ON THE $W$-ACTION ON $B$-SHEETS
IN POSITIVE CHARACTERISTIC

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Abstract. Let $G$ be a connected reductive group defined over an algebraically closed base field of characteristic $p \geq 0$, let $B \subseteq G$ be a Borel subgroup, and let $X$ be a $G$-variety. We denote the (finite) set of closed $B$-invariant irreducible subvarieties of $X$ that are of maximal complexity by $\mathcal{B}_0(X)$. The first named author has shown that for $p = 0$ there is a natural action of the Weyl group $W$ on $\mathcal{B}_0(X)$ and conjectured that the same construction yields a $W$-action whenever $p \neq 2$. In the present paper, we prove this conjecture.

1. Introduction

Let $G$ be a connected reductive group defined over an algebraically closed base field $k$ with Borel subgroup $B \subseteq G$. For any $G$-variety $X$ let $\mathcal{B}(X)$ be the set of all $B$-stable irreducible closed subvarieties of $X$. The complexity $c(Y)$ of $Y \in \mathcal{B}(X)$ is the codimension of a generic $B$-orbit in $Y$ or, equivalently, the transcendence degree of $k(Y)^B$. It is a result of Vinberg [Vi86] that the complexity takes its maximal value for $Y = X$. Of particular interest is therefore the subset $\mathcal{B}_0(X) := \{ Y \in \mathcal{B}(X) \mid c(Y) = c(X) \}$.

This set contains $X$ and is finite since it consists of the closures of all $B$-sheets with a maximal number of parameters (see [Kn95, Proposition 4.1]). The most important case is that of a spherical variety (i.e. $c(X) = 0$) when $\mathcal{B}_0(X) = \mathcal{B}(X)$ is just the set of all $B$-orbit closures.

Let $W$ be the Weyl group of $G$. In [Kn95], an action of $W$ on $\mathcal{B}_0(X)$ was constructed whenever the base field has characteristic zero. On the other side, in the same paper an example was given showing that the construction does not work in characteristic 2. In any other characteristic, the situation was unclear so far. The purpose of this paper is to close this gap by showing that the method of [Kn95] does indeed define a $W$-action on $\mathcal{B}_0(X)$ in every characteristic $\neq 2$.

It was already indicated [Kn95] that the problem can be reduced to the following special case: the characteristic $p$ of $k$ is $> 2$, the group $G$ is semisimple of rank 2, and the variety $X$ is of the form $X = G/H$ where $H$ is a connected non-spherical subgroup of $G$. Moreover, we may replace $\mathcal{B}_0(X)$ by a certain subset $\mathcal{B}_{00}(X)$ (see §2 for its definition).

It is then enough to consider only those $H$ for which $\mathcal{B}_{00}(G/H)$ consists of more than one element. Remarkably, under such assumptions the proof can be

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completed by showing that there exist a spherical subgroup $K \subseteq G$ and a bijection $\mathfrak{B}_0(G/H) \to \mathfrak{B}_0(G/K)$ that is compatible with the operation of the simple reflections of $W$.

While several subgroups require case-by-case considerations, others can be treated with general arguments, e.g., solvable subgroups.

Notation. All varieties are defined over an algebraically closed field $k$ of characteristic $p \geq 0$. We denote by $G$ a connected reductive group, we fix a Borel subgroup $B \subseteq G$, whose unipotent radical is denoted by $U$, and a maximal torus $T \subseteq B$. The opposite Borel subgroup with respect to $T$, and its unipotent radical, are denoted by $B^-$ and $U^-$, respectively.

Denote by $R$ the set of roots with respect to $T$, by $R^+$ the set of positive roots corresponding to $B$, and by $S \subseteq R$ the set of simple roots. If $G$ is simple, then the simple roots $\alpha_1, \alpha_2, \ldots$ and the fundamental dominant weights $\omega_1, \omega_2, \ldots$ will be numbered as in [Bou, Planches I–IX]. The 1-dimensional unipotent subgroup of $G$ associated to a root $\gamma$ is denoted by $U_\gamma$, and we choose once and for all an isomorphism $u_\gamma : G_\alpha \to U_\gamma$. The Weyl group of $G$ is denoted by $W$, its longest element is denoted by $w_0$, and the simple reflection associated to $\alpha \in S$ is denoted by $s_\alpha$.

If $\omega$ is a dominant weight, then $V_G(\omega)$ denotes the irreducible $G$-module of highest weight $\omega$. If no confusion arises, we simply write $V(\omega)$.

If $\alpha$ is a simple root of $G$, then $P_\alpha$ denotes the minimal parabolic subgroup of $G$ which is generated by $B$ and $U_{-\alpha}$, and $\pi_\alpha : G/B \to G/P_\alpha$ the natural map $gB \mapsto gP_\alpha$. If $g \in G$ and $H \subseteq G$, we use the notation $gHg^{-1}$. For any algebraic group $H$, we denote by $H'$ its commutator subgroup, by $H^r$ (resp. $H^u$) its radical (resp. unipotent radical), and by $\chi(H)$ its group of characters, i.e., the set of all algebraic group homomorphisms $H \to \mathbb{G}_m$.

2. The action of the Weyl group

We recall some definitions and facts from [Kn95]. Let $X$ be an algebraic variety equipped with a $G$-action. To avoid confusion, a $B$-stable irreducible closed subvariety $Z$ of $X$ is denoted sometimes by $(Z)$ if we are referring to it as an element of $\mathfrak{B}(X)$.

We define the character group $\chi(Z)$ of $Z$ as the group of $B$-eigenvectors of $B$-eigenvectors in $k(Z)$. The rank $r(Z)$ of $Z$ is the rank of the free abelian group $\chi(Z)$.

We also define the following subset of $\mathfrak{B}_0(X)$:

$$\mathfrak{B}_{00}(X) = \{ (Z) \in \mathfrak{B}(X) \mid c(Z) = c(X), r(Z) = r(X) \}.$$

If $X = G/H$ is homogeneous, then there is a canonical bijection between $\mathfrak{B}(X)$ and the set of $H$-stable $H$-irreducible closed subsets of $G/B$. Here “$H$-irreducible” means that $H$ acts transitively on the set of irreducible components. Throughout the paper we will sometimes implicitly make use of this bijection.

Let $\bar{W}$ be the group defined by generators $s_\alpha, \alpha \in S$ and relations $s_\alpha^2 = e, \alpha \in S$. In [Kn95] an action of $\bar{W}$ of $\mathfrak{B}_0(X)$ has been defined as follows. Let $\alpha$ be a simple root, and recall that $P_\alpha \supset B$ is the minimal parabolic subgroup of $G$ corresponding to $\alpha$. Then $Z \mapsto P_\alpha Z$ is an idempotent selfmap of $\mathfrak{B}_0(X)$. Its image $\mathfrak{B}_0^\alpha(X)$ consists of those $Z \in \mathfrak{B}_0(X)$ which are $P_\alpha$-stable. Thus, for $Y \in \mathfrak{B}_0^\alpha(X)$ the fibers

$$\mathfrak{B}_0(Y, P_\alpha) = \{ Z \in \mathfrak{B}_0(X) \mid P_\alpha Z = Y \}$$
form a partition of $\mathcal{B}_0(X)$. Now, the element $s_\alpha$ acts on each block $\mathcal{B}_0(Y, P_\alpha)$ as an involution according to the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>$\mathcal{B}<em>0(Y, P</em>\alpha)$</th>
<th>$s_\alpha$–action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(G)$</td>
<td>${Y}$</td>
<td>$s_\alpha \cdot (Y) = (Y)$</td>
</tr>
<tr>
<td>$(U)$</td>
<td>${Y, Z}$</td>
<td>$r(Z) = r(Y) \quad s_\alpha \cdot (Y) = (Z), \quad s_\alpha \cdot (Z) = (Y)$</td>
</tr>
<tr>
<td>$(N)$</td>
<td>${Y, Z}$</td>
<td>$r(Z) &lt; r(Y) \quad s_\alpha \cdot (Y) = (Y), \quad s_\alpha \cdot (Z) = (Z)$</td>
</tr>
<tr>
<td>$(T)$</td>
<td>${Y, Z_0, Z_\infty}$</td>
<td>$r(Z_0) = r(Z_\infty) &lt; r(Y) \quad s_\alpha \cdot (Y) = (Y), \quad s_\alpha \cdot (Z_0) = (Z_\infty)$</td>
</tr>
</tbody>
</table>

No other cases can occur (see [Kn95 §4]). This definition is based on a construction of Lusztig and Vogan in the case of symmetric spaces (see [LV83 §3]). Under the hypothesis that $B$ has a dense orbit on $G/H$ (in which case $G/H$ is called a spherical homogeneous space, and $H$ a spherical subgroup of $G$), the link is explained in more detail in [Kn95 §1 and §5].

