ON PRO-\(p\)-IWAHORI INVARIANTS OF \(R\)-REPRESENTATIONS OF REDUCTIVE \(p\)-ADIC GROUPS

N. ABE, G. HENNIART, AND M.-F. VIGNÉRAS

Abstract. Let \(F\) be a locally compact field with residue characteristic \(p\), and let \(G\) be a connected reductive \(F\)-group. Let \(U\) be a pro-\(p\) Iwahori subgroup of \(G = G(F)\). Fix a commutative ring \(R\). If \(\pi\) is a smooth \(R[G]\)-representation, the space of invariants \(\pi^U\) is a right module over the Hecke algebra \(H\) of \(U\) in \(G\).

Let \(P\) be a parabolic subgroup of \(G\) with a Levi decomposition \(P = MN\) adapted to \(U\). We complement a previous investigation of Ollivier-Vignéras on the relation between taking \(U\)-invariants and various functor like \(\text{Ind}^G_P\) and right and left adjoints. More precisely the authors’ previous work with Herzig introduced representations \(I_G(P, \sigma, Q)\) where \(\sigma\) is a smooth representation of \(M\) extending, trivially on \(N\), to a larger parabolic subgroup \(P(\sigma)\), and \(Q\) is a parabolic subgroup between \(P\) and \(P(\sigma)\). Here we relate \(I_G(P, \sigma, Q)^U\) to an analogously defined \(H\)-module \(I_H(P, \sigma^U_M, Q)\), where \(U_M = U \cap M\) and \(\sigma^U_M\) is seen as a module over the Hecke algebra \(H_M\) of \(U_M\) in \(M\). In the reverse direction, if \(V\) is a right \(H_M\)-module, we relate \(I_H(P, V, Q) \otimes c\text{-Ind}^G_{U_M} 1\) to \(I_G(P, V \otimes H_M c\text{-Ind}^M_{U_M} 1, Q)\). As an application we prove that if \(R\) is an algebraically closed field of characteristic \(p\), and \(\pi\) is an irreducible admissible representation of \(G\), then the contragredient of \(\pi\) is 0 unless \(\pi\) has finite dimension.

Contents

1. Introduction 119
2. Notation, useful facts, and preliminaries 122
3. Pro-\(p\) Iwahori invariants of \(I_G(P, \sigma, Q)\) 124
4. Hecke module \(I_H(P, V, Q)\) 134
5. Universal representation \(I_H(P, V, Q) \otimes_H R[U\backslash G]\) 150
6. Vanishing of the smooth dual 156
References 159

1. Introduction

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field \(F\) of residue characteristic \(p\) and a connected reductive \(F\)-group \(G\). We fix a commutative ring \(R\) and study the smooth \(R\)-representations of \(G = G(F)\).

Received by the editors March 14, 2018, and, in revised form, June 17, 2018.
2010 Mathematics Subject Classification. Primary 20C08; Secondary 11F70.
Key words and phrases. Parabolic induction, pro-\(p\) Iwahori Hecke algebra.
The first-named author was supported by JSPS KAKENHI Grant Number 26707001.

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In [AHHV17] the irreducible admissible \( R \)-representations of \( G \) are classified in terms of supersingular ones when \( R \) is an algebraically closed field of characteristic \( p \). That classification is expressed in terms of representations \( I_G(P, \sigma, Q) \), which make sense for any \( R \). In that notation, \( P \) is a parabolic subgroup of \( G \) with a Levi decomposition \( P = MN \) and \( \sigma \) a smooth \( R \)-representation of the Levi subgroup \( M \); there is a maximal parabolic subgroup \( P(\sigma) \) of \( G \) containing \( P \) to which \( \sigma \) inflated to \( P \) extends to a representation \( e_{P(\sigma)}(\sigma) \), and \( Q \) is a parabolic subgroup of \( G \) with \( P \subset Q \subset P(\sigma) \). Then

\[
I_G(P, \sigma, Q) = \text{Ind}^G_P(e_{P(\sigma)}(\sigma) \otimes \text{St}_{Q}^{P(\sigma)}),
\]

where \( \text{Ind} \) stands for parabolic induction and \( \text{St}_{Q}^{P(\sigma)} = \text{Ind}^{P(\sigma)}_{Q} R/ \sum \text{Ind}^{P(\sigma)}_{Q} R \), the sum being over parabolic subgroups \( Q' \) of \( G \) with \( Q \subset Q' \subset P(\sigma) \). Alternatively, \( I_G(P, \sigma, Q) \) is the quotient of \( Ind^G_P(e_Q(\sigma)) \) by \( \sum Ind^G_{Q'} e_{Q'}(\sigma) \) with \( Q' \) as above, where \( e_Q(\sigma) \) is the restriction of \( e_{P(\sigma)}(\sigma) \) to \( Q \), similarly for \( Q' \).

In [AHV] we mainly studied what happens to \( \text{Ind}^G_P(\sigma)\mid_{\sigma'} \) of \( U \)-invariants, as a right module over the Hecke algebra \( H = H_G \) of \( U \) in \( G \) - the convolution algebra on the double coset space \( \mathcal{U} \backslash G / \mathcal{U} \) - in terms of the module \( \sigma^{U, \mathcal{U}} \) over the Hecke algebra \( H_M \) of \( \mathcal{U}_M \) in \( M \). That goal is achieved in section 4 Theorem 4.17.

1.2. The initial work has been done in [OV17] §4 where \( (\text{Ind}^G_{P} \sigma)^{U, \mathcal{U}} \) is identified. Precisely, writing \( M^+ \) for the monoid of elements \( m \in M \) with \( m(U \cap N)m^{-1} \subset U \cap N \), the subalgebra \( H_{M^+} \) of \( H_M \) with support in \( M^+ \), has a natural algebra embedding \( \theta \) into \( H \) and [OV17] Proposition 4.4] identifies \( (\text{Ind}^G_{P} \sigma)^{U, \mathcal{U}} \) with \( \text{Ind}^H_{H_{M^+}}(\sigma^{U, \mathcal{U}}) = \sigma^{U, \mathcal{U}} \otimes H_{M^+} \). So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17] §5.

As \( I_G(P, \sigma, Q) \) is only a subquotient of \( \text{Ind}^G_{P} \sigma \) and taking \( U \)-invariants is only left exact, it is not straightforward to describe \( I_G(P, \sigma, Q)^{U, \mathcal{U}} \) from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume \( P(\sigma) = G \) so that \( I_G(P, \sigma, Q) = e_G(\sigma) \otimes \text{St}_{Q}^{G} \). The crucial case is when moreover \( \sigma \) is \( e \)-minimal, that is, not an extension \( e_{M}(\tau) \) of a smooth \( R \)-representation \( \tau \) of a proper Levi subgroup of \( M \). That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal \( F \)-split torus \( T \) in \( G \) and a minimal parabolic subgroup \( B = ZU \) with Levi component \( Z \) the \( G \)-centralizer of \( T \). We assume that \( P = MN \) contains \( B \) and \( M \) contains \( Z \), and that \( U \) corresponds to an alcove in the apartment associated to \( T \) in the adjoint building of \( G \). It turns out that when \( \sigma \) is \( e \)-minimal and \( P(\sigma) = G \), the set \( \Delta_M \) of simple roots of \( T \) in \( \text{Lie}(M \cap U) \) is orthogonal to its complement in the set \( \Delta \) of simple roots of \( T \) in \( \text{Lie}U \). We assume until the end of this section that \( \Delta_M \) and \( \Delta_2 = \Delta \setminus \Delta_M \) are orthogonal. If \( M_2 \) is the Levi subgroup - containing \( Z \) - corresponding to \( \Delta_2 \), both \( M \) and \( M_2 \) are normal in \( G \), \( M \cap M_2 = Z \) and \( G = MM_2 \). Moreover the normal subgroup \( M_2' \) of \( G \) generated by \( N \) is included in \( M_2 \) and \( G = MM_2' \).
We say that a right $H_M$-module $V$ is extensible to $H$ if $T_2^M$ acts trivially on $V$ for $z \in Z \cap M'_2$ (section 3.3). In this case, we show that there is a natural structure of right $H$-module $e_H(V)$ on $V$ such that $T_g \in H$ corresponding to $UgU$ for $g \in M'_2$ acts as in the trivial character of $G$ (section 3.4). We call $e_H(V)$ the extension of $V$ to $H$ though $H_M$ is not a subalgebra of $H$. That notion is already present in [Abe] in the case where $R$ has characteristic $p$. Here we extend the construction to any $R$ and prove some more properties. In particular we produce an $H$-equivariant embedding $e_H(V)$ into $\text{Ind}^H_{H_M}(V)$ (Lemma 3.10). If $Q$ is a parabolic subgroup of $G$ containing $P$, we go further and put on $e_H(V) \otimes_R (\text{Ind}^G_Q R)^{\mu}$ and $e_H(V) \otimes_R (\text{St}^G_Q)^{\mu}$ structures of $H$-modules (Proposition 3.15 and Corollary 3.17) - note that $H$ is not a group algebra and there is no obvious notion of tensor product of $H$-modules.

If $\sigma$ is an $R$-representation of $M$ extensible to $G$, then its extension $e_G(\sigma)$ is simply obtained by letting $M'_2$ act trivially on the space of $\sigma$; moreover it is clear that $\sigma^{\mu_M}$ is extensible to $H$, and one shows easily that $e_G(\sigma)^{\mu} = e_H(\sigma^{\mu_M})$ as an $H$-module (section 3.5). Moreover, the natural inclusion of $e_G(\sigma)$ into $\text{Ind}^G_Q \sigma$ induces on taking pro-$p$ Iwahori invariants an embedding $e_H(\sigma^{\mu_M}) \rightarrow (\text{Ind}^G_Q \sigma)^{\mu}$ which, via the isomorphism of [OV17], yields exactly the above embedding of $H$-modules of $e_H(\sigma^{\mu_M})$ into $\text{Ind}^H_{H_M}(\sigma^{\mu_M})$.

Then we show the $H$-modules $(e_G(\sigma) \otimes_R \text{Ind}^G_Q R)^{\mu}$ and $e_H(\sigma^{\mu_M}) \otimes_R (\text{Ind}^G_Q R)^{\mu}$ are equal, and similarly $(e_G(\sigma) \otimes_R \text{St}^G_Q)^{\mu}$ and $e_H(\sigma^{\mu_M}) \otimes_R (\text{St}^G_Q)^{\mu}$ are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal. Nevertheless, given a right $H_M$-module $V$, there exists a largest Levi subgroup $M(V)$ of $G$ containing $Z$ corresponding to $\Delta_M \cup \Delta_1$ where $\Delta_1$ is a subset of $\Delta \setminus \Delta_M$ orthogonal to $\Delta_M$, such that $V$ extends to a right $H_M(V)$-module $e_{M(V)}(V)$ with the notation of section 1.3. For any parabolic subgroup $Q$ between $P$ and $P(V) = M(V)U$ we put (Definition 4.12)

$$I_H(P, V, Q) = \text{Ind}^H_{H_M}(e_{M(V)}(V) \otimes_R (\text{St}^M_{Q \cap M(V)})^{\mu_{M(V)})}.$$  

We refer to Theorem 4.17 for the description of the right $H$-module $I_G(P, \sigma, Q)^{\mu}$ for any smooth $R$-representation $\sigma$ of $U$. As a special case, it says that when $\sigma$ is $e$-minimal then $P(\sigma) \supset P(\sigma^{\mu_M})$ and if moreover $P(\sigma) = P(\sigma^{\mu_M})$, then $I_G(P, \sigma, Q)^{\mu}$ is isomorphic to $I_H(P, \sigma^{\mu_M}, Q)$.

Remark 1.1. In [Abe] are attached similar $H$-modules to $(P, V, Q)$; here we write them as $CI_H(P, V, Q)$ because their definition uses, instead of $\text{Ind}^H_{H_M}$, a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when $R$ is an algebraically closed field of characteristic $p$, a classification of simple $H$-modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible $R$-representations of $G$ in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b] 4.3 for any $R$, our Corollary 4.24 expresses $CI_H(P, V, Q)$ as a module $I_H(P_1, V_1, Q_1)$; consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules $I_H(P, V, Q)$.

1.5. In a reverse direction one can associate to a right $H$-module $V$ a smooth $R$-representation $V \otimes_H R[U \setminus G]$ of $G$ (seeing $H$ as the endomorphism ring of the $R[G]$-module $R[U \setminus G]$).
If $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we construct, again using [OV17], a natural $R[G]$-map
\[
I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[U \setminus G] \rightarrow \text{Ind}^G_P(e_{M(\mathcal{V})}(\mathcal{V} \otimes_{\mathcal{H}_M} R[U \setminus M])) \otimes_{R} \text{St}^M_{Q(\mathcal{V})},
\]
with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the $\mathbb{Z}$-torsion in $\mathcal{V}$; in particular it is an isomorphism if $p = 0$ in $R$.

1.6. In the final section 6 we turn back to the case where $R$ is an algebraically closed field of characteristic $p$. We prove that the smooth dual of an irreducible admissible $R$-representation $V$ of $G$ is 0 unless $V$ is finite dimensional - that result is new if $F$ has positive characteristic, a case where the proof of Kohlhaase [Koh] for $\text{char}(F) = 0$ does not apply. Our proof first reduces to the case where $V$ is supercuspidal (by [AHIV17]) then uses again the $\mathcal{H}$-module $V^H$.

2. Notation, useful facts, and preliminaries

2.1. The group $G$ and its standard parabolic subgroups $P = MN$. In all that follows, $p$ is a prime number and $F$ is a local field with finite residue field $k$ of characteristic $p$. We denote an algebraic group over $F$ by a bold letter, like $\mathbf{H}$, and use the same ordinary letter for the group of $F$-points, $H = \mathbf{H}(F)$. We fix a connected reductive $F$-group $\mathbf{G}$. We fix a maximal $F$-split torus $\mathbf{T}$ and write $\mathbf{Z}$ for its $\mathbf{G}$-centralizer; we also fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi component $\mathbf{Z}$, so that $\mathbf{B} = \mathbf{ZU}$ where $U$ is the unipotent radical of $\mathbf{B}$. Let $X^*(\mathbf{T})$ be the group of $F$-rational characters of $\mathbf{T}$ and let $\Phi$ be the subset of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{G}$. Then $\mathbf{B}$ determines a subset $\Phi^+$ of positive roots - the roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$- and a subset of simple roots $\Delta$. The $\mathbf{G}$-normalizer $\mathbf{N}_G$ of $\mathbf{T}$ acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N}_G/\mathbf{Z}$ identifies with the Weyl group of the root system $\Phi$. Set $\mathcal{N} := \mathbf{N}_G(F)$ and note that $\mathbf{N}_G/\mathbf{Z} \simeq \mathcal{N}/\mathbf{Z}$; we write $\mathcal{W}$ for $\mathcal{N}/\mathbf{Z}$.

A standard parabolic subgroup of $\mathbf{G}$ is a parabolic $F$-subgroup containing $\mathbf{B}$. Such a parabolic subgroup $\mathbf{P}$ has a unique Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$, so that $\mathbf{P} = \mathbf{MN}$ where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ - we also call $\mathbf{M}$ standard. By a common abuse of language to describe the preceding situation, we simply say “let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic subgroup of $\mathbf{G}$”; we sometimes write $\mathbf{N}_P$ for $\mathbf{N}$ and $\mathbf{M}_P$ for $\mathbf{M}$. The parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{P}$ will be written $\mathbf{P}^\circ$ and its unipotent radical $\mathbf{N}_\mathbf{P}$, so that $\mathbf{P} = \mathbf{MN}$, but beware that $\mathbf{P}$ is not standard! We write $\mathcal{W}_M$ for the Weyl group $(\mathcal{M} \cap \mathcal{N})/\mathbf{Z}$.

If $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $\mathbf{G}$, then $\mathbf{M} \cap \mathbf{B}$ is a minimal parabolic subgroup of $\mathbf{M}$. If $\Phi_\mathbf{M}$ denotes the set of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{M}$, with respect to $\mathbf{M} \cap \mathbf{B}$ we have $\Phi^+_\mathbf{M} = \Phi_\mathbf{M} \cap \Phi^+$ and $\Delta_\mathbf{M} = \Phi_\mathbf{M} \cap \Delta$. We also write $\Delta_\mathbf{P}$ for $\Delta_\mathbf{M}$ as $\mathbf{P}$ and $\mathbf{M}$ determine each other, $\mathbf{P} = \mathbf{MU}$. Thus we obtain a bijection $\mathbf{P} \mapsto \Delta_\mathbf{P}$ from standard parabolic subgroups of $\mathbf{G}$ to subsets of $\Delta$, with $\mathbf{B}$ corresponding to $\Phi$ and $\mathbf{G}$ to $\Delta$. If $I$ is a subset of $\Delta$, we sometimes denote by $\mathbf{P}_I = \mathbf{M}_I \mathbf{N}_I$ the corresponding standard parabolic subgroup of $\mathbf{G}$. If $I = \{\emptyset\}$ is a singleton, we write $\mathbf{P}_\emptyset = \mathbf{M}_\emptyset \mathbf{N}_\emptyset$. We note a few useful properties. If $\mathbf{P}_1$ is another standard parabolic subgroup of $\mathbf{G}$, then $\mathbf{P} \subset \mathbf{P}_1$ if and only if $\Delta_\mathbf{P} \subset \Delta_{\mathbf{P}_1}$; we have $\Delta_{\mathbf{P} \cap \mathbf{P}_1} = \Delta_\mathbf{P} \cap \Delta_{\mathbf{P}_1}$ and the parabolic subgroup corresponding to $\Delta_\mathbf{P} \cup \Delta_{\mathbf{P}_1}$ is the subgroup $(\mathbf{P}, \mathbf{P}_1)$ of $G$ generated by $\mathbf{P}$ and $\mathbf{P}_1$. The standard parabolic subgroup of $\mathbf{M}$ associated to $\Delta_\mathbf{M} \cap \Delta_{\mathbf{M}_I}$ is $\mathbf{M} \cap \mathbf{P}_1 = (\mathcal{M} \cap \mathbf{M}_1)(\mathcal{M} \cap \mathbf{N}_1)$ [Car85] Proposition 2.8.9]. It is convenient to write $G'$ for the subgroup of $G$ generated by the unipotent
radicals of the parabolic subgroups; it is also the normal subgroup of $G$ generated by $U$, and we have $G = ZG'$. For future reference, we now give a useful lemma extracted from [AHHV17].

**Lemma 2.1.** The group $Z \cap G'$ is generated by the $Z \cap M'_\alpha$, $\alpha$ running through $\Delta$.

*Proof.* Take $I = \emptyset$ in [AHHV17 II.6.Proposition]. \hfill \Box

Let $v_F$ be the normalized valuation of $F$. For each $\alpha \in X^*(T)$, the homomorphism $x \mapsto v_F(\alpha(x)) : T \rightarrow \mathbb{Z}$ extends uniquely to a homomorphism $Z \rightarrow \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \twoheadrightarrow X_*(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = v_F(\alpha(z))$ for $z \in Z$, $\alpha \in X^*(T)$.

An interesting situation occurs when $\Delta = I \cup J$ is the union of two orthogonal subsets $I$ and $J$. In that case, $G' = M'_1 M'_J$, $M'_1$ and $M'_J$ commute with each other, and their intersection is finite and central in $G$ [AHHV17 II.7 Remark 4].

### 2.2. $I_G(P, \sigma, Q)$ and minimality

We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Let $\sigma$ be an $R$-representation of $M$ and let $P_\sigma$ be the standard parabolic subgroup with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{ \alpha \in \Delta \mid \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma \}.$$ 

We also let $P(\sigma)$ be the standard parabolic subgroup with $\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P$.

This is the largest parabolic subgroup $P(\sigma)$ containing $P$ to which $\sigma$ extends, here $N \subset P$ acts on $\sigma$ trivially. Clearly when $P \subset Q \subset P(\sigma)$, $\sigma$ extends to $Q$ and the extension is denoted by $e_Q(\sigma)$. The restriction of $e_{P(\sigma)}(\sigma)$ to $Q$ is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

**Definition 2.2.** An $R[G]$-triple is a triple $(P, \sigma, Q)$ made out of a standard parabolic subgroup $P = MN$ of $G$, a smooth $R$-representation of $M$, and a parabolic subgroup $Q$ of $G$ with $P \subset Q \subset P(\sigma)$. To an $R[G]$-triple $(P, \sigma, Q)$ is associated a smooth $R$-representation of $G$:

$$I_G(P, \sigma, Q) = \text{Ind}^G_{P(\sigma)}(e(\sigma) \otimes \text{St}_Q^{P(\sigma)}),$$

where $\text{St}_Q^{P(\sigma)}$ is the quotient of $\text{Ind}_Q^{P(\sigma)} 1$, $1$ denoting the trivial $R$-representation of $Q$, by the sum of its subrepresentations $\text{Ind}_Q^{P(\sigma)} 1$, the sum being over the set of parabolic subgroups $Q'$ of $G$ with $Q \subseteq Q' \subset P(\sigma)$.

Note that $I_G(P, \sigma, Q)$ is naturally isomorphic to the quotient of $\text{Ind}_Q^G(e_Q(\sigma))$ by the sum of its subrepresentations $\text{Ind}_{Q'}^G(e_Q(\sigma))$ for $Q \subset Q' \subset P(\sigma)$ by [AHHV Lemma 2.5].

It might happen that $\sigma$ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1 = M_1 N_1$ contained in $P$ and some $R$-representation $\sigma_1$ of $M_1$. In that case, $P(\sigma_1) = P(\sigma)$ and $e(\sigma) = e(\sigma_1)$. We say that $\sigma$ is $e$-minimal if $\sigma = e_P(\sigma_1)$ implies $P_1 = P, \sigma_1 = \sigma$. 
Lemma 2.3 ([AHV Lemma 2.9]). Let $P = MN$ be a standard parabolic subgroup of $G$ and let $\sigma$ be an $R$-representation of $M$. There exists a unique standard parabolic subgroup $P_{\min, \sigma} = M_{\min, \sigma}N_{\min, \sigma}$ of $G$ and a unique $e$-minimal representation of $\sigma_{\min}$ of $M_{\min, \sigma}$ with $\sigma = e_P(\sigma_{\min})$. Moreover $P(\sigma) = P(\sigma_{\min})$ and $e(\sigma) = e(\sigma_{\min})$.

Lemma 2.4. Let $P = MN$ be a standard parabolic subgroup of $G$ and let $\sigma$ be an $e$-minimal $R$-representation of $M$. Then $\Delta_P$ and $\Delta_P(\sigma)$ are orthogonal.

That comes from [AHHV17 II.7 Corollary 2]. That corollary of [AHHV17] also shows that when $R$ is a field and $\sigma$ is supercuspidal, then $\sigma$ is $e$-minimal. Lemma 2.4 shows that $\Delta_{P_{\min, \sigma}}$ and $\Delta_{P(\sigma_{\min}) \setminus \Delta_{P_{\min, \sigma}}}$ are orthogonal.

Note that when $\Delta_P$ and $\Delta_\sigma$ are orthogonal of union $\Delta = \Delta_P \sqcup \Delta_\sigma$, then $G = P(\sigma) = MM'$ and $e(\sigma)$ is the $R$-representation of $G$ simply obtained by extending $\sigma$ trivially on $M'$.

Lemma 2.5 ([AHV Lemma 2.11]). Let $(P, \sigma, Q)$ be an $R[G]$-triple. Then we have that $(P_{\min, \sigma}, \sigma_{\min}, Q)$ is an $R[G]$-triple and $I_G(P, \sigma, Q) = I_G(P_{\min, \sigma}, \sigma_{\min}, Q)$.

2.3. Pro-$p$ Iwahori Hecke algebras. We fix a standard parahoric subgroup $K$ of $G$ fixing a special vertex $x_0$ in the apartment $A$ associated to $T$ in the Bruhat-Tits building of the adjoint group of $G$. We let $B$ be the Iwahori subgroup fixing the alcove $C$ in $A$ with vertex $x_0$ contained in the Weyl chamber (of vertex $x_0$) associated to $B$. Let $U$ be the pro-$p$ radical of $B$ (the pro-$p$ Iwahori subgroup). The pro-$p$ Iwahori Hecke ring $H = H(G, U)$ is the convolution ring of compactly supported functions $G \to Z$ constant on the double classes of $G$ modulo $U$. We denote by $T(g)$ the characteristic function of $UgU$ for $g \in G$, seen as an element of $H$. Let $R$ be a commutative ring. The pro-$p$ Iwahori Hecke $R$-algebra $H_R$ is $R \otimes_Z H$. We will follow the custom to still denote by $h$ the natural image $1 \otimes h$ of $h \in H$ in $H_R$.

