CUSPIDAL REPRESENTATIONS OF REDUCTIVE P-ADIC GROUPS ARE RELATIVELY INJECTIVE AND PROJECTIVE

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ABSTRACT. Cuspidal representations of a reductive $p$-adic group $G$ over a field of characteristic different from $p$ are relatively injective and projective with respect to extensions that split by a $U$-equivariant linear map for any subgroup $U$ that is compact modulo the centre. The category of smooth representations over a field whose characteristic does not divide the pro-order of $G$ is the product of the subcategories of cuspidal representations and of subrepresentations of direct sums of parabolically induced representations.

1. Introduction

Let $G$ be a reductive linear algebraic group over a non-Archimedean local field with residue field of characteristic $p$; we briefly call $G$ a reductive $p$-adic group. Let $R$ be a commutative ring in which $p$ is invertible. Let $\text{Rep}_R(G)$ be the category of smooth representations of $G$ on $R$-modules.

A smooth representation is cuspidal if it is killed by the parabolic restriction functors for all proper parabolic subgroups. We call an extension $V' \hookrightarrow V \twoheadrightarrow V''$ in $\text{Rep}_R(G)$ cmc-split exact if it splits in $\text{Rep}_R(K)$ for any subgroup $K$ of $G$ that is compact modulo the centre $Z(G)$ of $G$ (cmc). A smooth representation is cmc-projective or cmc-injective, respectively, if it is projective or injective with respect to cmc-split extensions.

Theorem 1.1. Cuspidal representations are cmc-projective and cmc-injective.

We call $R$ a good field if it is a field whose characteristic $\ell$ does not divide the pro-order of $G$; that is, $\ell$ does not divide $[U_1 : U_2]$ for any compact open subgroups $U_2 \subseteq U_1 \subseteq G$. Our theorem implies:

Theorem 1.2. Any cuspidal representation over a good field is a quotient of a cuspidal projective representation and contained in a cuspidal injective representation.

This implies that the category of smooth representations of $G$ over a good field is the product of the subcategory of cuspidal representations and the subcategory of representations that are contained in a sum of parabolically induced representations. This splitting is a crucial part of the Bernstein decomposition.

Our main theorem follows quickly from the theory of support projections in [1]. We recall the relevant notation and results in Section 2. Section 3 proves the assertions involving relative projectivity and applies it to show that parabolically induced representations have no cuspidal subquotients if $R$ is a good field. Section 4

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proves the assertions about relative injectivity. This implies that any smooth representation over a good field is a direct sum of a cuspidal representation and a subrepresentation of a parabolically induced representation.

2. Support projections

Let $BT$ be the affine Bruhat–Tits building of $G$. We treat $BT$ as a partially ordered set of polysimplices, with the relation $\sigma \prec \tau$ if $\sigma$ is a facet of $\tau$. The group $G$ acts on $BT$. We denote the stabiliser of $\sigma \in BT$ by

$$G_\sigma := \{ g \in G \mid g\sigma = \sigma \};$$

its elements may permute the vertices of $\sigma$ non-trivially.

The group $G_\sigma$ is open and compact modulo the centre $Z(G)$ of $G$. Any subgroup that is compact modulo the centre is contained in $G_\sigma$ for some $\sigma \in BT$ because it fixes some point in the geometric realisation of $BT$. Hence an extension is cme-split exact (see Definition 3.3) if and only if it splits $G_\sigma$-equivariantly for each $\sigma \in BT$.

The normalised Haar measure on a compact, open, pro-$p$ subgroup $U \subseteq G$ gives an idempotent element $\langle U \rangle$ in the Hecke algebra $H = \mathcal{H}(G, R)$ of $G$ with coefficients in $R$ because $p^{-1} \in R$. Schneider and Stuhler [3] and Vignéras [5] use certain compact, open, pro-$p$ subgroups $U^n_\sigma$ for $\sigma \in BT$, $n \in \mathbb{N}$ to construct resolutions for smooth representations of $G$. Let $e^n_\sigma := \langle U^n_\sigma \rangle$. For fixed $\sigma \in BT$, the groups $(U^n_\sigma)_{n \in \mathbb{N}}$ form a neighbourhood basis of the unit in $G$. Hence $(e^n_\sigma)_{n \in \mathbb{N}}$ is an approximate unit in $\mathcal{H}$.