Let us further analyze the four above cases in case that $X = G/H$ is a homogeneous variety.

Suppose that $Y \in \mathcal{B}_0(G/H)$ is $P_\alpha$-stable. Then, considered as an $H$-stable subset of $G/B$, it satisfies $\pi^{-1}(\pi(Y)) = Y$ where $\pi = \pi_\alpha : G/B \rightarrow G/P_\alpha$.

The fiber $\pi^{-1}(x)$ over any $x \in \pi(Y)$ is isomorphic to $\mathbb{P}^1$ and equipped with a natural transitive action of $G_x$; we fix the isomorphism $\pi^{-1}(x) \cong \mathbb{P}^1$ and the corresponding homomorphism $\Phi : G_x \rightarrow \text{PGL}(2) = \text{Aut} \mathbb{P}^1$. The fiber $\pi^{-1}(x)$ intersects each $Z \in \mathcal{B}_0(Y, P_\alpha)$ in an $H_x$-stable subset. If $x$ is general in $\pi(Y)$, then the four cases of the above table correspond resp. to the following:

- $(G)$ $\Phi(H_x)$ is PGL(2) or finite,
- $(U)$ the unipotent radical of $\Phi(H_x)$ is a maximal unipotent subgroup of PGL(2),
- $(N)$ $\Phi(H_x)$ has trivial unipotent radical and $|\mathcal{B}_0(Y, P_\alpha)| = 2$,
- $(T)$ $\Phi(H_x)$ is a maximal torus of PGL(2).

In [Kn95] it is shown that the $\tilde{W}$-action on $\mathcal{B}_0(G/H)$ descends to an action of the Weyl group $W$ of $G$ under several different assumptions; see [Kn95] Theorem 4.2. In particular, this is true if $H$ is a spherical subgroup of $G$ and $\text{char } k \neq 2$.

We come to our main result.

2.1. **Theorem.** Let $\text{char } k > 2$ and let $G$ be semisimple of rank 2. For any connected subgroup $H \subseteq G$, the $\tilde{W}$-action defined above induces a $W$-action on $\mathcal{B}_{00}(G/H)$.

Sections [3][5] are devoted to the proof. Precisely, the theorem follows from Corollaries [3.3][5.2] and [5.6].

Thanks to [Kn95 §7], the above theorem implies the following.

2.2. **Corollary.** Let $\text{char } k \neq 2$. Then for any $G$-variety $X$ the $\tilde{W}$-action defined above induces a $W$-action on $\mathcal{B}_0(X)$.

Before going into the details of the proof of Theorem 2.1, we report two general results on $\mathcal{B}_{00}(G/H)$.

2.3. **Proposition.** Let $X = G/H$ be a homogeneous $G$-variety and $Z \in \mathcal{B}_{00}(X)$ with $Z \neq X$. Then there is $\alpha \in S$ such that $\dim s_\alpha \cdot (Z) = \dim Z + 1$. In particular:

- The $\tilde{W}$-action on $\mathcal{B}_{00}(X)$ is transitive.
- If $|\mathcal{B}_{00}(X)| > 1$, then there is $Z \in \mathcal{B}_{00}(X)$ with $\text{codim}_X Z = 1$. 
Proof. Suppose \( \dim s_\alpha \cdot (Z) \leq \dim Z \). Then the definition of the \( s_\alpha \)-action and the fact that \( Z \) is of maximal rank implies \( P_\alpha Z = Z \). Since \( Z \neq X \) and since \( X \) is homogeneous there is \( \alpha \in S \) with \( P_\alpha Z \neq Z \). Then \( \dim s_\alpha \cdot (Z) = \dim Z + 1 \) since \( s_\alpha \) increases dimension by at most 1.

This implies that the action is transitive since any \( Z \) can be moved in finitely many steps to \( (X) \). Finally, starting from any \( Z \neq X \) the next to the last step will be of codimension one.

2.4. Lemma. Let \( H \subseteq K \subseteq G \) be subgroups such that \( H \) is normal in \( K \) and the quotient \( K/H \) is diagonalizable. Let \( \pi: G/H \to G/K \) be the standard projection. Then the map \( Y \mapsto \pi^{-1}(Y) \) is a \( \tilde{W} \)-equivariant bijection \( \mathcal{B}_{00}(G/K) \to \mathcal{B}_{00}(G/H) \).

Proof. We show by induction on \( \text{codim}_{G/K} Z \) that \( \pi^{-1}(Z) \in \mathcal{B}_{00}(G/H) \) for all \( Z \in \mathcal{B}_{00}(G/K) \). This is true if \( Z = G/K \), hence we can suppose \( Z \subsetneq G/K \).

Let \( Y \in \mathcal{B}_{00}(G/K) \) and \( \alpha \in S \) be such that \( \dim Y = \dim Z + 1 \) and \( s_\alpha \cdot (Z) = (Y) \); notice that such elements exist thanks to Proposition 2.3. In particular, \( Z \) is the unique element of \( \mathcal{B}_{00}(G/K) \) such that \( P_\alpha Z = Y \).

By induction \( Y' = \pi^{-1}(Y) \) is in \( \mathcal{B}_{00}(G/H) \). Then for a general \( y \in Y \) the image of the stabilizer \( (P_\alpha)_y \) into \( \text{Aut}(P_\alpha/B) = \text{Aut}(\mathbb{P}^1) = \text{PGL}(2) \) is a proper subgroup that contains a maximal unipotent subgroup.

Let \( y' = gH \) be a general point of \( Y' \): its stabilizer \( (P_\alpha)_{y'} \) is equal to \( P_\alpha \cap gH \), and we may compare the latter with \( (P_\alpha)_y = P_\alpha \cap gK \) where \( gK = \pi(y') \). We have that \( P_\alpha \cap gH \) is normal in \( P_\alpha \cap gK \), and the quotient is a diagonalizable group. Therefore the image of \( (P_\alpha)_{y'} \) into \( \text{PGL}(2) \) is again proper and contains a maximal unipotent subgroup. Then the \( P_\alpha \)-orbit of \( y' \) in this case is the union of two \( B \)-orbits, and there exists a unique \( Z' \in \mathcal{B}(G/H) \) such that \( P_\alpha Z' = Y' \). It follows that \( P_\alpha \pi^{-1}(Z) = Y' \), so \( Z' = \pi^{-1}(Z) \), the latter lies in \( \mathcal{B}_{00}(G/H) \), and \( s_\alpha \cdot (Y') = (Z') \).

Denote with \( f: \mathcal{B}_{00}(G/K) \to \mathcal{B}_{00}(G/H) \) the map \( Y \mapsto \pi^{-1}(Y) \). We have shown that \( f \) is injective, and that whenever \( s_\alpha \) exchanges two elements \( Y, Z \in \mathcal{B}_{00}(G/K) \) then it exchanges \( f(Y) \) and \( f(Z) \).

