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CORRECTIONS TO: "A MURNAGHAN–NAKAYAMA RULE FOR VALUES OF UNIPOTENT CHARACTERS IN CLASSICAL GROUPS"

FRANK LÜBECK AND GUNTER MALLE

Abstract. We settle a missing case in the proof of one of the main applications of our results in [Frank Lübeck and Gunter Malle, *A Murnaghan–Nakayama rule for values of unipotent characters in classical groups*, Represent. Theory **20** (2016), 139–161, MR3466537].

In [5] we derived a Murnaghan–Nakayama formula for the values of unipotent characters of finite classical groups. As one application, we showed that the first Cartan invariant $c_{11}^{(\ell)}$ of a finite simple classical group G in non-defining characteristic ℓ is never equal to 2 if Sylow ℓ-subgroups of G are non-cyclic. But contrary to what was claimed in the first line of the proof of [5, Thm. 7.1], the same assertion for the case of defining characteristic does not follow from the work of Koshitani, Külshammer, and Sambale [2]. We thank Shigeo Koshitani and Jürgen Müller for pointing this out to us; see [3].

Here we give a short argument for the missing case. The argument also covers the analogous statement in [4, Prop. 6.3] pertaining to simple groups of exceptional Lie type. The crucial observation is that Sylow p-subgroups in groups of Lie type of characteristic p are large. Except for a few small rank cases, we do not need any knowledge about the ordinary character table of the group G.

Note that if $c_{11}^{(p)} = 2$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $1_G+\chi$ is the character of a projective module; in particular, $\chi(g) = -1$ for all p-singular elements $g \in G$.

Lemma 1. *Let G be a finite group and p a prime. Assume that $H \leq G$ is a subgroup such that $|G : H|$ is divisible by p, but $|G : H| < |G|_p$. Then $c_{11}^{(p)} > 2$.*

Proof. The permutation character π of G on the cosets of H has exactly one trivial constituent 1_G. On the other hand, as $|G : H|$ is divisible by p, the modular reduction of the permutation representation has (at least) two trivial modular constituents: one in the socle, one in the head. Thus, the p-modular reduction of one of the non-trivial ordinary constituents χ of π must contain the trivial p-modular character of G. By Brauer reciprocity, χ then occurs in the projective cover Ψ of the trivial character. If we had $c_{11}^{(p)} = 2$, then necessarily $\Psi = 1_G + \chi$. Since the degree of any projective character is divisible by $|G|_p$, this is a contradiction to

$$|G|_p \leq \Psi(1) = 1_G(1) + \chi(1) \leq \pi(1) = |G : H|. \quad \square$$

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Proposition 2. *Let G be a finite simple group of Lie type in characteristic p. Then $c_{11}^{(p)} > 2$ unless $G = \mathrm{L}_2(p)$.*

Proof. If the Lie-rank of G is not very small, then we can find a subgroup of G which fulfills the condition of Lemma 1. In Table 1 we give a subgroup $\hat{H}$ (see for example [6, §13] for the existence) of a suitable central extension $\hat{G}$ of G. From the order formulae for these groups (see for example [6, Tab. 24.1]) it is easy to check that $\hat{H} \leq \hat{G}$ fulfills the condition of Lemma 1. Since the center of $\hat{G}$ is of order prime to p, the same holds for $H \leq G$ where H is the image of $\hat{H}$ in G.

It remains to consider the groups of type A_1, A_2, ${}^2A_2$, ${}^3D_4$, ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$. For all of these the (generic) character tables are known and available in CHEVIE [1]. Here we look for all irreducible characters χ which have value -1 on all p-singular (non-semisimple) elements, so that $\chi + 1_G$ is a candidate for the 1-PIM of G. It is easy to see that, apart from two exceptions, all characters either: vanish on some p-singular element, or that a value on a non-trivial unipotent class is positive, or that the values on two non-trivial unipotent classes are different.

The two exceptions occur in type A_1, so $G = \mathrm{L}_2(q)$, which has a family of characters of degree $q-1$ with value -1 on all p-singular elements, and in type A_2, so $G = \mathrm{L}_3(q)$, which has a family of characters of degree $q^3 - 1$ which also take value -1 on all p-singular elements. By [7, Thm. 5.2] the 1-PIM of $G = \mathrm{L}_2(q)$ has degree $(2^f - 1)q$, where $q = p^f$. Since the maximal character degree for G is $q+1$, this shows that $c_{11}^{(p)} > 2$ unless $f = 1$; that is, unless $q = p$ is prime, in which case the Sylow p-subgroups of G are cyclic (and so indeed $c_{11}^{(p)} = 2$). Similarly, by [7, Thm. 5.3] the 1-PIM of $G = \mathrm{L}_3(q)$, $q = p^f$, has degree $(12^f - 6^f + 1)q^3$, so none of the candidate $1_G + \chi$ mentioned above can be the 1-PIM. □

Table 1. Large subgroups

$\hat{G}$		$\hat{H}$	$\hat{G}$	$\hat{H}$
$\mathrm{SL}_4^{\pm}(q)$		$\mathrm{Sp}_4(q)$	$G_2(q)$	${}^2A_2(q)$
$\mathrm{SL}_n^{\pm}(q)$	$(n \geq 5)$	$\mathrm{GL}_{n-1}^{\pm}(q)$	$F_4(q)$	$B_4(q)$
$\mathrm{Spin}_{2n+1}(q)$	$(n \geq 2)$	$\mathrm{Spin}_{2n}^{-}(q)$	$E_6^{\pm}(q)$	$F_4(q)$
$\mathrm{Sp}_{2n}(q)$	$(n \geq 3)$	$\mathrm{Sp}_{2n-2}(q)\mathrm{Sp}_2(q)$	$E_7(q)$	$E_6(q)$
$\mathrm{Spin}_{2n}^{\pm}(q)$	$(n \geq 4)$	$\mathrm{Spin}_{2n-1}(q)$	$E_8(q)$	$E_7(q)$

Observe that the case $p = 2$ was already dealt with in [2, Thm. 4.5].

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Lehrstuhl D für Mathematik, RWTH Aachen, Pontdriesch 14/16, 52062 Aachen, Germany.

E-mail address: `Frank.Luebeck@math.rwth-aachen.de`

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.

E-mail address: `malle@mathematik.uni-kl.de`