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CALCULUS OF ARCHIMEDEAN RANKIN–SELBERG INTEGRALS WITH RECURRENCE RELATIONS

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Abstract. Let n and n' be positive integers such that $n - n' \in \{0, 1\}$. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let K_n and $K_{n'}$ be maximal compact subgroups of $\mathrm{GL}(n, F)$ and $\mathrm{GL}(n', F)$, respectively. We give the explicit descriptions of archimedean Rankin–Selberg integrals at the minimal K_n- and $K_{n'}$-types for pairs of principal series representations of $\mathrm{GL}(n, F)$ and $\mathrm{GL}(n', F)$, using their recurrence relations. Our results for $F = \mathbb{C}$ can be applied to the arithmetic study of critical values of automorphic L-functions.

1. Introduction

The theory of automorphic L-functions via integral representations has its origin in the work of Hecke [8] for GL(2), and the works of Rankin [23], Selberg [24] for $\mathrm{GL}(2) \times \mathrm{GL}(2)$. As a direct outgrowth of their works, the theory of Rankin–Selberg integrals for $\mathrm{GL}(n) \times \mathrm{GL}(n')$ was developed by Jacquet, Piatetski-Shapiro, and Shalika [13]. Our interest here is the archimedean local theory of their Rankin–Selberg integrals.

Let F be either $\mathbb{R}$ or $\mathbb{C}$. We fix a maximal compact subgroup K_n of $\mathrm{GL}(n, F)$. Let Π and Π' be irreducible generic Casselman–Wallach representations of $\mathrm{GL}(n, F)$ and $\mathrm{GL}(n', F)$, respectively. We denote by $L(s, \Pi \times \Pi')$ the archimedean L-factor for $\Pi \times \Pi'$. The theory of archimedean Rankin–Selberg integrals for $\Pi \times \Pi'$ was developed by Jacquet and Shalika. In [16], they showed that any archimedean Rankin–Selberg integral for $\Pi \times \Pi'$ is extended to $\mathbb{C}$ as a holomorphic multiple of $L(s, \Pi \times \Pi')$, is bounded at infinity in vertical strips, and satisfies the local functional equation. In [15], Jacquet refined the proofs of the above results, and showed further that $L(s, \Pi \times \Pi')$ can be expressed as a linear combination of archimedean Rankin–Selberg integrals for $\Pi \times \Pi'$ if $n - n' \in \{0, 1\}$. Their results are sufficient for the proofs of important analytic properties of automorphic L-functions such as the analytic continuations, the functional equations and the converse theorems. However, in the studies of arithmetic properties of automorphic L-functions, the precise knowledge of archimedean Rankin–Selberg integrals at the special K_n- and $K_{n'}$-types is required. For example, Sun's nonvanishing result [28] at the minimal K_n- and K_{n-1}-types is vital to the arithmetic study of critical values of automorphic L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ by the cohomological method. The goal of this paper is to give explicit descriptions of archimedean Rankin–Selberg integrals at the

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minimal K_n- and $K_{n'}$-types for pairs of principal series representations of $\mathrm{GL}(n,F)$ and $\mathrm{GL}(n',F)$ with $n-n'\in\{0,1\}$. We generalize Stade's results [26], [27] (see [11] for a simplified proof) for the spherical case to general case.

Let us explain our main result for the $\mathrm{GL}(n)\times\mathrm{GL}(n-1)$-case. Assume that Π and Π' are irreducible principal series representations of $\mathrm{GL}(n,F)$ and $\mathrm{GL}(n-1,F)$, respectively. Let ψ be the standard additive character of F. We regard $\mathrm{GL}(n-1,F)$ as a subgroup of $\mathrm{GL}(n,F)$ via the embedding

$$\iota_n\colon \mathrm{GL}(n-1,F)\ni g\mapsto \begin{pmatrix} g & \\ & 1\end{pmatrix}\in \mathrm{GL}(n,F).$$

For $W\in\mathcal{W}(\Pi,\psi)$ and $W'\in\mathcal{W}(\Pi',\psi^{-1})$, we define the archimedean Rankin–Selberg integral $Z(s,W,W')$ by

$$Z(s,W,W')=\int_{N_{n-1}\backslash\mathrm{GL}(n-1,F)} W(\iota_n(g))W'(g)|\det g|_F^{s-1/2}\,dg \qquad (\mathrm{Re}(s)\gg 0),$$

where $\mathcal{W}(\Pi,\psi)$, $\mathcal{W}(\Pi',\psi^{-1})$ are the Whittaker models of Π, Π', respectively, N_{n-1} is the upper triangular unipotent subgroup of $\mathrm{GL}(n-1,F)$, and $|\cdot|_F$ is the usual norm on F. Let $(\tau_{\min},V_{\min})$ be the minimal K_n-type of Π, and we fix a K_n-embedding $\mathbf{W}\colon V_{\min}\to\mathcal{W}(\Pi,\psi)$. Let $(\tau'_{\min},V'_{\min})$ be the minimal K_{n-1}-type of Π', and we fix a K_{n-1}-embedding $\mathbf{W}'\colon V'_{\min}\to\mathcal{W}(\Pi',\psi^{-1})$. Here we give $\mathbf{W}$ and $\mathbf{W}'$ concretely by the Jacquet integrals (cf. §2.4). We note that

$$V_{\min}\otimes_{\mathbb{C}}V'_{\min}\ni v\otimes v'\mapsto Z(s,\mathbf{W}(v),\mathbf{W}'(v'))\in\mathbb{C}_{\mathrm{triv}}$$

defines an element of $\mathrm{Hom}_{K_{n-1}}(V_{\min}\otimes_{\mathbb{C}}V'_{\min},\mathbb{C}_{\mathrm{triv}})$, where $\mathbb{C}_{\mathrm{triv}}=\mathbb{C}$ is the trivial K_{n-1}-module. In the first main theorem (Theorem 2.7), we give the explicit description of this K_{n-1}-homomorphism. More precisely, under the assumption $\mathrm{Hom}_{K_{n-1}}(V_{\min}\otimes_{\mathbb{C}}V'_{\min},\mathbb{C}_{\mathrm{triv}})\neq\{0\}$, we show the equality

$$(1.1)\qquad Z(s,\mathbf{W}(v),\mathbf{W}'(v'))=L(s,\Pi\times\Pi')\Psi(v\otimes v') \qquad (v\in V_{\min},\ v'\in V'_{\min})$$

with some nonzero $\Psi\in\mathrm{Hom}_{K_{n-1}}(V_{\min}\otimes_{\mathbb{C}}V'_{\min},\mathbb{C}_{\mathrm{triv}})$ independent of s, and describe Ψ explicitly in terms of Gelfand–Tsetlin type bases of $V_{\min}$ and $V'_{\min}$. Here we remark that the integrals $Z(s,\mathbf{W}(v),\mathbf{W}'(v'))$ ($v\in V_{\min}$, $v'\in V'_{\min}$) vanish if $\mathrm{Hom}_{K_{n-1}}(V_{\min}\otimes_{\mathbb{C}}V'_{\min},\mathbb{C}_{\mathrm{triv}})=\{0\}$. In the second main theorem (Theorem 2.14), we give a similar description for the $\mathrm{GL}(n)\times\mathrm{GL}(n)$-case. Since the statement of Theorem 2.14 is slightly complicated, we leave it to §2.

We introduce some applications of our results (Theorems 2.7 and 2.14) for $F=\mathbb{C}$. In the arithmetic study of critical values of automorphic L-functions for $\mathrm{GL}(n)\times\mathrm{GL}(n')$ with $n-n'\in\{0,1\}$ by the cohomological method, the archimedean Rankin–Selberg integrals at the minimal K_n- and $K_{n'}$-types play important roles, and the hypothesis of the nonvanishing of them at critical points is called the nonvanishing hypothesis for $\mathrm{GL}(n,F)\times\mathrm{GL}(n',F)$. It is known that a local component at the complex place of irreducible regular algebraic cuspidal automorphic representation of $\mathrm{GL}(n)$ is a cohomological principal series representation (cf. [22, Proposition 2.14]). Hence, Theorem 2.7 gives another proof of the nonvanishing hypothesis for $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n-1,\mathbb{C})$ at all critical points, which were originally proved by Sun [28] and were used in Grobner–Harris [7] and Raghuram [22]. In [3], Dong and Xue proved the nonvanishing hypothesis for $\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$ at the central critical point, and they indicate that it is hard to generalize their result to all critical points by the technique of the translation functor. Theorem 2.14 proves the nonvanishing hypothesis for $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ at all critical points, and allows us to improve the archimedean part of Grenié's theorem [6, Theorem 2] into more explicit form (cf. Remark 2.17). We expect that our explicit results will be applied to deeper study of special values of automorphic L-functions.

There are some related results to be mentioned here. In the cases of $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ and $\mathrm{GL}(n) \times \mathrm{GL}(n)$, we expect that the archimedean Rankin–Selberg integrals for appropriate Whittaker functions are equal to the associated L-factors. This expectation was proved by Jacquet–Langlands [12] and Popa [21] for the $\mathrm{GL}(2) \times \mathrm{GL}(1)$-case; by Jacquet [14], Zhang [33] and the second author [20] for the $\mathrm{GL}(2) \times \mathrm{GL}(2)$-case; by Hirano and the authors [9] for the $\mathrm{GL}(3) \times \mathrm{GL}(2)$-case. The results of this paper, that is, the formula (1.1) and the analogous formula for $\mathrm{GL}(n) \times \mathrm{GL}(n)$ in Theorem 2.14 can be regarded as additional evidences of this expectation for the higher rank cases. On the other hand, it is somewhat widely believed that these results will not extend to the case of $\mathrm{GL}(n) \times \mathrm{GL}(n')$ with $n - n' \geq 2$ (cf. [2, Lecture 8, §4]).

Let us briefly explain the idea of the proofs of our main theorems. The key ingredients are two kinds of special sections for a principal series representation Π of $\mathrm{GL}(n,F)$. One is the Godement section, which is defined by Jacquet [15] as an integral transform of the standard section for some principal series representation of $\mathrm{GL}(n-1,F)$. It gives a recursive integral representation of a Whittaker function for Π. The other is defined as an integral transform of the standard section for the same representation Π of $\mathrm{GL}(n,F)$. It gives an integral representation of a Whittaker function for Π, which is related to the local theta correspondence in Watanabe [32, §2]. Using two kinds of the special sections, we construct the recurrence relations of the archimedean Rankin–Selberg integrals for pairs of principal series representations of $\mathrm{GL}(n,F)$ and $\mathrm{GL}(n',F)$ with $n - n' \in \{0,1\}$. Based on the representation theory of K_n, we write down these recurrence relations at the minimal K_n- and $K_{n'}$-types, explicitly, and prove the main theorems by induction. Here we remark that the explicit recurrence relations for the spherical case coincide with those in [11], which follow from explicit formulas of the radial parts of spherical Whittaker functions in [10].

This paper consists of five sections together with two appendices. In §2, we introduce basic notation and state the main theorems. In §3, we define two kinds of the special sections and give the recurrence relations of the archimedean Rankin–Selberg integrals. §4 is devoted to some preliminary results on the theory of finite dimensional representations of K_n and $\mathrm{GL}(n,\mathbb{C})$. In §5, we prove the main theorems using the results in §3 and §4. In Appendix A, we generalize the explicit formulas of the radial parts of Whittaker functions in [10] using the Godement section. In Appendix B, we give a list of symbols, because this paper contains a lot of notation and symbols.

2. Main results

In this section, we introduce basic notation and our main results. We describe each object explicitly as possible, although not all of them are necessary to state our main theorems. The authors believe that they are of interest and useful for further studies.

2.1. **Notation.** We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Let $\mathbb{R}_+^\times$ be the multiplicative group of positive real numbers. Let $\mathbb{N}_0$ be the set of non-negative integers. The real part, the imaginary part and the complex conjugate of a complex number z are denoted by $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $\overline{z}$, respectively.

Throughout this paper, F denotes the archimedean local field, that is, F is either $\mathbb{R}$ or $\mathbb{C}$. It is convenient to define the constant c_F by $\mathrm{c}_\mathbb{R} = 1$ and $\mathrm{c}_\mathbb{C} = 2$. We define additive characters $\psi_t\colon F \to \mathbb{C}^\times$ $(t \in F)$ and a norm $|\cdot|_F$ on F by

$$\psi_t(z) = \exp\bigl(\pi \mathrm{c}_F \sqrt{-1}(tz + \overline{tz})\bigr) = \begin{cases} \exp(2\pi\sqrt{-1}tz) & \text{if } F = \mathbb{R}, \\ \exp(2\pi\sqrt{-1}(tz+\overline{tz})) & \text{if } F = \mathbb{C}, \end{cases}$$

and $|z|_F = |z|^{\mathrm{c}_F}$ for $z \in F$, where $|\cdot|$ is the ordinary absolute value. When $t = \varepsilon \in \{\pm 1\}$, we call ψ_ε the standard character of F. We identify the additive group F with its dual group via the isomorphism $t \mapsto \psi_t$, and denote by $d_F z$ the self-dual additive Haar measure on F, that is, $d_\mathbb{R} z = dz$ is the ordinary Lebesgue measure on $\mathbb{R}$ and $d_\mathbb{C} z = 2dx\,dy$ $(z = x + \sqrt{-1}y)$ is twice the ordinary Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. For $m \in \mathbb{Z}$, we define a meromorphic function $\Gamma_F(s;m)$ of s in $\mathbb{C}$ by

$$\Gamma_F(s;m) = \mathrm{c}_F(\pi \mathrm{c}_F)^{-(s\mathrm{c}_F+m)/2}\Gamma\left(\frac{s\mathrm{c}_F+m}{2}\right) = \begin{cases} \Gamma_\mathbb{R}(s+m) & \text{if } F = \mathbb{R}, \\ \Gamma_\mathbb{C}(s+m/2) & \text{if } F = \mathbb{C}, \end{cases}$$

where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma(s)$ is the usual Gamma function.

Throughout this paper, n and n' are positive integers. The space of $n \times n'$ matrices over F is denoted by $\mathrm{M}_{n,n'}(F)$. When $n' = n$, we denote $\mathrm{M}_{n,n}(F)$ simply by $\mathrm{M}_n(F)$. We denote by $d_F z$ the measure on $\mathrm{M}_{n,n'}(F)$ defined by

$$d_F z = \prod_{i=1}^{n}\prod_{j=1}^{n'} d_F z_{i,j} \qquad (z = (z_{i,j}) \in \mathrm{M}_{n,n'}(F)).$$

Let $O_{n,n'}$ be the zero matrix in $\mathrm{M}_{n,n'}(F)$. Let 1_n be the unit matrix in $\mathrm{M}_n(F)$. Let $e_n = (O_{1,n-1}, 1) \in \mathrm{M}_{1,n}(F)$. When $n = 1$, we understand $e_1 = 1$.

2.2. **Groups and the invariant measures.** Let G_n be the general linear group $\mathrm{GL}(n,F)$ of degree n over F. We fix a maximal compact subgroup K_n of G_n by

$$K_n = \begin{cases} \mathrm{O}(n) & \text{if } F = \mathbb{R}, \\ \mathrm{U}(n) & \text{if } F = \mathbb{C}, \end{cases}$$

where $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are the orthogonal group and the unitary group of degree n, respectively. Let N_n and U_n be the groups of upper and lower triangular unipotent matrices in G_n, respectively, that is,

$$N_n = \{x = (x_{i,j}) \in G_n \mid x_{i,j} = 0 \ (1 \le j < i \le n), \ \ x_{k,k} = 1 \ (1 \le k \le n)\},$$
$$U_n = \{u = (u_{i,j}) \in G_n \mid u_{i,j} = 0 \ (1 \le i < j \le n), \ \ u_{k,k} = 1 \ (1 \le k \le n)\}.$$

We define subgroups M_n and A_n of G_n by

$$M_n = \{m = \mathrm{diag}(m_1, m_2, \cdots, m_n) \mid m_i \in G_1 = F^\times \quad (1 \le i \le n)\},$$
$$A_n = \{a = \mathrm{diag}(a_1, a_2, \cdots, a_n) \mid a_i \in \mathbb{R}_+^\times \quad (1 \le i \le n)\}.$$

Let Z_n be the center of G_n. Then we have $Z_n = \{t1_n \mid t \in G_1 = F^\times\}$. We denote by $C^\infty(G_n)$ the space of ($\mathbb{C}$-valued) smooth functions on G_n. We regard $C^\infty(G_n)$ as a G_n-module via the right translation

$$(R(g)f)(h) = f(hg) \qquad (g, h \in G_n,\ f \in C^\infty(G_n)).$$

Let dk, dx, du and da be the Haar measures on K_n, N_n, U_n and A_n, respectively. In this paper, we normalize these Haar measures by

$$\int_{K_n} dk = 1, \quad dx = \prod_{1\le i<j\le n} d_F x_{i,j}, \quad du = \prod_{1\le j<i\le n} d_F u_{i,j}, \quad da = \prod_{i=1}^{n} \frac{2c_F\, da_i}{a_i}$$

with $x = (x_{i,j}) \in N_n$, $u = (u_{i,j}) \in U_n$ and $a = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in A_n$. When $n = 1$, we understand $N_1 = U_1 = \{1\}$ and

$$\int_{N_1} f(x)\, dx = \int_{U_1} f(u)\, du = f(1)$$

for a function f on $\{1\}$. We normalize the Haar measure dg on G_n so that

$$\int_{G_n} f(g)\, dg = \int_{K_n}\int_{U_n}\int_{A_n} f(auk)\, da\, du\, dk = \int_{A_n}\int_{U_n}\int_{K_n} f(kua)\, dk\, du\, da \tag{2.1}$$

for any integrable function f on G_n. We normalize the Haar measure dh on Z_n so that

$$\int_{Z_n} f(h)\, dh = \int_{G_1} f(g1_n)\, dg$$

for any integrable function f on Z_n, where dg is the Haar measure on G_1 normalized by (2.1). We normalize the right G_n-invariant measure dg on $N_n\backslash G_n$ so that

$$\int_{G_n} f(g)\, dg = \int_{N_n\backslash G_n} \left(\int_{N_n} f(xg)\, dx \right) dg \tag{2.2}$$

for any integrable function f on G_n. We normalize the right G_n-invariant measure dg on $Z_nN_n\backslash G_n$ so that

$$\int_{N_n\backslash G_n} f(g)\, dg = \int_{Z_nN_n\backslash G_n} \left(\int_{Z_n} f(hg)\, dh \right) dg \tag{2.3}$$

for any integrable function f on $N_n\backslash G_n$.

2.3. **Principal series representations of G_n.** Following Jacquet [15], we will define principal series representations of G_n as representations induced from characters of the lower triangular Borel subgroup U_nM_n of G_n in this paper.

Let $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. For $l \in \mathbb{Z}$ and $t \in F^\times$, we set $\chi_l(t) = (t/|t|)^l$. We define characters χ_d and η_ν of M_n by

$$\chi_d(m) = \prod_{i=1}^{n} \chi_{d_i}(m_i) = \prod_{i=1}^{n} \left(\frac{m_i}{|m_i|} \right)^{d_i}, \qquad \eta_\nu(m) = \prod_{i=1}^{n} |m_i|_F^{\nu_i} = \prod_{i=1}^{n} |m_i|^{\nu_i c_F}$$

for $m = \mathrm{diag}(m_1, m_2, \cdots, m_n) \in M_n$. Let $\rho_n = (\rho_{n,1}, \rho_{n,2}, \cdots, \rho_{n,n}) \in \mathbb{Q}^n$ with

$$\rho_{n,i} = \tfrac{n+1}{2} - i \qquad (1 \le i \le n).$$

Let $I(d, \nu)$ be the subspace of $C^\infty(G_n)$ consisting of all functions f such that

$$f(umg) = \chi_d(m)\eta_{\nu-\rho_n}(m) f(g) \qquad (u \in U_n,\ m \in M_n,\ g \in G_n), \tag{2.4}$$

on which G_n acts by the right translation $\Pi_{d,\nu} = R$. We equip $I(d,\nu)$ with the usual Fréchet topology. We call $(\Pi_{d,\nu}, I(d,\nu))$ a (smooth) principal series representation of G_n. We denote by $I(d,\nu)_{K_n}$ the subspace of $I(d,\nu)$ consisting of all K_n-finite vectors. When $F = \mathbb{R}$, we note that

$$(2.5) \qquad \chi_{d+l} = \chi_d, \qquad I(d+l,\nu) = I(d,\nu) \qquad (l \in 2\mathbb{Z}^n).$$

When $I(d,\nu)$ is irreducible, for any element σ of the symmetric group $\mathfrak{S}_n$ of degree n, we have

$$(2.6) \qquad I(d,\nu) \simeq I((d_{\sigma(1)}, d_{\sigma(2)}, \cdots, d_{\sigma(n)}), (\nu_{\sigma(1)}, \nu_{\sigma(2)}, \cdots, \nu_{\sigma(n)}))$$

as representations of G_n (*cf.* [25, Corollary 2.8]).

Let $I(d)$ be the space of smooth functions f on K_n satisfying

$$f(mk) = \chi_d(m) f(k) \qquad (m \in M_n \cap K_n,\ k \in K_n),$$

and we equip this space with the usual Fréchet topology. Because of $G_n = U_n A_n K_n$ and (2.4), we can identify the space $I(d,\nu)$ with $I(d)$ via the restriction map $I(d,\nu) \ni f \mapsto f|_{K_n} \in I(d)$ to K_n. The inverse map $I(d) \ni f \mapsto f_\nu \in I(d,\nu)$ of the restriction map is given by

$$(2.7) \qquad f_\nu(uak) = \eta_{\nu-\rho_n}(a) f(k) \qquad (u \in U_n,\ a \in A_n,\ k \in K_n).$$

We regard $I(d)$ as a G_n-module via this identification, and we denote the action of G_n on $I(d)$ corresponding to $\Pi_{d,\nu}$ by Π_ν, that is,

$$(\Pi_\nu(g) f)(k) = f_\nu(kg) \qquad (g \in G_n,\ k \in K_n,\ f \in I(d)).$$

Here we note that $\Pi_\nu|_{K_n}$ is the right translation and does not depend on ν. We denote by $I(d)_{K_n}$ the subspace of $I(d)$ consisting of all K_n-finite vectors. For $f \in I(d)$, we call the map $\mathbb{C}^n \ni \nu \mapsto f_\nu \in C^\infty(G_n)$ defined by (2.7) the standard section corresponding to f.

Remark 2.1. For the study of automorphic forms such as the Eisenstein series, it is convenient to realize principal series representations of G_n as representations $(\Pi_{B_n,d,\nu}, I_{B_n}(d,\nu))$ induced from characters of the upper triangular Borel subgroup $B_n = N_n M_n$, that is, $I_{B_n}(d,\nu)$ is the subspace of $C^\infty(G_n)$ consisting of all functions f such that

$$f(xmg) = \chi_d(m) \eta_{\nu+\rho_n}(m) f(g) \qquad (x \in N_n,\ m \in M_n,\ g \in G_n),$$

and the action $\Pi_{B_n,d,\nu}$ of G_n is the right translation R. The results in this paper can be translated into this realization via the G_n-isomorphism

$$I_{B_n}(d,\nu) \ni f \mapsto f^{w_n} \in I((d_n, d_{n-1}, \cdots, d_1), (\nu_n, \nu_{n-1}, \cdots, \nu_1))$$

with $f^{w_n}(g) = f(w_n g)$ $(g \in G_n)$. Here w_n is the anti-diagonal matrix of size n with 1 at all anti-diagonal entries.

2.4. **Whittaker functions.** Let $\varepsilon \in \{\pm 1\}$, and let ψ_ε be the standard character of F defined in §2.1. Let $\psi_{\varepsilon,n}$ be a character of N_n defined by

$$\psi_{\varepsilon,n}(x) = \psi_\varepsilon(x_{1,2} + x_{2,3} + \cdots + x_{n-1,n}) \qquad (x = (x_{i,j}) \in N_n).$$

When $n = 1$, we understand that $\psi_{\varepsilon,1}$ is the trivial character of $N_1 = \{1\}$.

Let $d \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. A ψ_ε-form on $I(d,\nu)$ is a continuous $\mathbb{C}$-linear form $\mathcal{T}\colon I(d,\nu) \to \mathbb{C}$ satisfying

$$\mathcal{T}(\Pi_{d,\nu}(x) f) = \psi_{\varepsilon,n}(x) \mathcal{T}(f) \qquad (x \in N_n,\ f \in I(d,\nu)).$$

Kostant [19] shows that the space of ψ_ε-forms on $I(d,\nu)$ is one dimensional. Let us recall the construction of nonzero ψ_ε-forms on principal series representations of G_n, which are called the Jacquet integrals. If ν satisfies

$$\mathrm{Re}(\nu_{i+1}-\nu_i)>0 \qquad (1\le i\le n-1), \tag{2.8}$$

we define the Jacquet integral $\mathcal{J}_\varepsilon\colon I(d,\nu)\to\mathbb{C}$ by the integral

$$\mathcal{J}_\varepsilon(f)=\int_{N_n} f(x)\psi_{-\varepsilon,n}(x)\,dx \qquad (f\in I(d,\nu)),$$

which converges absolutely (see [30, Theorem 15.4.1]). When $n=1$, we understand $\mathcal{J}_\varepsilon(f)=f(1)$ $(f\in I(d,\nu))$. For $\nu\in\mathbb{C}^n$ satisfying (2.8), we set $\mathcal{J}_\varepsilon^{(d,\nu)}(f)=\mathcal{J}_\varepsilon(f_\nu)$ $(f\in I(d))$, where f_ν is the standard section corresponding to f. By [30, Theorem 15.4.1], we know that $\mathcal{J}_\varepsilon^{(d,\nu)}(f)$ has the holomorphic continuation to whole $\nu\in\mathbb{C}^n$ for every $f\in I(d)$, and $\mathbb{C}^n\times I(d)\ni(\nu,f)\mapsto\mathcal{J}_\varepsilon^{(d,\nu)}(f)\in\mathbb{C}$ is continuous. Furthermore, this extends $\mathcal{J}_\varepsilon^{(d,\nu)}$ to all $\nu\in\mathbb{C}^n$ as a nonzero continuous $\mathbb{C}$-linear form on $I(d)$ satisfying

$$\mathcal{J}_\varepsilon^{(d,\nu)}(\Pi_\nu(x)f)=\psi_{\varepsilon,n}(x)\mathcal{J}_\varepsilon^{(d,\nu)}(f) \qquad (x\in N_n,\ f\in I(d)).$$

We extend the Jacquet integral $\mathcal{J}_\varepsilon\colon I(d,\nu)\to\mathbb{C}$ to whole $\nu\in\mathbb{C}^n$ by

$$\mathcal{J}_\varepsilon(f)=\mathcal{J}_\varepsilon^{(d,\nu)}(f|_{K_n}) \qquad (f\in I(d,\nu))$$

which is a nonzero ψ_ε-form on $I(d,\nu)$. We set

$$\mathrm{W}_\varepsilon(f)(g)=\mathcal{J}_\varepsilon(\Pi_{d,\nu}(g)f) \qquad (f\in I(d,\nu),\ g\in G_n). \tag{2.9}$$

For $f\in I(d,\nu)$, $\mathrm{W}_\varepsilon(f)$ is called a Whittaker function for $(\Pi_{d,\nu},\psi_\varepsilon)$, and satisfies

$$\mathrm{W}_\varepsilon(f)(xg)=\psi_{\varepsilon,n}(x)\mathrm{W}_\varepsilon(f)(g) \qquad (x\in N_n,\ g\in G_n). \tag{2.10}$$

We note that $\mathrm{W}_\varepsilon(f_\nu)(g)=\mathcal{J}_\varepsilon^{(d,\nu)}(\Pi_\nu(g)f)$ is an entire function of ν for $g\in G_n$ and the standard section f_ν corresponding to $f\in I(d)_{K_n}$. Let

$$\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)=\{\mathrm{W}_\varepsilon(f)\mid f\in I(d,\nu)\}.$$

When $\Pi_{d,\nu}$ is irreducible, this is a Whittaker model of $\Pi_{d,\nu}$.

Remark 2.2. We give some remark for the topology on $\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$. Let $\mathcal{U}(\mathfrak{g}_{n\mathbb{C}})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_{n\mathbb{C}}=\mathfrak{gl}(n,F)\otimes_\mathbb{R}\mathbb{C}$ of the associated Lie algebra of G_n. Let $l>0$, and let $\mathcal{A}_l(G_n)$ be a subspace of $C^\infty(G_n)$ consisting of all functions W such that $\mathcal{Q}_{l,X}(W)<\infty$ $(X\in\mathcal{U}(\mathfrak{g}_{n\mathbb{C}}))$, where

$$\mathcal{Q}_{l,X}(W)=\sup_{g\in G_n}\|g\|^{-l}|(R(X)W)(g)|, \qquad \|g\|=\mathrm{Tr}(g^t\overline{g})+\mathrm{Tr}((g^{-1})^t\overline{(g^{-1})}).$$

We endow $\mathcal{A}_l(G_n)$ with the topology induced by the seminorms $\mathcal{Q}_{l,X}$ $(X\in\mathcal{U}(\mathfrak{g}_{n\mathbb{C}}))$. In [31, §2.7], it is proved that $(R,\mathcal{A}_l(G_n))$ is a smooth Fréchet representation of G_n of moderate growth. Assume that l is sufficiently large. Then $f\mapsto\mathrm{W}_\varepsilon(f)$ defines a continuous G_n-homomorphism from $I(d,\nu)$ to $\mathcal{A}_l(G_n)$ by [15, Proposition 3.2], and $\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$ is its image. Applying Casselman–Wallach's theorem [30, Theorem 11.6.7 (2)] to the continuous G_n-homomorphism $f\mapsto\mathrm{W}_\varepsilon(f)$ from $I(d,\nu)$ to the closure of $\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$ in $\mathcal{A}_l(G_n)$, we note that $\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$ coincides with its closure, that is, $\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$ is a closed subspace of $\mathcal{A}_l(G_n)$. Moreover, if $\Pi_{d,\nu}$ is irreducible, $I(d,\nu)\ni f\mapsto\mathrm{W}_\varepsilon(f)\in\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$ is a topological G_n-isomorphism.

