

DISTINGUISHED STRATA IN A REDUCTIVE GROUP

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Dedicated to the memory of Roger Carter (1934-2022)

ABSTRACT. The set of strata of a reductive group can be viewed as an enlargement of the set of unipotent classes. In this paper the notion of distinguished unipotent class is extended to this larger set. The strata of a Weyl group are also introduced and studied.

INTRODUCTION

0.1. Let G be a connected reductive algebraic group over \mathbf{C} , with Weyl group W . In [Lus15] we have defined a partition of G into finitely many strata. The set of strata of G is in natural bijection with the image of a map $cl(W) \rightarrow \text{Irr}(W)$ where $cl(W)$ is the set of conjugacy classes in W and $\text{Irr}(W)$ is a set of representatives for the isomorphism classes of irreducible representations of W over \mathbf{Q} . We define a partition of $cl(W)$ into subsets (called the *strata of $cl(W)$*): a stratum of $cl(W)$ is by definition a non-empty fibre of the map $cl(W) \rightarrow \text{Irr}(W)$. We define a partition of W into subsets (called the *strata of W*): a stratum of W is by definition the inverse image of a stratum of $cl(W)$ under the surjective map $W \rightarrow cl(W)$ which takes an element of W to its conjugacy class. The set of strata of W is in obvious bijection with the set of strata of $cl(W)$ which is in bijection with the set of strata of G .

In this paper we are interested in two problems:

- (i) How to parametrize the strata of $cl(W)$ (or W)?
- (ii) How to describe explicitly each individual stratum of $cl(W)$.

These problems are solved in [Lus15], but we would like to get a simpler and more direct approach. We shall reduce these problems to the same problems restricted to a much smaller set of cases.

The set of strata can be viewed as an enlargement of the set of unipotent classes of G (a unipotent class of G is contained in exactly one stratum); this enlargement is built from the sets of unipotent classes in groups like G but in all characteristics. According to [BC76], the classification of unipotent classes of G can be reduced to the classification of a smaller set of unipotent classes, the distinguished ones. We will show that, similarly, the classification of strata of G can be reduced to the classification of a much smaller set, that of distinguished strata. It turns out that the distinguished strata of G are indexed by a subset $CL_{dist}(W)$ of the set of elliptic conjugacy classes of W which may be called distinguished conjugacy classes. (It

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would be interesting to find a description of distinguished conjugacy classes which is purely in terms of W and is not case by case.) One of our results is that

(a) the set of strata of G (or W) has a simple parametrization in terms of the sets $CL_{dist}(W)$ where W is replaced by various parabolic subgroups of W (see Section 0.6(a)) to which “parabolic inclusion” (see 0.6) is applied.

Another result of this paper is that

(b) the strata of $cl(W)$ are precisely the connected components of an (oriented) graph with set of vertices $cl(W)$ (the graph structure on $cl(W)$ is defined in §2); the connected components of this graph are remarkably simple: they are products of Coxeter graphs of type A .

Our definition of the edges of the graph is by first defining (case by case) the edges for which one end is an elliptic conjugacy class and then applying “parabolic inclusion” to such elementary edges where W is replaced by various parabolic subgroups of W . It would be interesting to find a description of these elementary edges which is not case by case.

In §3 we show how

(c) $CL_{dist}(W)$ can be parametrized in terms of certain reflection subgroups of W .

Note that by combining (a), (b), (c) one can hope to understand the classification of conjugacy classes in W in different terms than those in the classification of Carter [Car72]. Namely, the subset $CL_{dist}(W)$ is classified by (c); next, the set of strata of $cl(W)$ is obtained from (a) by parabolic inclusion and finally the objects of $cl(W)$ should be described by their position in the graph (product of graphs of type A in (b)) associated to a stratum of $cl(W)$.

We note that the set $CL_{dist}(W)$ and (a), (b) above depend only on W as a Coxeter group (not on G); however, this is not so for (c).

0.2. Let $\mathcal{P} = \{2, 3, 5, \dots\}$ be the set of prime numbers. For any $r \in \mathcal{P}$ let $G^{(r)}$ be a connected reductive group over an algebraically closed field of characteristic r of the same type as G . We set $G^{(0)} = G$. For $r \in \{0\} \cup \mathcal{P}$ let $\mathcal{U}^{(r)}$ be the set of unipotent classes of $G^{(r)}$. By the Springer correspondence (extended in [Lus84] to small characteristic) there is a natural imbedding $\iota^{(r)} : \mathcal{U}^{(r)} \rightarrow \text{Irr}(W)$ whose image is denoted by $\mathbf{S}^{(r)}(W)$; it is known that $\mathbf{S}^{(0)}(W) \subset \mathbf{S}^{(r)}(W)$. Let

$$\mathbf{S}(W) = \cup_{r \in \{0\} \cup \mathcal{P}} \mathbf{S}^{(r)}(W) = \cup_{r \in \mathcal{P}} \mathbf{S}^{(r)}(W) \subset \text{Irr}(W).$$

In [Lus15] (where the notation $\mathcal{S}_2(W)$ is used instead of $\mathbf{S}(W)$) it is shown that $\mathbf{S}(W)$ depends only on W as a Coxeter group, not on the underlying root datum (but it is not clear whether $\mathbf{S}(W)$ makes sense for a finite non-crystallographic Coxeter group).

In [Lus15] we have defined for any $r \in \{0\} \cup \mathcal{P}$ a surjective map $\kappa^{(r)} : G^{(r)} \rightarrow \mathbf{S}(W)$ whose fibres are called the *strata* of $G^{(r)}$; each stratum is a union of conjugacy classes of the same dimension, independent of r and, according to [Car20], is locally closed in $G^{(r)}$. If $g \in G^{(r)}$ is unipotent, then $\kappa^{(r)}(g)$ is the same as the image of the conjugacy class of g under $\iota^{(r)}$. It follows that for any $E \in \mathbf{S}(W)$, there exists $r \in \{0\} \cup \mathcal{P}$ such that the stratum $(\kappa^{(r)})^{-1}(E)$ contains some unipotent element.

0.3. An element of $G^{(r)}$ is said to be distinguished if it is not contained in a Levi subgroup of a proper parabolic subgroup of $G^{(r)}$ (see [BC76]). Let $\mathcal{U}_{dist}^{(r)}$ be the set of unipotent classes in $G^{(r)}$ in which some/any element is distinguished. Let $\mathbf{S}_{dist}^{(r)}(W) = \iota^{(r)}(\mathcal{U}_{dist}^{(r)})$.

We say that $E \in \mathbf{S}(W)$ is distinguished if $E \in \cup_{r \in \{0\} \cup \mathcal{P}} \mathbf{S}_{dist}^{(r)}(W)$ or equivalently if there exists $r \in \{0\} \cup \mathcal{P}$ such that the stratum $(\kappa^{(r)})^{-1}(E)$ contains some distinguished unipotent element. Let $\mathbf{S}_{dist}(W)$ be the set of distinguished elements of $\mathbf{S}(W)$.

In §1 we will show:

(a) $E \in \mathbf{S}(W)$ is distinguished if and only if there exists $r \in \{0\} \cup \mathcal{P}$ such that the stratum $(\kappa^{(r)})^{-1}(E)$ contains some distinguished (not necessarily unipotent) element of $G^{(r)}$.

A stratum of $G^{(r)}$ (with $r \in \{0\} \cup \mathcal{P}$) is said to be distinguished if it is of the form $(\kappa^{(r)})^{-1}(E)$ where $E \in \mathbf{S}_{dist}(W)$ (such a stratum need not contain a distinguished unipotent element).

0.4. For $C \in cl(W)$ let $m(C)$ be the dimension of the 1-eigenspace of some/any $w \in C$ on the reflection representation of W . We shall write $\Phi : cl(W) \rightarrow \mathbf{S}(W)$ for what in [Lus15] is denoted by $\prime\Phi$ (a surjective map). In [Lus15] it is shown that

(a) for any $E \in \mathbf{S}(W)$ there is a unique $C_E \in \Phi^{-1}(E)$ which is as elliptic as possible, that is $m(C_E) \leq m(C)$ for any $C \in \Phi^{-1}(E)$;

thus $E \mapsto C_E$ is a cross section of the surjective map Φ . The following variant of (a) will be verified in §1.