**Theorem 2.1.** Let $\Sigma \subseteq BT$ be a finite, convex subcomplex. Define

$$u^n_\Sigma := \sum_{\sigma \in \Sigma} (-1)^{\dim(\sigma)} e^n_\sigma \in \mathcal{H}.$$ 

This element is idempotent and acts on any smooth representation such that

$$\text{im}(u^n_\Sigma) = \sum_{x \in \Sigma^0} \text{im}(e^n_x), \quad \ker(u^n_\Sigma) = \bigcap_{x \in \Sigma^0} \ker(e^n_x).$$

**Proof.** The idempotents $(e^n_x)_{x \in BT^0}$ satisfy the conditions in [1, Definition 2.1], and $e^n_\sigma = \prod_{x \prec \sigma} e^n_x$ for $\sigma \in BT$. Hence everything follows from [1, Theorem 2.12]. \qed

The idempotent $u^n_\Sigma$ is called the support projection of $\Sigma$ in [1].

3. Relative projectivity of cuspidal representations

**Definition 3.1.** A representation $V \in \text{Rep}_R(G)$ is cuspidal if it is killed by the parabolic restriction functor for any proper parabolic subgroup.

**Proposition 3.2 ([1 II.2.7]).** A representation $V$ is cuspidal if and only if for each compact open pro-$p$ subgroup $U \subseteq G$ and each $v \in V$, the set of $g \in G$ with $\langle U \rangle gv \neq 0$ is compact modulo $Z(G)$.

Since $e^n_\sigma = U^n_\sigma$ and the subgroups $U^n_\sigma$ form a neighbourhood basis in $G$, $V$ is cuspidal if and only if the set of $g \in G$ with $e^n_\sigma gv \neq 0$ is compact modulo the centre of $G$ for all $n \in \mathbb{N}$, $\sigma \in BT$, $v \in V$. The set of these $g \in G$ is closed under multiplication by elements of the centre $Z(G)$ because $e^n_\sigma zgv = ze^n_\sigma g v = 0$ if and only if $e^n_\sigma g v = 0$; and it is closed under left multiplication by elements of $U^n_\sigma$. Thus being cuspidal means that for each $n, \sigma, v$, the set of $g \in G$ with $e^n_\sigma g v \neq 0$ consists of only finitely many cosets $U^n_\sigma Z(G)g$. 

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Definition 3.3. An extension $W' \to W \to W''$ in $\text{Rep}_R(G)$ consists of smooth representations $W'$, $W$ and $W''$ of $G$ on $R$-modules and $G$-equivariant $R$-module maps $i: W' \to W$ and $p: W \to W''$ such that $i$ is injective, $p$ is surjective, and $i(W')$ is the kernel of $p$. This extension is $c$-split if there is a $U$-equivariant $R$-module map $s: W'' \to W$ with $p \circ s = \text{Id}_{W''}$ for each compact subgroup $U \subseteq G$, and $cmc$-split if the same happens for each subgroup $U \subseteq G$ for which $U/(Z(G) \cap U)$ is compact.

Definition 3.4. An object $V$ of $\text{Rep}_R(G)$ is projective, $c$-projective or $cmc$-projective if the functor $\text{Hom}_{R,G}(V, -)$ is exact on all extensions, all $c$-split extensions or all $cmc$-split extensions in $\text{Rep}_R(G)$, respectively. It is injective, $c$-injective or $cmc$-injective if $\text{Hom}_{R,G}(\omega, V)$ is exact on the appropriate extensions.

Theorem 3.5. Cuspidal representations in $\text{Rep}_R(G)$ are $cmc$-projective.

