

FINITE-DIMENSIONAL REPRESENTATIONS OF MINIMAL NILPOTENT W -ALGEBRAS AND ZIGZAG ALGEBRAS

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ABSTRACT. Let \mathfrak{g} be a simple finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} . To any nilpotent element $e \in \mathfrak{g}$ one can attach an associative (and noncommutative as a general rule) algebra $U(\mathfrak{g}, e)$ which is in a proper sense a “tensor factor” of $U(\mathfrak{g})$. In this article we consider the case in which e belongs to the minimal nonzero nilpotent orbit of \mathfrak{g} . Under these assumptions $U(\mathfrak{g}, e)$ was described explicitly in terms of generators and relations. One can expect that the representation theory of $U(\mathfrak{g}, e)$ would be very similar to the representation theory of $U(\mathfrak{g})$. For example one can guess that the category of finite-dimensional $U(\mathfrak{g}, e)$ -modules is semisimple.

The goal of this article is to show that this is the case if \mathfrak{g} is not simply-laced. We also show that, if \mathfrak{g} is simply-laced and is not of type A_n , then the regular block of finite-dimensional $U(\mathfrak{g}, e)$ -modules is equivalent to the category of finite-dimensional modules of a zigzag algebra.

1. INTRODUCTION

Let \mathfrak{g} be a simple finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} . To any nilpotent element $e \in \mathfrak{g}$ one can attach an associative (and noncommutative as a general rule) algebra $U(\mathfrak{g}, e)$ which is in a proper sense a “tensor factor” of $U(\mathfrak{g})$; see [Los2, Theorem 1.2.1], [Pet, Theorem 2.1]. The notion of W -algebras can be traced back to the work [Lyn]; see also [Kos]. The modern definition of it (which is valid for all nilpotent elements) was given by A. Premet [Pr1]. It turns out that the simple finite-dimensional modules of W -algebras are closely related to the primitive ideals of $U(\mathfrak{g}, e)$; see [Los1, Conjecture 1.2.1]. The simple finite-dimensional modules of W -algebras have attracted considerable attention in the last decade; see [BrG, Br, BK, Dodd, LO, Los1, PT].

In this article we focus on the case in which e belongs to the minimal nonzero nilpotent orbit of \mathfrak{g} . Under these assumptions $U(\mathfrak{g}, e)$ was described explicitly in terms of generators and relations in [Pr2, Theorem 1.1]. Moreover, a gap between primitive ideals of $U(\mathfrak{g})$ and primitive ideals of $U(\mathfrak{g}, e)$ is very small; see [Pr2, Theorem 5.3]. One can guess that the representation theory of $U(\mathfrak{g}, e)$ would be

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very similar to the representation theory of $U(\mathfrak{g})$. For example one can guess that the category of finite-dimensional $U(\mathfrak{g}, e)$ -modules is semisimple.

We will show that, under the assumption that $\mathfrak{g} \not\cong \mathfrak{sl}(n)$, this is true if and only if \mathfrak{g} is not simply-laced; see Theorems 2.1, 2.3. Moreover, we show that, if \mathfrak{g} is simple simply-laced and the Dynkin diagram Γ of \mathfrak{g} is not of type A_n , then the regular block of the category of finite-dimensional modules of $U(\mathfrak{g}, e)$ is equivalent to the category of finite-dimensional representations of a zigzag algebra $A(\Gamma)$ introduced by R. Huerfano and M. Khovanov [HK] (they also provided a description of indecomposable modules of $A(\Gamma)$). Explicit generators and relations for these algebras are presented in the statement of Theorem 7.9.

The cases of Γ of types C_n and G_2 were considered in [Pr2, Corollary 7.1]. Therefore the semisimplicity result is new if \mathfrak{g} is of type B_n and F_4 . If $\mathfrak{g} \cong \mathfrak{sl}(2)$, then $U(\mathfrak{g}, e)$ is isomorphic to an algebra of polynomials in one variable. The regular block of the category of finite-dimensional representations of this polynomial algebra is quite reasonable but it has not enough projective objects. Therefore this block is not equivalent to the category of finite-dimensional modules over a finite-dimensional algebra. One can expect a similar situation if $\mathfrak{g} \cong \mathfrak{sl}(n)$ ($n > 2$). Nevertheless, it is plausible that the category of finite-dimensional $U(\mathfrak{g}, e)$ -modules in this case can be described as locally nilpotent modules over a quiver with relations; see [GS, Theorem 1.1] for an example of such a category. Perhaps, this can be done through a reduction to a similar question on a proper version of a proper Hecke algebra; see [BK, Theorem A].

The paper is organized as follows. In Section 2 we state the main results, i.e., Theorems 2.1, 2.3, and introduce the notation which we need to do this. In Section 3 we recall several facts on W-algebras. In Section 4 we introduce the standard notation related to the simple Lie algebras and recall the notion of a cell in a reflection group. In Section 5 we study primitive ideals attached to the minimal nonzero nilpotent orbit of \mathfrak{g} . In Section 6 we recall the notion of a projective functor and several properties of it. In Section 7 we prove Theorem 2.1. In Section 8 we prove Theorem 2.3. In the appendix we write down a numerical result on maximal subalgebras which is needed in Section 8.

2. NOTATION AND THE MAIN RESULT

Let \mathfrak{g} be a simple finite-dimensional Lie algebra over an algebraically closed field \mathbb{F} of characteristic 0, let Γ be a Dynkin diagram of \mathfrak{g} , and let $e \in \mathfrak{g}$ be an element of the minimal nonzero nilpotent orbit $O \subset \mathfrak{g}$. We recall that to a pair (\mathfrak{g}, e) one can attach an associative algebra $U(\mathfrak{g}, e)$; see [Pr2] for details. For an \mathbb{F} -vector space V we denote by $\dim V$ the dimension V . For an ideal I of an algebra A we denote by $\text{Dim } I$ the Gelfand–Kirillov dimension of A/I . For a set S we denote by $|S|$ the number of elements in S .

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. Algebra $Z(\mathfrak{g})$ can be canonically identified with the center of $U(\mathfrak{g}, e)$ (see [Pr2, Corollary 5.1] and [Pr2, footnote 2] for the general case) and thus we also use notation $Z(\mathfrak{g})$ to denote the center of $U(\mathfrak{g}, e)$. Let m_0 be the intersection of the augmentation ideal (\mathfrak{g}) of $U(\mathfrak{g})$ with $Z(\mathfrak{g})$. It is clear that m_0 is a maximal ideal of $Z(\mathfrak{g})$. For any maximal ideal m of $Z(\mathfrak{g})$ we denote by

$$U(\mathfrak{g}, e) - f.d.\text{mod}^m$$

the category of finite-dimensional $U(\mathfrak{g}, e)$ -modules M such that

$$\forall x \in M \exists d \in \mathbb{Z}_{>0} (m^d x = 0).$$

Our main results for the simply-laced Lie algebras are as follows.

Theorem 2.1. *Assume that \mathfrak{g} is simply-laced and is not of type A_n . Then*

- a) $U(\mathfrak{g}, e) - f.d.mod^{m_0}$ contains exactly rank \mathfrak{g} simple objects,
- b) $U(\mathfrak{g}, e) - f.d.mod^{m_0}$ is equivalent to the category of representations of the zigzag algebra $A(\Gamma)$; see [HK].

The result of Theorem 2.1 can be enhanced by the following proposition. To state it we use a bit more notation; see Section 4.1. (In short, there is a surjective map from a certain vector space to the maximal ideals of $Z(\mathfrak{g})$, $\lambda \rightarrow m_\lambda$. Few of these λ 's are integral and all other λ 's are nonintegral.)

Proposition 2.2. *Assume that \mathfrak{g} is simply-laced and not of type A_n . If λ is nonintegral, then the category $U(\mathfrak{g}, e) - f.d.mod^{m_\lambda}$ contains no nonzero objects.*

The main result for the nonsimply-laced Lie algebras is as follows.

Theorem 2.3. *Assume that \mathfrak{g} is not simply-laced and is simple. Then $U(\mathfrak{g}, e) - f.d.mod$ is semisimple.*

Remark 2.4. Under the assumptions of Theorem 2.1 the category $U(\mathfrak{g}, e) - f.d.mod^{m_0}$ has finitely many isomorphism classes of its indecomposable objects which are parametrised by the roots of \mathfrak{g} ; see [HK, Corollary 1]. In his talk at the conference ‘‘Representation theory and symplectic singularities’’ in Edinburgh T. Arakawa provided a connection between these W-algebras and a proper class of vertex algebras. Being motivated by this connection he expressed a hope that these categories $U(\mathfrak{g}, e) - f.d.mod$ are semisimple. It turns out that this is exactly the case if \mathfrak{g} is not simply-laced, and if \mathfrak{g} is simply-laced the situation is almost as good as he expected.

Remark 2.5. We wish to mention that $U(\mathfrak{g}, e) - f.d.mod$ is equivalent to a subcategory of the category of \mathfrak{g} -modules; see Section 7. This allows one to apply to $U(\mathfrak{g}, e) - f.d.mod$ the technique of translation functors developed in [BJ]; see also [BeG]. Under the assumption that \mathfrak{g} is simply-laced this shows that any block of $U(\mathfrak{g}, e) - f.d.mod$ is either semisimple with a unique simple object or is equivalent to the category of representations of the zigzag algebra $A(\Gamma)$. Key examples of semisimple blocks are given in Proposition 7.6.

Remark 2.6. The dimensions of the simple finite-dimensional $U(\mathfrak{g}, e)$ -modules can be computed through the Goldie ranks of primitive ideals of $U(\mathfrak{g})$; see [Pr2, Theorem 5.3(2)]. If \mathfrak{g} is not simply-laced, then this fact leads to a very explicit answer; see [Pr2, Theorem 6.2].

To prove Theorem 2.1 we use a connection between simple $U(\mathfrak{g}, e)$ -modules and primitive ideals of $U(\mathfrak{g})$; see [Los1, Conjecture 1.2.1]. Using this approach, one can classify all simple finite-dimensional $U(\mathfrak{g}, e)$ -modules; see [LO].

3. PROPERTIES OF $U(\mathfrak{g}, e)$

Let $e \in \mathfrak{g}$ be a nilpotent element. For a general definition of W-algebra $U(\mathfrak{g}, e)$ see [Pr2]. Here we explore the features of this object which we need in this work.

