type; see [GePf, §6.5].) By [LuB, 4.14], every such family contains a unique special character (and this also holds for non-crystallographic types; see [GePf, §6.5]). Hence, $(\diamondsuit_1)$ follows from the known fact that the partition of $\mathrm{Irr}(W)$ into families coincides with the partition in Definition 2.5. For Weyl groups, this appeared in [LuB, Theorem 5.25]. A different argument based on certain "positivity" properties of the Kazhdan–Lusztig basis is given in [Lu5, Prop. 23.3]; the same argument also works for the non-crystallographic types, where the analogous "positivity" properties are known by explicit computation; see Alvis [Al], DuCloux [Du].

Now let $\mathfrak{C}$ be a two-sided cell. Then, if χ denotes the unique character in $\mathscr{S}(W) \cap \mathrm{Irr}_{\mathfrak{C}}(W)$, we have

$$(\diamondsuit_2) \qquad (-1)^{\mathbf{a}_\chi + \ell(w)} c_{w,\chi} > 0 \quad \text{for all } w \in C \cap C^{-1},$$

where C is any left cell contained in $\mathfrak{C}$. This holds by [Lu4, Prop. 3.14] for Weyl groups and by [Ge6, Rem. 5.12] for the remaining types. Note that, in the notation of [Lu4, §3], the special character χ corresponds to the pair $(1, 1) \in \mathscr{M}(G_{\mathfrak{C}})$ where $G_{\mathfrak{C}}$ is the finite group associated with $\mathfrak{C}$ (see also [LuB, 4.14.2]). The factor $(-1)^{\mathbf{a}_\chi + \ell(w)}$ comes from the identity [Lu4, 3.5(a)] which relates the leading coefficients to the characters of Lusztig's asymptotic algebra J.

Proposition 5.1. *Recall our assumption that we are in the equal parameter case. Let $\mathfrak{C}$ be a two-sided cell and C, C' be left cells of W which are contained in $\mathfrak{C}$. Let $\mathscr{C}$ be a conjugacy class of involutions in W. Then $\mathscr{C} \cap C \neq \varnothing$ if and only if $\mathscr{C} \cap C' \neq \varnothing$. In particular, we have $\mathscr{C}_2(C) = \mathscr{C}_2(C')$.*

Proof. We consider the $(\mathfrak{C}, C, \mathscr{C})$-identity in Lemma 4.1 with respect to the unique special character $\chi \in \mathscr{S}(W) \cap \mathrm{Irr}_{\mathfrak{C}}(W)$. Since the sign character of W is constant on $\mathscr{C}$, we can write this identity in the form

$$\langle [C], \chi \rangle_W \sum_{w \in \mathscr{C} \cap \mathfrak{C}} (-1)^{\ell(w)} c_{w,\chi} = \chi(1) \sum_{w \in \mathscr{C} \cap C} (-1)^{\ell(w)} c_{w,\chi}.$$

Multiplying both sides by $(-1)^{\mathbf{a}_\chi}$, we obtain

$$\langle [C], \chi\rangle_W \sum_{w\in\mathscr{C}\cap\mathfrak{C}} (-1)^{\mathbf{a}_\chi+\ell(w)} c_{w,\chi} = \chi(1) \sum_{w\in\mathscr{C}\cap C} (-1)^{\mathbf{a}_\chi+\ell(w)} c_{w,\chi}.$$

By $(\diamondsuit_2)$, we have $c_{d,\chi}\neq 0$, and so $\langle [C], \chi\rangle_W \neq 0$; see Proposition 3.5. Thus, we obtain

$$\sum_{w\in\mathscr{C}\cap\mathfrak{C}} (-1)^{\mathbf{a}_\chi+\ell(w)} c_{w,\chi} = \frac{\chi(1)}{\langle [C], \chi\rangle_W} \sum_{w\in\mathscr{C}\cap C} (-1)^{\mathbf{a}_\chi+\ell(w)} c_{w,\chi}.$$

Let us denote by $\Upsilon(\mathscr{C}, C)$ the expression on the right hand side of this identity. Since the left hand side does not depend on C, we have $\Upsilon(\mathscr{C}, C) = \Upsilon(\mathscr{C}, C')$. Consequently, we have

$$\sum_{w\in\mathscr{C}\cap C} (-1)^{\mathbf{a}_{\chi_\mathscr{C}}+\ell(w)} c_{w,\chi} \neq 0 \qquad \Longleftrightarrow \qquad \sum_{w\in\mathscr{C}\cap C'} (-1)^{\mathbf{a}_{\chi_\mathscr{C}}+\ell(w)} c_{w,\chi} \neq 0.$$

Finally, by $(\diamondsuit_2)$, we have

$$(-1)^{\mathbf{a}_\chi+\ell(w)} c_{w,\chi} > 0 \qquad \text{for all } w\in\mathscr{C}\cap C \text{ and for all } w\in\mathscr{C}\cap C'.$$

Thus, the left hand side of the above equivalence is non-zero if and only if $\mathscr{C}\cap C\neq\varnothing$, and, similarly, the right left hand side is non-zero if and only if $\mathscr{C}\cap C'\neq\varnothing$. $\square$

Definition 5.2. A character $\chi\in\mathrm{Irr}(W)$ is called *exceptional* if there exists some $w\in W$ such that $c_{w,\chi}\neq 0$ and $\mathbf{a}_\chi\not\equiv \ell(w) \bmod 2$.

Remark 5.3. One easily checks that there is a well-defined ring homomorphism $\alpha\colon \mathscr{H}\to\mathscr{H}$ such that $\alpha(v)=-v$ and $\alpha(r)=r$ for all $r\in\mathbb{R}$ and $\alpha(T_w) = (-1)^{\ell(w)}T_w$ for all $w\in W$. (See Lusztig [Lu2, 3.2].) Now, for $\chi\in\mathrm{Irr}(W)$, we have $\chi_\varphi(T_w)\in\mathbb{R}[v,v^{-1}]$ for all $w\in W$. Composing the action of $\mathscr{H}$ on a representation affording χ_φ with α, we see that there is a well-defined $\tilde\chi\in\mathrm{Irr}(W)$ such that

$$\tilde\chi_\varphi(T_w) = (-1)^{\ell(w)}\chi_\varphi(T_w)\big|_{v\mapsto -v} \qquad \text{for all } w\in W.$$

By the definition of $\mathbf{a}_\chi$ and $c_{w,\chi}$, this implies that

$$\mathbf{a}_{\tilde\chi} = \mathbf{a}_\chi \qquad \text{and} \qquad c_{w,\tilde\chi} = c^*_{w,\chi} \quad \text{for all } w\in W.$$

Thus, χ is exceptional if and only if $\chi\neq\tilde\chi$. Using Corollary 3.2 we see that, for a two-sided cell $\mathfrak{C}$ of W, we have

$$\chi\in\mathrm{Irr}_\mathfrak{C}(W) \qquad \Longleftrightarrow \qquad \tilde\chi\in\mathrm{Irr}_\mathfrak{C}(W).$$

Note that there do exist cases for which $\chi\neq\tilde\chi$. For example, let (W,S) be of type E_7. Then, by [LuB, 5.22.2], there exists an involution $x\in W$ such that $c_{x,\chi}\neq 0$ and $\mathbf{a}_\chi\not\equiv\ell(x) \bmod 2$ for the special character denoted $\chi = 512'_a$. In type E_8, examples are given by the special characters 4096_z and $4096'_x$; see [LuB, 5.23.2].

Example 5.4. Assume that (W,S) is irreducible. By the previous remark we see that if $v^{\ell(w)}\chi(T_w)\in\mathbb{R}[v^2]$ for all $w\in W$, then χ is non-exceptional. So, by [GePf, Example 9.3.4], all $\chi\in\mathrm{Irr}(W)$ are non-exceptional unless (W,S) is of type H_3, H_4, E_7, E_8 and χ is one of the characters listed in [GePf, Example 9.2.3]. (This list includes the characters $512'_a$, 4096_z, $4096'_x$ already mentioned in Remark 5.3.) The degree of such an exceptional character is a power of 2; furthermore, we have $v^{\ell(w_0)}\chi(T_{w_0})\notin\mathbb{R}[v^2]$, where $w_0\in W$ is the longest element.

In particular, if (W,S) is of classical type, then all $\chi\in\mathrm{Irr}(W)$ are non-exceptional.

