REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 673–697 (June 28, 2022)
https://doi.org/10.1090/ert/616

THE CONTRACTION CATEGORY OF GRAPHS

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Abstract. We study the category whose objects are graphs of fixed genus and whose morphisms are contractions. We show that the corresponding contravariant module categories are Noetherian and we study two families of modules over these categories. The first takes a graph to a graded piece of the homology of its unordered configuration space and the second takes a graph to an intersection homology group whose dimension is given by a Kazhdan–Lusztig coefficient; in both cases we prove that the module is finitely generated. This allows us to draw conclusions about torsion in the homology groups of graph configuration spaces, and about the growth of Betti numbers of graph configuration spaces and Kazhdan–Lusztig coefficients of graphical matroids. We also explore the relationship between our category and outer space, which is used in the study of outer automorphisms of free groups.

1. Introduction

We are interested in ways of assigning a vector space or abelian group to a graph that are contravariantly functorial with respect to contractions of graphs. A contraction, which is defined precisely in Section 2.1, preserves the genus (first Betti number) of a graph, so we consider the category $\mathcal{G}_g$ whose objects are graphs of genus g and whose morphisms are contractions. For any commutative ring k, we define $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ to be the category of functors from $\mathcal{G}_g^{\mathrm{op}}$ to k-modules. An object of this category is called a $\mathcal{G}_g^{\mathrm{op}}$*-module with coefficients in* k.

1.1. Noetherianity and growth.

For any category $\mathcal{C}$, a module $M \in \mathrm{Rep}_k(\mathcal{C})$ is called *finitely generated* if there exist finitely many objects $x_1, \ldots, x_r$ of $\mathcal{C}$ along with elements $v_i \in M(x_i)$ such that, for any object x of $\mathcal{C}$, $M(x)$ is spanned over k by the images of the elements v_i along the maps induced by all possible morphisms $f_i : x_i \to x$. If every submodule of a finitely generated module is itself finitely generated, the category $\mathrm{Rep}_k(\mathcal{C})$ is said to be *locally Noetherian*.

Sam and Snowden have developed powerful machinery for proving that module categories are locally Noetherian. They define what it means for $\mathcal{C}$ to be *quasi-Gröbner*, and they show that, if $\mathcal{C}$ is quasi-Gröbner, then $\mathrm{Rep}_k(\mathcal{C})$ is locally Noetherian for any Noetherian commutative algebra k [20]. The most prominent example of a quasi-Gröbner category is the category FI of finite sets with injections; the fact that $\mathrm{Rep}_k(\mathrm{FI})$ is locally Noetherian has been used to prove stability patterns in coinvariant algebras and in the cohomology groups of configuration spaces and

Received by the editors July 18, 2020, and, in revised form, January 18, 2022, and April 21, 2022.

2020 *Mathematics Subject Classification.* Primary 05C25, 05C83, 14F43, 55R80.

This work was supported by NSF grants DMS-1704811, DMS-1954050, DMS-2039316, DMS-2053243, and DMS-2137628.

other moduli spaces [4], in the homology groups of congruence subgroups [17], and in the syzygies of Segre embeddings [22].

In the prequel to this paper, the authors built on work of Barter [3] to prove that the opposite category $\mathcal{G}_0^{\mathrm{op}}$ of trees with contractions is quasi-Gröbner [14]. The technical heart of this paper is the extension of this result to arbitrary genus.

Theorem 1.1. *For any non-negative integer g, the category $\mathcal{G}_g^{\mathrm{op}}$ is quasi-Gröbner, and therefore the category $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is locally Noetherian for any Noetherian commutative algebra k.*

Theorem 1.1 is useful for proving that specific $\mathcal{G}_g^{\mathrm{op}}$-modules are finitely generated, and this gives some control over their dimension growth. More precisely, we say that a module is *finitely generated in degrees* $\leq d$ if the objects $x_1, \ldots, x_r$ in the definition of finite generation may be taken to be graphs with at most d edges. If k is a field and M is finitely generated in degrees $\leq d$, then the dimension of $M(G)$ is constrained by a polynomial of degree d in the number of edges of G (Proposition 4.3). Furthermore, if we fix a graph and modify it by either subdividing edges or "sprouting" new leaves at a fixed set of vertices, then the dimension of M evaluated on the modified graph behaves as a polynomial of degree at most d in the subdivision and sprouting parameters (Corollaries 4.5 and 4.7).

Sometimes we have no control of the generation degree of a finitely generated module, but we can still control its growth. We say that M is *d-small* if it is a subquotient of a module that is finitely generated in degrees $\leq d$, and *d-smallish* if it admits a filtration whose associated graded is d-small. Theorem 1.1 implies that d-small modules are finitely generated, and it is not hard to prove that the same is true for d-smallish modules (Proposition 4.2). The degree of generation of such modules may be much larger than d, but for the purposes of the results mentioned in the previous paragraphs, they grow as if they were finitely generated in degrees $\leq d$. This will be important for the two classes of examples that we study in detail, which we describe below.

1.2. **Homology of configuration spaces.** Given a graph G and a positive integer n, the *n-stranded unordered configuration space of G* is the topological space[1]

$$\mathrm{UConf}_n(G) := \big\{(x_1, \ldots, x_n) \in G^n \;\big|\; x_i \neq x_j\big\}/S_n.$$

The homology groups of these spaces have been extensively studied in settings both theoretical [1, 2, 9] and applied [8].

One powerful technique for studying these groups, which is applied for example in [2], is to fix the graph G and consider the direct sum of the homology groups of $\mathrm{UConf}_n(G)$ for all n. This direct sum is a module over a polynomial ring with generators indexed by the edges of G, where a variable acts by "adding a point" to the corresponding edge. An orthogonal approach is to fix n and vary G. This approach has been used in a number of recent works [10, 14, 18, 19], and it is the approach that we take here. In particular, the homology of $\mathrm{UConf}_n(G)$ is functorial with respect to contractions (Section 5.2), and therefore defines an object of $\mathrm{Rep}_{\mathbb{Z}}(\mathcal{G}_g^{\mathrm{op}})$.

[1]Later we give a formal combinatorial definition of a graph, but in the introduction we don't distinguish between a graph and its topological realization.

Theorem 1.2. *Fix natural numbers g, i, and n. The $\mathcal{G}_g^{\mathrm{op}}$-module*

$$G \mapsto H_i\big(\mathrm{UConf}_n(G);\mathbb{Z}\big)$$

is $(g+i+n)$-small. In particular, it is finitely generated.

One concrete consequence of Theorems 1.1 and 1.2 is that we obtain some control of the type of torsion that can appear in these homology groups. We know from the work of Ko and Park that the only torsion that can appear in $H_1\big(\mathrm{UConf}_n(G);\mathbb{Z}\big)$ is 2-torsion [9, Corollary 3.6]. Furthermore, this torsion carries extremely interesting information: it is trivial if and only if G is planar! The topological meaning of torsion in higher degree homology is more mysterious, but we can at least show that there is a bound on the type of torsion that can occur.

Corollary 1.3. *For any triple (g,i,n) of positive integers, there exists a constant $d_{g,i,n}$ such that for every graph G of genus g, the torsion part of $H_i\big(\mathrm{UConf}_n(G);\mathbb{Z}\big)$ has exponent at most $d_{g,i,n}$.*

Remark 1.4. In this work we only consider *unordered* configurations of points, mainly because the tools we use largely derive from the paper [2] and this is the setting in which they work. It is likely that one can obtain analogues of Theorem 1.2 and Corollary 1.3 for *ordered* configuration spaces, starting by reproving certain results from [2] in the ordered setting.

1.3. **Kazhdan–Lusztig coefficients.** Kazhdan–Lusztig polynomials of matroids are analogues of Kazhdan–Lusztig polynomials of Coxeter groups. Just as Kazhdan–Lusztig polynomials of Weyl groups can be interpreted as Poincaré polynomials of certain intersection homology groups, the same is true of Kazhdan–Lusztig polynomials of graphical (or, more generally, realizable) matroids. See [13] for a survey that explores this analogy in depth.

More precisely, given a graph G, we can define a complex variety X_G, called the *reciprocal plane*, with the property that the coefficient of t^i in the Kazhdan–Lusztig polynomial of G is equal to the dimension of $\mathit{IH}_{2i}(X_G)$. These homology groups are functorial with respect to contractions [16], thus we obtain an object of $\mathrm{Rep}_{\mathbb{C}}(\mathcal{G}_g^{\mathrm{op}})$.

Theorem 1.5. *Fix a natural number g and a positive integer i. The $\mathcal{G}_g^{\mathrm{op}}$-module*

$$G \mapsto \mathit{IH}_{2i}(X_G)$$

is $(2i-1+g)$-smallish. In particular, it is finitely generated.

For example, Theorem 1.5 combines with the results on subdivision described in Section 1.1 to imply that the i^{th} Kazhdan–Lusztig coefficient of the matroid associated with the n-cycle is a polynomial in n of degree at most i. Indeed, the formulas for these coefficients appearing in [15] demonstrate that this bound is sharp (Example 6.4).

1.4. **Outer automorphisms of free groups.** A further motivation for studying the category $\mathcal{G}_g$ and its modules is that this category is closely related to $\mathrm{Out}(F_g)$, the outer automorphism group of a free group on g generators. This group is in many ways analogous to various arithmetic groups and to mapping class groups of surfaces, and much work has gone into exploring its cohomology; see Vogtmann's ICM address [24] for a survey.

We call a graph G of genus $g \geq 2$ *reduced* if it has no bridges and no vertices of valence 2. If we consider the full subcategory of $\mathcal{G}_g$ consisting of reduced graphs and replace it with an equivalent small category, we obtain a category whose nerve is a classifying space for $\mathrm{Out}(F_g)$ (Corollary 7.5). This observation leads to Theorem 1.6.

Theorem 1.6. *Fix a non-negative integer g and a commutative ring k. Let $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ be the module that assigns k to every reduced graph and 0 to every non-reduced graph, with all nontrivial transition functions equal to the identity. Then there is a canonical k-algebra isomorphism*

$$\mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})}(M, M) \;\cong\; H^*(\mathrm{Out}(F_g); k).$$

Our proof of Theorem 1.6 relies on the very non-trivial theorem of Culler and Vogtmann that outer space is contractible [5]. Since $\mathrm{Out}(F_g)$ acts on outer space with finite stabilizers, the rational cohomology of the quotient coincides with the rational cohomology of $\mathrm{Out}(F_g)$. We stress, however, that Theorem 1.6 holds for arbitrary coefficients.

1.5. **Relationship to other work.** This paper generalizes the authors' previous paper [14], in which we prove Theorems 1.1 and 1.2 for the category $\mathcal{G}_0$ of trees. The proof of Theorem 1.1 takes the argument used in [14] as a starting point and builds on this argument in order to treat graphs of higher genus. While the idea of applying the techniques of [20] is the same, there is a significant additional layer of technical difficulty in the higher genus setting.