Theorem 3.6. For any cuspidal representation $W''$ in $\text{Rep}_R(G)$ there is a $c$-split extension $W' \to W \to W''$ with cuspidal and $c$-projective $W$.

We are going to prove Theorems 3.5 and 3.6 in Sections 3.1 and 3.2, respectively. In Section 3.3 we specialise to the case where $R$ is a good field and deduce an orthogonality result for cuspidal and parabolically induced representations.

3.1. Cuspidal representations are $cmc$-projective. We are going to prove Theorem 3.5 about cuspidal representations being $cmc$-projective. Let $V$ be a cuspidal representation, so it verifies the condition in Proposition 3.2. Fix $n \in \mathbb{N}$. Let $e^n_\sigma V \subseteq V$ for $\sigma \in \mathcal{B}T$ be the image of $e^n_\sigma$ on $V$. Let

$$F^n := \bigoplus_{\sigma \in \mathcal{B}T} e^n_\sigma V.$$ 

Elements of $F^n$ are functions $\psi: \mathcal{B}T \to V$ with finite support and $e^n_\sigma \psi(\sigma) = \psi(\sigma)$ for all $\sigma \in \mathcal{B}T$. We let $G$ act on $F^n$ by $(g \psi)(\sigma) := g \cdot (\psi(g^{-1} \sigma))$ for $g \in G$, $\sigma \in \mathcal{B}T$, $\psi \in F^n$. This belongs to $F^n$ again because $e^n_\sigma g = ge^n_\sigma g^{-1}$, and it defines a smooth representation, so $F^n \in \text{Rep}_R(G)$.

Lemma 3.7. The smooth representation $F^n$ is $cmc$-projective.

Proof. We may decompose $F^n$ as a sum over the finite set of $G$-orbits $G \backslash \mathcal{B}T$, where each summand is the subspace $F^n_\sigma$ of $F^n$ of functions supported on $G\sigma \cong G/G_\sigma$. When we apply compact induction from $G_\sigma$ to $G$ to the representation of $G_\sigma$ on the $G_\sigma$-invariant subspace $e^n_\sigma V \subseteq V$, we get the representation $F^n_\sigma$. Since compact induction for the open subgroup $G_\sigma \subseteq G$ is left adjoint to restriction, we get

$$\text{Hom}_G(F_\sigma, \omega) \cong \text{Hom}_{G_\sigma}(e^n_\sigma V, \omega).$$

Since the groups $G_\sigma/Z(G)$ are compact, this functor is exact on $cmc$-split extensions.

The inclusion maps $e^n_\sigma V \hookrightarrow V$ define a $G$-equivariant $R$-module homomorphism $\pi^n: F^n \to V$. Since $V$ is cuspidal, we also get a map in the opposite direction:

Lemma 3.8. Let $V$ be a cuspidal representation. There is a $G$-equivariant $R$-module homomorphism $\alpha^n: V \to F^n$ defined by

$$\alpha^n(v)(\sigma) := (-1)^{\dim(\sigma)} e^n_\sigma(v) \quad \text{for } v \in V, \sigma \in \mathcal{B}T.$$ 

The map $\pi^n \circ \alpha^n: V \to V$ is idempotent with image $\sum_{\sigma \in \mathcal{B}T} e^n_\sigma V$. 

Proposition 3.10. A smooth representation $V \in \mathcal{R}ep_R(G)$ is cuspidal if and only if for each $v \in V$ and $x \in \mathcal{H}$, the set of $g \in {}^0G$ with $xgv \neq 0$ is compact.

