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ON THE SOCLES OF CERTAIN PARABOLICALLY INDUCED REPRESENTATIONS OF p-ADIC CLASSICAL GROUPS

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Abstract. In this paper, we consider representations of p-adic classical groups parabolically induced from the products of shifted Speh representations and unitary representations of Arthur type of good parity. We describe how to compute the socles (the maximal semisimple subrepresentations) of these representations. As a consequence, we can determine whether these representations are reducible or not. In particular, our results produce many unitary representations, which appear in the complementary series.

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1. Introduction

One of the most important problems in representation theory is the classification of irreducible unitary representations of a given group. This is called the unitary dual problem. In this paper, we consider p-adic reductive groups. Let F be a non-archimedean local field of characteristic zero. The unitary duals of general linear groups $\mathrm{GL}_n(F)$ were explicitly described by Tadić [18] in the 1980's. According to his description, the union of the unitary duals of all $\mathrm{GL}_n(F)$ has two important classes; the *Speh representations* and their *complementary series*. Indeed, any irreducible unitary representation of $\mathrm{GL}_n(F)$ is obtained as the irreducible parabolically induced representation from a product of complementary series.

Now we consider classical groups. Let G_n be a split special odd orthogonal group $\mathrm{SO}_{2n+1}(F)$ or a symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n over F. There are several works on the unitary dual of G_n. For example:

- The generic unitary dual of G_n was explicitly described by Lapid–Muić–Tadić [8].

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- When $n \leq 3$, Tadić [21] computed the unitary dual of G_n completely.

However, the general case is still widely unknown.

The notion of *local A-packets* was introduced by Arthur [1]. They are (multi-)sets over the set of equivalence classes of irreducible unitary representations of G_n. We say that an irreducible representation is *of Arthur type* if it belongs to some local A-packet. One can regard representations of Arthur type as analogues of Speh representations in the sense that both of them are local factors of square-integrable automorphic representations. In fact, representations of Arthur type are expected to play an alternative role to Speh representations in the unitary dual problem (see [22, Conjectures 8.2, 8.3, 8.5]).

In this paper, we consider the parabolically induced representation of the form

$$\Pi_s = u_\rho(a,b)|\cdot|^s \rtimes \pi_A$$

where

- $u_\rho(a,b)$ is the (unitary) Speh representation with an irreducible self-dual supercuspidal representation ρ of $\mathrm{GL}_d(F)$ and positive integers a, b (see Section 2.1);
- π_A is an irreducible representation of Arthur type (see Section 4.2);
- $s \in \mathbb{R}$.

It is known that:

- Π_0 decomposes into a multiplicity-free direct sum of irreducible unitary representations of Arthur type ([1, Proposition 2.4.3]).
- If s_0 is the *first reducibility point* for Π_s, i.e., the minimal non-negative real number such that Π_{s_0} is reducible, then Π_s is irreducible and unitary for any $0 \leq s < s_0$ (see [19, Section 3 (b)]). In this case, Π_s is called a *complementary series representation.*
- All irreducible constituents of Π_{s_0} are also unitary (see [19, Section 3 (c)]).

In particular, the study of Π_s would produce many irreducible unitary representations.

In several restricted cases, the reducibility and the composition series of Π_s are already known. For example, several criteria for the irreducibility of Π_s were given by

- Muić [17] when $b = 1$ and π_A is a discrete series representation;
- Matić [10] when $a = 1$ and π_A is a discrete series representation;
- Lapid–Tadić [9, Theorems 1.1, 1.2] when (a, b) is arbitrary and π_A is supercuspidal.

All irreducible constituents of Π_s were obtained by

- Muić [16] when $b = 1$ and π_A is a strongly positive discrete series representation;
- Matić [11] when $a = 1$, $b+2s \in 2\mathbb{Z}$ and π_A is a discrete series representation;
- Bošnjak [4] when π_A is supercuspidal and $s \geq (a+b-1)/2$.

Finally, the semisimplification of Π_s for $s \in \{0, 1/2\}$ seems to be already given by Mœglin (watch the video of her talk [14]).

In this paper, we describe the *socle* $\mathrm{soc}(\Pi_s)$ of Π_s, i.e., the maximal semisimple subrepresentation of Π_s. The main theorem is as follows.

Theorem 1.1. *Let $\Pi_s = u_\rho(a,b)|\cdot|^s \rtimes \pi_A$ be as above. Then we can describe the socle* $\mathrm{soc}(\Pi_s)$ *algorithmically in terms of the Langlands classification. Moreover, the following hold.*

(1) *If $s > (a-1)/2$ or $s < -(b-1)/2$, or if $s \notin (1/2)\mathbb{Z}$, then the socle* $\mathrm{soc}(\Pi_s)$ *is irreducible. (See Proposition* 3.4.*)*
(2) *If $s \in (1/2)\mathbb{Z}$ and $0 < s \leq (a-1)/2$ (resp. $-(b-1)/2 \leq s < 0$), then any irreducible subrepresentation of Π_s is of the form $\pi' = \mathrm{soc}(u_\rho(2s,b)|\cdot|^{a/2} \rtimes \pi'_A)$ (resp. $\pi' = \mathrm{soc}(u_\rho(a,-2s)|\cdot|^{-b/2} \rtimes \pi'_A)$) for some irreducible summand π'_A of $u_\rho(a-2s,b) \rtimes \pi_A$ (resp. $u_\rho(a,b+2s) \rtimes \pi_A$). Moreover, for such an irreducible summand π'_A, one can determine whether π' is a subrepresentation of Π_s or not. (See Propositions* 3.6 *and* 3.7.*)*
(3) *If $s = 0$, then the decomposition of Π_0 is explicitly given in terms of extended multi-segments. (See Theorem* 4.4.*)*

We have several consequences.

Corollary 1.2 (Corollary 5.1). *Let $\Pi_s = u_\rho(a,b)|\cdot|^s \rtimes \pi_A$ be as above. Then any irreducible subrepresentation of Π_s appears in the composition series of Π_s with multiplicity one. In particular,* $\mathrm{soc}(\Pi_s)$ *is multiplicity-free.*

Corollary 1.3 (Corollary 5.2). *Let $\Pi_s = u_\rho(a,b)|\cdot|^s \rtimes \pi_A$ be as above. Then Π_s is irreducible if and only if all of the following conditions hold:*

- $\mathrm{soc}(\Pi_s)$ *is irreducible;*
- $\mathrm{soc}(\Pi_{-s})$ *is irreducible;*
- $\mathrm{soc}(\Pi_s) \cong \mathrm{soc}(\Pi_{-s})$.

In particular, one can compute the first reducibility point s_0 for Π_s (Corollary 5.4).

This paper is organized as follows. In Section 2, we review the Langlands classification of classical groups. In Section 3, we prove Theorem 1.1 (1) and (2). To do this, we refine the theory of derivatives used in [3]. Theorem 1.1 (3) is proven in Section 4 after reviewing the theory of extended multi-segments established in the previous paper [2]. Finally, in Section 5, we obtain several consequences about the irreducibility of Π_s, and we give some examples. To describe $\mathrm{soc}(\Pi_s)$ explicitly, we need to compute several derivatives. In Appendix A, we give an algorithm for the computations of certain derivatives, which are not obtained in [3].

Notation. Let F be a non-archimedean local field of characteristic zero. The normalized absolute value is denoted by $|\cdot|$, which is also regarded as a character of $\mathrm{GL}_d(F)$ via composing with the determinant map.

Let G_n be a split special odd orthogonal group $\mathrm{SO}_{2n+1}(F)$ or a symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n over F. For a smooth representation Π of G_n or $\mathrm{GL}_n(F)$ of finite length, we write $[\Pi]$ for its semisimplification. Similarly, we denote by $\mathrm{soc}(\Pi)$ the *socle* of Π, i.e., the maximal semisimple subrepresentation of Π. The set of equivalence classes of irreducible smooth representations of a group G is denoted by $\mathrm{Irr}(G)$.

We will often extend the set theoretical language to multi-sets. Namely, we write a multi-set as $\{x,\dots,x,y,\dots,y,\dots\}$. When we use a multi-set, we will mention it.

2. Langlands classification

In this section, we review the Langlands classification of classical groups.

2.1. **General linear groups.** First, we recall some notations for representations of $\mathrm{GL}_n(F)$. Let P be a standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi subgroup $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$. For representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_r}(F)$, respectively, we denote by

$$\tau_1 \times \cdots \times \tau_r := \mathrm{Ind}_P^{\mathrm{GL}_n(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)$$

the normalized parabolically induced representation.

Let $\mathrm{Cusp}_{\mathrm{unit}}(\mathrm{GL}_d(F))$ be the set of equivalence classes of irreducible unitary supercuspidal representations of $\mathrm{GL}_d(F)$, and $\mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$ be the subset consisting of self-dual elements.

A *segment* $[x, y]_\rho$ is a set of supercuspidal representations of the form

$$[x, y]_\rho := \{\rho|\cdot|^x, \rho|\cdot|^{x-1}, \dots, \rho|\cdot|^y\},$$

where $\rho \in \mathrm{Cusp}_{\mathrm{unit}}(\mathrm{GL}_d(F))$ and $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Z}$ and $x \geq y$. For a segment $[x, y]_\rho$, define the *Steinberg representation* $\Delta_\rho[x, y]$ as a unique irreducible subrepresentation of

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y.$$

This is an essentially discrete series representation of $\mathrm{GL}_{d(x-y+1)}(F)$. Similarly, we define $Z_\rho[y, x]$ as a unique irreducible quotient of the same induced representation. By convention, we set $\Delta_\rho[x, x+1]$ and $Z_\rho[x+1, x]$ to be the trivial representation of the trivial group $\mathrm{GL}_0(F)$.

The Langlands classification for $\mathrm{GL}_n(F)$ says that every $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$ is a unique irreducible subrepresentation of $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r]$, where $\rho_i \in \mathrm{Cusp}_{\mathrm{unit}}(\mathrm{GL}_{d_i}(F))$ for $i = 1, \dots, r$ such that $x_1 + y_1 \leq \cdots \leq x_r + y_r$. In this case, we write

$$\tau = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]).$$

When $(x_{i,j})_{1 \leq i \leq t, 1 \leq j \leq d}$ satisfies that $x_{i,j} = x_{1,1} - i + j$, the irreducible representation $L(\Delta_\rho[x_{1,1}, x_{t,1}], \dots, \Delta_\rho[x_{1,d}, x_{t,d}])$ is called a *(shifted) Speh representation* and is denoted by

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,d} \\ \vdots & \ddots & \vdots \\ x_{t,1} & \dots & x_{t,d} \end{pmatrix}_\rho := L(\Delta_\rho[x_{1,1}, x_{t,1}], \dots, \Delta_\rho[x_{1,d}, x_{t,d}]).$$

Note that it is isomorphic to the unique irreducible subrepresentation of $Z_\rho[x_{1,1}, x_{1,d}] \times \cdots \times Z_\rho[x_{t,1}, x_{t,d}]$. Especially, for positive integers a and b, set

$$u_\rho(a, b) = \begin{pmatrix} \frac{a-b}{2} & \dots & \frac{a+b}{2} - 1 \\ \vdots & \ddots & \vdots \\ -\frac{a+b}{2} + 1 & \dots & -\frac{a-b}{2} \end{pmatrix}_\rho .$$

It is an irreducible unitary representation. We often set $A := (a+b)/2 - 1$ and $B := (a-b)/2$.

2.2. **Classical groups.** Next we recall some notations for representations of the classical group G_n. Fix an F-rational Borel subgroup of G_n. Let P be a standard parabolic subgroup of G_n with Levi subgroup $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times G_{n_0}$. For representations $\tau_1, \dots, \tau_r$ and π_0 of $\mathrm{GL}_{n_1}(F), \dots, \mathrm{GL}_{n_r}(F)$ and of G_{n_0}, respectively, we denote by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 := \mathrm{Ind}_P^{G_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0)$$

the normalized parabolically induced representation.

The Langlands classification for G_n says that every $\pi \in \mathrm{Irr}(G_n)$ is a unique irreducible subrepresentation of $\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_0$, where

- $\rho_i \in \mathrm{Cusp}_{\mathrm{unit}}(\mathrm{GL}_{d_i}(F))$ for $i = 1, \dots, r$;
- $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$;
- π_0 is an irreducible tempered representation of G_{n_0}.

In this case, we write

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_0),$$

and call $(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_0)$ the *Langlands data* for π.

We say that $\pi \in \mathrm{Irr}(G_n)$ is *of Arthur type* if π belongs to an A-packet associated to some A-parameter. For the notion of A-parameters and properties of representations of Arthur type, see Sections 4.1 and 4.2. As basic properties, the following are known: Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type. Then π is a unitary representation. In particular, for $\rho \in \mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$, the parabolically induced representation $u_\rho(a, b) \rtimes \pi$ is also unitary so it is semisimple. By [1, Proposition 2.4.3] (see Proposition 4.2), we know that $u_\rho(a, b) \rtimes \pi$ is multiplicity-free.

3. Non-unitary inductions

Through this section, we fix

- $\rho \in \mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$;
- $\pi \in \mathrm{Irr}(G_n)$ of Arthur type; and
- positive integers a and b.

The purpose of this paper is to explain how to describe the socle of parabolically induced representation $u_\rho(a, b)|\cdot|^s \rtimes \pi$ for $s \in \mathbb{R}$. In this section, we do it for $s \neq 0$.

3.1. **Theory of derivatives.** In this subsection, we introduce the notion of *derivatives*, which is the main terminology.