Once we have established Theorem 1.1, the proof of Theorem 1.2 for arbitrary genus is nearly identical to the proof in the genus 0 case. Theorem 1.5 has no direct analogue in the genus 0 setting because Kazhdan–Lusztig polynomials of trees are trivial. The same goes for Theorem 1.6 because $\mathrm{Out}(F_0)$ is the trivial group.

In 2020, Miyata and the authors announced a proof of the *categorical graph minor theorem*, which is the analogue of Theorem 1.1 for the category $\mathcal{G}$ whose objects are connected graphs of arbitrary genus and whose morphisms are built out of contractions and edge deletions. The category $\mathcal{G}_g$ is the full subcategory of $\mathcal{G}$ consisting of graphs of genus g, and Theorem 1.1 would follow from this result. Unfortunately, that preprint contained a critical mistake, and the categorical graph minor theorem remains a conjecture.

2. Graph categories

We begin by fixing terminology and conventions about graphs and trees and defining all of the various categories of decorated graphs with which we will work in this paper. The reader may want to skim this section at first and refer back to it as needed.

2.1. **Graphs.** A *directed graph* is a quadruple (V, A, h, t), where V and A are finite sets (*vertices* and *arrows*), and h and t are maps from A to V (*head* and *tail*). We will always assume that our directed graphs are nonempty. A *graph* is a quintuple (V, A, h, t, σ), where (V, A, h, t) is a directed graph and σ is a free involution on A such that $h \circ \sigma = t$. An orbit of σ is called an *edge*. A graph is *connected* if it is possible to move from any vertex to any other vertex via a sequence of adjacent vertices. The *genus* of a connected graph is the number of edges minus the number of vertices plus one. A *tree* is a connected graph of genus 0. We will write $|G|$ to denote the number of edges of G.

Given a pair of graphs $G = (V, A, h, t, \sigma)$ and $G' = (V', A', h', t', \sigma')$, a *weak contraction* $\varphi : G \to G'$ is given by a map

$$\hat{\varphi} : V \sqcup A \to V' \sqcup A'$$

satisfying the following properties:

- If $\hat{\varphi}(a) \in A'$, then $h' \circ \hat{\varphi}(a) = \hat{\varphi} \circ h(a)$, $t' \circ \hat{\varphi}(a) = \hat{\varphi} \circ t(a)$, and $\sigma' \circ \hat{\varphi}(a) = \hat{\varphi} \circ \sigma(a)$.
- The preimage of every arrow arrow is a single arrow.
- The preimage of every vertex is a nonempty connected subgraph.

Note that the third condition implies that a maps to a vertex if and only if $\sigma(a)$ maps to a vertex; the edges that map to vertices are called *contracted edges*. If $F \subset G$ is a subgraph, we write G/F to denote the graph obtained from G by weakly contracting each component of F.

For any graph $G = (V, A, h, t, \sigma)$, we define the *topological realization* $\mathrm{Top}(G)$ by first taking the CW-complex with 0-cells given by V, 1-cells given by A, and attaching maps given by h and t, and then taking the quotient by the action of σ on this CW-complex. We note that a weak contraction $\varphi : G \to G'$ induces a map $\mathrm{Top}(\varphi) : \mathrm{Top}(G) \to \mathrm{Top}(G')$, making Top a functor from the category of graphs with weak contractions to the category of topological spaces.

We define a *contraction* $\varphi : G \to G'$ to be a weak contraction with the additional property that the preimage of every vertex is a tree. If $\varphi : G \to G'$ is a contraction and G is a graph of genus g, then so is G'. We denote by $\mathcal{G}_g$ the category whose objects are connected graphs of genus g and whose morphisms are contractions. Note that the weak contraction from G to G/F is a contraction if and only if F is a forest (that is, each component of F is a tree).

2.2. **Trees.** The definitions in Sections 2.2 and 2.3 will be used only in Section 3, where we prove Theorem 1.1. A *rooted tree* is a pair consisting of a tree and a vertex, which is called the *root*. The vertex set of a rooted tree is equipped with a natural partial order in which $v \leq w$ if and only if the unique directed path from v to the root passes through w (so the root is maximal). A *leaf* of a rooted tree is a minimal vertex with respect to this partial order.

For any vertex v, we define a *direct descendant* of v to be a vertex covered by v in the partial order. A *planar rooted tree* is a rooted tree along with a linear order on the set of direct descendants of each vertex v. This induces a depth-first linear order on the entire vertex set of the tree. A *contraction of rooted trees* is a contraction of trees that preserves the root, and a *contraction of planar rooted trees* is a contraction of rooted trees with the additional property that, if v comes before w in the depth-first order, then the first vertex in the preimage of v comes before the first vertex in the preimage of w. Let $\mathcal{RT}$ and $\mathcal{PT}$ be the contraction categories of rooted trees and planar rooted trees, respectively.

Remark 2.1. Barter [3] defines the category **RT** whose objects are rooted trees and whose morphisms are pointed order embeddings on vertex sets (embeddings compatible with the partial order), along with the category **PT** whose objects are planar rooted trees and whose morphisms are pointed order embeddings that preserve the depth-first linear order. In [14, Proposition 2.4], we prove that **RT** is equivalent to $\mathcal{RT}^{\mathrm{op}}$, and a similar argument shows that **PT** is equivalent to $\mathcal{PT}^{\mathrm{op}}$. We will make use of Barter's work, via these equivalences, in Section 3.

Finally, we will need a labeled version of the above definitions. Let S be a finite set. We define an *S-labeled planar rooted tree* to be a triple (T, v, ℓ), where (T, v) is a planar rooted tree and ℓ is a function from the set of vertices of T to S. The most naive way to define a contraction $\varphi : (T, v, \ell) \to (T', v', \ell')$ of labeled planar rooted trees would be to say that it is a contraction of planar rooted trees with the property that the pullback of ℓ' along φ is equal to ℓ. This, however, is not quite what we want. If $\varphi : (T, v) \to (T', v')$ is a contraction of planar rooted trees and $\varphi^* : (T', v') \to (T, v)$ is the corresponding pointed order embedding under the equivalence of Remark 2.1, we want to impose the condition that the pullback of ℓ along φ^* is equal to ℓ'. The proof of [14, Proposition 2.4] tells us that $\varphi^*(w') = \max \varphi^{-1}(w')$, so the appropriate condition for $\varphi : (T, v, \ell) \to (T', v', \ell')$ to be an S-labeled contraction is that $\ell'(w') = \ell(\max \varphi^{-1}(w'))$ for all $w' \in T'$. Equivalently, we say that a vertex w of T is *φ-maximal* if $u \leq w$ for all vertices u with $\varphi(u) = \varphi(w)$, and we say that φ is an S-labeled contraction if and only if $\ell' \circ \varphi(w) = \ell(w)$ for all φ-maximal vertices w.

2.3. **Rigidified graphs.** Given a nonempty connected graph G, a *spanning tree* for G is a subgraph which contains all of the vertices and is a tree. A *rigidified graph* of genus g is a graph of genus g along with a choice of spanning tree and an ordering and orientation (i.e. distinguished arrow) of each of the g *extra edges* that are not in the spanning tree. Equivalently, fix once and for all a graph R_g with one vertex and g loops, called the *rose of genus g*. Then a rigidified graph of genus g is a quadruple (G, T, v, τ), where G is a graph of genus g, (T, v) is a planar rooted spanning tree of G, and $\tau : G \to R_g$ is a contraction whose contracted edges are precisely the edges of T.

We denote by $\mathcal{PG}_g$ the category whose objects are rigidified graphs of genus g and whose morphisms are contractions that restrict to contractions of planar rooted trees (in particular, only edges in the spanning tree can be contracted) and are compatible with the order and orientations of the extra edges. We use the letter P in the notation because $\mathcal{PG}_0 \cong \mathcal{PT}$. The point of this definition is that rigid graphs are graphs with just enough extra structure to eliminate all nontrivial automorphisms.

2.4. **Reduced graphs.** Most of the definitions in Sections 2.4 and 2.5 will be used only in Section 7, where we discuss connections to outer automorphism groups of free groups. Fix a graph G. An edge of G is called a *bridge* if deleting the edge increases the number of connected components. For any vertex v, the *valence* of v is defined to be the number of arrows a with $h(a) = v$. We call a graph *reduced* if it has no bridges and no vertices of valence 2. We also consider the unique graph with one vertex and one edge to be reduced, even though the vertex has valence 2. Intuitively, the idea is that any connected graph may be obtained from a reduced graph by subdividing edges and "uncontracting" bridges, and there are finitely many isomorphism classes of reduced graphs of any fixed genus. For example, there are two reduced graphs of genus 2 up to isomorphism, namely the rose $R_2 = \infty$ and the melon $\oplus$.

Remark 2.2. If G is reduced and $\varphi : G \to G'$ is a contraction, then G' is also reduced. For example, all contractions with domain equal to the melon are either automorphisms or maps to the rose, and all contractions with domain equal to the rose are automorphisms.

We define $\mathcal{G}_{g,\mathrm{red}}$ to be the full subcategory of $\mathcal{G}_g$ whose objects are reduced graphs. In the next section, we will want to talk about the nerve of this category, but one can only define the nerve of a small category. For this reason, we choose a list $G_1, \dots, G_r$ that includes a unique representative of each isomorphism class of reduced graphs of genus g, and we let $\mathcal{G}^{\mathrm{small}}_{g,\mathrm{red}}$ be the full subcategory of $\mathcal{G}_{g,\mathrm{red}}$ with objects $G_1, \dots, G_r$. Thus $\mathcal{G}^{\mathrm{small}}_{g,\mathrm{red}}$ is a small category that is equivalent to $\mathcal{G}_{g,\mathrm{red}}$.

2.5. **Marked reduced graphs.** If G is a graph of genus g, a *marking* of G is an equivalence class of contractions from G to the rose R_g, where $\psi : G \to R_g$ is equivalent to $\psi' : G \to R_g$ if and only if the induced maps $\mathrm{Top}(\psi)$ and $\mathrm{Top}(\psi')$ on topological realizations are homotopy equivalent. We will denote the equivalence class of ψ by $[\psi]$. A *marked graph* of genus g is a pair $(G, [\psi])$, where G is a graph of genus g and $[\psi]$ is a marking of G. We observe that the set of all markings of G is a torsor for $\mathrm{Out}(F_g)$. A *contraction* from $(G, [\psi])$ to $(G', [\psi'])$ is a contraction $\varphi : G \to G'$ such that $[\psi] = [\psi' \circ \varphi]$. We define *outer category* $\mathcal{O}_g$ to be the category whose objects are marked reduced graphs of genus g and whose morphisms are contractions. The group $\mathrm{Out}(F_g)$ acts on $\mathcal{O}_g$ in a natural way, fixing the graph but changing the marking.