Example 5.5. Assume that (W, S) is irreducible and of classical type. (Also recall that we are in the equal parameter case). Let $\mathfrak{C}, C, \mathscr{C}$ be as in Lemma 4.1. Let $\chi \in \mathscr{S}(W)$ be the unique special character in $\mathrm{Irr}_{\mathfrak{C}}(W)$. Then we claim that

$$|\mathscr{C} \cap \mathfrak{C}| = \chi(1)|\mathscr{C} \cap C|.$$

This is seen as follows. As already noted in the proof of Proposition 5.1, we have $\langle [C], \chi\rangle_W \neq 0$. By [LuB, 12.13], every left cell module for W is multiplicity–free, and so $\langle [C], \chi\rangle_W = 1$. Consequently, the $(\mathfrak{C}, C, \mathscr{C})$-identity in Lemma 4.1 reduces to

$$\sum_{w \in \mathscr{C} \cap \mathfrak{C}} c_{w,\chi} = \chi(1) \sum_{w \in \mathscr{C} \cap C} c_{w,\chi}.$$

So it remains to show that

$$c_{w,\chi} = c^*_{w,\chi} = 1 \qquad \text{for all } w \in \mathscr{C} \cap \mathfrak{C}.$$

Now, the first equality holds since χ is non-exceptional; see Example 5.4. Furthermore, by [Lu4, 3.10(b)], we have $c_{w,\chi} \in \{0, \pm 1\}$ for all $w \in W$. Hence, the second equality immediately follows from $(\diamondsuit_2)$.

In particular, the equality $|\mathscr{C} \cap \mathfrak{C}| = \chi(1)|\mathscr{C} \cap C|$ shows that the cardinality $|\mathscr{C} \cap C|$ does not depend on C. This phenomenon is related to a conjecture of Kottwitz [Ko], which we shall now explain.

Definition 5.6 ([Ko], [LuVo], [Lu6])**.** Let $\mathscr{C}$ be a union of conjugacy classes of involutions in W. Let $V_{\mathscr{C}}$ be an $\mathbb{R}$-vector space with a basis $\{a_w \mid w \in \mathscr{C}\}$. Then, by [LuVo, 6.3] and [Lu6], there is a linear action of W on $V_{\mathscr{C}}$ such that, for any $s \in S$ and $w \in \mathscr{C}$, we have

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } \ell(sw) < \ell(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

Let $\rho_{\mathscr{C}}$ denote the character of this representation of W on $V_{\mathscr{C}}$.

Conjecture 5.7 (Kottwitz [Ko, §1])**.** *Let $\mathscr{C}$ be a union of conjugacy classes of involutions and C be a left cell of W. Then $\langle \rho_{\mathscr{C}}, [C]\rangle_W = |\mathscr{C} \cap C|$.*

Already Kottwitz [Ko] showed that his conjecture holds in type A_{n-1}; see Example 5.10 below. The aim of the following three sections is to deal with types B_n and D_n; see Theorem 7.3 and Corollary 8.6. This will rely in an essential way on the above identity in Example 5.5. As far as the exceptional types are concerned, Casselman [Ca] has verified the conjecture by explicit computation for F_4 and E_6; in [Ge6], this is extended to E_7; see [GeHa] for type E_8. Marberg [Ma] verified the conjecture for the non-crystallographic types H_3, H_4, $I_2(m)$. Thus, the results in this paper (together with [Ge7]) complete the proof of Kottwitz's Conjecture.

Remark 5.8. The fact that $\rho_{\mathscr{C}}$ indeed is equal to the character originally constructed in [Ko] is shown in [GeMa, Rem. 2.2]. Note also that if $\mathscr{C} = \mathscr{C}_1 \amalg \ldots \amalg \mathscr{C}_r$ is the partition of $\mathscr{C}$ into conjugacy classes, then we certainly have

$$\rho_{\mathscr{C}} = \rho_{\mathscr{C}_1} + \cdots + \rho_{\mathscr{C}_r}.$$

Hence, it is sufficient to prove the above conjecture for the case where $\mathscr{C}$ is a single conjugacy class of involutions.

A strong support is provided by the following general result.

Theorem 5.9 (Marberg [Ma, 1.7])**.** *Let* $\mathbf{I}$ *denote the set of all involutions in* W*. Then* $\langle \rho_{\mathbf{I}}, [C]\rangle_W = |\mathbf{I} \cap C|$ *for every left cell* C *in* W*.*

Example 5.10. Let $W = \mathfrak{S}_n$ be of type A_{n-1} with generators given by the basic transpositions $s_i = (i, i+1)$ for $1 \le i \le n-1$. A complete set of representatives of the conjugacy classes of involutions is given by the elements

$$\sigma_j := s_1 s_3 \cdots s_{2j-1} \in \mathfrak{S}_n, \qquad \text{where} \qquad 0 \le 2j \le n.$$

(Thus, σ_j is the product of j disjoint transpositions and σ_j has precisely $n-2j$ fixed points on $\{1, \ldots, n\}$.) Note that σ_j has minimal length in its conjugacy class; see [GePf, 3.1.16]. Let $\mathscr{C}_j$ be the conjugacy class containing σ_j and write $\rho_j = \rho_{\mathscr{C}_j}$. As in Example 4.4, we write $\mathrm{Irr}(\mathfrak{S}_n) = \{\chi^\alpha \mid \alpha \vdash n\}$. Then, by [Ko, 3.1], we have

$$\text{(a)} \qquad \langle \rho_j, \chi^\alpha \rangle_{\mathfrak{S}_n} = \delta_{n-2j=t},$$

where t is the number of odd parts of the conjugate partition α^*. In particular, if $\mathbf{I}$ denotes the set of all involutions in $\mathfrak{S}_n$, then

$$\rho_{\mathbf{I}} = \sum_j \rho_j = \sum_{\alpha \vdash n} \chi^\alpha.$$

For later reference, we explicitly note the following special case of (a). Let $\alpha = (1^n)$; then $\chi^\alpha = \varepsilon$ is the sign character of $\mathfrak{S}_n$. Then (a) yields:

$$\text{(b)} \qquad \langle \rho_j, \varepsilon \rangle_{\mathfrak{S}_n} = \begin{cases} 1 & \text{if } j = \lfloor n/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

We now have all ingredients in place to verify that Kottwitz's Conjecture holds. Indeed, first note that the longest element in $\mathfrak{S}_n$ has precisely one fixed point on $\{1, \ldots, n\}$ if n is odd, and no fixed point at all if n is even. Now let $\alpha \vdash n$ and let w_{α^*} be the longest element in the Young subgroup $\mathfrak{S}_{\alpha^*} \subseteq \mathfrak{S}_n$, as in Example 4.4. If $\alpha_1^*, \ldots, \alpha_r^*$ are the non-zero parts of α^*, then $\mathfrak{S}_{\alpha^*} \cong \mathfrak{S}_{\alpha_1^*} \times \ldots \times \mathfrak{S}_{\alpha_r^*}$, and so the number of fixed points of w_{α^*} on $\{1, \ldots, n\}$ is the number t of odd parts of α^*. Thus, w_{α^*} is conjugate to $\sigma_{(n-t)/2}$, and so we can reformulate (a) as follows. Let $\mathscr{C}$ be a conjugacy class of involutions in $\mathfrak{S}_n$. Then

$$\text{(c)} \qquad \langle \rho_{\mathscr{C}}, \chi^\alpha \rangle_{\mathfrak{S}_n} = \begin{cases} 1 & \text{if } w_{\alpha^*} \in \mathscr{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Comparison with Example 4.4 now shows that Conjecture 5.7 holds in this case.

6. An inductive approach to Kottwitz's Conjecture

We keep the basic assumptions of the previous section. The results in this section will provide some ingredients for an inductive proof of Kottwitz's Conjecture 5.7.