For a smooth representation π of G_n of finite length, denote by $\mathrm{Jac}_P(\pi)$ its Jacquet module along a standard parabolic subgroup P. Let P_d be the standard parabolic subgroup with Levi subgroup isomorphic to $\mathrm{GL}_d(F) \times G_{n-d}$. For $x \in \mathbb{R}$, the $\rho|\cdot|^x$*-derivative* $D_{\rho|\cdot|^x}(\pi)$ is a semisimple representation of G_{n-d} satisfying that

$$[\mathrm{Jac}_{P_d}(\pi)] = \rho|\cdot|^x \boxtimes D_{\rho|\cdot|^x}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where $\tau_i \boxtimes \pi_i$ is an irreducible representation of $\mathrm{GL}_d(F) \times G_{n-d}$ such that $\tau_i \not\cong \rho|\cdot|^x$. We say that π is $\rho|\cdot|^x$*-reduced* if $D_{\rho|\cdot|^x}(\pi) = 0$. For a segment $[x, y]_\rho$, we set

$$D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y}(\pi) = D_{\rho|\cdot|^y} \circ \cdots \circ D_{\rho|\cdot|^x}(\pi),$$
$$D_{\rho|\cdot|^y, \dots, \rho|\cdot|^x}(\pi) = D_{\rho|\cdot|^x} \circ \cdots \circ D_{\rho|\cdot|^y}(\pi).$$

Hence, with a suitable parabolic subgroup P, we have

$$[\mathrm{Jac}_P(\pi)] = \rho|\cdot|^x \boxtimes \cdots \boxtimes \rho|\cdot|^y \boxtimes D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y}(\pi) + (\text{others}),$$
$$[\mathrm{Jac}_P(\pi)] = \rho|\cdot|^y \boxtimes \cdots \boxtimes \rho|\cdot|^x \boxtimes D_{\rho|\cdot|^y, \dots, \rho|\cdot|^x}(\pi) + (\text{others}).$$

We also set $D^{(0)}_{\rho|\cdot|^x}(\pi) = \pi$ and

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k} D_{\rho|\cdot|^x} \circ D^{(k-1)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} \underbrace{D_{\rho|\cdot|^x} \circ \cdots \circ D_{\rho|\cdot|^x}}_{k}(\pi)$$

for $k > 0$. It satisfies that

$$[\mathrm{Jac}_{P_{dk}}(\pi)] = (\rho|\cdot|^x)^k \boxtimes D^{(k)}_{\rho|\cdot|^x}(\pi) + (\text{others}),$$

where $(\rho|\cdot|^x)^k = \rho|\cdot|^x \times \cdots \times \rho|\cdot|^x$ (k times). When $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ but $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$, we call $D^{(k)}_{\rho|\cdot|^x}(\pi)$ the *highest* $\rho|\cdot|^x$*-derivative* of π, and set $D^{\max}_{\rho|\cdot|^x}(\pi) := D^{(k)}_{\rho|\cdot|^x}(\pi)$. Note that if $|x - x'| \neq 1$, then $D^{(k)}_{\rho|\cdot|^x} \circ D^{(k')}_{\rho|\cdot|^{x'}}(\pi) = D^{(k')}_{\rho|\cdot|^{x'}} \circ D^{(k)}_{\rho|\cdot|^x}(\pi)$ (see [23, Lemma 5.6]).

Although it is difficult to describe $D_{\rho|\cdot|^x}(\pi)$, one can compute $D^{\max}_{\rho|\cdot|^x}(\pi)$ when $x \neq 0$.

Theorem 3.1 ([5, Lemma 3.1.3], [3, Propositions 3.3, 6.1, Theorem 7.1])**.** *Suppose that $x \neq 0$ so that $\rho|\cdot|^x$ is not self-dual. Let π be an irreducible representation of G_n. Then the highest $\rho|\cdot|^x$-derivative $D^{\max}_{\rho|\cdot|^x}(\pi)$ is irreducible. Moreover, the Langlands data for $D^{\max}_{\rho|\cdot|^x}(\pi)$ can be described from those for π explicitly, and vice versa.*

When $x = 0$, the ρ-derivative is more difficult. As alternatives of ρ-derivative, following [3], we introduce other two derivatives. We define the $\Delta_\rho[0,-1]$*-derivative* $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ and the $Z_\rho[0,1]$*-derivative* $D^{(k)}_{Z_\rho[0,1]}(\pi)$ as semisimple representations of G_{n-2dk} satisfying

$$[\mathrm{Jac}_{P_{2dk}}(\pi)] = \Delta_\rho[0,-1]^k \boxtimes D^{(k)}_{\Delta_\rho[0,-1]}(\pi) + Z_\rho[0,1]^k \boxtimes D^{(k)}_{Z_\rho[0,1]}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where $\tau_i \boxtimes \pi_i$ is an irreducible representation of $\mathrm{GL}_{2dk}(F) \times G_{n-2dk}$ such that $\tau_i \not\cong \Delta_\rho[0,-1]^k, Z_\rho[0,1]^k$. We set $D^{\max}_{\Delta_\rho[0,-1]}(\pi) = D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ (resp. $D^{\max}_{Z_\rho[0,1]}(\pi) = D^{(k)}_{Z_\rho[0,1]}(\pi)$) when $D^{(k)}_{\Delta_\rho[0,-1]}(\pi) \neq 0$ but $D^{(k+1)}_{\Delta_\rho[0,-1]}(\pi) = 0$ (resp. $D^{(k)}_{Z_\rho[0,1]}(\pi) \neq 0$ but $D^{(k+1)}_{Z_\rho[0,1]}(\pi) = 0$).

Theorem 3.2 ([3, Proposition 3.7], Section A)**.** *Let π be an irreducible representation of G_n. Suppose that π is $\rho|\cdot|^{-1}$-reduced (resp. $\rho|\cdot|^1$-reduced). Then the same assertions in Theorem* 3.1 *hold when $\rho|\cdot|^x$ is replaced with $\Delta_\rho[0,-1]$ (resp. $Z_\rho[0,1]$).*

To deal with these derivatives uniformly, we introduce the following notation.

Definition 3.3. Fix a segment $[x,y]_\rho$. Let π be a smooth representation of G_n of finite length.

(1) If $[x,y]_\rho$ does not contain ρ, then we set

$$D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi) := D^{\max}_{\rho|\cdot|^y} \circ \cdots \circ D^{\max}_{\rho|\cdot|^x}(\pi),$$
$$D^{\max}_{\rho|\cdot|^y,\dots,\rho|\cdot|^x}(\pi) := D^{\max}_{\rho|\cdot|^x} \circ \cdots \circ D^{\max}_{\rho|\cdot|^y}(\pi).$$

(2) Suppose that $[x,y]_\rho$ contains ρ, and that $y < 0$. Then we set

$$D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi) := D^{\max}_{\rho|\cdot|^y} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{-2}} \circ \left(D^{\max}_{\Delta_\rho[0,-1]} \circ D^{\max}_{\rho|\cdot|^{-1}}\right) \circ D^{\max}_{\rho|\cdot|^1} \circ \cdots \circ D^{\max}_{\rho|\cdot|^x}(\pi).$$

Moreover, if

$$D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi) = D^{(k_y)}_{\rho|\cdot|^y} \circ \cdots \circ D^{(k_{-2})}_{\rho|\cdot|^{-2}} \circ \left(D^{(k_0)}_{\Delta_\rho[0,-1]} \circ D^{(k_{-1})}_{\rho|\cdot|^{-1}}\right) \circ D^{(k_1)}_{\rho|\cdot|^1} \circ \cdots \circ D^{(k_x)}_{\rho|\cdot|^x}(\pi),$$

we formally write

$$D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi) = D^{(k_y)}_{\rho|\cdot|^y} \circ \cdots \circ D^{(k_x)}_{\rho|\cdot|^x}(\pi).$$

(3) Suppose that $[x,y]_\rho$ contains ρ, and that $x > 0$. Then we set

$$D^{\max}_{\rho|\cdot|^y,\dots,\rho|\cdot|^x}(\pi) := D^{\max}_{\rho|\cdot|^x} \circ \dots \circ D^{\max}_{\rho|\cdot|^2} \circ \left(D^{\max}_{Z_\rho[0,1]} \circ D^{\max}_{\rho|\cdot|^1}\right) \circ D^{\max}_{\rho|\cdot|^{-1}} \circ \dots \circ D^{\max}_{\rho|\cdot|^y}(\pi).$$

Moreover, if

$$D^{\max}_{\rho|\cdot|^y,\dots,\rho|\cdot|^x}(\pi) = D^{(k_x)}_{\rho|\cdot|^x} \circ \dots \circ D^{(k_2)}_{\rho|\cdot|^2} \circ \left(D^{(k_0)}_{Z_\rho[0,1]} \circ D^{(k_1)}_{\rho|\cdot|^1}\right) \circ D^{(k_{-1})}_{\rho|\cdot|^{-1}} \circ \dots \circ D^{(k_y)}_{\rho|\cdot|^y}(\pi),$$

we formally write

$$D^{\max}_{\rho|\cdot|^y,\dots,\rho|\cdot|^x}(\pi) = D^{(k_x)}_{\rho|\cdot|^x} \circ \dots \circ D^{(k_y)}_{\rho|\cdot|^y}(\pi).$$

By Theorems 3.1 and 3.2, if π is irreducible, then $D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi)$ is also irreducible whenever it is defined. Moreover, the Langlands data for $D^{\max}_{\rho|\cdot|^x,\dots,\rho|\cdot|^y}(\pi)$ can be described from those for π explicitly, and vice versa. Similar statements also hold for $D^{\max}_{\rho|\cdot|^y,\dots,\rho|\cdot|^x}(\pi)$.

3.2. **The case where $|s| \gg 0$.** In this subsection, we study $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ when $|s| \gg 0$ or $s \notin (1/2)\mathbb{Z}$.

Proposition 3.4. *Assume one of the following:*
- $s > (a-1)/2$;
- $s < -(b-1)/2$; *or*
- $s \notin (1/2)\mathbb{Z}$.

Then for any $\pi \in \mathrm{Irr}(G_n)$, the socle $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ is irreducible. Moreover, it appears in the semisimplification $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one.

Proof. Let π' be an irreducible subrepresentation of $u_\rho(a,b)|\cdot|^s \rtimes \pi$. Note that

$$u_\rho(a,b)|\cdot|^s = \begin{pmatrix} B+s & \dots & A+s \\ \vdots & \ddots & \vdots \\ -A+s & \dots & -B+s \end{pmatrix}_\rho$$

with $A = (a+b)/2-1$ and $B = (a-b)/2$. Hence the condition $s > (a-1)/2$ (resp. $s < -(b-1)/2$) implies that $-B+s > 0$ and $-(-B+s) < -A+s$ (resp. $-B+s < 0$ and $-(-B+s) > A+s$). Therefore, if $s > (a-1)/2$, then

$$\begin{aligned} &D^{\max}_{\rho|\cdot|^{-A+s},\dots,\rho|\cdot|^{-B+s}} \circ \dots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(\pi') \\ &= D^{\max}_{\rho|\cdot|^{-A+s},\dots,\rho|\cdot|^{-B+s}} \circ \dots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(\pi), \end{aligned}$$

whereas if $s < -(b-1)/2$, then

$$\begin{aligned} &D^{\max}_{\rho|\cdot|^{A+s},\dots,\rho|\cdot|^{-B+s}} \circ \dots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{-A+s}}(\pi') \\ &= D^{\max}_{\rho|\cdot|^{A+s},\dots,\rho|\cdot|^{-B+s}} \circ \dots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{-A+s}}(\pi). \end{aligned}$$

These equations determine π' uniquely. Moreover, it follows that π' appears in $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one. When $s \notin (1/2)\mathbb{Z}$, the same argument works. $\square$

Example 3.5. Let π be an irreducible representation of G_n. Then $Z_\rho[-1,2] \rtimes \pi$ has a unique irreducible subrepresentation. If

$$D^{\max}_{\rho|\cdot|^2} \circ \left(D^{\max}_{Z_\rho[0,1]} \circ D^{\max}_{\rho|\cdot|^1}\right) \circ D^{\max}_{\rho|\cdot|^{-1}}(\pi) = D^{(k_2)}_{\rho|\cdot|^2} \circ \left(D^{(k_0)}_{Z_\rho[0,1]} \circ D^{(k_1)}_{\rho|\cdot|^1}\right) \circ D^{(k_{-1})}_{\rho|\cdot|^{-1}}(\pi),$$

then $\pi' = \mathrm{soc}(Z_\rho[-1,2] \rtimes \pi)$ is uniquely determined by

$$D^{\max}_{\rho|\cdot|^2} \circ \left(D^{\max}_{Z_\rho[0,1]} \circ D^{\max}_{\rho|\cdot|^1} \right) \circ D^{\max}_{\rho|\cdot|^{-1}}(\pi') = D^{(k_2+1)}_{\rho|\cdot|^2} \circ \left(D^{(k_0+1)}_{Z_\rho[0,1]} \circ D^{(k_1)}_{\rho|\cdot|^1} \right) \circ D^{(k_{-1}+1)}_{\rho|\cdot|^{-1}}(\pi')$$
$$= D^{(k_2)}_{\rho|\cdot|^2} \circ \left(D^{(k_0)}_{Z_\rho[0,1]} \circ D^{(k_1)}_{\rho|\cdot|^1} \right) \circ D^{(k_{-1})}_{\rho|\cdot|^{-1}}(\pi).$$

Note that the last equation also determines π uniquely from π'.

3.3. **The middle case.** Next, we consider the case where $0 < s \leq (a-1)/2$ or $-(b-1)/2 \leq s < 0$. In this case, by Propositions 3.6 and 3.7, we reduce the problem to the case where $s = 0$.

Proposition 3.6. *Suppose that $s \in (1/2)\mathbb{Z}$. Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type, and π' be an irreducible subrepresentation of $u_\rho(a,b)|\cdot|^s \rtimes \pi$.*

(1) *If $0 < s \leq (a-1)/2$, then there exists a unique irreducible summand σ of $u_\rho(a-2s,b) \rtimes \pi$ such that $\pi' = \mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$.*
(2) *If $-(b-1)/2 \leq s < 0$, then there exists a unique irreducible summand σ of $u_\rho(a,b+2s) \rtimes \pi$ such that $\pi' = \mathrm{soc}(u_\rho(a,-2s)|\cdot|^{-\frac{b}{2}} \rtimes \sigma)$.*

Moreover, in the both cases, π' appears in the semisimplification $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one.

Proof. We only prove the case where $0 < s \leq (a-1)/2$. The other case is proven similarly.