As in Section 2.4, we would like to define a small subcategory of $\mathcal{O}_g$ that is equivalent to $\mathcal{O}_g$. We will do this in two subtly different ways, which we now describe. Recall that we have chosen representatives $G_1, \dots, G_r$ of the isomorphism classes of reduced graphs of genus g. Let $\mathcal{O}^{\mathrm{small}}_g$ be the full subcategory of $\mathcal{O}_g$ consisting of objects of the form $(G_i, [\psi_i])$ for some i and any marking $[\psi_i]$ of G_i. Note that there are still isomorphisms between distinct objects of $\mathcal{O}^{\mathrm{small}}_g$. Specifically, if $[\psi]$ is a marking of G and $\varphi : G \to G$ is a nontrivial automorphism of G, then $[\psi] \neq [\psi \circ \varphi]$, but $\varphi : (G, [\psi]) \to (G, [\psi \circ \varphi])$ is an isomorphism. To eliminate this phenomenon, we choose for each G_i a representative of each $\mathrm{Aut}(G_i)$ orbit in the set of markings of G_i, and we define $\mathcal{O}^{\mathrm{tiny}}_g$ to be the subcategory of $\mathcal{O}_g$ generated by these objects. Note that the natural inclusions

$$\mathcal{O}^{\mathrm{tiny}}_g \subset \mathcal{O}^{\mathrm{small}}_g \subset \mathcal{O}_g$$

are both equivalences.

Example 2.3. There is only one reduced graph of genus 1 up to isomorphism, namely the cycle R_1. A marking of R_1 is the same as an orientation of the loop. The category $\mathcal{O}^{\mathrm{small}}_1$ has two objects, related by the action $\mathrm{Out}(F_1) \cong S_2$, corresponding to the two choices of marking of R_1. Neither object has nontrivial automorphisms. The category $\mathcal{O}^{\mathrm{tiny}}_1$ has only one object, and it has no nontrivial automorphisms. We discuss the nerves of these categories in Example 7.6.

The advantage of working with $\mathcal{O}^{\mathrm{small}}_g$ is that the action of $\mathrm{Out}(F_g)$ on $\mathcal{O}_g$ restricts to an action on $\mathcal{O}^{\mathrm{small}}_g$, where it acts freely on the set of objects. The advantage of working with $\mathcal{O}^{\mathrm{tiny}}_g$ is that it is a poset category in the following sense.

Proposition 2.4. *If $(G, [\psi])$ and $(G', [\psi'])$ are objects of $\mathcal{O}^{\mathrm{tiny}}_g$, then*

$$|\, \mathrm{Mor}_{\mathcal{O}^{\mathrm{tiny}}_g} \big((G, [\psi]), (G', [\psi'])\big)| \leq 1.$$

Furthermore, if there exists a morphism in both directions, then $(G', [\psi']) = (G, [\psi])$. In particular, the set of objects of $\mathcal{O}^{\mathrm{tiny}}_g$ admits a poset structure with $(G', [\psi']) \leq (G, [\psi])$ if and only if there exists a morphism from $(G, [\psi])$ to $(G', [\psi'])$.

Proof. If $g \leq 1$, the proposition is trivial, so we assume that $g \geq 2$. We begin by proving the proposition when $G = G'$. In this case, the proposition says that, if ρ is an automorphism of G and $\mathrm{Top}(\rho)$ is homotopic to the identity, then ρ must in fact be equal to the identity. This is proved in [26, Lemma 1].

Next we consider the case where $G \neq G'$. Suppose that $[\psi']$ is a marking of G' and $\varphi_1, \varphi_2 : G \to G'$ are contractions with $[\psi' \circ \varphi_1] = [\psi' \circ \varphi_2]$. Since $\mathrm{Top}(\psi')$ is a homotopy equivalence, this implies that $\mathrm{Top}(\varphi_1)$ is homotopic to $\mathrm{Top}(\varphi_2)$. By [21, Lemma 1.3], φ_1 and φ_2 differ by an automorphism ρ of G'. We know that $\mathrm{Top}(\rho)$ is homotopic to the identity, therefore ρ is equal to the identity by the previous paragraph. Thus $\varphi_1 = \varphi_2$, as desired. $\square$

3. Local Noetherianity

The purpose of this section is to prove Theorem 1.1, which says that $\mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is locally Noetherian for any Noetherian commutative ring k.

3.1. Gröbner theory of categories.

Let $\mathcal{C}$ be an essentially small category and x an object of C. We define $\mathcal{C}_x$ to be the set of equivalence classes of morphisms out of x, where $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ is equivalent to $g \in \mathrm{Mor}_{\mathcal{C}}(x, y')$ if there exists an isomorphism h from y to y' such that $h \circ f = g$. The set $\mathcal{C}_x$ comes equipped with a natural quasi-order defined by putting

$$f \leq g \iff \text{there exists a morphism } h \text{ with } h \circ f = g.$$

Note that it is possible to have $f \leq g$ and $g \leq f$ even if the targets of f and g are not isomorphic, hence $\leq$ is only a quasi-order. An infinite sequence $f_0, f_1, f_2, \ldots$ of a quasi-ordered set is called *bad* if there is no pair of indices $i < j$ such that $f_i \leq f_j$. The category $\mathcal{C}$ is said to satisfy property **(G2)** if, for every object x of $\mathcal{C}$, $\mathcal{C}_x$ admits no bad sequences. The category $\mathcal{C}$ is said to satisfy property **(G1)** if, for every object x of $\mathcal{C}$, $\mathcal{C}_x$ admits a linear order $\prec$ that is compatible with post-composition in the following sense: if $f, g \in \mathrm{Mor}_{\mathcal{C}}(x, y)$, $h \in \mathrm{Mor}_{\mathcal{C}}(y, z)$, and $f \prec g$, then $h \circ f \prec h \circ g$. The category $\mathcal{C}$ is called *Gröbner* if it satisfies properties (G1) and (G2) and has no endomorphisms other than the identity maps.

Remark 3.1. Sam and Snowden [20] explain that the motivation for properties (G1) and (G2) is deeply rooted in Gröbner basis theory from commutative algebra, with $\leq$ playing the role of the natural divisibility order on monomials and $\prec$ playing the role of a term order such as the lexicographic order.

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories and let $\Phi : \mathcal{C}' \to \mathcal{C}$ be a functor. We say that Φ satisfies property **(F)** if, for all objects x of $\mathcal{C}$, there exists a finite collection of objects $y_1, \ldots, y_r$ of $\mathcal{C}'$ and morphisms $f_i : x \to \Phi(y_i)$ such that, for any object y of $\mathcal{C}'$ and any morphism $f : x \to \Phi(y)$, there exists a morphism $g : y_i \to y$ with $f = \Phi(g) \circ f_i$. We say $\mathcal{C}$ is *quasi-Gröbner* if there exist a Gröbner category $\mathcal{C}$ and an essentially surjective functor $\Phi : \mathcal{C}' \to \mathcal{C}$ satisfying property (F).

The motivation for these definitions comes from Theorems 3.2 and 3.3, both of which are of fundamental importance in our work.

Theorem 3.2 ([20, Proposition 3.2.3])**.** *If $\Phi : \mathcal{C} \to \mathcal{C}'$ has property (F) and M is a finitely generated $\mathcal{C}'$-module, then $\Phi^* M$ is a finitely generated $\mathcal{C}$-module.*

Theorem 3.3 ([20, Theorem 1.1.3])**.** *If $\mathcal{C}$ is quasi-Gröbner and k is a Noetherian commutative ring, then $\mathrm{Rep}_k(\mathcal{C})$ is locally Noetherian.*

3.2. **The category of rigidified graphs of fixed genus is Gröbner.** We begin with the following translation of Barter's work to our setting.

Theorem 3.4. *The category $\mathcal{PT}^{\mathrm{op}} \cong \mathcal{PG}_0^{\mathrm{op}}$ of planar rooted trees is Gröbner.*

Proof. Barter proves that **PT** is Gröbner [3], and the same is true of $\mathcal{PT}^{\mathrm{op}}$ by Remark 2.1. $\square$

Our goal in this section is to extend Theorem 3.4 to the category $\mathcal{PG}_g^{\mathrm{op}}$ for arbitrary genus g. We begin with Corollary 3.5 of Theorem 3.4.

Corollary 3.5. *For any natural number g, the category $\mathcal{PG}_g^{\mathrm{op}}$ satisfies property (G1).*

Proof. Fix a rigidified graph (G, T, v, τ) of genus g. We need to define a linear order on $(\mathcal{PG}_g^{\mathrm{op}})_{(G,T,v,\tau)}$ that is compatible with post-composition. Let $\Phi : \mathcal{PG}_g \to \mathcal{PT}$ denote the forgetful functor, which induces a poset map

$$\Phi_{(G,T,v,\tau)} : (\mathcal{PG}_g^{\mathrm{op}})_{(G,T,v,\tau)} \to (\mathcal{PT}^{\mathrm{op}})_{(T,v)}.$$

A key property of this map is that, if

$$\varphi_1, \varphi_2 : (G', T', v', \tau') \to (G, T, v, \tau)$$

represent different classes in $(\mathcal{PG}_g^{\mathrm{op}})_{(G,T,v,\tau)}$, then their classes remain different in $(\mathcal{PT}^{\mathrm{op}})_{(T,v)}$.

Choose any linear order $\prec_G$ on $(\mathcal{PG}_g^{\mathrm{op}})_{(G,T,v,\tau)}$ with the property that $\Phi_{(G,T,v,\tau)}$ is (weakly) order-preserving. In order to establish property (G1), assume that φ_1 and φ_2 are as above with $[\varphi_1] \prec_G [\varphi_2]$, and that

$$\psi : (G'', T'', v'', \tau'') \to (G', T', v', \tau')$$

is any contraction. Since $\Phi_{(G,T,v,\tau)}$ is order-preserving,

$$\Phi_{(G,T,v,\tau)}[\varphi_1] \preceq_T \Phi_{(G,T,v,\tau)}[\varphi_2],$$

and by the observation in the previous paragraph, it must be a strict inequality. By definition of $\prec_T$, this implies that $\Phi_{(G,T,v,\tau)}[\varphi_1 \circ \psi] \prec_T \Phi_{(G,T,v,\tau)}[\varphi_2 \circ \psi]$, and therefore that $[\varphi_1 \circ \psi] \prec_G [\varphi_2 \circ \psi]$. $\square$

Our next task is to prove that $\mathcal{PG}_g^{\mathrm{op}}$ satisfies property (G2). We begin by stating a version of Kruskal's tree theorem for labeled planar rooted trees. Let S be a finite set. If (T, v, ℓ) and (T', v', ℓ') are S-labeled planar rooted trees, we define $(T', v', \ell') \leq (T, v, \ell)$ if there exists a contraction from (T, v, ℓ) to (T', v', ℓ'). This defines a quasi-order on the set of isomorphism classes of S-labeled planar rooted trees.