(b) for any $E \in \mathbf{S}(W)$ there is a unique $C'_E \in \Phi^{-1}(E)$ which is as non-elliptic as possible, that is $m(C'_E) \geq m(C)$ for any $C \in \Phi^{-1}(E)$.

Let $CL(W)$ be the image of the map $E \mapsto C'_E$, $\mathbf{S}(W) \rightarrow cl(W)$. Note that Φ restricts to a bijection $CL(W) \xrightarrow{\sim} \mathbf{S}(W)$. Under this bijection, the subset $\mathbf{S}_{dist}(W)$ of $\mathbf{S}(W)$ corresponds to a subset $CL_{dist}(W)$ of $CL(W)$. The conjugacy classes of W contained in $CL_{dist}(W)$ are said to be distinguished. The following result will be proved in §1.

(c) Let $C \in CL(W)$. We have $C \in CL_{dist}(W)$ if and only if C is elliptic (that is, $m(C) = 0$).

0.5. For $r \in \{0\} \cup \mathcal{P}$ let $\psi^{(r)} : cl(W) \rightarrow \mathcal{U}^{(r)}$ be the surjective map defined in [Lus11a]. (In the case $r = 0$, an alternative definition of this map is given in [Yun21].) Let $\Phi^{(r)} : cl(W) \rightarrow \text{Irr}(W)$ be the composition of $\psi^{(r)}$ with $\iota^{(r)} : \mathcal{U}^{(r)} \rightarrow \text{Irr}(W)$. From the explicit description of $\psi^{(r)}$ in [Lus12] and the explicit description of $CL_{dist}(W)$ given in this paper we see that

(a) if $C \in CL_{dist}(W)$ then $\Phi^{(r)}(C)$ is independent of r . Hence, by the definition of Φ in [Lus15, 4.1], we have $\Phi(C) = \Phi^{(r)}(C)$ for all r .

0.6. Let $\{s_i; i \in I\}$ be the set of simple reflections of W . For $J \subset I$ let W_J be the subgroup of W generated by $\{s_i; i \in J\}$; this is the Weyl group of a Levi subgroup of a parabolic subgroup of G . Hence $CL(W_J)$ and its subset $CL_{dist}(W_J)$ are defined. For $C_1 \in cl(W_J)$ we define $\rho_J(C_1) \in cl(W)$ by the condition $C_1 \subset \rho_J(C_1)$. Now $C_1 \mapsto \rho_J(C_1)$ is an injective map $cl(W_J) \rightarrow cl(W)$. (We call it *parabolic inclusion*.) If $J \subset I, J' \subset I$, we say that J, J' are equivalent if $W_J, W_{J'}$ are conjugate under an

element of W . Let \mathcal{X} be a set of representatives for the equivalence classes of subsets $J \subset I$ for the equivalence relation above. The following result can be deduced from the explicit description of $CL(W)$ given in [Lus15] and that of $CL_{dist}(W)$ given in this paper.

$$(a) \quad CL(W) = \sqcup_{J \in \mathcal{X}} \rho_J(CL_{dist}(W_J)).$$

0.7. A bipartition is a sequence $\lambda_* = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in \mathbf{N} such that $\lambda_i = 0$ for i large and $\lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \dots, \lambda_2 \geq \lambda_4 \geq \lambda_6 \geq \dots$. We write $|\lambda_*| = \lambda_1 + \lambda_2 + \lambda_3 + \dots$. Let BP be the set of bipartitions.

Let T be the set of all $\lambda_* \in BP$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \dots$. Let R be the set of all $\lambda_* \in T$ such that λ_i is even for any i . Let P be the set of all $\lambda_* \in T$ such that $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \lambda_5 = \lambda_6, \dots$. For $\lambda_* \in T$ and $j > 0$ we set $\mu_j(\lambda_*) = \#\{k \geq 1; \lambda_k = j\}$. For $m \in \mathbf{N}$ let $BP^m = \{\lambda_* \in BP, |\lambda_*| = m\}$, $T^m = T \cap BP^m$. Let T_{ev} (resp. R_{ev}) be the subset of T (resp. R) consisting of the λ_* with an even number of > 0 terms. For a, b in \mathbf{Z} we write $a \gg b$ instead $a - b \geq 2$.

1. PROOF OF 0.3(A), 0.4(B), 0.4(C)

1.1. In this section we prove 0.3(a), 0.4(b), 0.4(c). To do this we can assume that G is almost simple. It is also enough to consider only one such G in each isogeny class. The case where G is of classical (resp. exceptional) type is considered in 1.2-1.9 (resp. 1.10-1.15).

1.2. Assume first that $G = SL_n(\mathbf{C}), n \geq 2$. In this case $\mathbf{S}^{(r)}(W) = \text{Irr}(W) = \mathbf{S}(W)$ for any r and the map Φ is a bijection $cl(W) \xrightarrow{\sim} \mathbf{S}(W)$. In this case 0.4(b) is obvious and we have $CL(W) = cl(W)$. Also 0.3(a) is immediate (an element is distinguished if and only if is regular unipotent times a central element). Note that $CL_{dist}(W)$ consists of a single element: the class of the Coxeter element; thus 0.4(c) holds.

1.3. Until the end of 1.5 we assume that $G = Sp_{2n}(\mathbf{C}), n \geq 2$. Then $\mathbf{S}(W) = \mathbf{S}^{(2)}(W)$ and Φ becomes a map $cl(W) \rightarrow \mathbf{S}^{(2)}(W)$. By [Lus12] we have bijections

$$(a) \quad \mathcal{U}^{(2)} \leftrightarrow (\text{set of all pairs } (c_*, \epsilon) \text{ where } c_* \in T^{2n} \text{ is such that } \mu_j(c_*) \in 2\mathbf{N} \text{ for any odd } j \text{ and } \epsilon : \{j \in 2\mathbf{N} + 2; \mu_j(c_*) \in 2\mathbf{N} + 2\} \rightarrow \{0, 1\}),$$

$$(b) \quad cl(W) \leftrightarrow (R \times P)^{2n} := \{(r_*, p_*) \in R \times P; |r_*| + |p_*| = 2n\}.$$

Via (a), (b), $(\iota^{(2)})^{-1}\Phi : cl(W) \rightarrow \mathcal{U}^{(2)}$ becomes the map

$$(c) \quad (r_*, p_*) \mapsto (c_*, \epsilon) \text{ where } \mu_j(c_*) = \mu_j(r_*) + \mu_j(p_*) \text{ for } j > 0 \text{ and for any } j \in 2\mathbf{N} + 2 \text{ such that } \mu_j(c_*) \in 2\mathbf{N} + 2, \text{ we have } \epsilon(j) = 1 \text{ if } j = r_i \text{ for some } i \text{ and } \epsilon(j) = 0, \text{ otherwise.}$$

Via (b), the map $cl(W) \rightarrow \mathbf{N}, C \mapsto m(C)$ becomes the map which to any $(r_*, p_*) \in (R \times P)^{2n}$ associates $|p_*|/2$. To prove 0.4(b) we must show that, if (c_*, ϵ) (as in (a)) is given, then there is a unique $(r_*, p_*) \in (R \times P)^{2n}$ which maps to it (as in (c)) and has $|p_*|$ maximum possible. Thus,

for $j \in 2\mathbf{N} + 2$ such that $\mu_j(c_*) \in 2\mathbf{N} + 2$, we must have that $\mu_j(r_*) = 2, \mu_j(p_*) = \mu_j(c_*) - 2$ (if $\epsilon(j) = 1$) and $\mu_j(r_*) = 0, \mu_j(p_*) = \mu_j(c_*)$ (if $\epsilon(j) = 0$); for $j \in 2\mathbf{N} + 2$ such that $\mu_j(c_*) \in 2\mathbf{N} + 1$, we must have that $\mu_j(r_*) = 1, \mu_j(p_*) = \mu_j(c_*) - 1$; for $j \in 2\mathbf{N} + 2$ such that $\mu_j(c_*) = 0$, we must have that $\mu_j(r_*) = 0, \mu_j(p_*) = 0$; for $j \in 2\mathbf{N} + 1$, we must have that $\mu_j(r_*) = 0, \mu_j(p_*) = \mu_j(c_*)$.