3.1. Skryabin’s equivalence. To an element $e \in \mathfrak{g}$ one can assign (in a noncanonical way) a Lie subalgebra

$$\mathfrak{m}(e) \subset (\mathfrak{g} \oplus \mathbb{F}) \subset U(\mathfrak{g})$$

such that the category $U(\mathfrak{g}, e) \text{-mod}$ of $U(\mathfrak{g}, e)$ -modules is equivalent to the category $(\mathfrak{g}, \mathfrak{m}(e)) \text{-l.n.mod}$ of \mathfrak{g} -modules with a locally nilpotent action of $\mathfrak{m}(e)$ (Skryabin’s equivalence; see [Pr2] or [Los1] for details). We use a particular choice of $\mathfrak{m}(e)$ defined by [Pr2, Subsection 2.1]. For a $U(\mathfrak{g}, e)$ -module M we denote by $\text{Skr}(M)$ the corresponding $(\mathfrak{g}, \mathfrak{m}(e))$ -module. This immediately defines a map

$$\mathcal{P} : M \rightarrow \text{Ann}_{U(\mathfrak{g})} \text{Skr}(M)$$

from the set of simple finite-dimensional $U(\mathfrak{g}, e)$ -modules to the set of primitive ideals of $U(\mathfrak{g})$. The following proposition describes the image of \mathcal{P} under the assumption that O is the minimal nilpotent orbit of \mathfrak{g} (for the general case see [LO]).

Proposition 3.1. *Assume that O is the minimal nonzero nilpotent orbit of \mathfrak{g} . Let m be a maximal ideal of $Z(\mathfrak{g})$. Then \mathcal{P} defines a bijection between the isomorphism classes of finite-dimensional simple $U(\mathfrak{g}, e)$ -modules which are annihilated by m , and the primitive ideals I of $U(\mathfrak{g})$ with $I \supset m$ and $\text{Var}(I) = \bar{O}$ where $\text{Var}(I)$ is the associated variety of I defined in [Pr2, Subsection 3.2]; see also Subsection 4.2.*

Proof. The proposition is equivalent to [Pr2, Theorem 5.3(5)]. □

3.2. Generators and relations for $U(\mathfrak{g}, e)$. Denote by \mathfrak{g}_e the centralizer of e in \mathfrak{g} . One can consider $U(\mathfrak{g}, e)$ as a deformation of the universal enveloping algebra $U(\mathfrak{g}_e)$. Using this approach one can provide $U(\mathfrak{g}, e)$ with a PBW-basis and evaluate the defining set of relations; see [Pr2, Subsection 1.1]. These generators and relations are known explicitly under the assumption that e belongs to the minimal nilpotent orbit O as we explain next (see also [Pr2, Theorem 1.1]).

From now on $e \in O \subset \mathfrak{g}$. There are elements $h, f \in \mathfrak{g}$ such that $\{e, h, f\}$ is a standard $\mathfrak{sl}(2)$ -triple. The semisimple element h acts semisimply on \mathfrak{g} with integral eigenvalues. We denote by $\mathfrak{g}_e(i)$ the respective eigenspaces. The associative algebra $U(\mathfrak{g}, e)$ is generated by copies $\widetilde{\mathfrak{g}_e(0)}, \widetilde{\mathfrak{g}_e(1)}$ of $\mathfrak{g}_e(0), \mathfrak{g}_e(1)$, and central element C modulo the following relations:

- 1) $\forall x, y \in \widetilde{\mathfrak{g}_e(0)}$ ($[x, y] = xy - yx \in \widetilde{\mathfrak{g}_e(0)}$); the respective Lie bracket on $\widetilde{\mathfrak{g}_e(0)}$ coincides with one induced from $\mathfrak{g}_e(0)$,
- 2) $\forall x \in \widetilde{\mathfrak{g}_e(0)} \forall y \in \widetilde{\mathfrak{g}_e(1)}$ ($[x, y] = xy - yx \in \widetilde{\mathfrak{g}_e(1)}$), i.e., $\widetilde{\mathfrak{g}_e(1)}$ is a $\widetilde{\mathfrak{g}_e(0)}$ -module; $\widetilde{\mathfrak{g}_e(1)}$ is isomorphic to $\mathfrak{g}_e(1)$ as a $\mathfrak{g}_e(0)$ -module,
- 3) a formula for $[x, y]$ ($x, y \in \widetilde{\mathfrak{g}_e(1)}$); see [Pr2, Theorem 1.1].

Further we will identify $\widetilde{\mathfrak{g}_e(0)}$ with $\mathfrak{g}_e(0)$ and $\widetilde{\mathfrak{g}_e(1)}$ with $\mathfrak{g}_e(1)$.

Using these formulas one can easily check that, if \mathfrak{g} is not of type A, then $U(\mathfrak{g}, e)$ has a unique one-dimensional module which is isomorphic to

$$U(\mathfrak{g}, e) / U(\mathfrak{g}, e)(\mathfrak{g}_e(0) \oplus \mathfrak{g}_e(1));$$

see [Pr2, Corollary 4.1]. The following proposition is crucial for the present work.

Proposition 3.2. *Assume that \mathfrak{g} is not of type A and that M is the one-dimensional $U(\mathfrak{g}, e)$ -module defined above. Then $\text{Ext}_{U(\mathfrak{g}, e)}^1(M, M) = 0$.*

Proof. It is enough to show that M has no nontrivial self-extensions. Indeed, let

$$0 \rightarrow M \rightarrow \tilde{M} \rightarrow M \rightarrow 0$$

be a self-extension of M . Then

$$(\text{Ann}_{U(\mathfrak{g},e)} M)^2 \subset \text{Ann}_{U(\mathfrak{g},e)} \tilde{M} \subset \text{Ann}_{U(\mathfrak{g},e)} M.$$

We claim that $\text{Ann}_{U(\mathfrak{g})} \tilde{M} = \text{Ann}_{U(\mathfrak{g},e)} M$. To show this we prove that $\text{Ann}_{U(\mathfrak{g},e)} M = (\text{Ann}_{U(\mathfrak{g},e)} M)^2$.

It is easy to see that $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] = \mathfrak{g}_e(0)$ and $[\mathfrak{g}_e(0), \mathfrak{g}_e(1)] = \mathfrak{g}_e(1)$; see [Pr2, Corollary 4.1]. This implies that $\text{Ann}_{U(\mathfrak{g},e)} M$ is generated by $\mathfrak{g}_e(0)$ as a two-sided ideal. Using once more that $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] = \mathfrak{g}_e(0)$ we see that $\text{Ann}_{U(\mathfrak{g},e)} M = (\text{Ann}_{U(\mathfrak{g},e)} M)^2$.

The claim implies that $\text{Ann}_{U(\mathfrak{g},e)} \tilde{M}$ is a $(U(\mathfrak{g}, e)/\text{Ann}_{U(\mathfrak{g},e)} M)$ -module. The fact that M is one-dimensional implies that $U(\mathfrak{g}, e)/\text{Ann}_{U(\mathfrak{g},e)} M \cong \mathbb{F}$. Therefore $\tilde{M} \cong M \oplus M$. Thus $\text{Ext}_{U(\mathfrak{g},e)}^1(M, M) = 0$. \square

4. ON THE CLASSIFICATION OF PRIMITIVE IDEALS OF $U(\mathfrak{g})$

We need a quite detailed description of the set of primitive ideals of $U(\mathfrak{g})$ together with the respective notation.

4.1. Notation. We assume that \mathfrak{g} is a simple Lie algebra. Denote by $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra of \mathfrak{g} and by $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra of \mathfrak{b} . We have

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{h}$, $\dim \mathfrak{g}_\alpha \leq 1$ if $\alpha \neq 0$, and if $\mathfrak{g}_\alpha \neq 0$, then \mathfrak{g}_α is a simple one-dimensional \mathfrak{h} -module with a character α . We put

$$\Delta := \{0 \neq \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}, \quad \Delta^+ := \{0 \neq \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \subset \mathfrak{b}\}, \quad \rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

We denote

- by (\cdot, \cdot) the Cartan-Killing form of \mathfrak{g} ,
- by Π the simple roots of Δ^+ , and by

$$\{\omega(\alpha)\}_{\alpha \in \Pi} \subset \mathfrak{h}^*$$

the corresponding fundamental weights,

- by Λ the lattice generated by $\{\omega(\alpha)\}_{\alpha \in \Pi}$,
- by Λ^+ the semigroup with 0 generated by $\{\omega(\alpha)\}_{\alpha \in \Pi}$,
- by W the subgroup generated by reflections with respect to the elements of Δ .

Note that (\cdot, \cdot) canonically identifies \mathfrak{g} and \mathfrak{g}^* and is nondegenerate after the restriction to \mathfrak{h} . Hence it also identifies \mathfrak{h} and \mathfrak{h}^* .

Fix $\lambda \in \Lambda$. Put

$$\Delta_\lambda := \{\alpha \in \Delta \mid \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} = 0\}.$$

Definition 4.1. We say that λ is *singular* if $\Delta_\lambda \neq \emptyset$, and we say that λ is *regular* otherwise.

Definition 4.2. We say that $\lambda \in \Lambda$ is *dominant* if $\lambda \in \Lambda^+$. We say that λ is ρ -*dominant* if $\lambda + \rho$ is dominant.

Definition 4.3. We say that two roots $\alpha, \beta \in \Pi$ are *adjacent* if $\alpha \neq \beta$ and $(\alpha, \beta) \neq 0$.

To any $\lambda \in \mathfrak{h}^*$ we assign a one-dimensional \mathfrak{h} -module \mathbb{F}_λ which we also consider as a \mathfrak{b} -module ($\mathfrak{h} \cong \mathfrak{b}/\mathfrak{n}$ where \mathfrak{n} is the nilpotent radical of \mathfrak{b}). Put

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{F}_\lambda, \quad m_\lambda := Z(\mathfrak{g}) \cap \text{Ann}_{U(\mathfrak{g})} M(\lambda).$$

The ideal m_λ is maximal in $Z(\mathfrak{g})$ for all λ and thus we have a map from \mathfrak{h}^* to the set of maximal ideals of $Z(\mathfrak{g})$. Moreover,

$$m_\lambda = m_\mu \Leftrightarrow \exists w \in W (w(\lambda + \rho) = \mu + \rho).$$

For any $\alpha \in \Delta$ we denote by s_α the corresponding reflection.

We set $L(\lambda)$ to be the unique simple quotient of $M(\lambda)$ and $I(\lambda) := \text{Ann}_{U(\mathfrak{g})} L(\lambda)$. According to Duflo’s theorem, for any primitive ideal I of $U(\mathfrak{g})$ there exists $\lambda \in \mathfrak{h}^*$ such that $I = I(\lambda)$. Put

$$\tau_L(w) := \{\alpha \in \Pi \mid w^{-1}\alpha \in \Delta^+\}, \quad \tau_R(w) := \{\alpha \in \Pi \mid w\alpha \in \Delta^+\}.$$

The following lemma provides a very useful invariant of primitive ideals.