When $0 < s \leq (a-1)/2$, since $u_\rho(a,b)|\cdot|^s \hookrightarrow u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \times u_\rho(a-2s,b)$, we have $\pi' \hookrightarrow u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma$ for some irreducible summand σ of $u_\rho(a-2s,b) \rtimes \pi$. Since $a/2 > (2s-1)/2$, by Proposition 3.4, the socle $\mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$ is irreducible. Hence $\pi' = \mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$. By this equation, σ is uniquely determined by π' using derivatives (see the proof of Proposition 3.4 and Example 3.5). Moreover, since $u_\rho(a-2s,b) \rtimes \pi$ is a multiplicity-free sum of irreducible representations ([1, Proposition 2.4.3]), and since π' appears in $[u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma]$ with multiplicity one, we conclude that π' appears in $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one. □

By Proposition 3.6, when $0 < s \leq (a-1)/2$, we obtained a well-defined injective map

$$\{\pi' \hookrightarrow u_\rho(a,b)|\cdot|^s \rtimes \pi\} \to \{\sigma \hookrightarrow u_\rho(a-2s,b) \rtimes \pi\}$$

characterized by $\pi' = \mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$. Similarly, when $-(b-1)/2 \leq s < 0$, we obtained a well-defined injective map

$$\{\pi' \hookrightarrow u_\rho(a,b)|\cdot|^s \rtimes \pi\} \to \{\sigma \hookrightarrow u_\rho(a,b+2s) \rtimes \pi\}$$

characterized by $\pi' = \mathrm{soc}(u_\rho(a,-2s)|\cdot|^{-\frac{b}{2}} \rtimes \sigma)$. We detemine the images of these maps.

Proposition 3.7. *Suppose that $s \in (1/2)\mathbb{Z}$, and that $\pi \in \mathrm{Irr}(G_n)$ is of Arthur type.*

(1) *When $0 < s \leq (a-1)/2$, for an irreducible summand σ of $u_\rho(a-2s,b) \rtimes \pi$, the following are equivalent.*
   (a) $\mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$ *is a subrepresentation of* $u_\rho(a,b)|\cdot|^s \rtimes \pi$*;*

(b) *if*

$$D^{\max}_{\rho|\cdot|^{B-s+1},\dots,\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(\pi)$$
$$= \left(D^{(k_{2s,b})}_{\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{(k_{2s,1})}_{\rho|\cdot|^{B-s+1}}\right) \circ \cdots \circ \left(D^{(k_{1,b})}_{\rho|\cdot|^{A+s}} \circ \cdots \circ D^{(k_{1,1})}_{\rho|\cdot|^{B+s}}\right)(\pi),$$

then

$$\left(D^{(k_{2s,b})}_{\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{(k_{2s,1})}_{\rho|\cdot|^{B-s+1}}\right) \circ \cdots \circ \left(D^{(k_{1,b})}_{\rho|\cdot|^{A+s}} \circ \cdots \circ D^{(k_{1,1})}_{\rho|\cdot|^{B+s}}\right)(\sigma) \neq 0.$$

(2) *When* $-(b-1)/2 \leq s < 0$, *for an irreducible summand* σ *of* $u_\rho(a, b+2s) \rtimes \pi$, *the following are equivalent.*
(a) $\mathrm{soc}(u_\rho(a,-2s)|\cdot|^{-\frac{b}{2}} \rtimes \sigma)$ *is a subrepresentation of* $u_\rho(a,b)|\cdot|^s \rtimes \pi$;
(b) *if*

$$D^{\max}_{\rho|\cdot|^{B-s-1},\dots,\rho|\cdot|^{-A-s-1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{-A+s}}(\pi)$$
$$= \left(D^{(k_{a,-2s})}_{\rho|\cdot|^{-A-s-1}} \circ \cdots \circ D^{(k_{1,-2s})}_{\rho|\cdot|^{B-s-1}}\right) \circ \cdots \circ \left(D^{(k_{a,1})}_{\rho|\cdot|^{-A+s}} \circ \cdots \circ D^{(k_{1,1})}_{\rho|\cdot|^{B+s}}\right)(\pi),$$

then

$$\left(D^{(k_{a,-2s})}_{\rho|\cdot|^{-A-s-1}} \circ \cdots \circ D^{(k_{1,-2s})}_{\rho|\cdot|^{B-s-1}}\right) \circ \cdots \circ \left(D^{(k_{a,1})}_{\rho|\cdot|^{-A+s}} \circ \cdots \circ D^{(k_{1,1})}_{\rho|\cdot|^{B+s}}\right)(\sigma) \neq 0.$$

Proof. We only prove (1). The proof of (2) is similar. From now, we assume that $0 < s \leq (a-1)/2$.

Note that

$$D^{\max}_{\rho|\cdot|^{B-s+1},\dots,\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$$
$$= u_\rho(a-2s,b) \rtimes D^{\max}_{\rho|\cdot|^{B-s+1},\dots,\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(\pi)$$

up to semisimplification. Hence, if $\mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$ is a subrepresentation of $u_\rho(a,b)|\cdot|^s \rtimes \pi$, then we must have

$$\left(D^{(k_{2s,b})}_{\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{(k_{2s,1})}_{\rho|\cdot|^{B-s+1}}\right) \circ \cdots \circ \left(D^{(k_{1,b})}_{\rho|\cdot|^{A+s}} \circ \cdots \circ D^{(k_{1,1})}_{\rho|\cdot|^{B+s}}\right)(\sigma) \neq 0.$$

This shows that (a) implies (b).

If we set $\pi' = \mathrm{soc}(u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \rtimes \sigma)$, then

$$\pi' \hookrightarrow u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \times u_\rho(a-2s,b) \rtimes \pi.$$

Note that $u_\rho(a,b)|\cdot|^s$ is a unique irreducible subrepresentation of $u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \times u_\rho(a-2s,b)$, which is characterized among its composition series by

$$\left(L_{\rho|\cdot|^{A-s+1}} \circ \cdots \circ L_{\rho|\cdot|^{B-s+1}}\right) \circ \cdots \circ \left(L_{\rho|\cdot|^{A+s}} \circ \cdots \circ L_{\rho|\cdot|^{B+s}}\right)(u_\rho(a,b)|\cdot|^s) \neq 0,$$

where $L_{\rho|\cdot|^x}$ is the left $\rho|\cdot|^x$-derivative, which is an analogue of $D_{\rho|\cdot|^x}$ for general linear groups (cf., see §A.1). Now if we assume (b), by considering the exponents of $D^{\max}_{\rho|\cdot|^{B-s+1},\dots,\rho|\cdot|^{A-s+1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+s},\dots,\rho|\cdot|^{A+s}}(\pi')$ (cf., see Example 3.5), we see that the inclusion $\pi' \hookrightarrow u_\rho(2s,b)|\cdot|^{\frac{a}{2}} \times u_\rho(a-2s,b) \rtimes \pi$ factors through $\pi' \hookrightarrow u_\rho(a,b)|\cdot|^s \rtimes \pi$. Hence we obtain (a). This completes the proof. □

4. Unitary inductions

Let

- $\rho \in \mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$;
- $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type; and
- a and b be positive integers.

In the previous section, we reduce the study of $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ for $s \in \mathbb{R}$ to the case where $s = 0$. In this section, we treat this case. To do this, we recall terminologies of A-parameters and A-packets.

4.1. ***A*-parameters.** Denote by $\widehat{G}_n$ the complex dual group of G_n. Namely, $\widehat{G}_n = \mathrm{Sp}_{2n}(\mathbb{C})$ if $G_n = \mathrm{SO}_{2n+1}(F)$, and $\widehat{G}_n = \mathrm{SO}_{2n+1}(\mathbb{C})$ if $G_n = \mathrm{Sp}_{2n}(F)$. Recall that an *$A$-parameter for G_n* is the $\widehat{G}_n$-conjugacy class of an admissible homomorphism

$$\psi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_n$$

such that the image of the Weil group W_F is bounded. By composing with the standard representation of $\widehat{G}_n$, we can regard ψ as a representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. It decomposes as

$$\psi = \bigoplus_{\rho} \left(\bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \right),$$

where

- ρ runs over $\cup_{d \geq 1} \mathrm{Cusp}_{\mathrm{unit}}(\mathrm{GL}_d(F))$, which is identified with an irreducible bounded representation of W_F by the local Langlands correspondence for the general linear groups;
- S_a is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a.

Notice that a_i and b_i depend on ρ, but we do not write it. We write $\rho \boxtimes S_a = \rho \boxtimes S_a \boxtimes S_1$ and $\rho = \rho \boxtimes S_1 \boxtimes S_1$ for short.

Let ψ be as above. We say that ψ is *of good parity* if $\rho \boxtimes S_{a_i} \boxtimes S_{b_i}$ is self-dual of the same type as ψ for any ρ and $i \in I_\rho$, i.e.,

- $\rho \in \mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$ is orthogonal and $a_i + b_i \equiv 0 \bmod 2$ if $G_n = \mathrm{Sp}_{2n}(F)$ (resp. $a_i + b_i \equiv 1 \bmod 2$ if $G_n = \mathrm{SO}_{2n+1}(F)$); or
- $\rho \in \mathrm{Cusp}^{\perp}(\mathrm{GL}_d(F))$ is symplectic and $a_i + b_i \equiv 1 \bmod 2$ if $G_n = \mathrm{Sp}_{2n}(F)$ (resp. $a_i + b_i \equiv 0 \bmod 2$ if $G_n = \mathrm{SO}_{2n+1}(F)$).

Let $\Psi(G_n) \supset \Psi_{\mathrm{gp}}(G_n)$ be the sets of equivalence classes of A-parameters and A-parameters of good parity, respectively. Also, we set $\Phi_{\mathrm{temp}}(G_n)$ to be the subset of $\Psi(G_n)$ consisting of *tempered* A-parameters, i.e., A-parameters ϕ which are trivial on the second $\mathrm{SL}_2(\mathbb{C})$. Finally, we set $\Phi_{\mathrm{gp}}(G_n) = \Psi_{\mathrm{gp}}(G_n) \cap \Phi_{\mathrm{temp}}(G_n)$.

For $\psi = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{\mathrm{gp}}(G_n)$, define the *enhanced component group* by

$$\mathcal{A}_\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} (\mathbb{Z}/2\mathbb{Z}) \alpha_{\rho,i},$$

i.e., $\mathcal{A}_\psi$ is a $(\mathbb{Z}/2\mathbb{Z})$-vector space with a canonical basis $\alpha_{\rho,i}$ corresponding to $\rho \boxtimes S_{a_i} \boxtimes S_{b_i}$. The *component group* $\mathcal{S}_\psi$ is the quotient of $\mathcal{A}_\psi$ by the subgroup generated by

- $\alpha_{\rho,i} + \alpha_{\rho,j}$ such that $\rho \boxtimes S_{a_i} \boxtimes S_{b_i} = \rho \boxtimes S_{a_j} \boxtimes S_{b_j}$; and

- $z_\psi = \sum_\rho \sum_{i \in I_\rho} \alpha_{\rho,i}$, which is called the *central element* of $\mathcal{A}_\psi$.

Let $\widehat{\mathcal{S}_\psi} \subset \widehat{\mathcal{A}_\psi}$ be the Pontryagin duals of $\mathcal{S}_\psi$ and $\mathcal{A}_\psi$, respectively. When $\varepsilon \in \widehat{\mathcal{A}_\psi}$, we write $\varepsilon(\rho \boxtimes S_{a_i} \boxtimes S_{b_i}) := \varepsilon(\alpha_{\rho,i}) \in \{\pm 1\}$.

4.2. *A*-**packets.** Let $\mathrm{Irr}_{\mathrm{unit}}(G_n)$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G_n)$) be the set of equivalence classes of irreducible unitary (resp. tempered) representations of G_n. To an A-parameter $\psi \in \Psi(G_n)$, Arthur [1, Theorem 1.5.1 (a)] associated an A-packet Π_ψ, which is a finite multi-set over $\mathrm{Irr}_{\mathrm{unit}}(G_n)$. We say that $\pi \in \mathrm{Irr}(G_n)$ is *of Arthur type* if $\pi \in \Pi_\psi$ for some $\psi \in \Psi(G_n)$. In particular, such π is unitary.

Mœglin [13] showed that Π_ψ is multiplicity-free, i.e., a subset of $\mathrm{Irr}_{\mathrm{unit}}(G_n)$. By [1, Theorem 1.5.1 (b)], if $\phi \in \Phi_{\mathrm{temp}}(G_n)$ is a tempered A-parameter, then Π_ϕ is a subset of $\mathrm{Irr}_{\mathrm{temp}}(G_n)$ and

$$\mathrm{Irr}_{\mathrm{temp}}(G_n) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G_n)} \Pi_\phi \quad \text{(disjoint union)}.$$

However, $\Pi_{\psi_1} \cap \Pi_{\psi_2} \neq \emptyset$ even if $\psi_1 \not\cong \psi_2$ in general.

If $\psi = \oplus_\rho (\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i})$, set

$$\tau_\psi = \bigtimes_\rho \bigtimes_{i \in I_\rho} u_\rho(a_i, b_i) = \bigtimes_\rho \bigtimes_{i \in I_\rho} \begin{pmatrix} \frac{a_i - b_i}{2} & \dots & \frac{a_i + b_i}{2} - 1 \\ \vdots & \ddots & \vdots \\ -\frac{a_i + b_i}{2} + 1 & \dots & -\frac{a_i - b_i}{2} \end{pmatrix}_\rho$$

to be a product of (unitary) Speh representations, which is an irreducible unitary representation of $\mathrm{GL}_m(F)$ with $m = \dim(\psi)$.

Proposition 4.1 ([12, Theorem 6], [24, Proposition 8.11])**.** *Any $\psi \in \Psi(G_n)$ can be decomposed as*

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee,$$

where

- $\psi_0 \in \Psi_{\mathrm{gp}}(G_{n_0})$;
- *ψ_1 is a direct sum of irreducible representations of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ which are not self-dual of the same type as ψ.*

For $\pi_0 \in \Pi_{\psi_0}$, the parabolically induced representation $\tau_{\psi_1} \rtimes \pi_0$ is irreducible and independent of the choice of ψ_1. Moreover,

$$\Pi_\psi = \{\tau_{\psi_1} \rtimes \pi_0 \mid \pi_0 \in \Pi_{\psi_0}\}.$$

Through this paper, we implicitly fix a Whittaker datum for G_n. Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$ so that we have defined the component group $\mathcal{S}_\psi$. Arthur [1, Theorem 1.5.1 (a)] gave a map

$$\Pi_\psi \to \widehat{\mathcal{S}_\psi}, \ \pi \mapsto \langle \cdot, \pi \rangle_\psi .$$

If $\psi = \phi \in \Phi_{\mathrm{gp}}(G_n)$ is tempered, this map is bijective. When $\pi \in \Pi_\phi$ corresponds to $\varepsilon \in \widehat{\mathcal{S}_\phi}$, we write $\pi = \pi(\phi, \varepsilon)$.