Let $\mathfrak{C}$ be a two-sided cell of W. We shall say that "*Kottwitz's Conjecture holds for* $\mathfrak{C}$" if, for any conjugacy class of involutions $\mathscr{C}$ in W, we have

$$\langle \rho_{\mathscr{C}}, [C] \rangle_W = |\mathscr{C} \cap C| \qquad \text{for all left cells } C \subseteq \mathfrak{C}.$$

Remark 6.1. Let $w_0 \in W$ be the longest element. Let C be a left cell of W. Then, by [LuB, 5.14], the set Cw_0 also is a left cell and we have

$$[Cw_0] = [C] \otimes \varepsilon, \qquad \text{where} \qquad \varepsilon = \text{sign character of } W.$$

Now let $\mathfrak{C}$ be a two-sided cell. Then $\mathfrak{C}w_0$ also is a two-sided cell and we have

$$\mathrm{Irr}_{\mathfrak{C}w_0}(W) = \mathrm{Irr}_{\mathfrak{C}}(W) \otimes \varepsilon := \{\chi \otimes \varepsilon \mid \chi \in \mathrm{Irr}_{\mathfrak{C}}(W)\}.$$

Lemma 6.2. *Assume that the longest element $w_0 \in W$ is central in W. Let $\mathscr{C}$ be a union of conjugacy classes of involutions in W. Then $\mathscr{C}w_0$ also is a union of conjugacy classes of involutions and we have $\rho_{\mathscr{C}w_0} = \rho_{\mathscr{C}} \otimes \varepsilon$.*

Proof. It is sufficient to prove this in the case where $\mathscr{C}$ is a single conjugacy class. Let $l_0 := \min\{\ell(w) \mid w \in \mathscr{C}\}$. Then $\ell(w) - l_0$ is even for every $w \in \mathscr{C}$. So, for any $w \in \mathscr{C}$, there is a well-defined integer $m(w)$ such that $\ell(w) - l_0 = 2m(w)$. Now we perform a change of basis in $V_{\mathscr{C}}$: we set $a'_w := (-1)^{m(w)} a_w$ for $w \in \mathscr{C}$. Then the action of W on $V_{\mathscr{C}}$ is given by the following formulae, where $s \in S$ and $w \in \mathscr{C}$:

$$s.a'_w = \begin{cases} -a'_w & \text{if } sw = ws \text{ and } \ell(sw) < \ell(w), \\ a'_w & \text{if } sw = ws \text{ and } \ell(sw) > \ell(w), \\ -a'_{sws} & \text{otherwise (that is, if } sw \neq ws). \end{cases}$$

Note that, since $w \in \mathscr{C}$ is an involution, we have $\ell(sw) > \ell(w)$ if and only if $\ell(ws) > \ell(w)$; hence, if $sw \neq ws$, then $\ell(sws) = \ell(w) \pm 2$ (see [GePf, 1.2.6]), and so $a'_{sws} = -a'_w$. Furthermore, it is well-known that $\ell(yw_0) = \ell(w_0) - \ell(y)$ for every $y \in W$. Hence, we can also write the above formulae in the following form:

$$s.a'_w = \begin{cases} a'_w & \text{if } sww_0 = ww_0s \text{ and } \ell(sww_0) < \ell(ww_0), \\ -a'_w & \text{if } sww_0 = ww_0s \text{ and } \ell(sww_0) > \ell(ww_0), \\ -a'_w & \text{otherwise.} \end{cases}$$

Tensoring with ε, we see that we obtain exactly the same formulae as for the action of W on $V_{\mathscr{C}w_0}$. □

Lemma 6.3. *Assume that the longest element $w_0 \in W$ is central in W. Let $\mathfrak{C}$ be a two-sided cell. Then Kottwitz's Conjecture holds for $\mathfrak{C}$ if and only if Kottwitz's Conjecture holds for $\mathfrak{C}w_0$.*

Proof. Assume that Kottwitz's Conjecture holds for $\mathfrak{C}$. Let $\mathscr{C}$ be a conjugacy class of involutions in W. Let C be a left cell contained in $\mathfrak{C}w_0$. Then Cw_0 is a left cell contained in $\mathfrak{C}$ and we obtain

$$\langle \rho_{\mathscr{C}}, [C] \rangle_W = \langle \rho_{\mathscr{C}} \otimes \varepsilon, [C] \otimes \varepsilon \rangle_W = \langle \rho_{\mathscr{C}w_0}, [Cw_0] \rangle_W,$$

where the last equality holds by Remark 6.1 and Lemma 6.2. Now, by assumption, the right hand side equals $|(\mathscr{C}w_0) \cap (Cw_0)| = |\mathscr{C} \cap C|$, as desired. The reverse implication is then clear. □

Definition 6.4 ([LuB, 8.1])**.** Let $\mathfrak{C}$ be a two-sided cell in W. We say that $\mathfrak{C}$ is *strongly non-cuspidal* if there exists a proper standard parabolic subgroup $W' \subsetneqq W$ and a two-sided cell $\mathfrak{C}'$ in W' such that the "truncated induction" $\mathrm{J}_{W'}^{W}$ (as defined in [LuB, 4.1.7]) establishes a bijection

$$\mathrm{Irr}_{\mathfrak{C}'}(W') \to \mathrm{Irr}_{\mathfrak{C}}(W), \qquad \chi' \mapsto \mathrm{J}_{W'}^{W}(\chi').$$

We say that $\mathfrak{C}$ is *non-cuspidal* if $\mathfrak{C}$ or $\mathfrak{C}w_0$ is strongly non-cuspidal (where $w_0 \in W$ is the longest element). Finally, we say that $\mathfrak{C}$ is *cuspidal* if $\mathfrak{C}$ is not non-cuspidal.

(Note that, in [LuB, 8.1], the formulation is in terms of "families" of $\mathrm{Irr}(W)$; however, as already mentioned at the beginning of Section 5, it is known that the sets $\mathrm{Irr}_{\mathfrak{C}}(W)$ are precisely the "families" of $\mathrm{Irr}(W)$.)

Remark 6.5. Let $\mathfrak{C}$ be a two-sided cell in W and assume that $\mathfrak{C}$ is *strongly non-cuspidal*. Let $W', \mathfrak{C}'$ be as in Definition 6.4. Let

$$\chi = \mathrm{J}_{W'}^{W}(\chi') \in \mathrm{Irr}_{\mathfrak{C}}(W), \qquad \text{where} \qquad \chi' \in \mathrm{Irr}_{\mathfrak{C}'}(W').$$

By the definition of the truncated induction, we have $\mathbf{a}_\chi = \mathbf{a}_{\chi'}$. Using [LuB, 4.1.6], one easily sees that also $f_{\chi'} = f_{\chi'}$. In particular, $\mathfrak{C}$ is smooth if and only if $\mathfrak{C}'$ is smooth (see Lemma 3.7).

Lemma 6.6. *Let $\mathfrak{C}$ be a strongly non-cuspidal two-sided cell in W. Let $W', \mathfrak{C}'$ be as in Definition 6.4. Then Kottwitz's Conjecture holds for $\mathfrak{C}$ if the following three conditions are satisfied:*

(K1) *For any conjugacy class of involutions $\mathscr{C}$ in W and any left cells $C_1, C_2 \subseteq \mathfrak{C}$ such that $[C_1] = [C_2]$, we have $|\mathscr{C} \cap C_1| = |\mathscr{C} \cap C_2|$.*

(K2) *Kottwitz's Conjecture holds for the two-sided cell $\mathfrak{C}'$ in W'.*

(K3) *For any conjugacy class of involutions $\mathscr{C}$ in W such that $\mathscr{C} \cap W' \neq \varnothing$, we have*

$$\langle \rho_{\mathscr{C}\cap W'}, \chi' \rangle_{W'} \leq \big\langle \rho_{\mathscr{C}}, \mathrm{J}_{W'}^{W}(\chi') \big\rangle_W \qquad \textit{for all } \chi' \in \mathrm{Irr}_{\mathfrak{C}'}(W').$$

Proof. Let $\mathscr{C}$ be any conjugacy class of involutions in W. First we show that

$$(*) \qquad \langle \rho_{\mathscr{C}}, [C] \rangle_W \geq |\mathscr{C} \cap C| \qquad \text{for all left cells } C \subseteq \mathfrak{C}.$$

Indeed, let C be a left cell of W which is contained in $\mathfrak{C}$. If $\mathscr{C} \cap C = \varnothing$, then $(*)$ is obvious. Now assume that $\mathscr{C} \cap C \neq \varnothing$. By [Lu3, §3] (see also [Ge2, Lemma 5.6]), there exists a left cell C' of W' which is contained in $\mathfrak{C}'$ and such that $[C] = \mathrm{J}_{W'}^{W}([C'])$. So we have

$$\langle \rho_{\mathscr{C}}, [C] \rangle_W = \langle \rho_{\mathscr{C}}, \mathrm{J}_{W'}^{W}([C']) \rangle_W.$$

Now using (K2) and (K3), we obtain

$$\langle \rho_{\mathscr{C}}, \mathrm{J}_{W'}^{W}([C']) \rangle_W \geq \langle \rho_{\mathscr{C}\cap W'}, [C'] \rangle_{W'} = |(\mathscr{C} \cap W') \cap C'| = |\mathscr{C} \cap C'|.$$