For $P = MN$ a standard parabolic subgroup of $G$, the similar objects for $M$ are indexed by $M$, we have $K_M = K \cap M, B_M = B \cap M, U_M = U \cap M$, the pro-$p$ Iwahori Hecke ring $H_M = H(M, U_M), T^M(m) \in H_M$ the characteristic function of $U_M m U_M$, for $m \in M$, seen as an element of $H_M$. The pro-$p$ Iwahori subgroup $U$ of $G$ satisfies the Iwahori decomposition with respect to $P$:

$$U = U_N U_M U_N,$$

where $U_N = U \cap N, U_N^{-1} = U \cap N$. The linear map

$$H_M \xrightarrow{\theta} H, \quad \theta(T^M(m)) = T(m) \quad (m \in M)$$

does not respect the product. But if we introduce the monoid $M^+$ of elements $m \in M$ contracting $U_N$, meaning $mU_Nm^{-1} \subset U_N$, and the submodule $H_{M^+} \subset H_M$ of functions with support in $M^+$, we have [Vig15b Theorem 1.4]:

$H_{M^+}$ is a subring of $H_M$ and $H_{M^+}$ is the localization of $H_{M^+}$ at an element $\tau^M \in H_M$ central and invertible in $H_M$, meaning $H_M = \bigcup_{n \in \mathbb{N}} H_{M^+}(\tau^M)^{-n}$. The map $H_M \xrightarrow{\theta} H$ is injective and its restriction $\theta|_{H_{M^+}}$ to $H_{M^+}$ respects the product.

These properties are also true when $(M^+, \tau^M)$ is replaced by its inverse $(M^-, (\tau^M)^{-1})$ where $M^- = \{m^{-1} \in M \mid m \in M^+\}$.

3. Pro-$p$ Iwahori invariants of $I_G(P, \sigma, Q)$

3.1. Pro-$p$ Iwahori Hecke algebras: Structures. Here we supplement the notation of sections 2.1 and 2.3. The subgroups $Z^0 = Z \cap K = Z \cap B$ and $Z^1 = Z \cap U$
are normal in $\mathcal{N}$ and we put

$$W = \mathcal{N}/Z^0, \quad W(1) = \mathcal{N}/Z^1, \quad \Lambda = Z/Z^0, \quad \Lambda(1) = Z/Z^1, \quad Z_k = Z^0/Z^1.$$  

We have $\mathcal{N} = (\mathcal{N} \cap \mathcal{K})Z$ so that we see the finite Weyl group $\mathbb{W} = \mathcal{N}/Z$ as the subgroup $(\mathcal{N} \cap \mathcal{K})/Z^0$ of $W$; in this way $W$ is the semidirect product $\Lambda \rtimes \mathbb{W}$. We put $\mathbb{N}_G = \mathcal{N} \cap G'$. The image $W_G = W'$ of $\mathbb{N}_G$ in $W$ is an affine Weyl group generated by the set $S^{\text{aff}}$ of affine reflections determined by the walls of the alcove $\mathcal{C}$. The group $W'$ is normal in $W$ and $W$ is the semidirect product $W' \rtimes \Omega$ where $\Omega$ is the image in $W$ of the normalizer $\mathbb{N}_G$ of $\mathcal{C}$ in $\mathcal{N}$. The length function $\ell$ on the affine Weyl system $(W', S^{\text{aff}})$ extends to a length function on $W$ such that $\Omega$ is the set of elements of length $0$. We also view $\ell$ as a function of $W(1)$ via the quotient map $W(1) \to W$. We write

$$\begin{align*}
(3.1) \quad \hat{w}, \hat{w}, w \in \mathcal{N} \times W(1) \times W \text{ corresponding via the quotient maps } \mathcal{N} \to W(1) \to W.
\end{align*}$$

When $w = s$ in $S^{\text{aff}}$ or more generally $w$ in $W_G$, we will most of the time choose $\hat{w}$ in $\mathcal{N} \cap G'$ and $\hat{w}$ in the image $\mathbb{N}_G$ of $\mathcal{N} \cap G'$ in $W(1)$.

We need now to describe the pro-$p$ Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, U)$ [Vig16]. We have $G = UNU$ and for $n, n' \in \mathcal{N}$ we have $UnU = Un'U$ if and only if $nZ^1 = n'Z^1$. For $n = \mathcal{N}$ of image $w \in W(1)$ and $g \in \mathbb{U}$ we denote $T_w = T(n) = T(g)$ in $\mathcal{H}$. The relations among the basis elements $(T_w)_{w \in W(1)}$ of $\mathcal{H}$ are:

(1) Braid relations: $T_wT_{w'} = T_{ww'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

(2) Quadratic relations: $T_{\hat{s}}^2 = q_sT_{\hat{s}} + c_sT_{\hat{s}}$ for $\hat{s} \in W(1)$ lifting $s \in S^{\text{aff}}$, where $q_s = q_G(s) = |U/U \cap \hat{s}U(\hat{s})^{-1}|$ depends only on $s$, and $c_s = \sum_{t \in Z_k} c_s(t)T_t$ for integers $c_s(t) \in \mathbb{N}$ summing to $q_s - 1$.

We shall need the basis elements $(T_w)_{w \in W(1)}$ of $\mathcal{H}$ defined by:

(1) $T_w = T_w$ for $w \in W(1)$ of length $\ell(w) = 0$.

(2) $T_{\hat{s}} = T_{\hat{s}} - c_s$ for $\hat{s} \in W(1)$ lifting $s \in S^{\text{aff}}$.

(3) $T_{ww'} = T_wT_{w'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

We need more notation for the definition of the admissible lifts of $S^{\text{aff}}$ in $\mathcal{N}_G$. Let $s \in S^{\text{aff}}$ fixing a face $\mathcal{C}_s$ of the alcove $\mathcal{C}$ and $\mathcal{K}_s$ the parahoric subgroup of $G$ fixing $\mathcal{C}_s$. The theory of Bruhat-Tits associates to $\mathcal{C}_s$ a certain root $\alpha_s \in \Phi^{+}$ [Vig16 §4.2]. We consider the group $G'_s$ generated by $U_{\alpha_s} \cup U_{-\alpha_s}$, where $U_{\pm \alpha_s}$, the root subgroup of $\pm \alpha_s$ (if $2\alpha_s \in \Phi$, then $U_{2\alpha_s} \subset U_{\alpha_s}$) and the group $G'_s$ generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm \alpha_s} = U_{\pm \alpha_s} \cap \mathcal{K}_s$. When $u \in U_{\alpha_s} \setminus \{1\}$, the intersection $\mathcal{N}_G \cap U_{-\alpha_s}uU_{-\alpha_s}$ (equal to $\mathcal{N}_G \cap U_{-\alpha_s}uU_{-\alpha_s}$ [BT72 6.2.1 (V5)], [Vig16 §3.3 (19)]) possesses a single element $n_\alpha(u)$. The group $Z'_k = Z \cap G'_s$ is contained in $Z \cap \mathcal{K}_s = Z^0$; its image in $Z_k$ is denoted by $Z'_{k,s}$.

The elements $n_\alpha(u)$ for $u \in U_{\alpha_s} \setminus \{1\}$ are the admissible lifts of $s$ in $\mathcal{N}_G$; their images in $W(1)$ are the admissible lifts of $s$ in $W(1)$. By [Vig16] Theorem 2.2, Proposition 4.4, when $\hat{s} \in W(1)$ is an admissible lift of $s$, $c_s(t) = 0$ if $t \in Z_k \setminus Z'_{k,s}$, and

$$c_s \equiv (q_s - 1)|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \mod p.$$  

The admissible lifts of $S$ in $\mathcal{N}_G$ are contained in $\mathcal{N}_G \cap \mathcal{K}$ because $\mathcal{K}_s \subset \mathcal{K}$ when $s \in S$. 
Definition 3.1. An admissible lift of the finite Weyl group \( \mathbb{W} \) in \( N_G \) is a map
\[ w \mapsto \tilde{w} : \mathbb{W} \to N_G \cap \mathcal{K} \]
such that \( \tilde{s} \) is admissible for all \( s \in S \) and \( \tilde{w} = \tilde{w}_1 \tilde{w}_2 \) for \( w_1, w_2 \in \mathbb{W} \) such that \( w = w_1 w_2 \) and \( \ell(w) = \ell(w_1) + \ell(w_2) \).

Any choice of admissible lifts of \( S \) in \( N_G \cap \mathcal{K} \) extends uniquely to an admissible lift of \( \mathbb{W} \) ([AHHV17 IV.6], [OV17 Proposition 2.7]).

Let \( P = MN \) be a standard parabolic subgroup of \( G \). The groups \( Z, Z^0 = Z \cap K_M = Z \cap B_M, Z^1 = Z \cap U_M \) are the same for \( G \) and \( M \), but \( N_M = N \cap M \) and \( M \cap G' \) are subgroups of \( N' \) and \( G' \). The monoid \( M^+ \) (section 2.3) contains \( N_M \cap K \) and is equal to \( M^+ = U_M N_M + U_M \) where \( N_M = N \cap M^+ \). An element \( z \in Z \) belongs to \( M^+ \) if and only if \( \nu_F(\alpha(z)) \geq 0 \) for all \( \alpha \in \Phi^+ \setminus \Phi^+_M \) (see [Vig15b Lemme 2.2]). Put \( W_M = N_M / Z^0 \) and \( W_M(1) = N_M / Z^1 \).

Let \( \epsilon = + \) or \( \epsilon = - \). We denote by \( W_{M^*}(1) \) the images of \( N_M \) in \( W_M, W_M(1) \). We see the groups \( W_M, W_M(1), W_{M^*} \) as subgroups of \( W, W(1), W_{G'} \). As \( \theta \) (section 2.3), the linear injective map
\[ \mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \theta^*(T_w^{m,s}) = T_w^{s}, \quad (w \in W_M(1)), \]
respects the product on the subring \( \mathcal{H}_M \). Here \( T_w^{m,s} \in \mathcal{H}_M \) is defined in the same way as \( T_w^m \) for \( \mathcal{H}_M \). Note that \( \theta \) and \( \theta^* \) satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

3.2. Orthogonal case. Let us examine the case where \( \Delta_M \) and \( \Delta \setminus \Delta_M \) are orthogonal, writing \( M_2 = M_{\Delta \setminus \Delta_M} \) as in section 2.3.

From \( M \cap M_2 \) we get \( W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1) \), the semisimple building of \( G \) is the product of those of \( M \) and \( M_2 \). The set \( S_{aff}^{M} \) is the disjoint union of \( S_{aff}^{M_2} \) and \( S_{aff}^{M_2} \). The group \( W_{G'} \) is the direct product of \( W_{M'} \) and \( W_{M_2} \). For \( s \in W_M(1) \) lifting \( s \in S_{aff}^{M} \), the elements \( T_{w}^{m,s} \in \mathcal{H}_M \) and \( T_{w}^{s} \) satisfy the same quadratic relations. A word of caution is necessary for the lengths \( \ell_M \) of \( W_M \) and \( \ell_{M_2} \) of \( W_{M_2} \) different from the restrictions of the length \( \ell \) of \( W_M \), for example \( \ell_M(\lambda) = 0 \) for \( \lambda \in \Lambda \cap W_{M_2} \).

Lemma 3.2. We have \( \Lambda = (W_{M^*} \cap \Lambda)(W_{M_2} \cap \Lambda) \).

Proof. We prove the lemma for \( \epsilon = - \). The case \( \epsilon = + \) is similar. The map \( v : Z \to X_*(T) \otimes \mathbb{Q} \) defined in section 2.4 is trivial on \( Z^0 \) and we also write \( v \) for the resulting homomorphism on \( \Lambda \). For \( \lambda \in \Lambda \) there exists \( \lambda_2 \in W_{M_2} \cap \Lambda \) such that \( \lambda \lambda_2 \in W_{M} \), or equivalently \( \alpha(v(\lambda \lambda_2)) \leq 0 \) for all \( \alpha \in \Phi_+ \setminus \Phi_+^{M_2} \). It suffices to have the inequality for all \( \alpha \in \Delta_{M_2} \). The matrix \( (\alpha(\beta'))_{\alpha, \beta \in \Delta_{M_2}} \) is invertible, hence there exists \( n_\beta \in Z \) such that \( \sum_{\beta \in \Delta_{M_2}} n_\beta \alpha(\beta') \leq -\alpha(v(\lambda_2)) \) for all \( \alpha \in \Delta_{M_2} \). As \( v(W_{M_2} \cap \Lambda) \) contains \( \bigoplus_{\alpha \in \Delta_{M_2}} \mathbb{Z} \alpha \) where \( \alpha' \) is the coroot of \( \alpha \) after formula (71), there exists \( \lambda_2 \in W_{M_2} \cap \Lambda \) with \( v(\lambda_2) = \sum_{\beta \in \Delta_{M_2}} n_\beta \beta' \).

The groups \( N \cap M' \) and \( N \cap M_2' \) are normal in \( N \), and
\[ N = (N \cap M')(N \cap M_2) = Z(N \cap M')(N \cap M_2), \]
and
\[ W = W_M \cap W_{M_2} = W_M W_{M_2} = W_M W_{M_2} W_M W_{M_2} W_M W_{M_2}. \]
The first two equalities are clear, the equality \( W_M W_{M_2} = W_{M^*} W_{M_2} \) follows from \( W_M = \mathbb{W}_M \Lambda, \mathbb{W}_M \subset W_{M^*} \) and the lemma. The inverse image in \( W(1) \) of these
Lemma 3.3. Let \( Y \) be a subgroup of \( N \cap M \). We recall the function \( q(n) = |\mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n)| \) on \( N \), Proposition 3.38] and we extend to \( N \) the functions \( q_M \) on \( N \cap M \) and \( q_{M'} \) on \( N \cap M_2 \):

\[
q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M \cap n^{-1} \mathcal{U}_M n)|, \quad q_{M'}(n) = |\mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n^{-1} \mathcal{U}_{M'} n)|.
\]

The functions \( q, q_M, q_{M'} \) descend to functions on \( W(1) \) and on \( W \), also denoted by \( q, q_M, q_{M'} \).

**Proof.** We put \( \mathcal{U}_{M'} = \mathcal{U} \cap M' \) and \( \mathcal{U}_{M_2} = \mathcal{U} \cap M_2 \). The product map

\[
(3.6) \quad Z^1 \prod_{\alpha \in \Phi_{M, \text{red}}} \mathcal{U}_\alpha \prod_{\alpha \in \Phi_{M_2, \text{red}}} \mathcal{U}_\alpha \to \mathcal{U}
\]

with \( \mathcal{U}_\alpha = \mathcal{U}_\alpha \cap \mathcal{U} \) is a homeomorphism. We have \( \mathcal{U}_M = Z^1 \mathcal{Y}_{M'} \), \( \mathcal{U}_{M'} = (Z^1 \cap M') \mathcal{Y}_{M'} \) where \( \mathcal{Y}_{M'} = \prod_{\alpha \in \Phi_{M, \text{red}}} \mathcal{U}_\alpha \) and \( N \cap M_2 \) commutes with \( \mathcal{Y}_{M'} \), in particular \( N \cap M_2 \) normalizes \( \mathcal{Y}_{M'} \). Similar results are true when \( M \) and \( M_2 \) are permuted, and \( \mathcal{U} = \mathcal{U}_M \mathcal{U}_{M_2} = \mathcal{U}_M \mathcal{U}_{M_2} \).

Writing \( N = Z(N \cap M') (N \cap M_2') \) (in any order), we see that the product map

\[
(3.7) \quad Z^1 (\mathcal{Y}_{M'} \cap n^{-1} \mathcal{Y}_{M'} n) (\mathcal{Y}_{M_2} \cap n^{-1} \mathcal{Y}_{M_2} n) \to \mathcal{U} \cap n^{-1} \mathcal{U} n
\]

is a homeomorphism. The inclusions induce bijections

\[
(3.8) \quad \mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1} \mathcal{Y}_{M'} n) \simeq \mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n^{-1} \mathcal{U}_{M'} n) \simeq \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1} \mathcal{U}_M n),
\]

similarly for \( M_2 \), and also a bijection

\[
(3.9) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n) \simeq (\mathcal{Y}_{M_2}'/(\mathcal{Y}_{M_2}' \cap n^{-1} \mathcal{Y}_{M_2}' n)) \times (\mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1} \mathcal{Y}_{M'} n)).
\]

From (3.8) and (3.9), we get

\[
(3.10) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n) \simeq (\mathcal{U}_{M_2}'/(\mathcal{U}_{M_2}' \cap n \mathcal{U}_{M_2}' n^{-1})) \times (\mathcal{U}_M/(\mathcal{U}_M \cap n \mathcal{U}_M n^{-1}))
\]

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since \( N \cap M_2' \) normalizes \( \mathcal{Y}_{M'} \); with (1), it implies the assertion (3).

A subgroup of \( N \) normalizes \( \mathcal{U}_M \) if and only if it normalizes \( \mathcal{Y}_{M'} \) by (3.8) if and only if \( q_{M'} = 1 \) on this group. The group \( N \cap M_2' \) normalizes \( \mathcal{Y}_{M'} \). Therefore the group \( (N \cap M_2') \mathcal{N}_C \) normalizes \( \mathcal{U}_M \). The coset \((N \cap M_2') \mathcal{N}_C\) contains an element \( n_{M'} \in M' \). For \( x \in (N \cap M_2') \mathcal{N}_C \), \((x n_{M'})^{-1} \mathcal{U} x n_{M'} = n_{M'}^{-1} \mathcal{U} n_{M'} \), hence \( q_M(x n_{M'}) = q_M(n_{M'}). \)

\[\square\]
3.3. Extension of an $\mathcal{H}_M$-module to $\mathcal{H}$. This section is inspired by similar results for the pro-$p$ Iwahori Hecke algebras over an algebraically closed field of characteristic $p$ [Abe Proposition 4.16]. We keep the setting of section 3.2 and we introduce ideals:

- $\mathcal{J}_e$ (resp., $\mathcal{J}_r$) the left (resp., right) ideal of $\mathcal{H}$ generated by $T_w^* - 1_{\mathcal{H}}$ for all $w \in 1W_{M_2}$.
- $\mathcal{J}_{M,e}$ (resp., $\mathcal{J}_{M,r}$) the left (resp., right) ideal of $\mathcal{H}_M$ generated by $T_{\lambda}^{M,*} - 1_{\mathcal{H}_M}$ for all $\lambda$ in $1W_{M_2} \cap W_M(1) = 1W_{M_2} \cap \Lambda(1)$.

The next proposition shows that the ideals $\mathcal{J}_e = \mathcal{J}_r$ are equal and similarly $\mathcal{J}_{M,e} = \mathcal{J}_{M,r}$. After the proposition, we will drop the indices $e$ and $r$.

**Proposition 3.4.** The ideals $\mathcal{J}_e$ and $\mathcal{J}_r$ are equal to the submodule $\mathcal{J}'$ of $\mathcal{H}$ generated by $T_w^* - T_{www}^*$ for all $w \in W(1)$ and $w_2 \in 1W_{M_2}$.

The ideals $\mathcal{J}_{M,e}$ and $\mathcal{J}_{M,r}$ are equal to the submodule $\mathcal{J}'_M$ of $\mathcal{H}_M$ generated by $T_{\lambda_2}^{M,*} - T_{\lambda_2}^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in \Lambda(1) \cap 1W_{M_2}$.

**Proof.**

(1) We prove $\mathcal{J}_e = \mathcal{J}'$. Let $w \in W(1), w_2 \in 1W_{M_2}$. We prove by induction on the length of $w_2$ that $T_w^*(T_{w_2}^* - 1) \in \mathcal{J}'$. This is obvious when $\ell(w_2) = 0$ because $T_w^*T_{w_2}^* = T_{www}^*$. Assume that $\ell(w_2) = 1$ and put $s = w_2$. If $\ell(ws) = \ell(w) + 1$, as before $T_w^*(T_s^* - 1) \in \mathcal{J}'$ because $T_w^*T_s^* = T_{ws}^*$. Otherwise $\ell(ws) = \ell(w) - 1$ and $T_w^* = T_{ws-1}^*$. Hence $T_w^*(T_s^* - 1) = T_{ws-1}^*(T_s^* - 1) = T_{ws}^*(q_sT_s^* - T_s^*c_s) - T_w^* = q_sT_{ws}^* - T_w^*(c_s + 1)$. Since $c_s + 1 = \sum_{t \in Z_k}c_s(t)T_t$ with $c_s(t) \in \mathbb{N}$ and $\sum_{t \in Z_k}c_s(t) = q_s$ [Vig16 Proposition 4.4],

$$q_sT_{ws}^* - T_{w_2}^*(c_s + 1) = \sum_{t \in Z_k}c_s(t)(T_{ws}^* - T_w^*T_t^*) = \sum_{t \in Z_k}c_s(t)(T_{ws}^* - T_w^*) \in \mathcal{J}'$$

Assume now that $\ell(w_2) > 1$. Then, we factorize $w_2 = xy$ with $x, y \in 1W_{M_2}$ of length $\ell(x), \ell(y) < \ell(w_2)$ and $\ell(w_2) = \ell(x) + \ell(y)$. The element $T_w^*(T_{w_2}^* - 1) = T_w^*T_x^*(T_{w_2}^* - 1) = T_w^*(T_x^* - 1) \in \mathcal{J}'$. By induction.

Conversely, we factorize $T_{www}^* - T_{e_1}^*$ in $\mathcal{J}_e$. We factorize $w = xy$ with $y \in 1W_{M_2}$ and $x \in 1W_{M_2} \Omega(1)$. Then, we have $\ell(w) = \ell(x) + \ell(y)$ and $\ell(w_2) = \ell(x) + \ell(yw_2)$. Hence $T_{ww}^* - T_e^* = T_x^*(T_{yw}^* - T_y^*) = T_x^*(T_{yw}^* - 1) - T_x^*(T_y^* - 1) \in \mathcal{J}_e$. This ends the proof of $\mathcal{J}_e = \mathcal{J}'$.

By the same argument, the right ideal $\mathcal{J}_r$ of $\mathcal{H}$ is equal to the submodule of $\mathcal{H}$ generated by $T_{ww}^* - T_{w}^*$ for all $w \in W(1)$ and $w_2 \in 1W_{M_2}$. But this latter submodule is equal to $\mathcal{J}'$ because $1W_{M_2}$ is normal in $W(1)$. Therefore we proved $\mathcal{J}' = \mathcal{J}_e = \mathcal{J}_r$.

(2) Proof of the second assertion. We prove $\mathcal{J}_{M,e} = \mathcal{J}'_M$. The proof is easier than in (1) because for $w \in W_M(1)$ and $\lambda_2 \in 1W_{M_2} \cap \Lambda(1)$, we have $\ell(w\lambda_2) = \ell(w) + \ell(\lambda_2)$ hence $T_{\lambda_2}^{M,*}(T_{\lambda_2}^{M,*} - 1) = T_{\lambda_2}^{M,*} - T_w^{M,*}$. We have also $\ell(\lambda_2w) = \ell(\lambda_2) + \ell(w)$ hence $(T_{\lambda_2}^{M,*} - 1)T_{w_2}^{M,*} = T_{\lambda_2}^{M,*} - T_w^{M,*}$ hence $\mathcal{J}_{M,r}$ is equal to the submodule of $\mathcal{H}_M$ generated by $T_{\lambda_2}^{M,*} - T_{\lambda_2}^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in 1W_{M_2} \cap \Lambda(1)$. This latter submodule is $\mathcal{J}'_M$, as $1W_{M_2} \cap \Lambda(1) = 1W_{M_2} \cap W_M(1)$ is normal in $W_M(1)$. Therefore $\mathcal{J}'_M = \mathcal{J}_{M,r} = \mathcal{J}_{M,e}$. 

\[ \square \]
By Proposition 3.4, a basis of $\mathcal{J}$ is $T_w^* - T_{w_2}^*$ for $w$ in a system of representatives of $W(1)/1W_{M'_2}$, and $w_2 \in 1W_{M'_2} \setminus \{1\}$. Similarly a basis of $\mathcal{J}_M$ is $T_{w_2}^* - T_{w_2}^{\lambda_2}$ for $w$ in a system of representatives of $W_M(1)/(\Lambda(1) \cap 1W_{M'_2})$ and $\lambda_2 \in (\Lambda(1) \cap 1W_{M'_2}) \setminus \{1\}$.

**Proposition 3.5.** The natural ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}_M$ and the ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}$ via $\theta^*$ induce ring isomorphisms

$$\mathcal{H}_M / \mathcal{J}_M \arrowsurj \mathcal{H}_M / (\mathcal{J}_M \cap \mathcal{H}_{M^-}) \arrowsurj \mathcal{H} / \mathcal{J}.$$ 

**Proof.**

(1) The left map is obviously injective. We prove the surjectivity. Let $w \in W_M(1)$. Let $\lambda_2 \in 1W_{M'_2} \cap \Lambda(1)$ such that $w\lambda_2^{-1} \in W_{M^-}(1)$ (see (3.4)). We have

$$T_{w\lambda_2^{-1}}^* \in \mathcal{H}_{M^-} \text{ and } T_w^{M^*} = T_{w\lambda_2^{-1}}^{M^*} + T_{w\lambda_2^{-1}}^*(T_{\lambda_2}^{M^*} - 1).$$

Therefore $T_w^{M^*} \in \mathcal{H}_{M^-} + \mathcal{J}_M$. As $w$ is arbitrary, $\mathcal{H}_M = \mathcal{H}_{M^-} + \mathcal{J}_M$.