Lemma 4.4. *Fix $w_1, w_2 \in W$. If $I(w_1\rho - \rho) = I(w_2\rho - \rho)$, then $\tau_R(w_1) = \tau_R(w_2)$. Thus we can define*

$$\tau(I) := \tau_R(w_1) = \tau_R(w_2).$$

Proof. See [BJ, Subsection 2.14]; see also [Vog, Theorem 2.4] and the text above it. □

4.2. Associated varieties of ideals. The universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} has natural degree filtration $\{U_i\}_{i \geq 0}$ where $U_{-1} = \{0\}$. The associated graded algebra

$$\text{gr } U(\mathfrak{g}) := \bigoplus_{i \geq 0} (U_i/U_{i-1})$$

is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$ of \mathfrak{g} . For a two-sided ideal I we put

$$\text{gr } I := \bigoplus_{i \geq 0} (I \cap U_i / I \cap U_{i-1}) \subset \text{gr } U(\mathfrak{g}) \cong S(\mathfrak{g}).$$

We put $\text{Var}(I) := \{x \in \mathfrak{g}^* \mid \forall f \in \text{gr } I (f(x) = 0)\}$. It is known that if I is a primitive ideal of $U(\mathfrak{g})$, then $\text{Var}(I)$ is the closure $\overline{O(I)}$ of a nilpotent coadjoint orbit $O(I)$ in \mathfrak{g}^* ; see [Jo1]. We put $O(\lambda) := O(I(\lambda))$. Note that $O(\lambda)$ is the minimal nilpotent orbit of \mathfrak{g} only for very exceptional λ ’s; see Section 5.

4.3. One-sided and two-sided cells. Next, we need some combinatorial data attached to the reflection group W . It has some definition through the Kazhdan–Lusztig polynomials (see, for example, [LO, Section 6]) and it has a much more explicit description for all classical Lie algebras; see [BV]. But we believe that the approach which we use here also makes sense; cf. with [BB]. We introduce two relations on W :

- 1) $w_1 \sim_L w_2 \Leftrightarrow I(w_1\rho - \rho) = I(w_2\rho - \rho) \ (w_1, w_2 \in W)$,
- 2) $w_1 \sim_R w_2 \Leftrightarrow I(w_1^{-1}\rho - \rho) = I(w_2^{-1}\rho - \rho) \ (w_1, w_2 \in W)$.

Clearly, \sim_L and \sim_R are equivalence relations on W , and we denote by \sim the smallest equivalence relation on W which includes both \sim_L, \sim_R . We denote by

- $\text{LCell}(w, \Pi)$ the equivalence class of \sim_L which contains w ,
- $\text{RCell}(w, \Pi)$ the equivalence class of \sim_R which contains w ,
- $\text{TCell}(w, \Pi)$ the equivalence class of \sim which contains w .

It is clear that the set of left cells can be naturally identified with the set of the primitive ideals of the form $I(w\rho - \rho)$ ($w \in W$). The following proposition gives a straightforward connection between the associated varieties of ideals and the two-sided cells.

Proposition 4.5 ([LO, Subsection 6.2]). *The two-sided ideals $I(w_1\rho - \rho)$ and $I(w_2\rho - \rho)$ have the same associated variety if and only if $w_1 \sim w_2$.*

Proposition 4.5 defines a map from the set of two-sided cells of W to the set of nilpotent orbits. An orbit which belongs to the image of this map is called *special*. The minimal nilpotent orbit O is special if and only if \mathfrak{g} is simply-laced; see [CM].

5. PRIMITIVE IDEALS I FOR WHICH $\text{Var}(I) = \bar{O}$

We recall that O denotes the minimal nilpotent orbit of \mathfrak{g} . Let $\lambda \in \Lambda$ be a weight. Denote by $\text{PrId}^\lambda(O)$ the set of primitive ideals I of $U(\mathfrak{g})$ such that $\text{Var}(I) = \bar{O}$ and $m_\lambda \subset I$. Further in this section we assume that \mathfrak{g} is simply-laced.

Lemma 5.1. *We have $|\text{PrId}^0(O)| = |\Pi|$.*

Proof. Denote by $\text{TCell}(O)$ the two-sided cell in W attached to O through Proposition 4.5. The desired number of ideals equals the number of left cells in $\text{TCell}(O)$; see Subsection 4.3. According to [Jo2, Subsection 4.3], $\text{TCell}(O)$ splits into exactly $|\Pi|$ left cells. □

Remark 5.2. One can deduce Lemma 5.1 from [Dou].

To work out the singular integral case we need the following proposition.

Proposition 5.3 ([BJ, Satz 2.14]). *Fix $\lambda \in \Lambda$. Then the following sets can be identified:*

- 1) *primitive ideals I of $U(\mathfrak{g})$ such that $I \supset m_\lambda$,*
- 2) *primitive ideals I of $U(\mathfrak{g})$ such that $I \supset m_0$ and $\tau(I) \cap \Delta_\lambda = \emptyset$.*

The associated varieties of the ideals identical under this correspondence are the same.

Using this proposition we can evaluate the desired classification of the singular cases which we need.

Proposition 5.4.

- a) *For $I \in \text{PrId}^0(O)$ there exists $\alpha \in \Pi$ such that $\tau(I) = \Pi \setminus \alpha$.*
- b) *For any $\alpha \in \Pi$ there exists a unique $I \in \text{PrId}^0(O)$ such that $\tau(I) = \Pi \setminus \alpha$.*
- c) *For any $\alpha \in \Pi$ there exists exactly one primitive ideal I such that $I \supset m_{-\omega(\alpha)}$.*

Proof. Part c) is implied by parts a) and b), and Proposition 5.3. Part a) is implied by part b) and Lemma 5.1. To prove part b) we first show that for any $\alpha \in \Pi$ there exists $I \in \text{PrId}^0(O)$ such that $\tau(I) = \Pi \setminus \alpha$.

Denote by α_0 the unique simple root with three neighbours. For any $\alpha \in \Pi$ put $\alpha_0, \alpha_1, \dots, \alpha_{n(\alpha)}$ to be the shortest sequence of adjacent roots which connects α_0 with $\alpha = \alpha_{n(\alpha)}$. Set

$$w_\alpha := s_{n(\alpha)} \dots s_{\alpha_1} s_{\alpha_0} \in W,$$

where $s_{\alpha_i} \in W$ is the reflection with respect to α_i . It follows from [Jo2, Subsection 4.3] that

$$\{w_\alpha\}_{\alpha \in \Pi}$$

is a left cell in $\text{TCell}(O)$. Thus

$$\{w_\alpha^{-1}\}_{\alpha \in \Pi}$$

is a right cell in $\text{TCell}(O)$. One can check that

$$\tau(I(w_\alpha^{-1}\rho - \rho)) = \tau_R(w_\alpha^{-1}) = \tau_L(w_\alpha) = \Pi \setminus \{\alpha\};$$

see also [Jo2, Subsection 4.3-4.4].

To complete the proof of part b) we mention that the ideals $I(w_\alpha^{-1}\rho - \rho)$ ($\alpha \in \Pi$) are distinct, and that according to Lemma 5.1 we have $|\text{PrId}^0(O)| = |\Pi|$. \square

Proof of Theorem 2.1a). It follows from a composition of Propositions 3.1, 5.4. \square

6. THE SEMIRING OF PROJECTIVE FUNCTORS

Let $\lambda \in \Lambda$ be a weight and let V be a finite-dimensional \mathfrak{g} -module. We say that a functor

$$F : \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\lambda} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod}$$

is *projective* if it is a direct summand of a functor

$$\cdot \otimes V : M \rightarrow M \otimes V \quad (\mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\lambda} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod});$$

cf. with [BeG]. It can be checked that $\cdot \otimes V$ maps $\mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\lambda}$ to $\bigoplus_{\mu+\rho \in \Lambda^+} \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\mu} \subset \mathbf{U}(\mathfrak{g}) - \text{mod}$ and therefore one can consider projective functors as endofunctors of the category $\bigoplus_{\mu+\rho \in \Lambda^+} \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\mu}$.

The projective functors enjoy the following properties:

- a projective functor is exact,
- a projective functor is a direct sum of finitely many indecomposable projective functors,
- the composition of projective functors is projective,
- the direct sum of projective functors is projective;

see [BeG, Section 3]. The indecomposable projective functors can be described as follows. For any pair $\chi, \xi \in \Lambda$ one can assign an indecomposable projective functor

$$F_{\chi, \xi} : \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\chi} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_\xi}.$$

Two such functors F_{χ_1, ξ_1} and F_{χ_2, ξ_2} are isomorphic if and only if there exists $w \in W$ such that

$$w(\chi_1 + \rho) = \chi_2 + \rho \text{ and } w(\xi_1 + \rho) = \xi_2 + \rho.$$

We will be particularly interested in the following collection of functors:

$$\psi_\alpha := F_{0, -\omega(\alpha)} : \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_0} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_{-\omega(\alpha)}} \quad (\alpha \in \Pi),$$

$$\phi_\alpha := F_{-\omega(\alpha), 0} : \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_{-\omega(\alpha)}} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_0} \quad (\alpha \in \Pi),$$

$$T_\alpha := F_{0, -\alpha} = F_{0, s_\alpha \rho - \rho} : \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_0} \rightarrow \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_{s_\alpha \rho - \rho}} = \mathbf{U}(\mathfrak{g}) - \text{mod}^{m_0} \quad (\alpha \in \Pi).$$

It is clear that the indecomposable projective functors form a semiring with respect to the direct sum (considered as an addition) and the composition (considered as

a multiplication). We denote this semiring by \mathcal{R} . The following lemma is quite standard and is pretty straightforward.

Lemma 6.1. *Let $\text{Ring}_0(\mathcal{R})$ be the set of pairs $(r_1, r_2) \in \mathcal{R} \times \mathcal{R}$ modulo the equivalence relation $(a, b) \sim (a', b')$ if and only if there exists $t \in \mathcal{R}$ such that $a + b' + t = a' + b + t$.*

- a) *The operations $(r_1, r_2) + (r_3, r_4) := (r_1 + r_3, r_2 + r_4)$, $(r_1, r_2) \cdot (r_3, r_4) := (r_1 r_3 + r_2 r_4, r_1 r_4 + r_2 r_3)$ define a structure of a ring on $\text{Ring}_0(\mathcal{R})$. We denote this ring $\text{Ring}(\mathcal{R})$. We denote by*

$$\phi_{\mathcal{R}} : \mathcal{R} \rightarrow \text{Ring}(\mathcal{R}) \quad (r \rightarrow (r + r, r))$$

the respective morphism of semirings.