2.5. **The Gelfand–Tsetlin type basis.** In this subsection, we give a Gelfand–Tsetlin type basis of an irreducible holomorphic finite dimensional representation of $\mathrm{GL}(n,\mathbb{C})$. Let $\mathfrak{gl}(n,\mathbb{C}) = \mathrm{M}_n(\mathbb{C})$ be the associated Lie algebra of $\mathrm{GL}(n,\mathbb{C})$. For $1 \le i,j \le n$, we denote by $E_{i,j}$ the matrix unit in $\mathfrak{gl}(n,\mathbb{C})$ with 1 at the (i,j)-th entry and 0 at other entries. We define the set Λ_n of dominant weights by

$$\Lambda_n = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n\}.$$

Let $(\tau_\lambda, V_\lambda)$ be an irreducible holomorphic finite dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ with highest weight $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$, and we fix a $\mathrm{U}(n)$-invariant hermitian inner product $\langle \cdot, \cdot \rangle$ on V_λ. By Weyl's dimension formula [18, Theorem 4.48], we have

$$\dim V_\lambda = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Let us recall the orthonormal basis on V_λ, which is constructed by Gel′fand and Tsetlin [4] (see Zhelobenko [34] for a detailed proof). We call

$$M = (m_{i,j})_{1 \le i \le j \le n} = \begin{pmatrix} m_{1,n} \quad m_{2,n} \quad\quad \cdots \quad\quad m_{n,n} \\ m_{1,n-1} \quad \cdots \quad m_{n-1,n-1} \\ \cdots \quad \cdots \quad \cdots \\ m_{1,2} \quad m_{2,2} \\ m_{1,1} \end{pmatrix} \qquad (m_{i,j} \in \mathbb{Z})$$

an integral triangular array of size n, and call $m_{i,j}$ the (i,j)-th entry of M. We denote by $\mathrm{G}(\lambda)$ the set of integral triangular arrays $M = (m_{i,j})_{1 \le i \le j \le n}$ of size n such that

$$m_{i,n} = \lambda_i \quad (1 \le i \le n), \qquad m_{j,k} \ge m_{j,k-1} \ge m_{j+1,k} \quad (1 \le j < k \le n). \tag{2.11}$$

For $M = (m_{i,j})_{1 \le i \le j \le n} \in \mathrm{G}(\lambda)$, we define $\gamma^M = (\gamma^M_1, \gamma^M_2, \cdots, \gamma^M_n)$ by

$$\gamma^M_j = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1} \qquad (1 \le j \le n). \tag{2.12}$$

We call γ^M the weight of M. Gelfand and Tsetlin construct an orthonormal basis $\{\zeta_M\}_{M \in \mathrm{G}(\lambda)}$ of V_λ with the following formulas of $\mathfrak{gl}(n,\mathbb{C})$-actions:

$$\tau_\lambda(E_{k,k})\zeta_M = \gamma^M_k \zeta_M \qquad (1 \le k \le n), \tag{2.13}$$

$$\tau_\lambda(E_{j,j+1})\zeta_M = \sum_{\substack{1 \le i \le j \\ M + \Delta_{i,j} \in \mathrm{G}(\lambda)}} \tilde{\mathrm{a}}^{+}_{i,j}(M)\zeta_{M+\Delta_{i,j}} \qquad (1 \le j \le n-1), \tag{2.14}$$

$$\tau_\lambda(E_{j+1,j})\zeta_M = \sum_{\substack{1 \le i \le j \\ M - \Delta_{i,j} \in \mathrm{G}(\lambda)}} \tilde{\mathrm{a}}^{-}_{i,j}(M)\zeta_{M-\Delta_{i,j}} \qquad (1 \le j \le n-1) \tag{2.15}$$

for $M=(m_{i,j})_{1\le i\le j\le n}\in \mathrm{G}(\lambda)$, where $\Delta_{i,j}$ is the integral triangular array of size n with 1 at the (i,j)-th entry and 0 at other entries, and

$$\tilde{\mathrm{a}}^{+}_{i,j}(M)=\left|\frac{\left(\prod_{h=1}^{j+1}(m_{h,j+1}-m_{i,j}-h+i)\right)\prod_{h=1}^{j-1}(m_{h,j-1}-m_{i,j}-h+i-1)}{\prod_{1\le h\le j,\ h\neq i}(m_{h,j}-m_{i,j}-h+i)(m_{h,j}-m_{i,j}-h+i-1)}\right|^{\frac{1}{2}},$$

$$\tilde{\mathrm{a}}^{-}_{i,j}(M)=\left|\frac{\left(\prod_{h=1}^{j+1}(m_{h,j+1}-m_{i,j}-h+i+1)\right)\prod_{h=1}^{j-1}(m_{h,j-1}-m_{i,j}-h+i)}{\prod_{1\le h\le j,\ h\neq i}(m_{h,j}-m_{i,j}-h+i)(m_{h,j}-m_{i,j}-h+i+1)}\right|^{\frac{1}{2}}.$$

We denote by $H(\lambda)$ a unique element of $\mathrm{G}(\lambda)$ whose weight is λ, that is,

$$H(\lambda)=(h_{i,j})_{1\le i\le j\le n}\in \mathrm{G}(\lambda)\qquad \text{with}\quad h_{i,j}=\lambda_i. \tag{2.16}$$

Then $\zeta_{H(\lambda)}$ is a highest vector in V_λ, that is,

$$\tau_\lambda(E_{i,i})\zeta_{H(\lambda)}=\lambda_i\zeta_{H(\lambda)}\quad (1\le i\le n),\qquad \tau_\lambda(E_{j,k})\zeta_{H(\lambda)}=0\quad (1\le j<k\le n).$$

There is a $\mathbb{Q}$-rational structure of V_λ associated to the highest weight vector $\zeta_{H(\lambda)}$. It comes from the natural $\mathbb{Q}$-rational structure of a tensor power of the standard representation of $\mathrm{GL}(n,\mathbb{C})$. We fix an embedding of V_λ into a tensor power of the standard representation of $\mathrm{GL}(n,\mathbb{C})$ so that the image of $\zeta_{H(\lambda)}$ is $\mathbb{Q}$-rational, and give a $\mathbb{Q}$-rational structure of V_λ via this embedding.

Let us construct a Gelfand–Tsetlin type $\mathbb{Q}$-rational basis of V_λ. We set

$$\xi_M=\sqrt{\mathrm{r}(M)}\zeta_M\qquad\qquad (M=(m_{i,j})_{1\le i\le j\le n}\in \mathrm{G}(\lambda)) \tag{2.17}$$

with the rational constant

$$\mathrm{r}(M)=\prod_{1\le i\le j<k\le n}\frac{(m_{i,k}-m_{j,k-1}-i+j)!(m_{i,k-1}-m_{j+1,k}-i+j)!}{(m_{i,k-1}-m_{j,k-1}-i+j)!(m_{i,k}-m_{j+1,k}-i+j)!}. \tag{2.18}$$

Then $\{\xi_M\}_{M\in\mathrm{G}(\lambda)}$ is an orthogonal basis of V_λ such that $\langle\xi_M,\xi_M\rangle=\mathrm{r}(M)$ ($M\in \mathrm{G}(\lambda)$). For an integral triangular array $M=(m_{i,j})_{1\le i\le j\le n}$, we define the dual triangular array $M^\vee=(m^\vee_{i,j})_{1\le i\le j\le n}$ of M by $m^\vee_{i,j}=-m_{j+1-i,j}$. The formulas corresponding to (2.13), (2.14) and (2.15) are given respectively by

$$\tau_\lambda(E_{k,k})\xi_M=\gamma_k^M\xi_M\qquad\qquad (1\le k\le n), \tag{2.19}$$

$$\tau_\lambda(E_{j,j+1})\xi_M=\sum_{\substack{1\le i\le j\\ M+\Delta_{i,j}\in\mathrm{G}(\lambda)}}\mathrm{a}_{i,j}(M)\xi_{M+\Delta_{i,j}}\qquad (1\le j\le n-1), \tag{2.20}$$

$$\tau_\lambda(E_{j+1,j})\xi_M=\sum_{\substack{1\le i\le j\\ M+\Delta^\vee_{i,j}\in\mathrm{G}(\lambda)}}\mathrm{a}_{i,j}(M^\vee)\xi_{M+\Delta^\vee_{i,j}}\qquad (1\le j\le n-1) \tag{2.21}$$

for $M=(m_{i,j})_{1\le i\le j\le n}\in \mathrm{G}(\lambda)$, where $\mathrm{a}_{i,j}(M)$ is a rational number given by

$$\mathrm{a}_{i,j}(M)=\frac{\prod_{h=1}^{i}(m_{h,j+1}-m_{i,j}-h+i)}{\prod_{h=1}^{i-1}(m_{h,j}-m_{i,j}-h+i)}\left(\prod_{h=2}^{i}\frac{m_{h-1,j-1}-m_{i,j}-h+i}{m_{h-1,j}-m_{i,j}-h+i}\right).$$

By these formulas and $\xi_{H(\lambda)}=\zeta_{H(\lambda)}$, we know that $\{\xi_M\}_{M\in\mathrm{G}(\lambda)}$ is a $\mathbb{Q}$-rational basis of V_λ.

Until the end of this subsection, we assume $n>1$. Let

$$\Xi^+(\lambda)=\{\mu=(\mu_1,\mu_2,\cdots,\mu_{n-1})\in\Lambda_{n-1}\mid \lambda_i\ge\mu_i\ge\lambda_{i+1}\ \ (1\le i\le n-1)\}.$$

We regard $\mathrm{GL}(n-1,\mathbb{C})$ as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ via the embedding

$$\iota_n\colon \mathrm{GL}(n-1,\mathbb{C}) \ni g \mapsto \begin{pmatrix} g & O_{n-1,1} \\ O_{1,n-1} & 1 \end{pmatrix} \in \mathrm{GL}(n,\mathbb{C}). \tag{2.22}$$

We set $\widehat{M} = (m_{i,j})_{1\le i\le j\le n-1}$ for $M = (m_{i,j})_{1\le i\le j\le n} \in \mathrm{G}(\lambda)$. By the construction of $\{\xi_M\}_{M\in\mathrm{G}(\lambda)}$, we know that V_λ has the irreducible decomposition

$$V_\lambda = \bigoplus_{\mu\in\Xi^+(\lambda)} V_{\lambda,\mu}, \qquad V_{\lambda,\mu} = \bigoplus_{M\in\mathrm{G}(\lambda;\mu)} \mathbb{C}\xi_M \simeq V_\mu, \tag{2.23}$$

as a $\mathrm{GL}(n-1,\mathbb{C})$-module, where

$$\mathrm{G}(\lambda;\mu) = \{M \in \mathrm{G}(\lambda) \mid \widehat{M} \in \mathrm{G}(\mu)\}.$$

Let $\mu \in \Xi^+(\lambda)$. For $M \in \mathrm{G}(\mu)$, we denote by $M[\lambda]$ the element of $\mathrm{G}(\lambda;\mu)$ characterized by $\widehat{M[\lambda]} = M$, that is,

$$M[\lambda] = \begin{pmatrix} \lambda \\ M \end{pmatrix} \in \mathrm{G}(\lambda;\mu). \tag{2.24}$$

Then we have $H(\mu)[\lambda] = \begin{pmatrix} \lambda \\ H(\mu) \end{pmatrix}$, and $\xi_{H(\mu)[\lambda]}$ is the highest weight vector in the $\mathrm{GL}(n-1,\mathbb{C})$-module $V_{\lambda,\mu}$. For later use, we prepare Lemma 2.3.

Lemma 2.3. *Retain the notation.*
(1) *We define a $\mathbb{C}$-linear map $\tilde{\mathrm{I}}^\lambda_\mu\colon V_\mu \to V_{\lambda,\mu}$ by*

$$\tilde{\mathrm{I}}^\lambda_\mu(\zeta_M) = \zeta_{M[\lambda]} \qquad (M \in \mathrm{G}(\mu)).$$

Then $\tilde{\mathrm{I}}^\lambda_\mu$ is a $\mathrm{GL}(n-1,\mathbb{C})$-isomorphism which preserves the inner products $\langle\cdot,\cdot\rangle$.
(2) *We define a surjective $\mathbb{C}$-linear map $\tilde{\mathrm{R}}^\lambda_\mu\colon V_\lambda \to V_\mu$ by*

$$\tilde{\mathrm{R}}^\lambda_\mu(\zeta_M) = \begin{cases} \zeta_{\widehat{M}} & \text{if } M \in \mathrm{G}(\lambda;\mu), \\ 0 & \text{otherwise} \end{cases} \qquad (M \in \mathrm{G}(\lambda)).$$

Then $\tilde{\mathrm{R}}^\lambda_\mu$ is a $\mathrm{GL}(n-1,\mathbb{C})$-homomorphism, and $\tilde{\mathrm{R}}^\lambda_\mu \circ \tilde{\mathrm{I}}^\lambda_\mu$ is the identity map on V_μ.
(3) *We define a surjective $\mathbb{C}$-linear map $\mathrm{R}^\lambda_\mu\colon V_\lambda \to V_\mu$ by*

$$\mathrm{R}^\lambda_\mu(\xi_M) = \begin{cases} \xi_{\widehat{M}} & \text{if } M \in \mathrm{G}(\lambda;\mu), \\ 0 & \text{otherwise} \end{cases} \qquad (M \in \mathrm{G}(\lambda)).$$

Then R^λ_μ is a $\mathrm{GL}(n-1,\mathbb{C})$-homomorphism.

Proof. Since $\{\zeta_M\}_{M\in\mathrm{G}(\lambda)}$ and $\{\zeta_N\}_{N\in\mathrm{G}(\mu)}$ are orthonormal basis, we obtain the statements (1) and (2) by (2.13), (2.14) and (2.15). The statement (3) follows from (2.19), (2.20) and (2.21). □

2.6. **Complex conjugate representations.** For a finite dimensional representation (τ, V_τ) of $\mathrm{GL}(n,\mathbb{C})$, we define the complex conjugate representation $(\overline{\tau}, \overline{V_\tau})$ of τ as follows:

- Let $\overline{V_\tau}$ be a set with a fixed bijective map $V_\tau \ni v \mapsto \overline{v} \in \overline{V_\tau}$. We regard $\overline{V_\tau}$ as a $\mathbb{C}$-vector space via the following addition and scalar multiplication:

  $$\overline{v_1} + \overline{v_2} = \overline{v_1 + v_2} \quad (v_1, v_2 \in V_\tau), \qquad c\overline{v} = \overline{\overline{c}v} \quad (c \in \mathbb{C},\ v \in V_\tau),$$

  where $\overline{c}$ is the complex conjugate of c.
- The action $\overline{\tau}$ is defined by $\overline{\tau}(g)\overline{v} = \overline{\tau(g)v} \quad (g \in \mathrm{GL}(n,\mathbb{C}),\ v \in V_\tau)$.

By definition, we obtain the following identifications and the natural maps for finite dimensional representations (τ, V_τ) and $(\tau', V_{\tau'})$ of $\mathrm{GL}(n,\mathbb{C})$:

- The complex conjugate representation $(\overline{\overline{\tau}}, \overline{\overline{V_\tau}})$ of $\overline{\tau}$ is naturally identified with (τ, V_τ) via the correspondence $\overline{\overline{v}} \leftrightarrow v$ $(v \in V_\tau)$.
- If $\langle \cdot, \cdot \rangle$ is a $\mathrm{U}(n)$-invariant hermitian inner product on V_τ, then
$$V_\tau \otimes_{\mathbb{C}} \overline{V_\tau} \ni v_1 \otimes \overline{v_2} \mapsto \langle v_1, v_2 \rangle \in \mathbb{C}$$
is a nondegenerate $\mathbb{C}$-bilinear $\mathrm{U}(n)$-invariant pairing.
- The complex conjugate representation $(\overline{\tau \otimes \tau'}, \overline{V_\tau \otimes_{\mathbb{C}} V_{\tau'}})$ of $\tau \otimes \tau'$ is naturally identified with $(\overline{\tau} \otimes \overline{\tau'}, \overline{V_\tau} \otimes_{\mathbb{C}} \overline{V_{\tau'}})$ via the correspondence
$$\overline{v_1 \otimes v_2} \leftrightarrow \overline{v_1} \otimes \overline{v_2} \qquad (v_1 \in V_\tau,\ v_2 \in V_{\tau'}).$$
- For any subgroup S of $\mathrm{GL}(n,\mathbb{C})$, there is a bijective $\mathbb{C}$-anti-linear map
$$\mathrm{Hom}_S(V_\tau, V_{\tau'}) \ni \Psi \mapsto \overline{\Psi} \in \mathrm{Hom}_S(\overline{V_\tau}, \overline{V_{\tau'}})$$
defined by $\overline{\Psi}(\overline{v}) = \overline{\Psi(v)} \in \overline{V_{\tau'}}$ $(v \in V_\tau)$.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$. We consider the complex conjugate representation $(\overline{\tau_\lambda}, \overline{V_\lambda})$ of τ_λ. We denote by $\mathfrak{u}(n)$ the associated Lie algebra of $\mathrm{U}(n)$. The complexification $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{u}(n)$ is isomorphic to $\mathfrak{gl}(n,\mathbb{C})$ via the correspondence $E^{\mathfrak{u}(n)}_{i,j} \leftrightarrow E_{i,j}$ $(1 \le i,j \le n)$ with
$$E^{\mathfrak{u}(n)}_{i,j} = \frac{1}{2}\{(E_{i,j} - E_{j,i}) \otimes 1 - \sqrt{-1}(E_{i,j} + E_{j,i}) \otimes \sqrt{-1}\} \in \mathfrak{u}(n)_{\mathbb{C}}.$$
For $1 \le i,j \le n$ and $v \in V_\lambda$, we have
$$\tau_\lambda(E^{\mathfrak{u}(n)}_{i,j})v = \tau_\lambda(E_{i,j})v, \qquad \overline{\tau_\lambda}(E^{\mathfrak{u}(n)}_{i,j})\overline{v} = -\overline{\tau_\lambda(E_{j,i})v}. \tag{2.25}$$
By the pairing $V_\lambda \otimes_{\mathbb{C}} \overline{V_\lambda} \ni v_1 \otimes \overline{v_2} \mapsto \langle v_1, v_2 \rangle \in \mathbb{C}$, we can identify $(\overline{\tau_\lambda}, \overline{V_\lambda})$ with the contragredient representation $(\tau_\lambda^\vee, V_\lambda^\vee)$ of τ_λ as a $\mathrm{U}(n)$-module. Let $\lambda^\vee = (-\lambda_n, -\lambda_{n-1}, \cdots, -\lambda_1) \in \Lambda_n$. Since $V_\lambda^\vee \simeq V_{\lambda^\vee}$ as $\mathrm{GL}(n,\mathbb{C})$-modules, we have $\overline{V_\lambda} \simeq V_{\lambda^\vee}$ as $\mathrm{U}(n)$-modules. In fact, by (2.19), (2.20), (2.21) and (2.25), we can confirm that the $\mathbb{C}$-linear map
$$\overline{V_\lambda} \ni \overline{\xi_M} \mapsto (-1)^{\sum_{1\le i\le j\le n} m_{i,j}} \xi_{M^\vee} \in V_{\lambda^\vee} \qquad (M = (m_{i,j})_{1\le i\le j\le n} \in \mathrm{G}(\lambda))$$
is a $\mathrm{U}(n)$-isomorphism. Via this isomorphism, we derive the $\mathbb{Q}$-rational structure of $\overline{V_\lambda}$ from that of $V_{\lambda^\vee}$. Then $\{\overline{\xi_M}\}_{M\in \mathrm{G}(\lambda)}$ is a $\mathbb{Q}$-rational basis of $\overline{V_\lambda}$.

Remark 2.4. We note that $\{E_{i,j} - E_{j,i}\}_{1\le i<j\le n}$ forms a basis of the associated Lie algebra $\mathfrak{o}(n)$ of $\mathrm{O}(n)$. By (2.20), (2.21) and (2.25), we know that
$$\overline{V_\lambda} \ni \overline{\xi_M} \mapsto \xi_M \in V_\lambda \qquad (M \in \mathrm{G}(\lambda))$$
defines a $\mathbb{Q}$-rational $\mathrm{O}(n)$-isomorphism.

2.7. **The minimal K_n- and $K_{n'}$-types.** We define a subset $\Lambda_{n,F}$ of Λ_n by $\Lambda_{n,\mathbb{R}} = \Lambda_n \cap \{0,1\}^n$ and $\Lambda_{n,\mathbb{C}} = \Lambda_n$. In §4.2, we study the $\mathrm{O}(n)$-module structure of V_λ for $\lambda \in \Lambda_{n,\mathbb{R}}$, and prove Lemma 2.5.

Lemma 2.5. *Let $\lambda \in \Lambda_{n,F}$. Then V_λ is an irreducible K_n-module. Moreover, for any $\lambda' \in \Lambda_{n,F}$ such that $\lambda' \ne \lambda$, we have $V_\lambda \not\simeq V_{\lambda'}$ as K_n-modules.*

Let $(\Pi_{d,\nu}, I(d,\nu))$ be a principal series representations of G_n with

$$d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n, \qquad \nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$$

such that $d \in \Lambda_{n,F}$. By (2.19) and the Frobenius reciprocity law [18, Theorem 1.14], we know that $\tau_d|_{K_n}$ is the minimal K_n-type of $\Pi_{d,\nu}$, and $\mathrm{Hom}_{K_n}(V_d, I(d,\nu))$ is 1 dimensional. Let $\mathrm{f}_{d,\nu}\colon V_d \to I(d,\nu)$ be the K_n-homomorphism normalized by $\mathrm{f}_{d,\nu}(\xi_{H(d)})(1_n) = 1$, that is,

$$\mathrm{f}_{d,\nu}(v)(uak) = \eta_{\nu-\rho_n}(a)\langle \tau_d(k)v, \xi_{H(d)}\rangle \tag{2.26}$$

for $u \in U_n$, $a \in A_n$, $k \in K_n$ and $v \in V_d$. Here $H(\lambda)$ $(\lambda \in \Lambda_n)$ are defined by (2.16). For $v \in V_d$, we note that $\mathrm{f}_{d,\nu}(v)$ is the standard section corresponding to $\mathrm{f}_d(v) \in I(d)$ defined by $\mathrm{f}_d(v)(k) = \langle \tau_d(k)v, \xi_{H(d)}\rangle$ $(k \in K_n)$.

Let $(\Pi_{d',\nu'}, I(d',\nu'))$ be a principal series representations of $G_{n'}$ with

$$d' = (d'_1, d'_2, \cdots, d'_{n'}) \in \mathbb{Z}^{n'}, \qquad \nu' = (\nu'_1, \nu'_2, \cdots, \nu'_{n'}) \in \mathbb{C}^{n'}$$

such that $-d' \in \Lambda_{n',F}$. By (2.19) and the Frobenius reciprocity law [18, Theorem 1.14], we know that $\overline{\tau_{-d'}}|_{K_{n'}}$ is the minimal $K_{n'}$-type of $\Pi_{d',\nu'}$, and the space $\mathrm{Hom}_{K_{n'}}(\overline{V_{-d'}}, I(d',\nu'))$ is 1 dimensional. Let $\overline{\mathrm{f}}_{d',\nu'}\colon \overline{V_{-d'}} \to I(d',\nu')$ be the $K_{n'}$-homomorphism normalized by $\overline{\mathrm{f}}_{d',\nu'}(\overline{\xi_{H(-d')}})(1_{n'}) = 1$, that is,

$$\overline{\mathrm{f}}_{d',\nu'}(\overline{v})(uak) = \eta_{\nu'-\rho_{n'}}(a)\overline{\langle \tau_{-d'}(k)v, \xi_{H(-d')}\rangle} \tag{2.27}$$

for $u \in U_{n'}$, $a \in A_{n'}$, $k \in K_{n'}$ and $v \in V_{-d'}$. For $v \in V_{-d'}$, we note that $\overline{\mathrm{f}}_{d',\nu'}(\overline{v})$ is the standard section corresponding to $\overline{\mathrm{f}}_{d'}(\overline{v}) \in I(d')$ defined by $\overline{\mathrm{f}}_{d'}(\overline{v})(k) = \overline{\langle \tau_{-d'}(k)v, \xi_{H(-d')}\rangle}$ $(k \in K_{n'})$.

We define the archimedean L-factor for $\Pi_{d,\nu} \times \Pi_{d',\nu'}$ by

$$L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) = \prod_{i=1}^{n}\prod_{j=1}^{n'} \Gamma_F(s + \nu_i + \nu'_j;\, |d_i + d'_j|),$$

where the functions $\Gamma_F(s;m)$ $(m \in \mathbb{Z})$ are defined in §2.1. Moreover, we set

$$\mathbf{\Gamma}_F(\nu; d) = \prod_{1 \le i < j \le n} \Gamma_F(\nu_j - \nu_i + 1;\, |d_i - d_j|),$$

$$\mathbf{\Gamma}_F(\nu'; d') = \prod_{1 \le i < j \le n'} \Gamma_F(\nu'_j - \nu'_i + 1;\, |d'_i - d'_j|).$$

In §5.1, we prove Proposition 2.6.

Proposition 2.6. *Retain the notation. Let* $\varepsilon \in \{\pm 1\}$.

(1) *Let* $g \in G_n$ *and* $v \in V_d$. *Then* $\mathbf{\Gamma}_F(\nu; d)\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v))(g)$ *is an entire function of* ν. *Moreover, we have* $1/\mathbf{\Gamma}_F(\nu; d) \neq 0$ *for* $\nu \in \mathbb{C}^n$ *such that* $\Pi_{d,\nu}$ *is irreducible.*

(2) *Let* $g \in G_{n'}$ *and* $v \in V_{-d'}$. *Then* $\mathbf{\Gamma}_F(\nu'; d')\mathrm{W}_\varepsilon(\overline{\mathrm{f}}_{d',\nu'}(\overline{v}))(g)$ *is an entire function of* ν'. *Moreover, we have* $1/\mathbf{\Gamma}_F(\nu'; d') \neq 0$ *for* $\nu' \in \mathbb{C}^{n'}$ *such that* $\Pi_{d',\nu'}$ *is irreducible.*

2.8. **Archimedean Rankin–Selberg integrals for $G_n \times G_{n-1}$.** In this subsection, we assume $n > 1$. Let $(\Pi_{d,\nu}, I(d,\nu))$ and $(\Pi_{d',\nu'}, I(d',\nu'))$ be principal series representations of G_n and G_{n-1}, respectively, with parameters

$$d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n, \qquad \nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n,$$

$$d' = (d'_1, d'_2, \cdots, d'_{n-1}) \in \mathbb{Z}^{n-1}, \quad \nu' = (\nu'_1, \nu'_2, \cdots, \nu'_{n-1}) \in \mathbb{C}^{n-1}.$$

We assume $d \in \Lambda_{n,F}$ and $-d' \in \Lambda_{n-1,F}$. If $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ are irreducible representations, these are not serious assumptions because of (2.5) and (2.6). We take $\mathrm{f}_{d,\nu}$, $\bar{\mathrm{f}}_{d',\nu'}$, $\mathbf{\Gamma}_F(\nu; d)$, $\mathbf{\Gamma}_F(\nu'; d')$ and $L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})$ as in §2.7 with $n' = n - 1$.

Let $\varepsilon \in \{\pm 1\}$, $W \in \mathcal{W}(\Pi_{d,\nu}, \psi_\varepsilon)$, $W' \in \mathcal{W}(\Pi_{d',\nu'}, \psi_{-\varepsilon})$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s)$ is sufficiently large. We define the archimedean Rankin–Selberg integral $Z(s, W, W')$ for $\Pi_{d,\nu} \times \Pi_{d',\nu'}$ by

$$Z(s, W, W') = \int_{N_{n-1} \backslash G_{n-1}} W(\iota_n(g)) W'(g) |\det g|_F^{s-1/2}\, dg,$$

where ι_n is defined by (2.22). Here we note

$$Z(s, R(\iota_n(k))W, R(k)W') = Z(s, W, W') \qquad (k \in K_{n-1}). \tag{2.28}$$

By (2.28), we know that

$$v_1 \otimes \overline{v_2} \mapsto Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2}))) \tag{2.29}$$

defines an element of $\mathrm{Hom}_{K_{n-1}}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}}, \mathbb{C}_{\mathrm{triv}})$. Here W_ε is defined by (2.9), and $\mathbb{C}_{\mathrm{triv}} = \mathbb{C}$ is the trivial K_{n-1}-module. Theorem 2.7 is the first main result of this paper, which gives the explicit expression of the K_{n-1}-homomorphism (2.29).

Theorem 2.7. *Retain the notation. For $v_1 \in V_d$ and $v_2 \in V_{-d'}$, we have*

$$\begin{aligned} &Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2}))) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i + d_i')}}{(\dim V_{-d'}) \mathbf{\Gamma}_F(\nu; d) \mathbf{\Gamma}_F(\nu'; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) \langle \mathrm{R}^d_{-d'}(v_1),\, v_2 \rangle \end{aligned}$$

if $-d' \in \Xi^+(d)$, and $Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2}))) = 0$ otherwise. Here $\mathrm{R}^d_{-d'}$ is given explicitly in Lemma 2.3(3). *In particular, we have*

$$\begin{aligned} &Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_{H(-d')[d]})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\xi_{H(-d')}}))) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i + d_i')}}{(\dim V_{-d'}) \mathbf{\Gamma}_F(\nu; d) \mathbf{\Gamma}_F(\nu'; d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}) \end{aligned} \tag{2.30}$$

if $-d' \in \Xi^+(d)$. Here $H(-d')$ and $H(-d')[d]$ are defined by (2.16) *and* (2.24).

Remark 2.8. Retain the notation, and assume that $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ are both irreducible. By Lemma 4.3 in §4.2, $\mathrm{Hom}_{K_{n-1}}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}}, \mathbb{C}_{\mathrm{triv}})$ is 1 dimensional if $-d' \in \Xi^+(d)$, and is equal to $\{0\}$ otherwise. Hence, Theorem 2.7 and Proposition 2.6 imply that (2.29) vanishes if and only if $\mathrm{Hom}_{K_{n-1}}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}}, \mathbb{C}_{\mathrm{triv}}) = \{0\}$.

Remark 2.9. We set $F = \mathbb{C}$. By [22, Proposition 2.14 and Theorem 2.21], we note that the compatible pairs of cohomological representations of G_n and G_{n-1} in Sun [28, §6] can be regarded as pairs of some irreducible principal series representations $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ with $d \in \Lambda_{n,\mathbb{C}}$ and $-d' \in \Xi^+(d)$. Hence, we have another proof of the nonvanishing result [28, Theorem C] for $G_n \times G_{n-1}$, using Theorem 2.7 instead of the analogue of [28, Proposition 4.1] for the complex case.

Corollary 2.10. *Retain the notation, and assume $-d' \in \Xi^+(d)$. Then*

$$\sum_{M \in \mathrm{G}(-d')} \mathrm{r}(M)^{-1}\, \xi_{M[d]} \otimes \overline{\xi_M}$$

is a unique $\mathbb{Q}$*-rational* K_{n-1}*-invariant vector in* $V_d \otimes_{\mathbb{C}} \overline{V_{-d'}}$ *up to scalar multiple, and its image under the* K_{n-1}*-homomorphism* (2.29) *is given by*

$$\sum_{M\in \mathrm{G}(-d')} \mathrm{r}(M)^{-1} Z(s, \mathrm{W}_{\varepsilon}(\mathrm{f}_{d,\nu}(\xi_{M[d]})), \mathrm{W}_{-\varepsilon}(\overline{\mathrm{f}}_{d',\nu'}(\overline{\xi_M})))$$
$$= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d'_i)}}{\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')} L(s, \Pi_{d,\nu}\times\Pi_{d',\nu'}).$$

Here $\mathrm{r}(M)$ *and* $M[d]$ *are defined by* (2.18) *and* (2.24)*, respectively.*

Corollary 2.10 follows from Theorem 2.7 with Lemma 4.3, and is an analogue of Corollary 2.16 which gives the explicit description of the archimedean part of [6, Theorem 2].