This proves 0.4(b) in our case.

Note that:

(e) $CL(W)$ is (via (b)) the set of all $(r_*, p_*) \in (R \times P)^{2n}$ such that $\mu_j(r_*) \leq 2$ for any $j > 0$.

1.4. As in [Lus15] we have a bijection

$$(a) \text{ Irr}(W) \leftrightarrow BP^n.$$

Using [Lus15, §3], we see that

(b) when $r \neq 2$, the subset $\iota^{(r)}(\mathcal{U}_{dist}^{(r)})$ of $\text{Irr}(W)$ becomes via (a) the subset of BP^n consisting of sequences of the form $(a_1 > a_2 > \dots > a_s > 0, 0, 0, \dots)$.

By [Wal63] (see also [LS12, 6.2]), the set $\mathcal{U}_{dist}^{(2)}$ can be identified via 1.3(a) with the subset of $\mathcal{U}^{(2)}$ consisting of

(c) all (c_*, ϵ) (as in 1.3(a)) such that $\mu_j(c_*) = 0$ for odd j , $\mu_j(c_*) \leq 2$ for even j and $\epsilon(j) = 1$ whenever j is even and $\mu_j(c_*) = 2$.

Using [Lus15, §3] we see that the subset $\iota^{(2)}(\mathcal{U}_{dist}^{(2)})$ of $\text{Irr}(W)$ becomes via (a) the subset of BP^n formed by the sequences $(c_1/2, c_2/2, c_3/2, \dots)$ for various (c_*, ϵ) as in (c). This is the same as the set of all $(a_1 \geq a_2 \geq a_3 \geq \dots) \in BP^n$ such that there are no consecutive equalities between the non-zero a_i . This set contains the set (b). It follows that

$$(d) \mathbf{S}_{dist}^{(r)}(W) \subset \mathbf{S}_{dist}^{(2)}(W) \text{ for any } r.$$

Under our bijection $CL_{dist}(W) \leftrightarrow \mathbf{S}_{dist}(W)$, the set of (c_*, ϵ) as in (c) corresponds to the set of $(r_*, p_*) \in (R \times P)^{2n}$ such that $\mu_j(r_*) \leq 2$ for $j > 0$ and $p_* = (0, 0, 0, \dots)$; this is the same as the set of all $(r_*, p_*) \in (R \times P)^{2n}$ which under 1.3(b) corresponds to elliptic conjugacy classes in W which are in $CL(W)$. This implies (by (d)) that 0.4(c) holds in our case.

1.5. Let $g \in G^{(r)}$ be a distinguished element. To prove 0.3(a) it is enough to show that $\kappa^{(r)}(g) \in \mathbf{S}_{dist}(W)$. If $r = 2$ then g is unipotent and the result is clear. Thus we can assume that $r \neq 2$. Using [Lus15, §3] we see that under the bijection 1.4(a), $\kappa^{(r)}(g)$ corresponds to a bipartition of the form

$$(a) ((a_1 + b_1)/2, (a_2 + b_2)/2, (a_3 + b_3)/2, \dots)$$

where

$$a_1 = \nu_1/2, a_2 = \nu_2/2, a_3 = \nu_3/2, \dots, a_s = \nu_s/2, a_{s+1} = 0, a_{s+2} = 0, \dots,$$

$$b_1 = \nu'_1/2, b_2 = \nu'_2/2, b_3 = \nu'_3/2, \dots, b_t = \nu'_t/2, b_{t+1} = 0, b_{t+2} = 0, \dots,$$

and $\nu_1 > \nu_2 > \nu_3 > \dots > \nu_s, \nu'_1 > \nu'_2 > \nu'_3 > \dots > \nu'_t$ are even integers ≥ 2 with $\sum_k \nu_k + \sum_k \nu'_k = 2n$.

Clearly, $(a_1 + b_1)/2 \geq (a_2 + b_2)/2 \geq (a_3 + b_3)/2 \geq \dots$. If $(a_i + b_i)/2 = (a_{i+2} + b_{i+2})/2$ then $a_i = a_{i+2}, b_i = b_{i+2}$ hence $a_{i+2} = 0, b_{i+2} = 0$ and $(a_{i+2} + b_{i+2})/2 = 0$. Thus (a) corresponds to an element of $\mathbf{S}_{dist}^{(2)}(W)$. This proves 0.3(a) in our case.

1.6. We now assume that $G = SO_{2n+1}(\mathbf{C}), n \geq 3$. Then the arguments in 1.3, 1.4 can be used word by word in the present case except that 1.4(b) must be replaced by the following statement:

(a) When $r \neq 2$, the subset $\iota^{(r)}(\mathcal{U}_{dist}^{(r)})$ of $\text{Irr}(W)$ becomes via 1.4(a) the subset of BP^n consisting of sequences (c_1, c_2, c_3, \dots) where

$$c_1 = (\nu_1 - 1)/2, c_2 = (\nu_2 + 1)/2, c_3 = (\nu_3 - 1)/2, \\ c_4 = (\nu_4 + 1)/2, \dots, c_{2s+1} = (\nu_{2s+1} - 1)/2, c_{2s+2} = 0, c_{2s+3} = 0, \dots$$

and $\nu_1 > \nu_2 > \nu_3 > \dots > \nu_{2s+1}$ are odd integers ≥ 1 with sum $2n + 1$.

(Note that $c_1, c_2, \dots) \in P$, with no two successive equalities between its non-zero terms hence it corresponds to an element in $\mathbf{S}_{dist}^{(2)}(W)$.)

The argument in 1.5 also continues to hold except that 1.5(a) must be replaced by

$$(b) ((a_1 + b_1)/2, (a_2 + b_2)/2, (a_3 + b_3)/2, \dots)$$

where

$$a_1 = (\nu_1 - 1)/2, a_2 = (\nu_2 + 1)/2, a_3 = (\nu_3 - 1)/2, \\ a_4 = (\nu_4 + 1)/2, \dots, a_{2s+1} = (\nu_{2s+1} - 1)/2, a_{2s+2} = 0, a_{2s+3} = 0, \dots,$$

$$b_1 = (\nu'_1 + 1)/2, b_2 = (\nu'_2 - 1)/2, b_3 = (\nu'_3 + 1)/2, \\ b_4 = (\nu'_4 - 1)/2, \dots, b_{2t} = (\nu'_{2t} - 1)/2, b_{2t+1} = 0, b_{2t+2} = 0, \dots,$$

and $\nu_1 > \nu_2 > \nu_3 > \dots > \nu_{2s+1}$, $\nu'_1 > \nu'_2 > \nu'_3 > \dots > \nu'_{2t}$ are odd integers ≥ 1 with $\sum_k \nu_k + \sum_k \nu'_k = 2n + 1$.

We have $a_1 \geq a_2 \geq a_3 \geq \dots$ and $b_1 \geq b_2 \geq b_3 \geq \dots$ hence

$$(a_1 + b_1)/2 \geq (a_2 + b_2)/2 \geq (a_3 + b_3)/2 \geq \dots$$

If $(a_i + b_i)/2 = (a_{i+2} + b_{i+2})/2$ then $a_i = a_{i+2}$, $b_i = b_{i+2}$ hence $a_i = b_i = 0$.

Thus (a) corresponds to an element of $\mathbf{S}_{dist}^{(2)}(W)$. This proves 0.3(a) in our case.

1.7. Until the end of 1.9 we assume that $G = SO_{2n}(\mathbf{C})$, $n \geq 4$. We have $\mathbf{S}(W) = \mathbf{S}^{(2)}(W)$ and Φ becomes a map $cl(W) \rightarrow \mathbf{S}^{(2)}(W)$. Now each of

$$cl(W), \text{Irr}(W), \mathbf{S}^{(2)}(W), \mathcal{U}^{(2)}$$

has a natural involution induced by conjugation by an element in the non-identity component of the full orthogonal group. We then have partitions

$$cl(W) = cl(W)' \sqcup cl(W)'', \text{Irr}(W) = \text{Irr}(W)' \sqcup \text{Irr}(W)'', \\ \mathbf{S}^{(2)}(W) = \mathbf{S}^{(2)}(W)' \sqcup \mathbf{S}^{(2)}(W)'', \mathcal{U}^{(2)} = \mathcal{U}^{(2)'} \sqcup \mathcal{U}^{(2)''},$$

where $()'$ denotes the set of fixed points of the involution and $()''$ denotes its complement.