- b) *If R' is a ring and $\phi' : \mathcal{R} \rightarrow R'$ is a morphism of semirings, then there exists a unique morphism of rings*

$$\psi : \text{Ring}(\mathcal{R}) \rightarrow R'$$

such that $\phi' := \psi \circ \phi_{\mathcal{R}}$.

Next, we note that \mathcal{R} naturally acts on the Grothendieck K -group of the category of \mathfrak{g} -modules (this is a straightforward check through the definitions of an exact functor and a K -group). Moreover, if \mathcal{C} is a subcategory of

$$\bigoplus_{\mu+\rho \in \Lambda^+} \text{U}(\mathfrak{g}) - \text{mod}^{m_\mu}$$

which is stable under $\cdot \otimes V$ for any finite-dimensional \mathfrak{g} -module V , then \mathcal{R} acts on the K -group $K(\mathcal{C})$ of \mathcal{C} . The endomorphisms of $K(\mathcal{C})$ form a ring and hence we have a natural action of $\text{Ring}(\mathcal{R})$ on $K(\mathcal{C})$.

6.1. Basis-dependent description of $\text{Ring}(\mathcal{R})$. We recall that one can attach to a projective functor $F_{\chi, \xi}$ an endomorphism $F_{\chi, \xi}^K$ of a free lattice generated by $\{\delta_\lambda\}_{\lambda \in \Lambda}$; see [BeG, Subsection 3.4]. (This corresponds to the action of $\text{Ring}(\mathcal{R})$ on the Grothendieck group $K(\mathcal{O})$ of category \mathcal{O} .) The assignment enjoys the following properties:

- $F_{\chi_1, \xi_1}^K = F_{\chi_2, \xi_2}^K$ implies that $F_{\chi_1, \xi_1} \cong F_{\chi_2, \xi_2}$,
- $(F_{\chi, \xi}^K)^K(\delta_\lambda) \neq 0$ if and only if $m_\chi = m_\lambda$,
- $F_{\chi_1, \xi_1}^K + F_{\chi_2, \xi_2}^K = (F_{\chi_1, \xi_1} \oplus F_{\chi_2, \xi_2})^K$,
- $F_{\chi_1, \xi_1}^K F_{\chi_2, \xi_2}^K = (F_{\chi_1, \xi_1} \circ F_{\chi_2, \xi_2})^K$,

$(\chi_1, \chi_2, \xi_1, \xi_2, \lambda \in \Lambda)$; see [BeG, Subsection 3.4]. This immediately implies that the map $(\cdot)^K$ defines the morphism from $\text{Ring}(\mathcal{R})$ to the endomorphisms of the lattice $\bigoplus_{\lambda \in \Lambda} \mathbb{F}\delta_\lambda$ and that this map is injective.

Next, we mention that the lattice $\bigoplus_{\lambda \in \Lambda} \mathbb{Z}\delta_\lambda$ carries a W -action defined by the formula

$$w \cdot \delta_\lambda := \delta_{w(\lambda+\rho)-\rho} \quad (w \in W, \lambda \in \Lambda)$$

and $F_{\chi, \xi}^K$ commutes with this action of W . One can use this fact to provide an action of a Weyl group on the Grothendieck group of the blocks of \mathfrak{g} -modules; see [KZ, Theorem C.2 of Appendix].

Lemma 6.2. *We have*

- a) $(\phi_\alpha)^K(\delta_{-\omega(\alpha)}) = \delta_0 + \delta_{s_\alpha \rho - \rho} = \delta_0 + \delta_{-\alpha}$,
- b) $(\psi_\alpha)^K(\delta_0) = \delta_{-\omega(\alpha)}$,
- c) $(T_\alpha)^K(\delta_0) = \delta_0 + \delta_{-\alpha}$.

Proof. It is easy to see that a) follows from a combination of [BeG, Subsections 1.12, 3.3, 3.4] and [Hum, Theorem 7.14(a)]; b) is implied by [BeG, 3.3, Theorem(ii)b)]; c) is a consequence of parts a) and b). \square

For $\lambda \in \Lambda$ denote by Id_λ the operator on $\bigoplus_{\mu \in \Lambda} \delta_\mu$ defined by the formula

$$Id_\lambda(\delta_\mu) := \begin{cases} \delta_\mu & \text{if } \mu + \rho = w(\lambda + \rho) \text{ for some } w \in W, \\ 0 & \text{otherwise.} \end{cases}$$

The following corollary will be very useful in the proof of Theorem 2.1.

- Corollary 6.3.**
- a) $(\psi_\alpha)^K(\phi_\alpha)^K = 2Id_{-\omega(\alpha)}$,
 - b) $(\phi_\alpha)^K(\psi_\alpha)^K = (T_\alpha)^K$,
 - c) $((T_\alpha)^K)^2 = 2(T_\alpha)^K$, $((T_\alpha)^K - Id_0)^2 = Id_0$,
 - d) $((T_\alpha)^K - Id_0)((T_\beta)^K - Id_0)^3 = Id_0$ if α and β are adjacent,
 - e) $((T_\alpha)^K - Id_0)((T_\beta)^K - Id_0)^2 = Id_0$ if α and β are not adjacent and $\alpha \neq \beta$.

7. PROOF OF THEOREM 2.1

To start with we note that $U(\mathfrak{g}, e) - f.d.mod$ is equivalent to the category \mathcal{C} of $(\mathfrak{g}, \mathfrak{m}(e))$ -l.n.modules of finite length and of Gelfand–Kirillov dimension $\frac{1}{2} \dim O$; see [Los2, Proposition 3.3.5]. We have

$$\mathcal{C} \subset \bigoplus_{\mu + \rho \in \Lambda^+} U(\mathfrak{g}) - mod^{m_\mu}.$$

This implies that $Ring(\mathcal{R})$ acts on $K(\mathcal{C})$. For any $F \in Ring(\mathcal{R})$ we denote by $(F)^{K(\mathcal{C})}$ the image of it in $End(K(\mathcal{C}))$.

For any $\lambda \in \Lambda$ we put $\mathcal{C}^\lambda := \mathcal{C} \cap U(\mathfrak{g}) - mod^{m_\lambda}$. Functors $T_\alpha, \psi_\alpha, \phi_\alpha$ act on these categories as follows:

$$\psi_\alpha : \mathcal{C}^0 \rightarrow \mathcal{C}^{-\omega(\alpha)}, \quad \phi_\alpha : \mathcal{C}^{-\omega(\alpha)} \rightarrow \mathcal{C}^0, \quad T_\alpha : \mathcal{C}^0 \rightarrow \mathcal{C}^0.$$

We proceed with a description of $K(\mathcal{C}^0)$ and $K(\mathcal{C}^{-\omega(\alpha)})$. For an object M of \mathcal{C} we denote by $[M]$ the image of it in $K(\mathcal{C})$. All objects of $\mathcal{C}, \mathcal{C}^0$ have finite length, and thus $K(\mathcal{C})$ (respectively, $K(\mathcal{C}^0)$) are generated by the images of the simple objects of $K(\mathcal{C})$ (respectively, of $K(\mathcal{C}^0)$).

Proposition 3.1 together with Proposition 5.4c) implies that $\mathcal{C}^{-\omega(\alpha)}$ has a unique simple object M_α^- which corresponds to the unique primitive ideal I of Gelfand–Kirillov dimension $\dim O$ such that $I \supset m_{-\omega(\alpha)}$.

Proposition 3.1 together with Proposition 5.4 defines a bijection between simple objects of \mathcal{C}^0 and elements of Π . For any $\alpha \in \Pi$ we denote by M_α the respective simple object of \mathcal{C}^0 .

We need the following lemma.

Lemma 7.1. *Let F be a projective functor and let M_1, M_2 be \mathfrak{g} -modules such that $\text{Ann}_{U(\mathfrak{g})} M_1 = \text{Ann}_{U(\mathfrak{g})} M_2$. Then $\text{Ann}_{U(\mathfrak{g})} F(M_1) = \text{Ann}_{U(\mathfrak{g})} F(M_2)$. In particular, $F(M_1) = 0$ if and only if $F(M_2) = 0$.*

Proof. It is implied by [Vog, Lemma 2.3]. \square

Assembling together Lemma 7.1, [BJ, Satz 2.14], and Proposition 5.4 we prove the following lemma.

Lemma 7.2. *We have $\psi_\alpha(M_\beta) = 0$ if and only if $\alpha \neq \beta \in \Pi$.*

As a corollary we have that $T_\alpha(M_\beta) = 0$ if and only if $\alpha \neq \beta$; $(T_\alpha)^{K(C)}[M_\alpha] = \Sigma_\beta c_{\alpha,\beta}[M_\beta]$ for some $c_{\alpha,\beta} \in \mathbb{Z}_{\geq 0}$.

Lemma 7.3. *We have $c_{\alpha,\alpha} = 2$ for all $\alpha \in \Pi$.*

Proof. It is implied by Lemma 7.2 and formula $((T_\alpha)^K)^2 = 2(T_\alpha)^K$. □

Lemma 7.4.

- a) *If $\alpha, \beta \in \Pi$ are adjacent, then $c_{\alpha,\beta} = c_{\beta,\alpha} = 1$.*
- b) *If $\alpha \neq \beta \in \Pi$ are not adjacent, then $c_{\alpha,\beta} = c_{\beta,\alpha} = 0$.*

Proof. Part a). We have that

- (1) $((T_\alpha)^K - Id_0)^{K(C)}((T_\beta)^K - Id_0)^{K(C)}[M_\alpha] = -[M_\alpha] - c_{\alpha,\beta}[M_\beta] - \Sigma_{\gamma \neq \alpha,\beta} c_{\alpha,\gamma}[M_\gamma]$,
- (2) $((T_\alpha)^K - Id_0)^{K(C)}((T_\beta)^K - Id_0)^{K(C)}[M_\beta] = c_{\beta,\alpha}[M_\alpha] + (c_{\beta,\alpha}c_{\alpha,\beta} - 1)[M_\beta] + \Sigma_{\gamma \neq \alpha,\beta} (c_{\beta,\alpha}c_{\alpha,\gamma} - c_{\beta,\gamma})[M_\gamma]$,
- (3) $((T_\alpha)^K - Id_0)^{K(C)}((T_\beta)^K - Id_0)^{K(C)}[M_\gamma] = [M_\gamma] \ (\gamma \neq \alpha, \beta)$.

