

ROOT COMPONENTS FOR TENSOR PRODUCT OF AFFINE KAC-MOODY LIE ALGEBRA MODULES

SAMUEL JERALDS AND SHRAWAN KUMAR

ABSTRACT. Let \mathfrak{g} be an affine Kac-Moody Lie algebra and let λ, μ be two dominant integral weights for \mathfrak{g} . We prove that under some mild restriction, for any positive root β , $V(\lambda) \otimes V(\mu)$ contains $V(\lambda + \mu - \beta)$ as a component, where $V(\lambda)$ denotes the integrable highest weight (irreducible) \mathfrak{g} -module with highest weight λ . This extends the corresponding result by Kumar from the case of finite dimensional semisimple Lie algebras to the affine Kac-Moody Lie algebras. One crucial ingredient in the proof is the action of Virasoro algebra via the Goddard-Kent-Olive construction on the tensor product $V(\lambda) \otimes V(\mu)$. Then, we prove the corresponding geometric results including the higher cohomology vanishing on the \mathcal{G} -Schubert varieties in the product partial flag variety $\mathcal{G}/\mathcal{P} \times \mathcal{G}/\mathcal{P}$ with coefficients in certain sheaves coming from the ideal sheaves of \mathcal{G} -sub-Schubert varieties. This allows us to prove the surjectivity of the Gaussian map.

1. INTRODUCTION

Let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra, and fix two dominant integral weights $\lambda, \mu \in \mathcal{P}^+$. To these, we can associate the integrable, highest weight (irreducible) representations $V(\lambda)$ and $V(\mu)$. Then, the content of the tensor decomposition problem is to express the product $V(\lambda) \otimes V(\mu)$ as a direct sum of irreducible components; that is, to find the decomposition

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \mathcal{P}^+} V(\nu)^{\oplus m_{\lambda, \mu}^{\nu}},$$

where $m_{\lambda, \mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ is the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. While this is a classical problem with a straightforward statement, determining the multiplicities $m_{\lambda, \mu}^{\nu}$ exactly—or even determining when $m_{\lambda, \mu}^{\nu} > 0$ —is a challenging endeavor. Various algebraic, geometric, and combinatorial methods have been developed to understand the tensor decomposition problem; see [Ku4] for a survey in the case of (finite dimensional) semisimple Lie algebras.

While having a complete description for the components of a tensor product is desirable, many significant results in the literature demonstrate the existence of “families” of components; that is, components that are uniformly described and exist for tensor product decompositions regardless of \mathfrak{g} . One such example is given by the *root components* $V(\lambda + \mu - \beta)$ for a positive root β .

We show the existence of root components for affine Lie algebras generalizing the corresponding result in the finite case (i.e., when \mathfrak{g} is a semisimple Lie algebra) as

Received by the editors July 14, 2021, and, in revised form, April 21, 2022.

2020 *Mathematics Subject Classification*. Primary 14M15, 14M17, 20G44, 22E47.

The second author was partially supported by the NSF grant number DMS-1802328.

in [Ku1]. Recall that by a *Wahl triple* (introduced in [Ku1] though not christened as *Wahl triple* there) we mean a triple $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi^+$ such that

- (P1) $\lambda + \mu - \beta \in \mathcal{P}^+$, and
- (P2) If $\lambda(\alpha_i^\vee) = 0$ or $\mu(\alpha_i^\vee) = 0$, then $\beta - \alpha_i \notin \Phi \sqcup \{0\}$,

where Φ (resp. Φ^+) is the set of all the roots (resp., positive roots).

The main representation theoretic result of the paper is Theorem I (cf. Theorem 9.1).

Theorem I. *For any affine Kac-Moody Lie algebra \mathfrak{g} and Wahl triple $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi^+$,*

$$V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu).$$

We construct the proof in three parts. First, notice that the conditions (P1) and (P2) are invariant under adding δ ; that is, if (λ, μ, β) is a Wahl triple, then so is $(\lambda, \mu, \beta + k\delta)$ for any $k \in \mathbb{Z}_{\geq 0}$. This allows us to make use of the Goddard-Kent-Olive construction of the Virasoro algebra action on the tensor product $V(\lambda) \otimes V(\mu)$ and we explore its action on the subspaces $W^{\lambda + \mu - \beta}$ (as in Definition 2.3). This reduces the problem to certain ‘maximal root components’ via Proposition 3.9. In the second part, closely following the construction of root components for simple Lie algebras as in [Ku1], we show the existence of the bulk of the maximal root components. Finally, in the third part, we construct the remaining maximal root components that are excluded from the previous methods explicitly using familiar, but ad hoc, constructions from the general tensor decomposition problem using the PRV components.