On the other hand, let C_1 be the left cell of W such that $C' \subseteq C_1$. Then we also have $[C_1] = \mathrm{J}_{W'}^{W}([C'])$; see [LuB, 5.28] (or the argument in the proof of Case 1 in [Lu5, Lemma 22.2]). Now, since $[C_1] = \mathrm{J}_{W'}^{W}([C'])$ and since $\mathrm{J}_{W'}^{W}$ establishes a bijection between $\mathrm{Irr}_{\mathfrak{C}'}(W)$ and $\mathrm{Irr}_{\mathfrak{C}}(W)$, we conclude that $[C']$ and $[C_1]$ have the same number of irreducible constituents (counting multiplicities). Hence, by Proposition 2.6, C' and C_1 contain the same number of involutions. Consequently, since $C' \subseteq C_1$, all the involutions in C_1 must be contained in C', and so $\mathscr{C} \cap C' = \mathscr{C} \cap C_1$. In particular, this shows that $|\mathscr{C} \cap C'| = |\mathscr{C} \cap C_1| = |\mathscr{C} \cap C|$, where the second equality holds by (K1). Thus, $(*)$ is proved. Once this is established, it actually follows that we must have equality in $(*)$. Indeed, let $\mathscr{C}_1, \ldots, \mathscr{C}_m$ be the conjugacy classes of involutions in W; then $\mathbf{I} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_m$ is the set of all involutions in W. By $(*)$, we have

$$\langle \rho_{\mathbf{I}}, [C] \rangle_W = \sum_{1 \leq i \leq m} \langle \rho_{\mathscr{C}_i}, [C] \rangle_W \geq \sum_{1 \leq i \leq m} |\mathscr{C}_i \cap C| = |\mathbf{I} \cap C|.$$

However, by Theorem 5.9, we know that the left hand side equals the right hand side. Hence, all the inequalities in $(*)$ must be equalities, as claimed. Thus, Kottwitz's Conjecture holds for $\mathfrak{C}$. □

Remark 6.7. We note that an analogous version of the inequality in (K3) always holds where $\mathrm{J}_{W'}^{W}(\chi')$ is replaced by $\mathrm{Ind}_{W'}^{W}(\chi')$. In fact, for any parabolic subgroup $W' \subseteq W$ and any conjugacy of involutions $\mathscr{C}$ in W such that $\mathscr{C}' \cap W' \neq \varnothing$, we have

$$\langle \rho_{\mathscr{C}\cap W'}, \chi' \rangle_{W'} \leq \big\langle \rho_{\mathscr{C}}, \mathrm{Ind}_{W'}^{W}(\chi') \big\rangle_W \qquad \text{for all } \chi' \in \mathrm{Irr}(W').$$

This is seen as follows. Let $V_{\mathscr{C}}$ be as in Definition 5.6. From the formulae for the action of W on $V_{\mathscr{C}}$, it is clear that the subspace $U \subseteq V_{\mathscr{C}}$ spanned by the basis elements $\{a_w \mid w \in \mathscr{C} \cap W'\}$ is a W'-submodule. Furthermore, the character of this W'-module is just $\rho_{\mathscr{C}\cap W'}$. Thus, we can write $\mathrm{Res}^W_{W'}(\rho_{\mathscr{C}}) = \rho_{\mathscr{C}\cap W'} + \psi$ for some character ψ of W'. This yields that

$$\langle \rho_{\mathscr{C}'\cap W'}, \chi' \rangle_{W'} \leq \langle \mathrm{Res}^W_{W'}(\rho_{\mathscr{C}}), \chi' \rangle_{W'} \qquad \text{for all } \chi' \in \mathrm{Irr}(W'),$$

and so the assertion immediately follows by Frobenius reciprocity.

Lemma 6.8. *Let $W' \subseteq W$ be a standard parabolic subgroup and $\mathscr{C}'$ be a conjugacy class of involutions in W'. Let $\mathscr{C}$ be the conjugacy class of W such that $\mathscr{C}' \subseteq \mathscr{C}$. Then*

$$\big\langle \rho_{\mathscr{C}}, \chi \big\rangle_W \leq \big\langle \mathrm{Ind}^W_{W'}(\rho_{\mathscr{C}'}), \chi \big\rangle_W \qquad \textit{for all } \chi \in \mathrm{Irr}(W).$$

Proof. We can find a representative $\sigma \in \mathscr{C}'$ such that σ is the longest element in a standard parabolic subgroup $W'' \subseteq W'$ and such that σ is central in W''; see [GePf, 3.2.10]. Then, by Kottwitz's original construction in [Ko], we have

$$\rho_{\mathscr{C}} = \mathrm{Ind}^W_{C_W(\sigma)}(\varepsilon_\sigma) \qquad \text{and} \qquad \rho_{\mathscr{C}'} = \mathrm{Ind}^{W'}_{C_{W'}(\sigma)}(\varepsilon'_\sigma),$$

where $\varepsilon_\sigma \colon C_W(\sigma) \to \{\pm 1\}$ and $\varepsilon'_\sigma \colon C_{W'}(\sigma) \to \{\pm 1\}$ are certain linear characters. To describe these characters explicitly, let Φ be the root system of W; let $\Phi = \Phi^+ \amalg \Phi^-$ be the decomposition into positive and negative roots (defined by the given set of generators S of W). Let Φ'' be the parabolic subsystem defined by W''. Then, for any $w \in C_W(\sigma)$, we have $\varepsilon_\sigma(w) = (-1)^k$, where k is the number of positive roots in Φ'' which are sent to negative roots by w (see also [GeMa, Rem. 2.2]). The definition of ε'_σ is analogous. By this description, it is clear that ε'_σ is the restriction of ε_σ to W'. Hence, we can write

$$\mathrm{Ind}^{C_W(\sigma)}_{C_{W'}(\sigma)}(\varepsilon'_\sigma) = \varepsilon_\sigma + \psi$$

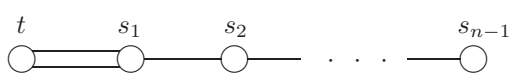
for some character ψ of $C_W(\sigma)$. By the transitivity of induction, this yields

$$\mathrm{Ind}^W_{W'}(\rho_{\mathscr{C}'}) = \mathrm{Ind}^W_{C_{W'}(\sigma)}(\varepsilon'_\sigma) = \mathrm{Ind}^W_{C_W(\sigma)}\big(\varepsilon_\sigma + \psi\big) = \rho_{\mathscr{C}} + \mathrm{Ind}^W_{C_W(\sigma)}(\psi),$$

which immediately implies the assertion. □

7. Kottwitz's Conjecture for type B_n

Throughout this section, let $W = W_n$ be of type B_n with generators $t, s_1, \ldots, s_{n-1}$ and a diagram given as follows:



The aim of this section is to prove that Conjecture 5.7 holds for W_n; here, we are in the equal parameter case where $\Gamma = \mathbb{Z}$ and $\varphi(t) = \varphi(s_1) = \ldots = \varphi(s_{n-1}) = 1$. For this purpose, we first need to recall some results from [Ko] concerning the decomposition of the character $\rho_{\mathscr{C}}$ into irreducibles.

Example 7.1. A complete set of representatives of the conjugacy classes of involutions in W_n is given as follows. Let l, j be non-negative integers such that $l + 2j \leq n$. Then set

$$\sigma_{l,j} := t_1 \cdots t_l s_{l+1} s_{l+3} \cdots s_{l+2j-1} \in W_n,$$

where $t_1 := t$ and $t_i := s_{i-1}t_{i-1}s_{s-i}$ for $2 \leq i \leq n$. Note that $\sigma_{l,j}$ is the longest element in a parabolic subgroup of W_n of type $B_l \times A_1 \times \ldots \times A_1$, where the A_1 factor is repeated j times. In particular, $\sigma_{l,j}$ has minimal length in its conjugacy class; see also [GePf, 3.2.10]. Let $\mathscr{C}_{l,j}$ be the conjugacy class containing $\sigma_{l,j}$ and write $\rho_{l,j} = \rho_{\mathscr{C}_{l,j}}$. The irreducible characters of W_n are parametrised by pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. We write this as

$$\mathrm{Irr}(W_n) = \{\chi^{(\alpha,\beta)} \mid (\alpha, \beta) \vdash n\}.$$

Now let $\chi \in \mathrm{Irr}(W)$. We associate with χ two invariants $d(\chi)$ and $j_0(\chi)$, as follows. Let $(\alpha, \beta) \vdash n$ be such that $\chi = \chi^{(\alpha,\beta)}$. Choose $m \geq 0$ such that we can write

$$\alpha = (0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{m+1}) \qquad \text{and} \qquad \beta = (0 \leq \beta_1 \leq \beta_2 \leq \ldots \leq \beta_m).$$

As in [LuB, §4.5], we have a corresponding "symbol",

$$\Lambda_m(\chi) := \begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_{m+1} \\ \mu_1, \mu_2, \ldots, \mu_m \end{pmatrix},$$

where $\lambda_i := \alpha + i - 1$ for $1 \leq i \leq m+1$ and $\mu_i = \beta_i + i - 1$ for $1 \leq i \leq m$. We set

$$d(\chi) := \text{ number of } i \in \{1, \ldots, m\} \text{ such that } \mu_i \notin \{\lambda_1, \lambda_2, \ldots, \lambda_{m+1}\},$$

$$j_0(\chi) := \sum_{1 \leq i \leq m} \min\{\alpha_{i+1}, \beta_i\}.$$

(Note that these definitions do not depend on the choice of m.) By [Lu5, 22.14], we have $f_\chi = 2^{d(\chi)}$; furthermore,

$$\chi \text{ is special} \qquad \Longleftrightarrow \qquad \lambda_i \leq \mu_i \leq \lambda_{i+1} \quad \text{for } 1 \leq i \leq m.$$

Now, by [Ko, (3.2.4)], the following hold:

(a) $\langle \rho_{l,j}, \chi \rangle_{W_n} = 0$ unless χ is special and $j + l = |\beta|$.