2.9. **Schwartz functions.** Let $\mathcal{S}(\mathrm{M}_{n,n'}(F))$ be the space of Schwartz functions on $\mathrm{M}_{n,n'}(F)$. We define $\mathbf{e}_{(n,n')} \in \mathcal{S}(\mathrm{M}_{n,n'}(F))$ by

$$\mathbf{e}_{(n,n')}(z) = \exp\bigl(-\pi \mathrm{c}_F \mathrm{Tr}({}^t\overline{z}z)\bigr) = \begin{cases} \exp(-\pi\mathrm{Tr}({}^tzz)) & \text{if } F=\mathbb{R},\\ \exp(-2\pi\mathrm{Tr}({}^t\overline{z}z)) & \text{if } F=\mathbb{C}\end{cases}$$

for $z \in \mathrm{M}_{n,n'}(F)$. We denote $\mathbf{e}_{(n,n)}$ simply by $\mathbf{e}_{(n)}$. Let $\mathcal{S}_0(\mathrm{M}_{n,n'}(F))$ be the subspace of $\mathcal{S}(\mathrm{M}_{n,n'}(F))$ consisting of all functions ϕ of the form

$$\phi(z) = p(z,\overline{z})\mathbf{e}_{(n,n')}(z) \qquad (z\in \mathrm{M}_{n,n'}(F)),$$

where p is a polynomial function. We call elements of $\mathcal{S}_0(\mathrm{M}_{n,n'}(F))$ standard Schwartz functions on $\mathrm{M}_{n,n'}(F)$.

Let $C(\mathrm{M}_{n,n'}(F))$ be the space of continuous functions on $\mathrm{M}_{n,n'}(F)$. We define actions of G_n and $G_{n'}$ on $C(\mathrm{M}_{n,n'}(F))$ by

$$(L(g)f)(z) = f(g^{-1}z), \qquad (R(h)f)(z) = f(zh)$$

for $g\in G_n$, $h \in G_{n'}$, $f \in C(\mathrm{M}_{n,n'}(F))$ and $z \in \mathrm{M}_{n,n'}(F)$. Since $\mathbf{e}_{(n,n')}$ is $K_n\times K_{n'}$-invariant, we note that $\mathcal{S}_0(\mathrm{M}_{n,n'}(F))$ is closed under the action $L\boxtimes R$ of $K_n \times K_{n'}$, and all elements of $\mathcal{S}_0(\mathrm{M}_{n,n'}(F))$ are $K_n\times K_{n'}$-finite.

Let $l\in\mathbb{N}_0$, and we consider the representation $(\tau_{(l,\mathbf{0}_{n-1})}, V_{(l,\mathbf{0}_{n-1})})$. Here we put $\mathbf{0}_{n-1} = (0,0,\cdots,0)\in\Lambda_{n-1}$ if $n>1$, and erase $\mathbf{0}_{n-1}$ if $n=1$. We set

$$(2.31)\qquad \ell(\gamma) = \gamma_1+\gamma_2+\cdots+\gamma_n \qquad (\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{Z}^n).$$

For $\gamma = (\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$, we define an integral triangular array $Q(\gamma)$ of size n by

$$(2.32)\qquad Q(\gamma) = (q_{i,j})_{1\le i\le j\le n} \quad \text{with} \quad q_{i,j} = \begin{cases}\sum_{k=1}^j \gamma_k & \text{if } i=1,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.11. *Retain the notation. For* $\gamma\in\mathbb{N}_0^n$ *such that* $\ell(\gamma)=l$*, the integral triangular array* $Q(\gamma)$ *is a unique element of* $\mathrm{G}((l,\mathbf{0}_{n-1}))$ *whose weight is* γ*. Moreover, we have*

$$\mathrm{G}((l,\mathbf{0}_{n-1})) = \{Q(\gamma) \mid \gamma\in\mathbb{N}_0^n,\ \ell(\gamma)=l\}.$$

Proof. By (2.11), for an integral triangular array $M=(m_{i,j})_{1\le i\le j\le n}$ of size n, we note that M is an element of $\mathrm{G}((l,\mathbf{0}_{n-1}))$ if and only if

$$l = m_{1,n}\ge m_{1,n-1}\ge\cdots\ge m_{1,1}\ge 0, \qquad m_{i,j}=0 \qquad (2\le i\le j\le n).$$

Hence, the assertion follows from the definition (2.12) of the weight. ☐

We define $\mathbb{C}$-linear maps $\varphi_{1,n}^{(l)} \colon V_{(l,\mathbf{0}_{n-1})} \to \mathcal{S}_0(\mathrm{M}_{1,n}(F))$ and $\overline{\varphi}_{1,n}^{(l)} \colon \overline{V_{(l,\mathbf{0}_{n-1})}} \to \mathcal{S}_0(\mathrm{M}_{1,n}(F))$ by

$$\varphi_{1,n}^{(l)}(\xi_{Q(\gamma)})(z) = z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n} \mathbf{e}_{(1,n)}(z), \tag{2.33}$$

$$\overline{\varphi}_{1,n}^{(l)}(\overline{\xi_{Q(\gamma)}})(z) = \overline{z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n}}\, \mathbf{e}_{(1,n)}(z) \tag{2.34}$$

for $z = (z_1, z_2, \cdots, z_n) \in \mathrm{M}_{1,n}(F)$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^n$ such that $\ell(\gamma) = l$. In §4.4, we prove Lemma 2.12.

Lemma 2.12. *Retain the notation and we regard $\mathcal{S}_0(\mathrm{M}_{1,n}(F))$ as a K_n-module via the action R. Then $\varphi_{1,n}^{(l)}$ and $\overline{\varphi}_{1,n}^{(l)}$ are K_n-homomorphisms.*

2.10. **Injector.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$ and $l \in \mathbb{N}_0$. In this subsection, we specify each irreducible component of the tensor product $V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$. Let

$$\Xi^\circ(\lambda) = \{\lambda' = (\lambda_1', \lambda_2', \cdots, \lambda_n') \in \Lambda_n \mid \lambda_1' \geq \lambda_1 \geq \lambda_2' \geq \lambda_2 \geq \cdots \geq \lambda_n' \geq \lambda_n\},$$

and $\Xi^\circ(\lambda; l) = \{\lambda' \in \Xi^\circ(\lambda) \mid \ell(\lambda' - \lambda) = l\}$. Then Pieri's rule [5, Corollary 9.2.4] asserts that $V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$ has the irreducible decomposition

$$V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})} \simeq \bigoplus_{\lambda' \in \Xi^\circ(\lambda;l)} V_{\lambda'} \tag{2.35}$$

as $\mathrm{GL}(n,\mathbb{C})$-modules. We define a $\mathrm{U}(n)$-invariant hermitian inner product on $V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$ by

$$\langle v_1 \otimes v_1',\ v_2 \otimes v_2' \rangle = \langle v_1, v_2 \rangle \langle v_1', v_2' \rangle \qquad (v_1, v_2 \in V_\lambda,\ v_1', v_2' \in V_{(l,\mathbf{0}_{n-1})}).$$

For $\lambda' = (\lambda_1', \lambda_2', \cdots, \lambda_n') \in \Xi^\circ(\lambda)$, we set

$$\mathrm{S}^\circ(\lambda', \lambda) = \frac{\prod_{1 \leq i \leq j \leq n} (\lambda_i' - \lambda_j - i + j)!}{\prod_{1 \leq i \leq j < n} (\lambda_i - \lambda_{j+1}' - i + j)!}, \tag{2.36}$$

$$\mathrm{C}^\circ(\lambda'; \lambda) = \prod_{1 \leq i < j \leq n} \frac{(\lambda_i' - \lambda_j' - i + j)!(\lambda_i - \lambda_j - i + j - 1)!}{(\lambda_i' - \lambda_j - i + j)!(\lambda_i - \lambda_j' - i + j - 1)!}. \tag{2.37}$$

For $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}_0^n$, we set

$$\mathrm{b}(\gamma) = \frac{(\gamma_1 + \gamma_2 + \cdots + \gamma_n)!}{\gamma_1! \gamma_2! \cdots \gamma_n!}. \tag{2.38}$$

When $n > 1$, for $\mu = (\mu_1, \mu_2, \cdots, \mu_{n-1}) \in \Xi^+(\lambda)$, we set

$$\mathrm{S}^+(\lambda, \mu) = \prod_{1 \leq i \leq j < n} \frac{(\lambda_i - \mu_j - i + j)!}{(\mu_i - \lambda_{j+1} - i + j)!}. \tag{2.39}$$

In §4.1, we prove Proposition 2.13 based on the result of Jucys [17].

Proposition 2.13. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$, $l \in \mathbb{N}_0$, $\lambda' = (\lambda_1', \lambda_2', \cdots, \lambda_n') \in \Xi^\circ(\lambda; l)$. Then there is a $\mathbb{Q}$-rational $\mathrm{GL}(n,\mathbb{C})$-homomorphism $\mathrm{I}_{\lambda'}^{\lambda,l} \colon V_{\lambda'} \to V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$ such that the following assertions* (i) *and* (ii) *hold:*

(i) *The explicit expression of $\mathrm{I}_{\lambda'}^{\lambda,l}$ is given by*

$$\mathrm{I}_{\lambda'}^{\lambda,l}(\xi_{M'}) = \sum_{M \in \mathrm{G}(\lambda)} \sum_{P \in \mathrm{G}((l,\mathbf{0}_{n-1}))} \mathrm{c}_{M'}^{M,P} \xi_M \otimes \xi_P \qquad (M' \in \mathrm{G}(\lambda')),$$

where $\mathrm{c}_{M'}^{M,P}$ ($M' \in \mathrm{G}(\lambda')$, $M \in \mathrm{G}(\lambda)$, $P \in \mathrm{G}((l, \mathbf{0}_{n-1}))$) are rational numbers determined by the following conditions, recursively:

- *When $n=1$, we have* $\mathrm{c}_{\lambda_1+l}^{\lambda_1,l}=1$.
- *When $n>1$, for $\mu'\in\Xi^+(\lambda')$, $M'\in \mathrm{G}(\lambda';\mu')$, $\mu\in\Xi^+(\lambda)$, $M\in\mathrm{G}(\lambda;\mu)$, $0\le q\le l$ and $P\in\mathrm{G}((l,\mathbf{0}_{n-1});(q,\mathbf{0}_{n-2}))$, we have*

$$\begin{aligned}\mathrm{c}_{M'}^{M,P} &=\mathrm{c}_{\widehat{M'}}^{\widehat{M},\widehat{P}}\,\mathrm{S}^\circ(\lambda',\lambda')\mathrm{S}^\circ(\mu,\mu)\frac{l!}{q!}\frac{\prod_{1\le i\le j<n}(\lambda_i-\lambda_{j+1}-i+j)!}{\prod_{1\le i\le j\le n}(\lambda'_i-\lambda_j-i+j)!}\\ &\times\left(\prod_{1\le i\le j<n}\frac{(\mu'_i-\mu_j-i+j)!(\mu'_i-\lambda'_{j+1}-i+j)!}{(\mu'_i-\mu'_j-i+j)!(\mu_i-\lambda_{j+1}-i+j)!}\right)\\ &\times\sum_{\substack{\alpha\in\Xi^+(\lambda)\cap\Xi^+(\lambda')\\ \mu'\in\Xi^\circ(\alpha),\ \alpha\in\Xi^\circ(\mu)}}\frac{(-1)^{\ell(\alpha-\mu)}\mathrm{S}^\circ(\alpha,\alpha)}{\mathrm{S}^\circ(\mu',\alpha)\mathrm{S}^\circ(\alpha,\mu)}\frac{\mathrm{S}^+(\lambda',\alpha)}{\mathrm{S}^+(\lambda,\alpha)}\end{aligned}\tag{2.40}$$

if $\mu'\in\Xi^\circ(\mu;q)$, and $\mathrm{c}_{M'}^{M,P}=0$ otherwise.

(ii) *We have the equalities*

$$\left\langle\,\mathrm{I}_{\lambda'}^{\lambda,l}(\xi_{H(\lambda')}),\ \xi_{H(\lambda)}\otimes\xi_{Q(\lambda'-\lambda)}\,\right\rangle=\mathrm{C}^\circ(\lambda';\lambda),\tag{2.41}$$

$$\left\langle\,\mathrm{I}_{\lambda'}^{\lambda,l}(v),\ \mathrm{I}_{\lambda'}^{\lambda,l}(v')\,\right\rangle=\mathrm{b}(\lambda'-\lambda)\mathrm{C}^\circ(\lambda';\lambda)\langle v,v'\rangle\qquad(v,v'\in V_{\lambda'}).\tag{2.42}$$

2.11. **Archimedean Rankin–Selberg integrals for $G_n\times G_n$.** Let $(\Pi_{d,\nu},I(d,\nu))$ and $(\Pi_{d',\nu'},I(d',\nu'))$ be principal series representations of G_n with parameters

$$\begin{aligned} d&=(d_1,d_2,\cdots,d_n)\in\mathbb{Z}^n, & \nu&=(\nu_1,\nu_2,\cdots,\nu_n)\in\mathbb{C}^n,\\ d'&=(d'_1,d'_2,\cdots,d'_n)\in\mathbb{Z}^n, & \nu'&=(\nu'_1,\nu'_2,\cdots,\nu'_n)\in\mathbb{C}^n.\end{aligned}$$

We assume $d\in\Lambda_{n,F}$ and $-d'\in\Lambda_{n,F}$. If $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ are irreducible representations, these are not serious assumptions because of (2.5) and (2.6). We take $\mathrm{f}_{d,\nu}$, $\overline{\mathrm{f}}_{d',\nu'}$, $\mathbf{\Gamma}_F(\nu;d)$, $\mathbf{\Gamma}_F(\nu';d')$ and $L(s,\Pi_{d,\nu}\times\Pi_{d',\nu'})$ as in §2.7 with $n'=n$.

Let $\varepsilon\in\{\pm1\}$, $W\in\mathcal{W}(\Pi_{d,\nu},\psi_\varepsilon)$, $W'\in\mathcal{W}(\Pi_{d',\nu'},\psi_{-\varepsilon})$ and $\phi\in\mathcal{S}(\mathrm{M}_{1,n}(F))$. Let $s\in\mathbb{C}$ such that $\mathrm{Re}(s)$ is sufficiently large. We define the archimedean Rankin–Selberg integral $Z(s,W,W',\phi)$ for $\Pi_{d,\nu}\times\Pi_{d',\nu'}$ by

$$Z(s,W,W',\phi)=\int_{N_n\backslash G_n}W(g)W'(g)\phi(e_ng)|\det g|_F^s\,dg,\tag{2.43}$$

where we put $e_n=(O_{1,n-1},1)\in\mathrm{M}_{1,n}(F)$ as in §2.1. Here we note

$$Z(s,W,W',\phi)=Z(s,W',W,\phi),\tag{2.44}$$

$$Z(s,R(k)W,R(k)W',R(k)\phi)=Z(s,W,W',\phi)\qquad(k\in K_n).\tag{2.45}$$

Let l be an integer determined by

$$\begin{cases} l\in\{0,1\}\text{ and }l\equiv-\ell(d+d')\bmod 2 & \text{if }F=\mathbb{R}\text{ and }\ell(d+d')\le0,\\ l\in\{0,-1\}\text{ and }l\equiv-\ell(d+d')\bmod 2 & \text{if }F=\mathbb{R}\text{ and }\ell(d+d')\ge0,\\ l=-\ell(d+d') & \text{if }F=\mathbb{C},\end{cases}$$

where $\ell(\gamma)$ $(\gamma\in\mathbb{Z}^n)$ are defined by (2.31). By (2.45), we know that

$$v_1\otimes\overline{v_2}\otimes v_3\mapsto Z(s,\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)),\mathrm{W}_{-\varepsilon}(\overline{\mathrm{f}}_{d',\nu'}(\overline{v_2})),\varphi_{1,n}^{(l)}(v_3))\tag{2.46}$$

defines an element of $\mathrm{Hom}_{K_n}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}}\otimes_{\mathbb{C}}V_{(l,\mathbf{0}_{n-1})},\mathbb{C}_{\mathrm{triv}})$ if $l\ge0$, and

$$v_1\otimes\overline{v_2}\otimes\overline{v_3}\mapsto Z(s,\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)),\mathrm{W}_{-\varepsilon}(\overline{\mathrm{f}}_{d',\nu'}(\overline{v_2})),\overline{\varphi}_{1,n}^{(-l)}(\overline{v_3}))\tag{2.47}$$

defines an element of $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} \overline{V_{(-l,\mathbf{0}_{n-1})}}, \mathbb{C}_{\mathrm{triv}})$ if $l \leq 0$. Here W_ε, $\varphi_{1,n}^{(l)}$, $\overline{\varphi}_{1,n}^{(-l)}$ are defined by (2.9), (2.33), (2.34), respectively, and $\mathbb{C}_{\mathrm{triv}} = \mathbb{C}$ is the trivial K_n-module. Theorem 2.14 is the second main result of this paper, which gives the explicit expressions of the K_n-homomorphisms (2.46) and (2.47).

Theorem 2.14. *Retain the notation.*

(1) *Assume $l \geq 0$. For $v_1 \in V_d$, $v_2 \in V_{-d'}$ and $v_3 \in V_{(l,\mathbf{0}_{n-1})}$, we have*

$$\begin{aligned} &Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})), \varphi_{1,n}^{(l)}(v_3)\big) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')}}{(\dim V_{-d'})\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})\, \langle v_1 \otimes v_3,\, \mathrm{I}_{-d'}^{d,l}(v_2) \rangle \end{aligned}$$

if $-d' \in \Xi^\circ(d)$, and $Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})), \varphi_{1,n}^{(l)}(v_3)\big) = 0$ otherwise. Here $\mathrm{I}_{-d'}^{d,l}$ is given explicitly in Proposition 2.13. In particular, if $-d' \in \Xi^\circ(d)$, we have

$$\begin{aligned} &Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_{H(d)})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\xi_{H(-d')}})), \varphi_{1,n}^{(l)}(\xi_{Q(-d-d')})\big) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')}\mathrm{C}^\circ(-d';d)}{(\dim V_{-d'})\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}). \end{aligned} \tag{2.48}$$

(2) *Assume $l \leq 0$. For $v_1 \in V_d$, $v_2 \in V_{-d'}$ and $v_3 \in V_{(-l,\mathbf{0}_{n-1})}$, we have*

$$\begin{aligned} &Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})), \overline{\varphi}_{1,n}^{(-l)}(\overline{v_3})) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')}}{(\dim V_d)\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})\, \langle \mathrm{I}_d^{-d',-l}(v_1),\, v_2 \otimes v_3 \rangle \end{aligned}$$

if $d \in \Xi^\circ(-d')$, and $Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})), \overline{\varphi}_{1,n}^{(-l)}(\overline{v_3})) = 0$ otherwise. Here $\mathrm{I}_d^{-d',-l}$ is given explicitly in Proposition 2.13. In particular, if $d \in \Xi^\circ(-d')$, we have

$$\begin{aligned} &Z(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_{H(d)})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\xi_{H(-d')}})), \overline{\varphi}_{1,n}^{(-l)}(\overline{\xi_{Q(d+d')}})) \\ &= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')}\mathrm{C}^\circ(d;-d')}{(\dim V_d)\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')} L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'}). \end{aligned} \tag{2.49}$$

Remark 2.15. Retain the notation, and assume that $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ are both irreducible.

(1) Assume $l \geq 0$. By Lemma 4.4 in §4.2, $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}, \mathbb{C}_{\mathrm{triv}})$ is 1 dimensional if $-d' \in \Xi^\circ(d)$, and is equal to $\{0\}$ otherwise. Hence, Theorem 2.14(1) and Proposition 2.6 imply that (2.46) vanishes if and only if $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}, \mathbb{C}_{\mathrm{triv}}) = \{0\}$.

(2) Assume $l \leq 0$. By Lemma 4.4 in §4.2, $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} \overline{V_{(-l,\mathbf{0}_{n-1})}}, \mathbb{C}_{\mathrm{triv}})$ is 1 dimensional if $d \in \Xi^\circ(-d')$, and is equal to $\{0\}$ otherwise. Hence, Theorem 2.14 (2) and Proposition 2.6 imply that (2.47) vanishes if and only if $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} \overline{V_{(-l,\mathbf{0}_{n-1})}}, \mathbb{C}_{\mathrm{triv}}) = \{0\}$.

Let P_n be a maximal parabolic subgroup of G_n defined by

$$P_n = \{p = (p_{i,j}) \in G_n \mid p_{n,j} = 0 \ (1 \leq j \leq n-1)\},$$

which contains the upper triangular Borel subgroup $B_n = N_n M_n$. We put $\chi_l(t) = (t/|t|)^l \ \ (t \in F^\times)$ as in §2.3, and set $\nu'' = -\sum_{i=1}^n(\nu_i + \nu_i')$. We define a subspace

$I_{P_n}(l,\nu'',s)$ of $C^\infty(G_n)$ consisting of all functions f such that

$$f(pg)=\chi_l(p_{n,n})|p_{n,n}|_F^{\nu''-ns}|\det p|_F^s f(g) \qquad (p=(p_{i,j})\in P_n,\ g\in G_n),$$

on which G_n acts by the right translation $\Pi_{P_n,l,\nu'',s}=R$. Then the representation $(\Pi_{P_n,l,\nu'',s}, I_{P_n}(l,\nu'',s))$ is called a degenerate principal series representation of G_n. Similar to the proof of [6, Proposition 7], we can specify the minimal K_n-type of $\Pi_{P_n,l,\nu'',s}$, which occurs in $\Pi_{P_n,l,\nu'',s}|_{K_n}$ with multiplicity 1. If $l\geq 0$, we know that $\tau_{(l,\mathbf{0}_{n-1})}|_{K_n}$ is the minimal K_n-type of $\Pi_{P_n,l,\nu'',s}$, and there is a K_n-homomorphism $\mathrm{f}_{P_n,l,\nu'',s}\colon V_{(l,\mathbf{0}_{n-1})}\to I_{P_n}(l,\nu'',s)$ characterized by

$$\mathrm{f}_{P_n,l,\nu'',s}(\xi_{Q(\gamma)})(g)=\frac{|\det g|_F^s\prod_{i=1}^n g_{n,i}^{\gamma_i}}{(\sum_{i=1}^n|g_{n,i}|^2)^{(ns c_F-\nu'' c_F+l)/2}} \qquad (g=(g_{i,j})\in G_n)$$

for $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$ such that $\ell(\gamma)=l$. If $l\leq 0$, we know that $\overline{\tau_{(-l,\mathbf{0}_{n-1})}}|_{K_n}$ is the minimal K_n-type of $\Pi_{P_n,l,\nu'',s}$, and there is a K_n-homomorphism $\bar{\mathrm{f}}_{P_n,l,\nu'',s}\colon \overline{V_{(-l,\mathbf{0}_{n-1})}}\to I_{P_n}(l,\nu'',s)$ characterized by

$$\bar{\mathrm{f}}_{P_n,l,\nu'',s}(\overline{\xi_{Q(\gamma)}})(g)=\frac{|\det g|_F^s\prod_{i=1}^n \overline{g_{n,i}}^{\gamma_i}}{(\sum_{i=1}^n|g_{n,i}|^2)^{(ns c_F-\nu'' c_F-l)/2}} \qquad (g=(g_{i,j})\in G_n)$$

for $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$ such that $\ell(\gamma)=-l$.

For $f\in I_{P_n}(l,\nu'',s)$, we define an integral

$$Z_{P_n}(W,W',f)=\int_{Z_nN_n\backslash G_n} W(g)W'(g)f(g)\,dg. \tag{2.50}$$

This integral is equivalent to (2.43) via the correspondence

$$Z(s,W,W',\phi)=Z_{P_n}(W,W',\mathrm{g}_{P_n,l,\nu'',s}(\phi))$$

with $\mathrm{g}_{P_n,l,\nu'',s}(\phi)\in I_{P_n}(l,\nu'',s)$ defined by

$$\mathrm{g}_{P_n,l,\nu'',s}(\phi)(g)=|\det g|_F^s\int_{G_1}\chi_{-l}(h)\phi(he_ng)|h|_F^{ns-\nu''}\,dh \qquad (g\in G_n).$$

For $g\in G_n$ and $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$ such that $\ell(\gamma)=|l|$, we have

$$\mathrm{g}_{P_n,l,\nu'',s}(\varphi_{1,n}^{(l)}(\xi_{Q(\gamma)}))(g)=\Gamma_F(ns-\nu'';l)\mathrm{f}_{P_n,l,\nu'',s}(\xi_{Q(\gamma)})(g) \qquad \text{if } l\geq 0,$$
$$\mathrm{g}_{P_n,l,\nu'',s}(\overline{\varphi}_{1,n}^{(-l)}(\overline{\xi_{Q(\gamma)}}))(g)=\Gamma_F(ns-\nu'';-l)\bar{\mathrm{f}}_{P_n,l,\nu'',s}(\overline{\xi_{Q(\gamma)}})(g) \qquad \text{if } l\leq 0$$

using

$$\int_0^\infty \exp(-\pi c_F r t^2)t^{s c_F+m}\,\frac{2c_F\,dt}{t}=\frac{\Gamma_F(s;m)}{r^{(s c_F+m)/2}} \tag{2.51}$$
$$(r\in\mathbb{R}_+^\times,\ m\in\mathbb{Z},\ \mathrm{Re}(s c_F+m)>0).$$

Hence, Theorem 2.14 gives the explicit descriptions of (2.50) at the minimal $K_n\times K_n\times K_n$-type of $\Pi_{d,\nu}\boxtimes\Pi_{d',\nu'}\boxtimes\Pi_{P_n,l,\nu'',s}$. We note that

$$v_1\otimes\overline{v_2}\otimes v_3\mapsto Z_{P_n}(\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})),\mathrm{f}_{P_n,l,\nu'',s}(v_3)) \tag{2.52}$$

defines an element of $\mathrm{Hom}_{K_n}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}}\otimes_{\mathbb{C}}V_{(l,\mathbf{0}_{n-1})},\mathbb{C}_{\mathrm{triv}})$ if $l\geq 0$, and

$$v_1\otimes\overline{v_2}\otimes\overline{v_3}\mapsto Z_{P_n}(\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(v_1)),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v_2})),\bar{\mathrm{f}}_{P_n,l,\nu'',s}(\overline{v_3})) \tag{2.53}$$

defines an element of $\mathrm{Hom}_{K_n}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}}\otimes_{\mathbb{C}}\overline{V_{(-l,\mathbf{0}_{n-1})}},\mathbb{C}_{\mathrm{triv}})$ if $l\leq 0$. By Theorem 2.14 with Lemma 4.4, we obtain Corollary 2.16.

Corollary 2.16. *Retain the notation.*

(1) *Assume* $-d' \in \Xi^\circ(d)$ *(this implies* $l \geq 0$*). Then*

$$\sum_{M \in \mathrm{G}(d)} \sum_{M' \in \mathrm{G}(-d')} \sum_{P \in \mathrm{G}((l, \mathbf{0}_{n-1}))} \mathrm{r}(M')^{-1} \mathrm{c}_{M'}^{M,P}\, \xi_M \otimes \overline{\xi_{M'}} \otimes \xi_P$$

is a unique $\mathbb{Q}$*-rational* K_n*-invariant vector in* $V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} V_{(l, \mathbf{0}_{n-1})}$ *up to scalar multiple, and its image under the* K_n*-homomorphism* (2.52) *is given by*

$$\sum_{M \in \mathrm{G}(d)} \sum_{M' \in \mathrm{G}(-d')} \sum_{P \in \mathrm{G}((l, \mathbf{0}_{n-1}))} \mathrm{r}(M')^{-1} \mathrm{c}_{M'}^{M,P}$$
$$\times Z_{P_n}(\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_M)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\xi_{M'}})), \mathrm{f}_{P_n,l,\nu'',s}(\xi_P))$$
$$= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')} \mathrm{b}(-d-d') \mathrm{C}^\circ(-d'; d)}{\mathbf{\Gamma}_F(\nu; d) \mathbf{\Gamma}_F(\nu'; d')} \frac{L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})}{\Gamma_F(ns - \nu''; l)}.$$

Here $\mathrm{b}(-d-d')$ *and* $\mathrm{C}^\circ(-d'; d)$ *are the nonzero rational constants, which are given by* (2.38) *and* (2.37)*, respectively.*

(2) *Assume* $d \in \Xi^\circ(-d')$ *(this implies* $l \leq 0$*). Then*

$$\sum_{M \in \mathrm{G}(d)} \sum_{M' \in \mathrm{G}(-d')} \sum_{P \in \mathrm{G}((-l, \mathbf{0}_{n-1}))} \mathrm{r}(M)^{-1} \mathrm{c}_{M}^{M',P}\, \xi_M \otimes \overline{\xi_{M'}} \otimes \overline{\xi_P}$$

is a unique $\mathbb{Q}$*-rational* K_n*-invariant vector in* $V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} \overline{V_{(-l, \mathbf{0}_{n-1})}}$ *up to scalar multiple, and its image under the* K_n*-homomorphism* (2.53) *is given by*

$$\sum_{M \in \mathrm{G}(d)} \sum_{M' \in \mathrm{G}(-d')} \sum_{P \in \mathrm{G}((-l, \mathbf{0}_{n-1}))} \mathrm{r}(M)^{-1} \mathrm{c}_{M}^{M',P}$$
$$\times Z_{P_n}(\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_M)), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\xi_{M'}})), \bar{\mathrm{f}}_{P_n,l,\nu'',s}(\overline{\xi_P}))$$
$$= \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d_i')} \mathrm{b}(d+d') \mathrm{C}^\circ(d; -d')}{\mathbf{\Gamma}_F(\nu; d) \mathbf{\Gamma}_F(\nu'; d')} \frac{L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})}{\Gamma_F(ns - \nu''; -l)}.$$

Here $\mathrm{b}(d+d')$ *and* $\mathrm{C}^\circ(d; -d')$ *are the nonzero rational constants, which are given by* (2.38) *and* (2.37)*, respectively.*

Remark 2.17. We set $F = \mathbb{C}$. By [3, Proposition 3.3], we note that the compatible pairs of cohomological representations of G_n in Grenié [6] can be regarded as pairs of some irreducible principal series representations $\Pi_{d,\nu}$ and $\Pi_{d',\nu'}$ with $d, -d' \in \Lambda_{n,F}$ such that either $-d' \in \Xi^\circ(d)$ or $d \in \Xi^\circ(-d')$ holds. Hence, Theorem 2.14 gives a proof of Grenié's conjecture [6, Conjecture 1] at all critical points (Dong and Xue [3] proved this conjecture only at the central critical point by another method). Moreover, Corollary 2.16 gives the explicit descriptions of the archimedean part of Grenié's theorem [6, Theorem 2].