We next prove the corresponding geometric results. Let \mathcal{G} be the ‘maximal’ Kac-Moody group associated to the symmetrizable Kac-Moody Lie algebra \mathfrak{g} and let \mathcal{P} be a standard parabolic subgroup of \mathcal{G} corresponding to a subset S of the set of simple roots $\{\alpha_1, \dots, \alpha_\ell\}$, i.e., S is the set of simple roots of the Levi group of \mathcal{P} . In the sequel, we abbreviate S as the subset of $\{1, \dots, \ell\}$. In particular, for $S = \emptyset$, we have the standard Borel subgroup \mathcal{B} (corresponding to the Borel subalgebra \mathfrak{b}). Let W be the Weyl group of \mathfrak{g} and let $W'_\mathcal{P}$ be the set of smallest length coset representatives in $W/W_\mathcal{P}$, where $W_\mathcal{P}$ is the subgroup of W generated by the simple reflections $\{s_k\}_{k \in S}$. For any finite dimensional pro-algebraic \mathcal{P} -module M , by $\mathcal{L}(M)$ we mean the corresponding homogeneous vector bundle on $\mathcal{X}_\mathcal{P} := \mathcal{G}/\mathcal{P}$ associated to the principal \mathcal{P} -bundle: $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{P}$ by the representation M of \mathcal{P} .

For any integral weight $\lambda \in \mathfrak{h}^*$ (where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} with the corresponding standard maximal torus \mathcal{H} of \mathcal{G}), such that $\lambda(\alpha_k^\vee) = 0$ for all $k \in S$, the one dimensional \mathcal{H} -module \mathbb{C}_λ (given by the character λ) admits a unique \mathcal{P} -module structure (extending the \mathcal{H} -module structure); in particular, we have the line bundle $\mathcal{L}(\mathbb{C}_\lambda)$ on $\mathcal{X}_\mathcal{P}$. We abbreviate the line bundle $\mathcal{L}(\mathbb{C}_{-\lambda})$ by $\mathcal{L}(\lambda)$ and its restriction to the Schubert variety $X_w^\mathcal{P} := \overline{\mathcal{B}w\mathcal{P}/\mathcal{P}}$ by $\mathcal{L}_w(\lambda)$ (for any $w \in W'_\mathcal{P}$). Given two line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$, we can form their external tensor product to get the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ on $\mathcal{X}_\mathcal{P} \times \mathcal{X}_\mathcal{P}$. A dominant integral weight μ is called *S-regular* if $\mu(\alpha_k^\vee) = 0$ if and only if $k \in S$. The set of such weights is denoted by \mathcal{P}_S^{+o} .

Then, we prove that, for any $\mu \in \mathcal{P}_S^{+o}$ and $w \in W'_\mathcal{P}$ such that $X_w^\mathcal{P}$ is \mathcal{P} -stable under the left multiplication:

$$H^p(X_w^\mathcal{P}, \mathcal{I}_e^k \otimes \mathcal{L}_w(\mu)) = H^p(X_w^\mathcal{P}, (\mathcal{O}_w/\mathcal{I}_e^k) \otimes \mathcal{L}_w(\mu)) = 0, \text{ for all } p > 0, k = 1, 2,$$

where \mathcal{I}_e is the ideal sheaf of $X_w^{\mathcal{P}}$ at the base point e and \mathcal{O}_w denotes the structure sheaf of $X_w^{\mathcal{P}}$. We further explicitly determine $H^0(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu))$ and $H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu))$ (cf. Proposition 10.1 and Corollary 10.2).