Proof. It makes no difference whether we use all $x \in \mathcal{H}$ or only $\langle U \rangle$ for compact open pro-$p$ subgroups $U \subseteq G$ because for any $x \in \mathcal{H}$ there is $U$ with $x = x\langle U \rangle$. Assume first that for each $v \in V$, the set of $g \in {}^0G$ with $xgv \neq 0$ is compact. Let $h_1, \ldots, h_n \in G$ be representatives for the finite quotient group $G/\langle G \rangle Z(G)$. So every element of $G$ is of the form $gh_i$ with $g \in {}^0G$, $z \in Z(G)$, $i \in \{1, \ldots, n\}$. For each $i$, the set $K_i$ of $g \in {}^0G$ with $xgh_i v \neq 0$ is compact. The set of $g \in G$ with $xgv \neq 0$ is $\bigcup_{i=1}^n K_i Z(G)$, which is compact modulo the centre. Hence $V$ verifies the criterion for being cuspidal in Proposition 3.2. Conversely, if $V$ verifies that criterion, then the set of $g \in {}^0G$ with $xgv \neq 0$ has to be compact because $Z(G)$ is compact.

If $V$ is finitely generated, then $V = \bigoplus_{n=0}^\infty e^n V$ for some $n \geq 0$, so that $\pi \circ \alpha = \text{Id}_V$ and $V$ is a direct summand of $F^n$. For general $V$, we let $F := \bigoplus_{n=0}^\infty F^n$. This is cmc-projective by Lemma 3.7. Let $\pi: F \to V$ be induced by the maps $\pi^n: F^n \to V$. Define $\alpha: V \to F$ by $\alpha(v) = (\alpha(v)_n)_{n \in \mathbb{N}}$ with

$$\alpha(v)_n := \alpha^n \circ (\text{Id}_V - \pi^{n-1} \circ \alpha^{n-1})(v)$$

for $n \in \mathbb{N}$, with the convention $\pi^{-1} \circ \alpha^{-1} = 0$ for $n = 0$. For any $v \in V$ and $\sigma \in \mathcal{B}T$, there is $n_0 \geq 0$ with $e^n v = v$ for all $n \geq n_0$. This implies $\langle \text{Id}_V - \pi^n \circ \alpha^n \rangle(v) = 0$ and hence $\alpha(v) = 0$ for $n \geq n_0$. Thus $\alpha(v) \in F$. Since the maps $\alpha^n$ and $\pi^n$ are $G$-equivariant $R$-module homomorphisms, so is $\alpha$. The sequence of idempotent maps $\pi^n \circ \alpha^n$ on $V$ is increasing, that is, $\pi^n \circ \alpha^n \circ \pi^n \circ \alpha^n = \pi^n \circ \alpha^n$. Hence

$$\pi \circ \alpha(v) = \sum_{n=0}^\infty (\pi^n \circ \alpha^n - \pi^n \circ \alpha^n)(v) = v.$$ 

This finishes the proof that cuspidal representations are cmc-projective.

3.2. Enough c-projective cuspidal representations. We are going to prove Theorem 3.6 about cuspidal representations being c-split quotients of c-projective cuspidal representations. An important tool that we will also use for other purposes is the subgroup $^0G \subseteq G$, that is, the intersection of the kernels of all unramified characters. The following is proved in [2, Section V.2.3–6]:

Proposition 3.9. The subgroup $^0G$ is open and normal in $G$ and contains any compact subgroup of $G$. The quotient group $G/^0G$ is free Abelian of finite rank. The image of $Z(G)$ in $G/^0G$ has finite index, and $^0Z := Z(G) \cap ^0G$ is compact.

The subgroup $^0G$ is equal to the subgroup generated by the compact elements of $G$ used in [4]. We modify the criterion for cuspidal representations in Proposition 3.2.
We use the criterion in Proposition 3.10 to define which smooth representations of \(0^G\) are cuspidal. Then Proposition 3.11 says that a representation of \(G\) is cuspidal if and only if its restriction to \(0^G\) is cuspidal.