(2) The right map is surjective: let $w \in W(1)$ and $w_2 \in 1W_{M'_2}$ such that $ww_2^{-1} \in W_{M^-}(1)$ (see (3.4)). Then $T_w^* - T_{w_2}^* \in \mathcal{J}$ with the same arguments as in (1), using Proposition 3.4. Therefore $\mathcal{H} = \theta^*(\mathcal{H}_{M^-}) + \mathcal{J}$.

We prove the injectivity: $\theta^*(\mathcal{H}_{M^-}) \cap \mathcal{J} = \theta^*(\mathcal{H}_M - \mathcal{J}_M)$. Let $\sum_{w \in W_{M^-}(1)} c_w T_w^{M^*}$, with $c_w \in \mathbb{Z}$, be an element of $\mathcal{H}_{M^-}$. Its image by $\theta^*$ is $\sum_{w \in W(1)} c_w T_w^*$ where we have set $c_w = 0$ for $w \in W(1) \setminus W_{M^-}(1)$. We have $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$ if and only if $\sum_{w_2 \in 1W_{M'_2}} c_{w_2} T_{w_2}^* = 0$ for all $w \in W(1)$. If $c_{w_2} \neq 0$, then $w_2 \in 1W_{M'_2} \cap W_M(1)$, that is, $w_2 \in 1W_{M'_2} \cap \Lambda(1)$. The sum $\sum_{w_2 \in 1W_{M'_2}} c_{w_2} T_{w_2}^*$ is equal to $\sum_{\lambda_2 \in 1W_{M'_2} \cap \Lambda(1)} c_{w_2} T_{w_2}^{\lambda_2}$. By Proposition 3.4, $\sum_{w \in W(1)} c_w T_w^{M^*} \in \mathcal{J}_M$ if and only if $\sum_{w \in W_{M^-}(1)} c_w T_w^{M^*} \in \mathcal{J}_M$.

We construct a ring isomorphism

$$e^*: \mathcal{H}_M / \mathcal{J}_M \arrowsurj \mathcal{H} / \mathcal{J}$$

by using Proposition 3.5. For any $w \in W(1)$, $T_w^* + \mathcal{J} = e^*(T_w^{M^*} + \mathcal{J}_M)$ where $w_{M^-} \in W_{M^-}(1) \cap w_1W_{M'_2}$ (see (3.4)), because by Proposition 3.4, $T_w^* + \mathcal{J} = T_w^{M^*} + \mathcal{J}$ and $T_w^{M^*} + \mathcal{J} = e^*(T_w^{M^*} + \mathcal{J}_M)$ by construction of $e^*$. We check that $e^*$ is induced by $\theta^*$.

**Theorem 3.6.** The linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ induces a ring isomorphism

$$e^*: \mathcal{H}_M / \mathcal{J}_M \arrowsurj \mathcal{H} / \mathcal{J}.$$ 

**Proof.** Let $w \in W_M(1)$. We have to show that $T_w^* + \mathcal{J} = e^*(T_w^{M^*} + \mathcal{J}_M)$. We saw above that $T_w^* + \mathcal{J} = e^*(T_w^{M^*} + \mathcal{J}_M)$ with $w = w_{M^-} - \lambda_2$ with $\lambda_2 \in 1W_{M'_2} \cap W_{M}(1)$. As $\ell_M(\lambda_2) = 0$, $T_w^{M^*} = T_w^{M^*} - T_{\lambda_2}^{M^*} \in T_w^{M^*} + \mathcal{J}_M$. Therefore $T_w^{M^*} + \mathcal{J}_M = T_w^{M^*} + \mathcal{J}_M$. This ends the proof of the theorem.

We now wish to compute $e^*$ in terms of the $T_w$ instead of the $T_w^*$.

**Proposition 3.7.** Let $w \in W(1)$. Then, $T_w + \mathcal{J} = e^*(T_{wM}^M q_{M_2}(w) + \mathcal{J}_M)$ for any $w_M \in 1W_{M}(1) \cap w_1W_{M'_2}$.
Proof. The element $w_M$ is unique modulo right multiplication by an element $\lambda_2 \in W_M(1) \cap W_{M'}'$ of length $\ell_M(\lambda_2) = 0$ and $T^{M*}_w q_M(w) + \mathcal{J}_M$ does not depend on the choice of $w_M$. We choose a decomposition (see (3.4)):
\[
w = \tilde{s}_1 \ldots \tilde{s}_a u \tilde{s}_{a+1} \ldots \tilde{s}_{a+b}, \quad \ell(w) = a + b,
\]
for $u \in \Omega(1)$, $\tilde{s}_i \in 1 W_{M'}$ lifting $s_i \in S^\text{aff}_M$ for $1 \leq i \leq a$ and $\tilde{s}_i \in 1 W_{M'}'$ lifting $s_i \in S^\text{aff}_{M_2}$ for $a+1 \leq i \leq a + b$, and we choose $u_M \in W_{M'}(1)$ such that $u \in u_M 1 W_{M_2'}$. Then
\[
w_M = \tilde{s}_1 \ldots \tilde{s}_a u_M \in W_{M'}(1) \cap w_1 W_{M_2'}
\]
and $q_{M_2}(w) = q_{M_2}(\tilde{s}_{a+1} \ldots \tilde{s}_{a+b})$ (Lemma 3.3 (4)). First we check the proposition in three simple cases:

**Case 1.** Let $w = \tilde{s} \in 1 W_{M'}$ lifting $s \in S^\text{aff}_M$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(T^M_{\tilde{s}} + \mathcal{J}_M)$ because $T^*_M - e^*(T^M_{\tilde{s}}) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_M + c_\tilde{s}$, $T_M^M = T^M_{\tilde{s}} + c_\tilde{s}$ and $1 = q_{M_2}(\tilde{s})$.

**Case 2.** Let $w = u \in W(1)$ of length $\ell(u) = 0$ and $u_M \in W_{M'}(1)$ such that $u \in u_M 1 W_{M_2'}$. We have $\ell_M(u_M) = 0$ and $q_{M_2}(w) = 1$ (Lemma 3.3). We deduce $T_u + \mathcal{J} = e^*(T^M_{u_M} + \mathcal{J}_M)$ because $T^*_M = T^*_M + \mathcal{J}_M$, and $T_u = T^*_M = T^M_{u_M}$.

**Case 3.** Let $w = \tilde{s} \in 1 W_{M_2'}$ lifting $s \in S^\text{aff}_{M_2}$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(q_{M_2}(\tilde{s}) + \mathcal{J}_M)$ because $T^*_M - 1, c_\tilde{s} = (q_s - 1) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_M + c_\tilde{s} \in q_s + \mathcal{J}$ and $q_s = q_{M_2}(\tilde{s})$.

In general, the braid relations $T_w = T_{\tilde{s}_1} \ldots T_{\tilde{s}_a} T_u T_{\tilde{s}_{a+1}} \ldots T_{\tilde{s}_{a+b}}$ give a similar product decomposition of $T_w + \mathcal{J}$, and the simple cases 1, 2, 3 imply that $T_w + \mathcal{J}$ is equal to
\[
e^*(T^M_{\tilde{s}_1} + \mathcal{J}_M) \ldots e^*(T^M_{\tilde{s}_a} + \mathcal{J}_M) e^*(T^M_{u_M} + \mathcal{J}_M) e^*(q_{M_2}(\tilde{s}_{a+1})) + \mathcal{J}_M) \ldots e^*(q_{M_2}(\tilde{s}_{a+b}) + \mathcal{J}_M)
\]
\[= e^*(T^M_{w_M} q_M(w) + \mathcal{J}_M).
\]

The proposition is proved. \qed

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring $R$ (instead of $Z$).

The two-sided ideal of $H_R$ generated by $T_w^* - 1$ for all $w \in 1 W_{M_2'}$ is $\mathcal{J}_R = \mathcal{J} \otimes Z R$, the two-sided ideal of $H_{M,R}$ generated by $T^*_M - 1$ for all $\lambda \in 1 W_{M_2'} \cap \Lambda(1)$ is $\mathcal{J}_{M,R} = \mathcal{J} \otimes Z R$, and we get as in Proposition 3.5 isomorphisms
\[H_{M,R}/\mathcal{J}_{M,R} \sim H_{M-,R}/(\mathcal{J}_{M,R} \cap H_{M-,R}) \sim H_R/\mathcal{J}_R,
\]
giving an isomorphism $H_{M,R}/\mathcal{J}_{M,R} \rightarrow H_R/\mathcal{J}_R$ induced by $\theta^*$. Therefore, we have an isomorphism from the category of right $H_{M,R}$-modules where $\mathcal{J}_M$ acts by 0 onto the category of right $H_R$-modules where $\mathcal{J}$ acts by 0.

**Definition 3.8.** A right $H_{M,R}$-module $V$ where $\mathcal{J}_{M}$ acts by 0 is called extensible to $H$. The corresponding $H_R$-module where $\mathcal{J}$ acts by 0 is called its extension to $H$ and denoted by $e_H(V)$ or $e(V)$.

With the element basis $T^*_w$, $V$ is extensible to $H$ if and only if
\[VT_{\lambda_2}^M = v \text{ for all } v \in V \text{ and } \lambda_2 \in 1 W_{M_2'} \cap \Lambda(1).
\]
The $H$-module structure on the $R$-module $e(V) = V$ is determined by
\[VT_w^* = v, \quad VT_w^* = VT_w^M \text{ for all } v \in V, w_2 \in 1 W_{M_2'}, w \in W_{M}(1).
\]
It is also determined by the action of $T_w^*$ for $w \in 1W_{M_2} \cup W_M(1)$ (or $w \in 1W_{M_2} \cup W_{M^+}(1)$). Conversely, a right $\mathcal{H}$-module $W$ over $R$ is extended from an $\mathcal{H}_M$-module if and only if

$$vT_{w_2}^* = v \quad \text{for all } v \in W, w_2 \in 1W_{M_2}. \tag{3.13}$$

In terms of the basis elements $T_w$ instead of $T_w^*$, this says the following.

**Corollary 3.9.** A right $\mathcal{H}_M$-module $V$ over $R$ is extensible to $\mathcal{H}$ if and only if

$$vT_{\lambda_2}^M = v \quad \text{for all } v \in V \text{ and } \lambda_2 \in 1W_{M_2} \cap \Lambda(1). \tag{3.14}$$

Then, the structure of an $\mathcal{H}$-module on the $R$-module $e(V) = V$ is determined by

$$vT_{w_2} = vq_{w_2}, \quad vT_w = vT_{w_2}^M q_{M_2}(w) \quad \text{for all } v \in V, w_2 \in 1W_{M_2}, w \in W_M(1). \tag{3.15}$$

$(W_M(1)$ or $W_{M^+}(1)$ instead of $W_M(1)$ is enough.) A right $\mathcal{H}$-module $W$ over $R$ is extended from an $\mathcal{H}_M$-module if and only if

$$vT_{w_2} = vq_{w_2} \quad \text{for all } v \in W, w_2 \in 1W_{M_2}. \tag{3.16}$$

### 3.4. $\sigma^{\mathcal{H}_M}$ is extensible to $\mathcal{H}$ of extension $e(\sigma^{\mathcal{H}_M}) = e(\sigma)^{\mathcal{H}_M}$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal, and let $\sigma$ be a smooth $R$-representation of $M$ extensible to $G$. Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of $G$ with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

Recall that $G = MM_2'$, that $M \cap M_2' = Z \cap M_2'$ acts trivially on $\sigma$, $e(\sigma)$ is the representation of $G$ equal to $\sigma$ on $M$ and trivial on $M_2'$. We will describe the $\mathcal{H}$-module $e(\sigma)^{\mathcal{H}_M}$ in this section. We first consider $e(\sigma)$ as a subrepresentation of $\text{Ind}_P^G \sigma$. For $v \in \sigma$, let $f_v \in (\text{Ind}_P^G \sigma)^{M_2'}$ be the unique function with value $v$ on $M_2'$.

Then, the map

$$v \mapsto f_v : \sigma \to \text{Ind}_P^G \sigma \tag{3.17}$$

is the natural $G$-equivariant embedding of $e(\sigma)$ in $\text{Ind}_P^G \sigma$. As $\sigma^{\mathcal{H}_M} = e(\sigma)^{\mathcal{H}_M}$ as $R$-modules, the image of $e(\sigma)^{\mathcal{H}_M}$ in $(\text{Ind}_P^G \sigma)^{\mathcal{H}_M}$ is made out of the $f_v$ for $v \in \sigma^{\mathcal{H}_M}$.

We now recall the explicit description of $(\text{Ind}_P^G \sigma)^{\mathcal{H}_M}$. For each $d \in W_{M_2}$, we fix a lift $\hat{d} \in 1W_{M_2}$ and for $v \in \sigma^{\mathcal{H}_M}$ let $f_{P,d,v} \in (\text{Ind}_P^G \sigma)^{\mathcal{H}_M}$ for the function with support contained in $Pd\mathcal{U}$ and value $v$ on $d\mathcal{U}$. As $Z \cap M_2'$ acts trivially on $\sigma$, the function $f_{P,d,v}$ does not depend on the choice of the lift $\hat{d} \in 1W_{M_2}$ of $d$. By [OV17] Lemma 4.5, recalling that $w \in W_{M_2}$ is of minimal length in its coset $w\mathcal{W}_M = \mathcal{W}_Mw$ as $\Delta_{M_2}$ and $\Delta_\mathcal{M}_2$ are orthogonal to each other:

The map $\bigoplus_{d \in W_{M_2}} \sigma^{\mathcal{H}_M} \to (\text{Ind}_P^G \sigma)^{\mathcal{H}_M}$ given on each $d$-component by $v \mapsto f_{P,d,v}$, is an $\mathcal{H}_{M^+}$-equivariant isomorphism where $\mathcal{H}_{M^+}$ is seen as a subring of $\mathcal{H}$ via $\theta$, and induces an $\mathcal{H}_R$-module isomorphism

$$v \otimes h \mapsto f_{P,d,v,h} : \sigma^{\mathcal{H}_M} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \to (\text{Ind}_P^G \sigma)^{\mathcal{H}_M}. \tag{3.18}$$

In particular for $v \in \sigma^{\mathcal{H}_M}$, $v \otimes T(\hat{d})$ does not depend on the choice of the lift $\hat{d} \in 1W_{M_2}$ of $d$ and

$$f_{P,d,v} = f_{P,d,v,T(\hat{d})}. \tag{3.19}$$
As \( G \) is the disjoint union of \( Pd\mathcal{U} \) for \( d \in \mathbb{W}_{M_2} \), we have \( f_v = \sum_{d \in \mathbb{W}_{M_2}} f_{Pd\mathcal{U},v} \) and \( f_v \) is the image of \( v \otimes e_{M_2} \) in \( 3.18 \), where
\[
(3.20) \quad e_{M_2} = \sum_{d \in \mathbb{W}_{M_2}} T(d).
\]
Recalling \( 3.17 \) we get the following.

**Lemma 3.10.** The map \( v \mapsto v \otimes e_{M_2} : e(\sigma)^\mathcal{U} \to \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M,+}} \mathcal{H} \) is an \( \mathcal{H}_R \)-equivariant embedding.

**Remark 3.11.** The trivial map \( v \mapsto v \otimes 1_{\mathcal{H}} \) is not an \( \mathcal{H}_R \)-equivariant embedding.

We describe the action of \( T(n) \) on \( e(\sigma)^\mathcal{U} \) for \( n \in \mathcal{N} \). By definition for \( v \in e(\sigma)^\mathcal{U} \),
\[
(3.21) \quad vT(n) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} \cap)} \sum_{y_1 \in \mathcal{U}_M} \sum_{y_2 \in \mathcal{U}_M} y_1 y_2 n^{-1} v.
\]

**Proposition 3.12.** We have \( vT(n) = vT^M(n_M)q_{M_2}(n) \) for any \( n_M \in \mathcal{N} \cap M \) is such that \( n = n_M(M \cap M'_2) \).

**Proof.** The description \( 3.10 \) of \( \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} \cap) \) gives
\[
vT(n) = \sum_{y_1 \in \mathcal{U}_M} \sum_{y_2 \in \mathcal{U}_M} y_1 y_2 n^{-1} v.
\]
As \( M'_2 \) acts trivially on \( e(\sigma) \), we obtain
\[
vT(n) = q_{M_2}(n) \sum_{y_1 \in \mathcal{U}_M} \sum_{y_2 \in \mathcal{U}_M} y_1 y_2 n^{-1} v = q_{M_2}(n) vT^M(n_M). \]

\[ \square \]

**Theorem 3.13.** Let \( \sigma \) be a smooth \( R \)-representation of \( M \). If \( P(\sigma) = G \), then \( \sigma^{\mathcal{U}_M} \) is extensible to \( \mathcal{H} \) of extension \( e(\sigma^{\mathcal{U}_M}) = e(\sigma)^\mathcal{U} \). Conversely, if \( \sigma^{\mathcal{U}_M} \) is extensible to \( \mathcal{H} \) and generates \( \sigma \), then \( P(\sigma) = G \).

**Proof.**

1. The \( \mathcal{H}_M \)-module \( \sigma^{\mathcal{U}_M} \) is extensible to \( \mathcal{H} \) if and only if \( Z \cap M'_2 \) acts trivially on \( \sigma^{\mathcal{U}_M} \). Indeed, for \( v \in \sigma^{\mathcal{U}_M} \), \( z_2 \in Z \cap M'_2 \),
\[
vT^M(z_2) = \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap z_2^{-1} \mathcal{U}_M z_2)} y_1 z_2^{-1} v = \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap z_2^{-1} \mathcal{U}_M z_2)} y_1 z_2^{-1} v = z_2^{-1} v,
\]
by \( 3.21 \), then \( 3.3 \), then the fact that \( z_2^{-1} \) commutes with the elements of \( \mathcal{Y}_M \).

2. \( P(\sigma) = G \) if and only if \( Z \cap M'_2 \) acts trivially on \( \sigma \) (the group \( Z \cap M'_2 \) is generated by \( Z \cap M'_2 \) for \( \alpha \in \Delta_{M_2} \) by Lemma 2.1). The \( R \)-submodule \( \sigma^{Z \cap M'_2} \) of elements fixed by \( Z \cap M'_2 \) is stable by \( M \), because \( M = ZM' \), the elements of \( M' \) commute with those of \( Z \cap M'_2 \) and \( Z \) normalizes \( Z \cap M'_2 \).

3. Apply (1) and (2) to get the theorem except the equality \( e(\sigma^{\mathcal{U}_M}) = e(\sigma)^\mathcal{U} \) when \( P(\sigma) = G \) which follows from Propositions 3.12 and 3.7 \[ \square \]

Let \( 1_M \) denote the trivial representation of \( M \) over \( R \) (or 1 when there is no ambiguity on \( M \)). The right \( \mathcal{H}_R \)-module \( (1_G)^\mathcal{U} = 1_{\mathcal{H}} \) (or 1 if there is no ambiguity) is the trivial right \( \mathcal{H}_R \)-module: for \( w \in W_M(1) \), \( T_w = q_w \text{id} \) and \( T_w^* = \text{id} \) on \( 1_{\mathcal{H}} \).
Example 3.14. The $\mathcal{H}$-module $(\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$ is the extension of the $\mathcal{H}_{M_2}$-module $(\text{Ind}\overline{G}_1)^{\mathcal{H}}_{M_2 \cap B}$. Indeed, the representation $\text{Ind}\overline{G}_1$ of $G$ is trivial on $N_2$, as $G = M M_2'$ and $N_2 \subset M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = m m_2' m_2$ with $m \in M, m_2' \in M_2'$ and $n_2 \in N_2$, we have $P g m_2 = P m_2' n_2 = P m_2' = P g$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of $M_2$ and $(\text{Ind}\overline{G}_1)^{\mathcal{H}}_{M_2} = \text{Ind}\overline{G}_{M_2 \cap B}$. Apply Theorem 3.13 as follows.

3.5. The $\mathcal{H}_{R}$-module $e(\mathcal{V}) \otimes_R (\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module which is extendible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $P_2 = M_2 N_2$ denote the standard parabolic subgroup of $G$ with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

We define on the $R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$ a structure of a right $\mathcal{H}_{R}$-module as follows.

Proposition 3.15.

1. The diagonal action of $T_w^*$ for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$ defines a structure of a right $\mathcal{H}_{R}$-module.

2. The action of the $T_w$ is also diagonal and satisfies:

$$((v \otimes f) T_w, (v \otimes f) T_w^*) = (v T_{uw_m} \otimes f T_{uw_{m_2}'}, v T_{uw_m}^* \otimes f T_{uw_{m_2}'},$$

where $w = uw_{m_2}' w_{m_2}$ with $u \in W(1), \ell(u) = 0, w_{m_2}' \in W_{m_2}'$, $w_{m_2} \in W_{m_2}'$. Thus $T_w^*$ for $w \in W(1)$ acts on $e(\mathcal{V}) \otimes_R (\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$ as in (1). The braid relations obviously hold. The quadratic relations hold because $T_s^*$ with $s \in S_{\text{aff}}$, acts trivially either on $e(\mathcal{V})$ or on $(\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$. Indeed, $S_{\text{aff}} = S_{\text{aff}} M \cup S_{\text{aff}} M_2$, $T_s^*$ for $s \in S_{\text{aff}} M_2$, acts trivially on $(\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$ which is extended from an $\mathcal{H}_{M_2}$-module (Example 3.14), and $T_s^*$ for $s \in S_{\text{aff}} M_2$, acts trivially on $e(\mathcal{V})$ which is extended from an $\mathcal{H}_M$-module. This proves (1).

We describe now the action of $T_w$ instead of $T_w^*$ on the $\mathcal{H}$-module $e(\mathcal{V}) \otimes_R (\text{Ind}\overline{G}_1)^{\mathcal{H}}_R$. Let $w \in W(1)$. We write $w = uw_{M_2}' w_{M_2}$, $u \in W(1), \ell(u) = 0, w_{M_2}' \in W_{M_2}', w_{M_2} \in W_{M_2}'$. We have $\ell(w) = \ell(w_{M_2}') + \ell(w_{M_2}')$ and hence $T_w = T_u T_{w_{M_2}'} T_{w_{M_2}'}$. For $u = w_{M_2}'$, we have $T_u = T_u^*$ and $(v \otimes f) T_u = (v \otimes f) T_u^* = v T_u^* \otimes f T_u^* = v T_u \otimes f T_u^*$. For $w = w_{M_2}'$, $(v \otimes f) T_w^* = v T_w \otimes f$; for $s \in S_{\text{aff}} M_2$, $c_s = \sum_{t \in Z_{M_2} \cap W_{M_2}} c_s(t) T_t^*$ in particular, we have $(v \otimes f) T_s = (v \otimes f) (T_s^* + c_s) = v T_s^* + c_s \otimes f = v T_s \otimes f$. Hence $(v \otimes f) T_w = v T_w \otimes f$. For $w = w_{M_2}'$, we have similarly $(v \otimes f) T_w = v \otimes f T_w^*$ and $(v \otimes f) T_w = v \otimes f T_w^*$. □

Example 3.16. Let $\mathcal{X}$ be a right $\mathcal{H}_{R}$-module. Then $1_\mathcal{H} \otimes_R \mathcal{X}$ where the $T_s^*$ acts diagonally is an $\mathcal{H}_{R}$-module isomorphic to $\mathcal{X}$. But the action of the $T_w$ on $1_\mathcal{H} \otimes_R \mathcal{X}$ is not diagonal.
It is known [Ly15] that \((\text{Ind}^G_Q 1)^H\) and \((\text{St}^G_Q)^H\) are free \(R\)-modules and that \((\text{St}^G_Q)^H\) is the cokernel of the natural \(H\)-map
\[
\bigoplus_{Q \subseteq Q'} (\text{Ind}^G_Q 1)^H \to (\text{Ind}^G_Q 1)^H
\]
although the invariant functor \((-)^H\) is only left exact.

**Corollary 3.17.** The diagonal action of \(T^*_w\) for \(w \in W(1)\) on \(e(V) \otimes_R (\text{St}^G_Q)^H\) defines a structure of a right \(H\)-module satisfying Proposition 3.15(2).

4. HECKE MODULE \(I_H(P,V,Q)\)

4.1. **Case \(V\) extensible to \(H\).** Let \(P = MN\) be a standard parabolic subgroup of \(G\) such that \(\Delta_P\) and \(\Delta \setminus \Delta_P\) are orthogonal, let \(V\) be a right \(H\)-module extensible to \(H_R\) of extension \(e(V)\), and let \(Q\) be a parabolic subgroup of \(G\) containing \(P\). As \(Q\) and \(M_Q\) determine each other: \(Q = M_Q U\), we denote also \(H_{M_Q} = H_Q\) and \(H_{M_Q,R} = H_{Q,R}\) when \(Q \neq P,G\). When \(Q = G\) we drop \(G\) and we denote \(e_H(V) = e(V)\).

**Lemma 4.1.** \(V\) is extensible to an \(H_{Q,R}\)-module \(e_{H_Q}(V)\).

**Proof.** This is straightforward. By Corollary 3.17, \(V\) extensible to \(H\) means that \(T^M(z)\) acts trivially on \(V\) for all \(z \in \mathcal{N}^{M_Q}_2 \cap Z\). We have \(M_Q = M M_{2,Q}'\) with \(M_{2,Q}' \subset M_Q \cap M'\) and \(\mathcal{N}^{M_{2,Q}'} \subset \mathcal{N}^{M_Q}\); hence \(T^M(z)\) acts trivially on \(V\) for all \(z \in \mathcal{N}^{M_{2,Q}} \cap Z\) meaning that \(V\) is extensible to \(H_Q\).