Theorem 3.6. *Let S be a finite set. The quasi-order on the set of isomorphism classes of S-labeled planar rooted trees admits no bad sequences.*

Proof. After using Remark 2.1 to translate between order embeddings and contractions, the case where S is a singleton is proved in [3, Lemma 10]. On the other hand, the theorem is proved for general S, but with rooted trees instead of planar rooted trees, in [6, Theorem 1.2]. Both proofs are essentially the same, and are in fact modeled on the original proof of Nash-Williams for unlabeled rooted trees [11]. These arguments can be trivially modified to cover the result stated above. $\square$

Corollary 3.7 is a relative version of Theorem 3.6. The case where S is a singleton is proved in [3, Theorem 9]. However, it turns out that the proof is greatly simplified by allowing labels, as we demonstrate below.

Corollary 3.7. *Let S be a finite set and let (T, v, ℓ) be an S-labeled planar rooted tree. The set $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$ admits no bad sequences.*

Proof. An element of $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$ is represented by a pair consisting of an S-labeled planar rooted tree (T', v', ℓ') and a contraction $\varphi' : (T', v', \ell') \to (T, v, \ell)$. Let V be the vertex set of T, let $U := S \times \big(V \sqcup \{0\}\big)$, and define a U-labeled planar rooted tree (T', v', ℓ'_U) by putting

$$\ell'_U(w') := \begin{cases} (\ell'(w'), \varphi'(w')) & \text{if } w' \text{ is } \varphi'\text{-maximal} \\ (\ell'(w'), 0) & \text{otherwise.} \end{cases}$$

Suppose that $\varphi' : (T', v', \ell') \to (T, v, \ell)$ and $\varphi'' : (T'', v'', \ell'') \to (T, v, \ell)$ represent two elements of $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$ and let (T', v', ℓ'_U) and (T'', v'', ℓ''_U) be the corresponding U-labeled planar rooted trees. We have $\varphi' \leq \varphi''$ with respect to the quasi-order on $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$ if and only if there exists an S-labeled contraction $\psi : (T'', v'', \ell'') \to (T', v', \ell')$ such that $\varphi'' = \varphi' \circ \psi$. On the other hand, we have $(T', v', \ell'_U) \leq (T'', v'', \ell''_U)$ with respect to the quasi-order on isomorphism classes of U-labeled planar rooted trees if and only if there exists a U-labeled contraction $\psi : (T'', v'', \ell''_U) \to (T', v', \ell'_U)$.

We claim that an S-labeled contraction ψ is a U-labeled contraction if and only if $\varphi'' = \varphi' \circ \psi$. The easiest way to see this is to use Remark 2.1 to translate from contractions to pointed order embeddings, as the statement becomes tautological in that setting. This implies that any bad sequence in $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$ induces a bad sequence of isomorphism classes of U-labeled planar rooted trees, and Theorem 3.6 tells us that no such sequences exist. $\square$

Let $S = \{0, 1\}^{2g}$. Given a ridigified graph (G, T, v, τ) of genus g, we construct an S-labeled planar rooted tree (T, v, ℓ) as follows. Recall that τ induces an ordering and an orientation on the g extra edges of G. For each $1 \leq i \leq g$, let w_{2i-1} be the vertex at which the i^{th} extra edge originates and let w_{2i} be the vertex at which the i^{th} extra edge terminates. Then for each vertex w and each $1 \leq j \leq 2g$, define the j^{th} component of $\ell(w)$ to be 1 if $w \geq w_j$ and 0 otherwise.

Lemma 3.8. *Let (G, T, v, τ) and (G', T', v', τ') be rigidified graphs of genus g, and let (T, v, ℓ) and (T', v', ℓ') be the associated S-labeled planar rooted trees. Let $\varphi : (T, v) \to (T', v')$ be a contraction of planar rooted trees. Then φ induces a contraction of rigidified graphs if and only if it is compatible with the S-labeling.*

Proof. On one hand, φ induces a contraction of rigidified graphs if and only if $\varphi(w_j) = w'_j$ for all j. On the other hand, φ is compatible with the S-labeling if and only if, for all φ-maximal vertices w, $w \geq w_j \iff \varphi(w) \geq w'_j$.

Assume first that φ induces a contraction of rigidified graphs, and let w be a φ-maximal vertex. If $w \geq w_j$, then $\varphi(w) \geq \varphi(w_j) = w'_j$. Conversely, if $\varphi(w) \geq w'_j$, then w lies above that unique φ-maximal preimage of w'_j, which in turn lies above w_j.

Assume next that φ is compatible with the S-labeling. For any j, we want to show that $\varphi(w_j) = w'_j$. To begin, let u_j be the unique φ-maximal preimage of w'_j.

Then

$$\varphi(u_j) = w'_j \Rightarrow u_j \geq w_j \Rightarrow w'_j = \varphi(u_j) \geq \varphi(w_j).$$

Assume for the sake of contradiction that $\varphi(w_j) \neq w'_j$, and let z_j be the φ-maximal preimage of $\varphi(w_j)$. Then we have that the label of z_j agrees with that of $\varphi(w_j)$, which must be 0 in the j-th coordinate by our assumption. But this would then force z_j to not be bigger than w_j. This directly contradicts maximality of z_j in the preimage of $\varphi(w_j)$. □

Corollary 3.9. *For any natural number g, the category $\mathcal{PG}_g^{\mathrm{op}}$ satisfies property (G2).*

Proof. Fix a rigidified graph (G, T, v, τ) of genus g, and let (T, v, ℓ) be its associated S-labeled planar rooted tree. We need to prove that the set $(\mathcal{PG}_g^{\mathrm{op}})_{(G,T,v,\tau)}$ admits no bad sequences. By Lemma 3.8, such a bad sequence induces a bad sequence in $(\mathcal{PT}_S^{\mathrm{op}})_{(T,v,\ell)}$, and Corollary 3.7 says that no such sequences exist. □

We are now ready to prove the main result of this section.

Theorem 3.10. *For any $g \geq 0$, the category $\mathcal{PG}_g^{\mathrm{op}}$ is Gröbner.*

Proof. This follows from Corollaries 3.5 and 3.9, along with the fact that rigidified graphs have no nontrivial automorphisms. □

3.3. **The category of graphs of fixed genus is quasi-Gröbner.**

Lemma 3.11. *The forgetful functor $\Phi : \mathcal{PG}_g^{\mathrm{op}} \to \mathcal{G}_g^{\mathrm{op}}$ is essentially surjective and has property (F).*

Proof. Essential surjectivity is clear. For any genus g graph G, we need to choose a finite collection of genus g rigidified graphs (G_i, T_i, v_i, τ_i) along with contractions $\varphi_i : G_i \to G$ such that, for every genus g rigidified graph (G', T', v', τ') and every contraction $\varphi : G' \to G$, there exist an index i and a contraction $\psi : (G', T', v', \tau') \to (G_i, T_i, v_i, \tau_i)$ such that $\varphi = \varphi_i \circ \psi$.

For our rigidified graphs (G_i, T_i, v_i, τ_i) and our contractions φ_i, we will choose a representative of every possible isomorphism class of such structures whose number of edges is at most $|G| + g$. Since there is a finite number of rigidified graphs with a fixed number of edges and finitely many contractions between any two graphs, there are only finitely many such choices.

Let (G', T', v', τ') and φ be given, and let E' be the set of edges of G' that are contracted by φ. Let ψ be the canonical contraction from (G', T', v', τ') to $(G'/(E' \cap T'), T'/(E' \cap T'), v', \tau')$. It is clear from the definition that φ factors through ψ. It thus remains only to show that the number of edges of $G'/(E' \cap T')$ is at most $|G| + g$. Indeed, we have $|E'| = |G'| - |G|$ and $|T'| = |G'| - g$, thus $|E' \cap T'| \geq |G'| - (|G| + g)$. From this it follows that $|G'/(E' \cap T')| = |G'| - |E' \cap T'| \leq |G| + g$. □

Proof of Theorem 1.1. By Theorem 3.10 and Lemma 3.11, $\mathcal{G}_g^{\mathrm{op}}$ is quasi-Gröbner. Theorem 1.1 then follows from Theorem 3.3. □

4. Smallness and growth

We define what it means for a module over $\mathcal{G}_g^{\mathrm{op}}$ to be generated in low degree, and see what this tells us about its dimension growth.

4.1. **Generation degree, smallness, and smallishness.** Fix a Noetherian commutative ring k. For any genus g graph G, let $P_G \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ be the *principal projective* module that assigns to a genus g graph G' the free k-module with basis $\mathrm{Mor}_{\mathcal{G}_g}(G', G)$. Note that a module M is finitely generated if and only if it is isomorphic to a quotient of a finite sum of principal projectives. We say that a module $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is *finitely generated in degree* $\leq d$ if we only need to use principal projectives corresponding to graphs with d or fewer edges. Lemma 4.1 illustrates this notion in a specific example.

Lemma 4.1. *Let E be the $\mathcal{G}_g^{\mathrm{op}}$-module that takes a graph G to the free k-module with basis indexed by edges of G, with maps given by the natural inclusions of bases. The $\mathcal{G}_g^{\mathrm{op}}$-module $E^{\otimes i}$ is generated in degrees $\leq g + i$.*

Proof. For any graph G of genus g, $E^{\otimes i}(G)$ has a basis given by an ordered i-tuple of edges, and any such basis element is in the image of the map induced by a contraction $\varphi : G \to G'$ if and only if none of the distinguished edges are contracted by φ. If G has more than $g + i$ edges, then it has more than i edges that are not loops, therefore for any given i-tuple, one can find a non-distinguished edge to contract. $\square$

We say that a module M is *d-small* if it is a subquotient of a module that is finitely generated in degrees $\leq d$. We say that M is *d-smallish* if it admits a filtration whose associated graded is d-small.

Proposition 4.2. *If M is d-smallish for some d, then M is finitely generated.*

Proof. Choose a filtration of M such that the associated graded $\mathrm{gr}\, M$ is d-small. Theorem 1.1 implies that $\mathrm{gr}\, M$ is finitely generated. This means that there is a finite collection $G_1, \ldots, G_r$ of genus g graphs, along with elements $v_i \in \mathrm{gr}\, M(G_i)$, such that, for any genus g graph G, the natural map

$$\bigoplus_{i=1}^{r} \bigoplus_{\varphi: G \to G_i} k \cdot e_{i,\varphi} \to \mathrm{gr}\, M(G)$$

taking $e_{i,\varphi}$ to $\varphi^* v_i$ is surjective. For each i, choose an arbitrary lift $\tilde{v}_i \in M(G_i)$ of v_i. The natural map

$$\bigoplus_{i=1}^{r} \bigoplus_{\varphi: G \to G_i} k \cdot e_{i,\varphi} \to M(G)$$

taking $e_{i,\varphi}$ to $\varphi^* \tilde{v}_i$ is also surjective, which means that M is finitely generated. $\square$

Proposition 4.3. *Let k be a field, and suppose that $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is d-smallish. Then there exists a polynomial $f_M(t) \in \mathbb{Z}[t]$ of degree at most d such that, for all G, $\dim_k M(G) \leq f_M(|G|)$.*

Proof. We may immediately reduce to the case where M is the principal projective $P_{G'}$ for some genus g graph G' with d edges. For any G, a contraction from G to G' is determined, up to automorphisms of G', by a choice of $|G| - d$ edges of G to contract. The number of such choices is $\binom{|G|}{d}$, so $\dim_k P_{G'}(G) \leq |\mathrm{Aut}(G')|\binom{|G|}{d}$. $\square$

4.2. **Subdivision.** Fix a graph G of genus g, a natural number r, and an ordered r-tuple $\underline{e} = (e_1, \ldots, e_r)$ of distinct directed non-loop edges of G. For any ordered r-tuple $\underline{m} = (m_1, \ldots, m_r)$ of natural numbers, let $G(\underline{e}, \underline{m})$ be the graph obtained from G by subdividing each edge e_i into m_i edges. The number m_i is allowed to be zero, and we adopt the convention that subdividing e_i into 0 edges means contracting e_i. For each i, the graph $G(\underline{e}, \underline{m})$ has a directed path of length m_i, where the directed edge e_i used to be, and we label the vertices of that path $v_i^0, \ldots, v_i^{m_i}$.