(b) If χ is special and $j + l = |\beta|$, then

$$\langle \rho_{l,j}, \chi \rangle_{W_n} = \begin{pmatrix} d(\chi) \\ j_0(\chi) - j \end{pmatrix} \qquad \text{(binomial coefficient);}$$

in particular, the multiplicity is zero unless $j_0(\chi) - d(\chi) \leq j \leq j_0(\chi)$.

Consequently, if $\mathbf{I}$ denotes the set of all involutions in W_n, then

$$\rho_{\mathbf{I}} = \sum_{l,j} \rho_{l,j} = \sum_{\chi \in \mathscr{S}(W_n)} 2^{d(\chi)}\, \chi.$$

Remark 7.2. By [LuB, 8.1], we have an explicit combinatorial description of the cuspidal and non-cuspidal two-sided cells in W_n. Let us briefly recall the main points of this description. Let $\mathfrak{C}$ be any two-sided cell and $\chi_0 \in \mathrm{Irr}_{\mathfrak{C}}(W_n)$ the unique special character. Let $(\alpha, \beta) \vdash n$ be such that $\chi_0 = \chi^{(\alpha,\beta)}$. Write

$$\alpha = (0 \leq \alpha_1 \leq \ldots \leq \alpha_{m+1}) \qquad \text{and} \qquad \beta = (0 \leq \beta_1 \leq \ldots \leq \beta_m)$$

for some $m \geq 0$. Consider the corresponding symbol

$$\Lambda_m(\chi_0) = \begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_{m+1} \\ \mu_1, \mu_2, \ldots, \mu_m \end{pmatrix}; \qquad \text{see Example 7.1.}$$

We assume that m is chosen such that 0 does not appear in both rows of $\Lambda_m(\chi)$.

First of all, $\mathfrak{C}$ is cuspidal if and only if $n = d^2 + d$ for some $d \geq 1$ and $\Lambda_m(\chi_0)$ contains each of the numbers $0, 1, \ldots, 2m$ exactly once.

Now consider the general case. Let t_0 be the largest entry in $\Lambda_m(\chi_0)$. Then $\mathfrak{C}$ is strongly non-cuspidal if there is some $i \in \{0, 1, \ldots, t_0 - 1\}$ which does not appear in any of the two rows of $\Lambda_m(\chi_0)$. Let us now assume that this is the case. Then there exists a parabolic subgroup $W' \subsetneqq W_n$ and a two-sided cell $\mathfrak{C}'$ of W' such that $\mathrm{J}_{W'}^{W_n}$ establishes a bijection

$$\mathrm{Irr}_{\mathfrak{C}'}(W') \to \mathrm{Irr}_{\mathfrak{C}}(W_n), \qquad \chi' \mapsto \mathrm{J}_{W'}^{W_n}(\chi').$$

More precisely, as discussed in [LuB, 8.1], the subgroup W' and the two-sided cell $\mathfrak{C}'$ can be chosen as follows, where $\chi_0' \in \mathrm{Irr}_{\mathfrak{C}'}(W')$ is the unique special character.

(a) There exists some $r \in \{1, \ldots, n\}$ such that $W' = W_{n-r} \times H_r$ where $W_{n-r} = \langle t, s_1, \ldots, s_{n-r-1} \rangle$ (of type B_{n-r}) and $H_r = \langle s_{n-r+1}, \ldots, s_{n-1} \rangle \cong \mathfrak{S}_r$.

(b) We have $\chi_0' = \psi_0 \boxtimes \varepsilon_r$, where $\psi_0 \in \mathrm{Irr}(W_{n-r})$ is special and ε_r denotes the sign character on H_r; furthermore, $\Lambda_m(\chi_0)$ is obtained by increasing the largest r entries in the symbol $\Lambda_m(\psi_0)$ by 1.

(c) We have $d(\chi_0) = d(\psi_0)$, $j_0(\chi_0) = j_0(\psi_0) + \lfloor r/2 \rfloor$ and $|\beta| = |\beta'| + \lfloor r/2 \rfloor$, where ψ_0 is labelled by the pair of partitions $(\alpha', \beta') \vdash n - r$.

Only (c) requires a proof here. (Both (a) and (b) are explicitly discussed in [LuB, 8.1].) As remarked in Example 7.1, we have $f_{\chi_0} = 2^{d(\chi_0)}$ and $f_{\psi_0} = 2^{d(\psi_0)}$. So the first equality follows from Remark 6.5. To see the second equality in (c), consider the symbol $\Lambda_m(\chi_0)$. Since χ_0 is special, the largest r entries in $\Lambda_m(\chi_0)$ are the last r terms in the sequence

$$\lambda_1, \quad \mu_1, \quad \lambda_2, \quad \mu_2, \quad \ldots, \quad \lambda_m, \quad \mu_m, \quad \lambda_{m+1}.$$

Now consider the pair of partitions $(\alpha', \beta') \vdash n - r$ such that $\psi_0 = \chi^{(\alpha', \beta')}$; we also write $\alpha' = (0 \leq \alpha_1' \leq \ldots \leq \alpha_{m+1}')$ and $\beta' = (0 \leq \beta_1' \leq \ldots \leq \beta_m')$. By (b), the symbol $\Lambda_m(\chi_0)$ is obtained by increasing the largest r entries in the symbol $\Lambda_m(\psi_0)$ by 1. Consequently, the sequence

$$\alpha_1, \quad \beta_1, \quad \alpha_2, \quad \beta_2, \quad \ldots, \quad \alpha_m, \quad \beta_m, \quad \alpha_{m+1}$$

is obtained from the sequence

$$\alpha_1', \quad \beta_1', \quad \alpha_2', \quad \beta_2', \quad \ldots, \quad \alpha_m', \quad \beta_m', \quad \alpha_{m+1}'$$

by increasing the last r terms in the latter sequence by 1. This then immediately yields the statements about $j_0(\chi_0)$ and $|\beta|$. Thus, (c) is proved.

Theorem 7.3. *Let $W = W_n$ be of type B_n, as above. Let $\mathfrak{C}$ be a two-sided cell of W_n and $\chi_0 \in \mathrm{Irr}_{\mathfrak{C}}(W_n)$ be the unique special character. Let $\mathscr{C}$ be a conjugacy class of involutions in W_n. Then*

$$\langle \rho_{\mathscr{C}}, [C] \rangle_{W_n} = \langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n} = |\mathscr{C} \cap C| \qquad \textit{for any left cell } C \subseteq \mathfrak{C}.$$

Thus, Kottwitz's Conjecture 5.7 *holds for W_n.*

Proof. The first equality is seen as follows. As already remarked in Example 5.5, we have $\langle [C], \chi_0 \rangle_{W_n} = 1$ for every left cell $C \subseteq \mathfrak{C}$. On the other hand, by Example 7.1(a), all constituents of $\rho_{\mathscr{C}}$ are special. Hence, we have $\langle \rho_{\mathscr{C}}, [C] \rangle_{W_n} = \langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n}$, as required.

We now show by induction on n that Kottwitz's Conjecture holds. If $n = 1$, then $W_2 \cong \mathfrak{S}_2$ and the assertion holds by Example 5.10. Now assume that $n \geq 2$.

First we consider the case where $\mathfrak{C}$ is strongly non-cuspidal. Let $W' = W_{n-r} \times H_r$ and $\psi_0 \in \mathrm{Irr}(W_{n-r})$ be as in Remark 7.2, where $r \in \{1, \ldots, n\}$. Then

$$\chi_0 = \mathrm{J}_{W'}^{W}(\chi_0'), \qquad \text{where} \qquad \chi_0' = \psi_0 \boxtimes \varepsilon_r.$$

Let $\mathfrak{C}'$ be the two-sided cell of W' such that $\chi_0' \in \mathrm{Irr}_{\mathfrak{C}'}(W')$. We now check that the assumptions (K1), (K2), (K3) in Lemma 6.6 are satisfied.

Assumption (K1) certainly holds by the identity in Example 5.5, while (K2) holds by our inductive hypothesis. Now consider (K3).