**Remark 4.2.** We cannot say that \(e_{H_Q}(V)\) is extensible to \(H\) of extension \(e(V)\) when the set of roots \(\Delta_Q\) and \(\Delta \setminus \Delta_Q\) are not orthogonal (Definition 3.8).

Let \(Q'\) be an arbitrary parabolic subgroup of \(G\) containing \(Q\). We are going to define an \(H_{Q,R}\)-embedding \(\text{Ind}^H_{H_{Q,R}}(e_{H_Q}(V)) \stackrel{i(Q,Q')_H}{\longrightarrow} \text{Ind}^H_{H_Q}(e_{H_Q}(V)) = e_{H_Q}(V) \otimes_{H_{M_Q}^+,\theta} H\), \(Q\) defining an \(H\)-homomorphism
\[
\bigoplus_{Q \subseteq Q' \subset G} \text{Ind}^H_{H_{Q,R}}(e_{H_Q}(V)) \to \text{Ind}^H_{H_Q}(e_{H_Q}(V))
\]
of cokernel isomorphic to \(e(V) \otimes_R (\text{St}^G_Q)^H\). In the extreme case \((Q,Q') = (P,G)\), the \(H_{Q,R}\)-embedding \(e(V) \stackrel{i(P,G)_H}{\longrightarrow} \text{Ind}^H_{H_M}(V)\) is given in the following lemma where \(f_G\) and \(f_{PL} \in (\text{Ind}^G_P 1)^H\) denote the characteristic functions of \(G\) and \(PL\), \(f_G = f_{PL} e_{M_2}\) (see 3.20).

**Lemma 4.3.** There is a natural \(H\)-isomorphism
\[
v \otimes 1_H \mapsto v \otimes f_{PL} : \text{Ind}^H_{H_M}(V) = V \otimes_{H_{M^+},\theta} H \xrightarrow{\kappa_P} e(V) \otimes_R (\text{Ind}^G_P 1)^H,
\]
and compatible \(H\)-embeddings
\[
v \mapsto v \otimes f_G : e(V) \to e(V) \otimes_R (\text{Ind}^G_P 1)^H,
\]
\[
v \mapsto v \otimes e_{M_2} : e(V) \xrightarrow{i(P,G)_H} \text{Ind}^H_{H_{M}}(V).
\]

**Proof.** We show first that the map
\[
v \mapsto v \otimes f_{PL} : V \to e(V) \otimes_R (\text{Ind}^G_P 1)^H
\]
is $\mathcal{H}_{M^+}$-equivariant. Let $w \in W_{M^+}(1)$. We write $w = uw_{M^+}w_{M_2}$ as in Proposition 3.15 (2), so that $f_{P\mathcal{U}}T_w = f_{P\mathcal{U}}T_{uw_{M_2}}$. We have $f_{P\mathcal{U}}T_{uw_{M_2}} = f_{P\mathcal{U}}$ because $1_{W_{M^+}} \subset W_{M^+}(1) \cap W_{M^-}(1)$ hence $uw_{M_2} = uw_{M_2}^{-1}$ is in $W_{M^+}(1)$ and in $1_{\mathcal{H}_M} \otimes 1_{\mathcal{H}_{M^+}}$ we have $(1 \otimes 1_{\mathcal{H}})T_{uw_{M_2}} = 1 T_{uw_{M_2}}^M \otimes 1_{\mathcal{H}}$, and $T_{uw_{M_2}}^M$ acts trivially in $1_{\mathcal{H}_M}$ because $\ell_M(uw_{M_2}) = 0$. We deduce $v \otimes f_{P\mathcal{U}}T_w = vT_w \otimes f_{P\mathcal{U}}T_w = vT_w \otimes f_{P\mathcal{U}}$.

By adjunction (4.3) gives an $\mathcal{H}_R$-equivariant linear map

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{P\mathcal{U}} : \mathcal{V} \otimes \mathcal{H}_{M^+} \mathcal{H} \xrightarrow{\kappa_\mathcal{P}} e(\mathcal{V}) \otimes_R (\text{Ind}^G_P 1)^M.$$

We prove that $\kappa_\mathcal{P}$ is an isomorphism. Recalling $\hat{d} \in N \cap M_2$, $\hat{d} \in 1_{W_{M_2}}$ lift $d$, one knows that

$$\mathcal{V} \otimes \mathcal{H}_{M^+} \mathcal{H} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes T_d \bigoplus \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes f_{P\mathcal{U}}(v),$$

where each summand is isomorphic to $\mathcal{V}$. The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $\mathbb{W}_{M^+}w = w\mathbb{W}_{M^+}$ as $\Delta_M$ and $\Delta_{M_2}$ are orthogonal; for the second equality see section 3.4 (3.19). We have $\mathcal{H}_R(v \otimes T_{\hat{d}}) = (v \otimes f_{P\mathcal{U}})T_{\hat{d}} = v \otimes f_{P\mathcal{U}}T_{\hat{d}}$ (Proposition 3.15). Hence $\kappa_\mathcal{P}$ is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{P\mathcal{U}}e_{M_2} : e(\mathcal{V}) \otimes_R \mathcal{V} \otimes 1_{\mathcal{H}} \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}^G_P 1)^M,$$

where the right map is the tensor product $e(\mathcal{V}) \otimes_R -$ of the $\mathcal{H}_R$-equivariant embedding $1_{\mathcal{H}} \rightarrow (\text{Ind}^G_P 1)^M$ sending $1_R$ to $f_{P\mathcal{U}}e_{M_2}$ (Lemma 3.10); this map is injective because $(\text{Ind}^G_P 1)^M/1$ is a free $R$-module; it is $\mathcal{H}_R$-equivariant for the diagonal action of the $T_w$ on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the $\mathcal{H}_R$-equivariant embedding $v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \text{Ind}_{\mathcal{H}}^R_{\mathcal{H}_{M^+}}(\mathcal{V})$. \hfill $\square$

For a general $(Q, Q')$ the $\mathcal{H}_R$-embedding $\text{Ind}_{\mathcal{H}_{M^+}}^R(e_{\mathcal{H}_{M^+}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \text{Ind}_{\mathcal{H}_{Q'}}^R(e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ is given in the next proposition generalizing Lemma 4.3. The element $e_{M_2}$ of $\mathcal{H}_R$ appearing in the definition of $\iota(P, G)$ is replaced in the definition of $\iota(Q, Q')$ by an element $\theta_{Q'}(e_{Q'}) \in \mathcal{H}_R$ that we define first.

Until the end of section 4, we fix an admissible lift $w \mapsto \hat{w} : \mathbb{W} \rightarrow N \cap K$ (Definition 3.1) and $\hat{w}$ denotes the image of $\hat{w}$ in $W(1)$. We denote $\mathbb{W}_{M^+} = \mathbb{W}_Q$ and by $\mathbb{W}_Q \mathbb{W}$ the set of $w \in \mathbb{W}$ of minimal length in their coset $\mathbb{W}_Q w$. The group $G$ is the disjoint union of $Q\mathcal{U}$ for $d$ running through $\mathbb{W}_Q \mathbb{W}$ [OV17 Lemma 2.15 (2)]: $G = \bigsqcup_{d \in \mathbb{W}_Q \mathbb{W}} Q\mathcal{U}_d$. Since $Q\mathcal{U}_d \subset Q'\mathcal{U}$ if and only if $d \in Q'$, namely $d \in \mathbb{W}_Q \mathbb{W}_{Q'}$, we have

$$Q'\mathcal{U} = \bigsqcup_{d \in \mathbb{W}_Q \mathbb{W}_{Q'}} Q\mathcal{U}_d.$$

Set

$$e_{Q'} = \sum_{d \in \mathbb{W}_Q \mathbb{W}_{Q'}} T_d M_{Q'}.$$

We write $e_Q = e_{Q'}$. We have $e_Q = \sum_{d \in \mathbb{W}_{M_2,Q}} T_d M_{S}$. 


Remark 4.4. Note that $W^M W = W_{M_2}$ and $e_P = e_{M_2}$, where $M_2$ is the standard Levi subgroup of $G$ with $\Delta_{M_2} = \Delta \setminus \Delta_M$, as $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal. More generally, $\hat{W} \varpi_\varpi_{M, \varpi_\varpi_{M, Q'}} = \hat{W}_\varpi_{M_2, \varpi_\varpi_{M_2, Q'}}$ where $M_{2, Q'} = M_2 \cap M_{Q'}$.

Note that $e_{Q'}^* \in H_{M, +} \cap H_{M, -}$. We consider the linear map

$$\theta_Q^* : H_Q \rightarrow H_{Q'} \quad T_{w}^{M_Q} \mapsto T_{w}^{M_{Q'}} \quad (w \in W_{M_Q}(1)).$$

We write $\theta_Q^* = \theta_Q$ so that $\theta_Q(T_{w}^{M_Q}) = T_{w}$. When $Q = P$ this is the map $\theta$ defined earlier. Similarly we denote by $\theta_{Q'}^*$ the linear map sending the $T_{w}^{M_{Q'}}$ to $T_{w}^{M_{Q'}}$ and $\theta_{Q'}^* = \theta_Q^*$. We have

$$\theta_Q^*(e_{Q'}) = \sum_{d \in \varpi_\varpi_{Q, \varpi_\varpi_{Q'}}} T_d, \quad \theta_Q^*(e_{Q'}) = \theta_Q(e_{P}) \theta_{Q'}^*(e_{Q'}).$$

Proposition 4.5. There exists an $H_R$-isomorphism

$$v \otimes 1_H \mapsto v \otimes f_{QH} : \text{Ind}_{H_Q}^H(e_{H_Q}(V)) = e_{H_Q}(V) \otimes H_{H_M} \rightarrow \otimes e(V) \otimes_R (\text{Ind}_{Q}^G 1)^H,$$

and compatible $H_R$-embeddings

$$v \otimes f_{QH} = v \otimes f_{QH}^* : e_{H_Q}(V) \otimes_R (\text{Ind}_{Q}^G 1)^H \rightarrow e_{H_Q}(V) \otimes_R (\text{Ind}_{Q}^G 1)^H,$$

$$v \otimes 1_H \mapsto v \otimes \theta_Q^*(e_{Q'}) : \text{Ind}_{H_{Q'}}^H(e_{H_Q}(V)) \rightarrow \text{Ind}_{H_{Q'}}^H(e_{H_Q}(V)).$$

Proof. We have the $H_{MQ, R}$-embedding

$$v \mapsto v \otimes e_{P}^* : e_{H_Q}(V) \rightarrow V \otimes_{H_{M, +, 0}} H_Q = \text{Ind}_{H_{M}}^H(V)$$

by Lemma 4.3.2 as $\Delta_M$ is orthogonal to $\Delta_{M_2}$ and $\Delta_M \setminus \Delta_M$. Applying the parabolic induction which is exact, we get the $H_{R}$-embedding

$$v \otimes 1_H \mapsto v \otimes e_{P}^* \otimes 1_H : \text{Ind}_{H_{Q}}^H(e_{H_Q}(V)) \rightarrow \text{Ind}_{H_{Q}}^H(e_{H_Q}(V)).$$

Note that $T_{d}^{M_Q} \in H_{MQ}^H$ for $d \in \varpi_\varpi_{MQ}$. By transitivity of the parabolic induction, it is equal to the $H_{R}$-embedding

$$v \otimes 1_H \mapsto v \otimes \theta_Q(e_{P}^*) : \text{Ind}_{H_{Q}}^H(e_{H_Q}(V)) \rightarrow \text{Ind}_{H_{M}}^H(V).$$

On the other hand we have the $H_{R}$-embedding

$$v \otimes f_{QH} \mapsto v \otimes \theta_Q(e_{P}^*) : e(V) \otimes_R (\text{Ind}_{Q}^G 1)^H \rightarrow \text{Ind}_{H_{R}}^H(V)$$

given by the restriction to $e(V) \otimes_R (\text{Ind}_{Q}^G 1)^H$ of the $H_{R}$-isomorphism given in Lemma 4.3 (4.1), from $e(V) \otimes_R (\text{Ind}_{Q}^G 1)^H$ to $V \otimes_{H_{M, +, 0}} H$ sending $v \otimes f_{QH}$ to $v \otimes 1_H$, noting that $v \otimes f_{QH} = (v \otimes f_{QH}) \theta_Q(e_{P})$ by Proposition 3.15, $f_{QH} = f_{QH} \theta_Q(e_{P})$ and $\theta_Q(e_{P})$ acts trivially on $e(V)$ (this is true for $T_{d}^{Q}$ for $d \in \varpi_\varpi_{MQ}$). Comparing the embeddings (4.12) and (4.13), we get the $H_{R}$-isomorphism (4.19).

We can replace $Q$ by $Q'$ in the $H_{R}$-homomorphisms (4.9), (4.12), and (4.13). With (4.12) we see $\text{Ind}_{H_{Q'}}^H(e_{H_{Q'}}(V))$ and $\text{Ind}_{H_{Q}}^H(e_{H_Q}(V))$ as $H_{R}$-submodules of $\text{Ind}_{H_{M}}^H(V)$. As seen in (4.8) we have $\theta_{Q'}(e_{Q'}) = \theta_Q(e_{P}) \theta_{Q'}(e_{Q'})$. We deduce the $H_{R}$-embedding (4.11).
By (3.19) for $Q$ and (4.6),
$$f_{Q'U} = \sum_{d \in \mathbb{G} \cap Q'} f_{QU} T_d = f_{QU} \theta_{Q'}(e_{Q'}^\ast)$$
in $(\text{Ind}_G^G 1)^\ast$. We deduce that the $H_R$-embedding corresponding to (4.11) via $\kappa_Q$ and $\kappa_{Q'}$ is the $H_R$-embedding (4.10).

We recall that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal and that $\mathcal{V}$ is extensible to $\mathcal{H}$ of extension $e(\mathcal{V})$.

**Corollary 4.6.** The cokernel of the $H_R$-map
$$\bigoplus_{Q \leq Q'} \text{Ind}_{H_{Q'}}^H(e_{H_{Q'}}(\mathcal{V})) \to \text{Ind}_{H_Q}^H(e_{H_Q}(\mathcal{V}))$$
defined by the $\iota(Q, Q')$, is isomorphic to $e(\mathcal{V}) \otimes_R (\text{St}_G^G)^\ast$ via $\kappa_Q$.

**4.2. Invariants in the tensor product.** We return to the setting where $P = MN$ is a standard parabolic subgroup of $G$, $\sigma$ is a smooth $R$-representation of $M$ with $P(\sigma) = G$ of extension $e(\sigma)$ to $G$, and $Q$ a parabolic subgroup of $G$ containing $P$. We still assume that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal.

The $H_R$-modules $e(\sigma)^{H_R} = e(\sigma)^\ast$ are equal (Theorem 3.13). We compute $I_G(P, \sigma, Q)^\ast = (e(\sigma) \otimes_R \text{St}_G^G)^\ast$.

**Theorem 4.7.** The natural linear maps $e(\sigma)^{H_R} \otimes_R (\text{Ind}_G^G 1)^\ast \to (e(\sigma) \otimes_R \text{Ind}_G^G 1)^\ast$ and $e(\sigma)^{H_R} \otimes_R (\text{St}_G^G)^\ast \to (e(\sigma) \otimes_R \text{St}_G^G)^\ast$ are isomorphisms.

**Proof.** We need some preliminaries. In [GK14, Ly15], are introduced a finite free $\mathbb{Z}$-module $\mathfrak{M}$ (depending on $\Delta_Q$) and a $\mathcal{B}$-equivariant embedding $\text{St}_G^G \to \mathbb{C}^\infty(\mathcal{B}, \mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $(\text{St}_G^G)^\ast \simeq \mathbb{C}^\infty(\mathcal{B}, \mathfrak{M})^\mathcal{B}$.

**Lemma 4.8.**

1. $(\text{Ind}_G^G \mathbb{Z})^\mathcal{B}$ is a direct factor of $\text{Ind}_G^G \mathbb{Z}$.
2. $(\text{St}_G^G \mathbb{Z})^\mathcal{B}$ is a direct factor of $\text{St}_G^G \mathbb{Z}$.

**Proof.**

1. [AHV] Example 2.2.
2. As $\mathfrak{M}$ is a free $\mathbb{Z}$-module, $C^\infty_e(\mathcal{B}, \mathfrak{M})^\mathcal{B}$ is a direct factor of $C^\infty_e(\mathcal{B}, \mathfrak{M})$. Consequently, $\iota((\text{St}_G^G \mathbb{Z})^\mathcal{B}) = C^\infty_e(\mathcal{B}, \mathfrak{M})^\mathcal{B}$ is a direct factor of $\iota(\text{St}_G^G \mathbb{Z})$. As $\iota$ is injective, we get (2). \hfill $\Box$

We now prove Theorem 4.7. We may and do assume that $\sigma$ is $e$-minimal (because $P(\sigma) = P(\sigma_{\text{min}})$, $e(\sigma) = e(\sigma_{\text{min}})$) so that $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in section 3.2 in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let $V$ be the space of $e(\sigma)$ on which $M_2$ acts trivially. The restriction of $\text{Ind}_G^G \mathbb{Z}$ to $M_2$ is $\text{Ind}_{G \cap M_2}^G \mathbb{Z}$, that of $\text{St}_G^G \mathbb{Z}$ is $\text{St}_{G \cap M_2}^G \mathbb{Z}$.

As in [AHV] Example 2.2], $((\text{Ind}_{G \cap M_2}^G \mathbb{Z}) \otimes V)^{\mathcal{U}M_2} \simeq (\text{Ind}_{G \cap M_2}^G \mathbb{Z})^{\mathcal{U}M_2} \otimes V$. We have
$$(\text{Ind}_{G \cap M_2}^G \mathbb{Z})^{\mathcal{U}M_2} = (\text{Ind}_{G \cap M_2}^G \mathbb{Z})^{\mathcal{U}M_2} = (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}}.$$ The first equality follows from $M_2 = (Q \cap M_2)\mathbb{W}_{M_2} \mathcal{U}_{M_2}$, $\mathcal{U}_{M_2} = Z^1 \mathcal{U}_{M_2}$ and $Z^1$ normalizes $\mathcal{U}_{M_2}$ and is normalized by $\mathbb{W}_{M_2}$. The second equality follows from $\mathcal{U} = \mathcal{U}_{M_2}$. A similar argument for $\text{St}_G^G \mathbb{Z}$.
\( \mathcal{U}_M \mathcal{U}_M \) and \( \text{Ind}^G_Q Z \) is trivial on \( M' \). Therefore ((\text{Ind}^G_Q Z) \otimes V)^{\mu_M'} \simeq (\text{Ind}^G_Q Z)^{\mu} \otimes V.

Now taking fixed points under \( \mathcal{U}_M \), as \( \mathcal{U} = \mathcal{U}_M' \mathcal{U}_M \),

\[
(\text{Ind}^G_Q Z) \otimes V)^{\mathcal{U}} \simeq ((\text{Ind}^G_Q Z) \otimes V)^{\mu_M} = (\text{Ind}^G_Q Z)^{\mu} \otimes V^{\mu_M}.
\]

The equality uses that the \( \mathcal{Z} \)-module \( \text{Ind}^G_Q Z \) is free. We get the first part of the theorem as (\( \text{Ind}^G_Q Z \)^{\mu} \otimes V)^{\mu_M} \simeq (\text{Ind}^G_Q R \)^{\mu} \otimes R^{V^{\mu_M}}.

Tensoring with \( R \) the usual exact sequence defining \( \text{St}^G_Q \mathcal{Z} \) gives an isomorphism \( \text{St}^G_Q \mathcal{Z} \otimes R \simeq \text{St}^G_Q R \) and in \( \text{GK14, Ly15} \), it is proved that the resulting map \( \text{St}^G_Q R \rightarrow C^\infty_\mathcal{B} (\mathcal{B}, \mathcal{M} \otimes R) \) is also injective. Their proof in no way uses the ring structure of \( R \), and for any \( \mathcal{Z} \)-module \( V \), tensoring with \( V \) gives a \( \mathcal{B} \)-equivariant embedding \( \text{St}^G_Q \mathcal{Z} \otimes V \rightarrow C^\infty_\mathcal{B} (\mathcal{B}, \mathcal{M} \otimes V) \). The natural map \( (\text{St}^G_Q \mathcal{Z})^B \otimes V \rightarrow \text{St}^G_Q \mathcal{Z} \otimes V \) is also injective by Lemma 14.3 (2). Taking \( \mathcal{B} \)-fixed points we get inclusions

\[
(\text{St}^G_Q \mathcal{Z})^B \otimes V \rightarrow (\text{St}^G_Q \mathcal{Z} \otimes V)^B \rightarrow C^\infty_\mathcal{B} (\mathcal{B}, \mathcal{M} \otimes V)^B \simeq \mathcal{M} \otimes V.
\]

The composite map is surjective, so the inclusions are isomorphisms. The image of \( \iota \) consists of functions which are left \( Z^0 \)-invariant, and \( \mathcal{B} = Z^0 \mathcal{U}' \) where \( \mathcal{U}' = G' \cap \mathcal{U} \). It follows that \( \iota \) yields an isomorphism \( (\text{St}^G_Q \mathcal{Z})^{\mu'} \simeq C^\infty_\mathcal{B} (Z_0 \mathcal{B}, \mathcal{M} \mu') \) again consisting of the constant functions. So that in particular \( (\text{St}^G_Q \mathcal{Z})^{\mu'} = (\text{St}^G_Q \mathcal{Z})^B \) and reasoning as previously we get isomorphisms

\[
(\text{St}^G_Q \mathcal{Z})^{\mu'} \otimes V \simeq (\text{St}^G_Q \mathcal{Z} \otimes V)^{\mu'} \simeq \mathcal{M} \otimes V.
\]

The equality \( (\text{St}^G_Q \mathcal{Z})^{\mu'} = (\text{St}^G_Q \mathcal{Z})^B \) and the isomorphisms remain true when we replace \( \mathcal{U}' \) by any group between \( \mathcal{B} \) and \( \mathcal{U}' \). We apply these results to \( \text{St}^M_{Q \cap M_2} \mathcal{Z} \otimes V \) to get that the natural map \( (\text{St}^M_{Q \cap M_2} \mathcal{Z})^{\mu_M'} \otimes V \rightarrow (\text{St}^M_{Q \cap M_2} \mathcal{Z} \otimes V)^{\mu_M'} \) is an isomorphism and also that \( (\text{St}^M_{Q \cap M_2} \mathcal{Z})^{\mu_M'} = (\text{St}^M_{Q \cap M_2} \mathcal{Z})^{\mu_M} \). We have \( \mathcal{U} = \mathcal{U}_M' \mathcal{U}_M \) so \( (\text{St}^G_Q \mathcal{Z})^{\mu} = (\text{St}^M_{Q \cap M_2} \mathcal{Z})^{\mu_M} \) and the natural map \( (\text{St}^G_Q \mathcal{Z})^{\mu} \otimes V \rightarrow (\text{St}^G_Q \mathcal{Z} \otimes V)^{\mu_M} \) is an isomorphism. The \( \mathcal{Z} \)-module \( (\text{St}^G_Q \mathcal{Z})^{\mu} \) is free and the \( V^{\mu_M} = V^{\mu} \), so taking fixed points under \( \mathcal{U}_M \), we get \( (\text{St}^G_Q \mathcal{Z})^{\mu} \otimes V \simeq (\text{St}^G_Q \mathcal{Z} \otimes V)^{\mu} \). We have \( \text{St}^G_Q \mathcal{Z} \otimes V = \text{St}^G_Q R \otimes R \mathcal{V} \) and \( (\text{St}^G_Q \mathcal{Z})^{\mu} \otimes V^{\mu} = (\text{St}^G_Q R)^{\mu} \otimes R V^{\mu} \). This ends the proof of the theorem.

**Theorem 4.9.** The \( \mathcal{H}_R \)-modules \( (e(\sigma) \otimes_R \text{Ind}^G_Q 1)^{\mu} \) and \( (\text{St}^G_Q \mathcal{Z})^{\mu} \) are equal. The \( \mathcal{H}_R \)-modules \( (e(\sigma) \otimes_R \text{Ind}^G_Q 1)^{\mu} = (e(\sigma)^{\mu} \otimes_R (\text{Ind}^G_Q 1)^{\mu}) \) are also equal.