By [Lus12] we have bijections

(a) (set of orbits of the involution on $\mathcal{U}^{(2)''}$) \leftrightarrow (set of all pairs (c_*, ϵ) as in 1.3(a) such that $\epsilon = 0$ and $\mu_j(c_*) = 0$ for j odd, $\mu_j(c_*)$ even for j even),

(b) (set of orbits of the involution on $cl(W)''$) \leftrightarrow (set of all pairs $(r_*, p_*) \in (R \times P)^{2n}$ such that $r_* = (0, 0, \dots)$ and $\mu_j(p_*) = 0$ for j odd),

(c) $\mathcal{U}^{(2)'}$ \leftrightarrow (set of all pairs (c_*, ϵ) as in 1.3(a) which are not as in (a) and are such that $c_* \in T_{ev}$),

(d) $cl(W)'$ \leftrightarrow (set of all pairs $(r_*, p_*) \in (R \times P)^{2n}$ which are not as in (b) and are such that $r_* \in R_{ev}$).

Now $(\iota^{(2)})^{-1}\Phi$ induces the (surjective) map $cl(W)' \rightarrow \mathcal{U}^{(2)'}$ which by (c), (d) becomes the map $(r_*, p_*) \mapsto (c_*, \epsilon)$ given by the same rule as in 1.3(c). The same proof as in 1.3 shows that if (c_*, ϵ) (as in (c)) is given, then there is a unique (r_*, p_*) (as in (d)) which maps to it and has $|p_*|$ maximum possible. This implies that 0.4(b) holds for any $E \in \mathbf{S}(W) = \mathbf{S}^{(2)}(W)$ that is contained in $\mathbf{S}^{(2)}(W)'$. If $E \in \mathbf{S}^{(2)}(W)''$ then $\Phi^{-1}(E)$ consists of a single element so that 0.4(b) holds automatically for such E . This proves 0.4(b) in our case.

In our case

(e) the set $CL(W)$ becomes the set of pairs (r_*, p_*) as in 1.3(e) such that $r_* \in R_{ev}$ and with each pair as in (b) repeated twice.

1.8. As in 1.4(a) we have a bijection

(a) $\text{Irr}(W)' \leftrightarrow$ set of all $(a_1, a_2, a_3, \dots) \in BP^n$ such that $a_1 - a_2, a_3 - a_4, \dots$ are not all zero.

Using [Lus15, §3] we see that

(b) when $r \neq 2$, the subset $\iota^{(r)}(\mathcal{U}_{dist}^{(r)})$ of $\text{Irr}(W)'$ becomes via (a) the subset of BP^n consisting of sequences of the form

$$((\nu_1 + 1)/2, (\nu_2 - 1)/2, (\nu_3 + 1)/2, \dots, (\nu_{2s} - 1)/2, 0, 0, 0, \dots),$$

where $\nu_1 > \nu_2 > \dots > \nu_{2s}$ are odd ≥ 1 .

By [Wal63] (see also [LS12, 6.2]), the set $\mathcal{U}_{dist}^{(2)}$ can be identified via 1.7(a) with the set consisting of

(c) all (c_*, ϵ) (as in 1.3(a)) where $c_* \in T_{ev}$ and such that $\mu_j(c_*) = 0$ for odd j , $\mu_j(c_*) \leq 2$ for even j and $\epsilon(j) = 1$ whenever j is even and $\mu_j(c_*) = 2$.

Using [Lus15, §3] we see that the subset $\iota^{(2)}(\mathcal{U}_{dist}^{(2)})$ of $\text{Irr}(W)'$ becomes via (a) the subset of BP^n formed by the sequences

$$((c_1 + 2)/2, (c_2 - 2)/2, (c_3 + 2)/2, (c_4 - 2)/2, \dots, (c_{2s} - 2)/2, 0, 0, 0, \dots)$$

for various (c_*, ϵ) as in (c) with $c_* = (c_1 \geq c_2 \geq \dots \geq c_{2s} > 0, 0, 0, \dots)$.

If $\nu_1 > \nu_2 > \dots > \nu_{2s}$ are odd ≥ 1 then

$$\begin{aligned} &((\nu_1 + 1)/2, (\nu_2 - 1)/2, (\nu_3 + 1)/2, \dots, (\nu_{2s} - 1)/2, 0, 0, 0, \dots) \\ &= ((c_1 + 2)/2, (c_2 - 2)/2, (c_3 + 2)/2, (c_4 - 2)/2, \dots, (c_{2s} - 2)/2, 0, 0, 0, \dots), \end{aligned}$$

where $c_1 = \nu_1 - 1, c_2 = \nu_2 + 1, c_3 = \nu_3 - 1, \dots, c_{2s} = \nu_{2s} + 1$ are even and $c_1 \geq c_2 \gg c_3 \geq c_4 \gg \dots \geq c_{2s} \gg 0$. From this we see that $\iota^{(r)}(\mathcal{U}_{dist}^{(r)}) \subset \iota^{(2)}(\mathcal{U}_{dist}^{(2)})$ for any r , that is

(d) $\mathbf{S}_{dist}^{(r)}(W) \subset \mathbf{S}_{dist}^{(2)}(W)$ for any r .

Under our bijection $CL_{dist}(W) \leftrightarrow \mathbf{S}_{dist}(W)$, the set of (c_*, ϵ) as in (c) corresponds to the set of (r_*, p_*) as in 1.7(d) such that $\mu_j(r_*) \leq 2$ for $j > 0$ and $p_* = (0, 0, 0, \dots)$; this is the same as the set of all (r_*, p_*) in 1.7(e) which correspond to elliptic conjugacy classes in W . This implies (by (d)) that 0.4(c) holds in our case.

1.9. Let $g \in G^{(r)}$ be a distinguished element. To prove 0.3(a) it is enough to show that $\kappa^{(r)}(g) \in \mathbf{S}_{dist}(W)$. If $r = 2$ then g is unipotent and the result is clear. Thus we can assume that $r \neq 2$. Using [Lus15, §3] we see that under the bijection 1.8(a), $\kappa^{(r)}(g)$ corresponds to a bipartition of the form

$$(a) \ ((a_1 + b_1)/2, (a_2 + b_2)/2, (a_3 + b_3)/2, \dots)$$

where

$$a_1 = (\nu_1 + 1)/2, a_2 = (\nu_2 - 1)/2, a_3 = (\nu_3 + 1)/2, a_4 = (\nu_4 - 1)/2, \dots,$$

$$a_{2s} = (\nu_{2s} - 1)/2, a_{2s+1} = 0, a_{2t+2} = 0, \dots,$$

$$b_1 = (\nu'_1 + 1)/2, b_2 = (\nu'_2 - 1)/2, b_3 = (\nu'_3 + 1)/2,$$

$$b_4 = (\nu'_4 - 1)/2, \dots, b_{2t} = (\nu'_{2t} - 1)/2, b_{2t+1} = 0, b_{2t+2} = 0, \dots,$$

and

$$\nu_1 > \nu_2 > \nu_3 > \dots > \nu_{2s}, \quad \nu'_1 > \nu'_2 > \nu'_3 > \dots > \nu'_{2t}$$

are odd integers ≥ 1 with $\sum_k \nu_k + \sum_k \nu'_k = 2n$. We can assume that $t \leq s$. We have

$$((a_1 + b_1)/2, (a_2 + b_2)/2, (a_3 + b_3)/2, \dots)$$

$$= ((c_1 + 2)/2, (c_2 - 2)/2, (c_3 + 2)/2, (c_4 - 2)/2, \dots, (c_{2s} - 2)/2, 0, 0, 0, \dots),$$

where

$$c_1 = \nu_1 + \nu'_1, c_2 = \nu_2 + \nu'_2, \dots, c_{2t} = \nu_{2t} + \nu'_{2t},$$

$$c_{2t+1} = \nu_{2t+1} - 1, c_{2t+2} = \nu_{2t+2} + 1, \dots, c_{2s} = \nu_{2s} + 1.$$

(The terms c_{2t+1}, \dots, c_{2s} are missing if $t = s$.) Note that c_1, c_2, \dots, c_{2s} are even, non-zero and $c_1 \geq c_2 \geq c_3 \geq c_4 \geq \dots \geq c_{2s}$ with no consecutive equalities. Thus (a) corresponds to an element of $\mathbf{S}_{dist}^{(2)}(W)$. This proves 0.3(a) in our case.