To show that \( f \) is \( \tilde{W} \)-equivariant, which also implies the surjectivity of \( f \) since both sets are \( \tilde{W} \)-orbits, it remains only to consider \( Y \in \mathcal{B}_{00}(G/K) \) and \( \alpha \in S \) such that \( Y \) is \( P_\alpha \)-stable and \( s_\alpha \cdot (Y) = (Y) \), and to check that \( s_\alpha \cdot (f(Y)) = (f(Y)) \). In this case, the image of \( P_\alpha \cap gK \) in \( \text{PGL}(2) \) is either the whole \( \text{PGL}(2) \), a maximal torus, the normalizer of a maximal torus, or a finite group. Since \( P_\alpha \cap gH \) is a normal subgroup of \( P_\alpha \cap gK \) such that the quotient is diagonalizable, we conclude that its image in \( \text{PGL}(2) \) also belongs to one of the above four possible types of subgroups. Hence \( s_\alpha \cdot (f(Y)) = (f(Y)) \). \( \square \)

3. Solvable subgroups

We assume from now on that \( p := \text{char } k > 2 \).

3.1. Lemma. Suppose that \( G \) is semisimple of rank 2, and let \( \alpha, \beta \) be the two simple roots. Let \( N \) be the normal subgroup of \( U \) generated by \( U_\alpha \). Then \( N = P^u_\beta \).

Proof. Clearly, \( N \subseteq P^u_\beta \). To show the reverse inclusion we recall that

\[
(1) \quad u_\beta(y)^{-1}u_\alpha(x)u_\beta(y) = \prod_{\gamma \in R^+ \setminus \{\beta\}} u_\gamma(f_\gamma(x, y)) \in N
\]
where each $f_\gamma(x, y)$ is a non-constant polynomial (see [SGA3] Exposé XXIII: Proposition 3.1.2(iii), Proposition 3.2.1(iii), Proposition 3.3.1(iii), Proposition 3.4.1(iii)) for groups of type $A_1 \times A_1, A_2, B_2$ and $G_2$, respectively). Since $N$ is normalized by $T$ we have $N = \prod_{\gamma \in R^+} U_\gamma$ where $R^+ \subseteq R^+$ is a subset. Thus, all factors of the right hand side of (1) are also in $N$. This shows $R^+ \setminus \{\beta\} \subseteq R'$ and therefore $P_\beta^u \subseteq N$, as claimed.

3.2. Proposition. Let $G$ be semisimple of rank 2 and $H \subseteq G$ be a connected solvable subgroup. Then $|S_0(G/H)| = 1$, or $H$ contains the unipotent radical of a parabolic subgroup of $G$.

Proof. Suppose $|S_0(G/H)| > 1$. Then there is a simple root $\alpha$ and $Z \in S_0(G/H)$ with $Z \neq G/H$ and $s_\alpha(Z) = (G/H)$ (Proposition 2.3). Let $Z' \subseteq G/B$ correspond to $Z$. The definition of the $s_\alpha$-action implies that $\pi_\alpha(Z') = G/P_\alpha$ and that the generic isotropy group of $H$ in $Z'$, hence in $G/P_\alpha$, contains a subgroup isomorphic to $G_\alpha$. Without loss of generality we may assume that $H \subseteq B^-$. The $U^+$-orbit of $eGP_\alpha$ is dense in $G/P_\alpha$ and, for $u \in U^-$, the isotropy group of $x = uP_\alpha$ in $B^\sim$ is $B^\sim_x = u(TU_{-\alpha}) = uT^uU_{-\alpha}$. Because of $H_x \subseteq B^\sim_x$ we conclude that $uU_{-\alpha} \subseteq H$ for $u \in U^-$ general and therefore, by continuity, for all $u \in U^-$. Then Lemma 3.1 implies $H \supseteq P_{-\beta}$ where $\beta$ is the simple root different from $\alpha$. □

3.3. Corollary. Theorem 2.1 holds for all connected solvable subgroups $H \subseteq G$.

Proof. If $|S_0(G/H)| = 1$ there is nothing to prove. Otherwise, without loss of generality, the unipotent radical $H^u$ is either $U$ or $P_\alpha^u$ for some simple root $\alpha$ (Proposition 3.2). The claim is also known to be true if $H$ is spherical. This leaves to check only $H = TP_\alpha^u$ where either $\overline{T} = \{e\}$ or $\overline{T} = (\ker \alpha)^0 \subseteq T$. In either case Lemma 2.4 applies to $H$ and $K = TP_\alpha^u$, since $TP_\alpha^u$ is spherical, the corollary follows. □

4. Reductive subgroups

In this section, we consider the case where $H$ is a reductive subgroups of $G$. Clearly, by the preceding section we may assume that $H$ is not a torus. So, its semisimple rank is one or two.

4.1. Proposition. Let $G$ be a semisimple group of rank 2 and $H \subseteq G$ a connected reductive subgroup of semisimple rank 2. Then $H$ is spherical.

Proof. The root system of $H$ is given by a subroot system of rank 2 of the root system of $G$. These are easily determined: in case $G = A_1 \times A_1$ or $G = A_2$ then only $H = G$ is possible. If $G = B_2$, then there is additionally $H = A_1 \times A_1$. Finally, if $G = G_2$ there is an extra-complication if $p = 3$. In general, $H = A_1 \times A_1$ and $H = A_2$ is possible, the latter corresponding to the set of long roots. If $p = 3$, then there is another subgroup of type $A_2$ corresponding to the short roots. But that is mapped by a special isogeny (see [BT73, §3.3]) to the former. We conclude that $H$ is a spherical subgroup in every case thanks to [Br98, Theorem 4.3]. □

We are left with the case where $H$ has semisimple rank one.

4.2. Lemma. Let $G$ be a semisimple group of rank 2 and $H \subseteq G$ a connected reductive subgroup with $\text{rk}_{ss} H = 1$ and $|S_0(G/H)| > 1$. Let $U_H \cong G_\alpha$ be a maximal connected unipotent subgroup of $H$. Then there is a simple root $\alpha \in S$
such that the fixed point set \((G/P_\alpha)^{U_H}\) has a component of codimension at most one in \(G/P_\alpha\).

**Proof.** Arguing as in the proof of Proposition 3.2, there is a simple root \(\alpha\) such that the isotropy group \(H_\gamma\) is not reductive for general \(y \in G/P_\alpha\). Then \(\text{rk}_\text{ss} H = 1\) implies that \(H_\gamma\) contains a conjugate of \(U_H\) for general \(y \in G/P_\alpha\) or, in other words, a general \(H\)-orbit in \(G/P_\alpha\) contains a \(U_H\)-fixed point. The normalizer of \(U_H\) in \(H\) is of codimension one. Thus, the \(U_H\)-fixed points are of codimension at most one in any general orbit \(H_\gamma\). We conclude that \((G/P_\alpha)^{U_H}\) has a component \(C\) which is of codimension at most one in \(G/P_\alpha\). \(\Box\)

Now we analyze the situation of the preceding lemma further.

4.3. **Lemma.** Let \(G, H, U_H\) be as in Lemma 4.2. If \(G\) is of type \(B_2\) or \(G_2\), then \(H\) is conjugate to either \(L\) or \(L'\) where \(L\) is a Levi subgroup of a parabolic subgroup of \(G\).

**Proof.** By replacing \(U_H\) by a conjugate, we may assume that \(U_H \subseteq B^-\). Let \(C\) be a component of \((G/P_\alpha)^{U_H}\) of codimension one. Then there are two possibilities: either \(C\) meets the open \(B^-\)-orbit in \(G/P_\alpha\) or not. We claim that in both cases \(U_H\) is conjugate in \(G\) to a root subgroup \(U_\gamma\) for some simple root \(\gamma\). Let \(y \in C\) be general.