Remark 2.18. Although we use the orthonormal basis $\{\zeta_M\}_{M \in \mathrm{G}(\lambda)}$ rather than $\{\xi_M\}_{M \in \mathrm{G}(\lambda)}$ in the proofs, we state the main theorems in terms of the $\mathbb{Q}$-rational basis $\{\xi_M\}_{M \in \mathrm{G}(\lambda)}$ because of the applications in Remark 2.17.

3. Recurrence relations

3.1. **The Godement section ($G_{n-1} \to G_n$).** Let us recall the Godement section, which is defined by Jacquet in [15, §7.1]. Assume $n > 1$. Let $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. We set $\widehat{d} = (d_1, d_2, \cdots, d_{n-1}) \in \mathbb{Z}^{n-1}$ and $\widehat{\nu} = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. Let $f \in I(\widehat{d})_{K_{n-1}}$, and we denote by $f_{\widehat{\nu}}$ the standard section corresponding to f. Let $\phi \in \mathcal{S}_0(\mathrm{M}_{n-1,n}(F))$. When $\mathrm{Re}(\nu_n - \nu_i) > -1$ $(1 \le i \le n-1)$, we define the Godement section $\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi)$ by the convergent integral

$$\begin{aligned}\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi)(g) = {}& \chi_{d_n}(\det g)|\det g|_F^{\nu_n+(n-1)/2}\\ &\times \int_{G_{n-1}} \phi((h, O_{n-1,1})g) f_{\widehat{\nu}}(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\, dh\end{aligned}$$

for $g \in G_n$. Here we set $\chi_l(t) = (t/|t|)^l$ $(l \in \mathbb{Z},\ t \in F^{\times})$ as in §2.3. Jacquet shows that $\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi)(g)$ extends to a meromorphic function of ν_n in $\mathbb{C}$, which is a holomorphic multiple of

$$\prod_{1\le i\le n-1} \Gamma_F(\nu_n - \nu_i + 1;\, |d_n - d_i|).$$

Moreover, $\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi)$ is an element of $I(d,\nu)_{K_n}$ if it is defined. For later use, we prepare Lemma 3.1.

Lemma 3.1. *Retain the notation. Then we have*

$$\Pi_{d,\nu}(k)\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi) = (\det k)^{d_n}\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, R(k)\phi) \qquad (k \in K_n), \tag{3.1}$$

$$(\det k')^{-d_n}\mathrm{g}^{+}_{d_n,\nu_n}\big(\Pi_{\widehat{d},\widehat{\nu}}(k')f_{\widehat{\nu}},\, L(k')\phi\big) = \mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi) \qquad (k' \in K_{n-1}). \tag{3.2}$$

Proof. When $\mathrm{Re}(\nu_n - \nu_i) > -1$ $(1 \le i \le n-1)$, the equalities (3.1) and (3.2) follow immediately from the definition. Hence, by the uniqueness of the analytic continuations, we obtain the assertion. □

Let $\varepsilon \in \{\pm 1\}$. In [15, §7.2], Jacquet gives convenient integral representations of Whittaker functions. If ν satisfies (2.8), then for $g \in G_n$, we have

$$\begin{aligned}\mathrm{W}_{\varepsilon}\big(\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi)\big)(g) = {}& \chi_{d_n}(\det g)|\det g|_F^{\nu_n+(n-1)/2}\\ &\times \int_{G_{n-1}} \left(\int_{\mathrm{M}_{n-1,1}(F)} \phi\left((h, hz)\, g\right) \psi_{-\varepsilon}(e_{n-1}z)\, dz\right)\\ &\times \mathrm{W}_{\varepsilon}(f_{\widehat{\nu}})(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\, dh,\end{aligned} \tag{3.3}$$

where $e_{n-1} = (O_{1,n-2}, 1) \in \mathrm{M}_{1,n-1}(F)$. Jacquet shows that the right hand side of (3.3) converges absolutely for all $\nu \in \mathbb{C}^n$, and defines an entire function of ν (see [15, Proposition 7.2]). Thus the equality holds for all ν. In Appendix A, we show that the integral representation (3.3) can be regarded as a generalization of the recursive formula [10, Theorem 14] of spherical Whittaker functions.

3.2. **The section $(G_n \to G_n)$.** In this subsection, we define another section, whose Whittaker function has appeared in Jacquet's formulas [15, (8.1) and (8.3)]. Let $d \in \mathbb{Z}^n$ and $\nu \in \mathbb{C}^n$. Let $f \in I(d)_{K_n}$ and $\phi \in \mathcal{S}_0(\mathrm{M}_n(F))$. We denote by f_ν the standard section corresponding to f. For $s \in \mathbb{C}$, $l \in \mathbb{Z}$ and $g \in G_n$, we set

$$\mathrm{g}^{\circ}_{l,s}(f_\nu, \phi)(g) = \int_{G_n} f_\nu(gh)\phi(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\, dh. \tag{3.4}$$

Proposition 3.2. *Let $d \in \mathbb{Z}^n$, $l \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$. Let Ω be an open relatively compact subset of $\mathbb{C}^n$. Then there is a constant c_0 such that, for any $f \in I(d)_{K_n}$ and $\phi \in \mathcal{S}_0(\mathrm{M}_n(F))$, the following assertions* (i) *and* (ii) *hold:*

(i) *On any compact subset of* $\{(s,\nu,g) \in \mathbb{C}\times\Omega\times G_n \mid \mathrm{Re}(s) > c_0\}$*, the integral* (3.4) *converges absolutely and uniformly.*
(ii) *Let* $\nu \in \Omega$ *and* $s \in \mathbb{C}$ *such that* $\mathrm{Re}(s) > c_0$*. Then* $\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi)$ *is an element of* $I(d,\nu)_{K_n}$ *satisfying*

$$\Pi_{d,\nu}(k)\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi) = (\det k)^{-l}\mathrm{g}^{\circ}_{l,s}(f_\nu, L(k)\phi) \qquad (k \in K_n), \tag{3.5}$$

$$(\det k')^l\, \mathrm{g}^{\circ}_{l,s}\big(\Pi_{d,\nu}(k')f_\nu,\, R(k')\phi\big) = \mathrm{g}^{\circ}_{l,s}(f_\nu,\phi) \qquad (k' \in K_n). \tag{3.6}$$

Moreover, for $g \in G_n$*, we have*

$$\mathrm{W}_\varepsilon(\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi))(g) = \int_{G_n} \mathrm{W}_\varepsilon(f_\nu)(gh)\phi(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\,dh. \tag{3.7}$$

Here f_ν *is the standard section corresponding to* f*.*

Proof. For $g \in G_n$, we set $\|g\| = \mathrm{Tr}(g\,{}^t\overline{g}) + \mathrm{Tr}((g^{-1})\,{}^t(\overline{g^{-1}}))$ and denote by

$$g = \mathrm{u}(g)\mathrm{a}(g)\mathrm{k}(g) \qquad (\mathrm{u}(g)\in U_n,\ \mathrm{a}(g)\in A_n,\ \mathrm{k}(g)\in K_n)$$

the decomposition of g according to $G_n = U_nA_nK_n$. It is easy to see that

$$\|\mathrm{a}(g)\| \le \|g\| = \|kgk'\|, \quad \|gh\| \le \|g\|\,\|h\|, \quad \mathrm{a}(gh) = \mathrm{a}(g)\mathrm{a}(\mathrm{k}(g)h) \tag{3.8}$$

for $g,h \in G_n$ and $k,k' \in K_n$. Since $G_n \ni g \mapsto \eta_{\nu-\rho_n}(\mathrm{a}(g)) \in \mathbb{C}$ is an element of $I(\mathbf{0}_n,\nu)$ with $\mathbf{0}_n = (0,0,\cdots,0) \in \mathbb{Z}^n$, we have

$$\int_{N_n} |\eta_{\nu-\rho_n}(\mathrm{a}(x))|\,dx < \infty \qquad (\nu \in \mathbb{C}^n \text{ satisfying (2.8)}) \tag{3.9}$$

by the absolute convergence of the Jacquet integral [30, Theorem 15.4.1].

We take d, l, ε and Ω as in the statement. Replacing Ω with its superset if necessary, we may assume that Ω contains an element ν satisfying (2.8). By (3.8) and [15, Proposition 3.2], there are a constant c_1 and a continuous seminorm $\mathcal{Q}$ on $I(d)$ such that, for any $\nu \in \Omega$, $g \in G_n$ and $f \in I(d)$, the following inequalities hold:

$$|\eta_{\nu-\rho_n}(\mathrm{a}(g))| \le \|g\|^{c_1}, \qquad |\mathrm{W}_\varepsilon(f_\nu)(g)| \le \|g\|^{c_1}\mathcal{Q}(f). \tag{3.10}$$

By [15, Lemma 3.3 (ii)], there is a positive constant c_0 such that, for any $t > c_0$ and $\phi \in \mathcal{S}(\mathrm{M}_n(F))$, the integral

$$\int_{G_n} \|h\|^{c_1}\phi(h)|\det h|_F^{t+(n-1)/2}\,dh \tag{3.11}$$

converges absolutely.

Let $f \in I(d)_{K_n}$ and $\phi \in \mathcal{S}_0(\mathrm{M}_n(F))$. By (3.8), (3.10) and the definition of $I(d,\nu)$, for $\nu \in \Omega$, $x \in N_n$ and $g,h \in G_n$, we have an estimate

$$|f_\nu(xgh)| \le |\eta_{\nu-\rho_n}(\mathrm{a}(x))|\,\|g\|^{c_1}\|h\|^{c_1}\sup_{k\in K_n}|f(k)|. \tag{3.12}$$

By the absolute convergence of (3.11) and (3.12) with $x = 1_n$, we obtain the assertion (i).

Let $\nu \in \Omega$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > c_0$. By definition, we have (3.5), (3.6) and

$$\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi)(umg) = \chi_d(m)\eta_{\nu-\rho_n}(m)\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi)(g) \tag{3.13}$$

for $u \in U_n$, $m \in M_n$ and $g \in G_n$. Since $\Pi_{d,\nu}$ is admissible and $\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi)$ is a continuous K_n-finite function on G_n satisfying (3.13), we know that $\mathrm{g}^{\circ}_{l,s}(f_\nu,\phi)$ is smooth and an element of $I(d,\nu)_{K_n}$ by [18, Propositions 8.4 and 8.5].

Let $g \in G_n$. If $\nu \in \Omega$ satisfies (2.8), we obtain the equality (3.7) as follows:

$$\begin{aligned}
\mathrm{W}_\varepsilon(\mathrm{g}^\circ_{l,s}(f_\nu,\phi))(g) &= \int_{N_n} \mathrm{g}^\circ_{l,s}(f_\nu,\phi)(xg)\psi_{-\varepsilon,n}(x)\,dx \\
&= \int_{N_n}\left(\int_{G_n} f_\nu(xgh)\phi(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\,dh\right)\psi_{-\varepsilon,n}(x)\,dx \\
&= \int_{G_n}\left(\int_{N_n} f_\nu(xgh)\psi_{-\varepsilon,n}(x)\,dx\right)\phi(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\,dh \\
&= \int_{G_n} \mathrm{W}_\varepsilon(f_\nu)(gh)\phi(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\,dh.
\end{aligned}$$

Here the third equality is justified by Fubini's theorem, since the double integral converges absolutely by (3.9), (3.12) and the absolute convergence of (3.11).

In order to complete the proof, it suffices to show that both sides of (3.7) are holomorphic functions of (s,ν) on a domain

$$\{(s,\nu) \in \mathbb{C}\times\Omega \mid \mathrm{Re}(s) > c_0\}. \tag{3.14}$$

By (3.8), (3.10) and the absolute convergence of (3.11), the integral on the right hand side of (3.7) converges absolutely and uniformly on any compact subset of the domain (3.14), and defines a holomorphic function on the domain (3.14).

Let $\mathcal{S}_{\phi,l}$ be a subspace of $\mathcal{S}_0(\mathrm{M}_n(F))$ spanned by $L(k)\phi$ $(k \in K_n)$, and we regard $\mathcal{S}_{\phi,l}$ as a K_n-module via the action $\det^{-l}\otimes L$. Let $I_{\phi,l}$ be a subspace of $I(d)_{K_n}$ spanned by $\{T(\phi') \mid \phi' \in \mathcal{S}_{\phi,l},\ T \in \mathrm{Hom}_{K_n}(\mathcal{S}_{\phi,l},\ I(d)_{K_n})\}$. Then we have $\mathrm{g}^\circ_{l,s}(f_\nu,\phi)|_{K_n} \in I_{\phi,l}$ by (3.5). Since ϕ is K_n-finite and $\Pi_{d,\nu}$ is admissible, the space $I_{\phi,l}$ is finite dimensional. Let $\{f_{\phi,i}\}_{i=1}^m$ be an orthonormal basis of $I_{\phi,l}$ with respect to the L^2-inner product

$$\langle f_1,f_2\rangle_{L^2} = \int_{K_n} f_1(k)\overline{f_2(k)}\,dk \qquad (f_1,f_2 \in I(d)).$$

Since $\mathrm{g}^\circ_{l,s}(f_\nu,\phi)|_{K_n} = \sum_{i=1}^m \langle \mathrm{g}^\circ_{l,s}(f_\nu,\phi)|_{K_n}, f_{\phi,i}\rangle_{L^2} f_{\phi,i}$, we have

$$\mathrm{W}_\varepsilon(\mathrm{g}^\circ_{l,s}(f_\nu,\phi))(g) = \sum_{i=1}^m \langle \mathrm{g}^\circ_{l,s}(f_\nu,\phi)|_{K_n},\ f_{\phi,i}\rangle_{L^2}\,\mathrm{W}_\varepsilon(f_{\phi,i,\nu})(g),$$

where $f_{\phi,i,\nu}$ is the standard section corresponding to $f_{\phi,i}$. By this expression and the statement (i), we know that the right hand side of (3.7) is holomorphic on the domain (3.14). □

Remark 3.3. The equality (3.7) with $l = 0$ can be regarded as the local theta correspondence for a principal series representation $\Pi_{d,\nu}$ in [32, §2].

3.3. **Recurrence relations with two kinds of the sections.** Let $\varepsilon \in \{\pm 1\}$. For $\phi \in \mathcal{S}(\mathrm{M}_{n,1}(F))$, we define $\mathcal{F}_\varepsilon(\phi) \in \mathcal{S}(\mathrm{M}_{1,n}(F))$ by

$$\mathcal{F}_\varepsilon(\phi)(t) = \int_{\mathrm{M}_{n,1}(F)} \phi(z)\psi_{-\varepsilon}(tz)\,d_F z \qquad (t \in \mathrm{M}_{1,n}(F)). \tag{3.15}$$

Let

$$d = (d_1,d_2,\cdots,d_n) \in \mathbb{Z}^n, \qquad \nu = (\nu_1,\nu_2,\cdots,\nu_n) \in \mathbb{C}^n,$$
$$d' = (d'_1,d'_2,\cdots,d'_{n'}) \in \mathbb{Z}^{n'}, \qquad \nu' = (\nu'_1,\nu'_2,\cdots,\nu'_{n'}) \in \mathbb{C}^{n'}.$$

If $n > 1$, we set $\widehat{d} = (d_1,d_2,\cdots,d_{n-1})$ and $\widehat{\nu} = (\nu_1,\nu_2,\cdots,\nu_{n-1})$. If $n' > 1$, we set $\widehat{d'} = (d'_1,d'_2,\cdots,d'_{n'-1})$ and $\widehat{\nu'} = (\nu'_1,\nu'_2,\cdots,\nu'_{n'-1})$.

Proposition 3.4 ($G_n \times G_n \to G_n \times G_{n-1}$)**.** *Retain the notation, and assume $n' = n > 1$. Let $f \in I(d)_{K_n}$ and $f' \in I(\widehat{d'})_{K_{n-1}}$. We denote by f_ν and $f'_{\widehat{\nu'}}$ the standard sections corresponding to f and f', respectively. Let $\phi_1 \in \mathcal{S}_0(\mathrm{M}_{n-1,n}(F))$ and $\phi_2 \in \mathcal{S}_0(\mathrm{M}_{1,n}(F))$. For $s \in \mathbb{C}$ such that* $\mathrm{Re}(s)$ *is sufficiently large, we have*

$$Z\big(s,\, \mathrm{W}_\varepsilon(f_\nu),\, \mathrm{W}_{-\varepsilon}\big(\mathrm{g}^{+}_{d'_n,\nu'_n}(f'_{\widehat{\nu'}}, \phi_1)\big),\, \phi_2\big) = Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^{\circ}_{d'_n, s+\nu'_n}(f_\nu, \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(f'_{\widehat{\nu'}})\big),$$

where $\phi_0 \in \mathcal{S}_0(\mathrm{M}_n(F))$ is defined by

$$\phi_0(z) = \phi_1((1_{n-1}, O_{n-1,1})z)\phi_2(e_n z) \qquad (z \in \mathrm{M}_n(F)).$$

Proof. Using (3.3), Jacquet shows the following equality [15, (8.1)]:

$$\begin{aligned} &Z\big(s,\, \mathrm{W}_\varepsilon(f_\nu),\, \mathrm{W}_{-\varepsilon}\big(\mathrm{g}^{+}_{d'_n,\nu'_n}(f'_{\widehat{\nu'}}, \phi_1)\big),\, \phi_2\big) \\ &= \int_{N_{n-1}\backslash G_{n-1}} \left(\int_{G_n} \mathrm{W}_\varepsilon(f_\nu)(\iota_n(h)g)\phi_0(g)\chi_{d'_n}(\det g)|\det g|_F^{s+\nu'_n+(n-1)/2}\, dg \right) \\ &\quad \times \mathrm{W}_{-\varepsilon}(f'_{\widehat{\nu'}})(h)|\det h|_F^{s-1/2} dh. \end{aligned}$$

Hence, we obtain the assertion by Proposition 3.2. □

Proposition 3.5 ($G_n \times G_{n-1} \to G_{n-1} \times G_{n-1}$)**.** *Retain the notation, and assume $n' = n-1$. Let $f \in I(\widehat{d})_{K_{n-1}}$ and $f' \in I(d')_{K_{n-1}}$. We denote by $f_{\widehat{\nu}}$ and $f'_{\nu'}$ the standard sections corresponding to f and f', respectively. Let $\phi_1 \in \mathcal{S}_0(\mathrm{M}_{n-1}(F))$ and $\phi_2 \in \mathcal{S}_0(\mathrm{M}_{n-1,1}(F))$. For $s \in \mathbb{C}$ such that* $\mathrm{Re}(s)$ *is sufficiently large, we have*

$$\begin{aligned} &Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(f'_{\nu'})\big) \\ &= Z\big(s,\, \mathrm{W}_\varepsilon(f_{\widehat{\nu}}),\, \mathrm{W}_{-\varepsilon}(\mathrm{g}^{\circ}_{d_n, s+\nu_n}(f'_{\nu'}, \phi_1)),\, \mathcal{F}_\varepsilon(\phi_2)\big), \end{aligned}$$

where $\phi_0 \in \mathcal{S}_0(\mathrm{M}_{n-1,n}(F))$ is defined by

$$\phi_0(z) = \phi_1(z\,{}^t(1_{n-1}, O_{n-1,1}))\phi_2(z\,{}^te_n) \qquad (z \in \mathrm{M}_{n-1,n}(F)).$$

Proof. Using (3.3), Jacquet shows the following equality [15, (8.3)]:

$$\begin{aligned} &Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^{+}_{d_n,\nu_n}(f_{\widehat{\nu}}, \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(f'_{\nu'})\big) \\ &= \int_{N_{n-1}\backslash G_{n-1}} \left(\int_{G_{n-1}} \mathrm{W}_{-\varepsilon}(f'_{\nu'})(gh)\phi_1(h)\chi_{d_n}(\det h)|\det h|_F^{s+\nu_n+(n-2)/2} dh \right) \\ &\quad \times \mathrm{W}_\varepsilon(f_{\widehat{\nu}})(g)\mathcal{F}_\varepsilon(\phi_2)(e_{n-1}g)|\det g|_F^{s}\, dg. \end{aligned}$$

Hence, we obtain the assertion by Proposition 3.2. □

4. Finite dimensional representations

In this section, we give some preliminary results on the theory of finite dimensional representations of K_n and $\mathrm{GL}(n, \mathbb{C})$. The most important results of this section are Lemmas 4.11, 4.12 and 4.13, which give computable expressions of (the polynomial parts of) the standard Schwartz functions which have appeared in the recurrence relations of the archimedean Rankin–Selberg integrals (Propositions 3.4 and 3.5).

For the readability, we explain the structure of this section. In §4.1, we recall Jucys's result for the Clebsch–Gordan coefficients, and prove Proposition 2.13. In §4.2, we prove Lemma 2.5 and prepare some lemmas for finite dimensional representations of K_n. In §4.3, we construct some polynomial functions on $\mathrm{M}_{n,n'}(\mathbb{C})$, concretely, and study the polynomial functions coming from the recurrence relations (Lemmas 4.11 and 4.12) using the Clebsch–Gordan coefficients. In §4.4, we construct the standard Schwartz functions on $\mathrm{M}_{n,n'}(F)$ based on the results in §4.3, and calculate the Fourier transform (3.15) in Lemma 4.13.

4.1. **The Clebsch–Gordan coefficients.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$, $l \in \mathbb{N}_0$ and $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_n) \in \Xi^\circ(\lambda; l)$. By Pieri's rule (2.35), we can take a $\mathrm{GL}(n, \mathbb{C})$-homomorphism $\tilde{\mathrm{I}}^{\lambda,l}_{\lambda'} \colon V_{\lambda'} \to V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$ satisfying

$$\left\langle \tilde{\mathrm{I}}^{\lambda,l}_{\lambda'}(v),\ \tilde{\mathrm{I}}^{\lambda,l}_{\lambda'}(v') \right\rangle = \langle v, v' \rangle \qquad (v, v' \in V_{\lambda'}). \tag{4.1}$$

Such $\tilde{\mathrm{I}}^{\lambda,l}_{\lambda'}$ is unique up to multiplication by scalars in $\mathrm{U}(1)$. We set

$$\tilde{\mathrm{I}}^{\lambda,l}_{\lambda'}(\zeta_{M'}) = \sum_{M \in \mathrm{G}(\lambda)} \sum_{P \in \mathrm{G}((l,\mathbf{0}_{n-1}))} \mathrm{C}^{M,P}_{M'} \zeta_M \otimes \zeta_P \qquad (M' \in \mathrm{G}(\lambda')),$$

and call $\mathrm{C}^{M,P}_{M'}$ ($M \in \mathrm{G}(\lambda)$, $P \in \mathrm{G}((l, \mathbf{0}_{n-1}))$, $M' \in \mathrm{G}(\lambda')$) the Clebsch–Gordan coefficients. When $n = 1$, we may normalize $\mathrm{C}^{\lambda_1,l}_{\lambda_1+l} = 1$, since

$$\Xi^\circ(\lambda_1; l) = \{\lambda_1 + l\}, \quad \mathrm{G}(\lambda_1) = \{\lambda_1\}, \quad \mathrm{G}(l) = \{l\}, \quad \mathrm{G}(\lambda_1 + l) = \{\lambda_1 + l\}.$$

We consider the case of $n > 1$. Let $\mu \in \Xi^+(\lambda)$ and $0 \le q \le l$. By the irreducible decomposition (2.35) and Lemma 2.3(1), (2), there are some constants

$$\left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \mu, & q & \mu' \end{array} \right) \qquad (\mu' \in \Xi^+(\lambda') \cap \Xi^\circ(\mu; q)) \tag{4.2}$$

such that, for any $\mu' \in \Xi^+(\lambda')$, the following equality holds:

$$\left(\tilde{\mathrm{R}}^\lambda_\mu \otimes \tilde{\mathrm{R}}^{(l,\mathbf{0}_{n-1})}_{(q,\mathbf{0}_{n-1})}\right) \circ \tilde{\mathrm{I}}^{\lambda,l}_{\lambda'} \circ \tilde{\mathrm{I}}^{\lambda'}_{\mu'} = \begin{cases} \left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \mu, & q & \mu' \end{array} \right) \tilde{\mathrm{I}}^{\mu,q}_{\mu'} & \text{if } \mu' \in \Xi^\circ(\mu; q), \\ 0 & \text{otherwise}, \end{cases} \tag{4.3}$$

where the symbols $\tilde{\mathrm{I}}^\lambda_\mu$, $\tilde{\mathrm{R}}^\lambda_\mu$ are defined in Lemma 2.3(1), (2), respectively. Then, for any $M' \in \mathrm{G}(\lambda'; \mu')$, $M \in \mathrm{G}(\lambda; \mu)$, $P \in \mathrm{G}((l, \mathbf{0}_{n-1}); (q, \mathbf{0}_{n-2}))$ and $\mu' \in \Xi^+(\lambda')$, we have

$$\mathrm{C}^{M,P}_{M'} = \begin{cases} \left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \mu, & q & \mu' \end{array} \right) \mathrm{C}^{\widehat{M},\widehat{P}}_{\widehat{M'}} & \text{if } \mu' \in \Xi^\circ(\mu; q), \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

since

$$\begin{aligned} &\left\langle \tilde{\mathrm{I}}^{\mu,q}_{\mu'}(\zeta_{\widehat{M'}}), \zeta_{\widehat{M}} \otimes \zeta_{\widehat{P}} \right\rangle = \mathrm{C}^{\widehat{M},\widehat{P}}_{\widehat{M'}} \qquad \text{if } \mu' \in \Xi^\circ(\mu; q), \\ &\left\langle \left(\left(\tilde{\mathrm{R}}^\lambda_\mu \otimes \tilde{\mathrm{R}}^{(l,\mathbf{0}_{n-1})}_{(q,\mathbf{0}_{n-1})}\right) \circ \tilde{\mathrm{I}}^{\lambda,l}_{\lambda'} \circ \tilde{\mathrm{I}}^{\lambda'}_{\mu'}\right)(\zeta_{\widehat{M'}}), \zeta_{\widehat{M}} \otimes \zeta_{\widehat{P}} \right\rangle = \mathrm{C}^{M,P}_{M'}. \end{aligned}$$

The constants (4.2) are called the isoscalar factors. In [17] (see also [1] and [29, Chapter 18]), Jucys gives the following expressions of them under some normalization of $\tilde{\mathrm{I}}_{\lambda'}^{\lambda;l}$:

$$
(4.5)\quad \left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \mu, & q & \mu' \end{array}\right) = \sqrt{\frac{(l-q)!\,\mathrm{S}^\circ(\lambda',\lambda')\mathrm{S}^+(\lambda,\mu)\mathrm{S}^\circ(\mu',\mu)\mathrm{S}^\circ(\mu,\mu)}{\mathrm{S}^\circ(\lambda',\lambda)\mathrm{S}^+(\lambda',\mu')}} \times \sum_{\substack{\alpha\in\Xi^+(\lambda)\cap\Xi^+(\lambda') \\ \mu'\in\Xi^\circ(\alpha),\ \alpha\in\Xi^\circ(\mu)}} \frac{(-1)^{\ell(\alpha-\mu)}\mathrm{S}^\circ(\alpha,\alpha)}{\mathrm{S}^\circ(\mu',\alpha)\mathrm{S}^\circ(\alpha,\mu)}\frac{\mathrm{S}^+(\lambda',\alpha)}{\mathrm{S}^+(\lambda,\alpha)}
$$

for $\mu'\in\Xi^+(\lambda')\cap\Xi^\circ(\mu;q)$, where the symbols $\mathrm{S}^\circ(\lambda',\lambda)$ and $\mathrm{S}^+(\lambda,\mu)$ are defined by (2.36) and (2.39), respectively. Hereafter, we assume that $\tilde{\mathrm{I}}_{\lambda'}^{\lambda;l}$ is normalized so that (4.5) holds. Then all the Clebsch–Gordan coefficients $\mathrm{C}_{M'}^{M,P}$ are real numbers, and we have

$$
(4.6)\quad \begin{aligned} \zeta_M\otimes\zeta_P &= \sum_{\lambda'\in\Xi^\circ(\lambda;l)}\ \sum_{M'\in\mathrm{G}(\lambda')} \left\langle \zeta_M\otimes\zeta_P,\ \tilde{\mathrm{I}}_{\lambda'}^{\lambda;l}(\zeta_{M'})\right\rangle \tilde{\mathrm{I}}_{\lambda'}^{\lambda;l}(\zeta_{M'}) \\ &= \sum_{\lambda'\in\Xi^\circ(\lambda;l)}\ \sum_{M'\in\mathrm{G}(\lambda')} \mathrm{C}_{M'}^{M,P}\,\tilde{\mathrm{I}}_{\lambda'}^{\lambda;l}(\zeta_{M'}) \end{aligned}
$$

for $M\in\mathrm{G}(\lambda)$ and $P\in\mathrm{G}((l,\mathbf{0}_{n-1}))$.

Lemma 4.1. *Retain the notation. We use the symbols* $H(\lambda)$, $Q(\gamma)$ $(\gamma\in\mathbb{N}_0^n)$ *and* $\mathrm{C}^\circ(\lambda';\lambda)$ *defined by* (2.16), (2.32) *and* (2.37), *respectively.*

(1) *Assume* $n>1$ *and* $\mu\in\Xi^+(\lambda')\cap\Xi^+(\lambda)$. *Let* $M\in\mathrm{G}(\lambda)$ *and* $M'\in\mathrm{G}(\lambda')$ *such that* $\widehat{M}=\widehat{M'}\in\mathrm{G}(\mu)$. *Then we have*

$$
\mathrm{C}_{M'}^{M,Q((\mathbf{0}_{n-1},l))} = \sqrt{\frac{l!\,\mathrm{S}^\circ(\lambda',\lambda')\mathrm{S}^+(\lambda',\mu)}{\mathrm{S}^\circ(\lambda',\lambda)\mathrm{S}^+(\lambda,\mu)}}.
$$

In particular, we have $\mathrm{C}_M^{M,Q(\mathbf{0}_n)}=1$ *if* $l=0$.

(2) *We have* $\mathrm{C}_{H(\lambda')}^{H(\lambda),Q(\lambda'-\lambda)} = \sqrt{\mathrm{C}^\circ(\lambda';\lambda)}$.