1.10. In the remainder of this section we assume that G is simple of exceptional type. In this case 0.4(b) can be deduced from tables in [Lus15]. In 1.11-1.15 we describe in each case, using notation of Carter [Car72] and that of [Lus15], the bijection between $CL(W)$ and $\mathbf{S}(W)$ in the form $C \leftrightarrow E$. Here we also use the description of distinguished unipotent elements in [Miz80], [LS12]. Now 0.3(a), 0.4(c) can be verified in each case using the definitions.

1.11. **Type G_2 .** $[G_2] \leftrightarrow 1_0; [A_2] \leftrightarrow 2_1; [A_1 + \tilde{A}_1] \leftrightarrow 2_2;$

$$[\tilde{A}_1] \leftrightarrow 1_3;$$

$$[A_1] \leftrightarrow 1_3;$$

$$[A_0] \leftrightarrow 1_6.$$

Here the first three items are in $CL_{dist}(W)$. Note that $2_2 \in \mathbf{S}_{dist}^{(r)}(W)$ for a single r namely $r = 3$.

1.12. **Type F_4 .** $[F_4] \leftrightarrow 1_0; [B_4] \leftrightarrow 4_1; [F_4(a_1)] \leftrightarrow 9_2; [D_4(a_1)] \leftrightarrow 12_4; [A_3 + \tilde{A}_1] \leftrightarrow 16_5; [\tilde{A}_2 + \tilde{A}_2] \leftrightarrow 6_6;$

$$[B_3] \leftrightarrow 8_3;$$

$$[C_3] \leftrightarrow 8_3;$$

$$[A_3] \leftrightarrow 9_6;$$

$$[B_2 + A_1] \leftrightarrow 9_6;$$

$$[A_2 + \tilde{A}_1] \leftrightarrow 4_7;$$

$$[\tilde{A}_2 + A_1] \leftrightarrow 4_7;$$

$$\begin{aligned}
[B_2] &\leftrightarrow 4_8; \\
[\tilde{A}_2] &\leftrightarrow 8_9; \\
[A_2] &\leftrightarrow 8_9; \\
[A_1 + \tilde{A}_1] &\leftrightarrow 9_{10}; \\
[2A_1] &\leftrightarrow 4_{13}; \\
[A_1] &\leftrightarrow 2_{16}; \\
[\tilde{A}_1] &\leftrightarrow 2_{16}; \\
[A_0] &\leftrightarrow 1_{24}.
\end{aligned}$$

Here the first six items are in $CL_{dist}(W)$.

Note that $16_5, 6_6$ are in $\mathbf{S}_{dist}^{(r)}(W)$ for a single r namely $r = 2$.

1.13. **Type E_6 .** $[E_6] \leftrightarrow 1_0$; $[E_6(a_1)] \leftrightarrow 6_1$; $[E_6(a_2)] \leftrightarrow 30_3$,

$$\begin{aligned}
[D_5] &\leftrightarrow 20_2; [D_5(a_1)] \leftrightarrow 64_4, \\
[A_5] &\leftrightarrow 15_4, \\
[A_4 + A_1] &\leftrightarrow 60_5, \\
[2A_2 + A_1] &\leftrightarrow 10_9, \\
[D_4] &\leftrightarrow 24_6; [D_4(a_1)] \leftrightarrow 80_7, \\
[A_4] &\leftrightarrow 81_6, \\
[A_3 + A_1] &\leftrightarrow 60_8, \\
[A_2 + 2A_1] &\leftrightarrow 60_{11}, \\
[2A_2] &\leftrightarrow 24_{12}, \\
[A_3] &\leftrightarrow 81_{10}, \\
[A_2 + A_1] &\leftrightarrow 64_{13}, \\
[3A_1] &\leftrightarrow 15_{16}, \\
[A_2] &\leftrightarrow 30_{15}, \\
[2A_1] &\leftrightarrow 20_{20}, \\
[A_1] &\leftrightarrow 6_{25}, \\
[A_0] &\leftrightarrow 1_{36}.
\end{aligned}$$

Here the first three items are in $CL_{dist}(W)$.

1.14. **Type E_7 .** $[E_7] \leftrightarrow 1_0$; $[E_7(a_1)] \leftrightarrow 7_1$; $[E_7(a_2)] \leftrightarrow 27_2$; $[E_7(a_3)] \leftrightarrow 56_3$;

$$\begin{aligned}
[A_7] &\leftrightarrow 189_5; [E_7(a_4)] \leftrightarrow 315_7; \\
[E_6] &\leftrightarrow 21_3; [E_6(a_1)] \leftrightarrow 120_4; [E_6(a_2)] \leftrightarrow 405_8; \\
[D_6] &\leftrightarrow 35_4; [D_6(a_1)] \leftrightarrow 210_6; [D_6(a_2)] \leftrightarrow 280_8; [2A_3] \leftrightarrow 378_{14}; \\
[A_6] &\leftrightarrow 105_6; \\
[D_5 + A_1] &\leftrightarrow 168_6; [D_5(a_1) + A_1] \leftrightarrow 378_9; \\
[(A_5 + A_1)'] &\leftrightarrow 70_9; \\
[A_4 + A_2] &\leftrightarrow 210_{10}; \\
[A_3 + A_2 + A_1] &\leftrightarrow 210_{13}; \\
[D_5] &\leftrightarrow 189_7; [D_5(a_1)] \leftrightarrow 420_{10}; \\
[A_5'] &\leftrightarrow 216_9; \\
[A_4 + A_1] &\leftrightarrow 512_{11}; \\
[A_5'] &\leftrightarrow 105_{12}; \\
[D_4 + A_1] &\leftrightarrow 84_{12}; [D_4(a_1) + A_1] \leftrightarrow 405_{15}; \\
[A_3 + A_2] &\leftrightarrow 84_{15}; \\
[(A_3 + 2A_1)'] &\leftrightarrow 216_{16}; \\
[2A_2 + A_1] &\leftrightarrow 70_{18}; \\
[A_2 + 3A_1] &\leftrightarrow 105_{21}; \\
[A_4] &\leftrightarrow 420_{13};
\end{aligned}$$

$$\begin{aligned}
[D_4] &\leftrightarrow 105_{15}; [D_4(a_1)] \leftrightarrow 315_{16}; \\
[(A_3 + A_1)'] &\leftrightarrow 280_{17}; \\
[(A_3 + A_1)'] &\leftrightarrow 189_{20}; \\
[2A_2] &\leftrightarrow 168_{21}; \\
[A_2 + 2A_1] &\leftrightarrow 189_{22}; \\
[(4A_1)'] &\leftrightarrow 15_{28}; \\
[A_3] &\leftrightarrow 210_{21}; \\
[A_2 + A_1] &\leftrightarrow 120_{25}; \\
[(3A_1)'] &\leftrightarrow 35_{31}; \\
[(3A_1)'] &\leftrightarrow 21_{36}; \\
[A_2] &\leftrightarrow 56_{30}; \\
[2A_1] &\leftrightarrow 27_{37}; \\
[A_1] &\leftrightarrow 7_{46}; \\
[A_0] &\leftrightarrow 1_{63}.
\end{aligned}$$

Here the first six items are in $CL_{dist}(W)$.

There are two conjugacy classes $4A_1$; one of them, $(4A_1)''$, comes from W_J of type D_4 . There are two conjugacy classes $A_5 + A_1$; one of them, $(A_5 + A_1)''$, comes from W_J of type E_6 .