We fix $\alpha, \beta \in \Pi$ such that α and β are adjacent. Formulas (1), (2), (3) together with Corollary 6.3e) imply that

$$(4) \quad \begin{pmatrix} -1 & -c_{\alpha,\beta} \\ c_{\beta,\alpha} & c_{\alpha,\beta}c_{\beta,\alpha} - 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We put

$$X := \begin{pmatrix} -1 & -c_{\alpha,\beta} \\ c_{\beta,\alpha} & c_{\alpha,\beta}c_{\beta,\alpha} - 1 \end{pmatrix}.$$

Equation (4) implies that all eigenvalues of X are roots of unity of degree 3, and therefore the trace $\text{tr } X$ of X equals the sum of two (not necessarily distinct) roots of unity of degree 3. On the other hand $\text{tr } X$ equals $c_{\alpha,\beta}c_{\beta,\alpha} - 2$, and hence $\text{tr } X$ is an integer. It can be easily checked that if such a sum is an integer, then it is equal to -1 or 2. Thus $c_{\alpha,\beta}c_{\beta,\alpha} \in \{1, 4\}$. Hence $c_{\alpha,\beta} = c_{\beta,\alpha} = 1$, or $c_{\alpha,\beta} = 4$ and $c_{\beta,\alpha} = 1$, or $c_{\alpha,\beta} = c_{\beta,\alpha} = 2$, or $c_{\alpha,\beta} = 1$ and $c_{\beta,\alpha} = 4$. It can be easily seen that $X^3 = \mathbf{1}$ if and only if $c_{\alpha,\beta} = c_{\beta,\alpha} = 1$.

Part b). We fix $\alpha, \beta \in \Pi$ such that α and β are adjacent. Formulas (1), (2), (3) together with Corollary 6.3d) imply that $X^2 = \mathbf{1}$. Further we have

$$\det(X) = (-1)(c_{\alpha,\beta}c_{\beta,\alpha} - 1) - (-c_{\alpha,\beta})c_{\beta,\alpha} = 1.$$

These two facts together imply that $X = \mathbf{1}$ or $X = -\mathbf{1}$. As a consequence we have $c_{\alpha,\beta} = c_{\beta,\alpha} = 0$. □

Lemma 7.5.

- a) *We have $\psi_\alpha M_\alpha \cong M_\alpha^-$.*
- b) *If $\alpha, \beta \in \Pi$ are adjacent, then we have $\psi_\beta(\phi_\alpha M_\alpha^-) \cong M_\beta^-$.*

Proof. Part a). Category $\mathcal{C}^{-\omega(\alpha)}$ has the unique simple object M_α^- and thus $[\psi_\alpha M_\alpha] = c[M_\alpha^-]$ for some $c \in \mathbb{Z}_{>0}$. Next,

$$[T_\alpha M_\alpha] = [\phi_\alpha \psi_\alpha M_\alpha] = c[\phi_\alpha M_\alpha^-].$$

In particular, this implies that, for all $\beta \in \Pi$, $c_{\alpha,\beta}$ must be divisible by c . This together with Lemma 7.4 implies that $c \mid 1$, and therefore that $c = 1$.

Part b). It follows from part a) that $\phi_\alpha M_\alpha^- \cong T_\alpha M_\alpha$. According to Lemma 7.4 we have that $c_{\alpha,\beta} = 1$. Therefore $[\psi_\beta(\phi_\alpha M_\alpha^-)] = [M_\beta^-]$, and we have $\psi_\beta(\phi_\alpha M_\alpha^-) \cong M_\beta^-$. \square

Proposition 7.6. *Categories $\mathcal{C}^{-\omega(\alpha)}$ are semisimple with a unique simple object.*

Proof. Recall that α_0 is the unique simple root with three neighbours. The statement of Proposition 7.6 for $\alpha = \alpha_0$ holds thanks to Proposition 3.2. Due to the fact that \mathfrak{g} is simple, it is enough to show that if the statement of Proposition 7.6 holds for $\alpha \in \Pi$ and β is adjacent to α , then the statement of Proposition 7.6 also holds for β .

Thus we assume that α and β are adjacent roots and $\mathcal{C}^{-\omega(\alpha)}$ is semisimple and has a unique up to an isomorphism simple object M_α^- . Thanks to Proposition 5.4c) and Proposition 3.1, it is enough to show that M_β^- is projective in $\mathcal{C}^{-\omega(\beta)}$.

We have

$$\text{Hom}_{\text{U}(\mathfrak{g})}(M_\beta^-, M) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(\psi_\beta \phi_\alpha M_\alpha^-, M) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(M_\alpha^-, \psi_\alpha \phi_\beta M)$$

for an object M of $\mathcal{C}^{-\omega(\beta)}$. Then the fact that ψ_α, ϕ_β are exact immediately implies that M_β^- is projective. \square

Corollary 7.7. *Put $P_\alpha := \phi_\alpha M_\alpha^-$. For any object M of \mathcal{C}^0 we have*

$$\dim \text{Hom}_{\text{U}(\mathfrak{g})}(M, P_\alpha) = \dim \text{Hom}_{\text{U}(\mathfrak{g})}(P_\alpha, M)$$

and it equals the Jordan–Hölder multiplicity of M_α in M . In particular, P_α is both projective and injective.

Proof. Fix an object M of $\text{U}(\mathfrak{g}) - f.d.\text{mod}^{m_0}$. We have

$$\text{Hom}_{\text{U}(\mathfrak{g})}(M_\alpha^-, \psi_\alpha M) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(\phi_\alpha M_\alpha^-, M) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(P_\alpha, M),$$

$$\text{Hom}_{\text{U}(\mathfrak{g})}(\psi_\alpha M, M_\alpha^-) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(M, \phi_\alpha M_\alpha^-) \cong \text{Hom}_{\text{U}(\mathfrak{g})}(M, P_\alpha).$$

The first two numbers in both rows are equal to the multiplicity of M_α in M thanks to Proposition 7.6, Lemma 7.5, and Lemma 7.2. \square

Proposition 7.8. *Module $\bigoplus_{\alpha \in \Pi} P_\alpha$ is a faithfully projective object of \mathcal{C}^0 ; see [Bass, Chapter II] and also [Mit, Theorem 4.1]. In particular, \mathcal{C}^0 is equivalent to the category of left finite-dimensional $\text{End}_{\text{U}(\mathfrak{g})}(\bigoplus_{\alpha \in \Pi} P_\alpha)$ -modules.*

Proof. All objects of \mathcal{C}^0 are of finite length and it is clear that $\text{Hom}_{\mathcal{C}^0}(\bigoplus_{\alpha \in \Pi} P_\alpha, \cdot)$ is an exact functor which preserves arbitrary coproducts, i.e., direct sums in \mathcal{C}^0 . This together with [Bass, Chapter II, Theorem 1.3] implies that it is enough to show that $\bigoplus_{\alpha \in \Pi} P_\alpha$ is faithful, i.e., that for any object M of \mathcal{C}^0 we have

$$\text{Hom}_{\text{U}(\mathfrak{g})}\left(\bigoplus_{\alpha \in \Pi} P_\alpha, M\right) \neq 0.$$

This follows from Corollary 7.7. \square

Put $A(\mathfrak{g}) := \text{End}_{\text{U}(\mathfrak{g})}(\bigoplus_{\alpha \in \Pi} P_\alpha)$.

7.1. Basis-dependent description of $A(\mathfrak{g})$. The goal of this subsection is to provide a convenient basis for the algebra $A(\mathfrak{g})$; see Theorem 7.9. To do this we work out a complete set of linearly independent elements together with multiplication rules for the elements of this set. In short, we have $4r - 2$ basis elements where r is the rank of \mathfrak{g} , and the product of two elements of the basis is equal to either 0 or an element of the basis. It can be easily seen from this presentation that $A(\mathfrak{g}) \cong A(\Gamma)$ where Γ is a Dynkin diagram of \mathfrak{g} and $A(\Gamma)$ is a zigzag algebra attached to Γ ; see [HK]. Altogether this will complete the proof of Theorem 2.1b).

Theorem 7.9. *There exists a basis π_α, π_α^0 ($\alpha \in \Pi$), $\varphi_{\alpha\beta}$ ($\alpha, \beta \in \Pi, \alpha$ and β are adjacent) of $A(\mathfrak{g})$ such that*

$$\begin{aligned} \pi_\alpha \pi_\beta &= \begin{cases} \pi_\alpha & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}, \quad \pi_\alpha \pi_\beta^0 = \pi_\beta^0 \pi_\alpha = \begin{cases} \pi_\beta^0 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}, \\ \pi_\alpha \varphi_{\beta\gamma} &= \begin{cases} \varphi_{\beta\gamma} & \text{if } \alpha = \gamma \\ 0 & \text{otherwise} \end{cases}, \quad \varphi_{\beta\gamma} \pi_\alpha = \begin{cases} \varphi_{\beta\gamma} & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \quad (\beta \text{ and } \gamma \text{ are adjacent}), \\ \varphi_{\gamma\tau} \varphi_{\alpha\beta} &= \begin{cases} \pi_\alpha^0 & \text{if } \alpha = \tau \text{ and } \beta = \gamma \\ 0 & \text{otherwise} \end{cases} \quad \left(\begin{array}{l} \alpha \text{ and } \beta \text{ are adjacent} \\ \gamma \text{ and } \tau \text{ are adjacent} \end{array} \right), \\ \pi_\alpha^0 \pi_\beta^0 &= \pi_\alpha^0 \varphi_{\beta\gamma} = \varphi_{\beta\gamma} \pi_\alpha^0 = 0 \quad (\beta \text{ and } \gamma \text{ are adjacent}) \end{aligned}$$

for all $\alpha, \beta, \gamma, \tau \in \Pi$.

To start with we compute $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta)$.

Lemma 7.10. *For $\alpha, \beta \in \Pi$ we have*

- a) $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\alpha) = 2$,
- b) $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta) = 1$ if α and β are adjacent,
- c) $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta) = 0$ if $\alpha \neq \beta$ are not adjacent.

Proof. Is implied by Corollary 7.7, Lemma 7.3, Lemma 7.4. □

Next, we fix an element $\varphi_{\alpha\beta} \in \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta)$ for the pair of adjacent roots α, β (this element is unique up to scaling). We have that $\dim \text{Hom}_{U(\mathfrak{g})}(M_\alpha, P_\alpha) = \dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, M_\alpha) = 1$. Thus the composition of nonzero morphisms

$$P_\alpha \rightarrow M_\alpha \rightarrow P_\alpha$$

is unique up to scaling and we denote by π_α^0 one such composition. We denote by π_α the identity morphism on P_α .

The following proposition is a first approximation to Theorem 7.9.