Proof. First, we will prove the statement (2) by induction with respect to n. In the case of $n=1$, the statement (2) follows from $\mathrm{C}^\circ(\lambda_1';\lambda_1)=1$ and our normalization $\mathrm{C}_{\lambda_1'}^{\lambda_1,\lambda_1'-\lambda_1}=1$. Let us consider the case of $n\geq 2$. Let $\widehat{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_{n-1})$ and $\widehat{\lambda'}=(\lambda_1',\lambda_2',\cdots,\lambda_{n-1}')$. By (4.5), we have

$$
\begin{aligned} \left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \widehat{\lambda}, & l-\lambda_n'+\lambda_n & \widehat{\lambda'} \end{array}\right) &= \sqrt{(\lambda_n'-\lambda_n)!\frac{\mathrm{S}^\circ(\lambda',\lambda')}{\mathrm{S}^+(\lambda',\widehat{\lambda'})}\frac{\mathrm{S}^\circ(\widehat{\lambda},\widehat{\lambda})}{\mathrm{S}^+(\lambda,\widehat{\lambda})}\frac{\mathrm{S}^+(\lambda',\widehat{\lambda})}{\mathrm{S}^\circ(\lambda',\lambda)}\frac{\mathrm{S}^+(\lambda',\widehat{\lambda})}{\mathrm{S}^\circ(\widehat{\lambda'},\widehat{\lambda})}} \\ &= \sqrt{\prod_{1\leq i\leq n-1}\frac{(\lambda_i'-\lambda_n'-i+n)!(\lambda_i-\lambda_n-i+n-1)!}{(\lambda_i'-\lambda_n-i+n)!(\lambda_i-\lambda_n'-i+n-1)!}} = \sqrt{\frac{\mathrm{C}^\circ(\lambda';\lambda)}{\mathrm{C}^\circ(\widehat{\lambda'};\widehat{\lambda})}}. \end{aligned}
$$

By this equality and (4.4), we have

$$
\mathrm{C}_{H(\lambda')}^{H(\lambda),Q(\lambda'-\lambda)} = \sqrt{\frac{\mathrm{C}^\circ(\lambda';\lambda)}{\mathrm{C}^\circ(\widehat{\lambda'};\widehat{\lambda})}}\,\mathrm{C}_{H(\widehat{\lambda'})}^{H(\widehat{\lambda}),Q(\widehat{\lambda'}-\widehat{\lambda})}.
$$

Hence, the statement (2) follows from the induction hypothesis and this relation.

Next, we will prove the statement (1) by induction with respect to n. Assume $n > 1$ and $\mu \in \Xi^+(\lambda') \cap \Xi^+(\lambda)$. Let $M \in \mathrm{G}(\lambda)$ and $M' \in \mathrm{G}(\lambda')$ such that $\widehat{M} = \widehat{M'} \in \mathrm{G}(\mu)$. By (4.4) and (4.5), we have

$$\mathrm{C}_{M'}^{M,Q((\mathbf{0}_{n-1},l))} = \left(\begin{array}{cc|c} \lambda, & l & \lambda' \\ \mu, & 0 & \mu \end{array}\right) \mathrm{C}_{\widehat{M'}}^{\widehat{M},Q(\mathbf{0}_{n-1})} = \sqrt{\frac{l!\,\mathrm{S}^\circ(\lambda',\lambda')\mathrm{S}^+(\lambda',\mu)}{\mathrm{S}^\circ(\lambda',\lambda)\mathrm{S}^+(\lambda,\mu)}}\,\mathrm{C}_{\widehat{M'}}^{\widehat{M},Q(\mathbf{0}_{n-1})}.$$

In the case of $n = 2$, the statement (1) follows from this relation and our normalization $\mathrm{C}_{\widehat{M'}}^{\widehat{M},0} = 1$. In the case of $n \geq 3$, the statement (1) follows from this relation and the induction hypothesis. □

Proof of Proposition 2.13. We define a $\mathrm{GL}(n,\mathbb{C})$-homomorphism $\mathrm{I}_{\lambda'}^{\lambda,l} : V_{\lambda'} \to V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}$ by

$$\mathrm{I}_{\lambda'}^{\lambda,l} = \sqrt{\mathrm{b}(\lambda'-\lambda)\mathrm{C}^\circ(\lambda';\lambda)}\,\tilde{\mathrm{I}}_{\lambda'}^{\lambda,l},$$

where $\mathrm{b}(\lambda'-\lambda)$ is defined by (2.38). We take constants $\mathrm{c}_{M'}^{M,P}$ ($M' \in \mathrm{G}(\lambda')$, $M \in \mathrm{G}(\lambda)$, $P \in \mathrm{G}((l,\mathbf{0}_{n-1}))$) so that

$$\mathrm{I}_{\lambda'}^{\lambda,l}(\xi_{M'}) = \sum_{M\in\mathrm{G}(\lambda)} \sum_{P\in\mathrm{G}((l,\mathbf{0}_{n-1}))} \mathrm{c}_{M'}^{M,P}\,\xi_M \otimes \xi_P \qquad (M' \in \mathrm{G}(\lambda')).$$

Then, for $M \in \mathrm{G}(\lambda)$, $P \in \mathrm{G}((l,\mathbf{0}_{n-1}))$ and $M' \in \mathrm{G}(\lambda')$, we have

$$\mathrm{c}_{M'}^{M,P} = \sqrt{\frac{\mathrm{b}(\lambda'-\lambda)\mathrm{C}^\circ(\lambda';\lambda)\mathrm{r}(M')}{\mathrm{r}(M)\mathrm{r}(P)}}\,\mathrm{C}_{M'}^{M,P},$$

where $\mathrm{r}(M)$ is defined by (2.18). Hence, the equality $\mathrm{c}_{\lambda_1+l}^{\lambda_1,l} = 1$ follows from our normalization $\mathrm{C}_{\lambda_1+l}^{\lambda_1,l} = 1$ in the case of $n = 1$. The recursive formula (2.40) follows from (4.4) and (4.5) in the case of $n > 1$. The equality (2.41) follows from $\mathrm{r}(H(\lambda)) = \mathrm{r}(H(\lambda')) = 1$, $\mathrm{b}(\gamma) = \mathrm{r}(Q(\gamma))^{-1}$ ($\gamma \in \mathbb{N}_0^n$) and Lemma 4.1(2). The equality (2.42) follows from (4.1). □

4.2. **Some lemmas for representations of K_n.** Let $\mathbb{C}_{\mathrm{triv}} = \mathbb{C}$ be the trivial $\mathrm{GL}(n,\mathbb{C})$-module. The purpose of this subsection is to give proofs of Lemma 2.5 and Lemmas 4.2, 4.3, and 4.4.

Lemma 4.2. *Let* $\lambda \in \Lambda_{n,F}$.

(1) *The space* $\mathrm{Hom}_{K_n}(V_\lambda \otimes_{\mathbb{C}} \overline{V_\lambda}, \mathbb{C}_{\mathrm{triv}})$ *is a* 1 *dimensional space spanned by the* $\mathbb{C}$*-linear map*

$$V_\lambda \otimes_{\mathbb{C}} \overline{V_\lambda} \ni v_1 \otimes \overline{v_2} \mapsto \langle v_1, v_2\rangle \in \mathbb{C}.$$

(2) *Let* $\lambda' \in \Lambda_{n,F} \cap \Xi^\circ(\lambda)$, *and set* $l = \ell(\lambda'-\lambda)$. *For* $\lambda'' \in \Xi^\circ(\lambda;l)$ *such that* $\lambda'' \neq \lambda'$, *we have* $\mathrm{Hom}_{K_n}(V_{\lambda'} \otimes_{\mathbb{C}} \overline{V_{\lambda''}}, \mathbb{C}_{\mathrm{triv}}) = \{0\}$.

Lemma 4.3. *Assume* $n > 1$, *and we regard* K_{n-1} *as a subgroup of* K_n *via* (2.22). *Let* $\lambda \in \Lambda_{n,F}$ *and* $\mu \in \Lambda_{n-1,F}$. *Then* $\mathrm{Hom}_{K_{n-1}}(V_\lambda \otimes_{\mathbb{C}} \overline{V_\mu}, \mathbb{C}_{\mathrm{triv}})$ *is a* 1 *dimensional space spanned by the* $\mathbb{C}$*-linear map*

$$V_\lambda \otimes_{\mathbb{C}} \overline{V_\mu} \ni v_1 \otimes \overline{v_2} \mapsto \langle \mathrm{R}_\mu^\lambda(v_1), v_2\rangle \in \mathbb{C}_{\mathrm{triv}}$$

if $\mu \in \Xi^+(\lambda)$, *and is equal to* $\{0\}$ *otherwise. Here* R_μ^λ *is defined in Lemma* 2.3(3). *Moreover, if* $\mu \in \Xi^+(\lambda)$, *then*

$$\sum_{M\in\mathrm{G}(\mu)} \mathrm{r}(M)^{-1}\,\xi_{M[\lambda]} \otimes \overline{\xi_M}$$

is a unique $\mathbb{Q}$*-rational* K_{n-1}*-invariant vector in* $V_\lambda \otimes_{\mathbb{C}} \overline{V_\mu}$ *up to scalar multiple.*

Lemma 4.4. *Let* $\lambda, \lambda' \in \Lambda_{n,F}$ *such that* $\ell(\lambda'-\lambda) \geq 0$*. Let* $l \in \mathbb{N}_0$*. If* $F = \mathbb{R}$*, we assume* $l \in \{0,1\}$*. Then* $\mathrm{Hom}_{K_n}(V_{\lambda'} \otimes_{\mathbb{C}} \overline{V_\lambda} \otimes_{\mathbb{C}} \overline{V_{(l,\mathbf{0}_{n-1})}}, \mathbb{C}_{\mathrm{triv}})$ *is a* 1 *dimensional space spanned by the* $\mathbb{C}$*-linear map*

$$V_{\lambda'} \otimes_{\mathbb{C}} \overline{V_\lambda} \otimes_{\mathbb{C}} \overline{V_{(l,\mathbf{0}_{n-1})}} \ni v_1 \otimes \overline{v_2} \otimes \overline{v_3} \mapsto \langle \mathrm{I}_{\lambda'}^{\lambda;l}(v_1),\, v_2 \otimes v_3 \rangle \in \mathbb{C}_{\mathrm{triv}}$$

if $\lambda' \in \Xi^\circ(\lambda;l)$*, and is equal to* $\{0\}$ *otherwise. Moreover, if* $\lambda' \in \Xi^\circ(\lambda;l)$*, then*

$$\sum_{M' \in \mathrm{G}(\lambda')} \sum_{M \in \mathrm{G}(\lambda)} \sum_{P \in \mathrm{G}((l,\mathbf{0}_{n-1}))} \frac{\mathrm{c}_{M'}^{M,P}}{\mathrm{r}(M')} \xi_{M'} \otimes \overline{\xi_M} \otimes \overline{\xi_P}, \tag{4.7}$$

is a unique $\mathbb{Q}$*-rational* K_n*-invariant vector in* $V_{\lambda'} \otimes_{\mathbb{C}} \overline{V_\lambda} \otimes_{\mathbb{C}} \overline{V_{(l,\mathbf{0}_{n-1})}}$ *up to scalar multiple.*

Since proofs of Lemmas 2.5, 4.2, 4.3 and 4.4 are easy for $F = \mathbb{C}$, the main concern is the case of $F = \mathbb{R}$. We have $\Lambda_{n,\mathbb{R}} = \{(\mathbf{1}_j, \mathbf{0}_{n-j}) \mid 0 \leq j \leq n\}$ with $\mathbf{1}_j = (1,1,\cdots,1) \in \mathbb{Z}^j$ and $\mathbf{0}_{n-j} = (0,0,\cdots,0) \in \mathbb{Z}^{n-j}$. Here we erase the symbol $\mathbf{1}_j$ if $j=0$, and erase the symbol $\mathbf{0}_{n-j}$ if $j=n$. Let $0 \leq l \leq n$, and we regard the l-th exterior power $\bigwedge^l(\mathrm{M}_{n,1}(\mathbb{C}))$ of $\mathrm{M}_{n,1}(\mathbb{C})$ as a $\mathrm{GL}(n,\mathbb{C})$-module via the action derived from the matrix multiplication. Then we have $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \simeq \bigwedge^l(\mathrm{M}_{n,1}(\mathbb{C}))$ as $\mathrm{GL}(n,\mathbb{C})$-modules via the correspondence

$$\zeta_M \leftrightarrow \mathfrak{e}_{i_1} \wedge \mathfrak{e}_{i_2} \wedge \cdots \wedge \mathfrak{e}_{i_l} \qquad (M \in \mathrm{G}((\mathbf{1}_l,\mathbf{0}_{n-l})))$$

with $1 \leq i_1 < i_2 < \cdots < i_l \leq n$ such that $\gamma_i^M = 1$ $(i \in \{i_1, i_2, \cdots, i_l\})$. Here $\mathfrak{e}_j$ is the matrix unit in $\mathrm{M}_{n,1}(\mathbb{C})$ with 1 at the $(j,1)$-th entry and 0 at other entries, for $1 \leq j \leq n$. We identify $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}$ with $\bigwedge^l(\mathrm{M}_{n,1}(\mathbb{C}))$ via this isomorphism.

We have $\mathrm{O}(n) = \mathrm{SO}(n) \sqcup \mathrm{SO}(n)k_0$ with $k_0 = \mathrm{diag}(1,1,\cdots,1,-1) \in \mathrm{O}(n)$ and $\mathrm{SO}(n) = \{k \in \mathrm{O}(n) \mid \det k = 1\}$. The complexification $\mathfrak{so}(n)_{\mathbb{C}}$ of the associated Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ is given by $\mathfrak{so}(n)_{\mathbb{C}} = \bigoplus_{1 \leq i < j \leq n} \mathbb{C}E_{i,j}^{\mathfrak{so}(n)}$ with $E_{i,j}^{\mathfrak{so}(n)} = E_{i,j} - E_{j,i}$. Here we understand $k_0 = -1$ and $\mathfrak{so}(1)_{\mathbb{C}} = \{0\}$ if $n=1$. Let us recall some facts in the highest weight theory [18, Theorem 4.28] for $\mathrm{SO}(n)$. Let m be the largest integer such that $2m \leq n$. When $n \geq 2$, for an irreducible representation (τ, V_τ) of $\mathrm{SO}(n)$, there is a nonzero vector v_0 in V_τ such that, for $1 \leq i \leq m$ and $2i+1 \leq j \leq n$,

$$\tau\big(E_{2i-1,2i}^{\mathfrak{so}(n)}\big)v_0 = \sqrt{-1}\lambda_{\tau,i}v_0, \qquad \tau\big(E_{2i-1,j}^{\mathfrak{so}(n)} + \sqrt{-1}E_{2i,j}^{\mathfrak{so}(n)}\big)v_0 = 0$$

with some $\lambda_\tau = (\lambda_{\tau,1}, \lambda_{\tau,2}, \cdots, \lambda_{\tau,m}) \in \mathbb{Z}^m$. Such vector v_0 is unique up to nonzero scalar multiple, and we call v_0 an $\mathrm{SO}(n)$-highest weight vector of weight λ_τ. The weight λ_τ is called the highest weight of τ, and $\tau \mapsto \lambda_\tau$ gives a bijection from the set of equivalence classes of irreducible representations of $\mathrm{SO}(n)$ to the set of $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{m-1}, \lambda_m) \in \Lambda_m$ satisfying

$$\begin{cases} (\lambda_1, \lambda_2, \cdots, \lambda_{m-1}, -\lambda_m) \in \Lambda_m & \text{if } n \text{ is even,} \\ \lambda_m \geq 0 & \text{if } n \text{ is odd.} \end{cases}$$

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{m-1}, \lambda_m) \in \Lambda_m$ such that $\lambda_m \geq 0$, we take a representation $(\tau_{\mathfrak{so}(n),\lambda}, V_{\mathfrak{so}(n),\lambda})$ of $\mathrm{SO}(n)$ as follows:

- Let $(\tau_{\mathfrak{so}(n),\lambda}, V_{\mathfrak{so}(n),\lambda})$ be an irreducible representation of $\mathrm{SO}(n)$ with highest weight λ unless $n = 2m$ and $\lambda_m > 0$.

- Let $(\tau_{\mathfrak{so}(n),\lambda}, V_{\mathfrak{so}(n),\lambda})$ be a direct sum of two irreducible representations of $\mathrm{SO}(n)$ with highest weights λ and $(\lambda_1, \lambda_2, \cdots, \lambda_{m-1}, -\lambda_m)$ if $n = 2m$ and $\lambda_m > 0$.

By Weyl's dimension formula [18, Theorem 4.48], we have

$$\dim V_{\mathfrak{so}(n),(i+1,\mathbf{1}_{h-1},\mathbf{0}_{m-h})} = \frac{(2i+n)}{(i+h)(i+n-h)} \frac{(i+n-1)!}{i!(n-1-h)!(h-1)!} \tag{4.8}$$

for $1 \le h \le m$ and $i \in \mathbb{N}_0$.

Lemma 4.5. *Retain the notation. Let $0 \le l \le n$. As an $\mathrm{O}(n)$-module, $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}$ is irreducible and $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \not\simeq V_{(\mathbf{1}_{l'},\mathbf{0}_{n-l'})}$ for any $0 \le l' \le n$ such that $l' \neq l$. We set $h = \min\{l, n-l\}$. When $n \ge 2$, we have*

$$V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \simeq V_{\mathfrak{so}(n),(\mathbf{1}_h,\mathbf{0}_{m-h})} \quad \text{as } \mathrm{SO}(n)\text{-modules.} \tag{4.9}$$

Proof. For $1 \le i_1 < i_2 < \cdots < i_l \le n$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n \in \{\pm 1\}$, we have

$$\tau_{(\mathbf{1}_l,\mathbf{0}_{n-l})}(\mathrm{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n))\, \mathfrak{e}_{i_1} \wedge \mathfrak{e}_{i_2} \wedge \cdots \wedge \mathfrak{e}_{i_l} = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_l}\, \mathfrak{e}_{i_1} \wedge \mathfrak{e}_{i_2} \wedge \cdots \wedge \mathfrak{e}_{i_l}.$$

By this equality, we know that $\mathrm{Hom}_{\mathrm{O}(n)}(V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}, V_{(\mathbf{1}_{l'},\mathbf{0}_{n-l'})}) = \{0\}$ for any $0 \le l' \le n$ such that $l' \neq l$. Hence, our task is to show (4.9) and the irreducibility of $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}$ as an $\mathrm{O}(n)$-module.

In the case of $n \ge 2$ and $n \neq 2l$, the isomorphism (4.9) follows from [18, Examples in Chapter IV, §7], and we note that $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}$ is an irreducible $\mathrm{O}(n)$-module. In the case of $n = 1$, the irreducibility of an $\mathrm{O}(1)$-module $V_{(\mathbf{1}_l,\mathbf{0}_{1-l})}$ is trivial. Let us consider the case of $n = 2l$. By direct computation, for $\varepsilon \in \{\pm 1\}$, we confirm that

$$v_\varepsilon = (\mathfrak{e}_1 + \sqrt{-1}\mathfrak{e}_2) \wedge (\mathfrak{e}_3 + \sqrt{-1}\mathfrak{e}_4) \wedge \cdots \wedge (\mathfrak{e}_{n-3} + \sqrt{-1}\mathfrak{e}_{n-2}) \wedge (\mathfrak{e}_{n-1} + \varepsilon\sqrt{-1}\mathfrak{e}_n)$$

is an $\mathrm{SO}(n)$-highest weight vector of weight $(\mathbf{1}_{l-1}, \varepsilon)$ in $V_{(\mathbf{1}_l,\mathbf{0}_l)}$ and satisfies the equality $\tau_{(\mathbf{1}_l,\mathbf{0}_l)}(k_0)v_\varepsilon = v_{-\varepsilon}$. Since $\dim V_{(\mathbf{1}_l,\mathbf{0}_l)} = \dim V_{\mathfrak{so}(n),\mathbf{1}_l}$ by (4.8), we know that (4.9) holds and $V_{(\mathbf{1}_l,\mathbf{0}_l)}$ is an irreducible $\mathrm{O}(n)$-module. □

Proof of Lemma 2.5. The assertion for $F = \mathbb{R}$ follows immediately from Lemma 4.5. The assertion for $F = \mathbb{C}$ follows immediately from the highest weight theory [18, Theorem 4.28] for $\mathrm{U}(n)$. □

Lemma 4.6. *Assume $n \ge 2$. Let $1 \le l \le n-1$. We set $h = \min\{l, n-l\}$. Let $\mathrm{I}_l \colon V_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})} \to V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})}$ be a $\mathbb{C}$-linear map defined by*

$$\mathrm{I}_l(\mathfrak{e}_{i_1} \wedge \mathfrak{e}_{i_2} \wedge \cdots \wedge \mathfrak{e}_{i_{l-1}}) = \sum_{j=1}^{n} (\mathfrak{e}_{i_1} \wedge \mathfrak{e}_{i_2} \wedge \cdots \wedge \mathfrak{e}_{i_{l-1}} \wedge \mathfrak{e}_j) \otimes \mathfrak{e}_j$$

for $i_1,\ i_2,\ \cdots,\ i_{l-1} \in \{1, 2, \cdots, n\}$. Here we understand $\mathrm{I}_1(1) = \sum_{j=1}^{n} \mathfrak{e}_j \otimes \mathfrak{e}_j$ if $l = 1$. Then I_l is an $\mathrm{O}(n)$-homomorphism. Moreover, there is an $\mathrm{SO}(n)$-submodule V' of $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})}$ such that $V' \simeq V_{\mathfrak{so}(n),(2,\mathbf{1}_{h-1},\mathbf{0}_{m-h})}$ and

(4.10)

$$V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})} = \mathrm{I}^{(\mathbf{1}_l,\mathbf{0}_{n-l}),1}_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})}(V_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})}) \oplus \mathrm{I}_l(V_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})}) \oplus V',$$

where $\mathrm{I}^{(\mathbf{1}_l,\mathbf{0}_{n-l}),1}_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})}$ is the $\mathrm{GL}(n,\mathbb{C})$-homomorphism in Proposition 2.13.

Proof. For $v \in V_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})}$ and $1 \le i < j \le n$, we have

$$\mathrm{I}_l(\tau_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})}(E^{\mathfrak{so}(n)}_{i,j})v) = (\tau_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes \tau_{(1,\mathbf{0}_{n-1})})(E^{\mathfrak{so}(n)}_{i,j})\mathrm{I}_l(v),$$
$$\mathrm{I}_l(\tau_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})}(k_0)v) = (\tau_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes \tau_{(1,\mathbf{0}_{n-1})})(k_0)\mathrm{I}_l(v)$$

by direct computation. Hence, I_l is an $\mathrm{O}(n)$-homomorphism.

For an $\mathrm{SO}(n)$-highest weight vector v of weight λ in $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})}$, we note that $v \otimes (\mathfrak{e}_1 + \sqrt{-1}\mathfrak{e}_2)$ is an $\mathrm{SO}(n)$-highest weight vector of weight $\lambda + (1, \mathbf{0}_{m-1})$ in $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})}$. Hence, by Lemma 4.5, there is an $\mathrm{SO}(n)$-submodule V' of $V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})}$ such that $V' \simeq V_{\mathfrak{so}(n),(2,\mathbf{1}_{h-1},\mathbf{0}_{m-h})}$, and we know that

$$\mathrm{I}^{(\mathbf{1}_l,\mathbf{0}_{n-l}),1}_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})}(V_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})}) + \mathrm{I}_l(V_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})}) + V'$$

is a direct sum. By (4.8), we know that $\dim V_{(\mathbf{1}_l,\mathbf{0}_{n-l})} \otimes_{\mathbb{C}} V_{(1,\mathbf{0}_{n-1})}$ is equal to

$$\dim V_{(\mathbf{1}_{l+1},\mathbf{0}_{n-l-1})} + \dim V_{(\mathbf{1}_{l-1},\mathbf{0}_{n-l+1})} + \dim V_{\mathfrak{so}(n),(2,\mathbf{1}_{h-1},\mathbf{0}_{m-h})}.$$

This implies that (4.10) holds. □

Lemma 4.7. *Let (τ, V_τ) be a finite dimensional representation of $\mathrm{GL}(n,\mathbb{C})$ with a $\mathrm{U}(n)$-invariant hermitian inner product $\langle\cdot,\cdot\rangle$ on V_τ. Let $\{v_i\}_{i=1}^d$ be an orthonormal basis of V_τ. Let $(\tau', V_{\tau'})$ be a finite dimensional representation of $\mathrm{GL}(n,\mathbb{C})$.*

(1) *A $\mathbb{C}$-linear map $\Psi_1\colon \mathrm{Hom}_{\mathbb{C}}(V_{\tau'}, V_\tau) \to \mathrm{Hom}_{\mathbb{C}}(V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau}, \mathbb{C}_{\mathrm{triv}})$ defined by*

$$\Psi_1(f)(v' \otimes \overline{v}) = \langle f(v'), v\rangle \qquad (f \in \mathrm{Hom}_{\mathbb{C}}(V_{\tau'}, V_\tau),\ v' \in V_{\tau'},\ v \in V_\tau)$$

is bijective, and its inverse map is given by

$$\Psi_1^{-1}(f)(v') = \sum_{i=1}^d f(v' \otimes \overline{v_i})\, v_i \qquad (f \in \mathrm{Hom}_{\mathbb{C}}(V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau}, \mathbb{C}_{\mathrm{triv}}),\ v' \in V_{\tau'}).$$

Moreover, we have $\Psi_1(\mathrm{Hom}_{K_n}(V_{\tau'}, V_\tau)) = \mathrm{Hom}_{K_n}(V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau}, \mathbb{C}_{\mathrm{triv}})$.

(2) *A $\mathbb{C}$-linear map $\Psi_2\colon V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau} \to \mathrm{Hom}_{\mathbb{C}}(V_\tau, V_{\tau'})$ defined by*

$$\Psi_2(v' \otimes \overline{v_1})(v_2) = \langle v_2, v_1\rangle v' \qquad (v_1, v_2 \in V_\tau,\ v' \in V_{\tau'})$$

is bijective, and its inverse map is given by

$$\Psi_2^{-1}(f) = \sum_{i=1}^d f(v_i) \otimes \overline{v_i} \qquad (f \in \mathrm{Hom}_{\mathbb{C}}(V_\tau, V_{\tau'})).$$

Moreover, we have $\Phi_2((V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau})^{K_n}) = \mathrm{Hom}_{K_n}(V_\tau, V_{\tau'})$, where $(V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau})^{K_n}$ is the subspace of $V_{\tau'} \otimes_{\mathbb{C}} \overline{V_\tau}$ consisting of all K_n-invariant vectors.

Proof. The former part of the statement (1) follows from definition. The latter part of the statement (1) follows from

$$\begin{aligned}\Psi_1(f)((\tau' \otimes \overline{\tau})(k)v' \otimes \overline{v}) &= \langle f(\tau'(k)v'), \tau(k)v\rangle = \langle \tau(k^{-1})f(\tau'(k)v'), v\rangle \\ &= \Psi_1(\tau(k^{-1}) \circ f \circ \tau'(k))(v' \otimes \overline{v})\end{aligned}$$

for $f \in \mathrm{Hom}_{\mathbb{C}}(V_{\tau'}, V_\tau)$, $v' \in V_{\tau'}$, $v \in V_\tau$ and $k \in K_n$.

The former part of the statement (2) follows from definition. The latter part of the statement (2) follows from

$$\begin{aligned}\Psi_2((\tau' \otimes \overline{\tau})(k)v' \otimes \overline{v_1})(v_2) &= \langle v_2, \tau(k)v_1\rangle \tau'(k)v' = \langle \tau(k^{-1})v_2, v_1\rangle \tau'(k)v' \\ &= \tau'(k)\Psi_2(v' \otimes \overline{v_1})(\tau(k^{-1})v_2)\end{aligned}$$

for $v_1, v_2 \in V_\tau$, $v' \in V_{\tau'}$ and $k \in K_n$. □

Proof of Lemma 4.2. Let $\lambda \in \Lambda_{n,F}$. By Lemma 2.5, we note that $\mathrm{Hom}_{K_n}(V_\lambda, V_\lambda)$ is a 1 dimensional space spanned by the identity map. Hence, we obtain the statement (1) by Lemma 4.7(1).

Let $\lambda' \in \Lambda_{n,F} \cap \Xi^\circ(\lambda)$, and we set $l = \ell(\lambda' - \lambda)$. By the decompositions (2.35), (4.10) and Lemma 4.5, we have $\mathrm{Hom}_{K_n}(V_{\lambda'}, V_{\lambda''}) = \{0\}$ for $\lambda'' \in \Xi^\circ(\lambda; l)$ such that $\lambda'' \neq \lambda'$. Hence, we obtain the statement (2) by Lemma 4.7(1). □

Proof of Lemma 4.3. By (2.23) and Lemma 2.5, we know that $\mathrm{Hom}_{K_{n-1}}(V_\lambda, V_\mu)$ is equal to $\mathbb{C}\,\mathrm{R}^\lambda_\mu$ if $\mu \in \Xi^+(\lambda)$, and is equal to $\{0\}$ otherwise. By Lemma 4.7, we obtain the former part of the assertion, and know that, if $\mu \in \Xi^+(\lambda)$,

$$\sum_{M \in \mathrm{G}(\lambda)} \mathrm{R}^\lambda_\mu(\zeta_M) \otimes \overline{\zeta_M} = \sum_{M \in \mathrm{G}(\mu)} \zeta_M \otimes \overline{\zeta_{M[\lambda]}}$$

is a unique K_{n-1}-invariant vector in $V_\mu \otimes_{\mathbb{C}} \overline{V_\lambda}$ up to scalar multiple. Hence, by (2.17) and the properties of complex conjugate representations in §2.6, we obtain the latter part of the assertion. □

Proof of Lemma 4.4. By the decompositions (2.35), (4.10) and Lemma 4.5, we know that the space $\mathrm{Hom}_{K_n}(V_{\lambda'}, V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})})$ is equal to $\mathbb{C}\,\mathrm{I}^{\lambda;l}_{\lambda'}$ if $\lambda' \in \Xi^\circ(\lambda; l)$, and is equal to $\{0\}$ otherwise. By Lemma 4.7, we obtain the former part of the assertion, and know that, if $\lambda' \in \Xi^\circ(\lambda; l)$,

$$\sum_{M' \in \mathrm{G}(\lambda')} \mathrm{I}^{\lambda;l}_{\lambda'}(\zeta_{M'}) \otimes \overline{\zeta_{M'}}$$

is a unique K_n-invariant vector in $V_\lambda \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})} \otimes_{\mathbb{C}} \overline{V_{\lambda'}}$ up to scalar multiple. Hence, by (2.17) and the properties of complex conjugate representations in §2.6, we obtain the latter part of the assertion. □

4.3. **Polynomial functions.** We set

$$\Lambda^{\mathrm{poly}}_n = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n \mid \lambda_n \geq 0\}.$$

We denote by $\mathcal{P}(\mathrm{M}_{n,n'}(\mathbb{C}))$ the subspace of $C(\mathrm{M}_{n,n'}(\mathbb{C}))$ consisting of all polynomial functions. Let $l \in \mathbb{N}_0$. We denote by $\mathcal{P}_l(\mathrm{M}_{n,n'}(\mathbb{C}))$ the subspace of $\mathcal{P}(\mathrm{M}_{n,n'}(\mathbb{C}))$ consisting of all degree l homogeneous polynomial functions. We regard $\mathcal{P}_l(\mathrm{M}_{n,n'}(\mathbb{C}))$ as a $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n',\mathbb{C})$-module via the action $L \boxtimes R$ which is defined in §2.9. Let $q = \min\{n, n'\}$. Then the $\mathrm{GL}(n)$-$\mathrm{GL}(n')$ duality [5, Theorem 5.6.7] asserts that

$$\mathcal{P}_l(\mathrm{M}_{n,n'}(\mathbb{C})) \simeq \bigoplus_{\lambda \in \Lambda^{\mathrm{poly}}_q,\ \ell(\lambda) = l} V^\vee_{(\lambda,\mathbf{0}_{n-q})} \boxtimes_{\mathbb{C}} V_{(\lambda,\mathbf{0}_{n'-q})}$$

as $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n',\mathbb{C})$-modules. Since $V^\vee_{(\lambda,\mathbf{0}_{n-q})} \simeq \overline{V_{(\lambda,\mathbf{0}_{n-q})}}$ as $\mathrm{U}(n)$-modules, we also have

$$\mathcal{P}_l(\mathrm{M}_{n,n'}(\mathbb{C})) \simeq \bigoplus_{\lambda \in \Lambda^{\mathrm{poly}}_q,\ \ell(\lambda) = l} \overline{V_{(\lambda,\mathbf{0}_{n-q})}} \boxtimes_{\mathbb{C}} V_{(\lambda,\mathbf{0}_{n'-q})} \tag{4.11}$$

as $\mathrm{U}(n) \times \mathrm{GL}(n',\mathbb{C})$-modules.