Proposition 4.2. *Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$. Suppose that $\psi = \psi_0 \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$. Then for $\varepsilon_0 \in \widehat{\mathcal{S}_{\psi_0}}$, we have*

$$\bigoplus_{\substack{\pi_0 \in \Pi_{\psi_0} \\ \langle \cdot, \pi_0 \rangle_{\psi_0} = \varepsilon_0}} u_\rho(a,b) \rtimes \pi_0 = \bigoplus_{\substack{\pi \in \Pi_\psi \\ \langle \cdot, \pi \rangle_\psi | \mathcal{S}_{\psi_0} = \varepsilon_0}} \pi.$$

In particular, $u_\rho(a,b) \rtimes \pi_0$ is multiplicity-free.

Proof. See (the proof of) [1, Proposition 2.4.3]. □

4.3. **Extended multi-segments.** To describe A-packets, in [2], we introduced the following notion.

Definition 4.3.

(1) An *extended segment* is a triple $([A,B]_\rho, l, \eta)$, where
- $[A,B]_\rho = \{\rho|\cdot|^A, \dots, \rho|\cdot|^B\}$ is a segment;
- $l \in \mathbb{Z}$ with $0 \le l \le \frac{b}{2}$, where $b := \#[A,B]_\rho = A-B+1$;
- $\eta \in \{\pm 1\}$.

(2) Two extended segments $([A,B]_\rho, l, \eta)$ and $([A',B']_{\rho'}, l', \eta')$ are *equivalent* if
- $[A,B]_\rho = [A',B']_{\rho'}$;
- $l = l'$; and
- $\eta = \eta'$ whenever $l = l' < \frac{b}{2}$.

Similarly, two multi-sets of extended segments $\{([A_i,B_i]_{\rho_i}, l_i, \eta_i)\}_{i\in I}$ and $\{([A_i',B_i']_\rho, l_i', \eta_i')\}_{i\in I}$ with the same index set I are *equivalent* if $([A_i,B_i]_\rho, l_i, \eta_i)$ and $([A_i',B_i']_\rho, l_i', \eta_i')$ are equivalent for all $i \in I$.

(3) An *extended multi-segment* for G_n is an equivalence class of multi-sets of extended segments

$$\mathcal{E} = \bigcup_\rho \{([A_i,B_i]_\rho, l_i, \eta_i)\}_{i\in(I_\rho,>)}$$

such that
- ρ runs over $\cup_{d\ge 1}\mathrm{Cusp}^\perp(\mathrm{GL}_d(F))$;
- I_ρ is a totally ordered finite set with a fixed order $>$ which is called admissible;
- $A_i + B_i \ge 0$ for all ρ and $i \in I_\rho$;
- as a representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$,

$$\psi_{\mathcal{E}} := \bigoplus_\rho \bigoplus_{i\in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}$$

belongs to $\Psi_{\mathrm{gp}}(G_n)$, where $a_i := A_i + B_i + 1$ and $b_i := A_i - B_i + 1$;
- a sign condition

$$\prod_\rho \prod_{i\in I_\rho} (-1)^{[\frac{b_i}{2}]+l_i} \eta_i^{b_i} = 1$$

holds.

In [2], to an extended multi-segment $\mathcal{E}$ for G_n, we associate a representation $\pi(\mathcal{E})$ of G_n. To describe $\pi(\mathcal{E})$ explicitly, it was important to consider several orders on I_ρ. Nevertheless, in this paper, we assume that the order $>$ on I_ρ always satisfies that

$$B_i < B_j \implies i < j.$$

We review the definition of $\pi(\mathcal{E})$. Let $\mathcal{E} = \cup_\rho \{([A_i,B_i]_\rho, l_i, \eta_i)\}_{i\in(I_\rho,>)}$. We say that

- $\mathcal{E}$ has a *discrete diagonal restriction (DDR)* if, for any ρ and $i,j \in I_\rho$ with $i \ne j$, the segments $[A_i,B_i]_\rho$ and $[A_j,B_j]_\rho$ have no intersection;
- $\mathcal{E}$ is *non-negative* if $B_i \ge 0$ for any ρ and $i \in I_\rho$.

When $\mathcal{E}$ is non-negative DDR, we define

$$\pi(\mathcal{E}) = \mathrm{soc}\left(\bigtimes_{\rho} \bigtimes_{i\in I_\rho} \begin{pmatrix} B_i & \ldots & B_i + l_i - 1 \\ \vdots & \ddots & \vdots \\ -A_i & \ldots & -(A_i - l_i + 1) \end{pmatrix}_{\rho} \rtimes \pi(\phi, \varepsilon) \right)$$

with

$$\phi = \bigoplus_{\rho} \bigoplus_{i\in I_\rho} \rho \boxtimes \left(S_{2(B_i+l_i)+1} \oplus \cdots \oplus S_{2(A_i-l_i)+1} \right)$$

and $\varepsilon(\rho \boxtimes S_{2(B_i+l_i+k)+1}) = (-1)^k \eta_i$ for $0 \le k \le b_i - 2l_i - 1$. In general, take a sequence of non-negative integers $\cup_\rho \{t_i\}_{i\in(I_\rho,>)}$ such that $\mathcal{E}' = \cup_\rho \{([A_i + t_i, B_i + t_i]_\rho, l_i, \eta_i)\}_{i\in(I_\rho,>)}$ is non-negative DDR, and define

$$\pi(\mathcal{E}) = \circ_\rho \circ_{i\in I_\rho} \left(D_{\rho|\cdot|^{B_i+1},\ldots,\rho|\cdot|^{A_i+1}} \circ \cdots \circ D_{\rho|\cdot|^{B_i+t_i},\ldots,\rho|\cdot|^{A_i+t_i}} \right) (\pi(\mathcal{E}')).$$

This definition does not depend on the choice of $\cup_\rho \{t_i\}_{i\in(I_\rho,>)}$.

Note that $\pi(\mathcal{E})$ is irreducible or zero by Theorems 3.1 and 3.2. The following properties were proven in [2, Theorems 1.2, 1.3, 1.4, 3.5]:

- There exists a non-vanishing criterion for $\pi(\mathcal{E})$.
- For $\psi = \oplus_\rho (\oplus_{i\in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{\mathrm{gp}}(G_n)$ with a fixed order $>$ on I_ρ, we have
$$\Pi_\psi = \{\pi(\mathcal{E}) \mid \psi_{\mathcal{E}} \cong \psi\} \setminus \{0\}.$$
- The character $\langle \cdot, \pi(\mathcal{E}) \rangle_{\psi_{\mathcal{E}}}$ is explicitly determined by $\mathcal{E}$.

4.4. **Decompositions of unitary inductions.** Now we describe the unitary induction $u_\rho(a,b) \rtimes \pi$ for π of Arthur type, i.e., $\pi \in \Pi_\psi$ for some $\psi \in \Psi(G_n)$. We decompose $\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$ as in Proposition 4.1. According to this proposition, $\pi = \tau_{\psi_1} \rtimes \pi_0$ for some $\pi_0 \in \Pi_{\psi_0}$. Since $u_\rho(a,b)$ and τ_{ψ_1} are both unitary, we have $u_\rho(a,b) \times \tau_{\psi_1} \cong \tau_{\psi_1} \times u_\rho(a,b)$. Hence the problem is reduced to the case where $\psi = \psi_0 \in \Psi_{\mathrm{gp}}(G_n)$. In this case, one can write $\pi = \pi(\mathcal{E})$ for some extended multi-segment $\mathcal{E}$ for G_n.

When $\psi_{\mathcal{E}} \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$ is not of good parity, by Proposition 4.1, $u_\rho(a,b) \rtimes \pi(\mathcal{E})$ is irreducible. Otherwise, we have the following:

Theorem 4.4. *Suppose that $\psi_{\mathcal{E}} \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$ is of good parity. For $(l,\eta) \in \mathbb{Z} \times \{\pm 1\}$ with $0 \le l \le b/2$, define $\mathcal{E}_{(l,\eta)}$ by adding $([A,B]_\rho, l, \eta)$ and $([A,B]_\rho, l, (-1)^{A-B}\eta)$ to $\mathcal{E}$, where $A = \frac{a+b}{2} - 1$ and $B = \frac{a-b}{2}$, such that if we let i_0 (resp. i_0') be the index for $([A,B]_\rho, l, \eta)$ (resp. $([A,B]_\rho, l, (-1)^{A-B}\eta)$), then $i_0 < i_0'$ are adjacent, and $j > i_0'$ if and only if $B_j > B$. Then*

$$u_\rho(a,b) \rtimes \pi(\mathcal{E}) \cong \bigoplus_{(l,\eta)} \pi(\mathcal{E}_{(l,\eta)}),$$

where (l,η) runs over the set $\{(l,\eta) \in \mathbb{Z} \times \{\pm 1\} \mid 0 \le l \le b/2\}/\sim$. Here, we write $(l,\eta) \sim (l',\eta')$ if $l = l'$ and if $\eta = \eta'$ whenever $l = l' < b/2$.

Proof. This theorem seems to be already known by Mœglin (watch the video of her talk [14]). For the sake of completeness, we give a proof.

Fix $\psi \in \Psi_{\mathrm{gp}}(G_n)$ and set $\psi' = \psi \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$. Since

$$\bigoplus_{\substack{\mathcal{E} \\ \psi_{\mathcal{E}} \cong \psi}} u_\rho(a,b) \rtimes \pi(\mathcal{E}) \cong \bigoplus_{\pi' \in \Pi_{\psi'}} \pi' \cong \bigoplus_{\substack{\mathcal{E} \\ \psi_{\mathcal{E}} \cong \psi}} \bigoplus_{(l,\eta)} \pi(\mathcal{E}_{(l,\eta)}),$$

it is enough to show that for any fixed ψ, one of two inclusions $u_\rho(a,b) \rtimes \pi(\mathcal{E}) \subset \oplus_{(l,\eta)}\pi(\mathcal{E}_{(l,\eta)})$ or $\oplus_{(l,\eta)}\pi(\mathcal{E}_{(l,\eta)}) \subset u_\rho(a,b) \rtimes \pi(\mathcal{E})$ holds whenever $\psi_{\mathcal{E}} \cong \psi$. We will prove this by considering several steps. Write $\mathcal{E} = \cup_{\rho'}\{([A_i, B_i]_{\rho'}, l_i, \eta_i)\}_{i\in(I_{\rho'},>)}$. We may assume that $\pi(\mathcal{E}_{(l,\eta)}) \neq 0$.

(1) We assume that $\psi = \psi_{\mathcal{E}}$ is a tempered A-parameter, i.e., $A_i = B_i$ for all ρ' and $i \in I_{\rho'}$. In this case, since the map $\Pi_\psi \to \widehat{\mathcal{S}_\psi}$, $\pi \mapsto \langle \cdot, \pi\rangle_\psi$ is injective, to prove $\pi(\mathcal{E}_{(l,\eta)}) \subset u_\rho(a,b) \rtimes \pi(\mathcal{E})$, by Proposition 4.2, it suffices to check that $\langle \cdot, \pi(\mathcal{E}_{(l,\eta)})\rangle_{\psi'}|_{\mathcal{S}_\psi} = \langle \cdot, \pi(\mathcal{E})\rangle_\psi$. It follows from [2, Theorem 3.5].

(2) Define $\mathcal{E}'_{(l,\eta)}$ by adding $([A,B]_\rho, l, \eta)$ and $([A+b, B+b]_\rho, l, (-1)^{A-B}\eta)$ to $\mathcal{E}$ as adjacent elements similar to $\mathcal{E}_{(l,\eta)}$. We assume that $\mathcal{E}'_{(l,\eta)}$ is non-negative DDR. In this case,

$$\pi(\mathcal{E}_{(l,\eta)}) = \mathrm{soc}\left(\tau \rtimes \pi(\mathcal{F}_{(l,\eta)})\right),$$

where $\mathcal{F}_{(l,\eta)}$ is defined from $\mathcal{E}_{(l,\eta)}$ by replacing $([A_i, B_i]_{\rho'}, l_i, \eta_i)$ with $([A_i - l_i, B_i + l_i]_{\rho'}, 0, \eta_i)$ for all ρ' and $i \in I_{\rho'}$, and we set

$$\tau = \bigtimes_{\rho'} \bigtimes_{i\in I_{\rho'}} \begin{pmatrix} B_i & \ldots & B_i + l_i - 1 \\ \vdots & \ddots & \vdots \\ -A_i & \ldots & -(A_i - l_i + 1)\end{pmatrix}_{\rho'}.$$

Moreover, by replacing $\{([A_i - l_i, B_i + l_i]_{\rho'}, 0, \eta_i)\}$ with

$$\bigcup_{k=0}^{A_i-B_i-2l_i} \{([B_i + l_i + k, B_i + l_i + k]_{\rho'}, 0, (-1)^k\eta_i)\}$$

and vice versa, one can apply the first case to $\pi(\mathcal{F}_{(l,\eta)})$. Hence $\oplus_{(l,\eta)}\pi(\mathcal{F}_{(l,\eta)}) \cong u_\rho(a,b) \rtimes \pi(\mathcal{F})$ with $\mathcal{F} = \cup_{\rho'}\{([A_i - l_i, B_i + l_i]_{\rho'}, 0, \eta_i)\}_{i\in(I_{\rho'},>)}$. Since $[B_i + l_i - 1, -A_i]_{\rho'}$ and $[A, -A]_\rho$ are always not linked, by [20, Theorem 1.1], we have $\tau \times u_\rho(a,b) \cong u_\rho(a,b) \times \tau$. Hence we have

$$\begin{aligned}\bigoplus_{(l,\eta)} \pi(\mathcal{E}_{(l,\eta)}) &\cong \mathrm{soc}\left(\tau \rtimes \bigoplus_{(l,\eta)}\pi(\mathcal{F}_{(l,\eta)})\right)\\ &\cong \mathrm{soc}\left(\tau \times u_\rho(a,b) \rtimes \pi(\mathcal{F})\right)\\ &\cong \mathrm{soc}\left(u_\rho(a,b) \times \tau \rtimes \pi(\mathcal{F})\right).\end{aligned}$$

Since $\pi(\mathcal{E}) = \mathrm{soc}(\tau \rtimes \pi(\mathcal{F}))$, we have

$$u_\rho(a,b) \rtimes \pi(\mathcal{E}) \hookrightarrow u_\rho(a,b) \times \tau \rtimes \pi(\mathcal{F}).$$

Since the left hand side is semisimple, we see that $u_\rho(a,b) \rtimes \pi(\mathcal{E}) \hookrightarrow \oplus_{(l,\eta)}\pi(\mathcal{E}_{(l,\eta)})$, as desired.