Proposition 7.11. *The elements π_α, π_α^0 ($\alpha \in \Pi$) and $\varphi_{\alpha\beta}$ ($\alpha, \beta \in \Pi, \alpha$ and β are adjacent) are a basis of $A(\mathfrak{g})$.*

Proof. It is clear that π_α, π_α^0 are not proportional. This together with Lemma 7.10a) implies that they form a basis of $\text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\alpha)$. Next, Lemma 7.10b) implies that $\varphi_{\alpha\beta}$ is a basis of $\text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta)$ for a pair of adjacent roots $\alpha, \beta \in \Pi$. We are left to mention that $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\beta) = 0$ if α and β are not adjacent according to Lemma 7.10c). □

To complete the proof of Theorem 7.9 we need to evaluate the products of the elements of the basis.

Lemma 7.12. *We have $(\pi_\alpha^0)^2 = 0$.*

Proof. Corollary 7.7 implies that P_α has the unique proper maximal submodule which is precisely the kernel of π_α^0 . This implies that P_α is indecomposable and hence all endomorphisms of P_α are either scalar or nilpotent. The image of π_α^0 is a simple submodule of P_α and therefore π_α^0 is not a scalar operator on P_α . Therefore π_α^0 is nilpotent, and the fact that the image of π_α^0 is simple implies that $(\pi_\alpha^0)^2 = 0$. \square

Lemma 7.13. *Let $\alpha, \beta \in \Pi$ be adjacent one to each other. Then*

- a) $\pi_\beta^0 \varphi_{\alpha\beta} = 0$,
- b) $\varphi_{\alpha\beta} \pi_\alpha^0 = 0$,
- c) $\varphi_{\alpha\beta} \varphi_{\beta\alpha}$ is proportional to π_β^0 .

Proof.

Part a). Corollary 7.7 implies that P_β has the unique proper maximal submodule $(P_\beta)_{sub}$ which is precisely the kernel of π_β^0 . Let $P_{\alpha\beta}$ be the image of $\varphi_{\alpha\beta}$. Either $P_{\alpha\beta} \subset (P_\beta)_{sub}$ or $P_{\alpha\beta} = P_\beta$. In the first case we have $\pi_\beta^0 \varphi_{\alpha\beta} = 0$. Thus we proceed to the second case.

Assume that $P_{\alpha\beta} = P_\beta$. Then the fact that P_β is projective implies that

$$1 = \dim \text{Hom}_{U(\mathfrak{g})}(P_\beta, P_\alpha) \geq \dim \text{Hom}_{U(\mathfrak{g})}(P_\beta, P_\beta) = 2.$$

This is a contradiction.

Part b). Corollary 7.7 implies that P_α has the unique simple submodule which is precisely the image of π_α^0 and which is isomorphic to M_α . We have

$$\dim \text{Hom}_{U(\mathfrak{g})}(M_\alpha, P_\beta) = \dim \text{Hom}_{U(\mathfrak{g})}(P_\beta, M_\alpha) = 0.$$

Part c). First, we show that $\varphi_{\alpha\beta} \varphi_{\beta\alpha} \neq 0$. Assume to the contrary that $\varphi_{\alpha\beta} \varphi_{\beta\alpha} = 0$. This means that the image $P_{\beta\alpha}$ of $\varphi_{\beta\alpha}$ belongs to the kernel $\text{Ker } \varphi_{\alpha\beta}$ of $\varphi_{\alpha\beta}$. Then we have a nonzero morphism

$$P_\alpha / P_{\beta\alpha} \rightarrow P_\beta.$$

This immediately implies that the Jordan–Hölder multiplicities of M_β in $P_{\beta\alpha}$ and $P_\alpha / P_{\beta\alpha}$ both are nonzero. Hence the Jordan–Hölder multiplicity of M_β in P_α is at least 2. This is not the case.

Proposition 7.11 implies that

$$\varphi_{\alpha\beta} \varphi_{\beta\alpha} = c_1 \pi_\beta + c_2 \pi_\beta^0.$$

If $c_1 \neq 0$, then the fact that $(\pi_\beta^0)^2 = 0$ implies that $\varphi_{\beta\alpha} \varphi_{\alpha\beta}$ is surjective and therefore that $\varphi_{\beta\alpha}$ is surjective. This is wrong; see case a). Therefore $\varphi_{\alpha\beta} \varphi_{\beta\alpha} = c_2 \pi_\beta^0$. \square

Lemma 7.14. *Let $\alpha, \beta, \gamma \in \Pi$ be such that α and β are adjacent, β and γ are adjacent, $\alpha \neq \gamma$. Then $\varphi_{\beta\gamma} \varphi_{\alpha\beta} = 0$.*

Proof. It is clear that α is not adjacent to γ . Thus $\dim \text{Hom}_{U(\mathfrak{g})}(P_\alpha, P_\gamma) = 0$. Hence $\varphi_{\beta\gamma} \varphi_{\alpha\beta} = 0$. \square

Proof of Theorem 7.9. Lemma 7.13 implies that, for a pair of adjacent roots $\alpha, \beta \in \Pi$, there exists a nonzero constant $c_{\alpha,\beta}$ such that $\varphi_{\beta\alpha} \varphi_{\alpha\beta} = c_{\alpha,\beta} \pi_\alpha^0$. Denote by α_0 the unique simple root with three neighbours. For any $\alpha \in \Pi$ put $\alpha_0, \alpha_1, \dots, \alpha_n(\alpha)$

to be the shortest sequence of adjacent roots which connects α_0 with $\alpha = \alpha_{n(\alpha)}$. Set

$$\begin{aligned} \underline{\pi}_\alpha &:= \pi_\alpha(\forall \alpha), \quad \underline{\varphi}_{\alpha_{n(\alpha)-1}\alpha} := \varphi_{\alpha_{n(\alpha)-1}\alpha} \quad (\alpha \neq \alpha_0) \\ \underline{\pi}_\alpha^0 &:= \begin{cases} \left(\prod_{0 \leq i < n(\alpha)} \frac{c_{\alpha_{i+1}\alpha_i}}{c_{\alpha_i\alpha_{i+1}}} \right) \pi_\alpha^0 & \text{if } \alpha \neq \alpha_0 \\ \pi_\alpha^0 & \text{otherwise} \end{cases}, \\ \underline{\varphi}_{\alpha_{n(\alpha)}\alpha_{n(\alpha)-1}} &:= \begin{cases} \left(\prod_{0 \leq i < n(\alpha)} \frac{c_{\alpha_{i+1}\alpha_i}}{c_{\alpha_i\alpha_{i+1}}} \right) \frac{\varphi_{\alpha_{n(\alpha)}\alpha_{n(\alpha)-1}}}{c_{\alpha_{n(\alpha)}\alpha_{n(\alpha)-1}}} & \text{if } \alpha \neq \alpha_0 \\ \text{undefined otherwise} \end{cases}. \end{aligned}$$

It can be easily checked through Lemmas 7.12, 7.13, 7.14 that $\underline{\pi}_\alpha, \underline{\pi}_\alpha^0, \underline{\varphi}_{\alpha\beta}$ satisfy all conditions of Theorem 7.9. □

8. PROOF OF THEOREM 2.3

We need to expand our notation on Weyl groups, root systems, etc., to the nonintegral case. We use the notation of Section 4. Fix $\lambda \in \mathfrak{h}^*$. Put

$$\Delta^{\mathbb{Z}} := \{ \alpha \in \Delta \mid \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \}.$$

It is clear that $\Delta_\lambda \subset \Delta^{\mathbb{Z}}$. We denote by $W^{\mathbb{Z}}$ the subgroup of W generated by the reflections with respect to elements of $\Delta^{\mathbb{Z}}$. To the root system $\Delta^{\mathbb{Z}}$ we attach a Lie algebra $\mathfrak{g}^{\mathbb{Z}}$ in such a way that \mathfrak{h} is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{Z}}$. We denote by $\mathfrak{b}^{\mathbb{Z}}$ the Borel subalgebra of $\mathfrak{g}^{\mathbb{Z}}$ attached to $\Delta^{\mathbb{Z}} \cap \Delta^+$. This allows us to define $M^{\mathbb{Z}}(\lambda), L^{\mathbb{Z}}(\lambda), I^{\mathbb{Z}}(\lambda)$, and $O^{\mathbb{Z}}(\lambda)$.

To any root $\alpha \in \Delta$ we attach a coroot $\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)} \in \mathfrak{h}^*$. The set of coroots Δ^\vee is a root system, and the Lie algebra \mathfrak{g}^\vee attached to Δ^\vee is called the *Langlands dual* to \mathfrak{g} . It can be easily seen that $\mathfrak{g}^{\mathbb{Z}}$ is not necessarily isomorphic to a subalgebra of \mathfrak{g} , but $(\mathfrak{g}^{\mathbb{Z}})^\vee$ is canonically isomorphic to the subalgebra of \mathfrak{g}^\vee (it is defined as the sum of \mathfrak{h} with the weight spaces of weights of $(\Delta^{\mathbb{Z}})^\vee$). It is worth mentioning that if \mathfrak{g} is simple, then $\mathfrak{g} \cong \mathfrak{g}^\vee$ if and only if \mathfrak{g} is not of type B_n, C_n ($n \geq 3$).

Definition 8.1. We say that a subalgebra \mathfrak{k} of \mathfrak{g} is an *r-subalgebra* if \mathfrak{k} and \mathfrak{g} have the same rank. By definition, $(\mathfrak{g}^{\mathbb{Z}})^\vee$ is an r-subalgebra of $(\mathfrak{g})^\vee$.

We denote by PrId^λ the set primitive ideals I of $U(\mathfrak{g})$ such that $I \cap Z(\mathfrak{g}) = m_\lambda$. The following theorem is a slight modification of [BV2, Theorem 2.5].

Theorem 8.2. *The map*

$$W^{\mathbb{Z}} \rightarrow \text{PrId}^\lambda \quad (w \rightarrow I(w(\lambda + \rho) - \rho))$$

is surjective.

Put $\rho^{\mathbb{Z}} := \frac{1}{2} \sum_{\alpha \in \Delta^{\mathbb{Z}} \cap \Delta^+} \alpha$. We say that λ is *dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$* if $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta^{\mathbb{Z}} \cap \Delta^+$. The following statement is a straightforward corollary of [BV2, Theorem 4.8, Proposition 2.28].

Theorem 8.3. *We have*

$$\dim \mathfrak{g} - \dim O(\lambda) = \dim \mathfrak{g}^{\mathbb{Z}} - \dim O^{\mathbb{Z}}(\lambda + \rho - \rho^{\mathbb{Z}}).$$

This theorem allows us to expand Theorem 2.1 beyond the integral case. Recall that

$$\text{PrId}^\lambda(O) = \{I \in \text{PrId}^\lambda \mid \text{Var}(I) = \bar{O}\}.$$

Recall that O is the minimal nilpotent orbit of \mathfrak{g} . Proposition 2.2 is implied by Proposition 3.1 and the following statement.