For any $w \in W'_{\mathcal{P}}$ such that the Schubert variety $X_w^{\mathcal{P}}$ is \mathcal{P} -stable, define the \mathcal{G} -Schubert variety:

$$\hat{\mathcal{X}}_w^{\mathcal{P}} := \mathcal{G} \times^{\mathcal{P}} X_w^{\mathcal{P}}.$$

Consider the isomorphism:

$$\delta : \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}} \simeq \mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, [g, x] \mapsto (g\mathcal{P}, gx), \text{ for } g \in \mathcal{G} \text{ and } x \in \mathcal{X}_{\mathcal{P}}.$$

We have the canonical embedding

$$\hat{\mathcal{X}}_w^{\mathcal{P}} \hookrightarrow \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}}.$$

In particular, we can restrict the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ via the above isomorphism δ to $\hat{\mathcal{X}}_w^{\mathcal{P}}$ to get the line bundle denoted $\mathcal{L}_w(\lambda \boxtimes \mu)$. Then, for $\lambda, \mu \in \mathcal{P}_S^{+\circ}, k = 1, 2, w \in W'_{\mathcal{P}}$ as above and $p \geq 0$, we determine (cf. Lemma 11.1):

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e^k \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \text{ and } H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^k) \otimes \mathcal{L}_w(\lambda \boxtimes \mu)),$$

in terms of the cohomology of the partial flag variety \mathcal{G}/\mathcal{P} with coefficients in explicit homogeneous vector bundles, where $\hat{\mathcal{I}}_e$ denotes the ideal sheaf of $\hat{\mathcal{X}}_e^{\mathcal{P}}$ in $\hat{\mathcal{X}}_w^{\mathcal{P}}$ and $\hat{\mathcal{O}}_w$ is the structure sheaf of $\hat{\mathcal{X}}_w^{\mathcal{P}}$. In fact, we show that

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

Let $\Phi^+ \subset \mathfrak{h}^*$ be the set of positive roots, $\Phi_S^+ := \Phi^+ \cap (\oplus_{k \in S} \mathbb{Z}_{\geq 0} \alpha_k)$ and $\Phi^+(S) := \Phi^+ \setminus \Phi_S^+$.

We next study the vanishing of the first cohomology $H^1(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu))$ and prove the following crucial result (cf. Proposition 11.4):

Proposition II. *Let \mathfrak{g} be an affine Kac-Moody Lie algebra. Then, for any $\lambda, \mu \in \mathcal{P}_S^{+\circ}$ (where S is an arbitrary subset of the simple roots of \mathfrak{g}) and any $w \in W'_{\mathcal{P}}$ such that the Schubert variety $X_w^{\mathcal{P}}$ is \mathcal{P} -stable, consider the following two conditions:*

- (a) $H^1(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0.$
- (b) *For all the real roots $\beta \in \Phi^+(S)$, satisfying $S \subset \{0 \leq i \leq \ell : \beta - \alpha_i \notin \Phi^+ \sqcup \{0\}\}$ and $\lambda + \mu - \beta \in \mathcal{P}^+$, there exists a $f_{\beta} \in \text{Hom}_{\mathfrak{b}}(\mathbb{C}_{\lambda + \mu - \beta} \otimes V(\lambda)^{\vee}, V(\mu))$ such that*

$$X_{\beta}(f_{\beta}(\mathbb{C}_{\lambda + \mu - \beta} \otimes v_{\lambda}^*)) \neq 0, \text{ for } X_{\beta} \neq 0 \in \mathfrak{g}_{\beta},$$

where $V(\lambda)^{\vee}$ is the restricted dual of $V(\lambda)$, and $v_{\lambda}^* \neq 0 \in [V(\lambda)^{\vee}]_{-\lambda}$.

Then, the condition (b) implies the condition (a).

Further, we show that under the assumptions of Proposition II, the condition (b) of Proposition II is satisfied in all the cases except possibly \mathfrak{g} of type F_4 or G_2 (cf. Proposition 11.6 for a more precise result). The proof of Proposition 11.6 relies on explicit constructions of root components obtained in the earlier sections including that of the GKO operator and Lemma III (cf. Lemma 11.5):

Lemma III. *Let (λ, μ, β) be a Wahl triple for a real root β and let $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal root component. Observe that $\beta \in \mathring{\Phi}^+$ or $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$. Then, for $\beta \in \mathring{\Phi}^+$, the validity of condition (b) of Proposition 11.4 for $V(\lambda + \mu - \beta)$ implies its validity for $V(\lambda + \mu - \beta - k\delta)$ for any $k \geq 0$.*

Moreover, if $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$, then we have an identity connecting the condition (b) of Proposition 11.4 for $V(\lambda + \mu - \delta + \gamma)$ with that of $V(\lambda + \mu - \beta - k\delta)$, $k \geq 1$ (see the identity (32)).