(3) We consider the general case. Take

$$\mathcal{E}'_{(l,\eta)} = \cup_{\rho'}\{([A_i + t_i, B_i + t_i]_{\rho'}, l_i, \eta_i)\}_{i\in(I'_{\rho'},>)}$$

such that it is non-negative DDR and

$$\pi(\mathcal{E}_{(l,\eta)}) = \circ_{\rho'} \circ_{i\in I'_{\rho'}} \left(D_{\rho'|\cdot|^{B_i+1},\ldots,\rho'|\cdot|^{A_i+1}} \circ \cdots \circ D_{\rho'|\cdot|^{B_i+t_i},\ldots,\rho'|\cdot|^{A_i+t_i}}\right)(\pi(\mathcal{E}'_{(l,\eta)})).$$

By construction, we have $I'_{\rho'} = I_{\rho'}$ unless $\rho' \cong \rho$, and in this case, $I'_\rho = I_\rho \cup \{i_0, i'_0\}$ such that $i_0 < i'_0$ are adjacent and that

$$([A_{i_0} + t_{i_0}, B_{i_0} + t_{i_0}]_\rho, l_{i_0}, \eta_{i_0}) = ([A + t, B + t]_\rho, l, \eta),$$
$$([A_{i'_0} + t_{i'_0}, B_{i'_0} + t_{i'_0}]_\rho, l_{i'_0}, \eta_{i'_0}) = ([A + t', B + t']_\rho, l, (-1)^{A-B}\eta)$$

for some $t < t'$. By the same argument as [2, Corollary 5.3], we may reset $t' = t$ by replacing

$$\left(D_{\rho|\cdot|^{B+1},\dots,\rho|\cdot|^{A+1}} \circ \cdots \circ D_{\rho|\cdot|^{B+t'},\dots,\rho|\cdot|^{A+t'}}\right) \circ \left(D_{\rho|\cdot|^{B+1},\dots,\rho|\cdot|^{A+1}} \circ \cdots \circ D_{\rho|\cdot|^{B+t},\dots,\rho|\cdot|^{A+t}}\right)$$

with

$$\left(D^{\max}_{\rho|\cdot|^{B+1},\dots,\rho|\cdot|^{A+1}} \circ \cdots \circ D^{\max}_{\rho|\cdot|^{B+t},\dots,\rho|\cdot|^{A+t}}\right).$$

Then we can use the second case so that $\pi(\mathcal{E}'_{(l,\eta)}) \hookrightarrow u_\rho(a + 2t, b) \rtimes \pi(\mathcal{E}')$, where $\mathcal{E}' = \cup_{\rho'}\{([A_i + t_i, B_i + t_i]_{\rho'}, l_i, \eta_i)\}_{i \in (I_{\rho'},>)}$. By computing derivatives, we conclude that $\pi(\mathcal{E}_{(l,\eta)}) \hookrightarrow u_\rho(a, b) \rtimes \pi(\mathcal{E})$.

This completes the proof. □

Combining Propositions 3.4, 3.6, 3.7 and Theorem 4.4, we obtain Theorem 1.1.

Corollary 4.5. *The length of $u_\rho(a, b) \rtimes \pi(\mathcal{E})$ is at most $\min\{a, b\} + 1$.*

Proof. By [2, Theorem 1.3], if $\pi(\mathcal{E}_{(l,\eta)}) \neq 0$ then $B + l \geq 0$, or $B + l = -1/2$ and $\eta \in \{\pm 1\}$ is uniquely determined. By a case-by-case calculation, we see that there are at most $\min\{a, b\} + 1$ such pairs (l, η). □

Corollary 4.6. *Suppose that $\pi(\mathcal{E}) \neq 0$. If $\mathcal{E}$ contains $([A, B]_\rho, l_0, \eta_0)$ for some (l_0, η_0), where $A = (a + b)/2 - 1$ and $B = (a - b)/2$, then $u_\rho(a, b) \rtimes \pi(\mathcal{E})$ is irreducible.*

Proof. If $\pi(\mathcal{E}_{(l,\eta)}) \neq 0$, by [2, Proposition 4.1], we see that $l = l_0$, and that η is determined uniquely. □

As in the following example, Corollary 4.5 is optimum.

Example 4.7. Suppose that $\psi = (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2} \in \Psi_{\mathrm{gp}}(\mathrm{SO}_{2abd+1}(F))$. Then

$$\bigoplus_{\pi \in \Pi_\psi} \pi \cong u_\rho(a, b) \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$$
$$\cong \bigoplus_{(l,\eta)} \pi(\{([A, B]_\rho, l, \eta), ([A, B]_\rho, l, (-1)^{A-B}\eta)\}).$$

By [2, Theorems 1.3, 1.4], $\pi(\{([A, B]_\rho, l, \eta), ([A, B]_\rho, l, (-1)^{A-B}\eta)\}) \neq 0$ if and only if $B + l \geq 0$, or $B + l = -1/2$ and $\eta = +1$. By a case-by-case calculation, we conclude that the length of $u_\rho(a, b) \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}$ is equal to $\min\{a, b\} + 1$.

5. Irreducibility and examples

In this section, we discuss when $u_\rho(a, b)|\cdot|^s \rtimes \pi$ is irreducible. Also, we give some examples.

5.1. **Irreducibility.** We give some consequences of the results in the previous sections.

Corollary 5.1. *Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type. Then for any $s \in \mathbb{R}$, any irreducible subrepresentation of $u_\rho(a,b)|\cdot|^s \rtimes \pi$ appears in the semisimplification $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one. In particular, the socle $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ is multiplicity-free.*

Proof. It follows from Propositions 3.4, 3.6 and 4.2. □

This corollary gives a criterion for the irreducibility.

Corollary 5.2. *Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type. Then $u_\rho(a,b)|\cdot|^s \rtimes \pi$ is irreducible if and only if all of the following conditions hold:*

- $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ *is irreducible;*
- $\mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$ *is irreducible;*
- $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi) \cong \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$.

Proof. The only if part is trivial. To prove the if part, we assume the three conditions. If $u_\rho(a,b)|\cdot|^s \rtimes \pi$ were to be reducible, since $\mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$ is a unique irreducible quotient of $u_\rho(a,b)|\cdot|^s \rtimes \pi$, we would have

$$\frac{u_\rho(a,b)|\cdot|^s \rtimes \pi}{\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)} \twoheadrightarrow \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi).$$

This contradicts that $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi) \cong \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$ appears in $[u_\rho(a,b)|\cdot|^s \rtimes \pi]$ with multiplicity one (Corollary 5.1). □

The following sufficient condition for the irreducibility is useful.

Theorem 5.3. *Let $\psi \in \Psi_{\mathrm{gp}}(G_n)$ and $\pi \in \Pi_\psi$. Suppose one of the following:*

- $s \notin (1/2)\mathbb{Z}$;
- $s \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$ *and* $\psi \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$ *is of good parity;*
- $s \in \mathbb{Z}$ *and* $\psi \oplus (\rho \boxtimes S_a \boxtimes S_b)^{\oplus 2}$ *is not of good parity.*

Then $u_\rho(a,b)|\cdot|^s \rtimes \pi$ is irreducible.

Proof. The case where $s = 0$ is Proposition 4.1. Since the irreducibility of $u_\rho(a,b)|\cdot|^s \rtimes \pi$ is equivalent to the one of $u_\rho(a,b)|\cdot|^{-s} \rtimes \pi$, we may assume that $s > 0$. Note that by Propositions 3.4 and 3.6, $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ and $\mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$ are both irreducible. Hence by Corollary 5.2, it is enough to show that $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi) \cong \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$. We prove this claim by several steps.

(1) We assume that $s \in (1/2)\mathbb{Z}$ and $b = 1$. Write

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi(\phi, \varepsilon)).$$

Since $\psi \in \Psi_{\mathrm{gp}}(G_n)$, we have $\phi \in \Phi_{\mathrm{gp}}(G_{n_0})$. By [25, Theorem 9.7] and [15, Théorème (i)] together with our assumption, we see that

- $u_\rho(a,1)|\cdot|^s \times \Delta_{\rho_i}[x_i, y_i] \cong \Delta_{\rho_i}[x_i, y_i] \times u_\rho(a,1)|\cdot|^s$;
- $u_\rho(a,1)|\cdot|^{-s} \times \Delta_{\rho_i}[x_i, y_i] \cong \Delta_{\rho_i}[x_i, y_i] \times u_\rho(a,1)|\cdot|^{-s}$;
- $u_\rho(a,1)|\cdot|^s \rtimes \pi(\phi, \varepsilon) \cong u_\rho(a,1)|\cdot|^{-s} \rtimes \pi(\phi, \varepsilon)$.

These isomorphisms and the characterization of $\mathrm{soc}(u_\rho(a,b)|\cdot|^{\pm s} \rtimes \pi)$ obtained in the proofs of Propositions 3.4 and 3.6, we see that

$$\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi) \cong \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi),$$

as desired.

(2) We assume that $s \in (1/2)\mathbb{Z}$ and $b \geq 2$. We prove the claim by induction on b. Set $\pi' = \mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$. Note that $u_\rho(a,b)|\cdot|^s \hookrightarrow \Delta_\rho[B+s,-A+s] \times u_\rho(a,b-1)|\cdot|^{s+1/2}$ with $A = (a+b)/2-1$ and $B = (a-b)/2$. By the induction hypothesis, we have $\pi' \hookrightarrow \Delta_\rho[B+s,-A+s] \times u_\rho(a,b-1)|\cdot|^{-s-\frac{1}{2}} \rtimes \pi$. We set

$$\begin{aligned}\tau &= \Delta_\rho[B+s,-A+s] \times u_\rho(a,b-1)|\cdot|^{-s-\frac{1}{2}} \\ &= \begin{pmatrix} B+s \\ \vdots \\ -A+s \end{pmatrix}_\rho \times \begin{pmatrix} B-s & \ldots & A-s-1 \\ \vdots & \ddots & \vdots \\ -A-s & \ldots & -B-s-1 \end{pmatrix}_\rho .\end{aligned}$$

If τ is irreducible, then $\pi' \hookrightarrow u_\rho(a,b-1)|\cdot|^{-s-1/2} \times \Delta_\rho[B+s,-A+s] \rtimes \pi$. In this case, by the previous case, we have

$$\pi' \hookrightarrow u_\rho(a,b-1)|\cdot|^{-s-1/2} \times \Delta_\rho[A-s,-B-s] \rtimes \pi.$$

By taking derivatives, we can conclude that $\pi' \cong \mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$.

Noting that $s > 0$, by [20, Theorem 1.1], τ is reducible if and only if $B+s > A-s-1$, $-A+s > -B-s-1$ and $-A+s \leq A-s$. In this case, τ is of length 2, and the socle $\mathrm{soc}(\tau)$ is isomorphic to

$$L(\Delta_\rho[B-s,-A-s],\ldots,\Delta_\rho[A-s-2,-B-s-2],\Delta_\rho[A-s-1,-A-s],\Delta_\rho[B+s,-B-s-1]).$$

This fact follows from [9, Lemma 2.7] by taking the Zelevinsky dual. Now suppose that $\pi' \hookrightarrow \mathrm{soc}(\tau) \rtimes \pi$. Then by the previous case,

$$\begin{aligned}\pi' &\hookrightarrow u_\rho(a,b-2)|\cdot|^{-s-1} \times \Delta_\rho[A-s-1,-A-s] \times \Delta_\rho[B+s,-B-s-1] \rtimes \pi \\ &\cong u_\rho(a,b-2)|\cdot|^{-s-1} \times \Delta_\rho[A-s-1,-A-s] \times \Delta_\rho[B+s+1,-B-s] \rtimes \pi.\end{aligned}$$

Since $B+s+1 > A-s$, by [20, Theorem 1.1], we see that $\rho|\cdot|^{B+s+1}$ commutes with $\Delta_\rho[A-s-1,-A-s]$ and $u_\rho(a,b-2)|\cdot|^{-s-1}$. This implies that $D_{\rho|\cdot|^{B+s+1}}(\pi') \neq 0$. This contradicts that $D_{\rho|\cdot|^{B+s+1}}(u_\rho(a,b)|\cdot|^s \rtimes \pi) = 0$. Therefore, we again have $\pi' \hookrightarrow u_\rho(a,b-1)|\cdot|^{-s-1/2} \times \Delta_\rho[B+s,-A+s] \rtimes \pi$, which implies the claim.

(3) We assume that $s \notin (1/2)\mathbb{Z}$. Using $u_\rho(a,b) \hookrightarrow u_\rho(a,1)|\cdot|^{-\frac{b-1}{2}} \times \cdots \times u_\rho(a,1)|\cdot|^{\frac{b-1}{2}}$, a similar argument to the first case works. In fact, we do not need to assume that ψ is of good parity in this case.

This completes the proof. □

Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type. We denote the minimal non-negative real number s such that $u_\rho(a,b)|\cdot|^s \rtimes \pi$ is reducible by s_0. We call s_0 the *first reducibility point* for $u_\rho(a,b)|\cdot|^s \rtimes \pi$. As in [19, Section 3 (b)], for $0 \leq s < s_0$, the irreducible induction $u_\rho(a,b)|\cdot|^s \rtimes \pi$ is unitary. Moreover, by [19, Section 3 (c)], all irreducible constituents of $u_\rho(a,b)|\cdot|^{s_0} \rtimes \pi$ are also unitary. Therefore, to attack the unitary dual problem for classical groups, it is important to compute s_0.