**Theorem 3.11.** Cuspidal representations of \(0^G\) are c-projective in \(\mathcal{R}ep_R(0^G)\).

*Proof.* Since \(0^Z\) is compact, cmc- and c-projectives in \(\mathcal{R}ep_R(0^G)\) are the same. Hence the assertion follows as in the proof of Theorem 3.5. □

The functor that restricts a smooth \(G\)-representation to \(0^G\) and then applies compact induction to \(G\), maps \(W'' \in \mathcal{R}ep_R(G)\) to \(W = \bigoplus_{[g] \in G/0^G} W''\), the space of finitely supported functions \(G/0^G \to W''\), with the (smooth) representation of \(G\) by \((g\psi)[h] = g \cdot (\psi[g^{-1}h])\) for \(g, h \in G\) and \(\psi: G/0^G \to W''\); here we use that \(G\) already acts on \(W''\) to simplify the result. Since this action of \(G\) on \(W\) restricts to the given representation of \(0^G\) on \(W''\) on each summand, Proposition 3.10 shows that \(W\) is cuspidal if and only if \(W''\) is.

Define \(\pi: W \to W'', \pi(\psi) = \sum_{[g] \in G/0^G} \psi[g]\). This map is \(G\)-equivariant and \(R\)-linear. It splits by the \(0^G\)-equivariant \(R\)-linear map \(s: W'' \to W\) defined by \(s(v)[1] := v\) and \(s(v)[g] := 0\) for \([g] \neq [1]\) in \(G/0^G\). Since all compact subgroups are contained in \(0^G\), \(\pi\) is a c-split surjection. The adjointness between compact induction and restriction for the open subgroup \(0^G \subseteq G\) gives

\[
\text{Hom}_G(W, \omega) \cong \text{Hom}_G(W'', \omega).
\]

This functor is exact on c-split extensions by Theorem 3.11 because \(W''\) is cuspidal. Thus \(W''\) is c-projective and cuspidal. Take \(W' := \ker(\pi)\) to finish the proof of Theorem 3.6.

**3.3. Representations over fields of good characteristic.** Now let \(R\) be a good field, that is, its characteristic does not divide the pro-order of \(G\).

**Corollary 3.12.** Let \(R\) be a good field. Cuspidal representations of \(G\) are projective as representations of \(0^G\), and quotients of projective, cuspidal representations of \(G\).

*Proof.* Any extension of vector spaces over the field \(R\) splits, and we can make the section \(U\)-invariant for a given compact subgroup \(U \subseteq G\) by averaging over the normalised Haar measure of \(U\); this measure has values in \(R\) because \([U : U']\) is invertible in \(R\) for any open subgroup \(U' \subseteq U\) by our assumption on the characteristic of \(R\). Thus all extensions are c-split exact, and c-projective objects are projective. Now everything follows from Theorems 3.11 and 3.6. □

**Definition 3.13.** A representation is *subinduced* if it is contained in a direct sum of parabolically induced representations from proper parabolic subgroups.

**Theorem 3.14.** Let \(R\) be a good field. Let \(V \in \mathcal{R}ep_R(G)\) be cuspidal and let \(W \in \mathcal{R}ep_R(G)\) be subinduced. Any map \(W' \to V\) or \(V \to W\) vanishes. Subinduced representations have no non-zero cuspidal subquotients.

*Proof.* The vanishing of maps \(V \to W\) is well known and follows from the (easy) left adjointness of parabolic restriction to parabolic induction (see [11, II.2.3] or [2, Section VI.7.2]). Assume there is a non-zero map \(f: W' \to V\). Since subrepresentations of cuspidal representations remain cuspidal, \(\text{im}(f)\) is cuspidal. By Corollary 3.12 \(\text{im}(f)\) is a quotient of a projective, cuspidal representation, \(p: V' \to \text{im}(f)\). Since \(V'\)
is projective, there is a map \( h : V' \to W \) with \( f \circ h = p \). Since \( \text{im}(f) \neq 0 \), we have \( p \neq 0 \) and thus \( h \neq 0 \). But since \( V' \) is cuspidal and \( W \) is subinduced, there is no non-zero map \( V' \to W \). Thus there cannot be a non-zero map \( W \to V \).