1.15. **Type E_8 .** $[E_8] \leftrightarrow 1_0$; $[E_8(a_1)] \leftrightarrow 8_1$; $[E_8(a_2)] \leftrightarrow 35_2$; $[E_8(a_4)] \leftrightarrow 112_3$;
 $[E_8(a_5)] \leftrightarrow 210_4$; $[D_8] \leftrightarrow 560_5$; $[E_8(a_3)] \leftrightarrow 700_6$; $[E_8(a_7)] \leftrightarrow 1400_7$;
 $[E_8(a_6)] \leftrightarrow 1400_8$; $[D_8(a_2)] \leftrightarrow 3240_9$; $[A_8] \leftrightarrow 2240_{10}$;
 $[D_8(a_3)] \leftrightarrow 1400_{11}$; $[A_7 + A_1] \leftrightarrow 4536_{13}$; $[E_8(a_8)] \leftrightarrow 4480_{16}$;
 $[E_7] \leftrightarrow 84_4$; $[E_7(a_1)] \leftrightarrow 567_6$; $[E_7(a_2)] \leftrightarrow 1344_8$; $[E_7(a_3)] \leftrightarrow 2268_{10}$;
 $[E_7(a_4)] \leftrightarrow 7168_{17}$; $[A_7'] \leftrightarrow 6075_{14}$;
 $[A_7''] \leftrightarrow 175_{12}$;
 $[A_6 + A_1] \leftrightarrow 2835_{14}$;
 $[D_5 + A_2] \leftrightarrow 840_{14}$; $[D_5(a_1) + A_2] \leftrightarrow 1344_{19}$;
 $[D_7] \leftrightarrow 400_7$; $[D_7(a_1)] \leftrightarrow 1050_{10}$; $[D_7(a_2)] \leftrightarrow 4200_{12}$; $[D_4 + A_3] \leftrightarrow 4200_{21}$;
 $[E_6 + A_1] \leftrightarrow 448_9$; $[E_6(a_1) + A_1] \leftrightarrow 4096_{11}$; $[E_6(a_2) + A_1] \leftrightarrow 3150_{18}$;
 $[A_4 + A_3] \leftrightarrow 420_{20}$;
 $[A_4 + A_2 + A_1] \leftrightarrow 2835_{22}$;
 $[E_6] \leftrightarrow 525_{12}$; $[E_6(a_1)] \leftrightarrow 2800_{13}$; $[E_6(a_2)] \leftrightarrow 5600_{21}$;
 $[A_6] \leftrightarrow 4200_{15}$;
 $[D_6] \leftrightarrow 972_{12}$; $[D_6(a_1)] \leftrightarrow 5600_{15}$; $[D_6(a_2)] \leftrightarrow 4200_{18}$; $[(2A_3)'] \leftrightarrow 3240_{31}$;
 $[D_5 + A_1] \leftrightarrow 3200_{16}$; $[D_5(a_1) + A_1] \leftrightarrow 6075_{22}$;
 $[(A_5 + A_1)'] \leftrightarrow 2016_{19}$;
 $[A_4 + A_2] \leftrightarrow 4536_{23}$;
 $[A_4 + 2A_1] \leftrightarrow 4200_{24}$;
 $[D_4 + A_2] \leftrightarrow 168_{24}$; $[D_4(a_1) + A_2] \leftrightarrow 2240_{28}$;
 $[(2A_3)'] \leftrightarrow 840_{26}$;
 $[A_3 + A_2 + A_1] \leftrightarrow 1400_{29}$;
 $[2A_2 + 2A_1] \leftrightarrow 175_{36}$;
 $[A_5] \leftrightarrow 3200_{22}$;
 $[D_5] \leftrightarrow 2100_{20}$; $[D_5(a_1)] \leftrightarrow 2800_{25}$;
 $[A_4 + A_1] \leftrightarrow 4096_{26}$;
 $[D_4 + A_1] \leftrightarrow 700_{28}$; $[D_4(a_1) + A_1] \leftrightarrow 1400_{32}$;
 $[A_3 + A_2] \leftrightarrow 972_{32}$;
 $[(A_3 + 2A_1)'] \leftrightarrow 1050_{34}$;

- $[2A_2 + A_1] \leftrightarrow 448_{39};$
- $[A_2 + 3A_1] \leftrightarrow 400_{43};$
- $[D_4] \leftrightarrow 525_{36}; [D_4(a_1)] \leftrightarrow 1400_{37};$
- $[A_4] \leftrightarrow 2268_{30};$
- $[A_3 + A_1] \leftrightarrow 1344_{38};$
- $[2A_2] \leftrightarrow 700_{42};$
- $[A_2 + 2A_1] \leftrightarrow 560_{47};$
- $[(4A_1)'] \leftrightarrow 50_{56};$
- $[A_3] \leftrightarrow 567_{46};$
- $[A_2 + A_1] \leftrightarrow 210_{52};$
- $[3A_1] \leftrightarrow 84_{64};$
- $[A_2] \leftrightarrow 112_{63};$
- $[2A_1] \leftrightarrow 35_{74};$
- $[A_1] \leftrightarrow 8_{91};$
- $[A_0] \leftrightarrow 1_{120}.$

Here the first fourteen items are in $CL_{dist}(W)$. There are two conjugacy classes A_7 ; one of them, A'_7 , comes from W_J of type E_7 . There are two conjugacy classes $2A_3$; one of them, $2A'_3$, comes from W_J of type D_6 . There are two conjugacy classes $A_5 + A_1$; one of them, $(A_5 + A_1)'$, comes from W_J of type E_6 . There are two conjugacy classes $A_3 + 2A_1$; one of them, $(A_3 + 2A_1)'$, comes from W_J of type D_5 . There are two conjugacy classes $4A_1$; one of them, $(4A_1)'$, comes from W_J of type D_4 .

Note that $3240_9, 4536_{11}$ are in $\mathbf{S}_{dist}^{(r)}(W)$ for a single r namely $r = 2$; 1400_{11} is in $\mathbf{S}_{dist}^{(r)}(W)$ for a single r namely $r = 3$.

2. A GRAPH STRUCTURE ON $cl(W)$

2.1. Let $cl_{ell}(W) = \{C \in cl(W); m(C) = 0\}$ (elliptic conjugacy classes). Let $cl_{s-ell}(W) = \{C \in cl(W); m(C) = 1\}$ (subelliptic conjugacy classes). We shall now define a subset $\mathcal{E}_{ell}(W)$ of $cl_{ell}(W) \times cl_{s-ell}(W)$. If W is a product $W_1 \times W_2$ of two Weyl groups and if $\mathcal{E}_{ell}(W_1), \mathcal{E}_{ell}(W_2)$ are already defined, then $\mathcal{E}_{ell}(W)$ consists of $(C_1 \times C_2, C'_1 \times C'_2)$ where either $(C_1, C'_1) \in \mathcal{E}_{ell}(W_1), C_2 = C'_2 \in cl_{ell}(W_2)$ or $(C_2, C'_2) \in \mathcal{E}_{ell}(W_2), C_1 = C'_1 \in cl_{ell}(W_1)$. In this way we see that it is enough to define $\mathcal{E}_{ell}(W)$ when W is irreducible.

When W is of type A , we set $\mathcal{E}_{ell}(W) = \emptyset$.

Assume that W is of type $B_n, n \geq 2$. With the identification 1.3(b), we define $\mathcal{E}_{ell}(W)$ to be the set of all $((r_*, p_*), (r'_*, p'_*)) \in (R \times P)^{2n} \times (R \times P)^{2n}$ such that for some $2t \in 2\mathbf{N} + 2$ the following hold:

- (a) r'_* is obtained from r_* by removing two consecutive terms equal to $2t$ and $2t$ appears at least once in r'_* ;
- (b) $p_* = (0, 0, \dots), p'_* = (2t, 2t, 0, 0, \dots)$.

Assume that W is of type $D_n, n \geq 4$. With the identification 1.7(d), we define $\mathcal{E}_{ell}(W)$ to be the set of all $((r_*, p_*), (r'_*, p'_*)) \in (R_{ev} \times P)^{2n} \times (R_{ev} \times P)^{2n}$ such that for some $2t \in 2\mathbf{N} + 2$, (a) and (b) hold.