Corollary 5.4. *Let $\pi \in \mathrm{Irr}(G_n)$ be of Arthur type. Then we can compute the first reducibility point s_0 for $u_\rho(a,b)|\cdot|^s \rtimes \pi$ algorithmically.*

Proof. By Theorem 5.3, s_0 belongs to $(1/2)\mathbb{Z}$. Moreover, by computing $\mathrm{soc}(u_\rho(a,b)|\cdot|^s \rtimes \pi)$ and $\mathrm{soc}(u_\rho(a,b)|\cdot|^{-s} \rtimes \pi)$ using Propositions 3.4, 3.6, 3.7 and Theorem 4.4, we can determine s_0 by Corollary 5.2. □

5.2. **Examples.** Now we give some examples. In this subsection, we set $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$. When $\phi = \rho \boxtimes (S_{2x_1+1} \oplus \cdots \oplus S_{2x_r+1})$ and $\varepsilon(\rho \boxtimes S_{2x_i+1}) = \varepsilon_i$, we write $\pi(\phi, \varepsilon) = \pi(x_1^{\varepsilon_1}, \dots, x_r^{\varepsilon_r})$.

Example 5.5. Let us consider

$$u_\rho(2,3)|\cdot|^s \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)},$$

which is a representation of $\mathrm{Sp}_{12}(F)$. We compute the socle of this representation for $s = \pm 1/2$.

(1) When $s = 1/2$, by Proposition 3.6,

$$\mathrm{soc}(u_\rho(2,3)|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}) \hookrightarrow Z_\rho[0,2] \times Z_\rho[-1,1] \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}.$$

Noting that $\mathbf{1}_{\mathrm{Sp}_0(F)} = \pi(\{([0,0]_\rho, 0, 1)\})$, by Theorem 4.4, we have

$$\begin{aligned} Z_\rho[-1,1] \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)} \cong& \pi(\{([1,-1]_\rho, 1, 1), ([1,-1]_\rho, 1, 1), ([0,0]_\rho, 0, 1)\}) \\ &\oplus \pi(\{([1,-1]_\rho, 1, -1), ([1,-1]_\rho, 1, -1), ([0,0]_\rho, 0, 1)\}) \\ \cong& L((\rho|\cdot|^{-1})^2; \pi(0^+, 0^+, 0^+)) \oplus L(\rho|\cdot|^{-1}; \pi(0^-, 0^-, 1^+)). \end{aligned}$$

By considering Proposition 3.7, we have

$$\begin{aligned} &\mathrm{soc}(u_\rho(2,3)|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}) \\ &\quad\cong \mathrm{soc}(Z_\rho[0,2] \rtimes L((\rho|\cdot|^{-1})^2; \pi(0^+, 0^+, 0^+))) \\ &\qquad\oplus \mathrm{soc}(Z_\rho[0,2] \rtimes L(\rho|\cdot|^{-1}; \pi(0^-, 0^-, 1^+))) \\ &\quad\cong L(\rho|\cdot|^{-1}, \Delta_\rho[0,-2]; \pi(0^+, 0^+, 1^+)) \oplus L(\Delta_\rho[0,-1]; \pi(0^-, 1^-, 2^+)). \end{aligned}$$

(2) When $s = -1/2$, by Proposition 3.6,

$$\mathrm{soc}(u_\rho(2,3)|\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}) \hookrightarrow \Delta_\rho[-1,-2] \times u_\rho(2,2) \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}.$$

By Theorem 4.4, we have

$$\begin{aligned} u_\rho(2,2) \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)} \cong& \pi(\{([0,0]_\rho, 0, 1), ([1,0]_\rho, 1, 1), ([1,0]_\rho, 1, -1)\}) \\ &\oplus \pi(\{([0,0]_\rho, 0, 1), ([1,0]_\rho, 0, 1), ([1,0]_\rho, 0, -1)\}) \\ \cong& L(\Delta_\rho[0,-1]^2; \pi(0^+)) \oplus L(\Delta_\rho[0,-1]; \pi(0^+, 0^+, 1^+)). \end{aligned}$$

By considering Proposition 3.7, we have

$$\begin{aligned} &\mathrm{soc}(u_\rho(2,3)|\cdot|^{-\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}) \\ &\quad\cong \mathrm{soc}(\Delta_\rho[-1,-2] \rtimes L(\Delta_\rho[0,-1]^2; \pi(0^+))) \\ &\qquad\oplus \mathrm{soc}(\Delta_\rho[-1,-2] \rtimes L(\Delta_\rho[0,-1]; \pi(0^+, 0^+, 1^+))) \\ &\quad\cong L(\Delta_\rho[-1,-2], \Delta_\rho[0,-1]^2; \pi(0^+)) \oplus L(\Delta_\rho[-1,-2], \Delta_\rho[0,-1]; \pi(0^+, 0^+, 1^+)). \end{aligned}$$

In particular, we see that the length of $u_\rho(2,3)|\cdot|^{1/2} \rtimes \mathbf{1}_{\mathrm{Sp}_0(F)}$ is at least 4.

Example 5.6. Let us consider $a = b = 4$ and

$$\mathcal{E} = \{([3,-1]_\rho, 2, -1), ([3,1]_\rho, 0, -1), ([2,2]_\rho, 0, -1)\}.$$

Note that

$$\pi(\mathcal{E}) = L(\Delta_\rho[-1,-3], \Delta_\rho[0,-2], \Delta_\rho[2,-3]; \pi(1^-, 1^-, 2^+)).$$

We determine the first reducibility point s_0 for $u_\rho(4,4)|\cdot|^s \rtimes \pi(\mathcal{E})$. To do this, we compute its socles for some $s \in \mathbb{Z}$.

(1) When $s=0$, we have

$$u_\rho(4,4) \rtimes \pi(\mathcal{E}) = \pi(\mathcal{E}^{(0)}_{(2,+1)}) \oplus \pi(\mathcal{E}^{(0)}_{(1,+1)}) \oplus \pi(\mathcal{E}^{(0)}_{(1,-1)}) \oplus \pi(\mathcal{E}^{(0)}_{(0,+1)}) \oplus \pi(\mathcal{E}^{(0)}_{(0,-1)}),$$

where $\mathcal{E}^{(0)}_{(l,\eta)} = \mathcal{E} \cup \{([3,0]_\rho, l, \eta), ([3,0]_\rho, l, -\eta)\}$. By [2, Theorems 1.3, 1.4], we have $\pi(\mathcal{E}^{(0)}_{(2,+1)}) = \pi(\mathcal{E}^{(0)}_{(1,+1)}) = \pi(\mathcal{E}^{(0)}_{(0,+1)}) = \pi(\mathcal{E}^{(0)}_{(0,-1)}) = 0$. Hence $u_\rho(4,4) \rtimes \pi(\mathcal{E}) = \pi(\mathcal{E}^{(0)}_{(1,-1)})$ is irreducible.

(2) When $s=1$, we have

$$u_\rho(4,4)|\cdot|^1 \rtimes \pi(\mathcal{E}) \hookrightarrow u_\rho(2,4)|\cdot|^2 \times u_\rho(2,4) \rtimes \pi(\mathcal{E}).$$

As in the previous case, we have $u_\rho(2,4) \rtimes \pi(\mathcal{E}) = \pi(\mathcal{E}^{(1)}_{(2,-1)}) \oplus \pi(\mathcal{E}^{(1)}_{(1,-1)})$ with both $\pi(\mathcal{E}^{(1)}_{(2,-1)})$ and $\pi(\mathcal{E}^{(1)}_{(1,-1)})$ being nonzero, where $\mathcal{E}^{(1)}_{(l,\eta)} = \mathcal{E} \cup \{([2,-1]_\rho, l, \eta), ([2,-1]_\rho, l, -\eta)\}$. However, since $D^{(1)}_{\rho|\cdot|^3} \circ D^{(2)}_{\rho|\cdot|^2} \circ D^{(1)}_{\rho|\cdot|^1}(\pi(\mathcal{E}))$ $\neq 0$ but $D^{(1)}_{\rho|\cdot|^3} \circ D^{(2)}_{\rho|\cdot|^2} \circ D^{(1)}_{\rho|\cdot|^1}(\pi(\mathcal{E}^{(1)}_{(2,-1)})) = 0$, by Proposition 3.6, we conclude that $\mathrm{soc}(u_\rho(4,4)|\cdot|^1 \rtimes \pi(\mathcal{E})) = \mathrm{soc}(u_\rho(2,4)|\cdot|^2 \rtimes \pi(\mathcal{E}^{(1)}_{(1,-1)}))$ is irreducible. Also, we note that

$$D^{(2)}_{|\cdot|^3} \circ D^{(3)}_{|\cdot|^2} \circ D^{(2)}_{|\cdot|^1} \left(\mathrm{soc}(u_\rho(2,4)|\cdot|^2 \rtimes \pi(\mathcal{E}^{(1)}_{(1,-1)}))\right) \neq 0.$$

(3) When $s=-1$, we have

$$u_\rho(4,4)|\cdot|^{-1} \rtimes \pi(\mathcal{E}) \hookrightarrow u_\rho(4,2)|\cdot|^{-2} \times u_\rho(4,2) \rtimes \pi(\mathcal{E}).$$

As above, we see that $u_\rho(4,2) \rtimes \pi(\mathcal{E}) = \pi(\mathcal{E}^{(-1)}_{(0,-1)})$ is irreducible, where $\mathcal{E}^{(-1)}_{(0,-1)} = \mathcal{E} \cup \{([2,1]_\rho, 0, -1), ([2,1]_\rho, 0, 1)\}$. In particular, $\mathrm{soc}(u_\rho(4,4)|\cdot|^{-1} \rtimes \pi(\mathcal{E})) = \mathrm{soc}(u_\rho(4,2)|\cdot|^{-2} \rtimes \pi(\mathcal{E}^{(-1)}_{(0,-1)}))$ is also irreducible. Note that

$$D^{(2)}_{|\cdot|^3} \circ D^{(3)}_{|\cdot|^2} \circ D^{(2)}_{|\cdot|^1} \left(\mathrm{soc}(u_\rho(4,2)|\cdot|^{-2} \rtimes \pi(\mathcal{E}^{(-1)}_{(0,-1)}))\right) = 0.$$

Hence we have

$$\mathrm{soc}(u_\rho(4,4)|\cdot|^{-1} \rtimes \pi(\mathcal{E})) \not\cong \mathrm{soc}(u_\rho(4,4)|\cdot|^1 \rtimes \pi(\mathcal{E})),$$

which means that $u_\rho(4,4)|\cdot|^1 \rtimes \pi(\mathcal{E})$ is reducible.

Since $s_0 \in \mathbb{Z}$ by Theorem 5.3, we conclude that $s_0 = 1$.

Example 5.7. Let $\psi = \rho \boxtimes (S_2 \boxtimes S_2 + S_5 \boxtimes S_3) \in \Psi_{\mathrm{gp}}(\mathrm{Sp}_{18}(F))$. Then $\Pi_\psi = \{\pi(\mathcal{E}_i) \mid 1 \leq i \leq 5\}$ with

$$\begin{aligned} \mathcal{E}_1 &= \{([1,0],1,1), ([3,1],1,1)\}, \\ \mathcal{E}_2 &= \{([1,0],0,-1), ([3,1],0,1)\}, \\ \mathcal{E}_3 &= \{([1,0],1,1), ([3,1],0,-1)\}, \\ \mathcal{E}_4 &= \{([1,0],0,1), ([3,1],1,-1)\}, \\ \mathcal{E}_5 &= \{([1,0],0,-1), ([3,1],1,-1)\}. \end{aligned}$$

Note that $\mathcal{S}_\psi$ has exactly two characters. By [2, Theorem 3.5], we have $\langle \cdot, \pi(\mathcal{E}_i) \rangle_\psi = \mathbf{1} \iff i = 1, 3$. Now, for $1 \leq i \leq 5$, let s_i be the first reducibility point for $u_\rho(4,2)|\cdot|^s \rtimes \pi(\mathcal{E}_i)$. Note that $s_i \in \mathbb{Z}$ by Theorem 5.3. We compute s_i for $1 \leq i \leq 5$.

(1) When $s = 0$, by Theorem 4.4 together with [2, Theorem 1.4], we have

$$\begin{aligned} u_\rho(4,2) \rtimes \pi(\mathcal{E}_1) \cong & \pi(\mathcal{E}_1 \cup \{([2,1],0,1),([2,1],0,-1)\}) \\ & \oplus \pi(\mathcal{E}_1 \cup \{([2,1],1,1),([2,1],1,-1)\}), \\ u_\rho(4,2) \rtimes \pi(\mathcal{E}_2) \cong & \pi(\mathcal{E}_2 \cup \{([2,1],0,1),([2,1],0,-1)\}), \\ u_\rho(4,2) \rtimes \pi(\mathcal{E}_3) \cong & \pi(\mathcal{E}_3 \cup \{([2,1],0,-1),([2,1],0,1)\}), \\ u_\rho(4,2) \rtimes \pi(\mathcal{E}_4) \cong & \pi(\mathcal{E}_4 \cup \{([2,1],1,1),([2,1],1,-1)\}) \\ & \oplus \pi(\mathcal{E}_4 \cup \{([2,1],0,-1),([2,1],0,1)\}), \\ u_\rho(4,2) \rtimes \pi(\mathcal{E}_5) \cong & \pi(\mathcal{E}_5 \cup \{([2,1],1,1),([2,1],1,-1)\}). \end{aligned}$$

In particular, $u_\rho(4,2) \rtimes \pi(\mathcal{E}_i)$ is reducible if and only if $i = 1,4$ so that $s_1 = s_4 = 0$.

(2) When $s = \pm 1$, by Propositions 3.6, 3.7 and 3.4, we have

$$\begin{aligned} \mathrm{soc}(u_\rho(4,2)|\cdot|^1 \rtimes \pi(\mathcal{E}_2)) &\cong L(\Delta_\rho[0,-1],\Delta_\rho[1,-3];\pi(0^-,1^-,2^-,2^-,3^+)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^{-1} \rtimes \pi(\mathcal{E}_2)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[1,-2];\pi(0^-,1^+,1^+,2^-,3^+)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^1 \rtimes \pi(\mathcal{E}_3)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[0,-1],\Delta_\rho[1,-2];\pi(1^-,2^+,3^-)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^{-1} \rtimes \pi(\mathcal{E}_3)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[0,-1],\Delta_\rho[1,-2];\pi(1^-,2^+,3^-)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^1 \rtimes \pi(\mathcal{E}_5)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[2,-3];\pi(0^-,1^+,1^+,1^+,2^-)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^{-1} \rtimes \pi(\mathcal{E}_5)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[1,-3],\Delta_\rho[1,-2];\pi(0^-,1^+,2^-)). \end{aligned}$$

In particular, for any $i = 2,3,5$, the socle $\mathrm{soc}(u_\rho(4,2)|\cdot|^{\pm 1} \rtimes \pi(\mathcal{E}_i))$ is irreducible. Since $\mathrm{soc}(u_\rho(4,2)|\cdot|^1 \rtimes \pi(\mathcal{E}_i)) \not\cong \mathrm{soc}(u_\rho(4,2)|\cdot|^{-1} \rtimes \pi(\mathcal{E}_i))$ for $i = 2,5$, we have $s_2 = s_5 = 1$. On the other hand, $u_\rho(4,2)|\cdot|^1 \rtimes \pi(\mathcal{E}_3)$ is irreducible.