In the first case, \(B^-_y\) is conjugate to \(TU_{-\alpha}\). Because of \(U_H \subseteq B^-\) we conclude that \(U_H\) is \(G\)-conjugate to \(U_\alpha\), which implies the claim with \(\gamma = \alpha\).

In the second case, \(C\) equals the Bruhat cell of codimension one. In that case \(B^-_y\) is \(B^-\)-conjugate to \(TU_0\) with \(U_0 = U_{-\beta}U_{-\gamma}\) where \(\beta\) is the simple root different from \(\alpha\) and \(\gamma = s_\beta(\alpha)\). But now \(U_H\) is contained in every \(B^-\)-conjugate of \(TU_0\), i.e., in the largest connected subgroup \(U_1\) of \(U_0\) such that \(U_1\) is normal in \(B^-\).

Since \(U_1\) is normalized by \(T\), it is either trivial, \(U_{-\beta}, U_{-\gamma}\), or \(U_0\). It can’t be trivial since it contains \(U_H\). Moreover, a short case-by-case consideration shows that if \(G = G_2\), then none of the other three groups are normal in \(B^-\).

Then \(G\) is of type \(B_2\). Another short case-by-case consideration shows that \(U_1\) is either trivial (which is impossible since \(U_H \subseteq U_1\)) or \(U_{-\gamma}\) (where in this case \(\gamma\) is long); proving the claim.

Now we may assume that \(U_H = U_\gamma\). Since both a maximal torus \(S\) of \(H\) and the maximal torus \(T\) of \(G\) are contained in the normalizer of \(U_\gamma\) in \(G\), we can replace \(H\) by a conjugate such that \(B \cap H\) contains both \(S\) and \(U_\gamma\). The intersection \(B \cap H\) is then a Borel subgroup of \(H\). Let \(L\) be the Levi subgroup corresponding to \(\gamma\). Then \(B \cap H \subseteq L \cap H\) is parabolic in \(H\). Thus \(H/L \cap H\) is a projective subvariety of the affine variety \(G/L\). Thus \(H \subseteq L\); proving the lemma. \(\Box\)

4.4. **Corollary.** Let \(G\) be a semisimple group of rank 2 and \(H \subset G\) a connected reductive subgroup of semisimple rank 1. Then \(H\) is spherical, or \(|B_0(G/H)| = 1\), or \(H = L\) or \(L'\) where \(L\) is a Levi subgroup of a proper parabolic subgroup \(P\) of \(G\).

**Proof.** If the rank of \(H\) is 2, then \(H\) is a Levi subgroup of a parabolic subgroup of \(G\).

If the rank of \(H\) is 1, then \(H\) is of type \(A_1\). If \(G = A_1 \times A_1\), then \(H\) is either one of the factors, in which case \(|B_0(G/H)| = 1\), or \(H\) is embedded diagonally, possibly by a power of the Frobenius morphism. Thus, there is an inseparable isogeny \(\phi\) of \(G\) such that \(\phi(H)\) is the diagonal subgroup of \(G\). This shows that \(H\) is spherical.
If $G = A_2$, then $H$ is embedded into $G$ via a 3-dimensional representation $V$. Because $p \neq 2$, there are no non-trivial extensions of a 2-dimensional representation $k^2$ and the trivial representation $k$ (see e.g. [SI10 Proposition 2.6]). Thus, $V = k^2 \oplus k$, or $V = k^3$ is irreducible. In the first case $H = \text{SL}(2) \times 1 \subset G = \text{SL}(3)$, in the second case $H = \text{SO}(3) \subset G = \text{SL}(3)$. In both cases $H$ is spherical.

It remains to check the case $G = B_2$ or $G = G_2$ and $H$ is of semisimple rank one. In this case the corollary follows from Lemma 4.3.

5. LEVI SUBGROUPS

In this section we discuss subgroups $H$ of $G$ that are Levi subgroups of some parabolic, up to a $k^*$-factor.

5.1. Proposition. Let $G$ be simple of rank 2, $p \geq 3$, let $P \supset B$ be a proper parabolic subgroup of $G$, and let $H$ be a Levi subgroup of $P$. If $G$ has type $A_1 \times A_1$, $A_2$ or $B_2$, then $H$ is spherical; if $G$ has type $G_2$, then $H$ is spherical or $|B_{00}(G/H)| = 1$.

Proof. We may assume that $G$ is simply connected. If it has type $A_1 \times A_1$, then $H$ is spherical because a maximal torus of $\text{SL}(2)$ is spherical in $\text{SL}(2)$ (see [Br98, Theorem 4.3]).

If $G$ has type $A_2$ or $B_2$, then $H$ itself appears in [Br98 Table 1], hence we apply again [Br98 Theorem 4.3].

Suppose that $G$ has type $G_2$, and consider the case $P = P_{\alpha_1}$. Assume also $p \geq 5$; indeed, if $p = 3$, then $P_{\alpha_1}$ is sent onto $P_{\alpha_2}$ by a special isogeny of $G$, so we refer to our later discussion of the case $P = P_{\alpha_2}$.

Thanks to the assumption $p \geq 5$ the module $V = V_G(\omega_1)$ is also the Weyl module of $G$ associated to $\omega_1$ (see e.g. [Pr88 Theorem 1]), and $G/P_{\alpha_1}$ is a subvariety of $\mathbb{P}(V)$. Let $U_H$ be a maximal unipotent subgroup of $H$. The weights of the $G$-module $V$ imply that the latter is the sum of three irreducible $H$-modules, therefore $(G/P_{\alpha_1})^{U_H}$ has components of dimension at most 2.

If $|B_{00}(G/H)| > 1$, then Lemma 4.2 applies: since $\dim G/B = 6$, then $(G/P_{\alpha_1})^{U_H}$ or $(G/P_{\alpha_2})^{U_H}$ has a component $C$ of dimension 4. The first case is excluded, so consider the second case. The unipotent group $U_H$ has at least one fixed point in $\pi_{\alpha_2}^{-1}(c)$ for all $c \in C$. It follows that $(G/P_{\alpha_1})^{U_H}$ has a component of dimension at least 3, which is also impossible. Hence $|B_{00}(G/H)| = 1$.

It remains the case where $P = P_{\alpha_2}$ and $p \geq 3$. Let $U_H$ be a maximal connected unipotent subgroup of $H$. We claim that the set of $U_H$-fixed points on $G/P_{\alpha_2}$ has components of dimension at most 3 for both $i = 1, 2$: from Lemma 4.2, we obtain $|B_{00}(G/H)| = 1$ as above.

To prove the claim, we use the commutation relations of [SGA3 Exposé XXIII: Proposition 3.4.1(iii)] as in the proof of Lemma 3.1. To simplify notation, write $\alpha = \alpha_1$ and $\beta = \alpha_2$.

Denoting $Q = P_{\alpha}$, we have

$$G/Q = (Q^u w_0 Q/Q) \cup (Q^u s_\beta w_0 Q/Q) \cup (\text{subvarieties of dimension } \leq 3).$$

Let us compute the $U_H$-fixed points on $Q^u w_0 Q/Q$. We may assume that $U_H$ is the set of elements of the form $u = u_\beta(x)$ for $x \in k$.

If $v \in Q^u$, then $vw_0 Q \in Q^u w_0 Q/Q$ is fixed under the action of $u$ if and only if $v^{-1} uv \in w_0 Q$. Now write $v^{-1}$ as a product:

$$v^{-1} = u_{3\alpha+\beta}(y_1) u_{2\alpha+\beta}(y_2) u_{\alpha+\beta}(y_3) u_{3\alpha+2\beta}(y_4) u_{\beta}(y_5).$$
Then:

\[ v^{-1}u_\beta(x)v = u_{3\alpha + 2\beta}(-xy_1)u_\beta(x). \]

This belongs to \( u_0P \) only if \( x = 0 \), therefore there are no \( U_H \)-fixed points on \( Q^n w_0/P \).