**Proof.** We already know that the \( R \)-modules are equal (Theorem 4.1). We show that they are equal as \( \mathcal{H} \)-modules. The \( \mathcal{H}_R \)-modules \( e(\sigma)^{\mu} \otimes_R (\text{Ind}^G_Q 1)^{\mu} \) are equal (Theorem 4.13, Proposition 4.5, to \( \text{Ind}^G_Q (e(\sigma)^{\mu}) \) (OV17), and to \( (e(\sigma) \otimes_R \text{Ind}^G_Q 1)^{\mu} \)). We deduce that the \( \mathcal{H}_R \)-modules \( e(\sigma)^{\mu} \otimes_R (\text{Ind}^G_Q 1)^{\mu} \) are also equal. The same is true when \( Q \) is replaced by a parabolic subgroup \( Q' \) of \( G \) containing \( Q \). The representation \( e(\sigma) \otimes_R \text{St}^G_Q 1 \) is the cokernel of the natural \( R[G] \)-map

\[
\bigoplus_{Q \subseteq Q'} e(\sigma) \otimes_R \text{Ind}^G_Q 1 \rightarrow e(\sigma) \otimes_R \text{Ind}^G_Q 1.
\]
and the $\mathcal{H}_R$-module $e(\sigma)^{U} \otimes_R (\text{St}_Q^G)^{U}$ is the cokernel of the natural $\mathcal{H}_R$-map
\[
\bigoplus_{Q \subseteq Q'} e(\sigma)^{U} \otimes_R (\text{Ind}_Q^G 1)^{U} \xrightarrow{\beta_Q} e(\sigma)^{U} \otimes_R (\text{Ind}_Q^G 1)^{U}
\]
obtained by tensoring (3.22) by $e(\sigma)^{U}$ over $R$, because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^{U}$ are equal and the $R$-modules $e(\sigma)^{U} \otimes_R (\text{St}_Q^G)^{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^{U}$ are equal. This implies that the $\mathcal{H}_R$-modules $e(\sigma)^{U} \otimes_R (\text{St}_Q^G)^{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^{U}$ are equal.

**Remark 4.10.** The proof shows that the representations $e(\sigma) \otimes_R \text{Ind}_Q^G 1$ and $e(\sigma) \otimes \text{St}_Q^G$ of $G$ are generated by their $U$-fixed vectors if the representation $\sigma$ of $M$ is generated by its $U_M$-fixed vectors. Indeed, the $R$-modules $e(\sigma)^{U} = \sigma^{U_M} \otimes_R (\text{Ind}_Q^G 1)^{U_M}$ are equal. If $\sigma^{U_M}$ generates $\sigma$, then $e(\sigma)$ is generated by $e(\sigma)^{U}$. The representation $\text{Ind}_Q^G 1_{M_1}$ is generated by $\text{Ind}_Q^G 1$ (this follows from the lemma below), we have $G = M M_2$ and $M_2$ acts trivially on $e(\sigma)$. Therefore the $R[G]$-module generated by $\sigma^{U} \otimes_R (\text{Ind}_Q^G 1)^{U}$ is $e(\sigma) \otimes_R \text{Ind}_Q^G 1$. As $e(\sigma) \otimes_R \text{St}_Q^G$ is a quotient of $e(\sigma) \otimes_R \text{Ind}_Q^G 1$, the $R[G]$-module generated by $\sigma^{U} \otimes_R (\text{St}_Q^G)^{U}$ is $e(\sigma) \otimes_R \text{St}_Q^G$.

**Lemma 4.11.** For any standard parabolic subgroup $P$ of $G$, the representation $\text{Ind}_P^G 1_{G'}$ is generated by its $U$-fixed vectors.

**Proof.** Because $G = PG'$ it suffices to prove that if $J$ is an open compact subgroup of $N$ the characteristic function $1_{PJ}$ of $PJ$ is a finite sum of translates of $1_{PLU}$ by $G'$. For $t \in T$ we have $PLUt = Pt^{-1} \mathcal{U}_T$ and we can choose $t \in T \cap J'$ such that $t^{-1} \mathcal{U}_T \subseteq J$.

4.3. **General triples.** Let $P = MN$ be a standard parabolic subgroup of $G$. We now investigate situations where $\Delta_P$ and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module.

**Definition 4.12.** Let $P(V) = M(V) \cap N(V)$ be the standard parabolic subgroup of $G$ with $\Delta_P(V) = \Delta_P \cup \Delta_V$.

\[
\Delta_V = \{ \alpha \in \Delta \text{ orthogonal to } \Delta_M, T^M(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M_\alpha' \}.
\]
If $Q$ is a parabolic subgroup of $G$ between $P$ and $P(V)$, the triple $(P, V, Q)$ called an $\mathcal{H}_R$-triple, defines a right $\mathcal{H}_R$-module $I_{H}(P, V, Q)$ equal to
\[
\text{Ind}_{H(M(V))}(e(V) \otimes_R (\text{St}_{Q \cap M(V)}^M)^{U_M(V)}) = (e(V) \otimes_R (\text{St}_{Q \cap M(V)}^M)^{U_M(V)}) \otimes_{\mathcal{H}_{M(V)}} \mathcal{H}_{R},
\]
where $e(V)$ is the extension of $\mathcal{V}$ to $\mathcal{H}_{M(V)}$.

This definition is justified by the fact that $M(V)$ is the maximal standard Levi subgroup of $G$ such that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}_{M(V)}$.

**Lemma 4.13.** $\Delta_V$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to $\Delta_P$ such that $T^{M,*}_\lambda$ acts trivially on $\mathcal{V}$ for all $\lambda \in \Lambda(1) \cap 1W_{M_\alpha'}$.

**Proof.** For $J \subseteq \Delta$ let $M_J$ denote the standard Levi subgroup of $G$ with $\Delta_{M_J} = J$. The group $Z \cap M_J'$ is generated by the $Z \cap M_\alpha'$ for all $\alpha \in J$ (Lemma 2.1). When $J$ is orthogonal to $\Delta_M$ and $\lambda \in \Lambda_{M_J}(1)$, $\ell_M(\lambda) = 0$ where $\ell_M$ is the length associated to $S^\text{aff}_M$, and the map $\lambda \mapsto T^{M,*}_\lambda = T^{M}_\lambda : M_J(1) \to \mathcal{H}_M$ is multiplicative.
The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let $Q'$ be a parabolic subgroup of $G$ with $Q \subset Q' \subset P(V)$. Applying the results of section 4.1 to $M(V)$ and its standard parabolic subgroups $Q \cap M(V) \subset Q' \cap M(V)$, we have an $\mathcal{H}_{M(V), R}$-isomorphism

$$\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(V)}}(e_{\mathcal{H}_Q}(V)) = e_{\mathcal{H}_Q}(V) \otimes_{\mathcal{H}_{\mathcal{M}_Q}} \theta_{\mathcal{H}_{M(V), R}}(\kappa_{Q \cap M(V)}) e(V) \otimes_R (\text{Ind}_{Q \cap M(V)}^M(V) \mathbf{1})_{\mathcal{M}(V)}$$

and an $\mathcal{H}_{M(V), R}$-embedding

$$\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{M(V)}}(e_{\mathcal{H}_{Q'}}(V)) \xrightarrow{\iota(Q \cap M(V), Q' \cap M(V))} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(V)}}(e_{\mathcal{H}_Q}(V))$$

Applying the parabolic induction $\text{Ind}_{\mathcal{H}_{M(V)}}^\mathcal{H}$ which is exact and transitive, we obtain an $\mathcal{H}_R$-isomorphism $\kappa_Q = \text{Ind}_{\mathcal{H}_{M(V)}}^\mathcal{H}(\kappa_{Q \cap M(V)})$,

$$\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(V)}}(e_{\mathcal{H}_Q}(V)) \xrightarrow{\kappa_Q} \text{Ind}_{\mathcal{H}_{M(V)}}^{\mathcal{H}}(e(\mathcal{V}) \otimes R (\text{Ind}_{Q \cap M(V)}^M(V) \mathbf{1})_{\mathcal{M}(V)})$$

and an $\mathcal{H}_R$-embedding $\iota(Q, Q') = \text{Ind}_{\mathcal{H}_{M(V)}}^{\mathcal{H}}(\iota(Q, Q') M(V))$

$$v \otimes 1_{\mathcal{H}_M(V)} \mapsto v \otimes f_{Q \cap M(V)} \otimes 1_{\mathcal{H}}$$

Applying Corollary 4.6 we obtain:

**Theorem 4.14.** Let $(P, V, Q)$ be an $\mathcal{H}_R$-triple. Then, the cokernel of the $\mathcal{H}_R$-map

$$\oplus_{Q \subset Q' \subset P(V)} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(V)}}(e_{\mathcal{H}_Q}(V)) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(V)),$$

defined by the $\iota(Q, Q')$ is isomorphic to $I_{\mathcal{H}}(P, V, Q)$ via the $\mathcal{H}_R$-isomorphism $\kappa_Q$.

Let $\sigma$ be a smooth $R$-representation of $M$ and let $Q$ be a parabolic subgroup of $G$ with $P \subset Q \subset P(\sigma)$.

**Remark 4.15.** The $\mathcal{H}_R$-module $I_{\mathcal{H}}(P, \sigma_{\mathcal{M}}, Q)$ is defined if $\Delta_Q \setminus \Delta_P$ and $\Delta_P$ are orthogonal because $Q \subset P(\sigma) \subset P(\sigma_{\mathcal{M}})$ (Theorem 3.13).

We denote here by $P_{\min} = M_{\min}, N_{\min}$ the minimal standard parabolic subgroup of $G$ contained in $P$ such that $\sigma = e_P(\sigma |_{M_{\min}})$ (Lemma 2.3) we drop the index $\sigma$. The sets of roots $\Delta_{P_{\min}}$ and $\Delta_P(\sigma |_{M_{\min}}) \setminus \Delta_{P_{\min}}$ are orthogonal (Lemma 2.4). The groups $P(\sigma) = P(\sigma |_{M_{\min}})$, the representations $e(\sigma) = e(\sigma |_{M_{\min}})$ of $M(\sigma)$, the representations $I_{\mathcal{G}}(P, \sigma, Q) = I_{\mathcal{G}}(P_{\min}, \sigma |_{M_{\min}}, Q) = \text{Ind}_{P(\sigma)}^P(\sigma \otimes R \text{St}_Q^{P(\sigma)})$ of $G$, and the $R$-modules $\sigma_{\mathcal{M}_{\min}} = \sigma_{\mathcal{M}}$ are equal. From Theorem 3.13

$$P(\sigma) \subset P(\sigma_{\mathcal{M}_{\min}}), \quad e_{\mathcal{H}_M(\sigma)}(\sigma_{\mathcal{M}_{\min}}) = e(\sigma_{\mathcal{M}_{\min}}),$$

and $P(\sigma_{\mathcal{M}_{\min}}) = P(\sigma)$ if $\sigma_{\mathcal{M}_{\min}}$ generates the representation $\sigma |_{M_{\min}}$. The $\mathcal{H}_R$-module

$$I_{\mathcal{H}}(P_{\min}, \sigma_{\mathcal{M}_{\min}}, Q) = \text{Ind}_{\mathcal{H}_{M(\sigma_{\mathcal{M}_{\min}})}}^{\mathcal{H}_M} (e(\sigma_{\mathcal{M}_{\min}}) \otimes_R (\text{St}_Q^{P(\sigma_{\mathcal{M}_{\min}})})_{\mathcal{M}(\sigma_{\mathcal{M}_{\min}})}$$

is defined because $\Delta_{P_{\min}}$ and $\Delta_P(\sigma_{\mathcal{M}_{\min}}) \setminus \Delta_{P_{\min}}$ are orthogonal and $P \subset Q \subset P(\sigma) \subset P(\sigma_{\mathcal{M}_{\min}})$. 

Remark 4.16. If $\sigma^\dagger_{M_{\text{min}}}$ generates the representation $\sigma|_{M_{\text{min}}}$ (in particular if $R$ is an algebraically closed field of characteristic $p$ and $\sigma$ is irreducible), then $P(\sigma) = P(\sigma^\dagger_{M_{\text{min}}})$ hence

$$I_H(P_{\text{min}}, \sigma^\dagger_{M_{\text{min}}}, Q) = \text{Ind}^H_{\mathcal{H}_M(\sigma)}(e_{\mathcal{H}_M(\sigma)}(\sigma^\dagger_{M_{\text{min}}}) \otimes_R (\text{St}_{Q\cap M(\sigma)}^M(\sigma)^\dagger_{M(\sigma)})].$$

Applying Theorem 4.19 to $(P_{\text{min}} \cap M(\sigma), \sigma|_{M_{\text{min}}}, Q \cap M(\sigma))$, the $M_{\mathcal{H}_M(\sigma), R}$-modules

$$e_{\mathcal{H}_M(\sigma)}(\sigma^\dagger_{M_{\text{min}}}) \otimes_R (\text{St}_{Q\cap M(\sigma)}^M(\sigma)^\dagger_{M(\sigma)}) = (e_{M(\sigma)}(\sigma) \otimes R \text{St}_{Q\cap M(\sigma)}^M(\sigma)^\dagger_{M(\sigma)})$$

are equal. We have the $\mathcal{H}_R$-isomorphism [OV17 Proposition 4.4]:

$$I_G(P, \sigma, Q)^\dagger = (\text{Ind}^G_{P(\sigma)}(e(\sigma) \otimes_R \text{St}_Q^P(\sigma)))^\dagger$$

$$\xrightarrow{\text{ov}} \text{Ind}^H_{\mathcal{H}_M(\sigma)}((e(\sigma) \otimes_R \text{St}_{Q\cap M(\sigma)}^M(\sigma)^\dagger_{M(\sigma)}))$$

$$f_P(\sigma)_{\dagger, x} \mapsto x \otimes 1_{\mathcal{H}}$$

We deduce the following.

Theorem 4.17. Let $(P, \sigma, Q)$ be an $R[G]$-triple. Then, we have the $\mathcal{H}_R$-isomorphism

$$I_G(P, \sigma, Q)^\dagger \xrightarrow{\text{ov}} \text{Ind}^H_{\mathcal{H}_M(\sigma)}((e_{\mathcal{H}_M(\sigma)}(\sigma^\dagger_{M_{\text{min}}}) \otimes_R (\text{St}_{Q\cap M(\sigma)}^M(\sigma)^\dagger_{M(\sigma)})].$$

In particular,

$$I_G(P, \sigma, Q)^\dagger \simeq \begin{cases} I_H(P_{\text{min}}, \sigma^\dagger_{M_{\text{min}}}, Q) & \text{if } P(\sigma) = P(\sigma^\dagger_{M_{\text{min}}}), \\ I_H(P, \sigma^\dagger_{M_{\text{min}}}, Q) & \text{if } P = P_{\text{min}}, P(\sigma) = P(\sigma^\dagger_{M}). \end{cases}$$

4.4. Comparison of the parabolic induction and coinduction. Let $P = MN$ be a standard parabolic subgroup of $G$, let $V$ be a right $\mathcal{H}_R$-module, and let $Q$ be a parabolic subgroup of $G$ with $Q \subset P(V)$. When $R$ is an algebraically closed field of characteristic $p$, in [Abe], we associated to $(P, V, Q)$ an $\mathcal{H}_R$-module using the parabolic coinduction

$$\text{Coind}^H_{\mathcal{H}_M Q}(-) = \text{Hom}_{\mathcal{H}_M Q^\vee, \sigma^*}(\mathcal{H}, -) : \text{Mod}_R(\mathcal{H}_M Q) \to \text{Mod}_R(\mathcal{H})$$

instead of the parabolic induction $\text{Ind}^H_{\mathcal{H}_M Q}(-) = - \otimes_{\mathcal{H}_M Q^\vee, \sigma} \mathcal{H}$. The index $\theta^*$ in the parabolic coinduction means that $\mathcal{H}_{M_{Q^\vee}}$ embeds in $\mathcal{H}$ by $\theta^*_{Q^\vee}$. Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup $Q'$ of $G$ with $Q \subset Q' \subset P(V)$, there is a natural inclusion of $\mathcal{H}_R$-modules

$$\text{Hom}_{\mathcal{H}_{M_{Q^\vee}}, \sigma^*}(\mathcal{H}, e_{\mathcal{H}_{Q^\vee}}(V)) \xrightarrow{i(Q, Q')} \text{Hom}_{\mathcal{H}_{M_{Q^\vee}}, \sigma^*}(\mathcal{H}, e_{\mathcal{H}_{Q}}(V))$$

because $\theta^*(\mathcal{H}_{M_{Q^\vee}}) \subset \theta^*(\mathcal{H}_{M_{Q^\vee}})$ as $W_{M_{Q^\vee}(1)} \subset W_{M_{Q^\vee}(1)}$, and $v T_{w, M_{Q^\vee}, \sigma} = v T_{w, M_{Q^\vee}, \sigma}$ for $w \in W_{M_{Q^\vee}(1)}$ and $v \in V$. (This is [Abe] Proposition 4.19 when $R$ is an algebraically closed field of characteristic $p$. This follows from our formulation of the extension for any $R$.)

Definition 4.18. Let $CI_H(P, V, Q)$ denote the cokernel of the map

$$\bigoplus_{Q \subseteq Q' \subset P(V)} \text{Hom}_{\mathcal{H}_{M_{Q^\vee}}, \sigma^*}(\mathcal{H}, e_{\mathcal{H}_{Q^\vee}}(V)) \to \text{Hom}_{\mathcal{H}_{M_{Q^\vee}}, \sigma^*}(\mathcal{H}, e_{\mathcal{H}_{Q}}(V))$$

defined by the $\mathcal{H}_R$-embeddings $i(Q, Q')$. 
When $R$ is an algebraically closed field of characteristic $p$, we showed that the $\mathcal{H}_R$-module $CI_{\mathcal{H}}(P, V, Q)$ is simple when $V$ is simple and supersingular (Definition 4.25), and that any simple $\mathcal{H}_R$-module is of this form for an $\mathcal{H}_R$-triple $(P, V, Q)$ where $V$ is simple and supersingular, $P, Q$ and the isomorphism class of $V$ are unique [Abe]. The aim of this section is to compare the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P, V, Q)$ with the $\mathcal{H}_R$-modules $CI_{\mathcal{H}}(P, V, Q)$ and to show that the classification is also valid with the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P, V, Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements $w$ of the finite Weyl group $\mathcal{W}$ to $\tilde{w} \in N_G \cap K$ as in [AHHV17 IV.6], [OV17 Proposition 2.7]: they satisfy the braid relations $\hat{\ell}(w_1) + \hat{\ell}(w_2) = \ell(w_1 w_2)$ when $\ell(w) = \ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ and when $s \in S$, $\tilde{s}$ is admissible, in particular lies in $1 W_G$.

Let $w, w_M, w_M^M$ denote, respectively, the longest elements in $M = \tilde{M}$. $w$ restricts to a group isomorphism $w_M : M = \tilde{M} \rightarrow M = \tilde{M}$, respecting the finite Weyl subgroups $w_M^M$ of $M = \tilde{M}$. It has a natural direct decomposition indexed by the set $\Delta_M = \Phi^+ \setminus \Phi^+_M$. The conjugation $\tilde{w} \mapsto \tilde{w}_M$ sends the positive part $\tilde{w}_M$ of $\tilde{w}$ onto $\tilde{w}_M$, and exchanging $w_M$ with $\tilde{w}_M$. $w$.

Let $w_M$ be the standard Levi subgroup of $G$ with $\Delta_M = \Phi^+ \setminus \Phi^+_M$ and $w.P$ the standard parabolic subgroup of $G$ with Levi $w.M$. We have

$$w.M = w_M^M w_M^{-1} = \tilde{w}_M w_M^{-1}, \quad w_{w.M} = w_M w = (w^M)^{-1}.$$ 

The conjugation $w \mapsto w^M w(W^M)^{-1}$ in $W$ gives a group isomorphism $w_M : W_M \rightarrow W_{w.M}$ sending $S_{w.M}$ onto $S_{w,M}$, respecting the finite Weyl subgroups $w_M^M w_M(W^M)^{-1} = W_{w,M} = w W_{w,M} w^{-1}$, and exchanging $W_{w,M}$ and $W_{(w,M)} = w W_{w,M} w^{-1}$. The conjugation by $\tilde{w}_M$ restricts to a group isomorphism $W_M(1) \rightarrow W_{w,M}(1)$ sending $W_M(1)$ onto $W_{w,M}(1)$. The linear isomorphism

$$\tilde{w}(\tilde{M}_M) \mapsto \tilde{w}_M w_M(\tilde{M}_M)^{-1}, \quad w \in W_M(1),$$

is a ring isomorphism between the pro-$p$-Iwahori Hecke rings of $M$ and $w.M$. It sends the positive part $\tilde{w}_M^+ \setminus \tilde{w}_M^+$ of $\tilde{w}_M$ onto the negative part $\tilde{w}_M^-$ of $\tilde{w}_M$ [Vig15b Proposition 2.20]. We have $\tilde{w} = \tilde{w}_M \tilde{w}_M w = \tilde{w}_M \tilde{w}_M, \quad (\tilde{w}_M)^{-1} = \tilde{w}_M w_M t_M$ where $t_M = \tilde{w}_M^2 \tilde{w}_M^{-2} \in Z_k$.

**Definition 4.19.** The twist $\tilde{w}_M^+ \tilde{w} M \tilde{w}^M \tilde{w} M$ of $\tilde{w}$ by $\tilde{w}_M$ is the right $\tilde{w}_M \tilde{w} M$-module deduced from the right $\tilde{w}_M \tilde{w} M$-module $\tilde{w} M \tilde{w}$ by functoriality: as $R$-modules $\tilde{w}_M \tilde{w} M \tilde{w} M = \tilde{w}_M \tilde{w} M \tilde{w} M$, (4.19) sending $w \mapsto (\tilde{w}_M)^{-1}$.

We can define the twist $\tilde{w}_M^+ \tilde{w} M \tilde{w} M$ of $\tilde{w}$ with the $T_{w.M}^{M}$ instead of $T_{w.M}$.

**Lemma 4.20.** For $v \in V, w \in W_M(1)$ we have $v T_{w.M}^{M, *}(w) = v T_{w.M}^{M, *}(w)$ in $\tilde{w}_M^+ \tilde{w} M \tilde{w} M$.

**Proof.** By the ring isomorphism $\tilde{w}_M \tilde{w}_M \tilde{w}_M \tilde{w}_M = c_{\tilde{w}}^M$ when $\tilde{w}_M(1)$ lifts $\tilde{w}_M(1)$, so the equality of the lemma is true for $w = \tilde{w}_M$. Apply the braid relations to get the equality for all $w \in W_M(1)$. \hfill \Box

We return to the $\mathcal{H}_R$-module Hom$_{\mathcal{H}_{M \tilde{w} M}}(\mathcal{H}, V)$ parabolically coinduced from $V$. It has a natural direct decomposition indexed by the set $\mathcal{W} w M$ of elements $d$ in
the finite Weyl group \( \mathbb{W} \) of minimal length in the coset \( d\mathbb{W}_M \). Indeed it is known that the linear map

\[
f \mapsto (f(T_d))_{d \in \mathbb{W}_M} : \text{Hom}_{\mathcal{H}_M}(-, \mathcal{V}) \to \bigoplus_{d \in \mathbb{W}_M} \mathcal{V}
\]
is an isomorphism. For \( v \in \mathcal{V} \) and \( d \in \mathbb{W}_M \), there is a unique element 

\[
f_{d,v} \in \text{Hom}_{\mathcal{H}_M}(-, \mathcal{V})
\]
satisfying \( f(T_d) = v \) and \( f(T_{d'}) = 0 \) for \( d' \in \mathbb{W}_M \setminus \{d\} \).

It is known that the map \( v \mapsto f_{\tilde{w}^M,v} : \tilde{w}^M \mathcal{V} \to \text{Hom}_{\mathcal{H}_M}(-, \mathcal{V}) \) is \( \mathcal{H}_{(w,M)} \)-equivariant: 

\[
f_{\tilde{w}^M,v} T_{\tilde{w}^M} = f_{\tilde{w}^M,v} T_w \quad \text{for all} \quad v \in \mathcal{V}, w \in W_{w,M} \setminus \{1\}. \]

By adjunction, this \( \mathcal{H}_{(w,M)} \)-equivariant map gives an \( \mathcal{H}_R \)-homomorphism from an induced module to a coinduced module:

\[
(4.21) \quad v \otimes 1_H \mapsto f_{\tilde{w}^M,v} : \tilde{w}^M \mathcal{V} \otimes_{\mathcal{H}_{(w,M)}} \mathcal{V} \to \text{Hom}_{\mathcal{H}_M}(-, \mathcal{V}).
\]

This is an isomorphism [Abc Proposition 4.15], [Vig15b Theorem 1.8].

The naive guess that a variant \( \mu_Q \) of \( \mu_P \) induces an \( \mathcal{H}_R \)-isomorphism between the \( \mathcal{H}_R \)-modules \( I_H(w.P, \tilde{w}^M \mathcal{V}, w.Q) \) and \( CI_{\mathcal{H}}(P, \mathcal{V}, Q) \) turns out to be true. The proof is the aim of the rest of this section.

The \( \mathcal{H}_R \)-module \( I_H(w.P, \tilde{w}^M \mathcal{V}, w.Q) \) is well defined because the parabolic subgroups of \( G \) containing \( w.P \) and contained in \( P(\tilde{w}^M \mathcal{V}) \) are \( w.Q \) for \( P \subset Q \subset P(\mathcal{V}) \), as follows from Lemma 4.21.