Let $\chi' \in \mathrm{Irr}_{\mathfrak{C}'}(W')$ and $\mathscr{C}$ be a conjugacy class of involutions in W_n such that $\mathscr{C} \cap W' \neq \varnothing$. If $\langle \rho_{\mathscr{C} \cap W'}, \chi' \rangle_{W'} = 0$, then the assertion is obvious. Now assume that $\langle \rho_{\mathscr{C} \cap W'}, \chi' \rangle_{W'} \neq 0$. Then there is a conjugacy class of involutions $\mathscr{C}'$ in W' such that

$$(\triangle) \qquad \mathscr{C}' \subseteq \mathscr{C} \cap W' \qquad \text{and} \qquad \langle \rho_{\mathscr{C}'}, \chi' \rangle_{W'} \neq 0.$$

(We shall see that $\mathscr{C}'$ is uniquely determined by this property.) Since W' is a direct product, we can write $\mathscr{C}'$ as a direct product of a conjugacy class in W_{n-r} and a conjugacy class in H_r. Thus, using the notation in Examples 5.10 and 7.1, we have

$$\mathscr{C}' = \mathscr{C}_{l,j'} \times \mathscr{C}_k, \qquad \text{where} \qquad l, j', k \geq 0, \quad l + 2j' \leq n - r, \quad 2k \leq r;$$

here, the class $\mathscr{C}_{l',j'} \subseteq W_{n-r}$ has a representative $\sigma_{l,j'}$ given by the expression in Example 7.1 and the class $\mathscr{C}_k \subseteq H_r$ has a representative σ_k as in Example 5.10. (Explicitly, we have $\sigma_k = s_{n-r+1} s_{n-r+3} \cdots s_{n-r+2k-1}$.) We note that $\sigma_{l,j'} \times \sigma_k \in \mathscr{C}'$ is the longest element in a parabolic subgroup of W_n of type $B_l \times A_1 \times \ldots \times A_1$, where the A_1 factor is repeated $j' + k$ times. Hence, since $\mathscr{C}' \subseteq \mathscr{C}$, we must have

$$\mathscr{C} = \mathscr{C}_{l,j}, \qquad \text{where} \qquad j = j' + k.$$

Now, we can also write $\chi' = \psi \boxtimes \varepsilon_r$ where $\psi \in \mathrm{Irr}(W_{n-r})$. Then we obtain

$$\langle \rho_{\mathscr{C}'}, \chi' \rangle_{W'} = \langle \rho_{l,j'}, \psi \rangle_{W_{n-r}} \langle \rho_k, \varepsilon_r \rangle_{H_r}.$$

Since this is assumed to be non-zero, we conclude that

$$\langle \rho_{l,j'}, \psi \rangle_{W_{n-r}} \neq 0 \qquad \text{and} \qquad \langle \rho_k, \varepsilon_r \rangle_{H_r} \neq 0.$$

By Example 7.1, the first condition implies that ψ is special and, hence, χ' is special. Thus, we must have $\chi' = \chi_0'$ and $\psi = \psi_0$. By Example 5.10(b), the second condition implies that $\langle \rho_k, \varepsilon_r \rangle_{H_r} = 1$ and $k = \lfloor r/2 \rfloor$. In particular, the class $\mathscr{C}'$ in $(\triangle)$ is uniquely determined. Combining these statements, we obtain that

$$\langle \rho_{\mathscr{C} \cap W'}, \chi' \rangle_{W'} = \langle \rho_{\mathscr{C}'}, \chi' \rangle_{W_n} = \langle \rho_{l,j'}, \psi_0 \rangle_{W_{n-r}}.$$

Since $\chi' = \chi_0'$, we have $\chi_0 = \mathrm{J}_{W'}^{W_n}(\chi')$; since $\mathscr{C} = \mathscr{C}_{l,j}$, we are finally reduced to showing that

$$\langle \rho_{l,j'}, \psi_0 \rangle_{W_{n-r}} \leq \langle \rho_{l,j}, \chi_0 \rangle_{W_n}, \qquad \text{where} \qquad j = j' + k \quad \text{and} \quad k = \lfloor r/2 \rfloor.$$

But, by Remark 7.2(c), we have $d(\chi_0) = d(\psi_0)$ and $j_0(\chi_0) = j_0(\psi_0) + \lfloor r/2 \rfloor$. Hence, the multiplicity formula in Example 7.1 shows that we actually have

$$\langle \rho_{l,j'}, \psi_0 \rangle_{W_{n-r}} = \langle \rho_{l,j}, \chi_0 \rangle_{W_n}.$$

Thus, (K3) is satisfied and so Kottwitz's Conjecture holds for $\mathfrak{C}$. Since the longest element $w_0 \in W_n$ is central in W_n, we can apply Lemma 6.3 which shows that Kottwitz's Conjecture will also hold for $\mathfrak{C}w_0$. By [LuB, 8.1], these arguments cover all non-cuspidal two-sided cells in W_n.

It remains to consider the case where $\mathfrak{C}$ is a cuspidal two-sided cell. By Remark 7.2, such a two-sided cell can only exist if $n = d^2 + d$ for some $d \geq 1$, in which case it is uniquely determined. So let us now assume that $n = d^2 + d$ where $d \geq 1$. Let $W_n = \coprod_{0 \leq i \leq N} \mathfrak{C}_i$ be the partition into two-sided cells where $\mathfrak{C}_0$ is the unique cuspidal two-sided cell. For $0 \leq i \leq N$, let $\chi_i \in \mathrm{Irr}_{\mathfrak{C}_i}(W_n)$ be the unique special character and let $C_i \subseteq \mathfrak{C}_i$ be a left cell. Let $\mathscr{C}$ be a conjugacy class of involutions. To obtain a statement about $\langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n}$, we consider

$$\sum_{0 \leq i \leq N} \chi_i(1) \langle \rho_{\mathscr{C}}, \chi_i \rangle_{W_n} = \Big\langle \rho_{\mathscr{C}}, \sum_{0 \leq i \leq N} \chi_i(1) \chi_i \Big\rangle_{W_n}.$$

Since all constituents of $\rho_{\mathscr{C}}$ are special, the sum on the right hand side can be extended over all $\chi \in \mathrm{Irr}(W_n)$, in which case we just obtain the character of the regular representation of W_n. Hence, the right hand side equals $\rho_{\mathscr{C}}(1)$. Now, for any $i \geq 1$, we already know that Kottwitz's Conjecture holds for $\mathfrak{C}_i$, and so $\langle \rho_{\mathscr{C}}, \chi_i \rangle_{W_n} = \langle \rho_{\mathscr{C}}, [C_i] \rangle_{W_n} = |\mathscr{C} \cap C_i|$. Hence, we find that

$$\chi_0(1) \langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n} = \rho_{\mathscr{C}}(1) - \sum_{1 \leq i \leq N} \chi_i(1) |\mathscr{C} \cap C_i|.$$

On the other hand, using the identity in Example 5.5, we obtain

$$\sum_{0 \leq i \leq N} \chi_i(1) |\mathscr{C} \cap C_i| = \sum_{0 \leq i \leq N} |\mathscr{C} \cap \mathfrak{C}_i| = |\mathscr{C}|.$$

Hence, we find that

$$\chi_0(1) |\mathscr{C} \cap C_0| = |\mathscr{C}| - \sum_{1 \leq i \leq N} \chi_i(1) |\mathscr{C} \cap C_i|.$$

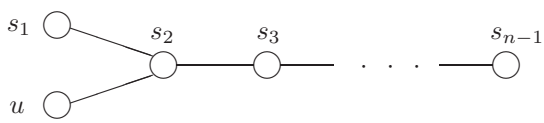
Since $\rho_{\mathscr{C}}(1) = |\mathscr{C}|$, we deduce that

$$\chi_0(1) \langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n} = \chi_0(1) |\mathscr{C} \cap C_0|,$$

and so $\langle \rho_{\mathscr{C}}, [C_0] \rangle_{W_n} = \langle \rho_{\mathscr{C}}, \chi_0 \rangle_{W_n} = |\mathscr{C} \cap C_0|$, as required. □

8. On Kottwitz's Conjecture for type D_n

Throughout this section, let $n \geq 2$ and $W = W'_n$ be of type D_n, with generators $u, s_1, \ldots, s_{n-1}$ and a diagram given as follows:



By convention, we will also set $W'_0 = W'_1 = \{1\}$. Assuming some results from [Ge7], the aim of this section is to prove that Conjecture 5.7 holds for W'_n. For this purpose, it will be convenient to use an embedding of W'_n into the group W_n of type B_n, with generators $t, s_1, \ldots, s_{n-1}$ and a diagram as in the previous section. Setting $u = ts_1t$ (and identifying the remaining generators $s_1, \ldots, s_{n-1}$), we can identify W'_n with a subgroup of W_n. Thus, we have $W_n \cong W'_n \rtimes \langle \theta \rangle$, where $\theta \colon W'_n \to W'_n$ is the automorphism given by conjugation with t. In this setting, a large part of the argument will be analogous to that for type B_n. However, when n is even, there are some particularly intricate questions to solve concerning the unique conjugacy class of involutions in W'_n which is not invariant under θ.