Let OI be the category whose objects are linearly ordered finite sets and whose morphisms are ordered inclusions. Every object of OI is isomorphic via a unique isomorphism to the finite set $[m]$ for some $m \in \mathbb{N}$. For any $\underline{m} \in \mathbb{N}^r$, let $[\underline{m}]$ denote the corresponding object of the product category OI^r.

Our goal in this section is to define a *subdivision functor* $\Phi_{G,\underline{e}} : \mathrm{OI}^r \to \mathcal{G}_g^{\mathrm{op}}$ and prove that $\Phi_{G,\underline{e}}$ has property (F). We define our functor on objects by putting $\Phi_{G,\underline{e}}([\underline{m}]) := G(\underline{e}, \underline{m})$. Let $\underline{f} = (f_1, \ldots, f_r)$ be a morphism in OI^r from $[\underline{m}]$ to $[\underline{n}]$. We define the corresponding contraction

$$\Phi_{G,\underline{e}}(\underline{f}) : G(\underline{e}, \underline{n}) \to G(\underline{e}, \underline{m})$$

by sending v_i^t to v_i^s, where s is the maximal element of the set $\{0\} \cup \{j \mid f_i(j) \leq t\} \subset \{0, 1, \ldots, m_i\}$.

For any $\underline{n} \in \mathbb{N}^r$, let $|\underline{n}| := \sum n_i$. We say that a contraction $\varphi : G(\underline{e}, \underline{n}) \to G'$ *factors nontrivially* if there exist a non-identity morphism $\underline{f} : [\underline{m}] \to [\underline{n}]$ in OI^r and a contraction $\psi : G(\underline{e}, \underline{m}) \to G'$ such that $\varphi = \psi \circ \Phi_{G,\underline{e}}(\underline{f})$.

Proposition 4.4. *The subdivision functor $\Phi_{G,\underline{e}} : \mathrm{OI}^r \to \mathcal{G}_g^{\mathrm{op}}$ has property (F).*

Proof. Property (F) says exactly that, for any graph G' of genus g, the set of contractions from some $G(\underline{e}, \underline{m})$ to G' that do not factor nontrivially is finite. Let $\varphi : G(\underline{e}, \underline{m}) \to G'$ be given. We have

$$|G(\underline{e}, \underline{m})| = |G| + |\underline{m}| - r,$$

so φ must contract $|G| + |\underline{m}| - r - |G'|$ edges. If $|\underline{m}|$ is sufficiently large, then at least one of those edges must be one of the subdivided edges. We may then factor φ nontrivially by first contracting that edge.

This tells us that, if we are looking for contractions from some $G(\underline{e}, \underline{m})$ to G' that do not factor nontrivially, we only need to consider finitely many r-tuples $\underline{m}$. The proposition then follows from the fact that all Hom sets in $\mathcal{G}_g^{\mathrm{op}}$ are finite. $\square$

Proposition 4.3 implies that, if $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is d-smallish, the dimension of $M(G(\underline{e}, \underline{m}))$ is bounded by a polynomial in $\underline{m}$ of degree at most d. Corollary 4.5 to Proposition 4.4 says that the dimension of $M(G(\underline{e}, \underline{m}))$ is in fact equal to a polynomial in $\underline{m}$ when each coordinate is sufficiently large.

Corollary 4.5. *Let k be a field, and suppose that $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is d-smallish. Then there exists a multivariate polynomial $f_{M,G,\underline{e}}(t_1, \ldots, t_r)$ of total degree at most d such that, if $\underline{m}$ is sufficiently large in every coordinate,*

$$\dim_k M(G(\underline{e}, \underline{m})) = f_{M,G,\underline{e}}(m_1, \ldots, m_r).$$

Proof. Proposition 4.2 tells us that M is finitely generated, though we have no control over the degree of generation. Theorem 3.2 and Proposition 4.4 combine to tell us that $\Phi_{G,\underline{e}}^* M$ is a finitely generated OI^r-module. By [20, Theorem 6.3.2, Proposition 6.3.3, and Theorem 7.1.2], this implies that there exists a multivariate polynomial $f_{M,G,\underline{e}}(t_1,\ldots,t_r)$ such that, if $\underline{m} \in \mathbb{N}^r$ is sufficiently large in every coordinate,

$$\dim_k M(G(\underline{e},\underline{m})) = \dim_k \Phi^*_{G,\underline{e}} M([\underline{m}]) = f_{M,G,\underline{e}}(m_1,\ldots,m_r).$$

Proposition 4.3 says that $\dim_k M(G(\underline{e},\underline{m}))$ is bounded above by a polynomial of degree d in the quantity $|G(\underline{e},\underline{m})| = |G| - r + |\underline{m}|$, thus the total degree of $f_{M,G,\underline{e}}(t_1,\ldots,t_r)$ can be at most d. $\square$

4.3. **Sprouting.** Fix a graph G of genus g, a natural number r, and an ordered r-tuple $\underline{v} := (v_1,\ldots,v_r)$ of distinct vertices of T. For any ordered r-tuple $\underline{m} = (m_1,\ldots,m_r)$ of natural numbers, let $G(\underline{v},\underline{m})$ be the tree obtained from G by attaching m_i new edges to the vertex v_i, each of which has a new leaf as its other endpoint. We will label the new leaves connected to the vertex v_i by the symbols $v_i^1,\ldots,v_i^{m_i}$.

Our goal in this section is to define a *sprouting functor* $\Psi_{G,\underline{v}} : \mathrm{OI}^r \to \mathcal{G}_g^{\mathrm{op}}$ and prove that $\Psi_{G,\underline{v}}$ has property (F). We define our functor on objects by putting $\Psi_{G,\underline{e}}([\underline{m}]) := G(\underline{v},\underline{m})$. Let $\underline{f} = (f_1,\ldots,f_r)$ be a morphism in OI^r from $[\underline{m}]$ to $[\underline{n}]$. We define the corresponding contraction

$$\Psi_{G,\underline{v}}(\underline{f}) : T(\underline{v},\underline{n}) \to T(\underline{v},\underline{m})$$

by fixing all of the vertices of T, sending v_i^t to v_i^s if $f_i(s) = t$, and sending v_i^t to v_i if t is not in the image of f_i.

As in Section 4.2, we say that a contraction $\varphi : G(\underline{v},\underline{n}) \to G'$ *factors nontrivially* if there exist a non-identity morphism $\underline{f} : [\underline{m}] \to [\underline{n}]$ in OI^r and a contraction $\psi : G(\underline{v},\underline{m}) \to G'$ such that $\varphi = \psi \circ \Psi_{T,\underline{v}}(\underline{f})$.

Proposition 4.6. *The sprouting functor* $\Phi_{G,\underline{v}} : \mathrm{OI}^r \to \mathcal{G}_g^{\mathrm{op}}$ *has property (F).*

Proof. The philosophy of the proof is nearly identical to that of Proposition 4.4. Property (F) says exactly that, for any graph G' of genus g, the set of contractions from some $G(\underline{v},\underline{m})$ to G' that do not factor nontrivially is finite. Let $\psi : G(\underline{v},\underline{m}) \to G'$ be given. We have

$$|G(\underline{v},\underline{m})| = |G| + |\underline{m}|,$$

so ψ must contract $|G| + |\underline{m}| - |G'|$ edges. If $|\underline{m}|$ is sufficiently large, then at least one of those edges must be one of the newly sprouted edges. We may then factor ψ nontrivially by first contracting that edge.

This tells us that, if we are looking for contractions from some $G(\underline{e},\underline{m})$ to G' that do not factor nontrivially, we only need to consider finitely many r-tuples $\underline{m}$. The proposition then follows from the fact that all Hom sets in $\mathcal{G}_g^{\mathrm{op}}$ are finite. $\square$

The proof of Corollary 4.7 is identical to the proof of Corollary 4.5, so we omit it.

Corollary 4.7. *Let k be a field, and suppose that $M \in \mathrm{Rep}_k(\mathcal{G}_g^{\mathrm{op}})$ is d-smallish. Then there exists a multivariate polynomial $f_{M,G,\underline{v}}(t_1,\ldots,t_r)$ of total degree at most d such that, if $\underline{m}$ is sufficiently large in every coordinate,*

$$\dim_k M(G(\underline{v},\underline{m})) = f_{M,G,\underline{v}}(m_1,\ldots,m_r).$$

4.4. **Combining small modules.** This section is devoted to stating and proving a lemma that we will need in Section 6.3.

Let $H = (V, A, h, t, \sigma)$ be a graph of genus h with no loops. For each vertex $v \in V$, fix a natural number g_v and a $\mathcal{G}_{g_v}^{\mathrm{op}}$-module N_v. Let $g := h + \sum_v g_v$. Consider the $\mathcal{G}_g^{\mathrm{op}}$-module N defined by putting

$$N(G) := \bigoplus_{\psi : G \to H} \bigotimes_{v \in V} N_v\big(\psi^{-1}(v)\big),$$

where the sum is over all weak contractions $\psi : G \to H$ with the property that $\psi^{-1}(v)$ has genus g_v for all v. If $\varphi : G \to G'$ is a contraction, the induced map $N(G') \to N(G)$ kills the ψ summand unless all of the edges contracted by φ are also contracted by ψ. If this is the case, then there is an induced weak contraction $\psi' : G' \to H$ whose fibers are contractions of the fibers of ψ, and these contractions induce a natural map from the ψ summand of $N(G)$ to the ψ' summand of $N(G')$.

Lemma 4.8. *In the above situation, suppose that N_v is d_v-small for all $v \in V$, and let $d := |H| + \sum_v d_v$. Then the $\mathcal{G}_g^{\mathrm{op}}$-module N is d-small.*

Proof. We may immediately reduce to the case where each N_v is a principal projective. That is, for all $v \in V$, we have $N_v = P_{G_v}$ for some fixed graph G_v of genus g_v with d_v edges. The k-module $N(G)$ is spanned by classes indexed by tuples of the form

$$(\psi; \varphi_v)_{v \in V},$$

where $\psi : G \to H$ is a weak contraction with the property that $\psi^{-1}(v)$ has genus g_v for all v and $\varphi_v : \psi^{-1}(v) \to G_v$ is a contraction. The set of edges of G that are *not* contracted by any of the maps φ_v has cardinality d and includes all of the loops. Thus, if G has more than d edges, at least one edge e is a non-loop that gets contracted by one of the maps φ_v. It follows that the class may be pulled back from the corresponding class in $N(G/e)$. □

5. Homology of configuration spaces

The purpose of this section is to prove Theorem 1.2. Our main technical tool is the reduced Świątkowski complex of An, Drummond-Cole, and Knudsen [2]. Sections 5.1 and 5.2 are reproduced from [14, Sections 3.1 and 3.2] for the reader's convenience.