Let us do the same with \( Q^n s_\beta w_0/Q \), and call \( w_1 = s_\beta w_0 \). This set is an affine space of dimension 4, and we may take \( y_1, \ldots, y_4 \) as its coordinates.

A point \( vw_1 \) is fixed by \( u \) if and only if \( v^{-1}wv \in w_1 \). This time, \( u_{3\alpha + 2\beta}(-xy_1)u_\beta(x) \) lies in \( w_1 \) for all \( x \) and only if \( y_1 = 0 \). It follows that the set of \( U_H \)-fixed points on \( Q^n s_\beta w_0/Q \) is irreducible of dimension 3.

Finally, we discuss the other parabolic \( P = P_\beta \) with the same procedure. Write

\[ G/P = (P^n w_0 P/P) \cup (P^n s_\alpha w_0 P/P) \cup (\text{subvarieties of dimension } \leq 3), \]

and consider \( v \in P^n \). The point \( vw_0 P \) is fixed by \( u \in U_H \) if and only if \( v^{-1}uv \in w_0 P \). If:

\[ v^{-1} = u_\alpha(y_1)u_{3\alpha + \beta}(y_2)u_{3\alpha + 2\beta}(y_3)u_{2\alpha + \beta}(y_4)u_{\alpha + \beta}(y_5) \]

and \( u = u_\beta(x) \), then:

\[ v^{-1}uv = (u_\alpha(-y_1)u_{3\alpha + \beta}(-y_2))u_\beta(x)(u_{3\alpha + \beta}(y_2)u_\alpha(y_1)) \]

\[ = u_\beta(x)u_{\alpha + \beta}(y_1)u_{2\alpha + \beta}(y_2 y_1^2)u_{3\alpha + \beta}(x y_1^2)u_{3\alpha + 3\beta}(y_1^3 - xy_2). \]

It belongs to \( w_0 P \) for all \( x \) if and only if \( y_1 = y_2 = 0 \), hence the set of \( U_H \)-fixed points on \( P^n w_0 P/P \) is 3-dimensional. On the other hand, for no \( v \in P^n \) we have \( v^{-1}uv \in s_\alpha w_0 P \) for all \( x \), therefore \( U_H \) has no fixed points on \( P^n s_\alpha w_0 P/P \), and the proof is complete.

\[ \square \]

5.2. Corollary. Let \( H \subseteq G \) be a connected subgroup. Suppose that \( L' \subseteq H \subseteq L \), where \( L \) is a Levi subgroup of some proper parabolic subgroup \( P \supseteq B \). Then Theorem 2.1 holds for \( H \).

Proof. Suppose that \( G \) has type \( A_1 \times A_1 \) and is simply connected, so \( G = SL(2) \times SL(2) \). Without loss of generality \( P = SL(2) \times B_{SL(2)} \), where \( B_{SL(2)} \) is a Borel subgroup of \( SL(2) \). It follows that \( H = SL(2) \times K \) where \( K \) is a subgroup of a maximal torus of \( B_{SL(2)} \). Therefore the reflection associated to one of the simple roots of \( G \) acts trivially on \( \mathfrak{b}_{00}(G/H) \), and Theorem 2.1 follows.

If \( G \) is simple, apply Lemma 2.4 to \( H \subseteq L \) and then Proposition 5.1 to \( L \): the corollary follows.

\[ \square \]

6. Other subgroups

In this section we finish the proof of Theorem 2.1 discussing the remaining connected subgroups \( H \) of \( G \).

The first result regards the representation theory of \( SL(2) \) and will be useful in subsequent proofs.

6.1. Lemma. Let \( V \) be a finite dimensional \( SL(2) \)-module, and \( R \subseteq V \) an \( SL(2) \)-stable additive subgroup. Suppose that one of the following two conditions is satisfied:

(1) The module \( V \) is non-trivial and simple.

(2) The characteristic \( p \) of \( k \) is \( \neq 2 \), the subgroup \( R \) is closed and connected, and \( V = V(0) \oplus V' \) where \( V' \) is non-trivial and simple.

Then \( R \) is a submodule of \( V \).
Proof. We may suppose that \( R \neq \{0\} \). Assume (1): then the union of the sets \( aR \) for all \( a \in k \) is a non-zero \( SL(2) \)-submodule of \( V \), therefore equal to \( V \). It follows that \( R \) contains a highest weight vector \( v \), therefore all its multiples and all linear combinations of elements of the form \( gv \) for \( g \in SL(2) \). Hence \( R = V \).

Now assume (2), and let \( R' \) be the projection of \( R \) on \( V' \) along \( V(0) \). Since it is an \( SL(2) \)-stable additive subgroup of \( V' \), as in the proof of the first part of the lemma we conclude that \( R' \) is either \( \{0\} \), or contains a highest weight vector \( v \in V' \).

In the first case \( R \) is either \( \{0\} \) or \( V(0) \), since it is closed and connected. In the second case, let \( r \in R \) project to \( v \): since \( p \neq 2 \) it is elementary to show that \( R \) also contains both projections of \( r \) in \( V(0) \) and \( V' \), and this completes the proof. \( \square \)

6.2. Lemma. Let \( G \) be semisimple of rank 2 and let \( p \geq 3 \). Then any connected subgroup \( H \) of \( G \) has a Levi subgroup. If \( H \) is contained in a parabolic subgroup \( P \) of \( G \), then any Levi subgroup of \( H \) is contained in a Levi subgroup of \( P \).

Proof. The proposition is true if \( H \) is solvable or very reductive, i.e., not contained in any proper parabolic subgroup of \( G \). Therefore we may assume that \( H \) is contained in a proper parabolic subgroup \( P \supset B \) but not in any Borel subgroup of \( G \). In this case \( H^u/P^u \) is not contained in any proper parabolic subgroup of \( P/P^u \), which implies \( H/H \cap P^u \cong H^u/P^u \) reductive, and hence \( H^u = H \cap P^u \).

Denote by \( L \supset T \) the standard Levi subgroup of \( P \); according to the decomposition \( P = L \ltimes P^u \) we define two projections \( \pi_\ell: P \rightarrow L \) and \( \pi_u: P \rightarrow P^u \). Notice that \( L \) is the quotient of \( SL(2) \times \mathbb{C}_m \) by a finite central subgroup scheme.

If \( H \) contains a maximal torus of \( G \), then it contains \( T \) up to conjugation by an element of \( P \); we may then suppose that \( H \supset T \). It follows that \( H \) contains \( \pi_\ell(H) \), because \( \pi_\ell(h) \in \{ t^{-1}h t \mid t \in T \} \) for any \( h \in H \). This implies that \( H = \pi_\ell(H) \ltimes (H \cap P^u) \), whence both statements of the proposition.

We are left with the case where \( H \) doesn’t contain any maximal torus of \( G \), hence \( \pi_\ell(H) \cong H/H^u \) is semisimple of rank 1. With this assumption, we show that the two statements of the proposition follow from the vanishing of \( H^1(L', R) \) and \( H^2(L', R) \) for certain subquotients \( R \) of \( P^u \).

Consider the lower central series \( P^u = P^u_0 \supseteq P^u_1 \supseteq P^u_2 \supseteq \ldots \) of \( P^u \), and the projection \( \pi_1: P \rightarrow P/P^u_1 \). Then \( \pi_1(H^u) \) is a \( \pi_\ell(H) \)-stable subgroup of \( P^u/P^u_1 \), which is a vector group where \( \pi_\ell(H) \) acts linearly by conjugation (see [SGA3, Exposé XXVI: Proposition 2.1]). If \( H^2(\pi_\ell(H), \pi_1(H^u)) = 0 \), then \( \pi_1(H) \) is isomorphic to the semidirect product of \( \pi_\ell(H) \) and \( \pi_1(H^u) \), thus it has a Levi subgroup \( L_1 \subseteq P/P^u_1 \). In particular \( L_1 \cap \pi_1(H^u) \) is trivial.