Combining Proposition II and Proposition 11.6, we obtain the following main geometric result of the paper (cf. Theorem 11.7):

Theorem IV. *Let \mathfrak{g} be an affine Kac-Moody Lie algebra and let $w \in W'_P$ be such that the Schubert variety X_w^P is \mathcal{P} -stable. Then, for any $\lambda, \mu \in \mathcal{P}_S^{+0}$ (where S is an arbitrary subset of the simple roots of $\mathring{\mathfrak{g}}$) such that the condition (b) of Proposition 11.4 is satisfied for all the Wahl triples (λ, μ, β) for any real root $\beta \in \Phi^+$,*

$$H^p(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

In particular, the canonical Gaussian map

$$H^0(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \rightarrow H^0(\hat{\mathcal{X}}_w^P, (\hat{\mathcal{I}}_e / \hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu))$$

is surjective.

In particular, the theorem holds for any simply-laced $\mathring{\mathfrak{g}}$ and $\mathring{\mathfrak{g}}$ of types B_ℓ, C_ℓ . Moreover, it also holds for $\mathring{\mathfrak{g}}$ of type F_4 in the case \mathcal{P} is the Borel subgroup \mathcal{B} .

As a fairly straight forward corollary of Theorem IV, taking inverse limits, we get the following (cf. Corollary 11.8):

Corollary V. *Under the notation and assumptions of Theorem IV, the canonical Gaussian map*

$$H^0(\mathcal{X}_P \times \mathcal{X}_P, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)) \rightarrow H^0(\mathcal{X}_P \times \mathcal{X}_P, (\mathcal{I}_D / \tilde{\mathcal{I}}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu))$$

is surjective, where \mathcal{I}_D is the ideal sheaf of the diagonal $D \subset \mathcal{X}_P \times \mathcal{X}_P$ and $\tilde{\mathcal{I}}_D^2$ is defined as $\varprojlim_w \hat{\mathcal{I}}_e(w)^2$.

2. AFFINE LIE ALGEBRAS (NOTATION AND PRELIMINARIES)

In this section, we recall the definition of affine Kac-Moody Lie algebras \mathfrak{g} and their root and weight lattices. For a more extensive treatment of \mathfrak{g} and its properties, see [Kac, Chapters 6 and 7].

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra over \mathbb{C} with a fixed Borel subalgebra $\mathring{\mathfrak{b}}$ and Cartan subalgebra $\mathring{\mathfrak{h}} \subset \mathring{\mathfrak{b}}$. We denote the rank of $\mathring{\mathfrak{g}}$ (which is, by definition, the dimension of $\mathring{\mathfrak{h}}$) by ℓ . Let $\mathring{\Phi} \subset \mathring{\mathfrak{h}}^*$ be the set of roots and $\mathring{\Phi}^+$ (resp. $\mathring{\Phi}^-$) be the subset of positive (resp. negative) roots. Then, the associated affine Kac-Moody Lie algebra is given, as a vector space, by

$$\mathfrak{g} = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is the central element and d the derivation or scaling element. The Lie bracket in \mathfrak{g} is given by

$$\begin{aligned} (1) \quad [x \otimes t^m + zK + \zeta d, x' \otimes t^n + z'K + \zeta' d] &= [x, x'] \otimes t^{m+n} + n\zeta x' \otimes t^n - m\zeta' x \otimes t^m \\ &\quad + m\delta_{n, -m}(x|x')_{\mathring{\mathfrak{g}}}K, \end{aligned}$$

for $x, x' \in \mathring{\mathfrak{g}}$, $n, m \in \mathbb{Z}$, $z, z', \zeta, \zeta' \in \mathbb{C}$, where $\delta_{m, -n}$ is the Kronecker delta and $(\cdot|\cdot)_{\mathring{\mathfrak{g}}}$ is the invariant form on $\mathring{\mathfrak{g}}$ normalized so that $(\theta|\theta)_{\mathring{\mathfrak{g}}} = 2$, where $\theta \in \mathring{\Phi}$ is the highest root of $\mathring{\mathfrak{g}}$. Through the paper we will always take this normalized form on $\mathring{\mathfrak{g}}$.