(3) When $s = \pm 2$, by Proposition 3.4, we have

$$\begin{aligned} \mathrm{soc}(u_\rho(4,2)|\cdot|^2 \rtimes \pi(\mathcal{E}_3)) &\cong L(\Delta_\rho[0,-3],\Delta_\rho[1,-2];\pi(1^-,3^+,4^-)), \\ \mathrm{soc}(u_\rho(4,2)|\cdot|^{-2} \rtimes \pi(\mathcal{E}_3)) &\cong L(\Delta_\rho[-1,-4],\Delta_\rho[0,-3],\Delta_\rho[0,-1];\pi(1^-,2^+,3^-)). \end{aligned}$$

Hence $u_\rho(4,2)|\cdot|^2 \rtimes \pi(\mathcal{E}_3)$ is reducible so that $s_3 = 2$.

Appendix A. Computations of certain derivatives

Recall that when $\pi \in \mathrm{Irr}(G_n)$ is $\rho|\cdot|^{-1}$-reduced (resp. $\rho|\cdot|^1$-reduced), the highest $\Delta_\rho[0,-1]$-derivative $D^{\max}_{\Delta_\rho[0,-1]}(\pi)$ (resp. the highest $Z_\rho[0,1]$-derivative $D^{\max}_{Z_\rho[0,1]}(\pi)$) is irreducible ([3, Proposition 3.7]). In [3], explicit formulas for $\Delta_\rho[0,-1]$-derivatives and for $Z_\rho[0,1]$-derivatives were given only for irreducible representations satisfying some specific conditions. The goal of this appendix is to compute these derivatives for π of good parity in general.

Here, we say that an irreducible representation π is *of good parity* if π is a subrepresentation of an induced representation of the form $\rho_1|\cdot|^{s_1} \times \cdots \times \rho_r|\cdot|^{s_r} \rtimes \sigma$, where

- $\rho_i \in \mathrm{Cusp}^\perp(\mathrm{GL}_{d_i}(F))$ and $s_i \in (1/2)\mathbb{Z}$;
- σ is an irreducible supercuspidal representation of G_{n_0};
- $\rho_i|\cdot|^{s_i+m_i} \rtimes \sigma$ is reducible for some $m_i \in \mathbb{Z}$.

A.1. **Derivatives for $\mathrm{GL}_n(F)$.** Before dealing with classical groups, we fix notations and recall some facts on representations of $\mathrm{GL}_n(F)$. For these facts, see [7] and its references.

Denote $P_{(m,n-m)}$ by the maximal standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi $\mathrm{GL}_m(F) \times \mathrm{GL}_{n-m}(F)$. For a smooth representation τ of $\mathrm{GL}_n(F)$ of finite length, define the *left $\rho|\cdot|^x$-derivative* $L^{(k)}_{\rho|\cdot|^x}(\tau)$ and the *right $\rho|\cdot|^x$-derivative* $R^{(k)}_{\rho|\cdot|^x}(\tau)$ by

$$[\mathrm{Jac}_{P_{(dk,n-dk)}}(\tau)] = (\rho|\cdot|^x)^k \boxtimes L^{(k)}_{\rho|\cdot|^x}(\tau) + (\text{others}),$$
$$[\mathrm{Jac}_{P_{(n-dk,dk)}}(\tau)] = R^{(k)}_{\rho|\cdot|^x}(\tau) \boxtimes (\rho|\cdot|^x)^k + (\text{others}).$$

The highest derivatives $L^{\max}_{\rho|\cdot|^x}(\tau)$ and $R^{\max}_{\rho|\cdot|^x}(\tau)$ are defined similar as in Section 3.1. It is known that if τ is irreducible, then $L^{\max}_{\rho|\cdot|^x}(\tau)$ and $R^{\max}_{\rho|\cdot|^x}(\tau)$ are also irreducible (see [7, Lemma 2.1]). Moreover, the Langlands data for $L^{\max}_{\rho|\cdot|^x}(\tau)$ (resp. $R^{\max}_{\rho|\cdot|^x}(\tau)$) can be described from those for τ explicitly, and vice versa (see, e.g., [7, Theorem 5.11]).

Similarly, following [3, Section 3.4], we can define

- the *highest left $\Delta_\rho[0,-1]$-derivative* $L^{\max}_{\Delta_\rho[0,-1]}(\tau)$;
- the *highest right $\Delta_\rho[0,-1]$-derivative* $R^{\max}_{\Delta_\rho[0,-1]}(\tau)$;
- the *highest left $Z_\rho[0,1]$-derivative* $L^{\max}_{Z_\rho[0,1]}(\tau)$;
- the *highest right $Z_\rho[0,1]$-derivative* $R^{\max}_{Z_\rho[0,1]}(\tau)$.

If τ is irreducible and left $\rho|\cdot|^1$-reduced, i.e., if $L^{(1)}_{\rho|\cdot|^1}(\tau) = 0$, then $L^{\max}_{Z_\rho[0,1]}(\tau)$ is also irreducible. In this case, if we write $L^{\max}_{\rho|\cdot|^1} \circ L^{\max}_{\rho}(\tau) = L^{(k_1)}_{\rho|\cdot|^1} \circ L^{(k_0)}_{\rho}(\tau) = \tau'$, then $k_0 \geq k_1$ and we have

$$L^{\max}_{Z_\rho[0,1]}(\tau) = L^{(k_1)}_{Z_\rho[0,1]}(\tau) = \mathrm{soc}\left(\rho^{k_0-k_1} \times \tau'\right).$$

Similar properties hold for other derivatives. These facts can be proven by the same argument as [3, Lemma 3.5].

On the other hand, for any irreducible representation τ of $\mathrm{GL}_n(F)$, the socle of $Z_\rho[0,1]^r \times \tau$ is irreducible, and it can be computed by

$$\mathrm{soc}(Z_\rho[0,1]^r \times \tau) = \mathrm{soc}\left(\rho^{k_0+r} \times \mathrm{soc}\left((\rho|\cdot|^1)^r \times L^{(k_0)}_\rho(\tau)\right)\right),$$

where we write $L^{\max}_\rho(\tau) = L^{(k_0)}_\rho(\tau)$. Similar properties hold for the socles of $\tau \times Z_\rho[0,1]^r$, $\Delta_\rho[0,-1]^r \times \tau$ and $\tau \times \Delta_\rho[0,-1]^r$. See [7, Proposition 5.6].

A.2. **$\Delta_\rho[0,-1]$-derivatives.** Let π be an irreducible representation of G_n. Suppose that π is $\rho|\cdot|^{-1}$-reduced. Then $D^{\max}_{\Delta_\rho[0,-1]}(\pi)$ is irreducible ([3, Proposition 3.7]). In a special case, an explicit formula for $D^{\max}_{\Delta_\rho[0,-1]}(\pi)$ was given in [3, Proposition 3.8]. In this subsection, we generalize this formula.

Proposition A.1. *Write $\pi = L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r];\pi_{\mathrm{temp}})$ as in the Langlands classification. Suppose that π is $\rho|\cdot|^{-1}$-reduced. Then*

$$D^{\max}_{\Delta_\rho[0,-1]}(\pi) \hookrightarrow L^{\max}_{\Delta_\rho[0,-1]}\left(L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r])\right) \rtimes \pi_{\mathrm{temp}}.$$

Proof. Write $\tau = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r])$ and $L^{\max}_{\Delta_\rho[0,-1]}(\tau) = L^{(k)}_{\Delta_\rho[0,-1]}(\tau)$. Note that $L^{\max}_{\Delta_\rho[0,-1]}(\tau)$ is irreducible since τ is left $\rho|\cdot|^{-1}$-reduced. Clearly, we have an inclusion

$$\pi \hookrightarrow \Delta_\rho[0,-1]^k \times L^{(k)}_{\Delta_\rho[0,-1]}(\tau) \rtimes \pi_{\mathrm{temp}}.$$

Since

- $L^{\max}_{\Delta_\rho[0,-1]}(\tau)$ is left $\rho|\cdot|^{-1}$-reduced;
- $x_i + y_i < 0$ so that $y_i \neq 0, 1$;
- π_{temp} is $\rho|\cdot|^{-1}$-reduced (Casselman's criterion),

we see that $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ is the highest $\Delta_\rho[0,-1]$-derivative, and

$$D^{(k)}_{\Delta_\rho[0,-1]}(\pi) \hookrightarrow L^{(k)}_{\Delta_\rho[0,-1]}(\tau) \rtimes \pi_{\mathrm{temp}}.$$

This completes the proof. □

A.3. $Z_\rho[0,1]$**-derivatives: A special case.** Let π be an irreducible representation of G_n. Suppose that π is of good parity and $\rho|\cdot|^1$-reduced. Then $D^{\max}_{Z_\rho[0,1]}(\pi)$ is irreducible ([3, Proposition 3.7]). When π is further $\rho|\cdot|^z$-reduced for any $z < 0$, an explicit formula for $D^{\max}_{Z_\rho[0,1]}(\pi)$ was given in [3, Theorem 8.1, Proposition 8.4]. In this and next subsections, we generalize this formula.

Here, we consider a special case, which is the main case. Suppose that π is of the form

$$\pi = L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi, \varepsilon))$$

for $s, t \geq 0$ and $\phi \in \Phi_{\mathrm{gp}}(G_{n_0})$. Set

$$\delta = \begin{cases} 1 & \text{if } \rho, \rho \boxtimes S_3 \subset \phi \text{ and } \varepsilon(\rho)\varepsilon(\rho \boxtimes S_3) \neq (-1)^t, \\ 0 & \text{otherwise.} \end{cases}$$

Then by [3, Theorem 7.1], we have $D^{\max}_{\rho|\cdot|^1}(\pi) = D^{(k)}_{\rho|\cdot|^1}(\pi)$ with

$$k = \min\{s - m_\phi(\rho) + \delta, 0\} + m_\phi(\rho \boxtimes S_3) - \delta,$$

where $m_\phi(\rho)$ (resp. $m_\phi(\rho \boxtimes S_3)$) denotes the multiplicity of ρ (resp. $\rho \boxtimes S_3$) in ϕ. In particular, π is $\rho|\cdot|^1$-reduced if and only if $m_\phi(\rho \boxtimes S_3) = \delta$ and $s \leq m_\phi(\rho) - \delta$. The following is a generalization of [3, Proposition 8.4].

Proposition A.2. *Let $\pi = L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi, \varepsilon))$ be as above. Suppose that π is $\rho|\cdot|^1$-reduced. Write $m = m_\phi(\rho)$ so that $s \leq m - \delta$.*

(1) *If $\delta = 1$ and $m \equiv s + 1 \bmod 2$, then the highest $Z_\rho[0,1]$-derivative of π is*

$$D^{(t)}_{Z_\rho[0,1]}(\pi) = \begin{cases} L((\rho|\cdot|^{-1})^s; \pi(\phi, \varepsilon)) & \text{if } t \equiv 0 \bmod 2, \\ L((\rho|\cdot|^{-1})^{s+1}; \pi(\phi + \rho - \rho \boxtimes S_3, \varepsilon)) & \text{if } t \equiv 1 \bmod 2. \end{cases}$$

(2) *If $\delta = 1$ and $m \equiv s \bmod 2$, then the highest $Z_\rho[0,1]$-derivative of π is*

$$D^{(t+1)}_{Z_\rho[0,1]}(\pi) = \begin{cases} \pi(\phi - \rho - \rho \boxtimes S_3, \varepsilon') & \text{if } t \equiv 0 \bmod 2, s = 0, \\ L((\rho|\cdot|^{-1})^{s-1}; \pi(\phi - \rho^2, \varepsilon)) & \text{if } t \equiv 0 \bmod 2, s > 0, \\ L((\rho|\cdot|^{-1})^s; \pi(\phi - \rho - \rho \boxtimes S_3, \varepsilon)) & \text{if } t \equiv 1 \bmod 2, \end{cases}$$

where ε' is given so that $\varepsilon'(\rho' \boxtimes S_d) \neq \varepsilon(\rho' \boxtimes S_d) \iff \rho' \boxtimes S_d = \rho \boxtimes S_1$.

(3) *If $\delta = 0$ and $m \equiv s+1 \bmod 2$, then the highest $Z_\rho[0,1]$-derivative of π is*

$$\begin{cases} D^{(0)}_{Z_\rho[0,1]}(\pi) = L((\rho|\cdot|^{-1})^s; \pi(\phi,\varepsilon)) & \text{if } t = 0, \\ D^{(t-1)}_{Z_\rho[0,1]}(\pi) = L((\rho|\cdot|^{-1})^{s+1}; \pi(\phi+\rho^2,\varepsilon)) & \text{if } t > 0, t \equiv 0 \bmod 2, \\ D^{(t-1)}_{Z_\rho[0,1]}(\pi) = L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]; \pi(\phi,\varepsilon)) & \text{if } t > 0, t \equiv 1 \bmod 2. \end{cases}$$

(4) *If $\delta = 0$ and $m \equiv s \bmod 2$, then the highest $Z_\rho[0,1]$-derivative of π is*

$$D^{(t)}_{Z_\rho[0,1]}(\pi) = \begin{cases} \pi(\phi,\varepsilon') & \text{if } t \equiv 1 \bmod 2, m > s = 0, \\ L((\rho|\cdot|^{-1})^{s-1}, \Delta_\rho[0,-1]; \pi(\phi-\rho^2,\varepsilon)) & \text{if } t \equiv 1 \bmod 2, m > s > 0, \\ L((\rho|\cdot|^{-1})^s; \pi(\phi,\varepsilon)) & \text{otherwise}, \end{cases}$$

where ε' is the same as in (2).

Proof. The proof is essentially the same as [3, Proposition 8.4]. We only give a detail for the proof of (2).