If subinduced representations may have non-zero cuspidal subquotients, they also may have non-zero cuspidal quotients. This would give a non-zero map from a subinduced representation to a cuspidal representation, which is impossible. □

An example mentioned in [4, II.2.5] shows that parabolically induced representations of \( \text{Gl}(2, \mathbb{Q}_5) \) over the field with 3 elements may have cuspidal subquotients. Hence the assertions of Theorem 3.14 and Corollary 3.12 become false in this case. Since 3 divides the pro-order of \( \text{Gl}(2, \mathbb{Q}_5) \), there is no contradiction.

Remark 3.15. Vigneras [5] proves that irreducible cuspidal representations are projective in the category of representations with a fixed central character; this is essentially the same as proving that they are projective as \( 0G \)-representations. The proof depends on Schur’s Lemma, which implies that any non-zero map on an irreducible representation is invertible. There are only finitely many isomorphism classes of cuspidal representations of \( 0G \) that contain a \( K \)-fixed vector for any fixed compact subgroup \( K \subseteq 0G \). This deep fact and the projectivity of irreducible cuspidal representations imply a product decomposition of the category of representations of \( 0G \) into one factor for each irreducible cuspidal representation and one factor for those representations that have no cuspidal subquotients (compare [2, Section VI.3.4]). This product decomposition shows that all cuspidal representations are projective as representations of \( 0G \) because this holds for irreducible cuspidal representations. In contrast, we directly prove that arbitrary cuspidal representations are projective and also injective as representations of \( 0G \). Much of the direct product decomposition in [2, Section VI.3.4] follows from such projectivity and injectivity results, compare Theorem 4.7 below.

4. Relative injectivity of cuspidal representations

Definition 4.1. A representation \( V \) is uniformly cuspidal if for each \( x, y \in \mathcal{H} \) the set of \( g \in G \) for which \( xgy \in \mathcal{H} \) acts non-trivially on \( V \) is compact modulo \( Z(G) \).

It is shown in [4, II.2.16] that cuspidal representations are uniformly cuspidal. This allows us to prove relative injectivity results for cuspidal representations.

4.1. Cmc-injectivity.

Theorem 4.2. Cuspidal representations in \( \text{Rep}_R(G) \) are cmc-injective.

For \( n \geq 0 \), let

\[
\tilde{I}^n := \prod_{\sigma \in \mathcal{B}T} e^n_\sigma V.
\]

Elements of \( \tilde{I}^n \) are functions \( \psi : \mathcal{B}T \to V \) with \( e^n_\sigma \psi(\sigma) = \psi(\sigma) \) for all \( \sigma \in \mathcal{B}T \), and \( G \) acts on such functions as above, by \( (g \psi)(\sigma) := g \cdot (\psi(g^{-1} \sigma)) \) for \( g \in G \), \( \sigma \in \mathcal{B}T \), \( \psi \in \tilde{I}^n \). This representation is not smooth. Let \( I^n \subseteq \tilde{I}^n \) be the subspace of smooth elements, that is, elements fixed by some compact open subgroup of \( G \).
Lemma 4.3. The smooth representation $I^n$ is cmc-injective.

Proof. This is proved in the same way as Lemma 3.7. Let $I^n_\sigma$ be the restriction of $I^n$ to the orbit of $\sigma$. Then $I^n \cong \prod_{\sigma \in G/\mathcal{B}T} I^n_\sigma$ because $G/\mathcal{B}T$ is finite. Thus $I^n$ is cmc-injective if and only if $I^n_\sigma$ is cmc-injective for all $\sigma \in G/\mathcal{B}T$. The induction functor from $G_\sigma$ to $G$ maps $V$ to $I^n_\sigma$ and is right adjoint to the restriction functor. Thus the same argument as in the proof of Lemma 3.7 shows that $I^n_\sigma$ is cmc-injective. □

Since $F^n \subseteq \tilde{I}^n$ is a smooth representation, we have $F^n \subseteq I^n$. Thus the map $\alpha^n$ constructed in Section 3.1 is also a $G$-equivariant $R$-module map $\alpha^n : V \to I^n$.