To use the same procedure to the group \( H_1 = H \cap P^{-1}_1(L_1) \): this group has unipotent radical contained in \( P^{-1}_1 \), and satisfies \( \pi_\ell(H) = \pi_\ell(H_1) \). We may go on applying the same procedure to the group \( H_1 \) using the projection onto the quotient \( P/P^u_1 \), provided that the corresponding cohomology groups vanish. We obtain a sequence \( H \supseteq H_1 \supseteq H_2 \supseteq \ldots \) of subgroups of \( H \) satisfying \( \pi_\ell(H) = \pi_\ell(H_1) \) and \( H^u_i \subseteq P^u_i \) for all \( i \). If \( n \) is large enough so that \( P^u_n \) is trivial, the subgroup \( H_n \) of \( H \) is reductive and isomorphic to \( \pi_\ell(H) \), hence it is a Levi subgroup of \( H \).

Now denote by \( L_H \) a Levi subgroup of \( H \), and consider the map \( (\pi_1 \circ \pi_u)|_{L_H}: L_H \rightarrow P^u/P^u_1 \). It is a 1-cocycle of \( L_H \) with values in the module \( P^u/P^u_1 \). If the group \( H^1(L_H, P^u/P^u_1) \) vanishes, then \( (\pi_1 \circ \pi_u)|_{L_H} \) is a coboundary, whence \( \pi_1(L_H) \) is contained in \( \pi_1(L) \) up to conjugation by an element of \( P^u/P^u_1 \).
Therefore we may assume that $L_H \subseteq L P_i$. Proceeding as above using the projections on $P_i P_i$ for $i \in \{1, 2, 3, \ldots \}$, provided that the needed cohomology groups vanish, we obtain that $L_H$ is contained in $L$ up to conjugation by an element of $P_i$.

To finish the proof we must show the vanishing of the cohomology groups involved. We notice that for all of them we may replace the group with its image under $\pi$, i.e., with $L'$. Then it remains to show that $H^n(L', R) = 0$, where $n \in \{1, 2\}$ and $R$ is an $L'$-stable additive subgroup of $P_i P_i$. Using the long exact sequence of group cohomology the problem is reduced to the case where $R$ is an $L'$-stable subgroup of a simple quotient $Q$ of $P_i P_i$.

If $Q$ is the trivial simple $L'$-module, then $L'$ acts trivially on $R$; this implies $H^1(L', R) = 0$ because $\text{SL}(2)$ has no non-trivial abelian quotient. In this case also $H^2(L', R) = 0$ follows, using the long exact sequence of group cohomology, the vanishing of $H^1(L', Q/R)$ by the above argument applied to $Q/R$ instead of $R$, and the vanishing of $H^2(L', Q)$ (see [St10, Theorem 1]). We may now assume that $Q$ is not the trivial simple $L'$-module, and that $R \neq \{0\}$.

From Lemma [6.3] part (1) it follows that $R = Q$. Moreover, by inspection on all rank 2 root systems, the $L'$-module $Q$ (viewed as an $\text{SL}(2)$-module) is isomorphic to $V(\omega_1), V(2\omega_1)$ or $V(3\omega_1)$. Finally, the vanishing of $H^1(L', Q)$ follows from [St10] Proposition 2.6, and the vanishing of $H^2(L', Q)$ follows from [St10] Theorem 1. Notice that here is the step where we use the assumption $p > 2$, since e.g. $H^1(\text{SL}(2), V(\omega_1)) \neq 0$ in characteristic 2.

6.3. Lemma. If $H \subseteq G$ has an open orbit on $G/P_\alpha$ for each simple root $\alpha$ but not on $G/B$, then $|B_{00}(G/H)| = 1$.

Proof. If $|B_{00}(G/H)| > 1$, then Proposition 2.3 implies the existence of an element $Y \in B_{00}(G/H)$ of codimension 1 in $G/H$ satisfying $s_\alpha \cdot (G/H) = (Y)$ for some simple root $\alpha$. Since $H$ has no open orbit on $G/B$, we have $c(Y) = c(G/H) > 0$. Then $s_\alpha \cdot (G/H) = (Y)$ implies that the generic $H$-sheet of $G/P_\alpha$ has positive complexity, which contradicts our assumptions.

6.4. Lemma. Let $P$ and $P_-$ be two opposite parabolic subgroups of $G$, set $L = P \cap P_-$ and let $I$ be either $L$ or $L'$. Also let $H \subseteq P$ be a connected subgroup containing $I$. Then $H_u \subseteq P_u$; if $P_u/H_u$ has an open $I$-orbit, then $H$ has an open orbit on $G/P_-$, and if $P_u/H_u$ is spherical under the action of $I$, then $H$ is a spherical subgroup of $G$.

Proof. The inclusion $H_u \subseteq P_u$ stems from the inclusion $H_u \subseteq P'$, which holds because $HP' = P'$ is reductive, therefore $H_u$ is in the kernel of the projection $P \to P'$. Consider a Borel subgroup $B_L \subseteq L$. Then $B_- = B_L P_u$ is a Borel subgroup of $G$, and its subgroup $P_u$ is a Borel subgroup of $P$. Then $L$ has an open orbit on $P/H$ if and only if $P_-$ has an open orbit on $G/H$, and $B_L$ has an open orbit on $P/H$ if and only if $B_-$ has an open orbit on $G/H$.

This completes the proof if $I = L$, because then $P/H = P_u/H_u$; in this case the two last statements of the lemma are even equivalences. If $I = L'$ but $H$ anyway contains $L$, then again $P/H = P_u/H_u$ and the two last statements of the lemma follow from the case $I = L$. Hence we can suppose $H \nsubseteq L$.

In this case $P/H = P'/H_u$; if $L'$ has an open orbit on $P_u/H_u$ then $L$ has an open orbit on $P'/H_u$ thus $P_-$ has an open orbit on $G/H$, and if $P_u/H_u$ is spherical
under the action of \( L' \), then \( P^u/H^u \) is spherical under the action of \( L \) thus \( H \) is a spherical subgroup of \( G \). \( \square \)

**6.5. Proposition.** Let \( G \) be semisimple of rank 2 and \( p \geq 3 \). Let \( P \supseteq B \) be a proper parabolic subgroup of \( G \) and \( H \) a connected non-reductive subgroup of \( P \) containing a Levi subgroup \( L \) of \( P \). Then \( H \) is spherical or \(|\mathcal{B}_0(G/H)| = 1\).

*Proof.* If \( G \) has type \( A_1 \times A_1, A_2 \) or \( B_2 \), then \( L \) is spherical (Proposition 5.1). This implies that \( H \) is also spherical.

So we suppose that \( G \) has type \( G_2 \), and also that \( L \supseteq T \). Write for brevity \( \alpha = \alpha_1 \) and \( \beta = \alpha_2 \).

Since \( H \supseteq L, P \) is minimal among the parabolic subgroups containing \( H \): it follows that \( H^u \subseteq P^u \). The quotient \( P^u/(P^u)' \) is a vector group where \( L \) acts linearly, and \( H^u(P^u)'/(P^u)' \) is an additive subgroup. Since the center of \( L \) acts on \( P^u/(P^u)' \) non-trivially by homotheties and \( H^u \) is \( L \)-stable, the group \( H^u(P^u)'/(P^u)' \) is an \( L \)-submodule of \( P^u/(P^u)' \).