The purpose of this subsection is to construct polynomial functions, explicitly. We define $\mathrm{U}(n) \times \mathrm{GL}(n,\mathbb{C})$-homomorphisms $\mathrm{P}^\circ_\lambda \colon \overline{V_\lambda} \boxtimes_{\mathbb{C}} V_\lambda \to \mathcal{P}(\mathrm{M}_n(\mathbb{C}))$ $(\lambda \in \Lambda^{\mathrm{poly}}_n)$ by Lemma 4.8.

Lemma 4.8. *Let* $\lambda \in \Lambda_n^{\mathrm{poly}}$*. Then there is a* $\mathrm{U}(n) \times \mathrm{GL}(n,\mathbb{C})$*-homomorphism* $\mathrm{P}_\lambda^\circ \colon \overline{V_\lambda} \boxtimes_{\mathbb{C}} V_\lambda \to \mathcal{P}(\mathrm{M}_n(\mathbb{C}))$ *characterized by*

$$\mathrm{P}_\lambda^\circ(\overline{v_1} \boxtimes v_2)(g) = \langle \tau_\lambda(g) v_2, v_1 \rangle \qquad (v_1, v_2 \in V_\lambda,\ g \in \mathrm{GL}(n,\mathbb{C})). \tag{4.12}$$

Proof. Because of the irreducible decomposition (4.11), there is a nonzero $\mathrm{U}(n) \times \mathrm{GL}(n,\mathbb{C})$-homomorphism $\mathrm{P} \colon \overline{V_\lambda} \boxtimes_{\mathbb{C}} V_\lambda \to \mathcal{P}(\mathrm{M}_n(\mathbb{C}))$. Since $\mathrm{GL}(n,\mathbb{C})$ is dense in $\mathrm{M}_n(\mathbb{C})$ and

$$\mathrm{P}(\overline{v_1} \boxtimes v_2)(g) = \mathrm{P}(\overline{v_1} \boxtimes \tau_\lambda(g) v_2)(1_n) \qquad (v_1, v_2 \in V_\lambda,\ g \in \mathrm{GL}(n,\mathbb{C})), \tag{4.13}$$

we note that

$$\overline{V_\lambda} \boxtimes_{\mathbb{C}} V_\lambda \ni \overline{v_1} \boxtimes v_2 \mapsto \mathrm{P}(\overline{v_1} \boxtimes v_2)(1_n) \in \mathbb{C}$$

is a nonzero $\mathbb{C}$-bilinear pairing. Because of

$$\mathrm{P}(\overline{\tau_\lambda}(k)\overline{v_1} \boxtimes \tau_\lambda(k) v_2)(1_n) = \mathrm{P}(\overline{v_1} \boxtimes v_2)(1_n) \qquad (k \in \mathrm{U}(n))$$

and Lemma 4.2(1) for $F = \mathbb{C}$, there is a nonzero constant c such that

$$\mathrm{P}(\overline{v_1} \boxtimes v_2)(1_n) = c\,\langle v_2, v_1 \rangle \qquad (v_1, v_2 \in V_\lambda). \tag{4.14}$$

By (4.13) and (4.14), we know that $\mathrm{P}_\lambda^\circ = c^{-1}\mathrm{P}$ satisfies (4.12). Since $\mathrm{GL}(n,\mathbb{C})$ is dense in $\mathrm{M}_n(\mathbb{C})$, we note that (4.12) characterizes P_λ°. □

When $n > 1$, we define $\mathrm{U}(n-1) \times \mathrm{GL}(n,\mathbb{C})$-homomorphisms $\mathrm{P}_\mu^+ \colon \overline{V_\mu} \boxtimes_{\mathbb{C}} V_{(\mu,0)} \to \mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C}))$ ($\mu \in \Lambda_{n-1}^{\mathrm{poly}}$) by Lemma 4.9.

Lemma 4.9. *Assume* $n > 1$ *and let* $\mu \in \Lambda_{n-1}^{\mathrm{poly}}$*. There is a* $\mathrm{U}(n-1) \times \mathrm{GL}(n,\mathbb{C})$*-homomorphism* $\mathrm{P}_\mu^+ \colon \overline{V_\mu} \boxtimes_{\mathbb{C}} V_{(\mu,0)} \to \mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C}))$ *characterized by*

$$\mathrm{P}_\mu^+(\overline{\zeta_M} \boxtimes v)((1_{n-1}, O_{n-1,1})z) = \mathrm{P}_{(\mu,0)}^\circ(\overline{\zeta_{M[(\mu,0)]}} \boxtimes v)(z) \tag{4.15}$$

for $M \in \mathrm{G}(\mu)$*,* $v \in V_{(\mu,0)}$ *and* $z \in \mathrm{M}_n(\mathbb{C})$*. Here* $M[(\mu,0)]$ *is defined by* (2.24)*. Furthermore, we have*

$$\mathrm{P}_\mu^+(\overline{v} \boxtimes \zeta_{M[(\mu,0)]})(z) = \mathrm{P}_\mu^\circ(\overline{v} \boxtimes \zeta_M)(z\,{}^t(1_{n-1}, O_{n-1,1})) \tag{4.16}$$

for $v \in V_\mu$*,* $M \in \mathrm{G}(\mu)$ *and* $z \in \mathrm{M}_{n-1,n}(\mathbb{C})$*.*

Proof. We regard $\mathrm{GL}(n-1,\mathbb{C})$ as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ via the embedding ι_n defined by (2.22). By the irreducible decomposition (4.11) and Lemma 2.3(1), the image of a $\mathrm{U}(n-1) \times \mathrm{GL}(n,\mathbb{C})$-homomorphism

$$\overline{V_\mu} \boxtimes_{\mathbb{C}} V_{(\mu,0)} \ni \overline{\zeta_M} \boxtimes v \mapsto \mathrm{P}_{(\mu,0)}^\circ\big(\overline{\zeta_{M[(\mu,0)]}} \boxtimes v\big) \in \mathcal{P}(\mathrm{M}_n(\mathbb{C}))$$

is contained in the image of an injective $\mathrm{U}(n-1) \times \mathrm{GL}(n,\mathbb{C})$-homomorphism

$$\mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C})) \ni p \mapsto \big(z \mapsto p((1_{n-1}, O_{n-1,1})z)\big) \in \mathcal{P}(\mathrm{M}_n(\mathbb{C})).$$

Hence, there is a $\mathrm{U}(n-1) \times \mathrm{GL}(n,\mathbb{C})$-homomorphism

$$\mathrm{P}_\mu^+ \colon \overline{V_\mu} \boxtimes_{\mathbb{C}} V_{(\mu,0)} \to \mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C}))$$

characterized by (4.15). By the irreducible decompositions (4.11) and Lemma 2.3(1), two injective $\mathrm{U}(n-1) \times \mathrm{GL}(n-1,\mathbb{C})$-homomorphisms

$$\overline{V_\mu} \boxtimes_{\mathbb{C}} V_\mu \ni \overline{v} \boxtimes \zeta_M \mapsto \mathrm{P}_\mu^+\big(\overline{v} \boxtimes \zeta_{M[(\mu,0)]}\big) \in \mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C})),$$

$$\overline{V_\mu} \boxtimes_{\mathbb{C}} V_\mu \ni \overline{v_1} \boxtimes v_2 \mapsto \big(z \mapsto \mathrm{P}_\mu^\circ(\overline{v_1} \boxtimes v_2)(z\,{}^t(1_{n-1}, O_{n-1,1}))\big) \in \mathcal{P}(\mathrm{M}_{n-1,n}(\mathbb{C}))$$

coincide up to scalar multiple. Hence, (4.16) follows from the equalities

$$\mathrm{P}^{+}_{\mu}\big(\overline{\zeta_M}\boxtimes\zeta_{M[(\mu,0)]}\big)((1_{n-1},O_{n-1,1}))=\langle\zeta_{M[(\mu,0)]},\zeta_{M[(\mu,0)]}\rangle=1$$

and $\mathrm{P}^{\circ}_{\mu}(\overline{\zeta_M}\boxtimes\zeta_M)(1_{n-1})=\langle\zeta_M,\zeta_M\rangle=1$ for $M\in\mathrm{G}(\mu)$. □

Let $l\in\mathbb{N}_0$. We define two $\mathbb{C}$-linear maps $\mathrm{p}^{(l)}_{1,n}\colon V_{(l,\mathbf{0}_{n-1})}\to\mathcal{P}(\mathrm{M}_{1,n}(\mathbb{C}))$ and $\mathrm{p}^{(l)}_{n,1}\colon\overline{V_{(l,\mathbf{0}_{n-1})}}\to\mathcal{P}(\mathrm{M}_{n,1}(\mathbb{C}))$ by

$$\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma)})(z)=\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma)}})({}^tz)=\sqrt{\mathrm{b}(\gamma)}\,z_1^{\gamma_1}z_2^{\gamma_2}\cdots z_n^{\gamma_n}$$

for $z=(z_1,z_2,\cdots,z_n)\in\mathrm{M}_{1,n}(\mathbb{C})$ and $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$ such that $\ell(\gamma)=l$. Here $Q(\gamma)$ and $\mathrm{b}(\gamma)$ are defined by (2.32) and (2.38), respectively.

Lemma 4.10. *Let $l\in\mathbb{N}_0$.*

(1) *The group $\mathrm{GL}(n,\mathbb{C})$ acts on $\mathcal{P}(\mathrm{M}_{1,n}(\mathbb{C}))$ by R. Then $\mathrm{p}^{(l)}_{1,n}$ is a $\mathrm{GL}(n,\mathbb{C})$-homomorphism such that, for $z\in\mathrm{M}_n(\mathbb{C})$ and $v\in V_{(l,\mathbf{0}_{n-1})}$,*

$$\mathrm{p}^{(l)}_{1,n}(v)(e_nz)=\mathrm{P}^{\circ}_{(l,\mathbf{0}_{n-1})}(\overline{\zeta_{Q((\mathbf{0}_{n-1},l))}}\boxtimes v)(z). \tag{4.17}$$

(2) *The group $\mathrm{U}(n)$ acts on $\mathcal{P}(\mathrm{M}_{n,1}(\mathbb{C}))$ by L. Then $\mathrm{p}^{(l)}_{n,1}$ is a $\mathrm{U}(n)$-homomorphism such that, for $z\in\mathrm{M}_n(\mathbb{C})$ and $v\in V_{(l,\mathbf{0}_{n-1})}$,*

$$\mathrm{p}^{(l)}_{n,1}(\overline{v})(z\,{}^te_n)=\mathrm{P}^{\circ}_{(l,\mathbf{0}_{n-1})}(\overline{v}\boxtimes\zeta_{Q((\mathbf{0}_{n-1},l))})(z). \tag{4.18}$$

Proof. Let $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{N}_0^n$. By direct computation, we have

$$\begin{aligned}
&R(E_{i,i})\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma)})=\gamma_i\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma)}),\\
&R(E_{j,j+1})\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma)})=\sqrt{\gamma_{j+1}(\gamma_j+1)}\,\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma+\delta_j-\delta_{j+1})}),\\
&R(E_{j+1,j})\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma)})=\sqrt{\gamma_j(\gamma_{j+1}+1)}\,\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma-\delta_j+\delta_{j+1})}),\\
&L(E_{i,i})\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma)}})=-\gamma_i\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma)}}),\\
&L(E_{j,j+1})\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma)}})=-\sqrt{\gamma_j(\gamma_{j+1}+1)}\,\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma-\delta_j+\delta_{j+1})}}),\\
&L(E_{j+1,j})\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma)}})=-\sqrt{\gamma_{j+1}(\gamma_j+1)}\,\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma+\delta_j-\delta_{j+1})}})
\end{aligned}$$

for $1\le i\le n$ and $1\le j\le n-1$. Here we put $\mathrm{p}^{(l)}_{1,n}(\zeta_{Q(\gamma')})=\mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q(\gamma')}})=0$ if $\gamma'\notin\mathbb{N}_0^n$, and denote by δ_i the element of $\mathbb{Z}^n$ with 1 at i-th entry and 0 at other entries for $1\le i\le n$. Comparing these formulas with (2.13), (2.14) and (2.15), we know that $\mathrm{p}^{(l)}_{1,n}$ is a $\mathrm{GL}(n,\mathbb{C})$-homomorphism. Using (2.25) and

$$L(E^{\mathfrak{u}(n)}_{i,j})=L(E_{i,j})\quad(1\le i,j\le n)\qquad\text{on }\mathcal{P}(\mathrm{M}_{n,1}(\mathbb{C})),$$

we further know that $\mathrm{p}^{(l)}_{n,1}$ is a $\mathrm{U}(n)$-homomorphism.

Next, we will prove the equality (4.17). When $n=1$, this equality follows from $\mathrm{G}(l)=\{l\}$ and $\mathrm{p}^{(l)}_{1,1}(\zeta_l)(g)=\langle\tau_l(g)\zeta_l,\zeta_l\rangle=g^l$ $(g\in\mathrm{GL}(1,\mathbb{C}))$. Assume $n>1$. We regard $\mathrm{GL}(n-1,\mathbb{C})$ as a subgroup of $\mathrm{GL}(n,\mathbb{C})$ via the embedding ι_n defined by

(2.22). Because of the irreducible decompositions (4.11) and Lemma 2.3(1), two injective $\mathrm{U}(n-1)\times\mathrm{GL}(n,\mathbb{C})$-homomorphisms

$$\overline{V_{\mathbf{0}_{n-1}}}\boxtimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}\ni\overline{\zeta_{Q(\mathbf{0}_{n-1})}}\boxtimes v\mapsto(z\mapsto \mathrm{p}_{1,n}^{(l)}(v)(e_nz))\in\mathcal{P}(\mathrm{M}_n(\mathbb{C})),$$
$$\overline{V_{\mathbf{0}_{n-1}}}\boxtimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}\ni\overline{\zeta_{Q(\mathbf{0}_{n-1})}}\boxtimes v\mapsto \mathrm{P}^{\circ}_{(l,\mathbf{0}_{n-1})}(\overline{\zeta_{Q((\mathbf{0}_{n-1},l))}}\boxtimes v)\in\mathcal{P}(\mathrm{M}_n(\mathbb{C}))$$

coincide up to scalar multiple. Hence, (4.17) follows from the equalities

$$\mathrm{p}_{1,n}^{(l)}(\zeta_{Q((\mathbf{0}_{n-1},l))})(e_n)=1,\qquad \mathrm{P}^{\circ}_{(l,\mathbf{0}_{n-1})}(\overline{\zeta_{Q((\mathbf{0}_{n-1},l))}}\boxtimes\zeta_{Q((\mathbf{0}_{n-1},l))})(1_n)=1.$$

The proof of the equality (4.18) is similar. □

Lemmas 4.11 and 4.12 play important roles to give the explicit description of the recurrence relations of the archimedean Rankin–Selberg integrals. The polynomial function p_0 in Lemma 4.11 (resp. Lemma 4.12) comes from the polynomial part of the standard Schwartz function ϕ_0 in Proposition 3.4 (resp. Proposition 3.5).

Lemma 4.11. *Assume $n>1$. Let $\mu\in\Lambda_{n-1}^{\mathrm{poly}}$ and $\gamma\in\mathbb{N}_0^n$. We set $l=\ell(\gamma)$ and*

$$p_0(z)=\mathrm{P}^{+}_{\mu}\big(\overline{\zeta_{H(\mu)}}\boxtimes\zeta_{H((\mu,0))}\big)((1_{n-1},O_{n-1,1})z)\mathrm{p}_{1,n}^{(l)}(\zeta_{Q(\gamma)})(e_nz)$$

for $z\in\mathrm{M}_n(\mathbb{C})$. Then we have

$$p_0=\sum_{\lambda'\in\Xi^{\circ}((\mu,0);l)}\ \sum_{N,N'\in\mathrm{G}(\lambda')}\mathrm{C}_N^{H((\mu,0)),Q((\mathbf{0}_{n-1},l))}\,\mathrm{C}_{N'}^{H((\mu,0)),Q(\gamma)}\,\mathrm{P}^{\circ}_{\lambda'}(\overline{\zeta_N}\boxtimes\zeta_{N'}),$$

where $\mathrm{C}_{M'}^{M,P}$ is the Clebsch–Gordan coefficient in §4.1.

Proof. We set $Q_0=Q((\mathbf{0}_{n-1},l))$ and $Q_1=Q(\gamma)$. Let $g\in\mathrm{GL}(n,\mathbb{C})$. By Lemmas 4.8, 4.9 and 4.10, we have

$$\begin{aligned}p_0(g)=&\mathrm{P}^{\circ}_{\mu}\big(\overline{\zeta_{H((\mu,0))}}\boxtimes\zeta_{H((\mu,0))}\big)(g)\mathrm{P}^{\circ}_{(l,\mathbf{0}_{n-1})}(\overline{\zeta_{Q_0}}\boxtimes\zeta_{Q_1})(g)\\ =&\big\langle\tau_{(\mu,0)}(g)\zeta_{H((\mu,0))},\zeta_{H((\mu,0))}\big\rangle\big\langle\tau_{(l,\mathbf{0}_{n-1})}(g)\zeta_{Q_1},\ \zeta_{Q_0}\big\rangle\\ =&\big\langle(\tau_{(\mu,0)}\otimes\tau_{(l,\mathbf{0}_{n-1})})(g)\zeta_{H((\mu,0))}\otimes\zeta_{Q_1},\ \zeta_{H((\mu,0))}\otimes\zeta_{Q_0}\big\rangle.\end{aligned}$$

By (4.1), (4.6) and Lemma 4.8, we have

$$\begin{aligned}p_0(g)=&\sum_{\lambda'\in\Xi^{\circ}((\mu,0);l)}\ \sum_{N,N'\in\mathrm{G}(\lambda')}\mathrm{C}_N^{H((\mu,0)),Q_0}\,\mathrm{C}_{N'}^{H((\mu,0)),Q_1}\,\big\langle\tau_{\lambda'}(g)\zeta_{N'},\zeta_N\big\rangle\\ =&\sum_{\lambda'\in\Xi^{\circ}((\mu,0);l)}\ \sum_{N,N'\in\mathrm{G}(\lambda')}\mathrm{C}_N^{H((\mu,0)),Q_0}\,\mathrm{C}_{N'}^{H((\mu,0)),Q_1}\,\mathrm{P}^{\circ}_{\lambda'}(\overline{\zeta_N}\boxtimes\zeta_{N'})(g).\end{aligned}$$

Since $\mathrm{GL}(n,\mathbb{C})$ is dense in $\mathrm{M}_n(\mathbb{C})$, we obtain the assertion. □

Lemma 4.12. *Assume $n>1$. Let $\mu\in\Lambda_{n-1}^{\mathrm{poly}}$ and $\gamma\in\mathbb{N}_0^{n-1}$. We set $l=\ell(\gamma)$ and*

$$p_0(z)=\mathrm{P}^{\circ}_{\mu}\big(\overline{\zeta_{H(\mu)}}\boxtimes\zeta_{H(\mu)}\big)(z\,{}^t(1_{n-1},O_{n-1,1}))\mathrm{p}_{n-1,1}^{(l)}(\overline{\zeta_{Q(\gamma)}})(z\,{}^te_n)$$

for $z\in\mathrm{M}_{n-1,n}(\mathbb{C})$. Then we have

$$p_0=\sum_{\mu'\in\Xi^{\circ}(\mu;l)}\ \sum_{\substack{N\in\mathrm{G}((\mu',0);\mu')\\ N'\in\mathrm{G}((\mu',0))}}\mathrm{C}_N^{H((\mu,0)),Q((\gamma,0))}\,\mathrm{C}_{N'}^{H((\mu,0)),Q((\mathbf{0}_{n-1},l))}\,\mathrm{P}^{+}_{\mu'}(\overline{\zeta_{\widehat{N}}}\boxtimes\zeta_{N'}),$$

where $\mathrm{C}_{M'}^{M,P}$ is the Clebsch–Gordan coefficient in §4.1.

Proof. We set $Q_0 = Q((\mathbf{0}_{n-1}, l))$ and $Q_1 = Q((\gamma, 0))$. Let $z = (1_{n-1}, O_{n-1,1})g$ with $g \in \mathrm{GL}(n, \mathbb{C})$. Then we have $\mathrm{p}^{(l)}_{n-1,1}(\overline{\zeta_{Q(\gamma)}})(z\,{}^te_n) = \mathrm{p}^{(l)}_{n,1}(\overline{\zeta_{Q_1}})(g\,{}^te_n)$ by definition. Hence, by Lemmas 4.8, 4.9 and 4.10, we have

$$\begin{aligned} p_0(z) =&\mathrm{P}^\circ_{(\mu,0)}\big(\overline{\zeta_{H((\mu,0))}} \boxtimes \zeta_{H((\mu,0))}\big)(g)\mathrm{P}^\circ_{(l,\mathbf{0}_{n-1})}(\overline{\zeta_{Q_1}} \boxtimes \zeta_{Q_0})(g)\\ =&\big\langle \tau_{(\mu,0)}(g)\zeta_{H((\mu,0))}, \zeta_{H((\mu,0))}\big\rangle \big\langle \tau_{(l,\mathbf{0}_{n-1})}(g)\zeta_{Q_0},\ \zeta_{Q_1}\big\rangle\\ =&\big\langle (\tau_{(\mu,0)} \otimes \tau_{(l,\mathbf{0}_{n-1})})(g)\zeta_{H((\mu,0))} \otimes \zeta_{Q_0},\ \zeta_{H((\mu,0))} \otimes \zeta_{Q_1}\big\rangle. \end{aligned}$$

By (4.1) and (4.6), we have

$$p_0(z) = \sum_{\lambda' \in \Xi^\circ((\mu,0);l)} \ \sum_{N,N' \in \mathrm{G}(\lambda')} \mathrm{C}^{H((\mu,0)),Q_1}_N \mathrm{C}^{H((\mu,0)),Q_0}_{N'} \big\langle \tau_{\lambda'}(g)\zeta_{N'}, \zeta_N\big\rangle.$$

Because of $H((\mu, 0)) \in \mathrm{G}((\mu, 0); \mu)$, $Q_1 \in \mathrm{G}((l, \mathbf{0}_{n-1}); (l, \mathbf{0}_{n-2}))$ and (4.4), for $\lambda' \in \Xi^\circ((\mu, 0); l)$ and $N \in \mathrm{G}(\lambda')$, we have $\mathrm{C}^{H((\mu,0)),Q_1}_N = 0$ unless $\lambda' = (\mu', 0)$ and $N \in \mathrm{G}((\mu', 0); \mu')$ with some $\mu' \in \Xi^\circ(\mu; l)$. Hence, we have

$$\begin{aligned} p_0(z) =& \sum_{\mu' \in \Xi^\circ(\mu;l)} \ \sum_{\substack{N \in \mathrm{G}((\mu',0);\mu')\\ N' \in \mathrm{G}((\mu',0))}} \mathrm{C}^{H((\mu,0)),Q_1}_N \mathrm{C}^{H((\mu,0)),Q_0}_{N'} \big\langle \tau_{(\mu',0)}(g)\zeta_{N'}, \zeta_N\big\rangle\\ =& \sum_{\mu' \in \Xi^\circ(\mu;l)} \ \sum_{\substack{N \in \mathrm{G}((\mu',0);\mu')\\ N' \in \mathrm{G}((\mu',0))}} \mathrm{C}^{H((\mu,0)),Q_1}_N \mathrm{C}^{H((\mu,0)),Q_0}_{N'} \mathrm{P}^+_{\mu'}\big(\overline{\zeta_{\widehat{N}}} \boxtimes \zeta_{N'}\big)(z) \end{aligned}$$

by Lemmas 4.8 and 4.9. Since $\{(1_{n-1}, O_{n-1,1})g \mid g \in \mathrm{GL}(n, \mathbb{C})\}$ is dense in $\mathrm{M}_{n-1,n}(\mathbb{C})$, we obtain the assertion. □

4.4. **Standard Schwartz functions.** For $\lambda \in \Lambda^{\mathrm{poly}}_n$, we define two $\mathbb{C}$-linear maps

$$\begin{aligned} \Phi^\circ_\lambda &\colon \overline{V_\lambda} \boxtimes_{\mathbb{C}} V_\lambda \ni \overline{v_1} \boxtimes v_2 \mapsto \mathrm{P}^\circ_\lambda(\overline{v_1} \boxtimes v_2)(z)\mathbf{e}_{(n)}(z) \in \mathcal{S}_0(\mathrm{M}_n(F)),\\ \overline{\Phi^\circ_\lambda} &\colon V_\lambda \boxtimes_{\mathbb{C}} \overline{V_\lambda} \ni v_1 \boxtimes \overline{v_2} \mapsto \overline{\mathrm{P}^\circ_\lambda(\overline{v_1} \boxtimes v_2)(z)}\mathbf{e}_{(n)}(z) \in \mathcal{S}_0(\mathrm{M}_n(F)) \end{aligned}$$

with $z \in \mathrm{M}_n(F)$. By the $K_n \times K_n$-invariance of $\mathbf{e}_{(n)}$ and Lemma 4.8, we know that these are $K_n \times K_n$-homomorphisms.

When $n > 1$, for $\mu \in \Lambda^{\mathrm{poly}}_{n-1}$, we define two $\mathbb{C}$-linear maps

$$\begin{aligned} \Phi^+_\mu &\colon \overline{V_\mu} \boxtimes_{\mathbb{C}} V_{(\mu,0)} \ni \overline{v_1} \boxtimes v_2 \mapsto \mathrm{P}^+_\mu(\overline{v_1} \boxtimes v_2)(z)\mathbf{e}_{(n-1,n)}(z) \in \mathcal{S}_0(\mathrm{M}_{n-1,n}(F)),\\ \overline{\Phi^+_\mu} &\colon V_\mu \boxtimes_{\mathbb{C}} \overline{V_{(\mu,0)}} \ni v_1 \boxtimes \overline{v_2} \mapsto \overline{\mathrm{P}^+_\mu(\overline{v_1} \boxtimes v_2)(z)}\mathbf{e}_{(n-1,n)}(z) \in \mathcal{S}_0(\mathrm{M}_{n-1,n}(F)) \end{aligned}$$

with $z \in \mathrm{M}_{n-1,n}(F)$. By the $K_{n-1} \times K_n$-invariance of $\mathbf{e}_{(n-1,n)}$ and Lemma 4.9, we know that these are $K_{n-1} \times K_n$-homomorphisms.

We regard $\mathcal{S}_0(\mathrm{M}_{1,n}(F))$ and $\mathcal{S}_0(\mathrm{M}_{n,1}(F))$ as K_n-modules via the actions R and L, respectively. Let $l \in \mathbb{N}_0$. We define two $\mathbb{C}$-linear maps $\varphi^{(l)}_{n,1} \colon \overline{V_{(l,\mathbf{0}_{n-1})}} \to \mathcal{S}_0(\mathrm{M}_{n,1}(F))$ and $\overline{\varphi}^{(l)}_{n,1} \colon V_{(l,\mathbf{0}_{n-1})} \to \mathcal{S}_0(\mathrm{M}_{n,1}(F))$ by

$$\begin{aligned} \varphi^{(l)}_{n,1} &\colon \overline{V_{(l,\mathbf{0}_{n-1})}} \ni \overline{v} \mapsto \mathrm{p}^{(l)}_{n,1}(\overline{v})(z)\mathbf{e}_{(n,1)}(z) \in \mathcal{S}_0(\mathrm{M}_{n,1}(F)),\\ \overline{\varphi}^{(l)}_{n,1} &\colon V_{(l,\mathbf{0}_{n-1})} \ni v \mapsto \overline{\mathrm{p}^{(l)}_{n,1}(\overline{v})(z)}\mathbf{e}_{(n,1)}(z) \in \mathcal{S}_0(\mathrm{M}_{n,1}(F)) \end{aligned}$$

with $z \in \mathrm{M}_{n,1}(F)$. By the K_n-invariance of $\mathbf{e}_{(n,1)}$ and Lemma 4.10, we know that these are K_n-homomorphisms.

Proof of Lemma 2.12. Since $\mathrm{b}(\gamma)=\mathrm{r}(Q(\gamma))^{-1}$ for $\gamma\in\mathbb{N}_0^n$, we have

$$\varphi_{1,n}^{(l)}(v)(z)=\mathrm{p}_{1,n}^{(l)}(v)(z)\mathbf{e}_{(1,n)}(z),\qquad \overline{\varphi}_{1,n}^{(l)}(\overline{v})(z)=\overline{\mathrm{p}_{1,n}^{(l)}(v)(z)}\mathbf{e}_{(1,n)}(z)$$

for $v\in V_{(l,\mathbf{0}_{n-1})}$ and $z\in\mathrm{M}_{1,n}(F)$. By the K_n-invariance of $\mathbf{e}_{(1,n)}$ and Lemma 4.10, we obtain the assertion. □

In Lemma 4.13, we consider the Fourier transforms of $\varphi_{n,1}^{(l)}(\overline{v})$, $\overline{\varphi}_{n,1}^{(l)}(v)$ ($v\in V_{(l,\mathbf{0}_{n-1})}$), which we need to describe the recurrence relation in Proposition 3.5, explicitly.