**Lemma 4.21.** \( \Delta_{\tilde{w}^M,\mathcal{V}} = -w(\Delta_{\mathcal{V}}) \).

**Proof.** Recall that \( \Delta_{\mathcal{V}} \) is the set of simple roots \( \alpha \in \Delta \setminus \Delta_M \) orthogonal to \( \Delta_M \) and \( T_M(z) \) acts trivially on \( \mathcal{V} \) for all \( z \in Z \cap M'_\alpha \), and the corresponding standard parabolic subgroup \( P_{\mathcal{V}} = M_{\mathcal{V}} N_{\mathcal{V}} \). The \( Z \cap M'_\alpha \) for \( \alpha \in \Delta_{\mathcal{V}} \) generate the group \( Z \cap M'_\alpha \). A root \( \alpha \in \Delta \setminus \Delta_M \) orthogonal to \( \Delta_M \) is fixed by \( w.M \) so \( \tilde{w}^M(\alpha) = w(\alpha) \) and

\[
\tilde{w}^M w.M \tilde{w}^M(\alpha) = \tilde{w}^M w.M \tilde{w}^M(\alpha) = \tilde{w}^M w.M (\tilde{w}^M)^{-1} = \tilde{w}^M w.M (\tilde{w}^M)^{-1}.
\]
The proof of Lemma 4.21 is straightforward as \( \Delta = -w(\Delta) \), \( \Delta_{w.M} = -w(\Delta_M) \).

Before going further, we check the commutativity of the extension with the twist. As \( Q = M_Q U \) and \( M_Q \) determine each other we denote \( w.M_Q = w.Q, w.M_Q = w.Q \) when \( Q \neq P, G \).

**Lemma 4.22.** \( e_{H_{w,Q}}(\tilde{w}^M \mathcal{V}) = \tilde{w}^Q e_{H_{w,Q}}(\mathcal{V}) \).

**Proof.** As \( \mathcal{H}_R \)-modules \( \mathcal{V} = e_{H_{w,Q}}(\tilde{w}^M \mathcal{V}) = \tilde{w}^Q e_{H_{w,Q}}(\mathcal{V}) \). A direct computation shows that the Hecke element \( T_{w,Q} \) acts in the \( \mathcal{H}_R \)-module \( e_{H_{w,Q}}(\tilde{w}^M \mathcal{V}) \), by the identity if \( w \in \tilde{w}^Q \), \( W_{1/2} \), \( (\tilde{w}^M) \), and by \( T_{w,Q}^{-1} \) if \( w \in \tilde{w}^Q \), \( W_{1/2} \), \( (\tilde{w}^M) \), where \( M \) denotes the standard Levi subgroup with \( \Delta_M = \Delta_Q \setminus \Delta_P \). Whereas in the \( \mathcal{H}_R \)-module \( \tilde{w}^Q e_{H_{w,Q}}(\mathcal{V}) \), the Hecke element \( T_{w,Q} \) acts by the identity if \( w \in W_{1/2} \) and by \( T_{w,Q}^{-1} \) if \( w \in W_{1/2} \). So the lemma means that

\[
1 W_{w,M} = \tilde{w}^Q 1 W_{w,M} (\tilde{w}^M)^{-1}, \quad (\tilde{w}^Q)^{-1} w \tilde{w}^Q = (\tilde{w}^M)^{-1} w \tilde{w}^M \quad \text{if} \quad w \in W_{w,M} \setminus \{1\}.
\]

These properties are easily proved using that \( 1 W_{G} \) is normal in \( W(1) \) and that the sets of roots \( \Delta_P \) and \( \Delta_Q \setminus \Delta_P \) are orthogonal: \( w.Q = w.M w.M \), the elements \( w.M \) and \( w.M \) normalize \( W_M \) and \( W_{M_2} \), the elements of \( \tilde{w}^M_2 \) commutes with the elements of \( \mathbb{W}_M \).
We return to our guess. The variant $\mu_Q$ of $\mu_P$ is obtained by combining the commutativity of the extension with the twist and the isomorphism \((1.21)\) applied to \((Q, e_{\mathcal{H}_Q}(V))\) instead of \((P, V)\). The $\mathcal{H}_R$-isomorphism $\mu_Q$ is
\[(4.22) \quad v \otimes 1_\mathcal{H} \mapsto f_{\tilde{w}^M,v} : \text{Ind}^\mathcal{H}_{\tilde{\mathcal{H}}_{w,Q}}(e_{\tilde{\mathcal{H}}_{w,Q}}(\tilde{w}^M,V)) \xrightarrow{\mu_Q} \text{Hom}_{\tilde{\mathcal{H}}_{w,Q}}(\mathcal{H}, e_{\mathcal{H}_Q}(V)).\]

Our guess is that $\mu_Q$ induces an $\mathcal{H}_R$-isomorphism from the cokernel of the $\mathcal{H}_R$-map
\[\bigoplus_{Q \subseteq Q' \subset P(V)} \text{Ind}^\mathcal{H}_{\tilde{\mathcal{H}}_{w,Q'}}(e_{\tilde{\mathcal{H}}_{w,Q'}}(\tilde{w}^M,V)) \to \text{Ind}^\mathcal{H}_{\tilde{\mathcal{H}}_{w,Q}}(e_{\tilde{\mathcal{H}}_{w,Q}}(\tilde{w}^M,V))\]
defined by the $\mathcal{H}_R$-embeddings $i(w.Q, w.Q')$, isomorphic to $I_{\mathcal{H}}(w.P, \tilde{w}^M.V, w.Q)$ via $\kappa_{w,Q}$ (Theorem 4.13), onto the cokernel $CI_{\mathcal{H}}(P, V, Q)$ the $\mathcal{H}_R$-map
\[\bigoplus_{Q \subseteq Q' \subset P(V)} \text{Hom}_{\tilde{\mathcal{H}}_{w,Q'}}(\mathcal{H}, e_{\mathcal{H}_Q'}(V)) \to \text{Hom}_{\tilde{\mathcal{H}}_{w,Q}}(\mathcal{H}, e_{\mathcal{H}_Q}(V))\]
defined by the $\mathcal{H}_R$-embeddings $i(Q, Q')$. This is true if $i(Q, Q')$ corresponds to $i(w.Q, w.Q')$ via the isomorphisms $\mu_Q'$ and $\mu_Q$. This is the content of the next proposition.

**Proposition 4.23.** For all $Q \subseteq Q' \subset P(V)$ we have
\[i(Q, Q') \circ \mu_Q' = \mu_Q \circ i(w.Q, w.Q').\]

We postpone to section 4.6 the rather long proof of the proposition.

**Corollary 4.24.** The $\mathcal{H}_R$-isomorphism $\mu_Q \circ \kappa_{w,Q}^{-1}$ induces an $\mathcal{H}_R$-isomorphism
\[I_{\mathcal{H}}(w.P, \tilde{w}^M.V, w.Q) \to CI_{\mathcal{H}}(P, V, Q).\]

4.5. **Supersingular $\mathcal{H}_R$-modules, classification of simple $\mathcal{H}_R$-modules.** We recall first the notion of supersingularity based on the action of the center of $\mathcal{H}$.

The center of $\mathcal{H}$ [Vig14, Theorem 1.3] contains a subalgebra $Z_{T^+}$ isomorphic to $\mathbb{Z}[T^+/T_1]$ where $T^+$ is the monoid of dominant elements of $T$ and $T_1$ is the pro-$p$-Sylow subgroup of the maximal compact subgroup of $T$.

Let $t \in T$ of image $\mu_t \in W(1)$ and let \((E_o(w))_{w \in W(1)}\) denote the alcove walk basis of $\mathcal{H}$ associated to a closed Weyl chamber $o$ of $\mathbb{W}$. The element
\[E_o(C(\mu_t)) = \sum_{\mu'} E_o(\mu')\]
is the sum over the elements in $\mu'$ in the conjugacy class $C(\mu_t)$ of $\mu_t$ in $W(1)$. It is a central element of $\mathcal{H}$ and does not depend on the choice of $o$. We write also $z(t) = E_o(C(\mu_t))$.

**Definition 4.25.** A non-zero right $\mathcal{H}_R$-module $V$ is called supersingular when, for any $v \in V$ and any non-invertible $t \in T^+$, there exists a positive integer $n \in \mathbb{N}$ such that $v(z(t))^n = 0$. If one can choose $n$ independent on $(v, t)$, then $V$ is called uniformly supersingular.

**Remark 4.26.** One can choose $n$ independent on $(v, t)$ when $V$ is finitely generated as a right $\mathcal{H}_R$-module. If $R$ is a field and $V$ is simple we can take $n = 1$.

When $G$ is compact modulo the center, $T^+ = T$, and any non-zero $\mathcal{H}_R$-module is supersingular.
The induction functor \( \text{Ind}^H_{H,M} : \text{Mod}(H,M,R) \to \text{Mod}(H,R) \) has a left adjoint \( \mathcal{L}^H_{H,M} \) and a right adjoint \( \mathcal{R}^H_{H,M} \) [Vig15b] for \( \mathcal{V} \in \text{Mod}(H,R) \),

\[
(4.23) \quad \mathcal{L}^H_{H,M} (\mathcal{V}) = \check{w}^{w,M} \circ (\mathcal{V} \otimes H_{(w,M)}, \theta^* \check{H}_{w,M}), \quad \mathcal{R}^H_{H,M} (\mathcal{V}) = \text{Hom}_{H,M+} (H,M, \mathcal{V}).
\]

In the left adjoint, \( \mathcal{V} \) is seen as a right \( H_{(w,M)}^- \)-module via the ring homomorphism \( \theta^*_{w,M} : H_{(w,M)}^- \to H \); in the right adjoint, \( \mathcal{V} \) is seen as a right \( H_{M+} \)-module via the ring homomorphism \( \theta_M : H_{M+} \to H \) (section 2.3).

**Proposition 4.27.** Assume that \( \mathcal{V} \) is a supersingular right \( H_R \)-module and that \( p \) is nilpotent in \( \mathcal{V} \). Then \( \mathcal{L}^H_{H,M} (\mathcal{V}) = 0 \), and if \( \mathcal{V} \) is uniformly supersingular \( \mathcal{R}^H_{H,M} (\mathcal{V}) = 0 \).

**Proof.** This is a consequence of three known properties:

1. \( H_M \) is the localization of \( H_{M+} \) (resp., \( H_{M-} \)) at \( \mathcal{T}^M_\mu \) for any element \( \mu \in \Lambda^+_T(1) \), central in \( W_M(1) \) and strictly \( N \)-positive (resp., \( N \)-negative), and \( \mathcal{T}^M_\mu = \mathcal{T}^{M,*}_\mu \). See [Vig15b] Theorem 1.4].
2. When \( o \) is anti-dominant, \( E_o(\mu) = T^*_\mu \) if \( \mu \in \Lambda^+(1) \) and \( E_o(\mu) = T^*_\mu \) if \( \mu \in \Lambda^-(1) \).
3. Let an integer \( n > 0 \) and \( \mu \in \Lambda(1) \) such that the \( \mathcal{W} \)-orbit of \( v(\mu) \in X_c(T) \otimes \mathbf{Q} \) (definition in section 2.1] and of \( \mu \) have the same number of elements. Then

\[
(E_o(C(\mu)))^n E_o(\mu) - E_o(\mu)^{n+1} \in pH.
\]

See [Vig15a] Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let \( \mu \in \Lambda^+_T(1) \) satisfying (1) for \( M^+ \) and (3), similarly let \( w, \mu \in \Lambda^-_T(1) \) satisfying (1) for \( (w,M)^- \) and (3). For \( (R, \mathcal{V}) \) as in the proposition, let \( v \in \mathcal{V} \) and \( n > 0 \) such that \( v E_o(C(\mu))^n = n E_o(C(w,\mu))^n = 0 \). Multiplying by \( E_o(\mu) \) or \( E_o(w,\mu) \), and applying (3) and (2) for \( o \) anti-dominant we get:

\[
v E_o(\mu)^{n+1} = v T^*_\mu^{n+1} \in p \mathcal{V}, \quad v E_o(w,\mu)^{n+1} = v T^*_w^{n+1} \in p \mathcal{V}.
\]

The proposition follows from: \( v T^*_\mu^{n+1}, v T^*_w^{n+1} \) in \( p \mathcal{V} \) (as explained in [Abc16 Proposition 5.17] when \( p = 0 \) in \( R \)). From \( v(T^*_w \mu)^{n+1} \) in \( p \mathcal{V} \), we get \( v \otimes (T^*_w \mu)^{n+1} \) in \( p \mathcal{V} \otimes H_{(w,M)}^-, \theta^* H_{w,M} \). As \( T^*_w \mu \) is invertible in \( H_{w,M} \) we get \( v \otimes 1_{H_{w,M}} \) in \( p \mathcal{V} \otimes H_{(w,M)}^-, \theta^* H_{w,M} \). As \( v \) was arbitrary, \( \mathcal{V} \otimes H_{(w,M)}^-, \theta^* H_{w,M} \subset p \mathcal{V} \otimes H_{(w,M)}^-, \theta^* H_{w,M} \). If \( p \) is nilpotent in \( \mathcal{V} \), then \( \mathcal{V} \otimes H_{(w,M)}^-, \theta^* H_{w,M} = 0 \). Suppose now that there exists \( n > 0 \) such that \( \mathcal{V}(z(t))^n = 0 \) for any non-invertible \( t \in T^+ \); then \( VT^*_\mu^{n+1} \subset p \mathcal{V} \) where \( \mu = \mu_t \) and hence \( \varphi(h) = \varphi(hT^*_\mu^{n-1}) T^*_\mu^{n+1} \) in \( p \mathcal{V} \) for an arbitrary \( \varphi \in \text{Hom}_{H_{M+}}(H_{M}, \mathcal{V}) \) and an arbitrary \( h \in H_{M} \). We deduce \( \text{Hom}_{H_{M+}}(H_{M}, \mathcal{V}) \subset \text{Hom}_{H_{M+}}(H_{M}, p \mathcal{V}) \). If \( p \) is nilpotent in \( \mathcal{V} \), then \( \text{Hom}_{H_{M+}}(H_{M}, \mathcal{V}) = 0 \). \( \square \)

Recalling that \( \check{w}^{M} \mathcal{V} \) is obtained by functoriality from \( \mathcal{V} \) and the ring isomorphism \( \iota(\check{w}^{M}) \) defined in (4.20], the equivalence between \( \mathcal{V} \) supersingular and \( \check{w}^{M} \mathcal{V} \) supersingular follows from Lemma 4.28

**Lemma 4.28.**

1. Let \( t \in T \). Then \( t \) is dominant for \( U_M \) if and only if \( \check{w}^{M} t (\check{w}^{M})^{-1} \in T \) is dominant for \( U_{w,M} \).
Lemma 4.31. The $R$-algebra isomorphism $\mathcal{H}_{M,R} \xrightarrow{\iota(\tilde{w}^M)} \mathcal{H}_{w,M,R}$, $T_w^M \mapsto T_{\tilde{w}^M}^M$ for $w \in W_M(1)$ sends $\varepsilon(t)$ to $\varepsilon(w^M(t))$ for $t \in T$ dominant for $U_M$.

Proof. The conjugation by $\tilde{w}^M$ stabilizes $T$, sends $U_M$ to $U_{w,M}$, and sends the $W_M$-orbit of $t \in T$ to the $W_{w,M}$-orbit of $\tilde{w}^M(t)$, as $w^M W_M(w^M) = w^M$. It is known that $\iota(\tilde{w}^M)$ respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E^M(w)$ to $E^w(w^M(w^M)^{-1})$ for $w \in W_M(1)$. 

We deduce the following.

**Corollary 4.29.** Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module. Then $\mathcal{V}$ is supersingular if and only if the right $\mathcal{H}_{w,M,R}$-module $w^M \mathcal{V}$ is supersingular.

Assume $R$ is an algebraically close field of characteristic $p$. The supersingular simple $\mathcal{H}_{M,R}$-modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple $\mathcal{H}_R$-modules in [Abe] remains valid with the $\mathcal{H}_R$-modules $I_H(P, V, Q)$ instead of $CI_H(P, V, Q)$:

**Corollary 4.30** (Classification of simple $\mathcal{H}_R$-modules). Assume $R$ is an algebraically closed field of characteristic $p$. Let $(P, V, Q)$ be an $\mathcal{H}_R$-triple where $V$ is simple and supersingular. Then, the $\mathcal{H}_R$-module $I_H(P, V, Q)$ is simple. A simple $\mathcal{H}_R$-module is isomorphic to $I_H(P, V, Q)$ for an $\mathcal{H}_R$-triple $(P, V, Q)$ where $V$ is simple and supersingular, $P, Q$ and the isomorphism class of $V$ are unique.

4.6. A commutative diagram. We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(V)$ we show by an explicit computation that

$$
\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'} \text{Ind}_{\mathcal{H}_{w,Q'}}^\mathcal{H}((\tilde{w}^M, \mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{w,Q}}^\mathcal{H}(e_{\mathcal{H}_{w,Q}}(\tilde{w}^M, \mathcal{V}))
$$

is equal to $\iota(w, Q, w, Q')$. The $R$-module $e_{\mathcal{H}_{w,Q'}}(\tilde{w}^M, \mathcal{V}) \otimes 1_H$ generates the $\mathcal{H}_R$-module $e_{\mathcal{H}_{w,Q'}}(\tilde{w}^M, \mathcal{V}) \otimes \otimes_{\mathcal{H}_R}(\tilde{w}^M, \mathcal{V})$ and by (4.17)

$$(4.24) \quad \iota(w, Q, w, Q')(v \otimes 1_H) = v \otimes \sum_{d \in W_{w,Q} \mathcal{V}} T_d
$$

for $v \in V$ seen as an element of $e_{\mathcal{H}_{w,Q'}}(\tilde{w}^M, \mathcal{V})$ in the LHS and an element of $e_{\mathcal{H}_{w,Q}}(\tilde{w}^M, \mathcal{V})$ in the RHS.

**Lemma 4.31.** $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_H) = v \otimes \sum_{d \in W_{w,Q} \mathcal{V}} q_d T_{w}^{w}(\tilde{w}^M, d^{-1})$.

Proof. $\mu_{Q'}(v \otimes 1_H)$ is the unique homomorphism $f_{\tilde{w}^M} : v \in \hom_{\tilde{w}^M, \mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ sending $T_{w}^{w}$ to $v$ and vanishing on $T_d$ for $d' \in W_{w,M} \mathcal{V} \setminus \{wQ'\}$ by (4.22). By (4.19), $i(Q, Q')$ is the natural embedding of $\hom_{\tilde{w}^M, \mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ in $\hom_{\tilde{w}^M, \mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ therefore $i(Q, Q')(f_{\tilde{w}^M, \mathcal{V}})$ is the unique homomorphism $\hom_{\tilde{w}^M, \mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ sending $T_{w}^{w}$ to $v$ and vanishing on $T_d$ for $d' \in W_{w,M} \mathcal{V} \setminus \{wQ'\}$. As $W_{w,M} = W_{w,Q'} W_{M,Q'}$, this homomorphism vanishes on $T_{w}^{w}$ for $w$ not
in $\mathbb{W}^{M_{Q'}}\mathbb{W}^{W_{M_{Q'}}}_{M_{Q'}}$. By [Abe6] Lemma 2.22, the inverse of $\mu_Q$ is the $H_R$-isomorphism

$$\text{Hom}_{H_{M_{Q}}}^{H}(\mathcal{H}, e_{H_{M_{Q}}} (V)) \overset{H_{Q}}{\rightarrow} \text{Ind}_{H_{M_{Q}}}^{H}(e_{H_{M_{Q}}} (\tilde{w}^M_{M} V))$$

$$f \mapsto \sum_{d \in \mathbb{W}^{W_{M}}} f(T_d) \otimes T_{\tilde{w}^M_{M} d^{-1}},$$

where $\mathbb{W}^{W_{M}}$ is the set of $d \in \mathbb{W}$ with minimal length in the coset $d\mathbb{W}_{M}$. We deduce the explicit formula

$$(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_Q')(v \otimes 1_H) = \sum_{w \in \mathbb{W}^{W_{M}}_{Q}} i(Q,Q')(f_{w_{M_{Q}},v}^Q)(T_{\tilde{w}}) \otimes T_{\tilde{w}^M_{M} Q}^{-1}.$$

Some terms are zero: the terms for $w \in \mathbb{W}^{W_{M}}_{Q}$ not in $\mathbb{W}^{M_{Q'}}\mathbb{W}^{W_{M}}_{M_{Q'}}$, we analyze the other terms for $w$ in $\mathbb{W}^{W_{M}}_{Q'} \cap \mathbb{W}^{M_{Q'}}\mathbb{W}^{W_{M}}_{M_{Q'}}$: this set is $\mathbb{W}^{M_{Q'}}\mathbb{W}^{W_{M}}_{M_{Q'}}$. Let $w = w_{M_{Q}} d, d \in \mathbb{W}^{W_{M}}_{Q'}$, and $\tilde{w} = \tilde{w}^M_{M} \tilde{d}$ with $\tilde{d} \in 1 W_{M}$ lifting $d$. By the braid relations $T_{\tilde{w}} = T_{\tilde{w}^M_{M}} T_{\tilde{d}}$. We have $T_{\tilde{d}} = \theta^*(T_{\tilde{d}})$ by the braid relations because $d \in \mathbb{W}^{M_{Q'}}$, $S_{M_{Q'}} \subset S_{\text{aff}}^{M_{Q'}}$ and $\theta^*(c_{s}) = c_s$ for $s \in S_{M_{Q'}}$. As $\mathbb{W}^{M_{Q'}}_{Q'} \subset W_{M_{Q'}} \cap W_{M_{Q'}}$, we deduce:

$$i(Q,Q')(f_{w_{M_{Q}},v}^Q)(T_{\tilde{w}}) = i(Q,Q')(f_{w_{M_{Q}},v}^Q)(T_{\tilde{w}^M_{M} Q}^{-1} T_{\tilde{d}}) = i(Q,Q')(f_{w_{M_{Q}},v}^Q)(T_{\tilde{w}^M_{M} Q}^{-1}) T_{\tilde{d}}^M_{Q'}$$

$$= v T_{\tilde{d}} = q_d v.$$

Corollary 3.3 gives the last equality.}

The formula for $(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_Q')(v \otimes 1_H)$ given in Lemma 4.31 is different from the formula (4.24) for $i(w,Q,w')(v \otimes 1_H)$. It needs some work to prove that they are equal.

A first reassuring remark is that $\mathbb{W}^{M_{Q'}}_{M_{Q}} \mathbb{W}^{M_{Q'}}_{M_{Q'}} = \{ w d^{-1}w \mid d \in \mathbb{W}^{W_{M}}_{Q} \}$, so the two summation sets have the same number of elements. But better,

$$\mathbb{W}^{M_{Q'}}_{M_{Q}} \mathbb{W}^{M_{Q'}}_{M_{Q'}} = \{ w^Q(w Q' d)^{-1} \mid d \in \mathbb{W}^{W_{M}}_{Q} \}$$

because $w Q' \mathbb{W}^{W_{M}}_{Q'} w Q = \mathbb{W}^{W_{M}}_{Q'}$. To prove the latter equality, we apply the criterion: $w \in \mathbb{W}^{W_{M}_{Q'}}$ lies in $\mathbb{W}^{W_{M}}_{Q'}$ if and only if $w(\alpha) > 0$ for all $\alpha \in \Delta_{Q}$ noticing that $d \in \mathbb{W}^{W_{M}}_{Q'}$ implies $w Q(\alpha) \in -\Delta_{Q}$, $d w Q(\alpha) \in -\Phi_{M_{Q'}}$, $w Q d w Q(\alpha) > 0$. Let $x_d = w Q(w Q' d)^{-1}$. We have $\tilde{w}^M_{M}(w Q' \tilde{d})^{-1} = \tilde{x}_d$ because the lifts $\tilde{w}$ of the elements $w \in \mathbb{W}$ satisfy the braid relations and $\ell(x_d) = \ell(w Q d^{-1} w Q') = \ell(w Q') - \ell(w Q d^{-1}) = \ell(w Q') - \ell(w Q) - \ell(d) = \ell(w Q') + \ell(w Q) - \ell(d)$. We have $q_d = q_{w, Q, x_d w, Q}$ because $w d^{-1}w = w_{Q, x_d, w, Q'}$, and $q_d = q_{d^{-1}} = q_{w d^{-1}, w}$. So

$$\sum_{d \in \mathbb{W}^{W_{M}}_{Q'}} q_d T_{\tilde{w}^M_{M} Q}^{-1} = \sum_{x_d \in \mathbb{W}^{M_{Q'}}_{Q} \mathbb{W}_{M_{Q'}}^{W_{M}}_{Q'}} q_{w, Q, x_d w, Q} T_{\tilde{x}_d}.$$
In the RHS, only $\tilde{\mathbb{w}}^{M}.V, w.Q, w.Q'$ appear. The same holds true in the formula (4.24). The map $(P, V, Q, Q') \mapsto (w.P, \tilde{\mathbb{w}}^{M}.V, w.Q, w.Q')$ is a bijection of the set of triples $(P, V, Q, Q')$ where $P = MN, Q, Q'$ are standard parabolic subgroups of $G$, $\mathbb{V}$ a right $\mathcal{H}_R$-module, $Q \subset Q' \subset P(\mathbb{V})$ by Lemma 4.21. So we can replace $(w.P, \tilde{\mathbb{w}}^{M}.V, w.Q, w.Q')$ by $(P, V, Q, Q')$. Our task is reduced to prove in $e_{\mathcal{H}_Q}(\mathbb{V}) \otimes_{\mathcal{H}^+_{M Q}} \mathcal{H}_R$:

\[(4.26)\] 
\[v \otimes \sum_{d \in \mathcal{W}_{M Q}} T_d = v \otimes \sum_{d \in \mathcal{W}_{M Q}} q_{w_Q d w_{Q'}} T^*_d.\]

A second simplification is possible: we can replace $Q \subset Q'$ by the standard parabolic subgroups $Q_2 \subset Q'_2$ of $G$ with $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$ and $\Delta_{Q'_2} = \Delta_{Q'} \setminus \Delta_P$, because $\Delta_P$ and $\Delta_{P(\mathbb{V})} \setminus \Delta_P$ are orthogonal. Indeed, $\mathcal{W}_{M Q} = \mathcal{W}_M \times \mathcal{W}_{M Q_2}$ and $\mathcal{W}_{M Q} = \mathcal{W}_M \times \mathcal{W}_{M Q'_2}$ are direct products, the longest elements $w_{Q'} = w_M w_{Q_2}, w_Q = w_M w_{Q_2}$ are direct products and

$\mathcal{W}_{M Q} \mathcal{W}_{M Q_2} = \mathcal{W}_{M Q_2} \mathcal{W}_{M Q'_2}, \quad w_Q d w_{Q'} = w_{Q_2} d w_{Q'_2}.$

Once this is done, we use the properties of $e_{\mathcal{H}_Q}(\mathbb{V})$: $vh \otimes 1_\mathcal{H} = v \otimes \theta_Q(h)$ for $h \in \mathcal{H}^+_{M Q_2}$, and $T^*_d$ acts trivially on $e_{\mathcal{H}_Q}(\mathbb{V})$ for $w \in 1 \mathcal{W}_{M Q_2} \cup (\Lambda(1) \cap 1 \mathcal{W}_{M Q'_2})$.