Example 8.1. Let $\mathscr{C}'$ be a conjugacy class of involutions in W_n'. If $\theta(\mathscr{C}') = \mathscr{C}'$, then $\mathscr{C}'$ is a conjugacy class in W_n and the decomposition of $\rho_{\mathscr{C}'}$ into irreducible characters of W_n' is given by formulae similar to those for type W_n in Example 7.1; see [Ko, §3.3]. In particular, we have

(a) $\langle \rho_{\mathscr{C}'}, \chi \rangle_{W_n'} = 0$ unless $\chi \in \mathrm{Irr}(W_n')$ is special and can be extended to W_n.

Classes which are not θ-invariant can only exist if n is even, and then we will also encounter characters which cannot be extended to W_n. So let us now assume that n is even. Let $\mathscr{C}_0'$ be the conjugacy class of W_n' containing the element

$$\sigma_{0,n/2} := s_1 s_3 s_5 \cdots s_{n-1}.$$

Then $\theta(\mathscr{C}_0') \neq \mathscr{C}_0'$ and $\{\mathscr{C}_0', \theta(\mathscr{C}_0')\}$ is the only pair of conjugacy classes of involutions with this property; see [GePf, 3.4.12]. To describe the decomposition of $\rho_{\mathscr{C}_0'}$ into irreducible characters, we introduce some further notation. For every partition $\alpha \vdash n/2$, we define two characters $\chi^{\alpha,\pm 1} \in \mathrm{Irr}(W_n')$, as follows. Let $H_n = \langle s_1, \ldots, s_{n-1} \rangle \cong \mathfrak{S}_n$. Let $2\alpha^*$ denote the partition of $2n$ obtained by multiplying all parts of the conjugate partition α^* by 2 and consider the corresponding Young subgroup $H_{2\alpha^*} \subseteq H_n$. (We have $H_{2\alpha^*} \cong \mathfrak{S}_{2\alpha^*}$.) Let $\varepsilon_{2\alpha^*}$ be the sign character of $H_{2\alpha^*}$ and let $w_{2\alpha^*}$ be the longest element in $H_{2\alpha^*}$. Then, by [GePf, 5.3.2], there is a unique $\chi^{\alpha,\pm 1} \in \mathrm{Irr}(W_n')$ such that $\mathbf{b}_\chi = \ell(w_{2\alpha^*})$ and

$$\mathrm{Ind}_{H_{2\alpha^*}}^{W_n'}(\varepsilon_{2\alpha^*}) = \chi^{\alpha,+1} + \text{ sum of various } \chi \in \mathrm{Irr}(W_n') \text{ with } \mathbf{b}_\chi > \ell(w_{2\alpha^*});$$

furthermore, $\chi^{\alpha,-1}$ is defined as the conjugate of $\chi^{\alpha,+}$ under θ. It is well-known that $\{\chi^{\alpha,\pm 1} \mid \alpha \vdash n/2\}$ are precisely the irreducible characters of W_n' which cannot be extended to W_n; see [LuB, 4.6], [GePf, §5.6]. By [Ko, §3.3], the decompositions of $\rho_{\mathscr{C}_0'}$ and $\rho_{\theta(\mathscr{C}_0')}$ into irreducible characters are given as follows:

(b) $$\rho_{\mathscr{C}_0'} = \sum_{\alpha \vdash n/2} \chi^{\alpha,\nu_\alpha} \qquad \text{and} \qquad \rho_{\theta(\mathscr{C}_0')} = \sum_{\alpha \vdash n/2} \chi^{\alpha,-\nu_\alpha},$$

where $\nu_\alpha \in \{\pm 1\}$ for all $\alpha \vdash n/2$.

Note that the above signs have not been determined in [Ko]. It will be essential to fix these signs in order to prove Kottwitz's Conjecture. In fact, the following example shows that this conjecture can only hold if $\nu_\alpha = +1$ for all $\alpha \vdash n/2$.

Example 8.2. Assume that n is even and let $\mathscr{C}_0'$ be the conjugacy class of the element $\sigma_{0,n/2}$, as above. By [LuB, 4.6.10], we have for any $\chi = \chi^{\alpha,\pm 1}$, where $\alpha \vdash n/2$:

(a) $$f_\chi = 1 \qquad \text{and} \qquad \mathbf{a}_\chi = \mathbf{b}_\chi = \ell(w_{2\alpha^*})$$

Consequently, each $\chi^{\alpha,\pm 1}$ is special and we have

(b) $$\chi^{\alpha,+1} = \mathrm{J}_{H_{2\alpha^*}}^{W_n'}(\varepsilon_{2\alpha^*}) \qquad \text{for any } \alpha \vdash n/2;$$

see [LuB, 4.6.2]. Let $\mathfrak{C}_\alpha^\pm$ denote the two-sided cell such that $\chi^{\alpha,\pm 1} \in \mathrm{Irr}_{\mathfrak{C}_\alpha^\pm}(W_n')$. Then $\mathfrak{C}_\alpha^\pm$ is smooth, by (a) and Lemma 3.7. We also have

(c) $$\mathscr{C}_0' \cap \mathfrak{C}_\alpha^+ \neq \varnothing \qquad \text{and} \qquad \theta(\mathscr{C}_0') \cap \mathfrak{C}_\alpha^- \neq \varnothing \qquad \text{for all } \alpha \vdash n/2.$$

Indeed, (a), (b) and Example 3.3 show that $w_{2\alpha^*} \in \mathfrak{C}_\alpha^+$. Now recall that $H_{2\alpha^*}$ is isomorphic to a direct product of various symmetric groups of even degrees, where the sum of all these degrees is n. Since the longest element in $\mathfrak{S}_{2m}$ (any $m \geq 1$)

is a product of m disjoint 2-cycles, we see that $w_{2\alpha^*}$ is a product of $n/2$ disjoint 2-cycles, and so we have $w_{2\alpha^*} \in \mathscr{C}_0'$. Hence, $\mathscr{C}_0'$ is the unique conjugacy class of involutions in W_n' such that $\mathscr{C}_0' \cap \mathfrak{C}_\alpha^+ \neq \varnothing$ (see Corollary 4.3). Similarly, $\theta(\mathscr{C}_0')$ is the unique conjugacy class of involutions in W_n' such that $\theta(\mathscr{C}_0') \cap \mathfrak{C}_\alpha^- \neq \varnothing$.

In particular, if C is a left cell contained in $\mathfrak{C}_\alpha^+$, then $[C] = \chi^{\alpha,+1}$ and $|\mathscr{C}_0' \cap C| = 1$. So, if Kottwitz's Conjecture holds for W_n', then we must have $\langle \rho_{\mathscr{C}_0'}, \chi^{\alpha,+1}\rangle = 1$.

Now, determining the signs in Example 8.1(b) is related to the subtle issue of distinguishing the two characters $\chi^{\alpha,+1}$ and $\chi^{\alpha,-1}$ for a given partition $\alpha \vdash n/2$. We shall need the following version of the "branching rule" for the characters of W_n'.

Lemma 8.3. *Assume that $n \geq 2$ is even. Consider the parabolic subgroup $W' = W_{n-2}' \times H_2$, where $W_{n-2}' = \langle u, s_1, \ldots, s_{n-3}\rangle$ (type D_{n-2}) and $H_2 = \langle s_{n-1}\rangle$. Let $\alpha' \vdash (n-2)/2$ and denote by ε_1 the sign character on the factor H_2. Then*

$$\mathrm{Ind}_{W'}^{W_n'}\big(\chi^{\alpha',+1} \boxtimes \varepsilon_1\big) = \sum_{\alpha} \chi^{\alpha,+1} \quad + \quad \text{“further terms”},$$

where the sum runs over all partitions $\alpha \vdash n/2$ such that α is obtained by increasing one part of α' by 1; the expression "further terms" stands for a sum of various $\chi \in \mathrm{Irr}(W_n')$ which can be extended to W_n. In particular,

$$\Big\langle \mathrm{Ind}_{W'}^{W_n'}\big(\chi^{\alpha',+1} \boxtimes \varepsilon_1\big), \chi^{\alpha,-1}\Big\rangle_{W_n'} = 0 \qquad \text{for all } \alpha \vdash n/2.$$

A proof can be found in [Ge7, §3].