Lemma 4.13. *Let $\varepsilon\in\{\pm1\}$ and $l\in\mathbb{N}_0$. Assume $l\in\{0,1\}$ if $F=\mathbb{R}$. For $v\in V_{(l,\mathbf{0}_{n-1})}$, we have*

$$\mathcal{F}_\varepsilon(\varphi_{n,1}^{(l)}(\overline{v}))=(-\varepsilon\sqrt{-1})^l\overline{\varphi}_{1,n}^{(l)}(\overline{v}),\qquad \mathcal{F}_\varepsilon(\overline{\varphi}_{n,1}^{(l)}(v))=(-\varepsilon\sqrt{-1})^l\varphi_{1,n}^{(l)}(v),$$

where the Fourier transform $\mathcal{F}_\varepsilon$ is defined in (3.15).

Proof. It suffices to show the assertion for $v=\zeta_{Q(\gamma)}$ with $\gamma\in\mathbb{N}_0^n$ such that $\ell(\gamma)=l$. For $t=(t_1,t_2,\cdots,t_n)\in\mathrm{M}_{1,n}(F)$, we have

$$\begin{aligned}\mathcal{F}_\varepsilon(\varphi_{n,1}^{(l)}(\overline{\zeta_{Q(\gamma)}}))(t)&=\int_{\mathrm{M}_{n,1}(F)}\varphi_{n,1}^{(l)}(\overline{\zeta_{Q(\gamma)}})(z)\psi_{-\varepsilon}(tz)\,d_Fz\\&=\sqrt{\mathrm{b}(\gamma)}\prod_{i=1}^n\int_F z_i^{\gamma_i}\exp\bigl(-\pi\mathrm{c}_F\overline{z_i}z_i-\pi\varepsilon\mathrm{c}_F\sqrt{-1}(t_iz_i+\overline{t_iz_i})\bigr)\,d_Fz_i\\&=\sqrt{\mathrm{b}(\gamma)}(-\varepsilon\sqrt{-1}\,\overline{t_i})^{\gamma_i}\exp(-\pi\mathrm{c}_F\overline{t_i}t_i)=(-\varepsilon\sqrt{-1})^l\overline{\varphi}_{1,n}^{(l)}(\overline{\zeta_{Q(\gamma)}})(t).\end{aligned}$$

Here the third equality follows from the elementary formula

$$\int_F z^m\exp\bigl(-\pi\mathrm{c}_F\overline{z}z+\pi\mathrm{c}_F\sqrt{-1}(zt+\overline{zt})\bigr)d_Fz=(\sqrt{-1}\,\overline{t})^m\exp(-\pi\mathrm{c}_F\overline{t}\,t)\tag{4.19}$$

for ($t\in\mathbb{R}$, $m\in\{0,1\}$) or ($t\in\mathbb{C}$, $m\in\mathbb{N}_0$) according as $F=\mathbb{R}$ or $F=\mathbb{C}$. Moreover, we have

$$\begin{aligned}\mathcal{F}_\varepsilon(\overline{\varphi}_{n,1}^{(l)}(\zeta_{Q(\gamma)}))(t)&=\overline{\mathcal{F}_{-\varepsilon}(\varphi_{n,1}^{(l)}(\overline{\zeta_{Q(\gamma)}}))(t)}=\overline{(\varepsilon\sqrt{-1})^l\overline{\varphi}_{1,n}^{(l)}(\overline{\zeta_{Q(\gamma)}})(t)}\\&=(-\varepsilon\sqrt{-1})^l\varphi_{1,n}^{(l)}(\zeta_{Q(\gamma)})(t),\end{aligned}$$

which completes the proof. □

5. The proofs of the main theorems

In this section, we prove our main theorems (Theorems 2.7 and 2.14) using the results in §3 and §4.

5.1. **Explicit calculations for the sections.** In this subsection, we calculate the sections in §3, explicitly, at the minimal K_n-types of principal series representations.

Lemma 5.1. *Let $a=\mathrm{diag}(a_1,a_2,\cdots,a_n)\in A_n$, $u\in U_n$, $\lambda\in\Lambda_n$ and $M\in\mathrm{G}(\lambda)$. Then we have the following equalities*

$$\bigl\langle\tau_\lambda(ua)\zeta_M,\zeta_M\bigr\rangle=\bigl\langle\tau_\lambda(au)\zeta_M,\zeta_M\bigr\rangle=\prod_{i=1}^n a_i^{\gamma_i^M},\tag{5.1}$$

$$\eta_{\rho_n}(a)\int_{U_n}\mathbf{e}_{(n)}(ua)\,du=\eta_{-\rho_n}(a)\int_{U_n}\mathbf{e}_{(n)}(au)\,du=\frac{\mathbf{e}_{(n)}(a)}{|\det a|_F^{(n-1)/2}},\tag{5.2}$$

where $\gamma^M = (\gamma_1^M, \gamma_2^M, \cdots, \gamma_n^M)$ *is the weight of* M *defined by* (2.12)*.*

Proof. By (2.13) and (2.15), we have

$$\tau_\lambda(a)\zeta_M = \left(\prod_{i=1}^n a_i^{\gamma_i^M}\right)\zeta_M, \qquad \tau_\lambda(u)\zeta_M = \zeta_M + \sum_{N\in \mathrm{G}(\lambda),\ \gamma^M >_{\mathrm{lex}} \gamma^N} p_{M,N}(u)\zeta_N,$$

where $p_{M,N}$ is some polynomial function on U_n and $>_{\mathrm{lex}}$ is the lexicographical order. The equality (5.1) follows from these equalities and the orthonormality of $\{\zeta_M\}_{M\in\mathrm{G}(\lambda)}$. The equality (5.2) follows from direct computation

$$\begin{aligned}\eta_{\rho_n}(a)\int_{U_n}\mathbf{e}_{(n)}(ua)\,du &= \eta_{-\rho_n}(a)\int_{U_n}\mathbf{e}_{(n)}(au)\,du\\ &= \prod_{i=1}^n a_i^{-(n+1-2i)\mathrm{c}_F/2}\exp(-\pi \mathrm{c}_F a_i^2)\prod_{j=1}^{i-1}\int_F \exp(-\pi\mathrm{c}_F a_i^2\overline{u_{i,j}}u_{i,j})\,d_F u_{i,j}\\ &= \prod_{i=1}^n a_i^{-(n-1)\mathrm{c}_F/2}\exp(-\pi\mathrm{c}_F a_i^2) = \frac{\mathbf{e}_{(n)}(a)}{|\det a|_F^{(n-1)/2}}.\end{aligned}$$

Here the first equality follows from the substitution $u \to aua^{-1}$, and the third equality follows from the substitution $u_{i,j} \to a_i^{-1}u_{i,j}$ and the elementary formula (4.19) with $t = m = 0$. □

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$, $M = (m_{i,j})_{1\le i\le j\le n} \in \mathrm{G}(\lambda)$ and $l \in \mathbb{Z}$, we define $\lambda + l \in \Lambda_n$ and $M + l \in \mathrm{G}(\lambda + l)$ by

$$\lambda + l = (\lambda_1 + l, \lambda_2 + l, \cdots, \lambda_n + l), \qquad M + l = (m_{i,j} + l)_{1\le i\le j\le n},$$

and denote $\lambda + (-l)$ and $M + (-l)$ simply by $\lambda - l$ and $M - l$, respectively. For $\lambda \in \Lambda_n$, $l \in \mathbb{Z}$, $g \in \mathrm{GL}(n,\mathbb{C})$ and $M, N \in \mathrm{G}(\lambda)$, we have

$$(\det g)^l\langle\tau_\lambda(g)\zeta_M, \zeta_N\rangle = \langle\tau_{\lambda+l}(g)\zeta_{M+l}, \zeta_{N+l}\rangle.$$

Lemma 5.2. *Let* $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$ *and* $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$*.*

(1) *Assume* $n > 1$*. We take* $\widehat{d}$ *and* $\widehat{\nu}$ *as in* §3.1*. If* $d \in \Lambda_{n,F}$*, we have*

$$\begin{aligned}&\mathrm{g}^+_{d_n,\nu_n}\big(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \Phi^+_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}}\boxtimes\zeta_{M-d_n})\big)\\ &= \frac{1}{\dim V_{\widehat{d}}}\left(\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\,d_i-d_n)\right)\mathrm{f}_{d,\nu}(\zeta_M)\end{aligned}\tag{5.3}$$

for $M \in \mathrm{G}(d)$*. If* $-d \in \Lambda_{n,F}$*, we have*

$$\begin{aligned}&\mathrm{g}^+_{d_n,\nu_n}\big(\overline{\mathrm{f}}_{\widehat{d},\widehat{\nu}}(\overline{\zeta_{H(-\widehat{d})}}), \overline{\Phi^+_{-\widehat{d}+d_n}}(\zeta_{H(-\widehat{d})+d_n}\boxtimes\overline{\zeta_{M+d_n}})\big)\\ &= \frac{1}{\dim V_{-\widehat{d}}}\left(\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\,d_n-d_i)\right)\overline{\mathrm{f}}_{d,\nu}(\overline{\zeta_M})\end{aligned}\tag{5.4}$$

for $M \in \mathrm{G}(-d)$*. Here* $\mathrm{f}_{d,\nu}$ *and* $\overline{\mathrm{f}}_{d,\nu}$ *are defined by* (2.26) *and* (2.27)*, respectively.*

(2) *Let $l \in \mathbb{Z}$ and $s \in \mathbb{C}$ such that* $\mathrm{Re}(s)$ *is sufficiently large. If $d \in \Lambda_{n,F}$ and $d+l \in \Lambda_n^{\mathrm{poly}}$, we have*

$$\begin{aligned}&\mathrm{g}_{l,s}^{\circ}\big(\,\mathrm{f}_{d,\nu}(\zeta_{H(d)}),\,\overline{\Phi_{d+l}^{\circ}(\zeta_{M+l}\boxtimes\overline{\zeta_{H(d)+l}})}\,\big)\\ &=\frac{1}{\dim V_d}\left(\prod_{i=1}^{n}\Gamma_F(s+\nu_i;\,d_i+l)\right)\mathrm{f}_{d,\nu}(\zeta_M)\qquad(M\in\mathrm{G}(d)).\end{aligned}\tag{5.5}$$

If $-d \in \Lambda_{n,F}$ and $-d-l \in \Lambda_n^{\mathrm{poly}}$, we have

$$\begin{aligned}&\mathrm{g}_{l,s}^{\circ}\big(\,\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(-d)}}),\,\Phi_{-d-l}^{\circ}(\overline{\zeta_{M-l}}\boxtimes\zeta_{H(-d)-l})\,\big)\\ &=\frac{1}{\dim V_{-d}}\left(\prod_{i=1}^{n}\Gamma_F(s+\nu_i;\,-d_i-l)\right)\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_M})\qquad(M\in\mathrm{G}(-d)).\end{aligned}\tag{5.6}$$

Proof. First, we consider the proof of the statement (1). Since the proofs of (5.3) and (5.4) are similar, here we will prove only (5.3). Assume $n>1$ and $d \in \Lambda_{n,F}$. We define a $\mathbb{C}$-linear map $\mathrm{g}_+ \colon V_d \to I(d,\nu)$ by

$$\mathrm{g}_+(\zeta_M)=\mathrm{g}^{+}_{d_n,\nu_n}\big(\,\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}),\Phi^{+}_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}}\boxtimes\zeta_{M-d_n})\,\big)\qquad(M\in\mathrm{G}(d)).$$

Then g_+ is a K_n-homomorphism because of (3.1). Since $\mathrm{Hom}_{K_n}(V_d, I(d,\nu))$ is 1 dimensional, there is a constant c_+ such that $\mathrm{g}_+ = c_+\mathrm{f}_{d,\nu}$. Let us calculate c_+. Since (4.12) and (4.16) imply

$$\Phi^{+}_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}}\boxtimes\zeta_{H(d)-d_n})((h,O_{n-1,1}))=\big\langle\tau_{\widehat{d}-d_n}(h)\zeta_{H(\widehat{d})-d_n},\zeta_{H(\widehat{d})-d_n}\big\rangle\mathbf{e}_{(n-1)}(h)$$

for $h \in G_{n-1}$, we have

$$\begin{aligned}c_+ &=c_+\mathrm{f}_{d,\nu}(H(d))(1_n)=\mathrm{g}_+(\zeta_{H(d)})(1_n)\\ &=\int_{G_{n-1}}\big\langle\tau_{\widehat{d}-d_n}(h)\zeta_{H(\widehat{d})-d_n},\zeta_{H(\widehat{d})-d_n}\big\rangle\mathbf{e}_{(n-1)}(h)\\ &\qquad\times\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})})(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\,dh.\end{aligned}$$

Decomposing $h = kua$ ($k \in K_{n-1}$, $u \in U_{n-1}$, $a \in A_{n-1}$) and applying Schur's orthogonality [18, Corollary 1.10] for the integration on K_{n-1} with the equalities

$$\chi_{d_n}(\det h)\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})})(h^{-1})=\eta_{\widehat{\nu}-\rho_{n-1}}(a^{-1})\overline{\big\langle\tau_{\widehat{d}-d_n}(k)\zeta_{H(\widehat{d})-d_n},\zeta_{H(\widehat{d})-d_n}\big\rangle}$$

and $\dim V_{\widehat{d}-d_n}=\dim V_{\widehat{d}}$, we have

$$\begin{aligned}c_+ =&\frac{1}{\dim V_{\widehat{d}}}\int_{A_{n-1}}\left(\int_{U_{n-1}}\big\langle\tau_{\widehat{d}-d_n}(ua)\zeta_{H(\widehat{d})-d_n},\zeta_{H(\widehat{d})-d_n}\big\rangle\mathbf{e}_{(n-1)}(ua)\,du\right)\\ &\times\eta_{\widehat{\nu}-\rho_{n-1}}(a^{-1})|\det a|_F^{\nu_n+n/2}\,da.\end{aligned}$$

By Lemma 5.1 and (2.51), we have

$$\begin{aligned}c_+ =&\frac{1}{\dim V_{\widehat{d}}}\prod_{i=1}^{n-1}\int_0^{\infty}\exp(-\pi \mathrm{c}_F a_i^2)a_i^{(\nu_n-\nu_i+1)\mathrm{c}_F+d_i-d_n}\frac{2\mathrm{c}_F\,da_i}{a_i}\\ =&\frac{1}{\dim V_{\widehat{d}}}\prod_{i=1}^{n-1}\Gamma_F\big(\nu_n-\nu_i+1;\,d_i-d_n\big).\end{aligned}$$

Hence, the equality (5.3) follows from $\mathrm{g}_+ = c_+\mathrm{f}_{d,\nu}$.

Next, we consider the proof of the statement (2). Since the proofs of (5.5) and (5.6) are similar, here we will prove only (5.5). Assume $d \in \Lambda_{n,F}$ and $d+l \in \Lambda_n^{\mathrm{poly}}$. We define a $\mathbb{C}$-linear map $\mathrm{g}_\circ \colon V_d \to I(d,\nu)$ by

$$\mathrm{g}_\circ(\zeta_M) = \mathrm{g}_{l,s}^\circ\big(\,\mathrm{f}_{d,\nu}(\zeta_{H(d)}),\, \overline{\Phi_{d+l}^\circ}(\zeta_{M+l} \boxtimes \overline{\zeta_{H(d)+l}})\,\big) \qquad (M \in \mathrm{G}(d)).$$

Then $\mathrm{g}_\circ$ is a K_n-homomorphism because of (3.5). Since $\mathrm{Hom}_{K_n}(V_d, I(d,\nu))$ is 1 dimensional, there is a constant $c_\circ$ such that $\mathrm{g}_\circ = c_\circ \mathrm{f}_{d,\nu}$. Let us calculate $c_\circ$. By (4.12), we have

$$\begin{aligned} c_\circ &= c_\circ \mathrm{f}_{d,\nu}(H(d))(1_n) = \mathrm{g}_\circ(\zeta_{H(d)})(1_n) \\ &= \int_{G_n} \mathrm{f}_{d,\nu}(\zeta_{H(d)})(h) \overline{\langle \tau_{d+l}(h)\zeta_{H(d)+l}, \zeta_{H(d)+l}\rangle}\, \mathbf{e}_{(n)}(h)\chi_l(\det h)|\det h|_F^{s+(n-1)/2}\, dh. \end{aligned}$$

Decomposing $h = auk$ ($a \in A_n$, $u \in U_n$, $k \in K_n$) and applying Schur's orthogonality [18, Corollary 1.10] for the integration on K_n with the equalities

$$\begin{aligned} &\chi_l(\det h)\mathrm{f}_{d,\nu}(\zeta_{H(d)})(h) = \eta_{\nu-\rho_n}(a)\langle \tau_{d+l}(k)\zeta_{H(d)+l}, \zeta_{H(d)+l}\rangle, \\ &\tau_{d+l}(h)\zeta_{H(d)+l} = \sum_{M \in \mathrm{G}(d)} \langle \tau_{d+l}(k)\zeta_{H(d)+l}, \zeta_{M+l}\rangle\, \tau_{d+l}(au)\zeta_{M+l} \end{aligned}$$

and $\dim V_{d+l} = \dim V_d$, we have

$$\begin{aligned} c_\circ =& \frac{1}{\dim V_d} \int_{A_n} \left(\int_{U_n} \overline{\langle \tau_{d+l}(au)\zeta_{H(d)+l}, \zeta_{H(d)+l}\rangle} \mathbf{e}_{(n)}(au)\, du \right) \\ & \times \eta_{\nu-\rho_n}(a)|\det a|_F^{s+(n-1)/2}\, da. \end{aligned}$$

By Lemma 5.1 and (2.51), we have

$$\begin{aligned} c_\circ &= \frac{1}{\dim V_d} \prod_{i=1}^n \int_0^\infty \exp(-\pi \mathrm{c}_F a_i^2) a_i^{(s+\nu_i)\mathrm{c}_F + d_i + l}\, \frac{2\mathrm{c}_F\, da_i}{a_i} \\ &= \frac{1}{\dim V_d} \prod_{i=1}^n \Gamma_F(s+\nu_i;\, d_i + l). \end{aligned}$$

Hence, the equality (5.5) follows from $\mathrm{g}_\circ = c_\circ \mathrm{f}_{d,\nu}$. □

Corollary 5.3. *We use the notation in Lemma* 5.2(1). *If* $d \in \Lambda_{n,F}$, *we have*

$$\begin{aligned} \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_M))(g) =& \frac{\big(\dim V_{\widehat{d}}\big)\chi_{d_n}(\det g)|\det g|_F^{\nu_n+(n-1)/2}}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n - \nu_i + 1;\, d_i - d_n)} \\ & \times \int_{G_{n-1}} \left(\int_{\mathrm{M}_{n-1,1}(F)} \Phi_{\widehat{d}-d_n}^+ (\overline{\zeta_{H(\widehat{d})-d_n}} \boxtimes \zeta_{M-d_n})\, ((h, hz)\, g)\, \psi_{-\varepsilon}(e_{n-1}z)\, dz \right) \\ & \times \mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\, dh \end{aligned}$$

for $M \in \mathrm{G}(d)$ and $g \in G_n$. If $-d \in \Lambda_{n,F}$, we have

$$\begin{aligned}\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_M}))(g) = &\frac{\big(\dim V_{-\widehat{d}}\big)\chi_{d_n}(\det g)|\det g|_F^{\nu_n+(n-1)/2}}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\, d_n-d_i)}\\ &\times \int_{G_{n-1}}\left(\int_{\mathrm{M}_{n-1,1}(F)}\overline{\Phi^+_{-\widehat{d}+d_n}}(\zeta_{H(-\widehat{d})+d_n}\boxtimes\overline{\zeta_{M+d_n}})\,((h,hz)\,g)\,\psi_{-\varepsilon}(e_{n-1}z)\,dz\right)\\ &\times \mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{\widehat{d},\widehat{\nu}}(\overline{\zeta_{H(-\widehat{d})}}))(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\,dh\end{aligned}$$

for $M \in \mathrm{G}(-d)$ and $g \in G_n$.

Proof. The assertion follows immediately from (3.3) and Lemma 5.2(1). □

In order to prove Proposition 2.6, we prepare Lemma 5.4 of complex analysis.

Lemma 5.4. *Let Ω_1 and Ω_2 be open relatively compact subsets of $\mathbb{C}^2$ such that Ω_2 contains the closure of Ω_1. Let Ω_3 be an open relatively compact subset of $\mathbb{C}$ which contains the closure of $\{s_1-s_2 \mid (s_1,s_2)\in\Omega_2\}$. Let $\beta(z)$ be a meromorphic function on Ω_3. Then there is a constant c_0 which depends only on $(\Omega_1,\Omega_2,\beta(z))$ and satisfies the inequality*

$$\sup_{(s_1,s_2)\in\Omega_1}|\beta(s_1-s_2)f(s_1,s_2)| \le c_0 \sup_{(s_1,s_2)\in\Omega_2}|f(s_1,s_2)|$$

for any bounded holomorphic function $f(s_1,s_2)$ on Ω_2 such that $\beta(s_1-s_2)f(s_1,s_2)$ is holomorphic on Ω_2.

Proof. We take a compact subset Ω_3' of Ω_3 so that $\{s_1-s_2 \mid (s_1,s_2)\in\Omega_1\}\subset\Omega_3'$ and $\beta(z)$ is holomorphic at any point of the boundary of Ω_3'. Then there is a finite subset S of the interior of Ω_3' such that $\beta(z)$ is holomorphic at any point of Ω_3' which is not in S. Take a sufficiently small $r_0>0$ so that, for any $(s_1,s_2)\in\Omega_1$ and any $a\in S$,

$$\{(z,s_2)\mid z\in\mathcal{D}(s_1;3r_0)\}\subset\Omega_2,\qquad \mathcal{D}(a;3r_0)\subset\Omega_3',\qquad \mathcal{D}(a;3r_0)\cap S=\{a\},$$

where $\mathcal{D}(t;r)=\{z\in\mathbb{C}\mid |z-t|<r\}$ for $t\in\mathbb{C}$ and $r>0$. Let

$$\Omega_3''=\{z\in\Omega_3'\mid |z-a|\ge r_0 \text{ for any } a\in S\}.$$

Since $\beta(z)$ is holomorphic at any point of the compact set Ω_3'', we know that $\beta(z)$ is bounded on Ω_3''. Let $c_0=\sup_{z\in\Omega_3''}|\beta(z)|$. Let $f(s_1,s_2)$ be a bounded holomorphic function on Ω_2 such that $\beta(s_1-s_2)f(s_1,s_2)$ is holomorphic on Ω_2. For $(s_1,s_2)\in\Omega_1$ such that $s_1-s_2\in\Omega_3''$, we have

$$|\beta(s_1-s_2)f(s_1,s_2)|=|\beta(s_1-s_2)|\times|f(s_1,s_2)|\le c_0\sup_{(t_1,t_2)\in\Omega_2}|f(t_1,t_2)|.$$

For $(s_1,s_2)\in\Omega_1$ such that $s_1-s_2\notin\Omega_3''$, we have

$$\begin{aligned}|\beta(s_1-s_2)f(s_1,s_2)| =&\left|\frac{1}{2\pi}\int_0^{2\pi}\beta(s_1-s_2+2r_0e^{\sqrt{-1}\theta})f(s_1+2r_0e^{\sqrt{-1}\theta},s_2)\,d\theta\right|\\ \le&\frac{1}{2\pi}\int_0^{2\pi}|\beta(s_1-s_2+2r_0e^{\sqrt{-1}\theta})|\times|f(s_1+2r_0e^{\sqrt{-1}\theta},s_2)|\,d\theta\\ \le&\frac{1}{2\pi}\int_0^{2\pi}c_0\sup_{(t_1,t_2)\in\Omega_2}|f(t_1,t_2)|\,d\theta=c_0\sup_{(t_1,t_2)\in\Omega_2}|f(t_1,t_2)|\end{aligned}$$

by the mean value theorem for holomorphic functions and the choice of r_0. Therefore, we complete the proof. □

Proof of Proposition 2.6. We will prove the statement (1) by induction with respect to n. In the case of $n = 1$, the statement (1) holds, since $\mathbf{\Gamma}_F(\nu_1; d_1) = 1$, $\mathrm{W}_\varepsilon(\mathrm{f}_{d_1,\nu_1}(v))(g) = \chi_{d_1}(g)|g|_F^{\nu_1}$. Let us consider the case of $n \geq 2$. Let $M \in \mathrm{G}(d)$. By Corollary 5.3, we have

(5.7)
$$\begin{aligned}\mathbf{\Gamma}_F(\nu; d)\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_M))(g) &= \big(\dim V_{\widehat{d}}\big)\chi_{d_n}(\det g)|\det g|_F^{\nu_n+(n-1)/2} \\ &\times \int_{G_{n-1}} \left(\int_{\mathrm{M}_{n-1,1}(F)} \Phi^+_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}} \boxtimes \zeta_{M-d_n})\,((h, hz)\,g)\,\psi_{-\varepsilon}(e_{n-1}z)\,dz \right) \\ &\times \mathbf{\Gamma}_F(\widehat{\nu}; \widehat{d})\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))(h^{-1})\chi_{d_n}(\det h)|\det h|_F^{\nu_n+n/2}\,dh.\end{aligned}$$

By the induction hypothesis, $\mathbf{\Gamma}_F(\widehat{\nu}; \widehat{d})\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))(h^{-1})$ is an entire function of $\widehat{\nu}$ for any $h \in G_{n-1}$. Applying Lemma 5.4 for $\beta(z) = \Gamma_F(z + 1; |d_i - d_j|)$ $(1 \leq i < j \leq n - 1)$, we know that the majorization [15, Proposition 3.3 with $X = 1$] for $\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))$ is also valid for $\mathbf{\Gamma}_F(\widehat{\nu}; \widehat{d})\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))$. Hence, similar to the proof of [15, Proposition 7.2], we know that the right hand side of (5.7) converges absolutely and is an entire function of ν. Hence, we obtain the former part of the statement (1). For $\nu \in \mathbb{C}^n$ such that $\Pi_{d,\nu}$ is irreducible, there is some $g \in G_n$ such that $\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_M))(g) = \mathcal{J}_\varepsilon(\Pi_{d,\nu}(g)\mathrm{f}_{d,\nu}(\zeta_M)) \neq 0$, since the Jacquet integral $\mathcal{J}_\varepsilon$ is a nonzero continuous $\mathbb{C}$-linear form on $I(d, \nu)$. Hence, the latter part of the statement (1) follows from the former part, and we complete the proof of the statement (1). The proof of the statement (2) is similar. □

5.2. **Explicit recurrence relations.** Let $(\Pi_{d,\nu}, I(d, \nu))$ and $(\Pi_{d',\nu'}, I(d', \nu'))$ be principal series representations of G_n and $G_{n'}$, respectively, with parameters

$$\begin{aligned} d &= (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n, & \nu &= (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n, \\ d' &= (d'_1, d'_2, \cdots, d'_{n'}) \in \mathbb{Z}^{n'}, & \nu' &= (\nu'_1, \nu'_2, \cdots, \nu'_{n'}) \in \mathbb{C}^{n'}. \end{aligned}$$

Let $\varepsilon \in \{\pm 1\}$. Let $s \in \mathbb{C}$ such that $\mathrm{Re}(s)$ is sufficiently large.