Set $1 \mathcal{W}_{M Q_2} = \{w \in 1 \mathcal{W}_{M Q_2} : w \text{ is a lift of some element in } \mathcal{W}_{M Q_2}\}$ and $1 \mathcal{W}_{M Q'_2}$ similarly. Then $Z_k \cap 1 \mathcal{W}_{M Q'_2} \subset (\Lambda(1) \cap 1 \mathcal{W}_{M Q'_2}) \cap 1 \mathcal{W}_{M Q_2}$ and $1 \mathcal{W}_{M Q'_2} \subset 1 \mathcal{W}_{M Q'_2} \cap 1 \mathcal{W}_{M Q_2}$. This implies that (4.26) where $Q \subset Q'$ has been replaced by $Q_2 \subset Q'_2$ follows from a congruence

\[(4.27)\] 
\[\sum_{d \in \mathcal{W}_{M Q_2} \mathcal{W}_{M Q'_2}} T_d \equiv \sum_{d \in \mathcal{W}_{M Q_2} \mathcal{W}_{M Q'_2}} q_{w_Q d w_{Q'}} T^*_d\]

in the finite subring $H(1 \mathcal{W}_{M Q_2})$ of $\mathcal{H}$ generated by $\{T_{w} : w \in 1 \mathcal{W}_{M Q_2}\}$ modulo the right ideal $J_2$ with generators $\{\theta_Q(T^*_d) - 1 \mid w \in (Z_k \cap 1 \mathcal{W}_{M Q'_2}) \cup 1 \mathcal{W}_{M Q'_2}\}$.

Another simplification concerns $T^*_d$ modulo $J_2$ for $d \in \mathcal{W}_{M Q'_2}$. We recall that for any reduced decomposition $d = s_1 \ldots s_n$ with $s_i \in S \cap \mathcal{W}_{M Q'_2}$ we have $T^*_d = (T_{\hat{s}_1} - c_{\hat{s}_1}) \ldots (T_{\hat{s}_n} - c_{\hat{s}_n})$ where the $\tilde{s}_i$ are admissible. For $\tilde{s}$ admissible, by (3.2)

$\tilde{c}_\tilde{s} \equiv q_\tilde{s} - 1.$

Therefore

$T^*_d \equiv (T_{\tilde{s}_1} - q_{\tilde{s}_1} + 1) \ldots (T_{\tilde{s}_n} - q_{\tilde{s}_n} + 1)$.

Let $J' \subset J_2$ be the ideal of $H(1 \mathcal{W}_{M Q'_2})$ generated by $\{T_t - 1 \mid t \in Z_k \cap 1 \mathcal{W}_{M Q'_2}\}$. Then the ring $H(1 \mathcal{W}_{M Q'_2})/J'$ and its right ideal $J_2/J'$ are the specialization of the generic finite ring $H(\mathcal{W}_{M Q'_2})^g$ over $\mathbb{Z}[[q_s]_{s \in S_{M Q'_2}}]$ where the $q_s$ for $s \in S_{M Q'_2} = S \cap \mathcal{W}_{M Q'_2}$ are indeterminates, and of its right ideal $J_2^g$ with the same generators. The similar congruence modulo $J_2^g$ in $H(\mathcal{W}_{M Q'_2})^g$ (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where $H$ is the generic Hecke ring of a finite Coxeter system $(\mathcal{W}, S)$ and parameters $(q_s)_{s \in S}$ such that $q_s = q_{s'}$ when $s, s'$ are conjugate in $\mathcal{W}$. The Hecke ring $H$ is a $\mathbb{Z}[[q_s]_{s \in S}]$-free
Lemma 4.32. A basis of \((T_w)_{w \in \mathcal{W}}\) for all \(s \in S\). The other basis \((T_w^*)_{w \in \mathcal{W}}\) satisfies the braid relations and the quadratic relations \((T_w^*)^2 = q_s + (q_s - 1)T_w^*\) for \(s \in S\), and is related to the first basis by \(T_w^* = T_s - (q_s - 1)\) for \(s \in S\), and more generally \(T_wT_{w^{-1}} = T_{w^{-1}}T_w = q_w\) for \(w \in \mathcal{W}\) [Vig16, Proposition 4.13]. Let \(J \subset S\) and \(J\) is the right ideal of \(H\) with generators \(T_w^* - 1\) for all \(w\) in the group \(\mathcal{W}_J\) generated by \(J\).

** Lemma 4.33.** In the generic Hecke ring \(H\), the congruence modulo \(J\)

\[
\sum_{d \in \mathcal{W}_J \mathcal{W}} T_d \equiv \sum_{d \in \mathcal{W}_J \mathcal{W}} q_{w,J,dw}T_d^* \quad \text{for all } w \in \mathcal{W}. 
\]

holds true.

**Proof.**

**Step 1.** We show

\[
\mathcal{W}_J \mathcal{W} = w_J \mathcal{W} \mathcal{W} w, \quad q_{w,J,dw}T_d^* = T_{w,J}T_{w,J,dw}T_w^*.
\]

The equality between the groups follows from the characterization of \(\mathcal{W}_J \mathcal{W}\) in \(\mathcal{W}\): an element \(d \in \mathcal{W}\) has minimal length in \(\mathcal{W}_J d\) if and only if \(\ell(ud) = \ell(u) + \ell(d)\) for all \(u \in \mathcal{W}_J\). An easy computation shows that \(\ell(udw) = \ell(u) + \ell(wdw)\) for all \(u \in \mathcal{W}_J, d \in \mathcal{W}_J \mathcal{W}\) (both sides are equal to \(\ell(u) + \ell(w) - \ell(w_J) - \ell(d)\)). The second equality follows from \(q_{w,J,dw} = q_{dw}w\) because \((w_J)^2 = 1\) and \(\ell(w_J) + \ell(w_J dw) = \ell(dw)\) (both sides are \(\ell(w) = \ell(d)\)) and from \(q_{dw}T_d^* = T_{dw}T_{w^{-1}}T_d = T_{dw}T_w^*\). We also have \(T_{dw} = T_{w,J}T_{w,J,dw}\).

**Step 2.** The multiplication by \(q_{w,J}\) on the quotient \(H/J\) is injective (Lemma 4.32) and \(q_{w,J} \equiv T_{w,J}\). By Step 1 \(q_{w,J,dw}T_d^* = T_{w,J,dw}T_w^*\) and

\[
\sum_{d \in \mathcal{W}_J \mathcal{W}} q_{w,J,dw}T_d^* \equiv \sum_{d \in \mathcal{W}_J \mathcal{W}} T_d T_w^* \quad \text{for all } s \in S.
\]

The congruence

\[
(4.28) \quad \sum_{d \in \mathcal{W}_J \mathcal{W}} T_d \equiv \sum_{d \in \mathcal{W}_J \mathcal{W}} T_d T_s^*
\]

for all \(s \in S\) implies the lemma because \(T_w^* = T_{s_1}^* \ldots T_{s_n}^*\) for any reduced decomposition \(w = s_1 \ldots s_n\) with \(s_i \in S\).
Step 3. When $J = \emptyset$, the congruence (4.28) is an equality

\[(4.29) \quad \sum_{w \in W} T_w = \sum_{w \in W} T_w T_s^*.
\]

It holds true because $\sum_{w \in W} T_w = \sum_{w < w_s} T_w (T_s + 1)$ and $(T_s + 1) T_s^* = T_s T_s^* + T_s^* = q_s + T_s^* = T_s + 1$.

Step 4. Conversely the congruence (4.28) follows from (4.29) because

\[
\sum_{w \in W} T_w = (\sum_{u \in W} T_u) \sum_{d \in \mathcal{W} \cap J} T_d \equiv (\sum_{u \in W} q_u) \sum_{d \in \mathcal{W} \cap J} T_d
\]

(recall $q_u = T_{u^{-1}} u \equiv T_u$) and we can simplify by $\sum_{u \in W} q_u$ in $H/J$. \hfill \Box

This ends the proof of Proposition 4.23.

5. Universal representation

$I_H(P, V, Q) \otimes H R[U\backslash G]$

The invariant functor $(-)^{\mathcal{U}}$ by the pro-$p$-Iwahori subgroup $U$ of $G$ has a left adjoint

\[\otimes R[U\backslash G] : \text{Mod}_R(H) \rightarrow \text{Mod}_R^\infty(G).\]

The smooth $R$-representation $V \otimes_{H_R} R[U\backslash G]$ of $G$ constructed from the right $H_R$-module $V$ is called universal. We write

\[R[U\backslash G] = X.\]

**Question 5.1.** Does $V \neq 0$ imply $V \otimes_{H_R} X \neq 0$ or does $v \otimes 1_U = 0$ for $v \in V$ imply $v = 0$? We have no counterexample. If $R$ is a field and the $H_R$-module $V$ is simple, the two questions are equivalent: $V \otimes_{H_R} X \neq 0$ if and only if the map $v \mapsto v \otimes 1_U$ is injective. When $R$ is an algebraically closed field of characteristic $p$, $V \otimes_{H_R} X \neq 0$ for all simple $H_R$-modules $V$ if this is true for $V$ simple supersingular (this is a consequence of Corollary 5.13).

The functor $- \otimes_{H_R} X$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P = MN$ be a standard parabolic subgroup and $X_M = R[U_M \backslash M]$. We have functor isomorphisms

\[(5.1) \quad (- \otimes_{H_R} X) \circ \text{Ind}^H_{H_M} \rightarrow \text{Ind}^G_P (- \otimes_{H_{M,R}} X_M),
\]

\[(5.2) \quad (-)_N \circ (- \otimes_{H_R} X) \rightarrow (- \otimes_{H_{M,R}} X_M) \circ \text{Ind}^H_{H_M}.\]

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism $\text{Ind}^H_{H_M} \circ (-)^{\mathcal{U}_M} \rightarrow (-)^{\mathcal{U}} \circ \text{Ind}^G_P$ [OV17, formula (4.14)]. If $V$ is a right $H_R$-supersingular module and $p$ is nilpotent in $V$, then $L^H_{H_M}(V) = 0$ if $M \neq G$ (Proposition 4.27). Applying (5.2) we deduce the following.

**Proposition 5.2.** If $p$ is nilpotent in $V$ and $V$ supersingular, then $V \otimes_{H_R} X$ is left cuspidal.
Remark 5.3. For a non-zero smooth $R$-representation $\tau$ of $M$, $\Delta_\tau$ is orthogonal to $\Delta_P$ if $\tau$ is left cuspidal. Indeed, we recall from [AHHV17 II.7 Corollary 2] that $\Delta_\tau$ is not orthogonal to $\Delta_P$ if and only if there exists a proper standard parabolic subgroup $X$ of $M$ such that $\sigma$ is trivial on the unipotent radical of $X$; moreover $\tau$ is a subrepresentation of $\text{Ind}_X^M(\tau|_X)$, so the image of $\tau$ by the left adjoint of $\text{Ind}_X^M$ is not 0.

From now on, $\mathcal{V}$ is a non-zero right $\mathcal{H}_{M,R}$-module and

$$\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} X_M.$$

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of $G$ with $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp,\sigma}$ where $\Delta_{\perp,\sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to $\Delta_P$.

**Proposition 5.4.**

1. $P(\mathcal{V}) \subset P_\perp(\sigma)$ if $\sigma \neq 0$.
2. $P(\mathcal{V}) = P_\perp(\sigma)$ if the map $v \mapsto v \otimes 1_{\mathcal{U}_M}$ is injective.
3. $P(\mathcal{V}) = P(\sigma)$ if the map $v \mapsto v \otimes 1_{\mathcal{U}_M}$ is injective, $p$ nilpotent in $\mathcal{V}$ and $\mathcal{V}$ supersingular.
4. $P(\mathcal{V}) = P(\sigma)$ if $\sigma \neq 0$, $R$ is a field of characteristic $p$ and $\mathcal{V}$ simple supersingular.

**Proof.**

1. $P(\mathcal{V}) \subset P_\perp(\sigma)$ means that $Z \cap M^\vee_\mathcal{V}$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}_M}$, where $M_\mathcal{V}$ is the standard Levi subgroup such that $\Delta_{M_\mathcal{V}} = \Delta_\mathcal{V}$. Let $z \in Z \cap M'_\mathcal{V}$ and $v \in \mathcal{V}$. As $\Delta_\mathcal{M}$ and $\Delta_\mathcal{V}$ are orthogonal, we have $T^M \ast (z) = T^M(z)$ and $\mathcal{U}_M z \mathcal{U}_M = \mathcal{U}_M z$. We have $v \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes T^M(z) 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M} = v \otimes z^{-1} 1_{\mathcal{U}_M} = z^{-1} (v \otimes 1_{\mathcal{U}_M})$. 

2. If $v \otimes 1_{\mathcal{U}_M} = 0$ for $v \in \mathcal{V}$ implies $v = 0$, then $\sigma = 0$ because $\mathcal{V} \neq 0$. By (1) $P(\mathcal{V}) \subset P_\perp(\sigma)$. As in the proof of (1), for $z \in Z \cap M'_\perp,\sigma$ we have $v T^M \ast(z) \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M}$ and our hypothesis implies $v T^M \ast(z) = v$ hence $P(\mathcal{V}) \supset P_\perp(\sigma)$.

3. Proposition 5.2, Remark 5.3 and (2).

4. Question 5.1 and (3). □

Let $Q$ be a parabolic subgroup of $G$ with $P \subset Q \subset P(\mathcal{V})$. In this chapter we will compute $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ where $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^R(e(\mathcal{V}) \otimes (\text{St}_{Q \cap M(\mathcal{V})}^M(\mathcal{V})))^{\mathcal{U}(\mathcal{V})}$. The smooth $R$-representation $I_G(P, \sigma, Q)$ of $G$ is well defined: it is 0 if $\sigma = 0$ and $\text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^R(e(\sigma) \otimes \text{St}_Q^P(\sigma))$ if $\sigma \neq 0$ because $(P, \sigma, Q)$ is an $R[G]$-triple by Proposition 5.4. We will show that the universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ is isomorphic to $I_G(P, \sigma, Q)$, if $P(\mathcal{V}) = P(\sigma)$ and $p = 0$, or if $\sigma = 0$ (Corollary 5.12). In particular, $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$ when $R$ is an algebraically closed field of characteristic $p$ and $\mathcal{V}$ is supersingular.

5.1. $Q = G$. We consider first the case $Q = G$. We are in the simple situation where $\mathcal{V}$ is extensible to $\mathcal{H}$ and $P(\mathcal{V}) = P(\sigma) = G$, $I_{\mathcal{H}}(P, \mathcal{V}, G) = e(\mathcal{V})$ and $I_G(P, \sigma, G) = e(\sigma)$. We recall that $\Delta \setminus \Delta_P$ is orthogonal to $\Delta_P$ and that $M_2$ denotes the standard Levi subgroup of $G$ with $\Delta_{M_2} = \Delta \setminus \Delta_P$. 

The $\mathcal{H}_R$-morphism $e(V) \to e(\sigma)_{M} = \sigma^M$ sending $v$ to $v \otimes 1_{U_M}$ for $v \in V$, given by adjunction an $R[G]$-homomorphism

$$v \otimes 1_U \mapsto v \otimes 1_{U_M} : e(V) \otimes_{\mathcal{H}_R} X \xrightarrow{\Phi^G} e(\sigma).$$

If $\Phi^G$ is an isomorphism, then $e(V) \otimes_{\mathcal{H}_R} X$ is the extension to $G$ of $(e(V) \otimes_{\mathcal{H}_R} X)|_M$, meaning that $M_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. The converse is true.

**Lemma 5.6.** If $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$, then $\Phi^G$ is an isomorphism.

**Proof.** Suppose that $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. Then $e(V) \otimes_{\mathcal{H}_R} X$ is the extension to $G$ of $(e(V) \otimes_{\mathcal{H}_R} X)|_M$, and by Theorem 3.13 $(e(V) \otimes_{\mathcal{H}_R} X)^{M}$ is the extension of $(e(V) \otimes_{\mathcal{H}_R} X)^{M'}$. Therefore, by (3.12),

$$(v \otimes 1_U)T_w = (v \otimes 1_U)T_{w^*}^{M} \quad \text{for all} \quad v \in V, \quad w \in W_M(1).$$

As $V$ is extensible to $H$, the natural map $v \mapsto v \otimes 1_U : V \xrightarrow{\Psi} (e(V) \otimes_{\mathcal{H}_R} X)^{M}$ is $\mathcal{H}_M$-equivariant, i.e.,

$$vT_{w}^{M^*} \otimes 1_U = (v \otimes 1_U)T_{w}^{M^*} \quad \text{for all} \quad v \in V, \quad w \in W_M(1),$$

because (3.12) $vT_{w}^{M^*} \otimes 1_U = vT_{w} \otimes 1_U = v \otimes T_{w}^{*} = (v \otimes 1_U)T_{w}^{*}$ in $(e(V) \otimes_{\mathcal{H}_R} X)$.

We recall that $- \otimes_{\mathcal{H}_{M,R}} X_M$ is the left adjoint of $(-)^{M}$. The adjoint $R[M]$-homomorphism $\sigma = V \otimes_{\mathcal{H}_{M,R}} X_M \to e(V) \otimes_{\mathcal{H}_R} X$ sends $v \otimes 1_{U_M}$ to $v \otimes 1_U$ for all $v \in V$. The $R[M]$-module generated by the $v \otimes 1_U$ for all $v \in V$ is equal to $e(V) \otimes_{\mathcal{H}_R} X$ because $M'_2$ acts trivially. Hence we obtained an inverse of $\Phi^G$. □

Our next move is to determine if $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. It is equivalent to see if $M'_2$ acts trivially on $e(V) \otimes 1_U$ as this set generates the representation $e(V) \otimes_{\mathcal{H}_R} X$ of $G$ and $M'_2$ acts as $M'_2$ and $M$ commute and $G = ZM'_2$. Obviously, $U \cap M'_2$ acts trivially on $e(V) \otimes 1_U$. The group of double classes $(U \cap M'_2) \setminus M'_2/(U \cap M'_2)$ is generated by the lifts $s \in N \cap M'_2$ of the simple affine roots $s$ of $W_{M'_2}$. Therefore, $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$ if and only if for any simple affine root $s \in S_{M'_2}^\text{aff}$ of $W_{M'_2}$, any $s \in N \cap M'_2$ lifting $s$ acts trivially on $e(V) \otimes 1_U$.

**Lemma 5.6.** Let $v \in V$, $s \in S_{M'_2}^\text{aff}$ and $\hat{s} \in N \cap M'_2$ lifting $s$. We have

$$(q_s + 1)(v \otimes 1_U - \hat{s}(v \otimes 1_U)) = 0.$$

**Proof.** We compute:

$$T_s(\hat{s}1_U) = \hat{s}(T_s1_U) = 1_{U_M\hat{s}U(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_U = \sum_{u^o} u^{op}1_U,$$

$$T_s(\hat{s}1_U) = \hat{s}^2(T_s1_U) = 1_{U_M\hat{s}U(\hat{s})^{-2}} = \sum_u u\hat{s}1_U$$

for $u$ in the group $U/(\hat{s}^{-1}U\hat{s} \cap U)$ and $u^{op}$ in the group $\hat{s}U(\hat{s})^{-1}/(\hat{s}\hat{s}U(\hat{s})^{-1} \cap U)$; the reason is that $\hat{s}^2$ normalizes $U$, $U\hat{s}U\hat{s}^{-1}$ is the disjoint union of the sets $U\hat{s}u^{-1}(\hat{s})^{-1}$ and $\hat{s}U(\hat{s})^{-1}U$ is the disjoint union of the sets $U\hat{s}^{-1}u^{-1}$, we introduce now a natural bijection

$$u \mapsto u^{op} : U/(\hat{s}^{-1}U\hat{s} \cap U) \to \hat{s}U(\hat{s})^{-1}/(\hat{s}\hat{s}U(\hat{s})^{-1} \cap U)$$

which is not a group homomorphism. We recall the finite reductive group $G_{k,s}$ quotient of the parahoric subgroup $\mathbb{R}_s$ of $G$ fixing the face fixed by $s$ of the alcove $C$. The Iwahori groups $Z^0U$ and $Z^0\hat{s}U(\hat{s})^{-1}$ are contained in $\mathbb{R}_s$ and their images
in \(G_{s,k}\) are opposite Borel subgroups \(Z_kU_{s,k}\) and \(Z_kU_{s,k}^{op}\). Via the surjective maps 
\[ u \mapsto \overline{u} : \mathcal{U} \to U_{s,k} \text{ and } u^{op} \mapsto \overline{u^{op}} : \mathcal{U}(\hat{s})^{-1} \to U_{s,k}^{op} \] we identify the groups 
\[ \mathcal{U}/(\hat{s}^{-1}U\hat{s} \cap \mathcal{U}) \simeq U_{s,k} \text{ and similarly } \hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U}) \simeq U_{s,k}^{op}. \]
Let \(G'_{k,s}\) be the group generated by \(U_{s,k}\) and \(U_{s,k}^{op}\), and let \(B'_{s,k} = G'_{k,s} \cap Z_kU_{s,k} = (G'_{k,s} \cap Z_k)U_{s,k}\). We suppose (as we can) that \(\hat{s} \in \mathfrak{s}_k\) and that its image \(\hat{s}_k\) in \(G_{s,k}\) lies in \(G'_{k,s}\). We have \(\hat{s}_kU_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{op}\) and the Bruhat decomposition \(G'_{k,s} = B'_{k,s} \cup U_{s,k}\hat{s}_kB_{k,s}\) implies the existence of a canonical bijection \(\overline{u^{op}} \mapsto \overline{u : (U_{k,s} \setminus \{1\})} \to (U_{k,s} \setminus \{1\})\) respecting the cosets \(\overline{u^{op}B'_{k,s}} = \overline{u\hat{s}_kB'_{k,s}}\). Via the preceding identifications we get the wanted bijection \(\overline{5.3}\).

For \(v \in e(\mathcal{V})\) and \(z \in Z^0 \cap M_2^\prime\) we have \(vT_z = v\), \(z1_{\mathcal{U}} = T_z1_{\mathcal{U}}\) and \(v \otimes T_z1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}}\) therefore \(Z^0 \cap M_2^\prime\) acts trivially on \(\mathcal{V} \otimes 1_{\mathcal{U}}\). The action of the group \((Z^0 \cap M_2^\prime)\mathcal{U}\) on \(\mathcal{V} \otimes 1_{\mathcal{U}}\) is also trivial. As the image of \(Z^0 \cap M_2^\prime\) in \(G_{s,k}\) contains \(Z_k \cap G'_{s,k}\),
\[ u\hat{s}(v \otimes 1_{\mathcal{U}}) = u^{op}(v \otimes 1_{\mathcal{U}}) \]
when \(u\) and \(u^{op}\) are not units and correspond via the bijection \(\overline{5.3}\). So we have
\[ v \otimes T_s(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_s(\hat{s}^21_{\mathcal{U}}) - v \otimes \hat{s}1_{\mathcal{U}}. \]
We can move \(T_s\) on the other side of \(\otimes\) and as \(vT_s = q_s v\) (Corollary \(3.9\)), we can replace \(T_s\) by \(q_s\). We have \(v \otimes \hat{s}^21_{\mathcal{U}} = v \otimes T_{s^2}1_{\mathcal{U}}\) because \(\hat{s}^2 \in Z^0 \cap M_2^\prime\) normalizes \(\mathcal{U}\); as we can move \(T_{s^2}\) on the other side of \(\otimes\) and as \(vT_{s^2} = v\) we can forget \(\hat{s}^2\). So \(\overline{5.4}\) is equivalent to \((q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0. \)

Combining the two lemmas we obtain the following.