Assume that $\delta = 1$ and $m \equiv s \bmod 2$. Write $m = s + 2u$ so that $u > 0$. Note that $\pi \in \Pi_\psi$ with

$$\psi = \phi - \rho^s + (\rho \boxtimes S_1 \boxtimes S_3)^s + (\rho \boxtimes S_2 \boxtimes S_2)^t.$$

Since ψ contains ρ with multiplicity $2u$, by Theorem 4.4, we see that

$$\begin{aligned} \pi &\hookrightarrow \rho^u \rtimes L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi-\rho^{2u},\varepsilon)) \\ &\hookrightarrow \rho^{u+t} \rtimes L((\rho|\cdot|^{-1})^{s+t}; \pi(\phi-\rho^{2u},\varepsilon)). \end{aligned}$$

Since $L((\rho|\cdot|^{-1})^{s+t}; \pi(\phi-\rho^{2u},\varepsilon)) = (\rho|\cdot|^{-1})^t \rtimes L((\rho|\cdot|^{-1})^s; \pi(\phi-\rho^{2u},\varepsilon))$ is irreducible, and since $L((\rho|\cdot|^{-1})^s; \pi(\phi-\rho^{2u},\varepsilon))$ belongs to Π_{ψ_0} with $\psi_0 = \phi - \rho^m + (\rho \boxtimes S_1 \boxtimes S_3)^s$, by [24, Proposition 8.3 (ii)], we see that $D^{\max}_\rho(\pi) = L((\rho|\cdot|^{-1})^{s+t}; \pi(\phi-\rho^{2u},\varepsilon))$ up to a multiplicity.

When t is odd, since $\varepsilon(\rho \boxtimes S_3) = \varepsilon(\rho)$, we have

$$\pi \hookrightarrow \rho^{u+t} \times (\rho|\cdot|^1)^{t+1} \rtimes L((\rho|\cdot|^{-1})^s; \pi(\phi-\rho^{2u-1}-\rho\boxtimes S_3,\varepsilon)).$$

Hence

$$\begin{aligned} \pi &\hookrightarrow Z_\rho[0,1]^{t+1} \times \rho^{u-1} \rtimes L((\rho|\cdot|^{-1})^s; \pi(\phi-\rho^{2u-1}-\rho\boxtimes S_3,\varepsilon)) \\ &\cong Z_\rho[0,1]^{t+1} \rtimes L((\rho|\cdot|^{-1})^s; \pi(\phi-\rho-\rho\boxtimes S_3,\varepsilon)). \end{aligned}$$

On the other hand, when t is even and $s > 0$, since $\varepsilon(\rho \boxtimes S_3) \neq \varepsilon(\rho)$, we have

$$\pi \hookrightarrow \rho^{u+t} \times (\rho|\cdot|^1)^{t+1} \rtimes L((\rho|\cdot|^{-1})^{s-1}; \pi(\phi-\rho^{2u},\varepsilon)).$$

Hence

$$\begin{aligned} \pi &\hookrightarrow Z_\rho[0,1]^{t+1} \times \rho^{u-1} \rtimes L((\rho|\cdot|^{-1})^{s-1}; \pi(\phi-\rho^{2u},\varepsilon)) \\ &\cong Z_\rho[0,1]^{t+1} \rtimes L((\rho|\cdot|^{-1})^{s-1}; \pi(\phi-\rho^2,\varepsilon)). \end{aligned}$$

The last isomorphism follows from Theorem 4.4. The case where $s = 0$ was proven in [3, Proposition 8.4]. Therefore, we obtain (2). □

The converse of this proposition is given as follows.

Corollary A.3. *Let $\pi = L((\rho|\cdot|^{-1})^s, \Delta_\rho[0,-1]^t; \pi(\phi,\varepsilon))$ be as above. Suppose that π is $\rho|\cdot|^1$-reduced. Write $D^{\max}_{Z_\rho[0,1]}(\pi) = D^{(k)}_{Z_\rho[0,1]}(\pi) = L((\rho|\cdot|^{-1})^{s'}, \Delta_\rho[0,-1]^{t'}; \pi(\phi',\varepsilon'))$. Assume that $k > 0$. Set $m' = m_{\phi'}(\rho)$.*

(1) *If k is even and $t' = 1$, then*

$$(s, t, \phi, \varepsilon) = (s', k+1, \phi', \varepsilon').$$

(2) *If k is even, $t' = 0$ and $m' \equiv s' \bmod 2$, then*

$$(s, t, \phi, \varepsilon) = (s', k, \phi', \varepsilon').$$

(3) *If k is even, $t' = 0$, $m' \equiv s' + 1 \bmod 2$ and $\phi' \supset \rho \boxtimes S_3$, then*

$$(s, t, \phi, \varepsilon) = (s', k, \phi', \varepsilon').$$

(4) *If k is even, $t' = 0$, $m' \equiv s' + 1 \bmod 2$ and $\phi' \not\supset \rho \boxtimes S_3$, then $m' > 0$ and*

$$(s, t, \phi, \varepsilon) = (s', k-1, \phi' + \rho + \rho \boxtimes S_3, \varepsilon)$$

with $\varepsilon(\rho) = \varepsilon'(\rho)$ and $\varepsilon(\rho \boxtimes S_3) = (-1)^k \varepsilon(\rho)$.

(5) *If k is odd and $t' = 1$, then $m' > 0$ and*

$$(s, t, \phi, \varepsilon) = (s'+1, k, \phi' + \rho^2, \varepsilon)$$

with $\varepsilon(\rho) = \varepsilon'(\rho)$.

(6) *If k is odd, $t' = 0$ and $m' = s'$, then*

$$(s, t, \phi, \varepsilon) = (s', k, \phi', \varepsilon').$$

(7) *If k is odd, $t' = 0$, $s' = 0 < m'$ and $m' \equiv 0 \bmod 2$, then*

$$(s, t, \phi, \varepsilon) = (0, k, \phi', \varepsilon)$$

with $\varepsilon(\rho) \neq \varepsilon'(\rho)$.

(8) *If k is odd, $t' = 0$ $s' = 0 < m'$, $m' \equiv 1 \bmod 2$ and $\phi' \supset \rho \boxtimes S_3$, then $m' > 0$ and*

$$(s, t, \phi, \varepsilon) = (1, k-1, \phi' + \rho^2, \varepsilon').$$

(9) *If k is odd, $t' = 0$ $s' = 0 < m'$, $m' \equiv 1 \bmod 2$ and $\phi' \not\supset \rho \boxtimes S_3$, then*

$$(s, t, \phi, \varepsilon) = (0, k-1, \phi' + \rho + \rho \boxtimes S_3, \varepsilon)$$

with $\varepsilon(\rho) \neq \varepsilon'(\rho)$ and $\varepsilon(\rho \boxtimes S_3) = (-1)^k \varepsilon(\rho)$.

(10) *If k is odd, $t' = 0$, $0 < s' < m'$ and $m' \equiv s' \bmod 2$, then*

$$(s, t, \phi, \varepsilon) = (s'-1, k+1, \phi' - \rho^2, \varepsilon').$$

(11) *If k is odd, $t' = 0$, $0 < s' < m'$, $m' \equiv s' + 1 \bmod 2$ and $\phi' \supset \rho \boxtimes S_3$, then*

$$(s, t, \phi, \varepsilon) = (s'+1, k-1, \phi' + \rho^2, \varepsilon)$$

with $\varepsilon(\rho) = \varepsilon'(\rho)$.

(12) *If k is odd, $t' = 0$, $0 < s' < m'$, $m' \equiv s' + 1 \bmod 2$ and $\phi' \not\supset \rho \boxtimes S_3$, then*

$$(s, t, \phi, \varepsilon) = (s'-1, k, \phi' - \rho + \rho \boxtimes S_3, \varepsilon)$$

with $\varepsilon(\rho) = \varepsilon'(\rho)$ and $\varepsilon(\rho \boxtimes S_3) = (-1)^{k-1} \varepsilon(\rho)$.

A.4. $Z_\rho[0,1]$**-derivatives: The general case.** We continue to study $Z_\rho[0,1]$-derivatives. Here, we consider the general case. The following is an algorithm to compute $D^{\max}_{Z_\rho[0,1]}(\pi)$, which is analogue to Jantzen's one [6, Section 3.3]. The proof is also similar and we omit it.

Algorithm A.4. Let $\pi \in \mathrm{Irr}(G_n)$ be of good parity. Assume that π is $\rho|\cdot|^1$-reduced.

(1) We write $\pi = L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r],(\rho|\cdot|^{-1})^s,\Delta_\rho[0,-1]^t;\pi(\phi,\varepsilon))$ as in the Langlands classification, where
- $\phi \in \Phi_{\mathrm{gp}}(G_{n_0})$;
- $s,t \geq 0$;
- $x_1+y_1 \leq \dots \leq x_r+y_r < 0$;
- $\Delta_{\rho_i}[x_i,y_i] \not\cong \rho|\cdot|^{-1}, \Delta_\rho[0,-1]$ for $i = 1,\dots,r$.

Remark that if $\rho_i \cong \rho$ and $x_i+y_i = -1/2$, then $\rho|\cdot|^{-1} \in [x_i,y_i]_\rho$ so that $\Delta_{\rho_i}[x_i,y_i] \times \rho|\cdot|^{-1} \cong \rho|\cdot|^{-1} \times \Delta_{\rho_i}[x_i,y_i]$. Note that $y_i \neq -1$ if $\rho_i \cong \rho$.

(2) Set
$$\begin{aligned}\pi_A &= L((\rho|\cdot|^{-1})^s,\Delta_\rho[0,-1]^t;\pi(\phi,\varepsilon)),\\ \pi'_A &= D^{\max}_{\rho|\cdot|^1}(\pi_A) = D^{(l_1)}_{\rho|\cdot|^1}(\pi_A),\\ \pi''_A &= D^{\max}_{Z_\rho[0,1]}(\pi'_A) = D^{(k_1)}_{Z_\rho[0,1]}(\pi'_A).\end{aligned}$$
Note that π'_A and π''_A are of the same form as π_A.

(3) We have $\pi \hookrightarrow \tau \rtimes \pi''_A$, where
$$\begin{aligned}\tau &= \mathrm{soc}(L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r],(\rho|\cdot|^1)^{l_1}) \times Z_\rho[0,1]^{k_1})\\ &\cong L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r],\rho^{k_1},(\rho|\cdot|^1)^{k_1+l_1}).\end{aligned}$$

(4) Remark that τ is left $\rho|\cdot|^1$-reduced since π is $\rho|\cdot|^1$-reduced. Compute $\tau' = L^{\max}_{Z_\rho[0,1]}(\tau) = L^{(k)}_{Z_\rho[0,1]}(\tau)$. It is of the form
$$\tau' = L(\Delta_{\rho_1}[x'_1,y_1],\dots,\Delta_{\rho_r}[x'_r,y_r],\rho^{k_2},(\rho|\cdot|^1)^{k_2+l_2})$$
with $l_2 \leq l_1$, $k_2 \leq k_1$ and $x'_1+y_1 \leq \dots \leq x'_r+y_r < 0$. Then
$$D^{\max}_{Z_\rho[0,1]}(\pi) \hookrightarrow \tau' \rtimes \pi''_A.$$

(5) Compute
$$\begin{aligned}\pi'_B &= \mathrm{soc}\left(Z_\rho[0,1]^{k_2} \rtimes \pi''_A\right),\\ \pi_B &= \mathrm{soc}\left((\rho|\cdot|^1)^{l_2} \rtimes \pi'_B\right).\end{aligned}$$
Then
$$D^{\max}_{Z_\rho[0,1]}(\pi) \hookrightarrow L(\Delta_{\rho_1}[x'_1,y_1],\dots,\Delta_{\rho_r}[x'_r,y_r]) \rtimes \pi_B.$$

(6) Note that π_B is of the form $\pi_B = L((\rho|\cdot|^{-1})^{s'},\Delta_\rho[0,-1]^{t'};\pi(\phi',\varepsilon'))$. We conclude that
$$D^{\max}_{Z_\rho[0,1]}(\pi) = L(\Delta_{\rho_1}[x'_1,y_1],\dots,\Delta_{\rho_r}[x'_r,y_r],(\rho|\cdot|^{-1})^{s'},\Delta_\rho[0,-1]^{t'};\pi(\phi',\varepsilon')).$$

Finally, we state an algorithm to compute $\mathrm{soc}(Z_\rho[0,1]^k \rtimes \pi)$.

Algorithm A.5. Let $\pi \in \mathrm{Irr}(G_n)$ be of good parity. Assume that π is $\rho|\cdot|^1$-reduced.

(1) Write $\pi = L(\Delta_{\rho_1}[x_1,y_1],\dots,\Delta_{\rho_r}[x_r,y_r],(\rho|\cdot|^{-1})^s,\Delta_\rho[0,-1]^t;\pi(\phi,\varepsilon))$ as in Algorithm A.4 (1).

(2) Let π_A, $\pi'_A = D^{(l_1)}_{\rho|\cdot|^1}(\pi_A)$, $\pi''_A = D^{(k_1)}_{Z_\rho[0,1]}(\pi'_A)$ be as in Algorithm A.4 (2), and $\tau = L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r], \rho^{k_1}, (\rho|\cdot|^1)^{k_1+l_1})$ be as in Algorithm A.4 (3).

(3) Compute $\tau' = \mathrm{soc}(Z_\rho[0,1]^k \rtimes \tau)$. It is of the form
$$\tau' = L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r], \rho^{k_2}, (\rho|\cdot|^1)^{k_2+l_2})$$
with $x'_1 + y_1 \leq \cdots \leq x'_r + y_r < 0$.

(4) Compute
$$\pi'_B = \mathrm{soc}\left(Z_\rho[0,1]^{k_2} \rtimes \pi''_A\right),$$
$$\pi_B = \mathrm{soc}\left((\rho|\cdot|^1)^{l_2} \rtimes \pi'_B\right).$$

Then π_B is of the form $\pi_B = L((\rho|\cdot|^{-1})^{s'}, \Delta_\rho[0,-1]^{t'}; \pi(\phi', \varepsilon'))$. We conclude that
$$\mathrm{soc}(Z_\rho[0,1]^k \rtimes \pi) = L(\Delta_{\rho_1}[x'_1, y_1], \dots, \Delta_{\rho_r}[x'_r, y_r], (\rho|\cdot|^{-1})^{s'}, \Delta_\rho[0,-1]^{t'}; \pi(\phi', \varepsilon')).$$

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