Moreover, we may suppose that it is a proper submodule, otherwise \( H^u \) is the whole \( P^u \), and hence \( H = P \) is spherical. Notice that here \( (P^u)' \) is abelian, so also a vector group where \( L \) acts linearly. Let us then recall the structure of the \( L \)-modules under consideration (viewed as \( SL(2) \times G_m \)-modules):

\[
\begin{align*}
P^u_\alpha/(P^u_\alpha)' &\cong \text{Sym}^3(V(\omega_1)) \otimes V(\beta|G_m), \\
(P^u_\alpha)' &\cong V(0) \otimes V(2\beta|G_m),
\end{align*}
\]

and

\[
\begin{align*}
P^u_\beta/(P^u_\beta)' &\cong V(\omega_1) \otimes V(\alpha|G_m), \\
(P^u_\beta)' &\cong (V(0) \otimes V(2\alpha|G_m)) \oplus (V(\omega_1) \otimes V(3\alpha|G_m)).
\end{align*}
\]

If now \( p > 3 \) or if \( P = P_\beta \), then \( H^u(P^u)'/(P^u)' \) is trivial, because with either of these two assumptions \( P^u/(P^u)' \) is a simple \( L \)-module. For \( p = 3 \) and \( P = P_\alpha \), the \( L \)-module \( P^u/(P^u)' \) contains a unique non-trivial and proper \( L \)-submodule (of dimension 2). However, if \( H^u(P^u)'/(P^u)' \) contains this submodule, then \( P^u/H^u \) is spherical under the action of \( L \), therefore \( H \) is spherical thanks to Lemma 6.4.

As a consequence, we may suppose from now on that \( H^u \subseteq (P^u)' \). Now \( (P^u)' \) as an \( SL(2) \)-module is either \( V(0) \) or \( V(0) \oplus V(\omega_1) \), and we observe that in both cases \( H^u \) is an \( L \)-submodule. This is obvious in the first case since \( H^u \) is connected, and in the second case it follows from Lemma 6.1 part (2).

In addition, if \( P = P_\beta \) and \( H^u \) contains the 2-dimensional \( L \)-submodule of \( (P^u)' \), then again \( P^u/H^u \) is \( L \)-spherical, so \( H \) is spherical in \( G \).

This leaves only one subgroup \( H \) for each of the two choices of \( P \), namely the one where \( H^u \) is equal to the 1-dimensional summand of \( (P^u)' \). We claim that in both cases \( H \) has an open orbit on \( G/P_\alpha \) and \( G/P_\beta \). This proves the proposition thanks to Lemma 6.3 since \( H \) has no open orbit on \( G/B \) for dimension reasons.

Let us prove the claim for \( P = P_\alpha \), and consider first \( G/P_\alpha \). Thanks to Lemma 6.3 to prove that \( H \subset P_\alpha \) has an open orbit on \( G/P_\alpha \) it is enough to prove that \( L \) has an open orbit on \( P_\alpha^u/H^u \). Both \( L \) and \( P_\alpha^u/H^u \cong \text{Sym}^3(V(\omega_1)) \otimes V(\beta|G_m) \) have dimension 4, so our claim follows from the fact that the point \( e_1e_2(e_1 + e_2) \) has finite stabilizer in \( L \), where \( e_1, e_2 \) is the standard basis of \( V(\omega_1) \cong k^2 \).

We show now that \( H \subset P_\alpha \) has an open orbit on \( G/P_\beta \). Notice that a Levi subgroup of \( P_\beta \) and a Levi subgroup of \( P_\alpha \) are not conjugated in \( G \), whence the group \((P_\beta \cap H)^o \) is solvable for all \( g \in G \). It is then contained in a Borel subgroup
of $H$. Since the flag variety of $H$ has dimension $1$ and $B \cap H$ is a Borel of $H$, the inclusion
\[(9P_\beta \cap H)^0 \subseteq B \cap H\]
holds for all $g$ such that $gP_\beta \in D$, where $D$ is a subvariety of $G/P_\beta$ of codimension $1$. If $H$ has no open orbit on $G/P_\beta$, then $(9P_\beta \cap H)$ has positive dimension for all $g \in G$, therefore for all $g$ such that $gP_\beta \in D$ we have
\[(2) \quad \dim(9P_\beta \cap B \cap H) > 0.\]
We prove that this is impossible, by checking that the intersection of the locus where $[2]$ is satisfied with any Schubert cell of $G/P_\beta$ has codimension at least $2$ in $G/P_\beta$.

We first consider the open Schubert cell, i.e., $g$ is of the form $g = uw_0$ with $u \in U$. Then
\[(9P_\beta \cap B) \cap H = (uw_0P_\beta \cap B) \cap H = u(TU_\beta) \cap H.\]

We fix an isomorphism of the open Schubert cell with $k^5$ in such a way that
\[u = u_{\alpha+\beta}(x_1)u_{2\alpha+\beta}(x_2)u_{3\alpha+\beta}(x_3)\]
where the parameters $x_1$, $x_2$ and $x_3$ are coordinate functions of $k^5$. This is possible since $U_\alpha U_{3\alpha+2\beta} \subset H$. An elementary computation shows that $u(TU_\beta)\cap H$ is infinite only if two of $x_1$, $x_2$, $x_3$ are zero.

Now consider the codimension $1$ Schubert cell, i.e., $g$ is of the form $uw_0s_\alpha$. Then
\[(9P_\beta \cap B) \cap H = (uw_0s_\alpha P_\beta \cap B) \cap H = u(TU_\alpha U_{3\alpha+\beta}) \cap H.\]
Here we can assume that
\[u = u_\beta(x_1)u_{\alpha+\beta}(x_2)u_{2\alpha+\beta}(x_3)\]
where $x_1$, $x_2$, $x_3$ are coordinate functions on the $4$-dimensional Schubert cell. Consider also elements $t \in T$, $u_\alpha(y_1)$ and $u_{3\alpha+\beta}(y_2)$ with $y_1, y_2 \in k$. Then $u(tu_\alpha(y_1)u_{3\alpha+\beta}(y_2))$ is equal to
\[tu_\beta(A_\beta)u_{\alpha+\beta}(A_{2\alpha+\beta})u_{2\alpha+\beta}(A_{3\alpha+\beta}u_{3\alpha+\beta}(A_{3\alpha+2\beta})u_{3\alpha+2\beta}(A_{3\alpha+2\beta})u_{\alpha}(A_\alpha)\]
where
\[A_\beta = (b-1)x_1, \quad A_{\alpha+\beta} = (ab-1)x_2 + x_1y_1, \quad A_{2\alpha+\beta} = (a^2b-1)x_3 + 2x_2y_1 - x_1y_1^2, \quad A_{3\alpha+\beta} = y_2 - 6x_2y_1^2 + x_1y_1^3.\]
and $a = \alpha(t^{-1})$, $b = \beta(t^{-1})$. This element is in $H$ only if $A_\beta = A_{\alpha+\beta} = A_{2\alpha+\beta} = A_{3\alpha+\beta} = 0$. For general $x_1$, $x_2$ and $x_3$ the resulting system has only finitely many solutions in $a, b, y_1, y_2$. This finishes the proof that \cite{2} cannot be satisfied for $gP_\beta$ lying on a subvariety of $G/P_\beta$ of codimension $1$.

Finally, we prove the claim for $P = P_\beta$ using the same method. Here $H \subset P_\beta$, and we consider first $G/P_\beta$. The quotient $P_\beta^u/H^u$ is abelian and, as an $\text{SL}(2) \times \mathbb{C}^m$-module, isomorphic to the sum
\[(V(\omega_1) \otimes V(\alpha|_{\mathbb{C}^m})) \oplus (V(\omega_1) \otimes V(3\alpha|_{\mathbb{C}^m})) \cong k^2 \oplus k^2.\]
The group $L$ has the open orbit
\[\{(v, w) \in k^2 \oplus k^2 | v \neq 0 \neq w, kv \neq kw\},\]
hence $H$ has an open orbit on $G/P_\beta$.  