5.1. **The reduced Świątkowski complex.** Let $G = (V, A, h, t, \sigma)$ be a graph, and let $\mathcal{R}_G$ be the integral polynomial ring generated by the edges of G. For any vertex v, let

$$A_v := \{a \in A \mid h(a) = v\},$$

and let $S(v)$ denote the free $\mathcal{R}_G$-module generated by the set $A_v \sqcup \{\emptyset\}$. We equip $S(v)$ with a bigrading by defining an element of A_v to have degree $(1,1)$, $\emptyset$ to have degree $(0,0)$, and an edge to have degree $(0,1)$. Let $\widetilde{S}(v) \subset S(v)$ be the submodule generated by the elements $\emptyset$ and $a - a'$ for all $a, a' \in A_v$. We equip $\widetilde{S}(v)$ with an $\mathcal{R}_G$-linear differential ∂_v of degree $(-1, 0)$ by putting

$$\partial\emptyset = 0 \qquad \text{and} \qquad \partial(a - a') := \big([a] - [a']\big),$$

where $[a]$ denotes the edge $\{a, \sigma(a)\}$. We then define the *reduced Świątkowski complex*

$$\widetilde{S}(G) := \bigotimes_{v \in V} \widetilde{S}(v),$$

where the tensor product is taken over the ring $\mathcal{R}_G$; this is a bigraded free $\mathcal{R}_G$-module with a differential ∂.

For any graph G and any natural number n, let $\mathrm{UConf}_n(G)$ denote the configuration space of n distinct unordered points in $\mathrm{Top}(G)$. Let $H_\bullet\big(\mathrm{UConf}_\star(G)\big)$ denote the bigraded abelian group

$$H_\bullet\big(\mathrm{UConf}_\star(G)\big) := \bigoplus_{(i,n)} H_i\big(\mathrm{UConf}_n(G); \mathbb{Z}\big).$$

Theorem 5.1 ([2, Theorem 4.5 and Proposition 4.9])**.** *If G has no isolated vertices, then there is an isomorphism of bigraded abelian groups*

$$H_\bullet\big(\mathrm{UConf}_\star(G)\big) \cong H_\bullet\big(\widetilde{S}(G)\big).$$

Remark 5.2. If G is connected, then the only way that G can have isolated vertices is if G has one vertex and no edges. In this case, $H_\bullet\big(\widetilde{S}(G)\big) = \widetilde{S}(G) = \mathbb{Z}$, concentrated in bidegree $(0,0)$, whereas $H_\bullet\big(\mathrm{UConf}_\star(G)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$, concentrated in bidegrees $(0,0)$ and $(0,1)$. Thus the reduced Świątkowski complex fails only to recognize that the degree zero homology of $\mathrm{UConf}_1(G)$ is nontrivial.

5.2. **Functoriality.** Suppose that we have two graphs $G = (V, A, h, t, \sigma)$ and $G' = (V', A', h', t', \sigma')$ along with a contraction $\varphi : G \to G'$. There is a natural map of differential bigraded modules

$$\widetilde{\varphi}^* : \widetilde{S}(G') \to \widetilde{S}(G),$$

which induces a map

$$\varphi^* : H_i\big(\mathrm{UConf}_n(G'); \mathbb{Z}\big) \to H_i\big(\mathrm{UConf}_n(G); \mathbb{Z}\big)$$

by passing to homology [2, Lemma C.7]. To describe $\widetilde{\varphi}^*$, we first consider the case where the number of edges of G is one greater than the number of edges of G'; we call such a contraction φ a *simple contraction*. Let $e = \{a_0, a_1\}$ denote the unique contracted edge, and write

$$0 = h(a_0) = t(a_1) \qquad \text{and} \qquad 1 = t(a_0) = h(a_1).$$

Let

$$w' := \hat{\varphi}(e) = \hat{\varphi}(0) = \hat{\varphi}(1) \in V'.$$

We have a canonical ring homomorphism $\mathcal{R}_{G'} \to \mathcal{R}_G$ along with an $\mathcal{R}_{G'}$-module homomorphism

$$\bigotimes_{v' \in V' \smallsetminus \{w'\}} \widetilde{S}(v') \quad \to \bigotimes_{v \in V \smallsetminus \{0,1\}} \widetilde{S}(v).$$

Given $a' \in A'_{w'}$, let $a \in A_0 \sqcup A_1$ be the unique arrow mapping to a'. We then define an $\mathcal{R}_{G'}$-module homomorphism

$$\widetilde{S}(w') \to \widetilde{S}(0) \otimes \widetilde{S}(1)$$

by the formula

$$\emptyset \mapsto \emptyset \otimes \emptyset \qquad \text{and} \qquad a' \mapsto \begin{cases} (a - a_0) \otimes \emptyset & \text{if } h(a) = 0 \\ \emptyset \otimes (a - a_1) & \text{if } h(a) = 1. \end{cases}$$

Tensoring these two maps together, we obtain the homomorphism $\widetilde{\varphi}^* : \widetilde{S}(G') \to \widetilde{S}(G)$, and it is straightforward to check that this homomorphism respects the differential. Arbitrary contractions may be obtained as compositions of simple contractions, and the induced homomorphism is independent of choice of factorization into simple contractions. To summarize, we have the following result.

Theorem 5.3. [2] *There is a bigraded differential $\mathcal{G}_g^{\mathrm{op}}$-module that assigns to each graph G the reduced Świątkowski complex $\widetilde{S}(G)$. The homology of this bigraded differential $\mathcal{G}_g^{\mathrm{op}}$-module is the bigraded $\mathcal{G}_g^{\mathrm{op}}$-module that assigns to each graph G the bigraded Abelian group $H_\bullet\big(\mathrm{UConf}_\star(G)\big)$.*

5.3. **Smallness.** We are now ready to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. Given a graph G and a pair of natural numbers i and n, let $\widetilde{S}(G)_{i,n}$ be the degree (i,n) summand of the reduced Świątkowski complex. We will show that the $\mathcal{G}_g^{\mathrm{op}}$-module taking G to the abelian group $\widetilde{S}(G)_{i,n}$ is generated in degrees $\leq g+i+n$. Theorem 5.3 says that the $\mathcal{G}_g^{\mathrm{op}}$-module taking G to $H_i\big(\mathrm{UConf}_n(G)\big)$ is a subquotient of this module, so this will imply that it is $(g+i+n)$-small.

The group $\widetilde{S}(G)_{i,n}$ is generated by elements of the form

$$\zeta := e_1 \cdots e_{n-i} \bigotimes_{j=1}^{i} (a_{j0} - a_{j1}) \otimes \bigotimes_{v \notin \{v_1,\ldots,v_i\}} \emptyset,$$

where $e_1,\ldots,e_{n-i}$ are edges (not necessarily distinct), $v_1,\ldots,v_i$ are vertices (distinct), and, for each j, $a_{j0}, a_{j1} \in A_{v_j}$. For a particular ζ of this form, we will call $\{v_1,\ldots,v_i\}$ the set of *distinguished vertices*. Without loss of generality, we may assume that there is some integer r with $0 \leq r \leq i$ such that v_j is adjacent to some distinguished vertex (possibly itself) if and only if $j \leq r$. We may also assume that, if $j \leq r$, $t(a_{j1})$ is distinguished. If not, then ζ may be written as a difference of classes of this form.

We call an edge e a *distinguished edge* if one of the following five conditions hold:

- e is a loop
- e connects two distinguished vertices
- $e = e_k$ for some $k \leq n-i$
- $e = [a_{j0}]$ for some $j \leq i$
- $e = [a_{j1}]$ for some $j \leq i$.

We will now argue that there are at most $g+i+n$ distinguished edges. Let l be the number of loops that are not at distinguished vertices. Let H be the induced subgraph on $\{v_1,\ldots,v_r\}$, which in particular contains all of the loops that are at distinguished vertices. Since H is a subgraph of G and is missing l loops, the sum of the genera of the components of H is at most $g-l$, and therefore H has at most $r+g-l$ edges. (Equality is achieved if and only if $r=0$ and G is obtained by attaching g loops to a tree, in which case H is empty and $l=g$.) This means that the total number of distinguished edges is at most

$$l + (r+g-l) + (n-i) + i + (i-r) = g+i+n.$$

Let G be given with $|G| > g+i+n$. Since there are at most $g+i+n$ distinguished edges, we may choose an edge e which is not distinguished. Let $G' := G/e$ be the graph obtained from G by contracting e, and let $\varphi : G \to G'$ be the canonical simple contraction. Let e'_k be the image of e_k in G', v'_j the image of v_j in G', a'_{j0} the image of a_{j0} in G', and a'_{j1} the image of a_{j1} in G'. Let

$$\zeta' := e'_1 \cdots e'_{n-i} \bigotimes_{j=1}^{i} (a'_{j0} - a'_{j1}) \quad \otimes \bigotimes_{v' \notin \{v'_1,\dots,v'_i\}} \emptyset \quad \in \quad \widetilde{S}(G')_{i,n}.$$

We claim that $\zeta = \widetilde{\varphi}^* \zeta'$.

If e is not incident to any vertex v_j, this is clear. The interesting case occurs when e is incident to one of the distinguished vertices. Assume without loss of generality that it is incident to v_1, and let w be the other end point of e. Consider the unique $a \in A_{v_1}$ with $[a] = e$ (this uniquely characterizes a because e is not a loop). Applying the map φ^* replaces each e'_k with e_k. When $j > 1$, it replaces a'_{j0} with a_{j0} and a'_{j1} with a_{j1}. It replaces a'_{10} with $a_{10} - a$ and a'_{11} with $a_{11} - a$. This means that it replaces $a'_{j0} - a'_{j1}$ with $a_{j0} - a_{j1}$, and therefore that $\widetilde{\varphi}^* \zeta' = \zeta$.