Lemma 4.4. If $V$ is cuspidal, then the map $\pi^n : F^n \to V$ extends to a $G$-equivariant $R$-module map $\pi^n : I^n \to V$.

Proof. Let $\psi \in I^n \subseteq \tilde{I}^n$. Since $\psi$ is a smooth element, it is fixed by some compact open pro-$p$ subgroup $U$ of $G$. Let $\sigma \in \mathcal{B}T$. The set of $g \in G$ with $\langle U \rangle g^{-1} \psi \neq 0$ is compact modulo $Z(G)$ because $V$ is uniformly cuspidal (see [4, II.2.16]). Since $\psi(g\sigma) \in e^n_{g\sigma}V = ge^n_{\sigma}g^{-1}V = ge^n_{\sigma}V$, the set of $g \in G$ with $\langle U \rangle g \psi \neq 0$ is compact modulo $Z(G)$. Thus $\langle U \rangle g \psi = 0$ for all but finitely many elements in the orbit $G\sigma$ because the stabiliser $G_{\sigma}$ of $\sigma$ contains $Z(G)$. Since $G/\mathcal{B}T$ is finite, the sum

$$\tilde{\pi}^n_U(\psi) := \sum_{\sigma \in \mathcal{B}T} \langle U \rangle \psi(\sigma)$$

is finite. If $\psi$ has finite support, then we may pull $\langle U \rangle$ out of the sum and get $\tilde{\pi}^n_U(\psi) = \langle U \rangle \pi^n(\psi) = \pi^n(g \psi)$ because $\pi^n$ is $G$-equivariant.

We claim that $\tilde{\pi}^n_U$ does not depend on $U$, so we get a well-defined map $\pi^n : I^n \to V$. Taking this for granted, it is routine to check that $\tilde{\pi}^n_U$ is a $G$-equivariant $R$-module homomorphism. We checked already that it extends $\pi^n$, so this finishes the proof.

It remains to prove that $\tilde{\pi}^n_U$ does not depend on $U$. If $U_1, U_2$ are compact open pro-$p$ subgroups fixing $\psi$, then there is an open subgroup $U_3 \subseteq U_1 \cap U_2$ which is normal in $U_1$, and $U_3$ contains an open subgroup $U_4$ that is normal in $U_2$ and hence in $U_3$. Thus we are done if we show that $\tilde{\pi}^n_{U_4} = \tilde{\pi}^n_{U_3}$ if $U$ is a normal open subgroup of $U'$. Then $\langle U' \rangle = [U' : U]^{-1} \sum_{g \in U'/U} \langle U \rangle g$. We compute

$$\tilde{\pi}^n_{U'}(\psi) = \frac{1}{[U' : U]} \sum_{g \in U'/U, \sigma \in \mathcal{B}T} \langle U \rangle g \cdot (\psi(\sigma)) = \frac{1}{[U' : U]} \sum_{g \in U'/U, \sigma \in \mathcal{B}T} \langle U \rangle g \cdot (\psi(g^{-1}\sigma))$$

$$= \frac{1}{[U' : U]} \sum_{g \in U'/U, \sigma \in \mathcal{B}T} \langle U \rangle (g\psi)(\sigma) = \tilde{\pi}^n_{U}(\psi);$$

the second step reindexes the sum over $\sigma$; the third step is the definition of the $G$-action on $I^n$, and the last step uses $g\psi = \psi$ for $g \in U'$.

Lemma 3.8 shows that $\pi^n \circ \alpha^n = \pi^n \circ \alpha^n$ is idempotent with image $\sum_{x \in \mathcal{B}T^0} e^n_x V$. If $V = \sum_{x \in \mathcal{B}T^0} e^n_x V$, then Lemmas 3.8 and 4.4 show that $V$ is cmc-injective because it is a direct summand in the cmc-injective smooth representation $I^n$.