**Proposition 5.7.** When \(\mathcal{V}\) is extensible to \(\mathcal{H}\) and has no \(q_s + 1\)-torsion for any \(s \in S^\text{aff}_{M_2^\prime}\), then \(M_2^\prime\) acts trivially on \(e(\mathcal{V}) \otimes_\mathcal{H} \mathbb{X}\) and \(\Phi^G\) is an \(R[G]\)-isomorphism.

**Example 5.8.** Let \(G = GL(2, F)\) and let \(R\) be an algebraically closed field where \(q_{s_0} + 1 = q_{s_1} + 1 = 0\) and \(S^\text{aff} = \{s_0, s_1\}\). (Note that \(q_{s_0} = q_{s_1}\) is the order of the residue field of \(F\).) Then the dimension of \(1_\mathcal{H} \otimes_\mathcal{H} \mathbb{X}\) is infinite, in particular \(1_\mathcal{H} \otimes_\mathcal{H} \mathbb{X} \neq 1_G\).

Indeed, the Steinberg representation \(\text{St}_G = (\text{Ind}_G^Z 1_Z) / 1_G\) of \(G\) is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation \(\pi\) with \(\pi^U = 0\) of quotient the character \((-1)\text{val}^\text{det}\). This follows from the proof of Theorem 3 and from Proposition 24 in [Vig80]. The kernel of the quotient map \(\text{St}_G \otimes (-1)\text{val}^\text{det} \to 1_G\) is infinite dimensional without a non-zero \(\mathcal{U}\)-invariant vector. As the characteristic of \(R\) is not \(p\), the functor of \(\mathcal{U}\)-invariants is exact hence \((\text{St}_G \otimes (-1)\text{val}^\text{det})^\mathcal{U} = 1_\mathcal{H}\). As \(- \otimes_\mathcal{H} R[U \setminus G]\) is the left adjoint of \((-)^\mathcal{U}\) there is a non-zero homomorphism
\[ 1_\mathcal{H} \otimes_\mathcal{H} \mathbb{X} \to \text{St}_G \otimes (-1)\text{val}^\text{det} \]
with image generated by its \(\mathcal{U}\)-invariants. The homomorphism is therefore surjective.
5.2. $\mathcal{V}$ extensible to $\mathcal{H}$. Let $P = MN$ be a standard parahoric subgroup of $G$ with $\Delta_P$ and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$, but now $P \subset Q \subset G$. So we have $I_\mathcal{H}(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (St_Q^G)^{1^U}$ and $I_G(P, \sigma, Q) = e(\sigma) \otimes_R St_Q^G$ where $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} X_M$. We compare the images by $- \otimes_{\mathcal{H}_R} X$ of the $\mathcal{H}_R$-modules $e(\mathcal{V}) \otimes_R (\text{Ind}^G_Q 1)^{1^U}$ and $e(\mathcal{V}) \otimes_R (St_Q^G)^{1^U}$ with the smooth $R$-representations $e(\sigma) \otimes \text{Ind}^G_Q 1$ and $e(\sigma) \otimes St_Q^G$ of $G$.

As $- \otimes_{\mathcal{H}_R} X$ is left adjoint of $(-)^{1^U}$, the $\mathcal{H}_R$-homomorphism $v \otimes f \mapsto v \otimes 1_{U_M} \otimes f : e(\mathcal{V}) \otimes_R (\text{Ind}^G_Q 1)^{1^U} \to (e(\sigma) \otimes_R \text{Ind}^G_Q 1)^{1^U}$ gives by adjunction an $R[G]$-homomorphism

$$v \otimes f \otimes 1_{U_M} \mapsto v \otimes 1_{U_M} : e_{\mathcal{V}}(\mathcal{V}) \otimes_{\mathcal{H}_{M,R}} X_{M,\mathcal{H}} \xrightarrow{\Phi^G_Q} e(\sigma) \otimes_{R} \text{Ind}^G_Q 1.$$

When $Q = G$ we have $\Phi^G_Q = \Phi^G$. By Remark 4.10 $\Phi^G_Q$ is surjective. Proposition 5.7 applies with $M_Q$ instead of $G$ and gives the $R[M_Q]$-homomorphism

$$v \otimes 1_{U_{M,Q}} \mapsto v \otimes 1_{U_M} : e_{\mathcal{V}}(\mathcal{V}) \otimes_{\mathcal{H}_{M,Q,R}} X_{M,Q} \xrightarrow{\Phi^Q} e(\sigma).$$

**Proposition 5.9.** The $R[G]$-homomorphism $\Phi^G_Q$ is an isomorphism if $\Phi^Q$ is an isomorphism, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\text{aff}}_{M,Q}.$

**Proof.** The proposition follows from another construction of $\Phi^G_Q$ that we now describe. Proposition 4.5 gives the $\mathcal{H}_R$-module isomorphism

$$v \otimes f_{QU} \mapsto v \otimes 1_{\mathcal{H}} : (e(\mathcal{V}) \otimes_R (\text{Ind}^G_Q 1)^{1^U}) \to \text{Ind}^R_{\mathcal{H}_Q}(e_{\mathcal{V}}(\mathcal{V})) = e_{\mathcal{V}}(\mathcal{V}) \otimes_{\mathcal{H}_{M,Q,R}} \mathcal{H}.$$

We have the $R[G]$-isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_{\mathcal{H}} \otimes 1_{U_M} \mapsto f_{QU,v \otimes 1_{U_{M,Q}}} : \text{Ind}^H_{\mathcal{H}_Q}(e_{\mathcal{V}}(\mathcal{V})) \otimes_{\mathcal{H}_R} X \to \text{Ind}^G_{\mathcal{V}}(e_{\mathcal{V}}(\mathcal{V}) \otimes_{\mathcal{H}_{M,Q,R}} X_{M,Q})$$

and the $R[G]$-isomorphism

$$f_{QU,v \otimes 1_{U_{M}}} \mapsto v \otimes 1_{U_M} \otimes f_{QU} : \text{Ind}^G_Q(e(\sigma)(\mathcal{V})) \to e(\sigma) \otimes \text{Ind}^G_Q 1.$$

From $\Phi^Q$ and these three homomorphisms, there exists a unique $R[G]$-homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{Ind}^G_Q 1)^{1^U}) \otimes_{\mathcal{H}_R} X \to e(\sigma) \otimes_{R} \text{Ind}^G_Q 1$$

sending $v \otimes f_{QU} \otimes 1_{U}$ to $v \otimes 1_{U_M} \otimes f_{QU}$. We deduce: this homomorphism is equal to $\Phi^G_Q$, $\mathcal{V} \otimes 1_{U_1} \otimes 1_{U_2} \otimes f_{QU}$ generates $(e(\mathcal{V}) \otimes_R (\text{Ind}^G_Q 1)^{1^U}) \otimes_{\mathcal{H}_R} X$, if $\Phi^Q$ is an isomorphism, then $\Phi^G_Q$ is an isomorphism. By Proposition 5.7 if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\text{aff}}_{M,Q}$, then $\Phi^Q$ and $\Phi^G_Q$ are isomorphisms. \hfill $\Box$

We recall that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$.

**Proposition 5.10.** The $R[G]$-homomorphism $\Phi^G_Q$ induces an $R[G]$-homomorphism

$$(e(\mathcal{V}) \otimes_R (St_Q^G)^{1^U}) \otimes_{\mathcal{H}_R} X \to e(\sigma) \otimes_{R} St_Q^G,$$

It is an isomorphism if $\Phi^G_Q$ is an $R[G]$-isomorphism for all parabolic subgroups $Q'$ of $G$ containing $Q$, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S^{\text{aff}}_{M,Q}$. 


Proof. The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations \( e(\sigma) \otimes_R \text{St}_Q^G \) and \( (e(V) \otimes_R (\text{St}_Q^G)_{U}) \otimes_{\mathcal{H}_R} X \) of \( G \) are the cokernels of the natural \( R[G] \)-homomorphisms

\[
\oplus_{Q \subseteq Q'} e(\sigma) \otimes_R \text{Ind}_{Q}^{G} 1 \xrightarrow{\text{id} \otimes \alpha} e(\sigma) \otimes_R \text{Ind}_{Q}^{G} 1,
\]

\[
\oplus_{Q \subseteq Q'} (e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)_{U}) \otimes_{\mathcal{H}_R} X \xrightarrow{\text{id} \otimes \alpha' \otimes \text{id}} (e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)_{U}) \otimes_{\mathcal{H}_R} X.
\]

These \( R[G] \)-homomorphisms make a commutative diagram with the \( R[G] \)-homomorphisms \( \oplus_{Q \subseteq Q'} \Phi^G_Q \) and \( \Phi^G_Q \) going from the lower line to the upper line. Indeed, let \( v \otimes f_{Q'U} \otimes 1_U \in (e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)_{U}) \otimes_{\mathcal{H}_R} X \). On the one hand, it goes to \( v \otimes f_{Q'U} e(Q'_{1}) \otimes 1_U \in (e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)_{U}) \otimes_{\mathcal{H}_R} X \) by the horizontal map, and then to \( v \otimes 1_{U_M} \otimes f_{Q'U} e(Q'_{1}) \) by the vertical map. On the other hand, it goes to \( v \otimes 1_{U_M} \otimes f_{Q'U} e(Q'_{1}) \) by the horizontal map, and then to \( v \otimes 1_{U_M} \otimes f_{Q'U} e(Q'_{1}) \) by the horizontal map. One deduces that \( \Phi^G_Q \) induces an \( R[G] \)-homomorphism \( (e(V) \otimes_R (\text{St}_Q^G)_{U}) \otimes_{\mathcal{H}_R} X \rightarrow e(\sigma) \otimes_R \text{St}_Q^G \), which is an isomorphism if \( \Phi^G_Q \) is an \( R[G] \)-isomorphism for all \( Q \subseteq Q' \).

5.3. General. We consider now the general case: let \( P = MN \subset Q \) be two standard parabolic subgroups of \( G \) and let \( \mathcal{V} \) be a non-zero right \( \mathcal{H}_{M,R} \)-module with \( Q \subset P(\mathcal{V}) \). We recall \( I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\text{St}_Q^P)_{U(\mathcal{V})}) \) and \( \sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathcal{X}_M \) (Proposition 5.4). There is a natural \( R[G] \)-homomorphism

\[
I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} X \xrightarrow{\phi^G_P} \text{Ind}_{P(\mathcal{V})}^{\mathcal{G}}(e(\mathcal{V}) \otimes_R \text{St}_Q^P) \]

obtained by composition of the \( R[G] \)-isomorphism [OV17, Corollary 4.7] (proof of Proposition 3.3):

\[
I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} X \rightarrow \text{Ind}_{P(\mathcal{V})}^{\mathcal{G}}((e(\mathcal{V}) \otimes_R (\text{St}_Q^P)_{U(\mathcal{V})}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathcal{X}_M(\mathcal{V}))
\]

with the \( R[G] \)-homomorphism

\[
\text{Ind}_{P(\mathcal{V})}^{\mathcal{G}}((e(\mathcal{V}) \otimes_R (\text{St}_Q^P)_{U(\mathcal{V})}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathcal{X}_M(\mathcal{V})) \rightarrow \text{Ind}_{P(\mathcal{V})}^{\mathcal{G}}(e(\mathcal{V}) \otimes_R \text{St}_Q^P)
\]

image by the parabolic induction \( \text{Ind}_{P(\mathcal{V})}^{\mathcal{G}} \) of the homomorphism

\[
(e(\mathcal{V}) \otimes_R (\text{St}_Q^P)_{U(\mathcal{V})}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathcal{X}_M(\mathcal{V}) \rightarrow e(\mathcal{V}) \otimes_R \text{St}_Q^P
\]

induced by the \( R[M(\mathcal{V})] \)-homomorphism \( \Phi^P_Q = \Phi^M_{Q \cap M(\mathcal{V})} \) of Proposition 5.10 applied to \( M(\mathcal{V}) \) instead of \( G \).

This homomorphism \( \Phi^G_P \) is an isomorphism if \( \Phi^P_Q \) is an isomorphism, in particular if \( \mathcal{V} \) has no \( q_s + 1 \)-torsion for any \( s \in S^\text{aff}_{M_2} \) where \( \Delta_{M_2} = \Delta_{M(\mathcal{V})} \setminus \Delta_M \) (Proposition 5.10). We get the main theorem of this section.

Theorem 5.11. Let \( (P = MN, \mathcal{V}, Q) \) be an \( \mathcal{H}_R \)-triple and \( \sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M] \). Then, \( (P, \sigma, Q) \) is an \( R[G] \)-triple. The \( R[G] \)-homomorphism

\[
I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \setminus G] \xrightarrow{\Phi^G_P} \text{Ind}_{P(\mathcal{V})}^{\mathcal{G}}(e(\mathcal{V}) \otimes_R \text{St}_Q^P)
\]

is an isomorphism if \( \Phi^P_Q \) is an isomorphism. In particular \( \Phi^G_P \) is an isomorphism if \( \mathcal{V} \) has no \( q_s + 1 \)-torsion for any \( s \in S^\text{aff}_{M_2} \).
Recalling $I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{p(\sigma)})$ when $\sigma \neq 0$, we deduce the following.

**Corollary 5.12.** We have the following:
$I_H(P, V, Q) \otimes_{H_r} R[U\backslash G] \simeq I_G(P, \sigma, Q)$, if $\sigma \neq 0$, $P(V) = P(\sigma)$ and $V$ has no $p_s+1$-torsion for any $s \in S_{M_2}$.
$I_H(P, V, Q) \otimes_{H_r} R[U\backslash G] = I_G(P, \sigma, Q) = 0$, if $\sigma = 0$.

Recalling $P(V) = P(\sigma)$ if $\sigma \neq 0$, $R$ is a field of characteristic $p$ and $V$ simple supersingular (Proposition 5.4 (4)), we deduce the following.

**Corollary 5.13.** $I_H(P, V, Q) \otimes_{H_r} R[U\backslash G] \simeq I_G(P, \sigma, Q)$ if $R$ is a field of characteristic $p$ and $V$ simple supersingular.

6. **Vanishing of the smooth dual**

Let $V$ be an $R[G]$-module. The dual $\text{Hom}_R(V, R)$ of $V$ is an $R[G]$-module for the contragredient action: $gL(gv) = L(v)$ if $g \in G$, $L \in \text{Hom}_R(V, R)$ is a linear form and $v \in V$. When $V \in \text{Mod}_R^p(G)$ is a smooth $R$-representation of $G$, the dual of $V$ is not necessarily smooth. A linear form $L$ is smooth if there exists an open subgroup $H \subset G$ such that $L(hv) = L(v)$ for all $h \in H, v \in V$; the space $\text{Hom}_R(V, R)_{\text{fin}}$ of smooth linear forms is a smooth $R$-representation of $G$, called the **smooth dual** (or smooth contragredient) of $V$. The smooth dual of $V$ is contained in the dual of $V$.

**Example 6.1.** When $R$ is a field and the dimension of $V$ over $R$ is finite, the dual of $V$ is equal to the smooth dual of $V$ because the kernel of the action of $G$ on $V$ is an open normal subgroup $H \subset G$; the action of $G$ on the dual $\text{Hom}_R(V, R)$ is trivial on $H$.

We assume in this section that $R$ is a field of characteristic $p$. Let $P = MN$ be a parabolic subgroup of $G$ and $V \in \text{Mod}_R^p(M)$. Generalizing the proof given in [Vig07, 8.1] when $G = GL(2, F)$ and the dimension of $V$ is 1, we show the following.

**Proposition 6.2.** If $P \neq G$, the smooth dual of $\text{Ind}_P^G(V)$ is 0.

**Proof.** Let $L$ be a smooth linear form on $\text{Ind}_P^G(V)$ and let $K$ be an open pro-$p$-subgroup of $G$ which fixes $L$. Let $J$ be an arbitrary open subgroup of $K$, $g \in G$ and $f \in (\text{Ind}_P^G(V))^J$ with support $PgJ$. We want to show that $L(f) = 0$. Let $J'$ be any open normal subgroup of $J$ and let $\varphi$ denote the function in $(\text{Ind}_P^G(V))^{J'}$ with support $PgJ'$ and value $\varphi(g) = f(g)$ at $g$. For $j \in J$ we have $L(j\varphi) = L(\varphi)$, and the support of $j\varphi(x) = \varphi(xj)$ is $P_gJ'j^{-1}$. The function $f$ is the sum of translates $j\varphi$, where $j$ ranges through the left cosets of the image $X$ of $g^{-1}Pg \cap J$ in $J/J'$, so that $L(f) = rL(\varphi)$ where $r$ is the order of $X$ in $J/J'$. We can certainly find $J'$ such that $r \neq 1$, and then $r$ is a positive power of $p$. As the characteristic of $R$ is $p$ we have $L(f) = 0$. \hfill \Box

The module $R[U\backslash G]$ is contained in the module $R^{U\backslash G}$ of functions $f : U\backslash G \to R$. The actions of $H$ and of $G$ on $R[U\backslash G]$ extend to $R^{U\backslash G}$ by the same formulas. The pairing

$$ (f, \varphi) \mapsto \langle f, \varphi \rangle = \sum_{g \in U\backslash G} f(g)\varphi(g) : R^{U\backslash G} \times R[U\backslash G] \to R $$
identifies $R^d \backslash G$ with the dual of $R[U \backslash G]$. Let $h \in H$ and $\tilde{h} \in H$, $\tilde{h}(g) = h(g^{-1})$ for $g \in G$. We have

$$
\langle f, h \varphi \rangle = \langle \tilde{h} f, \varphi \rangle.
$$

**Proposition 6.3.** When $R$ is an algebraically closed field of characteristic $p$, $G$ is not compact modulo the center and $V$ is a simple supersingular right $H_R$-module, the smooth dual of $V \otimes H_R R[U \backslash G]$ is 0.

**Proof.** Let $H_R^{\text{aff}}$ be the subalgebra of $H_R$ of basis $(T_w)_{w \in W'}(1)$ where $W'(1)$ is the inverse image of $W'$ in $W(1)$. The dual of $V \otimes H_R R[U \backslash G]$ is contained in the dual of $V \otimes H_R^{\text{aff}} R[U \backslash G]$; the $H_R^{\text{aff}}$-module $V|_{H_R^{\text{aff}}}$ is a finite sum of supersingular characters. Let $\chi : H_R^{\text{aff}} \rightarrow R$ be a supersingular character. The dual of $\chi \otimes H_R^{\text{aff}} R[U \backslash G]$ is contained in the dual of $R[U \backslash G]$ isomorphic to $R^d \backslash G$. It is the space of $f \in R^d \backslash G$ with $\tilde{h} f = \chi(h)f$ for all $h \in H_R^{\text{aff}}$. The smooth dual of $\chi \otimes H_R^{\text{aff}} R[U \backslash G]$ is 0 if the dual of $\chi \otimes H_R^{\text{aff}} R[U \backslash G]$ has no non-zero element fixed by $U$. Let us take $f \in R^d \backslash G/U$ with $\tilde{h} f = \chi(h)f$ for all $h \in H_R^{\text{aff}}$. We shall prove that $f = 0$. We have $\tilde{T}_w = T_{w^{-1}}$ for $w \in W(1)$.

Let $\prec$ denote the Bruhat order of $W(1)$ associated to $S^{\text{aff}}$. The elements $(T_t)_{t \in Z_k}$ and $(T_s)_{s \in S^{\text{aff}}}$ where $\tilde{s}$ is an admissible lift of $s$ in $W^{\text{aff}}(1)$, generate the algebra $H_R^{\text{aff}}$ and

$$
T_t T_w = T_{tw}, \quad T_s T_w = \begin{cases} T_{\tilde{w} s}, & \tilde{s}w > w, \\ c_s T_w, & \tilde{s}w < w, \end{cases}
$$

with $c_s = -|Z_k, s| \sum_{t \in Z_k, s} T_t$ because the characteristic of $R$ is $p$. Expressing $f = \sum_{w \in W(1)} a_w T_w$, $a_w \in R$, as an infinite sum, we have

$$
T_t f = \sum_{w \in W(1)} a_{t^{-1} w} T_w, \quad T_s f = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1} w} + a_w c_s) T_w.
$$

A character $\chi$ of $H_R^{\text{aff}}$ is associated to a character $\chi_k : Z_k \rightarrow R^*$ and a subset $J$ of

$$
S^{\text{aff}}_\chi = \{ s \in S^{\text{aff}} \mid (\chi_k)|_{Z_k^{\text{aff}}} \text{ trivial } \}
$$

[Vig15a] Definition 2.7. We have

$$
\left\{ \begin{array}{ll}
(\chi(t)) = \chi_k(t), & t \in Z_k, \\
(\chi(\tilde{s})) = \left\{ \begin{array}{ll}
0, & s \in S^{\text{aff}} \setminus J, \\
-1, & s \in J.
\end{array} \right.
\end{array} \right.
$$

Therefore $\chi_k(t)f = \tilde{T}_t f = T_{t^{-1}} f$ hence $\chi_k(t)a_{tw} = a_{tw}$. We have $\chi(T_s)f = \tilde{T}_s f = T_{(\tilde{s})^{-1}} f = T_{(\tilde{s})^{-1}} T_{(\tilde{s})^{-2}} f = \chi_k((\tilde{s})^2) T_{\tilde{s}} f$; as $(\tilde{s})^2 \in Z_k, s$ three lines before Proposition 4.4] and $J \subset S^{\text{aff}}_\chi$, we obtain

$$
T_{\tilde{s}} f = \left\{ \begin{array}{ll}
0, & s \in S^{\text{aff}} \setminus J, \\
-f, & s \in J.
\end{array} \right.
$$
Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_\tilde{f}$, we get
\[
\sum_{w \in W(1), \tilde{s}w < w} a_w c_\tilde{s} T_w = -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_w T_{tw} \\
= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_{t^{-1}w} T_{tw} \\
= -|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_{tw} \\
= \chi_k(c_\tilde{s}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_{w}.
\]

\[
T_\tilde{f} = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_\tilde{s})) T_w
\]

\[
= \begin{cases} 
\sum_{w \in W(1), \tilde{s}w < w} a_{(\tilde{s})^{-1}w} T_w, & s \in S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k}, \\
\sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} - a_w) T_w, & s \in S^{\text{aff}}_{\chi_k}.
\end{cases}
\]

From the last equality and (6.2) for $T_\tilde{f}$, we get:

\[
a_{\tilde{s}w} = \begin{cases} 
0, & s \in J \cup (S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k}), \tilde{s}w < w, \\
a_w, & s \in S^{\text{aff}}_{\chi_k} \setminus J.
\end{cases}
\]

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J \cup (S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k})$. The character $\chi$ is supersingular if for each irreducible component $X$ of $S^{\text{aff}}$, the intersection $X \cap J$ is not empty and different from $X \ [\text{Vig15a}]$ Definition 2.7, Theorem 6.18]. This implies that the group generated by the $s \in S^{\text{aff}}_{\chi_k} \setminus J$ is finite. If $\chi$ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s \in S^{\text{aff}}_{\chi_k}$. But there is no such element if $S^{\text{aff}}_{\chi_k}$ is not empty. \hfill \Box

**Theorem 6.4.** Let $\pi$ be an irreducible admissible $R$-representation of $G$ with a non-zero smooth dual where $R$ is an algebraically closed field of characteristic $p$. Then $\pi$ is finite dimensional.

**Proof.** Let $(P, \sigma, Q)$ be an $R[G]$-triple with $\sigma$ supercuspidal such that $\pi \simeq I_G(P, \sigma, Q)$. The representation $I_G(P, \sigma, Q)$ is a quotient of $\text{Ind}^G_Q e_Q(\sigma)$ hence the smooth dual of $\text{Ind}^G_Q e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P, \sigma, G) = e(\sigma)$. The smooth dual of $\sigma$ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As $\sigma$ is supercuspidal, the $\mathcal{H}_M$-module $\sigma^{\text{mut}}$ contains a simple supersingular submodule $\mathcal{V}$ [Vig15a] Proposition 7.10, Corollary 7.11]. The functor $- \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ being the right adjoint of $(-)^{\text{mut}}$, the irreducible representation $\sigma$ is a quotient of $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$, hence the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ is not zero. By Proposition 6.3, $M = Z$. Hence $\sigma$ is finite dimensional and the same is true for $e(\sigma) = I_G(B, \sigma, G) \simeq \pi$. \hfill \Box

**Remark 6.5.** When the characteristic of $F$ is 0, Theorem 6.4 was proved by Kohlhaase for a field $R$ of characteristic $p$. He gives two proofs [Koh Propostion 3.9, Remark 3.10], but none of them extends to $F$ of characteristic $p$. Our proof is valid without restriction on the characteristic of $F$ and does not use the results of Kohlhaase. Our assumption that $R$ is an algebraically closed field of characteristic $p$ comes from the classification theorem in [AHHV17].
References


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