Proposition 8.4. *Assume that \mathfrak{g} is simply-laced and not of type A_n . If λ is nonintegral, then $\text{PrId}^\lambda(O)$ is empty.*

Proof. Assume to the contrary that λ is nonintegral, and there exists an ideal I in $\text{PrId}^\lambda(O)$. Without loss of generality we can assume that $I = I(\lambda)$ and $O = O(\lambda)$.

The assumptions on λ imply that

$$\dim \mathfrak{g}^{\mathbb{Z}} < \dim \mathfrak{g}.$$

Theorem 8.3 implies that

$$\dim O = \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathbb{Z}} + \dim O^{\mathbb{Z}}(\lambda + \rho - \rho^{\mathbb{Z}}),$$

and thus

$$\dim O \geq \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathbb{Z}}.$$

The dimensions of $\dim O$ are written in Table 1 (they are evaluated through [CM, Lemma 4.3.5]).

TABLE 1

$A_n (n \geq 2)$	$B_n (n \geq 2)$	$C_n (n \geq 2)$	$D_n (n \geq 4)$	E_6	E_7	E_8	F_4	G_2
$2n$	$4n - 4$	$2n$	$4n - 6$	22	34	58	16	6

Comparing these numbers with Proposition 9.1 of the appendix applied to $(\mathfrak{g}^{\mathbb{Z}})^\vee \subset (\mathfrak{g})^\vee$ we see that

$$\dim O < \dim(\mathfrak{g})^\vee - \dim(\mathfrak{g}^{\mathbb{Z}})^\vee$$

due to the fact that \mathfrak{g} is simply-laced and not of type A . This is a contradiction. \square

We wish to mention that cases C_n, G_2 were considered in [Pr2, Corollary 7.1]. Theorem 2.3 is implied by the following proposition.

Proposition 8.5. *Assume that \mathfrak{g} is not simply-laced and is simple. Then the following statements are equivalent:*

- a) $U(\mathfrak{g}, e) - f.d.\text{mod}^{m_\lambda}$ has a nonzero object,
- b) $U(\mathfrak{g}, e) - f.d.\text{mod}^{m_\lambda}$ is semisimple with a unique simple object.

Proof of Proposition 8.5. It is enough to show that a) implies b). From now on we assume a). We need the following lemmas.

Lemma 8.6. *Assume that $O(\lambda) = O$. Then*

- a) $(\mathfrak{g}^{\mathbb{Z}})^\vee$ is a maximal proper r -subalgebra of \mathfrak{g}^\vee ,
- b) $\lambda + \rho - \rho^{\mathbb{Z}}$ is dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$,
- c) $W_\lambda = \{e\}$.

Proof. The minimal nilpotent orbit O is not special under the assumption that \mathfrak{g} is not simply-laced; see [CM]. Thus $(\mathfrak{g}^{\mathbb{Z}})^{\vee}$ is a proper r-subalgebra of \mathfrak{g}^{\vee} . Theorem 8.3 implies that

$$\dim O = \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathbb{Z}} + \dim O^{\mathbb{Z}}(\lambda + \rho - \rho^{\mathbb{Z}}),$$

and thus that

$$\dim O \geq \dim \mathfrak{g} - \dim \mathfrak{g}^{\mathbb{Z}}.$$

The dimensions of $\dim O$ are written in Table 1. Comparing these numbers with Proposition 9.1 of the appendix applied to $(\mathfrak{g}^{\mathbb{Z}})^{\vee} \subset (\mathfrak{g}^{\vee})^{\vee}$ we see that

$$\dim O \leq \dim(\mathfrak{g}^{\vee})^{\vee} - \dim(\mathfrak{g}^{\mathbb{Z}})^{\vee}$$

due to the fact that \mathfrak{g} is not simply-laced. Therefore Proposition 9.1 and the fact that $(\mathfrak{g}^{\mathbb{Z}})^{\vee}$ is an r-subalgebra of \mathfrak{g}^{\vee} , implies that $(\mathfrak{g}^{\mathbb{Z}})^{\vee}$ is a maximal r-subalgebra of \mathfrak{g}^{\vee} . Also we have that

$$\dim O^{\mathbb{Z}}(\lambda + \rho - \rho^{\mathbb{Z}}) = 0,$$

and thus that $\lambda + \rho - \rho^{\mathbb{Z}}$ is dominant with respect to $\Delta^{\mathbb{Z}}$. The latter condition implies that W_{λ} is trivial. This completes the proof. \square

The lemma below is well known, and we provide the proof of it only for the convenience of the reader.

Lemma 8.7. *Let $\Delta_1, \Delta_2 \subset \Delta$ be subsets of Δ such that*

$$\mathfrak{g}_1 := \mathfrak{h} \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_{\alpha}, \quad \mathfrak{g}_2 := \mathfrak{h} \bigoplus_{\alpha \in \Delta_1} \mathfrak{g}_{\alpha},$$

are r-subalgebras of \mathfrak{g} . Then the following conditions are equivalent:

- a) \mathfrak{g}_1 is conjugate to \mathfrak{g}_2 by the adjoint group $\text{Adj}(\mathfrak{g})$ of \mathfrak{g} ,
- b) there exists $w \in W$ such that $w(\Delta_1) = \Delta_2$.

Proof. First, we show that a) implies b). We fix $g \in \text{Adj}(\mathfrak{g})$ such that $g(\mathfrak{g}_1) = \mathfrak{g}_2$. By definition, \mathfrak{h} is a Cartan subalgebra of both \mathfrak{g}_1 and \mathfrak{g}_2 . Thus $g(\mathfrak{h})$ is a Cartan subalgebra of \mathfrak{g}_2 . Therefore there exists an element g_2 of the adjoint group $\text{Adj}(\mathfrak{g}_2)$ of \mathfrak{g}_2 such that $g_2(g(\mathfrak{h})) = \mathfrak{h}$. Put $g' := g_2g$. We have

$$g'(\mathfrak{g}_1) = \mathfrak{g}_2, \quad g'(\mathfrak{h}) = \mathfrak{h}.$$

The second condition implies that g' can be represented by an element $w \in W$ and the first condition implies that this w is as desired in b).

Next, we show that b) implies a). The Weyl group can be identified with the quotient of the normalizer of \mathfrak{h} in $\text{Adj}(\mathfrak{g})$ by the centralizer of \mathfrak{h} in $\text{Adj}(\mathfrak{g})$. Thus there exists $g \in \text{Adj}(\mathfrak{g})$ such that

$$g(\mathfrak{h}) = \mathfrak{h}, \quad g(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{w(\alpha)}.$$

It is clear that $g(\mathfrak{g}_1) = \mathfrak{g}_2$. \square

Lemma 8.8. *Assume that $U(\mathfrak{g}, e)\text{-}f.d.\text{mod}^{m_{\lambda}}$ has a nonzero object. Then $U(\mathfrak{g}, e)\text{-}f.d.\text{mod}^{m_{\lambda}}$ has a unique simple object $M_{f.d.}(\lambda)$. Moreover, there exists $w \in W^{\mathbb{Z}}$ such that $M_{f.d.}(\lambda) = \mathcal{P}^{-1}(I(w(\lambda + \rho) - \rho))$.*

Proof. Theorem 8.2 together with Proposition 3.1 implies that any simple object of $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$ is of the form $\mathcal{P}^{-1}(I(w(\lambda + \rho) - \rho))$ for $w \in W^{\mathbb{Z}}$.

We consider $w \in W^{\mathbb{Z}}$ such that $\mathcal{P}^{-1}(I(w(\lambda + \rho) - \rho))$ belongs to $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$. The latter condition is equivalent to $O(w(\lambda + \rho) - \rho) = O$. This implies that $w(\lambda + \rho) - \rho^{\mathbb{Z}}$ is dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$. This implies that w is unique. Thus we have shown that $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$ has a unique simple object. We denote this object by $M_{f.d.}(\lambda)$. \square

We are left to prove that $\text{Ext}^1(M_{f.d.}(\lambda), M_{f.d.}(\lambda))$. If $\mathfrak{g} \cong G_2$ or $\mathfrak{g} \cong C_n$, then the statement of Proposition 8.5 follows immediately from [Pr2, Corollary 7.1]. From now on we assume that \mathfrak{g} is not of type C_n, G_2 , i.e., that \mathfrak{g} is of type B_n, F_4 . We use notation for roots and weights of [Bou], and consider both cases simultaneously. The underlying calculations can be easily done through [Bou].

Set μ_0 as in [Jo2, Table 3]. Put

$$\Delta(\mathfrak{k}) := \{\alpha \in \Delta \mid (\alpha^\vee, \mu_0) \in \mathbb{Z}\}, \quad \mathfrak{k}^\vee := \mathfrak{h} \bigoplus_{\alpha \in \Delta(\mathfrak{k})} (\mathfrak{g}^\vee)_{\alpha^\vee}.$$

Dimension arguments together with Proposition 9.1 imply that \mathfrak{k}^\vee is a maximal r-subalgebra of \mathfrak{g}^\vee . We have that $(\mathfrak{g}^{\mathbb{Z}})^\vee$ is also a maximal r-subalgebra of \mathfrak{g}^\vee . Lemma 8.7 implies that there exists $w' \in W$ such that $w'(\Delta^{\mathbb{Z}}) = \Delta(\mathfrak{k})$. Put $\lambda' := w'(\lambda + \rho) - \rho$. We have that $U(\mathfrak{g}, e) - f.d.mod^{m\lambda} = U(\mathfrak{g}, e) - f.d.mod^{m\lambda'}$. Thus we can assume that $\lambda = \lambda'$. This immediately implies that $\Delta^{\mathbb{Z}} = \Delta(\mathfrak{k})$.

Let $\gamma \in \Pi$ be the unique simple root such that $(\gamma, \mu_0) \neq 0$ (in both cases $\gamma = \alpha_1$). Then $\Pi \setminus \gamma \in \Delta^{\mathbb{Z}}$.

It is easy to check that $2\gamma^\vee$ belongs to the lattice generated by $(\Delta^{\mathbb{Z}})^\vee$. Thus

$$2(\gamma^\vee, \lambda) \in \mathbb{Z}.$$

Therefore either $(\gamma^\vee, \lambda) \in \mathbb{Z}$ or $(\gamma^\vee, \lambda) \in \frac{1}{2} + \mathbb{Z}$. We have that $(\alpha^\vee, \lambda) \in \mathbb{Z}$ for all $\alpha \in (\Pi \setminus \gamma) \subset \Delta^{\mathbb{Z}}$. Thus if $(\gamma^\vee, \lambda) \in \mathbb{Z}$, then $\Pi \subset \Delta^{\mathbb{Z}}$, and hence $\Delta^{\mathbb{Z}} = \Delta$. This is a contradiction.