To show that \( H \subset P_\beta \) has an open orbit on \( G/P_\alpha \) we proceed as above, showing that the locus where \( gP_\alpha \) satisfies
\[
\dim(\langle gP_\alpha \cap B \cap H \rangle) > 0
\]
has codimension at least 2 in \( G/P_\alpha \).

Suppose first that \( gP_\alpha \) is in the open Schubert cell, i.e., \( g \) is of the form \( uw_0 \) for \( u \in U \). Then
\[
(\langle gP_\alpha \cap B \rangle \cap H) = (\langle uw_0P_\alpha \cap B \rangle \cap H) = u(TU_\alpha) \cap H.
\]
We can write
\[
u = u_{\alpha+\beta}(x_1)u_{3\alpha+\beta}(x_2)u_{3\alpha+2\beta}(x_3)
\]
since here \( U_\beta U_{2\alpha+\beta} \subset H \). Again, the intersection \( u(TU_\alpha) \cap H \) is infinite only if two of \( x_1, x_2, x_3 \) are zero.

Now consider the codimension 1 Schubert cell, i.e., \( g \) is of the form \( uw_0s_\beta \). Then
\[
(\langle gP_\alpha \cap B \rangle \cap H) = (\langle uw_0s_\beta P_\beta \cap B \rangle \cap H) = u(TU_\beta U_{\alpha+\beta}) \cap H.
\]
Here we can assume that
\[
u = u_\alpha(x_1)u_{3\alpha+\beta}(x_2)u_{3\alpha+2\beta}(x_3)
\]
where \( x_1, x_2, x_3 \) are coordinate functions on the 4-dimensional Schubert cell. Consider also elements \( t \in T \), \( u_\beta(y_1) \) and \( u_{\alpha+\beta}(y_2) \) with \( y_1, y_2 \in \mathbb{k} \). Then \( u(tu_\beta(y_1)u_{\alpha+\beta}(y_2)) \) is equal to
\[
tu_\alpha(A_\alpha)u_{3\alpha+\beta}(A_{3\alpha+\beta})u_{3\alpha+2\beta}(A_{3\alpha+2\beta})u_{2\alpha+\beta}(A_{2\alpha+\beta})u_\beta(A_\beta)u_{\alpha+\beta}(A_{\alpha+\beta})
\]
where
\[
A_\alpha = (a-1)x_1,
A_{3\alpha+\beta} = (a^3b-1)x_2 + x_1^3y_1 - 3x_1^2y_2,
A_{3\alpha+2\beta} = (a^3b^2 - 1)x_3 - x_2y_1 - 9x_1^2y_1y_2 - x_1^3y_1^2 - 3x_1y_2^2,
A_{\alpha+\beta} = x_1y_1 + y_2,
\]
and \( a = \alpha(t^{-1}), b = \beta(t^{-1}) \). This element is in \( H \) only if \( A_\alpha = A_{3\alpha+\beta} = A_{3\alpha+2\beta} = A_{\alpha+\beta} = 0 \). For general \( x_1, x_2 \) and \( x_3 \) the resulting system has finitely many solutions in \( a, b, y_1, y_2 \). This finishes the proof.

6.6. Corollary. Theorem 2.1 holds for every connected subgroup \( H \) of \( G \) such that \( H \) is neither solvable nor very reductive, and does not satisfy \( L' \subset H \subset L \) where \( P \supset B \) is a proper parabolic subgroup of \( G \) and \( L \) is a Levi of \( P \).

Proof. Since \( H \) is not solvable nor very reductive, it is contained in a proper parabolic subgroup \( P \), which may be assumed to properly contain \( B \). Denote by \( L \) a Levi subgroup of \( P \). Thanks to Lemma 6.2 \( H \) has a Levi subgroup inside \( L \) and containing \( L' \). Moreover, the inclusion \( H^u \subset P^u \) holds thanks to Lemma 6.4 and \( H^u \) is not trivial thanks to our assumptions.

To prove the corollary it is enough to show that \( H \) is spherical or \( L^r \) normalizes \( H \). Indeed, if \( L^r \) normalizes \( H \), then by Lemma 2.3 the sets \( \mathcal{B}_{00}(G/H) \) and \( \mathcal{B}_{00}(G/L'H) \) are \( \mathbb{W} \)-equivariantly isomorphic. In this case the corollary stems from Proposition 6.5 since \( L^r \) contains \( L \).

Suppose first that \( P^u \) is abelian and a trivial or simple \( L' \)-module, which is the case if \( G \) has type \( A_1 \times A_1 \) or \( A_2 \), or \( G \) has type \( B_2 \) and \( P = P_{12} \). Then by Lemma 6.2 part (1), we conclude that \( H^u \) is an \( L \)-stable submodule of \( P^u \), which implies that \( H \) is normalized by \( L^r \).
If $G$ has type $B_2$ and $P = P_{a_1}$, then $P^u/(P^u)'$ under the action of $L'$ is the SL(2)-module $V(\omega_1)$ and the image $K$ of $H^u$ in the quotient $P^u/(P^u)'$ is an $L'$-stable additive subgroup. Lemma 6.1, part (1) implies that $K$ is either the whole $V(\omega_1)$ or trivial. In the first case then $H^u = P^u$. In the second case $H^u \subseteq (P^u)'$, and since the former is non-trivial while the latter is 1-dimensional we have $H^u = (P^u)'$. In any case $H$ is both spherical and normalized by $L'$, and this concludes the proof in the case $G$ is not of type $G_2$.

We may now suppose that $G$ has type $G_2$. We start with the case $P = P_{a_1}$.

If $p = 3$, then $P^u$ is abelian, therefore SL(2)-isomorphic to $\text{Sym}^3(V(\omega_1)) \oplus V(0)$.

The first summand has a composition series $\text{Sym}^3(V(\omega_1)) \supseteq V_1 \supseteq \{0\}$ of SL(2)-submodules with both $V_1$ and $V/V_1$ simple and 2-dimensional.

Consider the projection $K$ of $H^u$ in $\text{Sym}^3(V(\omega_1))$. By Lemma 6.1, part (1), the intersection $K \cap V_1$ and the projection of $K$ on $V/V_1$ are SL(2)-submodules, therefore either 0- or 2-dimensional.

If both are 0-dimensional, then $H^u$ is non-trivial and contained in $V(0)$, therefore $H^u = V(0)$. This implies that $H$ is normalized by $L'$. Suppose that at least one is 2-dimensional, and consider the projection $J$ of $H^u$ on $V(0) \subset P^u$. Then either $J$ is $V(0)$, which implies that $P^u/H^u$ is SL(2)-spherical and thus $H$ is a spherical subgroup of $G$, or $J$ is trivial, which implies that $L'$ normalizes $H$.

If $p > 3$, then $V = P^u/(P^u)' \cong \text{Sym}^3(V(\omega_1))$ is irreducible under the action of SL(2). It follows from Lemma 6.1, part (1) that the projection of $H^u$ on $V$ is an SL(2)-submodule, so either trivial, and hence $H$ is normalized by $L'$, or the full $V$, which implies that $H^u = P^u$ and that $H$ is spherical.

We deal now with the case $P = P_{a_2}$. If $H^u \subseteq (P^u)'$, since the latter is abelian and the sum of a trivial and a 2-dimensional simple SL(2)-module, we conclude from Lemma 6.1, part (2) that $H^u$ is an SL(2)-submodule, thus normalized by $L'$.

Otherwise $H^u$ projects surjectively on $P^u/(P^u)'$, because the latter is a simple SL(2)-module. Then $H^u = P^u$ and $H$ is a spherical subgroup of $G$.  

\begin{thebibliography}{10}


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