In general, let $V^n := (\pi^n \alpha^n - \pi^n - 1 \alpha^{n-1}) V$ for $n \in \mathbb{N}$. These are direct summands in $V$ with $V = \bigoplus_{n \in \mathbb{N}} V^n$. We claim that any element of $\prod_{n \in \mathbb{N}} V^n$ that is smooth for
the $G$-action already belongs to the direct sum $\bigoplus_{n \in \mathbb{N}} V^n$. Indeed, a smooth vector is in the image of $e^n$ for some $m \in \mathbb{N}$ because these form an approximate unit, and then it is killed by $\pi^n \alpha^n - \pi^{n-1} \alpha^{n-1}$ for $n > m$. Hence $V$ is also the product of the $V^n$ in the category of smooth representations. The proof above shows that the factors $V^n$ are c-injective; so $V$ is also because products of c-injective representations remain c-injective. This finishes the proof of Theorem 4.2.

4.2. Enough c-injective cuspidal representations.

**Theorem 4.5.** For any cuspidal representation $W'$ in $\mathcal{R}ep_R(G)$ there is a c-split extension $W' \to W \to W''$ with cuspidal and c-injective $W$. Cuspidal representations of $^0G$ are c-injective in $\mathcal{R}ep_R(^0G)$.

**Proof.** This is analogous to the proofs of Theorem 3.6 and 3.11. For the first statement, we restrict the representation on $W'$ to $^0G$ and now apply ordinary induction to define $W$; this produces a c-injective representation of $G$ because induction is right adjoint to restriction. There is a canonical $G$-equivariant map $W' \to W$, which splits $^0G$-equivariantly by restriction to $^0G \subseteq G$. Letting $W'' := W/W'$, we get the desired c-split extension. \hfill $\Box$

**Corollary 4.6.** Let $R$ be a good field. Any cuspidal representation in $\mathcal{R}ep_R(G)$ is contained in a cuspidal, injective representation.

**Proof.** See the proof of Corollary 3.12. \hfill $\Box$

**Theorem 4.7.** Let $R$ be a good field. The category $\mathcal{R}ep_R(G)$ is the product of the two subcategories of cuspidal and subinduced representations.

**Proof.** Theorem 3.14 shows that there are no arrows in either direction between the subcategories of cuspidal and subinduced representations. It remains to show that every representation is a product of a cuspidal and a subinduced representation.

Let $S$ be the set of proper standard parabolic subgroups of $G$. Let $i_P$ and $r_P$ for $P \in S$ be the parabolic induction and restriction functors. The right adjointness of $i_P$ to $r_P$ gives natural maps $\beta_P: V \to i_P r_P(V)$ for $V \in \mathcal{R}ep_R(G)$, which we put together into a natural map $\beta: V \to \bigoplus_{P \in S} i_P r_P(V)$. Let $V' := \ker(\beta)$, $V'' := \operatorname{im}(\beta)$. These form an extension $V' \to V \to V''$ in $\mathcal{R}ep_R(G)$ with subinduced $V''$. Inspection shows that $\beta(v) = 0$ for $v \in V$ if and only if for each unipotent subgroup $N \subseteq G$ there is a compact open subgroup $U_N$ with $\langle U_N \rangle v = 0$; thus $V'$ is cuspidal.

We may embed $V'$ in an injective, cuspidal representation $W$ by Corollary 4.6. The embedding $i: V' \to W$ extends to a map $\varphi: V \to W$ because $W$ is injective. Since $\varphi(V') = i(V')$, $\varphi$ induces a map $\hat{\varphi}: V'' \to W/V'$. The quotient $W/V'$ is again cuspidal, so Theorem 3.14 gives $\hat{\varphi} = 0$. Thus $\varphi(V) \subseteq i(V')$, so $\varphi: V \to V'$ is a section for our extension. Hence $V \cong V' \oplus V''$. \hfill $\Box$

**References**


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