Proposition 8.4. *Assume that $n \geq 2$ is even. Then, with the notation in Example 8.1, we have*

$$\rho_{\mathscr{C}_0'} = \sum_{\alpha \vdash n/2} \chi^{\alpha,+1} \qquad \text{and} \qquad \rho_{\theta(\mathscr{C}_0')} = \sum_{\alpha \vdash n/2} \chi^{\alpha,-1}.$$

Proof. We prove this by induction on $n/2$. If $n = 2$, then the assertion is easily checked directly. The character table of $W_2' = \langle u, s_1\rangle$ with the appropriate labelling of the characters is given as follows:

	1	s_1	u	s_1u
$\chi^{(2,\varnothing)}$	1	1	1	1
$\chi^{(11,\varnothing)}$	1	-1	-1	1
$\chi^{(1),+}$	1	-1	1	-1
$\chi^{(1),-}$	1	1	-1	-1

$$\rho_{\mathscr{C}_0'} = \chi^{(1),+},$$

$$\rho_{\theta(\mathscr{C}_0')} = \chi^{(1),-}.$$

Now assume that $n \geq 4$. Let $W' = W_{n-2}' \times H_2$ be as in Lemma 8.3. As already noted in the above proof, the intersection $\mathscr{C}_0' \cap W'$ is just the conjugacy class of W' containing $\sigma_{0,n/2}$. Hence, we are in the setting of Lemma 6.8, and so

$$\big\langle \rho_{\mathscr{C}_0'}, \chi^{\alpha,-}\big\rangle_{W_n'} \leq \Big\langle \mathrm{Ind}_{W'}^{W_n'}(\rho_{\mathscr{C}_0' \cap W'}), \chi^{\alpha,-}\Big\rangle_{W_n'} \qquad \text{for all } \alpha \vdash n/2.$$

It will now be sufficient to show that the scalar product on the right hand side is zero for all $\alpha \vdash n/2$. Now, since $\sigma_{0,n/2} = \sigma_{0,(n-2)/2} \times s_{n-1} \in W_{n-2}' \times H_2$, we can apply induction and obtain

$$\rho_{\mathscr{C}_0' \cap W'} = \Big(\sum_{\alpha' \vdash n/2} \chi^{\alpha',+1}\Big) \boxtimes \varepsilon_1.$$

Then Lemma 8.3 implies that

$$\left\langle \mathrm{Ind}_{W'}^{W'_n}\big(\rho_{\mathscr{C}'_0\cap W'}\big), \chi^{\alpha,-1}\right\rangle_{W'_n} = 0 \qquad \text{for all } \alpha \vdash n/2,$$

as required. □

Remark 8.5. Let $\mathscr{C}$ be any conjugacy classes of involutions in W_n. We can associate with $\mathscr{C}$ a character $\tilde{\rho}_{\mathscr{C}}$ of W_n, as follows. If $\mathscr{C}$ is contained in W'_n, let $\rho_{\mathscr{C}}$ be the character of W'_n as defined in Definition 5.6. Then $\tilde{\rho}_{\mathscr{C}}$ will be the canonical extension described in [GeMa, §2]. If $\mathscr{C}$ is contained in the coset $W'_n t$, then we consider a similar extension of the "twisted" character defined in [Ko, 4.2]. Then one can show that

$$\big\langle \tilde{\rho}_{\mathscr{C}}, \mathrm{Ind}_{W'_n}^{W_n}([C])\big\rangle_{W_n} = |\mathscr{C} \cap (C \cup tC)| \qquad \text{for any left cell } C \subseteq W'_n.$$

The proof of this equality, although quite similar to that of Theorem 7.3, requires a number of preparations concerning "twisted" involutions with respect to the non-trivial graph automorphism of W'_n. Furthermore, the sets $C \cup tC$ can actually be interpreted as left cells for W_n, but with respect to the non-constant weight function with value 0 on t and value 1 on all s_i; the characters of the corresponding left cell modules of W_n are given by the induced characters on the left hand side. The whole argument is worked out in [Ge7, §5].

Corollary 8.6. *Let $W = W'_n$ be of type D_n, as above. Let $\mathfrak{C}$ be a two-sided cell of W'_n and $\chi_0 \in \mathrm{Irr}_{\mathfrak{C}}(W'_n)$ be the unique special character. Let $\mathscr{C}'$ be a conjugacy class of involutions in W'_n. Then*

$$\langle \rho_{\mathscr{C}'}, [C]\rangle_{W'_n} = \langle \rho_{\mathscr{C}'}, \chi_0\rangle_{W'_n} = |\mathscr{C}' \cap C| \qquad \textit{for any left cell } C \subseteq \mathfrak{C}.$$

Thus, Kottwitz's Conjecture 5.7 *holds for W'_n. In particular, if n is even and $\mathscr{C}'_0$ denotes the conjugacy class of the element $\sigma_{0,n/2} = s_1 s_3 \cdots s_{n-1}$, then*

$$\langle \rho_{\mathscr{C}'_0}, [C]\rangle_{W'_n} = |\mathscr{C}'_0 \cap C| = 1 \qquad \textit{for any left cell } C \subseteq \mathfrak{C}^+_\alpha \textit{ and any } \alpha \vdash n/2,$$

where $\mathfrak{C}^+_\alpha$ is the smooth two-sided cell as in Example 8.2.

Proof. The equality $\langle \rho_{\mathscr{C}'}, [C]\rangle_{W'_n} = \langle \rho_{\mathscr{C}'}, \chi_0\rangle_{W'_n}$ is shown as in the proof of Theorem 7.3, using Example 8.1(a), (b).

To prove Kottwitz's Conjecture, let us first deal with the case where $\mathscr{C}'$ is a conjugacy class of involutions in W'_n such that $\theta(\mathscr{C}') = \mathscr{C}'$. Then $\mathscr{C}'$ is a conjugacy class in W_n and we can use the identity in Remark 8.5. By Frobenius reciprocity we obtain:

(a) $$\langle \rho_{\mathscr{C}'}, [C]\rangle_{W'_n} = |\mathscr{C}' \cap C| \qquad \text{for any left cell } C \subseteq W'_n.$$

It now remains to deal with the case where n is even and $\mathscr{C}'$ is such that $\theta(\mathscr{C}') \neq \mathscr{C}'$. Let $\mathscr{C}' = \mathscr{C}'_0$ be the conjugacy class in Example 8.1. First we will show that

(b) $$\langle \rho_{\mathscr{C}'_0}, [C]\rangle_{W'_n} \leq |\mathscr{C}'_0 \cap C| \qquad \text{for any left cell } C \subseteq W'_n.$$

Indeed, let C be a left cell in W'_n. If $\langle \rho_{\mathscr{C}'_0}, [C]\rangle_{W'_n} = 0$, then (a) is obvious. Now assume that $\langle \rho_{\mathscr{C}'_0}, [C]\rangle_{W'_n} \neq 0$. Then, by Proposition 8.4, there exists some $\alpha \vdash n/2$ such that $\langle \chi^{\alpha,+1}, [C]\rangle_{W'_n} \neq 0$. So, using the notation in Example 8.2, we have $C \subseteq \mathfrak{C}^+_\alpha$ and $\mathscr{C}'_0 \cap \mathfrak{C}^+_\alpha \neq \varnothing$. Since $\mathfrak{C}^+_\alpha$ is smooth, we have $[C] = \chi^{\alpha,+1}$ (see Lemma 3.7) and $\mathscr{C}'_0 \cap C \neq \varnothing$ (see Corollary 4.3). Consequently, we see that (b) holds. Once

this is established, it actually follows that we must have equality in (b). Indeed, let $W = \coprod_{1\le i\le m} C_i$ be the partition into left cells. By (b), we have

$$\Big\langle \rho_{\mathscr{C}_0'}, \sum_{1\le i\le m} [C_i] \Big\rangle_{W_n'} = \sum_{1\le i\le m} \langle \rho_{\mathscr{C}_0'}, [C_i]\rangle_{W_n'} \le \sum_{1l\le i\le m} |\mathscr{C}_0' \cap C_i| = |\mathscr{C}_0'|.$$

But $\sum_{1\le i\le m}[C_i]$ is the character of the regular representation of W_n', and so the left hand side also equals $|\mathscr{C}_0'| = \rho_{\mathscr{C}_0'}(1)$. So all the inequalities in (b) must be equalities, as required. The argument for $\theta(\mathscr{C}_0')$ is completely analogous. Note that, by Example 8.1(b), the character $\rho_{\theta(\mathscr{C}_0')}$ is the conjugate of $\rho_{\mathscr{C}_0'}$ under θ. $\square$

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