We thus conclude that every element of $\widetilde{S}(G)_{i,n}$ is a linear combination of elements in the images of map associated with simple contractions; this completes the proof. □

Proof of Corollary 1.3. Let $T_{g,i,n} \in \mathrm{Rep}_{\mathbb{Z}}(\mathcal{G}_g^{\mathrm{op}})$ be the module that assigns to each graph G the torsion subgroup of $H_i\big(\mathrm{UConf}_n(G); \mathbb{Z}\big)$. By Theorem 1.2, $T_{g,i,n}$ is a submodule of a finitely generated module, and is therefore itself finitely generated. We may then take $d_{g,i,n}$ to be the least common multiple of the exponents of the generators. □

6. Kazhdan–Lusztig coefficients

For each graph $G = (V, A, h, t, \sigma)$, let R_G be the $\mathbb{C}$-subalgebra of rational functions in the variables $\{x_v \mid v \in V\}$ generated by the elements

$$\left\{ \frac{1}{x_v - x_w} \;\middle|\; v \neq w \text{ adjacent} \right\},$$

and let $X_G := \operatorname{Spec} R_G$. The ring R_G is called the *Orlik-Terao algebra* of G and the variety X_G is called the *reciprocal plane* of G. We will be interested in the intersection homology group $IH_{2i}(X_G)$ with coefficients in the complex numbers.

If $\varphi : G \to G'$ is a contraction, we obtain a canonical map from $IH_{2i}(X_{G'})$ to $IH_{2i}(X_G)$, and these maps compose in the expected way [16, Theorem 3.3(1,3)]. The purpose of this section is to study the $\mathcal{G}^{\mathrm{op}}$-module IH_{2i} that takes G to $IH_{2i}(X_G)$, and in particular to prove Theorem 1.5.

6.1. **Orlik-Solomon algebras.**

For each G, let $OS^\bullet(G)$ be the *Orlik-Solomon algebra* [12] of the matroid associated with G, with coefficients in the complex numbers. For any natural number i, we will denote the linear dual of $OS^i(G)$ by $OS_i(G)$. For the purposes of this paper, we will need to know four things about the Orlik-Solomon algebra:

- $OS^1(G)$ is spanned by classes x_e indexed by the edges of G, with relations $x_e = x_f$ if e and f are parallel and $x_e = 0$ if e is a loop.
- $OS^\bullet(G)$ is generated as a $\mathbb{C}$-algebra by $OS^1(G)$.
- If G' is a contraction of G, we obtain a functorial map $OS^\bullet(G) \to OS^\bullet(G')$ by killing the generators indexed by contracted edges. This in turn induces a map $OS_\bullet(G') \to OS_\bullet(G)$.

- If G is the disjoint union of G_1 and G_2, then $OS^\bullet(G) \cong OS^\bullet(G_1) \otimes OS^\bullet(G_2)$.

By the third bullet point above, OS_i is a $\mathcal{G}_g^{\mathrm{op}}$-module for any natural number i.

Lemma 6.1. *For any natural number i, OS_i is $(g+i)$-small.*

Proof. Recall from Lemma 4.1 the $\mathcal{G}_g^{\mathrm{op}}$-module E that assigns to any graph the $\mathbb{C}$-vector space with basis given by the edges. By the first two bullet points above, $OS^i(G)$ is a quotient of the i^{th} tensor power of $E(G)^*$, therefore OS_i is a submodule of $E^{\otimes i}$. Lemma 4.1 says that $E^{\otimes i}$ is generated in degrees $\leq g+i$, therefore OS_i is $(g+i)$-small. $\square$

6.2. **The spectral sequence.** A subgraph $F \subset G$ with the same vertex set is called a *flat* of G if the graph G/F has no loops. The *rank* of F is defined as the number of vertices minus the number of connected components, and the *corank* of F, denoted $\operatorname{crk} F$, is the number of connected components minus 1. Theorem 6.2 was proved in [16, Theorems 3.1 and 3.3]; see also [14, Theorem 4.2].

Theorem 6.2. *For any graph G and natural number i, there is a first quadrant homological spectral sequence $E(-,i)$ in the category of $\mathcal{G}_g^{\mathrm{op}}$-modules converging to IH_{2i}, with*

$$E(G,i)^1_{p,q} = \bigoplus_{\operatorname{crk} F = p} OS_{2i-p-q}(F) \otimes IH_{2(i-q)}(X_{G/F}).$$

If $\varphi : G \to G'$ is a contraction, the induced map $E(G',i)^1_{p,q} \to E(G,i)^1_{p,q}$ kills the F-summand unless F contains all of the edges contracted by φ. In this case, the image of F in G' is a flat F' of G', and G'/F' is canonically isomorphic to G/F. The map takes the F-summand of $E(G,i)^1_{p,q}$ to the F'-summand of $E(G',i)^1_{p,q}$ by the canonical map $OS_{2i-p-q}(F) \to OS_{2i-p-q}(F')$ tensored with the identity map on $IH_{2(i-q)}(X_{G/F})$.

6.3. **Smallness.**

Proof of Theorem 1.5. By Theorem 6.2, IH_{2i} admits a filtration whose associated graded is isomorphic to the infinity page of $E(-,i)$, therefore it is sufficient to show that, for all p and q, $E(-,i)^1_{p,q}$ is $(2i-1+g)$-small.

The set of flats of G is in bijection with equivalence classes of weak contractions with source G for which the target has no loops, where two such weak contractions are equivalent if they differ by an automorphism of the target. We therefore have

$$E(G,i)^1_{p,q} \cong$$
$$\bigoplus_{|\operatorname{Vert}(H)|=p+1} \left(\bigoplus_{\substack{\psi: G \to H \\ \text{weak}}} \left(\bigotimes_{v \in \operatorname{Vert}(H)} OS_*\big(\psi^{-1}(v)\big) \right)_{2i-p-q} \bigotimes IH_{2(i-q)}(X_H) \right)^{\operatorname{Aut}(H)}.$$

If we fix H and require that the graph $\psi^{-1}(v)$ has genus g_v, Lemmas 4.8 and 6.1 together imply that $E(-,i)^1_{p,q}$ is d-small, where $d = |H| + 2i - p - q + \sum_v g_v$. If h is the genus of H, then $|H| = p + h$ and $\sum_v g_v = g - h$, so $d = 2i + g - q$. Since this is independent of the choice of H or of the numbers g_v, we can conclude that $E(-,i)^1_{p,q}$ is $(2i+g-q)$-small.

Finally, we note that $IH_{2(i-q)}(X_H) = 0$ unless $2(i-q) < p$ or $q = i$ and $p = 0$ [7, Proposition 3.4], while $OS_{2i-p-q}(F) = 0$ unless $p + q \leq 2i$. In particular $E(-,i)^1_{p,0} = 0$ for all p, which implies that each $E(-,i)^1_{p,q}$ is $(2i-1+g)$-small. $\square$

Remark 6.3. The module $IH_0 = E(-,0)^1_{0,0}$ is the constant module taking every graph to $\mathbb{C}$ and every morphism to the identity. This module is g-small rather than $(g-1)$-small, which is why we required that i be positive in the statement of Theorem 1.5. Indeed, one can see that the last sentence of the proof fails when $p = q = i = 0$.

Example 6.4. When $g = 1$, Theorem 1.5 and Corollary 4.5 combine to say that the i^{th} Kazhdan–Lusztig coefficient of the n-cycle should eventually agree with a polynomial in n of degree at most $2i$. In fact, it is equal to [15, Theorem 1.2(1)]

$$\frac{1}{i+1}\binom{n-i-2}{i}\binom{n}{i},$$

so our result is sharp.

Example 6.5. Let $G_g(a_1,\ldots,a_{g+1})$ be the genus $g > 0$ graph obtained by taking the graph with two vertices and $g+1$ edges between them and subdividing the i^{th} edge into a_i pieces. Theorem 1.5 and Corollary 4.5 say that the first Kazhdan–Lusztig coefficient of $G_g(a_1,\ldots,a_{g+1})$ should eventually agree with a multivariate polynomial of total degree at most $g+1$ in $a_1,\ldots,a_{g+1}$.

The first Kazhdan–Lusztig coefficient is equal to the number of corank 1 flats minus the number of rank 1 flats [7, Proposition 2.12]. If $a_i > 1$ for all i, this is equal to

$$\prod_{i=1}^{g+1} a_i + \sum_{i=1}^{g+1}\binom{a_i}{2} - \sum_{i=1}^{g+1} a_i.$$

Thus our result is again sharp.

7. Outer Category

The purpose of this section is to describe how one may use the category $\mathcal{G}_{g,\text{red}}$ to compute cohomology groups of $\text{Out}(F_g)$ with arbitrary coefficients.

7.1. **Nerves of categories.** We begin by briefly reviewing some facts about small categories and their nerves. Let $\mathcal{C}$ be a small category. Then we define the *nerve* $|\mathcal{C}|$ of $\mathcal{C}$ to be the geometric realization of the simplicial set defined as follows. The 0-simplices are in bijection with the objects of $\mathcal{C}$, while the i-simplices for $i > 0$ are in bijection with i-tuples of morphisms

$$(f_1,\ldots,f_i)$$

such that, for each $0 \le j \le i$, the codomain of f_{j+1} agrees with the domain of f_j. For each $i > 0$ and $1 \le j \le i+1$ the face map ∂_j is defined by

$$\partial_j(f_1,\ldots,f_i) = \begin{cases} (f_2,\ldots,f_i) & \text{if } j = 0 \\ (f_1,\ldots,f_{i-1}) & \text{if } j = i+1 \\ (f_1,\ldots,f_{j-2},f_{j-1}\circ f_j, f_{j+1},\ldots,f_i) & \text{otherwise.} \end{cases}$$

The degeneracy map σ_j is defined by

$$\sigma_j(f_0,\ldots,f_i) = (f_0,\ldots,f_{j-1},\text{id},f_j,\ldots,f_i),$$

where id is the identity map on the domain of f_{j-1} (or the codomain of f_j).

Remark 7.1. We immediately see that there is a canonical homeomorphism $|\mathcal{C}| \cong |\mathcal{C}^{\mathrm{op}}|$. A functor between two categories induces a map between their nerves, and an equivalence of categories induces a homotopy equivalence between the nerves.

Let k be a commutative ring, and let $\underline{k} \in \mathrm{Rep}_k(\mathcal{C})$ be the module that takes every object to the 1-dimensional vector space k and every morphism to the identity map. The following standard result can be found, for example, in [25, Theorem 5.3].

Theorem 7.2. *There is a canonical graded k-algebra isomorphism* $\mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{C})}(\underline{k},\underline{k}) \cong H^*(|\mathcal{C}|;k)$.

7.2. **Outer category and the cohomology of** $\mathrm{Out}(F_g)$**.** We begin with the following result, which relies heavily on Culler and Vogtmann's work on outer space [5].

Theorem 7.3. *The nerves* $|\mathcal{O}_g^{\mathrm{small}}|$ *and* $|\mathcal{O}_g^{\mathrm{tiny}}|$ *are contractible.*

Proof. The categories $\mathcal{O}_g^{\mathrm{small}}$ and $\mathcal{O}_g^{\mathrm{tiny}}$ are equivalent, therefore Remark 7.1 tells us that it is sufficient to prove that $|\mathcal{O}_g^{\mathrm{tiny}}|$ is contractible. By Proposition 2.4, $\mathcal{O}_g^{\mathrm{tiny}}$ is a poset category, which implies that $|\mathcal{O}_g^{\mathrm{tiny}}|$ is homeomorphic to the order complex of the poset structure on the set of objects. This order complex is called the *spine of outer space*, and it is known to be contractible [5, Corollary 6.1.2]. □

Recall that we have an action of the group $\mathrm{Out}(F_g)$ on the category $\mathcal{O}_g^{\mathrm{small}}$, which induces an action on the nerve. We also have a functor $\Phi : \mathcal{O}_g^{\mathrm{small}} \to \mathcal{G}_{g,\mathrm{red}}^{\mathrm{small}}$ given by forgetting the marking, and this functor induces a map $\Phi_* : |\mathcal{O}_g^{\mathrm{small}}| \to |\mathcal{G}_{g,\mathrm{red}}^{\mathrm{small}}|$ of nerves.