For conjugacy classes in W of exceptional type we use the notation of Carter [Car72] except that we sometimes write $D_4 + A_3$ for what Carter denotes by $D_6(a_2) + A_1$.

2.2. If W is of type G_2 we set $\mathcal{E}_{ell}(W) = \emptyset$.

2.3. If W is of type F_4 , $\mathcal{E}_{ell}(W)$ consists of:

$$([D_4], [B_3]); ([C_3 + A_1], [C_3]); ([4A_1], [3A_1]).$$

2.4. If W is of type E_6 , $\mathcal{E}_{ell}(W)$ consists of:

$$([A_5 + A_1], [A_5]); ([3A_2], [2A_2 + A_1]).$$

2.5. If W is of type E_7 , $\mathcal{E}_{ell}(W)$ consists of:

$$([D_6 + A_1], [D_6]); ([D_6(a_2) + A_1], [D_6(a_2)]); ([D_4 + 3A_1], [D_4 + 2A_1]);$$

$$([A_5 + A_2], [(A_5 + A_1)']); ([2A_3 + A_1], [A_3 + A_2 + A_1]); ([7A_1], [6A_1]).$$

2.6. If W is of type E_8 , $\mathcal{E}_{ell}(W)$ consists of:

$$([E_7 + A_1], [E_7]); ([E_7(a_2) + A_1], [E_7(a_2)]); ([E_7(a_4) + A_1], [E_7(a_4)]);$$

$$([E_6 + A_2], [E_6 + A_1]); ([E_6(a_2) + A_2], [E_6(a_2) + A_1]);$$

$$([D_8(a_1)], [D_7]); ([D_6 + 2A_1], [D_6 + A_1]);$$

$$([D_5(a_1) + A_3], [D_5(a_1) + A_2]); ([2D_4], [D_4 + A_3]);$$

$$([2D_4(a_1)], [D_4(a_1) + A_3]); ([D_4 + 4A_1], [D_4 + 3A_1]);$$

$$([A_5 + A_2 + A_1], [A_5 + A_2]); ([A_5 + A_2 + A_1], [A_5 + 2A_1]);$$

$$([2A_4], [A_4 + A_3]);$$

$$([2A_3 + 2A_2], [A_3 + A_2 + 2A_1]);$$

$$([2A_3 + 2A_2], [2A_3 + A_1]);$$

$$([4A_2], [3A_2 + A_1]);$$

$$([8A_1], [7A_1]).$$

2.7. We define a subset $\mathcal{E}(W)$ of $cl(W) \times cl(W)$ as follows. We say that $(C, C') \in cl(W) \times cl(W)$ is in $\mathcal{E}(W)$ if there exists $J \subset I$ and $(C_1, C'_1) \in \mathcal{E}_{ell}(W_J)$ such that $C = \rho_J(C_1)$, $C' = \rho_J(C'_1)$, (ρ_J as in 0.6). Note that $\mathcal{E}_{ell}(W) \subset \mathcal{E}(W)$ and that if $(C, C') \in \mathcal{E}(W)$ then $m(C') = m(C) + 1$. We can regard $\mathcal{E}(W)$ as the set of edges of a graph with vertices $cl(W)$. This graph is oriented: the edge (C, C') is oriented from C to C' . From the results of [Lus15] one can verify that:

(a) *the strata of $cl(W)$ (or W) are exactly the connected components of this graph.*

Now each stratum of $cl(W)$ (or W) can be viewed as the set of vertices of an oriented graph (restriction of the graph above to the stratum). From the results of [Lus15] one can verify the following strengthening of 0.4(a) and 0.4(b):

(b) *this oriented graph is a product of finitely many Coxeter graphs of type A (with the usual orientation).*

Recall that an oriented Coxeter graph of type A is of the form

$$\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet.$$

For example, if W is of type E_8 , then the graph attached to a stratum of $cl(W)$ (or W) is of one of the types $A_5, A_4, A_3, A_2 \times A_2, A_2, A_1$. (Type A_5 appears for a unique stratum; type $A_2 \times A_2$ appears for two strata.) If W is of type E_7 , then the graph attached to a stratum of $cl(W)$ (or W) is of one of the types A_4, A_3, A_2, A_1 . (Type A_4 appears for a unique stratum.) If W is of type E_6 , then the graph attached to a stratum of $cl(W)$ (or W) is of one of the types A_2, A_1 . If W is of type F_4 , then the graph attached to a stratum of $cl(W)$ (or W) is of one of the types A_4, A_2, A_1 . (Type A_4 appears for a unique stratum.) If W is of type G_2 . then the graph attached to a stratum of $cl(W)$ (or W) is of type A_1 .

3. COMPLEMENTS

3.1. Let $r \in \{0\} \cup \mathcal{P}$. We state two properties similar to 0.4(a), (b).

(a) For any $E \in \mathbf{S}^{(r)}(W)$ there is a unique $C_E^{(r)} \in (\Phi^{(r)})^{-1}(E)$ which is as elliptic as possible, that is $m(C_E^{(r)}) \leq m(C)$ for any $C \in (\Phi^{(r)})^{-1}(E)$;

(b) For any $E \in \mathbf{S}^{(r)}(W)$ there is a unique $C_E^{\prime(r)} \in (\Phi^{(r)})^{-1}(E)$ which is as non-elliptic as possible, that is $m(C_E^{\prime(r)}) \leq m(C)$ for any $C \in (\Phi^{(r)})^{-1}(E)$.

Now (a) is proved in [Lus12]; the proof of (b) is similar. Let $CL^{(r)}(W)$ be the image of the map $E \mapsto C_E^{\prime(r)}, \mathbf{S}^{(r)}(W) \rightarrow cl(W)$. Note that $\Phi^{(r)}$ restricts to a bijection $CL^{(r)}(W) \xrightarrow{\sim} \mathbf{S}^{(r)}(W)$. Let $CL_{dist}^{(r)}(W)$ be the subset of $CL^{(r)}(W)$ corresponding to $\mathbf{S}_{disc}^{(r)} \subset \mathbf{S}^{(r)}(W)$ under this bijection. We have the following analogue of 0.4(c).

(c) *Let $C \in CL^{(r)}(W)$. We have $C \in CL_{dist}^{(r)}(W)$ if and only if C is elliptic.*

Note that $CL_{dist}(W) = \cup_r CL_{dist}^{(r)}(W)$.

3.2. For any semisimple element $s \in G = G^{(0)}$ let $E_s = j_{W_s}^W(\text{sign})$ (j -induction) where W_s is the Weyl group of the connected centralizer of s viewed as a subgroup of W . We have $E_s \in \mathbf{S}(W)$; the subset of $\mathbf{S}(W)$ formed by the E_s for various s as above is denoted by $\text{Irr}_{ss}(W)$.

Now let $E \in \mathbf{S}_{dist}(W)$ and let $C \in CL_{dist}(W)$ be the corresponding conjugacy class. According to 0.5(a) we have $E = \Phi^{(0)}(C)$ hence $E = \iota^{(0)}(\gamma)$ where $\gamma = \psi^{(0)}(C) \in \mathcal{U}^{(0)}$. (The elements of γ need not be distinguished.) From [Lus11b] there exists a semisimple element $s \in G^{(0)}$ such that $E = E_s$; thus, we have $\mathbf{S}_{dist}(W) \subset \text{Irr}_{ss}(W)$. (We can assume that G is almost simple; then the statement in the previous sentence holds for $E = \Phi^{(0)}(C)$ for any elliptic $C \in cl(W)$ with a single exception in type E_8 ; but that exception is not distinguished, see 1.15.) Note that s belongs to the stratum of $G^{(0)}$ that contains γ .

Thus, to $E \in \mathbf{S}_{dist}(W)$ (or $C \in CL_{dist}(W)$) one can associate a collection of reflection subgroups W_s (for various s as above).

Now, [Lus11a, 4.4(b)] implies that the minimum length of an element in $C \in CL_{dist}(W)$ is equal to the dimension of the centralizer of s (as above) in G modulo the centre of G .

3.3. In 3.4-3.10 we describe explicitly a correspondence

(a) $C \mapsto W_s$

which to any $C \in CL_{dist}(W)$ associates a reflection subgroup W_s of W (up to conjugacy) as in 3.2 (assuming that W is irreducible of type $\neq A$); if W is of type A , then C is the Coxeter class and the corresponding W_s is $\{1\}$.