Proposition 5.5. *Retain the notation. Assume $n' = n > 1$, $-d' \in \Lambda_{n,F}$ and $d \in \Xi^\circ(-d') \cap \Lambda_{n,F}$. Let $l = \ell(d + d')$. Then we have*

$$\begin{aligned} &Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(d)})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})), \overline{\varphi}^{(l)}_{1,n}(\overline{\zeta_{Q(d+d')}})\big) \\ &= \mathrm{C}^{H(-d')+d'_n, Q((\mathbf{0}_{n-1}, l))}_{H(-\widehat{d'})[d]+d'_n}\, \mathrm{C}^{H(-d')+d'_n, Q(d+d')}_{H(d)+d'_n} \\ &\times \frac{\dim V_{-\widehat{d'}}}{\dim V_d} \frac{\prod_{i=1}^n \Gamma_F(s + \nu_i + \nu'_n;\, d_i + d'_n)}{\prod_{i=1}^{n-1} \Gamma_F(\nu'_n - \nu'_i + 1;\, d'_n - d'_i)} \\ &\times Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(-\widehat{d'})[d]})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}))\big). \end{aligned}$$

Proof. Let $\phi_1 = \overline{\Phi^+_{-\widehat{d'}+d'_n}}(\zeta_{H(-\widehat{d'})+d'_n} \boxtimes \overline{\zeta_{H(-d')+d'_n}})$ and $\phi_2 = \overline{\varphi}^{(l)}_{1,n}(\overline{\zeta_{Q(d+d')}})$. By Proposition 3.4, we have

$$\begin{aligned} &Z\big(s,\, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(d)})),\, \mathrm{W}_{-\varepsilon}\big(\mathrm{g}^+_{d'_n,\nu'_n}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}), \phi_1)\big),\, \phi_2\big) \\ &= Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^\circ_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}))\big), \end{aligned}$$

where $\phi_0(z) = \phi_1((1_{n-1}, O_{n-1,1})z)\phi_2(e_n z)$ $(z \in \mathrm{M}_n(F))$. Since we have

$$\mathrm{g}^{+}_{d'_n,\nu'_n}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}), \phi_1) = \frac{\prod_{i=1}^{n-1}\Gamma_F(\nu'_n - \nu'_i + 1;\, d'_n - d'_i)}{\dim V_{-\widehat{d'}}}\,\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})$$

by (5.4), it suffices to prove the equality

$$\begin{aligned}(5.8)\quad & Z\big(s,\,\mathrm{W}_\varepsilon\big(\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \phi_0)\big),\,\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}))\big)\\ &= \mathrm{C}^{H(-d')+d'_n,Q((\mathbf{0}_{n-1},l))}_{H(-\widehat{d'})[d]+d'_n}\,\mathrm{C}^{H(-d')+d'_n,Q(d+d')}_{H(d)+d'_n}\,\frac{\prod_{i=1}^{n}\Gamma_F(s+\nu_i+\nu'_n;\,d_i+d'_n)}{\dim V_d}\\ &\quad\times Z\big(s,\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(-\widehat{d'})[d]})),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}))\big).\end{aligned}$$

By Lemma 4.11, we have

$$\begin{aligned}(5.9)\quad & \mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \phi_0) = \sum_{\lambda'\in\Xi^{\circ}(-d'+d'_n;l)}\ \sum_{N,N'\in\mathrm{G}(\lambda')}\mathrm{C}^{H(-d')+d'_n,Q((\mathbf{0}_{n-1},l))}_{N}\\ &\qquad\times\mathrm{C}^{H(-d')+d'_n,Q(d+d')}_{N'}\,\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \overline{\Phi^{\circ}_{\lambda'}}(\zeta_N\boxtimes\overline{\zeta_{N'}})).\end{aligned}$$

By (3.6), we note that

$$v\otimes\overline{\zeta_M}\mapsto \mathrm{g}^{\circ}_{d'_n,s+\nu'_n}\big(\mathrm{f}_{d,\nu}(v),\,\overline{\Phi^{\circ}_{\lambda'}}(v_1\boxtimes\overline{\zeta_{M+d'_n}})\big)(g)$$

defines an element of $\mathrm{Hom}_{K_n}(V_d\otimes_{\mathbb{C}}\overline{V_{\lambda'-d'_n}}, \mathbb{C}_{\mathrm{triv}})$ for $\lambda'\in\Xi^{\circ}(-d'+d'_n;l)$, $v_1\in V_{\lambda'}$ and $g\in G_n$. Hence, by Lemma 4.2, for $\lambda'\in\Xi^{\circ}(-d'+d'_n;l)$ and $N,N'\in\mathrm{G}(\lambda')$, we have

$$\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}\big(\mathrm{f}_{d,\nu}(\zeta_{H(d)}),\,\overline{\Phi^{\circ}_{\lambda'}}(\zeta_N\boxtimes\overline{\zeta_{N'}})\big)(g) = 0$$

unless $\lambda' = d+d'_n$ and $N' = H(d)+d'_n$. By (5.9) and this equality, we have

$$\begin{aligned}(5.10)\quad & \mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \phi_0) = \sum_{N\in\mathrm{G}(d+d'_n)}\mathrm{C}^{H(-d')+d'_n,Q((\mathbf{0}_{n-1},l))}_{N}\,\mathrm{C}^{H(-d')+d'_n,Q(d+d')}_{H(d)+d'_n}\\ &\qquad\times\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}), \overline{\Phi^{\circ}_{d+d'_n}}(\zeta_N\boxtimes\overline{\zeta_{H(d)+d'_n}})).\end{aligned}$$

By (2.28) and (3.5), we note that

$$\zeta_M\otimes\overline{v}\mapsto Z\big(s,\mathrm{W}_\varepsilon\big(\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(v_1),\overline{\Phi^{\circ}_{d+d'_n}}(\zeta_{M+d'_n}\boxtimes\overline{v_2}))\big),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{v}))\big)$$

defines an element of $\mathrm{Hom}_{K_{n-1}}(V_d\otimes_{\mathbb{C}}\overline{V_{-\widehat{d'}}}, \mathbb{C}_{\mathrm{triv}})$ for $v_1\in V_d$ and $v_2\in V_{d+d'_n}$. Hence, by Lemma 4.3, for $N\in\mathrm{G}(d+d'_n)$, we have

$$Z\big(s,\mathrm{W}_\varepsilon\big(\mathrm{g}^{\circ}_{d'_n,s+\nu'_n}(\mathrm{f}_{d,\nu}(\zeta_{H(d)}),\overline{\Phi^{\circ}_{d+d'_n}}(\zeta_N\boxtimes\overline{\zeta_{H(d)+d'_n}}))\big),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}}))\big) = 0$$

unless $\widehat{N} = H(-\widehat{d'})+d'_n$. By (5.5), (5.10) and this equality, we obtain (5.8). □

Proposition 5.6. *Retain the notation. Assume $n' = n-1$, $d\in\Lambda_{n,F}$ and $-d'\in\Xi^{+}(d)\cap\Lambda_{n-1,F}$. Let $l = \ell(\widehat{d}+d')$. Then we have*

$$\begin{aligned}& Z\big(s,\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(-d')[d]})),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big)\\ &= (-\varepsilon\sqrt{-1})^{l}\left(\mathrm{C}^{H((-d'-d_n,0)),\,Q((\widehat{d}+d',0))}_{H(d)-d_n}\,\mathrm{C}^{H((-d'-d_n,0)),\,Q((\mathbf{0}_{n-1},l))}_{H(-d')[d]-d_n}\right)^{-1}\\ &\quad\times\frac{\dim V_{\widehat{d}}}{\dim V_{-d'}}\,\frac{\prod_{i=1}^{n-1}\Gamma_F(s+\nu_n+\nu'_i;\,-d_n-d'_i)}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\,-d_n+d_i)}\\ &\quad\times Z\big(s,\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})})),\mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})),\overline{\varphi}^{(l)}_{1,n-1}(\overline{\zeta_{Q(\widehat{d}+d')}})\big).\end{aligned}$$

Proof. Let $\phi_1 = \Phi^\circ_{-d'-d_n}\big(\overline{\zeta_{H(-d')-d_n}} \boxtimes \zeta_{H(-d')-d_n}\big)$ and $\phi_2 = \varphi^{(l)}_{n-1,1}(\overline{\zeta_{Q(\widehat{d}+d')}})$. By Proposition 3.5, we have

$$\begin{aligned}&Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big)\\ &= Z\big(s,\, \mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})})),\, \mathrm{W}_{-\varepsilon}(\mathrm{g}^\circ_{d_n,s+\nu_n}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}), \phi_1)),\, \mathcal{F}_\varepsilon(\phi_2)\big),\end{aligned}$$

where $\phi_0(z) = \phi_1(z\,{}^t(1_{n-1}, O_{n-1,1}))\phi_2(z\,{}^te_n)\quad (z \in \mathrm{M}_{n-1,n}(F))$. We have

$$\mathrm{g}^\circ_{d_n,s+\nu_n}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}),\, \phi_1) = \frac{\prod_{i=1}^n \Gamma_F(s+\nu_n+\nu'_i;\, -d_n-d'_i)}{\dim V_{-d'}}\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})$$

by (5.6). Because of these equalities and Lemma 4.13, it suffices to prove

$$\begin{aligned}&Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \phi_0)\big),\, \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big)\\ &= \mathrm{C}^{H((-d'-d_n,0)),\, Q((\widehat{d}+d',0))}_{H(d)-d_n}\, \mathrm{C}^{H((-d'-d_n,0)),\, Q((\mathbf{0}_{n-1},l))}_{H(-d')[d]-d_n}\\ &\quad\times \frac{\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\, -d_n+d_i)}{\dim V_{\widehat{d}}}\\ &\quad\times Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(-d')[d]})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big).\end{aligned} \tag{5.11}$$

By Lemma 4.12, we have

$$\begin{aligned}&\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \phi_0) = \sum_{\lambda'\in\Xi^\circ(-d'-d_n;l)}\ \sum_{\substack{N\in\mathrm{G}((\lambda',0);\lambda')\\ N'\in\mathrm{G}((\lambda',0))}} \mathrm{C}^{H((-d'-d_n,0)),Q((\widehat{d}+d',0))}_{N}\\ &\quad\times \mathrm{C}^{H((-d'-d_n,0)),Q((\mathbf{0}_{n-1},l))}_{N'}\, \mathrm{g}^+_{d_n,\nu_n}\big(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \Phi^+_{\lambda'}\big(\overline{\zeta_{\widehat{N}}} \boxtimes \zeta_{N'}\big)\big).\end{aligned} \tag{5.12}$$

By (3.2), we note that

$$v\otimes\overline{\zeta_M} \mapsto \mathrm{g}^+_{d_n,\nu_n}\big(\mathrm{f}_{\widehat{d},\widehat{\nu}}(v), \Phi^+_{\lambda'}\big(\overline{\zeta_{M-d_n}}\boxtimes v_1\big)\big)(g)$$

defines an element of $\mathrm{Hom}_{K_{n-1}}(V_{\widehat{d}}\otimes_{\mathbb{C}}\overline{V_{\lambda'+d_n}}, \mathbb{C}_{\mathrm{triv}})$ for $\lambda'\in\Xi^\circ(-d'-d_n;l)$, $v_1\in V_{(\lambda',0)}$ and $g\in G_n$. Hence, by Lemma 4.2, for $N\in\mathrm{G}((\lambda',0);\lambda')$, $N'\in\mathrm{G}((\lambda',0))$ and $\lambda'\in\Xi^\circ(-d'-d_n;l)$, we have

$$\mathrm{g}^+_{d_n,\nu_n}\big(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \Phi^+_{\lambda'}\big(\overline{\zeta_{\widehat{N}}}\boxtimes\zeta_{N'}\big)\big) = 0$$

unless $\lambda' = \widehat{d}-d_n$ and $\widehat{N} = H(\widehat{d})-d_n$. By (5.12) and this equality, we have

$$\begin{aligned}&\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \phi_0)\\ &= \sum_{N'\in\mathrm{G}(d-d_n)} \mathrm{C}^{H((-d'-d_n,0)),Q((\widehat{d}+d',0))}_{H(d)-d_n}\, \mathrm{C}^{H((-d'-d_n,0)),Q((\mathbf{0}_{n-1},l))}_{N'}\\ &\quad\times \mathrm{g}^+_{d_n,\nu_n}\big(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \Phi^+_{\widehat{d}-d_n}\big(\overline{\zeta_{H(\widehat{d})-d_n}}\boxtimes\zeta_{N'}\big)\big).\end{aligned} \tag{5.13}$$

By (2.28) and (3.1), we note that

$$\zeta_M\otimes\overline{v} \mapsto Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(v_1), \Phi^+_{\widehat{d}-d_n}(\overline{v_2}\boxtimes\zeta_{M-d_n}))\big),\, \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{v}))\big)$$

defines an element of $\mathrm{Hom}_{K_{n-1}}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}}, \mathbb{C}_{\mathrm{triv}})$ for $v_1\in V_{\widehat{d}}$ and $v_2\in V_{\widehat{d}-d_n}$. Hence, by Lemma 4.3, for $N'\in\mathrm{G}(d-d_n)$, we have

$$Z\big(s,\, \mathrm{W}_\varepsilon\big(\mathrm{g}^+_{d_n,\nu_n}(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}), \Phi^+_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}}\boxtimes\zeta_{N'}))\big),\, \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big) = 0$$

unless $\widehat{N'} = H(-d')-d_n$. By (5.3), (5.13) and this equality, we obtain (5.11). □

Theorem 5.7. *Retain the notation. Assume* $d \in \Lambda_{n,F}$ *and* $-d' \in \Lambda_{n',F}$. *We take* $\mathbf{\Gamma}_F(\nu; d)$ *and* $\mathbf{\Gamma}_F(\nu'; d')$ *as in §2.7.*

(1) *Assume* $n' = n$ *and* $d \in \Xi^\circ(-d')$. *Let* $l = \ell(d + d')$. *Then we have*

$$Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(d)})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})), \overline{\varphi}^{(l)}_{1,n}(\overline{\zeta_{Q(d+d')}})\big) = \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d'_i)}\sqrt{\mathrm{b}(d+d')}\mathrm{C}^\circ(d; -d')L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})}{(\dim V_d)\mathbf{\Gamma}_F(\nu; d)\mathbf{\Gamma}_F(\nu'; d')}.$$

(2) *Assume* $n' = n - 1$ *and* $-d' \in \Xi^+(d)$. *Then we have*

$$Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(-d')[d]})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}}))\big) = \frac{(-\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d'_i)}L(s, \Pi_{d,\nu} \times \Pi_{d',\nu'})}{(\dim V_{-d'})\sqrt{\mathrm{r}(H(-d')[d])}\mathbf{\Gamma}_F(\nu; d)\mathbf{\Gamma}_F(\nu'; d')}.$$

Here $\mathrm{r}(H(-d')[d])$ *is defined by* (2.18).

Proof. Let us prove the statement (1) by induction with respect to n. First, we consider the case of $n = 1$. Since

$$\mathrm{W}_\varepsilon\big(\mathrm{f}_{d_1,\nu_1}(\zeta_{d_1})\big)(ak) = a^{\nu_1 \mathrm{c}_F} k^{d_1}, \quad \mathrm{W}_{-\varepsilon}\big(\bar{\mathrm{f}}_{d'_1,\nu'_1}(\overline{\zeta_{-d'_1}})\big)(ak) = a^{\nu'_1 \mathrm{c}_F} k^{d'_1},$$
$$\overline{\varphi}^{(d_1+d'_1)}_{1,1}(\overline{\zeta_{d_1+d'_1}})(ak) = (a\overline{k})^{d_1+d'_1} \exp(-\pi \mathrm{c}_F a^2)$$

for $a \in A_1 = \mathbb{R}_+^\times$ and $k \in K_1$, we have

$$\begin{aligned} &Z\big(s, \mathrm{W}_\varepsilon\big(\mathrm{f}_{d_1,\nu_1}(\zeta_{d_1})\big), \mathrm{W}_{-\varepsilon}\big(\bar{\mathrm{f}}_{d'_1,\nu'_1}(\overline{\zeta_{-d'_1}})\big), \overline{\varphi}^{(d_1+d'_1)}_{1,1}(\overline{\zeta_{d_1+d'_1}})\big) \\ &= \left(\int_0^\infty \exp(-\pi \mathrm{c}_F a^2) a^{(s+\nu_1+\nu'_1)\mathrm{c}_F + d_1 + d'_1} \frac{2\mathrm{c}_F\, da}{a}\right)\left(\int_{K_1} dk\right) \\ &= \Gamma_F(s + \nu_1 + \nu'_1;\, d_1 + d'_1) = L(s, \Pi_{d_1,\nu_1} \times \Pi_{d'_1,\nu'_1}). \end{aligned}$$

Here the second equality follows from (2.51). Next, we consider the case of $n \geq 2$. Let $q = \ell(\widehat{d} + \widehat{d'})$. By Propositions 5.5 and 5.6, we have

$$\begin{aligned} &Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(d)})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{d',\nu'}(\overline{\zeta_{H(-d')}})), \overline{\varphi}^{(l)}_{1,n}(\overline{\zeta_{Q(d+d')}})\big) \\ &= (-\varepsilon\sqrt{-1})^q \frac{\mathrm{C}^{H(-d')+d'_n, Q(d+d')}_{H(d)+d'_n}}{\mathrm{C}^{H((-\widehat{d'}-d_n,0)),\, Q((\widehat{d}+\widehat{d'},0))}_{H(d)-d_n}} \frac{\mathrm{C}^{H(-d')+d'_n, Q((\mathbf{0}_{n-1},l))}_{H(-\widehat{d'})[d]+d'_n}}{\mathrm{C}^{H((-\widehat{d'}-d_n,0)),\, Q((\mathbf{0}_{n-1},q))}_{H(-\widehat{d'})[d]-d_n}} \\ &\quad \times \frac{\dim V_{\widehat{d}}}{\dim V_d} \frac{\prod_{i=1}^n \Gamma_F(s + \nu_i + \nu'_n;\, d_i + d'_n)}{\prod_{i=1}^{n-1} \Gamma_F(\nu'_n - \nu'_i + 1;\, d'_n - d'_i)} \frac{\prod_{i=1}^{n-1} \Gamma_F(s + \nu_n + \nu'_i;\, -d_n - d'_i)}{\prod_{i=1}^{n-1} \Gamma_F(\nu_n - \nu_i + 1;\, -d_n + d_i)} \\ &\quad \times Z\big(s, \mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}})(\zeta_{H(\widehat{d})})), \mathrm{W}_{-\varepsilon}(\bar{\mathrm{f}}_{\widehat{d'},\widehat{\nu'}}(\overline{\zeta_{H(-\widehat{d'})}})), \overline{\varphi}^{(q)}_{1,n-1}(\overline{\zeta_{Q(\widehat{d}+\widehat{d'})}})\big). \end{aligned}$$

Moreover, by Lemma 4.1, we have

$$\begin{aligned} &\frac{\mathrm{C}^{H(-d')+d'_n, Q(d+d')}_{H(d)+d'_n}}{\mathrm{C}^{H((-\widehat{d'}-d_n,0)),\, Q((\widehat{d}+\widehat{d'},0))}_{H(d)-d_n}} \frac{\mathrm{C}^{H(-d')+d'_n, Q((\mathbf{0}_{n-1},l))}_{H(-\widehat{d'})[d]+d'_n}}{\mathrm{C}^{H((-\widehat{d'}-d_n,0)),\, Q((\mathbf{0}_{n-1},q))}_{H(-\widehat{d'})[d]-d_n}} \\ &= \sqrt{\frac{l!}{q!(d_n + d'_n)!}} \prod_{h=1}^{n-1} \frac{(d_h - d_n - h + n)!(-d'_h + d'_n - h + n - 1)!}{(d_h + d'_n - h + n)!(-d'_h - d_n - h + n - 1)!}. \end{aligned}$$

By the above equalities and the induction hypothesis, we obtain the statement (1).

The statement (2) follows from Lemma 4.1, Proposition 5.6, the statement (1) and

$$\frac{1}{\sqrt{\mathrm{r}(H(\mu)[\lambda])}} = \sqrt{\prod_{1\le i\le j\le n-1} \frac{(\mu_i-\mu_j-i+j)!(\lambda_i-\lambda_{j+1}-i+j)!}{(\lambda_i-\mu_j-i+j)!(\mu_i-\lambda_{j+1}-i+j)!}}$$

for $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\Lambda_n$ and $\mu=(\mu_1,\mu_2,\cdots,\mu_{n-1})\in\Xi^+(\lambda)$. □

Proof of Theorem 2.7. The equality (2.30) follows from Theorem 5.7(2). Since (2.29) is an element of $\mathrm{Hom}_{K_{n-1}}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}},\mathbb{C}_{\mathrm{triv}})$, we complete the proof by Lemma 4.3. □

Proof of Theorem 2.14(2). The equality (2.49) follows from Theorem 5.7(1). Since (2.47) is an element of $\mathrm{Hom}_{K_n}(V_d\otimes_{\mathbb{C}}\overline{V_{-d'}}\otimes_{\mathbb{C}}\overline{V_{(-l,\mathbf{0}_{n-1})}},\mathbb{C}_{\mathrm{triv}})$, we complete the proof by Lemma 4.4. □

Similar to Propositions 5.5 and 5.6, we obtain Propositions 5.8 and 5.9.

Proposition 5.8. *Retain the notation. Assume $n'=n>1$, $d'\in\Lambda_{n,F}$ and $-d\in\Xi^\circ(d')\cap\Lambda_{n,F}$. Let $l=\ell(-d-d')$. Then we have*

$$\begin{aligned}
&Z\big(s,\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(-d)}})),\mathrm{W}_{-\varepsilon}(\mathrm{f}_{d',\nu'}(\zeta_{H(d')})),\varphi^{(l)}_{1,n}(\zeta_{Q(-d-d')})\big)\\
&=\mathrm{C}^{H(d')-d'_n,Q((\mathbf{0}_{n-1},l))}_{H(\widehat{d'})[-d]-d'_n}\,\mathrm{C}^{H(d')-d'_n,Q(-d-d')}_{H(-d)-d'_n}\\
&\quad\times\frac{\dim V_{\widehat{d'}}}{\dim V_{-d}}\frac{\prod_{i=1}^n\Gamma_F(s+\nu_i+\nu'_n;\,-d_i-d'_n)}{\prod_{i=1}^{n-1}\Gamma_F(\nu'_n-\nu'_i+1;\,-d'_n+d'_i)}\\
&\quad\times Z\big(s,\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(\widehat{d'})[-d]}})),\mathrm{W}_{-\varepsilon}(\mathrm{f}_{\widehat{d'},\widehat{\nu'}}(\zeta_{H(\widehat{d'})}))\big).
\end{aligned}$$

Proposition 5.9. *Retain the notation. Assume $n'=n-1$, $-d\in\Lambda_{n,F}$ and $d'\in\Xi^+(-d)\cap\Lambda_{n-1,F}$. Let $l=\ell(-\widehat{d}-d')$. Then we have*

$$\begin{aligned}
&Z\big(s,\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(d')[-d]}})),\mathrm{W}_{-\varepsilon}(\mathrm{f}_{d',\nu'}(\zeta_{H(d')}))\big)\\
&=(-\varepsilon\sqrt{-1})^l\left(\mathrm{C}^{H((d'+d_n,0)),\,Q((-\widehat{d}-d',0))}_{H(-d)+d_n}\,\mathrm{C}^{H((d'+d_n,0)),\,Q((\mathbf{0}_{n-1},l))}_{H(d')[-d]+d_n}\right)^{-1}\\
&\quad\times\frac{\dim V_{-\widehat{d}}}{\dim V_{d'}}\frac{\prod_{i=1}^{n-1}\Gamma_F(s+\nu_n+\nu'_i;\,d_n+d'_i)}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\,d_n-d_i)}\\
&\quad\times Z\big(s,\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{\widehat{d},\widehat{\nu}}(\overline{\zeta_{H(-\widehat{d})}})),\mathrm{W}_{-\varepsilon}(\mathrm{f}_{d',\nu'}(\zeta_{H(d')})),\varphi^{(l)}_{1,n-1}(\zeta_{Q(-\widehat{d}-d')})\big).
\end{aligned}$$

Similar to Theorem 5.7, we obtain Theorem 5.10 using Propositions 5.8 and 5.9.

Theorem 5.10. *Retain the notation. Assume $-d\in\Lambda_{n,F}$ and $d'\in\Lambda_{n',F}$. We take $\mathbf{\Gamma}_F(\nu;d)$ and $\mathbf{\Gamma}_F(\nu';d')$ as in* §2.7.

(1) *Assume $n'=n$ and $-d\in\Xi^\circ(d')$. Let $l=\ell(-d-d')$. Then we have*

$$\begin{aligned}
&Z\big(s,\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(-d)}})),\mathrm{W}_{-\varepsilon}(\mathrm{f}_{d',\nu'}(\zeta_{H(d')})),\varphi^{(l)}_{1,n}(\zeta_{Q(-d-d')})\big)\\
&=\frac{(\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d'_i)}\sqrt{\mathrm{b}(-d-d')}\mathrm{C}^\circ(-d;d')L(s,\Pi_{d,\nu}\times\Pi_{d',\nu'})}{(\dim V_{-d})\mathbf{\Gamma}_F(\nu;d)\mathbf{\Gamma}_F(\nu';d')}.
\end{aligned}$$

(2) *Assume $n' = n-1$ and $d' \in \Xi^+(-d)$. Then we have*

$$Z\big(s, \mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\zeta_{H(d')[-d]}})), \mathrm{W}_{-\varepsilon}(\mathrm{f}_{d',\nu'}(\zeta_{H(d')}))\big) = \frac{(\varepsilon\sqrt{-1})^{\sum_{i=1}^{n-1}(n-i)(d_i+d'_i)} L(s, \Pi_{d,\nu}\times\Pi_{d',\nu'})}{(\dim V_{d'})\sqrt{\mathrm{r}(H(d')[-d])}\,\boldsymbol{\Gamma}_F(\nu;d)\boldsymbol{\Gamma}_F(\nu';d')}.$$

Proof of Theorem 2.14(1). The equality (2.48) follows from Theorem 5.10(1) and (2.44). Since (2.46) is an element of $\mathrm{Hom}_{K_n}(V_d \otimes_{\mathbb{C}} \overline{V_{-d'}} \otimes_{\mathbb{C}} V_{(l,\mathbf{0}_{n-1})}, \mathbb{C}_{\mathrm{triv}})$, we complete the proof by Lemma 4.4 and the properties of complex conjugate representations in §2.6. □

Appendix A. Explicit formulas of Whittaker functions

In this appendix, we consider the explicit formulas of the radial parts of Whittaker functions on G_n. Let $\varepsilon \in \{\pm 1\}$, $d = (d_1, d_2, \cdots, d_n) \in \mathbb{Z}^n$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{C}^n$. Assume that either $d \in \Lambda_{n,F}$ or $-d \in \Lambda_{n,F}$ holds. We set

$$\widetilde{W}^{(\varepsilon)}_{d,\nu}(a) = \begin{cases} \eta_{-\rho_n}(a)\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\xi_{H(d)}))(a) & \text{if } d \in \Lambda_{n,F}, \\ \eta_{-\rho_n}(a)\mathrm{W}_\varepsilon(\bar{\mathrm{f}}_{d,\nu}(\overline{\xi_{H(-d)}}))(a) & \text{if } -d \in \Lambda_{n,F} \end{cases} \qquad (a \in A_n).$$

Then we have Theorem A.1, which is the generalization of the explicit formulas [10, Theorem 14] of spherical Whittaker functions on $\mathrm{GL}(n,\mathbb{R})$.

Theorem A.1. *Retain the notation, and we assume $n > 1$. We take $\widehat{d}$ and $\widehat{\nu}$ as in §3.1. Let $a = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in A_n$. Then we have*

$$\widetilde{W}^{(\varepsilon)}_{d,\nu}(a) = \frac{\prod_{i=1}^n a_i^{\nu_n \mathrm{c}_F + |d_i - d_n|}}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n - \nu_i + 1;\, |d_i - d_n|)} \times \int_{(\mathbb{R}_+^\times)^{n-1}} \widetilde{W}^{(\varepsilon)}_{\widehat{d},\widehat{\nu}}(t) \prod_{i=1}^{n-1} \exp\left(-\pi \mathrm{c}_F\left(\frac{t_i^2}{a_{i+1}^2} + \frac{a_i^2}{t_i^2}\right)\right) t_i^{-\nu_n \mathrm{c}_F - |d_i - d_n|}\,\frac{2\mathrm{c}_F\, dt_i}{t_i}$$

with $t = \mathrm{diag}(t_1, t_2, \cdots, t_{n-1}) \in A_{n-1}$.

Proof. We will prove here only the case of $d \in \Lambda_{n,F}$, since the proof for the case of $-d \in \Lambda_{n,F}$ is similar. By Lemmas 4.8 and 4.9, we have

$$\Phi^+_{\widehat{d}-d_n}(\overline{\zeta_{H(\widehat{d})-d_n}} \boxtimes \zeta_{H(d)-d_n})((h, hz)a) = \left(\prod_{i=1}^{n-1} a_i^{d_i - d_n}\right)\big\langle \tau_{\widehat{d}-d_n}(h)\zeta_{H(\widehat{d})-d_n}, \zeta_{H(\widehat{d})-d_n}\big\rangle \mathbf{e}_{(n-1,n)}((h,hz)a)$$

for $h \in G_{n-1}$ and $z \in \mathrm{M}_{n-1,1}(F)$. Hence, by Corollary 5.3, we have

$$\widetilde{W}^{(\varepsilon)}_{d,\nu}(a) = \eta_{-\rho_n}(a)\mathrm{W}_\varepsilon(\mathrm{f}_{d,\nu}(\zeta_{H(d)}))(a) = \frac{(\dim V_{\widehat{d}})\prod_{i=1}^n a_i^{(\nu_n + i - 1)\mathrm{c}_F + d_i - d_n}}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n - \nu_i + 1;\, d_i - d_n)}$$
$$\times \int_{G_{n-1}}\left(\int_{\mathrm{M}_{n-1,1}(F)} \mathbf{e}_{(n-1,n)}((h,hz)a)\psi_{-\varepsilon}(e_{n-1}z)\,dz\right)$$
$$\times \mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))(h^{-1})\big\langle \tau_{\widehat{d}-d_n}(h)\zeta_{H(\widehat{d})-d_n}, \zeta_{H(\widehat{d})-d_n}\big\rangle \chi_{d_n}(\det h)|\det h|_F^{\nu_n + n/2}\,dh.$$

Decomposing $h^{-1}=xtk$ ($x\in N_{n-1}$, $t=\mathrm{diag}(t_1,t_2,\cdots,t_{n-1})\in A_{n-1}$, $k\in K_{n-1}$) and applying Schur's orthogonality [18, Corollary 1.10] for the integration on K_{n-1} together with the equalities

$$\begin{aligned}&\left\langle \tau_{\widehat{d}-d_n}(h)\zeta_{H(\widehat{d})-d_n},\, \zeta_{H(\widehat{d})-d_n}\right\rangle \chi_{d_n}(\det h)\\ &=\left(\prod_{i=1}^{n-1} t_i^{-d_i+d_n}\right)\overline{\left\langle \tau_{\widehat{d}}(k)\zeta_{H(\widehat{d})},\, \zeta_{H(\widehat{d})}\right\rangle} \qquad \text{(by Lemma 5.1)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_{H(\widehat{d})}))(h^{-1}) &= \psi_{\varepsilon,n-1}(x)\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\tau_{\widehat{d}}(k)\zeta_{H(\widehat{d})}))(t)\\ &=\sum_{M\in \mathrm{G}(\widehat{d})}\langle \tau_{\widehat{d}}(k)\zeta_{H(\widehat{d})},\zeta_M\rangle\ \psi_{\varepsilon,n-1}(x)\mathrm{W}_\varepsilon(\mathrm{f}_{\widehat{d},\widehat{\nu}}(\zeta_M))(t),\end{aligned}$$

we have

$$\begin{aligned}\widetilde{W}^{(\varepsilon)}_{d,\nu}(a) =&\frac{\prod_{i=1}^{n} a_i^{(\nu_n+i-1)\mathrm{c}_F+d_i-d_n}}{\prod_{i=1}^{n-1}\Gamma_F(\nu_n-\nu_i+1;\, d_i-d_n)}\\ &\times\int_{(\mathbb{R}_+^\times)^{n-1}}\left(\int_{N_{n-1}}\int_{\mathrm{M}_{n-1,1}(F)}\mathbf{e}_{(n-1,n)}((t^{-1}x^{-1},t^{-1}x^{-1}z)a)\psi_{\varepsilon,n-1}(x)\right.\\ &\left.\times\,\psi_{-\varepsilon}(e_{n-1}z)\,dz\,dx\right)\widetilde{W}^{(\varepsilon)}_{\widehat{d},\widehat{\nu}}(t)\prod_{i=1}^{n-1} t_i^{-(\nu_n+n-i)\mathrm{c}_F-d_i+d_n}\,\frac{2\mathrm{c}_F\,dt_i}{t_i}.\end{aligned}$$

Let us consider the integral

$$\int_{N_{n-1}}\int_{\mathrm{M}_{n-1,1}(F)}\mathbf{e}_{(n-1,n)}((t^{-1}x^{-1},t^{-1}x^{-1}z)a)\psi_{\varepsilon,n-1}(x)\psi_{-\varepsilon}(e_{n-1}z)\,dz\,dx.$$

Substituting $\begin{pmatrix} x^{-1} & x^{-1}z\\ O_{1,n-1} & 1\end{pmatrix}\to x$, this integral becomes

$$\int_{N_n}\mathbf{e}_{(n-1,n)}((1_{n-1},O_{n-1,1})\iota_n(t^{-1})xa)\psi_{-\varepsilon,n}(x)\,dx. \tag{A.1}$$

By the elementary formula (4.19) and the equality

$$\begin{aligned}&\mathbf{e}_{(n-1,n)}((1_{n-1},O_{n-1,1})\,\iota_n(t^{-1})xa)\\ &=\prod_{i=1}^{n-1}\exp(-\pi\mathrm{c}_F t_i^{-2}a_i^2)\prod_{j=i+1}^{n}\exp(-\pi\mathrm{c}_F t_i^{-2}a_j^2 x_{i,j}\overline{x_{i,j}}),\end{aligned}$$

we know that (A.1) is equal to

$$\prod_{i=1}^{n-1}\exp\left(-\pi\mathrm{c}_F\left(\frac{t_i^2}{a_{i+1}^2}+\frac{a_i^2}{t_i^2}\right)\right)t_i^{(n-i)\mathrm{c}_F}a_{i+1}^{-i\mathrm{c}_F}.$$

Therefore, we obtain the assertion. □

Appendix B. The list of symbols

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