Proposition 7.4. *The action of* $\mathrm{Out}(F_g)$ *on* $|\mathcal{O}_g^{\mathrm{small}}|$ *is free and proper, and* $\Phi_* : |\mathcal{O}_g^{\mathrm{small}}| \to |\mathcal{G}_{g,\mathrm{red}}^{\mathrm{small}}|$ *is the quotient map.*

Proof. The fact that the action is free and proper follows from the fact that it is free on the set of objects (which correspond to 0-simplices) and each group element acts by a simplicial map. To see that Φ_* is the quotient map, we need to show that it is surjective and its fibers coincide with the orbits of $\mathrm{Out}(F_g)$. This follows from the fact that $\mathrm{Out}(F_g)$ acts transitively on the set of markings of a reduced graph of genus g. □

Corollary 7.5. *The nerve* $|\mathcal{G}_{g,\mathrm{red}}^{\mathrm{small}}|$ *is a classifying space for the group* $\mathrm{Out}(F_g)$.

Example 7.6. Let us consider the very simple case where $g = 1$, which we began discussing in Example 2.3. The category $\mathcal{O}_1^{\mathrm{tiny}}$ has only one object (an oriented loop) and no nontrivial morphisms, so its nerve is a point. The category $\mathcal{O}_1^{\mathrm{small}}$ has two objects, namely a loop with two different orientations, and these two objects are uniquely isomorphic. The nerve of $\mathcal{O}_1^{\mathrm{small}}$ is an infinite-dimensional sphere S^∞, and the group $\mathrm{Out}(F_1) \cong S_2$ acts via the antipodal map with quotient $\mathbb{R}P^\infty$. The category $\mathcal{G}_{1,\mathrm{red}}^{\mathrm{small}}$ has a single object with automorphism group S_2, so its nerve is homeomorphic to $\mathbb{R}P^\infty$, which is a classifying space for S_2.

Corollary 7.7. *For any commutative ring k, we have*

$$\mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{g,\mathrm{red}})}(\underline{k},\underline{k}) \cong H^*(\mathrm{Out}(F_g);k).$$

Proof. To compute $\mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{g,\mathrm{red}})}(\underline{k},\underline{k})$, we may replace $\mathcal{G}_{g,\mathrm{red}}$ with the equivalent category $\mathcal{G}^{\mathrm{small}}_{g,\mathrm{red}}$. The result then follows from Remark 7.1, Theorem 7.2, and Corollary 7.5. □

Proof of Theorem 1.6. Given a pair of modules

$$M \in \mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_g) \qquad \text{and} \qquad N \in \mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{g,\mathrm{red}}),$$

we will write $\overline{M}$ to denote the restriction of M to $\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{g,\mathrm{red}})$ and $N^!$ to denote the extension of N by zero to $\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_g)$. The functors $M \mapsto \overline{M}$ and $N \mapsto N^!$ are exact and the former is left adjoint to the latter, therefore

$$\mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_{g,\mathrm{red}})}(\overline{M}, N) \cong \mathrm{Ext}^*_{\mathrm{Rep}_k(\mathcal{G}^{\mathrm{op}}_g)}(M, N^!).$$

If we apply this fact with $M = \underline{k}^!$ and $N = \underline{k}$, we see that Theorem 1.6 is equivalent to Corollary 7.7. □

7.3. **A sample calculation.** We now use Corollary 7.7 to compute the first cohomology of $\mathrm{Out}(F_2) \cong \mathrm{GL}(2;\mathbb{Z})$ with coefficients in an arbitrary field k. In particular, we illustrate the extent to which the representation theory of finite groups (namely automorphism groups of graphs) can be used to aid our calculations.

As in Section 2.4, there are exactly two reduced graphs of genus 2 up to isomorphism, namely the rose ∞ and the melon $\varTheta$. The automorphism group of the rose is D_4, while the automorphism group of the melon is $S_3 \times S_2$. Let φ_1, φ_2, and φ_3 be the three contractions from the melon to the rose obtained by cyclically permuting the edges and then contracting the middle one. Up to post-composition by an automorphism of the rose, every contraction is of this form.

Let $P_\infty \in \mathrm{Rep}_k((\mathcal{G}^{\mathrm{small}}_{2,\mathrm{red}})^{\mathrm{op}})$ be the principal projective module corresponding to the rose, and consider the surjection $P_\infty \to \underline{k}$ that sends every basis element to 1. Let K be the kernel of this homomorphism. Applying the functor $\mathrm{Hom}(-,\underline{k})$ gives us the long exact sequence

$$0 \to \mathrm{Hom}(\underline{k},\underline{k}) \to \mathrm{Hom}(P_\infty,\underline{k}) \to \mathrm{Hom}(K,\underline{k}) \to \mathrm{Ext}^1(\underline{k},\underline{k}) \to \mathrm{Ext}^1(P_\infty,\underline{k}).$$

An element of $\mathrm{Hom}(P_\infty,\underline{k})$ is determined by its value on the identity morphism of ∞, which implies that the first map $\mathrm{Hom}(\underline{k},\underline{k}) \to \mathrm{Hom}(P_\infty,\underline{k})$ is an isomorphism. The fact that P_∞ is projective implies that $\mathrm{Ext}^1(P_\infty,\underline{k}) = 0$, thus $\mathrm{Hom}(K,\underline{k}) \to \mathrm{Ext}^1(\underline{k},\underline{k})$ must also be an isomorphism. We therefore want to compute $\mathrm{Hom}(K,\underline{k})$.

An element of $\mathrm{Hom}(K,\underline{k})$ is a pair[2]

$$(f,g) \in \mathrm{Hom}_{D_4}(K(\infty),k) \times \mathrm{Hom}_{S_3\times S_2}(K(\varTheta),k)$$

satisfying the condition that, if we pre-compose g with any of the three inclusions $K(\infty) \to K(\varTheta)$ induced by φ_1, φ_2, and φ_3, we obtain f.

Let's start by computing $\mathrm{Hom}_{S_3\times S_2}(K(\varTheta),k)$ and $\mathrm{Hom}_{D_4}(K(\infty),k)$. The group $S_3 \times S_2$ acts freely on the set of contractions from the melon to the rose with two orbits, which we will call the *untwisted contractions* and the *twisted contractions*. The untwisted contractions consist of the orbit that includes the three maps φ_i, and the twisted contractions consist of untwisted contractions followed by an automorphism of the rose that fixes one of the two loops and reverses the orientation of the other loop. We therefore have $P_\infty(\varTheta) \cong k[S_3 \times S_2] \oplus k[S_3 \times S_2]$ as representations

[2] Here we are using the symbol k to denote the 1-dimensional trivial representations of both D_4 and $S_3 \times S_2$.

of $S_3 \times S_2$. The space of homomorphisms from $P_\infty(\obar)$ to k is 2-dimensional, with a basis given by the homomorphisms that take the sum of the coefficients of the twisted or untwisted maps. Applying $\mathrm{Hom}_{S_3\times S_2}(-,k)$ to the short exact sequence $0 \to K(\obar) \to P_\infty(\obar) \to k \to 0$ and noting that $P_\infty(\obar)$ is a projective representation of $S_3 \times S_2$, we obtain the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{S_3\times S_2}(k,k) \to \mathrm{Hom}_{S_3\times S_2}(P_\infty(\obar),k) \to \mathrm{Hom}_{S_3\times S_2}(K(\obar),k) \\ &\to \mathrm{Ext}^1_{S_3\times S_2}(k,k) \to 0. \end{aligned}$$

Since the abelianization of $S_3 \times S_2$ is $S_2 \times S_2$, we have $\dim \mathrm{Ext}^1_{S_3\times S_2}(k,k) = 2$ if k has characteristic 2 and 0 otherwise. Hence $\dim \mathrm{Hom}_{S_3\times S_2}(K(\obar),k) = 3$ if k has characteristic 2 and 1 otherwise. A similar argument for the rose tells us that $\dim \mathrm{Hom}_{D_4}(K(\infty),k) = 2$ if k has characteristic 2 and 0 otherwise.

Let's find explicit bases for our Hom spaces. Let $h_1 : K(\obar) \to k$ be the homomorphism that adds the coefficients of the untwisted maps in $K(\obar) \subset P_\infty(\obar)$. This homomorphism is well defined and nonzero for any field k. Let $h_2 : K(\obar) \to k$ be the homomorphism that adds the coefficients of $C_3 \times S_2 \subset S_3 \times S_2$ for both the twisted and untwisted maps and let $h_3 : K(\obar) \to k$ be the homomorphism that adds the coefficients of $S_3 \times \{\mathrm{id}\} \subset S_3 \times S_2$ for both the twisted and untwisted maps. Each of these homomorphisms is well defined if and only if the characteristic of k is 2, in which case it is straightforward to check that $\{h_1, h_2, h_3\}$ is a basis for $\mathrm{Hom}_{S_3\times S_2}(K(\obar),k)$. Let $f_1 : K(\infty) \to k$ add the coefficients of the untwisted automorphisms of the rose (those generated by horizontal and vertical reflections), and let $f_2 : K(\infty) \to k$ add the coefficients of the automorphisms that keep the left loop on the left and the right loop on the right. Each of these homomorphisms is well defined if and only if the characteristic of k is 2, in which case it is straightforward to check that $\{f_1, f_2\}$ is a basis for $\mathrm{Hom}_{S_3\times S_2}(K(\obar),k)$.

Finally, we observe that h_1 restricts to f_1 and h_2 restricts to f_2 under all three inclusions of $K(\infty)$ into $K(\obar)$. On the other hand, the restriction of h_3 to $K(\infty)$ fails to be D_4-equivariant *and* depends on the choice of inclusion of $K(\infty)$ into $K(\obar)$. We therefore conclude that

$$\dim H^1(\mathrm{Out}(F_2);k) = \begin{cases} 2 & \text{if } \mathrm{char}(k) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 7.8. This result can also be obtained by working directly with a presentation for $\mathrm{Out}(F_2)$, such as the one in [23, Section 2.1]. This presentation can be used to compute the abelianization, and $H^1(\mathrm{Out}(F_2);k)$ is isomorphic to the vector space of group homomorphisms from the abelianization to k.

Acknowledgments

The authors would like to thank Melody Chan, Jim Davis, Dan Dugger, Steven Sam, Paul Seymour, Dev Sinha, and Karen Vogtmann for valuable conversations.

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