3.4. We now assume that $G = Sp_{2n}(\mathbf{C}), n \geq 2$ or that $G = SO_{2n+1}(\mathbf{C}), n \geq 2$. According to 1.4, 1.6, $CL_{dist}(W)$ can be identified with the set of pairs $(r_*, p_*) \in (R \times P)^{2n}$ such that $p_* = (0, 0, \dots)$ and r_* is a sequence $r_1 \geq r_2 \geq r_3 \geq \dots \geq r_\sigma$ of even integers > 0 without two consecutive equalities. For such (r_*, p_*) we define $k = r_1/2$. We define $\bar{r}_1 \geq \bar{r}_2 \geq \dots \geq \bar{r}_k > 0$ by $\bar{r}_i = \sharp\{t \in [1, \sigma]; r_t/2 \geq i\}$ for $i \in [1, k]$; note that $\bar{r}_1 = \sigma$ and $\bar{r}_1 + \bar{r}_2 + \dots + \bar{r}_k = n$. Using 0.5(a) and [Lus11b], we see that the reflection subgroup W_s corresponding to (r_*, p_*) is a product of symmetric groups

$$S_{\bar{r}_k} \times S_{\bar{r}_{k-1}} \times \dots \times S_{\bar{r}_1}$$

(a subgroup of S_n which is itself naturally a subgroup of W of the form W_J for some J).

We see that $CL_{dist}(W)$ is in natural bijection with the set of sequences $\bar{r}_1 \geq \bar{r}_2 \geq \bar{r}_3 \geq \dots$ of integers ≥ 0 such that $\bar{r}_1 + \bar{r}_2 + \dots = n$ and $\bar{r}_i - \bar{r}_{i+1} \leq 2$ for all $i \geq 1$.

3.5. We now assume that $G = SO_{2n}(\mathbf{C})$, $n \geq 4$. According to 1.8, $CL_{dist}(W)$ can be identified with the set of pairs $(r_*, p_*) \in (R \times P)^{2n}$ such that $p_* = (0, 0, \dots)$ and r_* is a sequence $r_1 \geq r_2 \geq r_3 \geq \dots \geq r_\sigma$ of even integers > 0 without two consecutive equalities with σ even. For such (r_*, p_*) we define $\bar{r}_1 \geq \bar{r}_2 \geq \dots \geq \bar{r}_k > 0$ as in 3.4. Note that \bar{r}_1 is even and ≥ 2 . Using 0.5(a) and [Lus11b], we see that the reflection subgroup W_s corresponding to (r_*, p_*) is of the form

$$S_{\bar{r}_k} \times S_{\bar{r}_{k-1}} \times \dots \times S_{\bar{r}_2} \times W_{D_{\bar{r}_1/2}} \times W_{D_{\bar{r}_1/2}}.$$

Here W_{D_m} denotes a Weyl group of type D_m (for $m = 1$ this is taken to be $\{1\}$; for $m = 2$ this taken to be $S_2 \times S_2$). We view

$$S_{\bar{r}_k} \times S_{\bar{r}_{k-1}} \times \dots \times S_{\bar{r}_2} \times W_{D_{\bar{r}_1}}$$

as a subgroup of W of the form W_J for some J and $W_{D_m} \times W_{D_m}$ as a reflection subgroup of $W_{D_{2m}}$ in the standard way.

We see that $CL_{dist}(W)$ is in natural bijection with the set of sequences $\bar{r}_1 \geq \bar{r}_2 \geq \bar{r}_3 \geq \dots$ of integers ≥ 0 such that $\bar{r}_1 + \bar{r}_2 + \dots = n$, \bar{r}_1 is even and $\bar{r}_i - \bar{r}_{i+1} \leq 2$ for all $i \geq 1$.

3.6. Until the end of 3.10, W is of exceptional type. In each case the reflection subgroup W_s attached by 3.3(a) to $C \in CL_{dist}(W)$ is specified by its type. (We use 0.5(a) and [Lus11b].)

If W is of type G_2 , the correspondence 3.3(a) is:

$$[G_2] \mapsto A_0; [A_2] \mapsto A_1; [A_1 + \tilde{A}_1] \mapsto 2A_1.$$

Note that the group W_s is of the form W_J for some $J \subset I$ except for the last case.

3.7. If W is of type F_4 , the correspondence 3.3(a) is:

$$[F_4] \mapsto A_0; [B_4] \mapsto A_1; [F_4(a_1)] \mapsto 2A_1; [D_4(a_1)] \mapsto A_2 + A_1;$$

$$[A_3 + \tilde{A}_1] \mapsto B_2 + A_1; [\tilde{A}_2 + \tilde{A}_2] \mapsto A_2 + A_2.$$

Note that the group W_s is of the form W_J for some $J \subset I$ except for the last two cases.

3.8. If W is of type E_6 , the correspondence 3.3(a) is:

$$[E_6] \mapsto A_0; [E_6(a_1)] \mapsto A_1; [E_6(a_2)] \mapsto 3A_1.$$

Note that the group W_s is of the form W_J for some $J \subset I$.

3.9. If W is of type E_7 , the correspondence 3.3(a) is:

$$[E_7] \mapsto A_0; [E_7(a_1)] \mapsto A_1; [E_7(a_2)] \mapsto 2A_1; [E_7(a_3)] \mapsto 3A_1;$$

$$[A_7] \mapsto A_2 + 2A_1; [E_7(a_4)] \mapsto 2A_2 + A_1.$$

Note that the group W_s is of the form W_J for some $J \subset I$.

3.10. If W is of type E_8 , the correspondence 3.3(a) is:

$$\begin{aligned} [E_8] &\mapsto A_0; [E_8(a_1)] \mapsto A_1; [E_8(a_2)] \mapsto 2A_1; [E_8(a_4)] \mapsto 3A_1; \\ [E_8(a_5)] &\mapsto 4A_1; [D_8] \mapsto A_2 + 2A_1; [E_8(a_3)] \mapsto A_2 + 3A_1; \\ [E_8(a_7)] &\mapsto 2A_2 + A_1; [E_8(a_6)] \mapsto 2A_2 + 2A_1; \\ [A_8] &\mapsto A_3 + A_2 + A_1; [E_8(a_8)] \mapsto A_4 + A_3; \\ [D_8(a_2)] &\mapsto A_3 + 3A_1; [D_8(a_3)] \mapsto A_3 + A_2 + 2A_1; [A_7 + A_1] \mapsto 2A_3 + A_1. \end{aligned}$$

Note that the group W_s is of the form W_J for some $J \subset I$ except for the last three cases.

3.11. Let K be a maximal compact subgroup of $G^{(0)}$. The following result was stated in [Lus21, 5.2]:

(a) *Let X be a stratum of $G^{(0)}$ and let E be the corresponding element of $\mathbf{S}(W)$. We have $X \cap K \neq \emptyset$ if and only if $E \in \text{Irr}_{ss}(W)$.*

By the results in 3.2, we have $E \in \text{Irr}_{ss}(W)$ if and only if X contains a semisimple element of $G^{(0)}$. This last condition is clearly satisfied when $X \cap K \neq \emptyset$. Conversely, assume that X contains a semisimple element s of $G^{(0)}$. It is well known that we can find $s' \in K$ such that the connected centralizers of s and s' are conjugate. It follows that $E_s = E_{s'}$ hence s, s' belongs to the same stratum. Since $s \in X$ we have $s' \in X$ so that $X \cap K \neq \emptyset$. This proves (a).

We show:

(b) *Let X, E be as in (a). If E is distinguished then $X \cap K \neq \emptyset$.*

By (a) it is enough to show that $E \in \text{Irr}_{ss}(W)$. This follows from $\mathbf{S}_{dist}(W) \subset \text{Irr}_{ss}(W)$, see 3.2.

Errata to [Lus15].

p. 355, line containing 378₁₄: replace $[A_3 + A_2]$ by $[2A_3]$.

p. 356, line containing 3200₂₂: replace $[(A_5 + A_1)']$ by $[(A_5 + A_1)', A_5]$.

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