Thus $(\gamma^\vee, \lambda) \in \frac{1}{2} + \mathbb{Z}$. The fact that $(\alpha^\vee, \lambda) \in \mathbb{Z}$ for all $\alpha \in (\Pi \setminus \gamma) \subset \Delta^{\mathbb{Z}}$ implies that

$$(\lambda - \mu_0, \alpha^\vee) \in \mathbb{Z}.$$

As a consequence $\lambda - \mu_0 \in \Lambda, \lambda - \mu_0 + \rho \in \Lambda$. Lemma 8.6 implies that $\lambda + \rho - \rho^{\mathbb{Z}}$ is dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$. It is straightforward to check that $\mu_0 - \rho^{\mathbb{Z}}$ is dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$.

To finish the proof we need the following lemma.

Lemma 8.9. *Assume that*

- a) *both $\lambda + \rho - \rho^{\mathbb{Z}}$ and $\mu_0 - \rho^{\mathbb{Z}}$ are dominant with respect to $\Delta^{\mathbb{Z}} \cap \Delta^+$,*
- b) *$\lambda - \mu_0 + \rho \in \Lambda$,*
- c) *$W_\lambda = W_{\mu_0}$.*

Then $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$ is equivalent to $U(\mathfrak{g}, e) - f.d.mod^{m\mu_0 - \rho}$.

Proof. We recall that $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$ is equivalent to a category of \mathfrak{g} -modules \mathcal{C}^λ ; see Section 7. Conditions a), b), c) imply together with [BeG, Theorem, Section 4.1] that \mathcal{C}^λ is equivalent to $\mathcal{C}^{\mu_0 - \rho}$. Hence $U(\mathfrak{g}, e) - f.d.mod^{m\lambda}$ is equivalent to $U(\mathfrak{g}, e) - f.d.mod^{m\mu_0 - \rho}$. \square

Thanks to Lemmas 8.8, 8.9 we have that

$$\dim \text{Ext}^1(M_{f.d.}(\lambda), M_{f.d.}(\lambda)) = \dim \text{Ext}^1(M_{f.d.}(\mu_0 - \rho), M_{f.d.}(\mu_0 - \rho)).$$

Thus we are left to show that $\dim \text{Ext}^1(M, M) = 0$ for one nonzero simple finite-dimensional $U(\mathfrak{g}, e)$ -module M . This was done in Proposition 3.2. \square

9. APPENDIX: MAXIMAL REDUCTIVE ROOT SUBALGEBRAS

Let \mathfrak{g} be a reductive Lie algebra. By definition, the rank of \mathfrak{g} equals the dimension of a Cartan subalgebra of \mathfrak{g} . We say that a subalgebra \mathfrak{k} of \mathfrak{g} is an *r-subalgebra* if the rank of \mathfrak{k} equals the rank of \mathfrak{g} . The description of maximal *by inclusion* r-subalgebras of simple Lie algebras are very well known; see [BS] and also [Bou], [Dyn]. We used a description of maximal *by dimension* r-subalgebras in Section 8. Of course, the second one is a straightforward consequence of the first one, but this consequence requires plenty of not very conceptual computations. We decided to provide these computations in this appendix.

Proposition 9.1. *Let \mathfrak{g} be a simple Lie algebra. Then the conjugacy classes of maximal by dimension proper r-subalgebras \mathfrak{k} of \mathfrak{g} are listed in Table 2. In particular, such a conjugacy class is unique if $\mathfrak{g} \not\cong D_4$.*

TABLE 2

\mathfrak{g}	\mathfrak{k}	$\text{codim}(\mathfrak{g}/\mathfrak{k})$	
A_l	$A_{l-1} \oplus \mathbb{F}$	$2l - 2$	$l \geq 1$
B_l	D_l	$2l$	$l \geq 2$
C_l	$C_{l-1} \oplus C_1$	$4l - 4$	$l \geq 2$
D_l	$D_{l-1} \oplus \mathbb{F}$	$4l - 4$	$l \geq 5$
D_4	$D_3 \oplus \mathbb{F}, A_3 \oplus \mathbb{F}, A'_3 \oplus \mathbb{F}$	12	
E_6	$D_5 \oplus \mathbb{F}$	32	
E_7	$E_6 \oplus \mathbb{F}$	54	
E_8	$E_7 \oplus A_1$	112	
F_4	B_4	16	
G_2	A_2	6	

Proof. The conjugacy classes of such subalgebras can be identified with the conjugacy classes of subgroups of maximal rank of the compact group attached to the root system of \mathfrak{g} . The latter conjugacy classes are described in [BS, Table]. We reproduce the needed part of [BS, Table] in Table 3.

We are left to figure out which subalgebras of this list are maximal by dimension. For the exceptional groups this is very straightforward because the list of maximal by inclusion subalgebras is finite. We proceed with the classical cases one-by-one.

Case A. We claim that $A_{l-1} \oplus \mathbb{F}$ has the maximal dimension in A_l , and thus that

$$\dim(A_{l-1} \oplus \mathbb{F}) > \dim(A_i \oplus A_{l-i-1} \oplus \mathbb{F})$$

if $0 < i < l - 1$. This is equivalent to the following statements:

$$\begin{aligned} l^2 &> (i + 1)^2 + (l - i)^2 - 1, \\ 2li - 2i^2 - 2i &\geq 0, & (0 < i < l - 1), \\ 2i(l - i - 1) &\geq 0. \end{aligned}$$

The latter statement is trivial.

TABLE 3

\mathfrak{g}	\mathfrak{k}	
A_l	$A_i \oplus A_{l-i-1} \oplus \mathbb{F}$	$1 \leq i < l, l \geq 1$
B_l	D_l	$l \geq 2$
	$B_i \oplus D_{l-i}$	$1 \leq i < l, l \geq 2$
	$B_{l-1} \oplus \mathbb{F}$	$j \geq 2$
C_l	$C_i \oplus C_{l-i}$	$1 \leq i < l, l \geq 2$
	$A_{l-1} \oplus \mathbb{F}$	$l \geq 2$
D_l	$D_i \oplus D_{l-i}$	$1 \leq i \leq l, l \geq 4$
	$A_{l-1} \oplus \mathbb{F}$	$l \geq 4$
	$D_{l-1} \oplus \mathbb{F}$	$l \geq 4$
E_6	$A_1 \oplus A_5, A_2 \oplus A_2 \oplus A_2$ $D_5 \oplus \mathbb{F}$	
E_7	$A_1 \oplus D_6, A_7, A_2 \oplus A_5$ $E_6 \oplus \mathbb{F}$	
E_8	$D_8, A_1 \oplus E_7, A_8$ $A_2 \oplus E_6, A_4 \oplus A_4$	
F_4	$A_1 \oplus C_3, B_4, A_2 \oplus A_2$	
G_2	$A_1 \oplus A_1, A_2$	

Case B. We claim that D_l has the maximal dimension in B_l , and thus that

$$(B1) \quad \dim D_l > \dim(B_i \oplus D_{l-i}), \quad 0 < i < l, l \geq 3,$$

$$(B2) \quad \dim D_l > \dim(B_{l-1} \oplus \mathbb{F}), \quad l \geq 3.$$

We now verify statements (B1) and (B2).

Statement (B1) is equivalent to the following inequalities

$$\begin{aligned} 2l^2 - l &> (2i^2 + i) + (2(l-i)^2 - (l-i)), \\ 2i(2l-i) - i &> 2i^2 + i, \quad (0 < i < l), \\ 62i(2l-2i-1) &> 0. \end{aligned}$$

The latter statement is trivial.

Statement (B2) is equivalent to the following inequalities:

$$\begin{aligned} 2l^2 - l &> (2(l-1)^2 + (l-1)) + 1, \quad (l \geq 2), \\ 2(l-1) &> 0. \end{aligned}$$

The latter statement is trivial.

Case C. We claim that $C_{l-1} \oplus C_1$ has the maximal dimension in C_l , and thus that

$$(C1) \quad \dim(C_{l-1} \oplus C_1) > \dim(C_i \oplus C_{l-i}), \quad 1 < i < l-1, l \geq 2,$$

$$(C2) \quad \dim(C_{l-1} \oplus C_1) > \dim(A_{l-1} \oplus \mathbb{F}), \quad l \geq 2.$$

We now verify statements (C1) and (C2).

Statement (C1) is equivalent to the following inequalities:

$$\begin{aligned} 2(l-1)^2 + (l-1) + 3 &> (2i^2 + i) + (2(l-i)^2 - (l-i)), \\ 2(i-1)(2l-i-1) - 2(i-1)(i+1) &> 0, \quad (1 < i < l-1), \\ 4(i-1)(l-i-1) &> 0. \end{aligned}$$

The latter statement is trivial.

Statement (C2) is equivalent to the following inequalities:

$$\begin{aligned} 2(l-1)^2 + (l-1) + 3 &> l^2, \\ (l-2)^2 + l &> 0, \end{aligned} \quad (l \geq 2),$$

The latter statement is trivial.

Case D. We claim that $D_{l-1} \oplus \mathbb{F}$ has the maximal dimension in D_l , and thus that

$$\begin{aligned} \text{(D1)} \quad \dim(D_{l-1} \oplus \mathbb{F}) &> \dim(D_i \oplus D_{l-i}), \quad 1 < i < l-1, l \geq 4, \\ \text{(D2)} \quad \dim(D_{l-1} \oplus \mathbb{F}) &> \dim(A_{l-1} \oplus \mathbb{F}), \quad l \geq 4. \end{aligned}$$

We now verify statements (D1) and (D2).

Statement (D1) is equivalent to the following inequalities:

$$\begin{aligned} 2(l-1)^2 - (l-1) + 1 &> (2i^2 - i) + (2(l-i)^2 - (l-i)), \\ 2(i-1)(2l-i-1) - (i-1) - 2(i-1)(i+1) + (i-1) &> 0. \end{aligned} \quad (1 < i < l-1),$$

The latter statement is trivial.

Statement (D2) is equivalent to the following inequalities:

$$\begin{aligned} 2(l-1)^2 - (l-1) + 1 &> l^2, \\ (l-1)(l-4) &> 0, \end{aligned} \quad (l \geq 4).$$

If $l \geq 5$ the latter statement is trivial. If $l = 4$, then $D_3 = A_3$ and the subalgebras having the same dimension $D_3 \oplus \mathbb{F}$, $A_3 \oplus \mathbb{F}$, and $A'_3 \oplus \mathbb{F}$ are not conjugate in D_4 . \square

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