The Cartan subalgebra of \mathfrak{g} is given by $\mathfrak{h} := \mathring{\mathfrak{h}} \oplus (\mathbb{C}K + \mathbb{C}d)$. We thus have

$$\mathfrak{h}^* := \mathring{\mathfrak{h}}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0),$$

where δ is defined by $\delta(d) = 1$, $\delta|_{\mathring{\mathfrak{h}} \oplus \mathbb{C}K} = 0$ and $\Lambda_0(K) = 1$, $\Lambda_0|_{\mathring{\mathfrak{h}} \oplus \mathbb{C}d} = 0$ and $\mathring{\mathfrak{h}}^*$ is the dual of $\mathring{\mathfrak{h}}$ taking d and K to zero. Setting $\alpha_0 := \delta - \theta$, $\alpha_0^\vee := K - \theta^\vee$ for θ as above, and $\{\alpha_i\}_{i=1}^\ell$, $\{\alpha_i^\vee\}_{i=1}^\ell$ the set of simple roots and coroots of $\mathring{\mathfrak{g}}$, we have that $\{\alpha_i\}_{i=0}^\ell$, $\{\alpha_i^\vee\}_{i=0}^\ell$ are the simple roots and coroots of \mathfrak{g} .

Let Φ be the set of roots of \mathfrak{g} and Φ^+ (resp. Φ^-) be the subset of positive (resp. negative) roots. Then, we can separate $\Phi = \Phi_{Re} \sqcup \Phi_{Im}$, a disjoint union of real and imaginary roots. These are given precisely by

$$\begin{aligned} \Phi_{Re} &= \{\beta + k\delta : \beta \in \mathring{\Phi}, k \in \mathbb{Z}\}, \\ \Phi_{Im} &= \{k\delta : k \in \mathbb{Z} \setminus 0\}. \end{aligned}$$

Finally, let $\{\Lambda_i\}_{i=0}^\ell$ be the set of fundamental weights of \mathfrak{g} defined by $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$ (determined up to a multiple of δ). Then, the dominant integral weights of \mathfrak{g} are given by $\mathcal{P}^+ = \left(\bigoplus_{i=0}^\ell \mathbb{Z}_{\geq 0} \Lambda_i\right) \oplus \mathbb{C}\delta$. The integrable highest weight (irreducible) \mathfrak{g} -modules are parameterized by \mathcal{P}^+ . For $\lambda \in \mathcal{P}^+$, let $V(\lambda)$ be the corresponding integrable highest weight \mathfrak{g} -module.

Fix dominant weights $\lambda, \mu \in \mathcal{P}^+$, and consider the tensor product $V(\lambda) \otimes V(\mu)$ of representations. Note that, as $\delta(\alpha_i^\vee) = 0$ for all i , the representation $V(k\delta)$ is one-dimensional for all $k \in \mathbb{C}$, and $V(\lambda) \otimes V(k\delta) \cong V(\lambda + k\delta)$. Therefore, throughout the remainder of this paper, we can assume without loss of generality that $\lambda(d) = \mu(d) = 0$ up to an appropriate twist by $V(k\delta)$.

Definition 2.1. For $\lambda \in \mathcal{P}^+$, the integer $\lambda(K)$ is called the *level* of λ . Further, for any $v \in V(\lambda)$ (not necessarily a highest weight vector), we have $K.v = \lambda(K)v$, since K is a central element of \mathfrak{g} . So, we say that $V(\lambda)$ has *level* $\lambda(K)$.

If λ is of level l and μ is of level m , then each irreducible component of $V(\lambda) \otimes V(\mu)$ is of level $l + m$. Indeed, the Cartan component $V(\lambda + \mu)$ clearly has level $l + m$, and if $V(\nu) \subset V(\lambda) \otimes V(\mu)$, then necessarily $\lambda + \mu - \nu \in \bigoplus_{i=0}^\ell \mathbb{Z}_{\geq 0} \alpha_i$, and $\alpha_i(K) = 0$ for all i .

We also recall Definition 2.2 taken from [BrKu] of δ -maximal components of the tensor product.

Definition 2.2 (δ -Maximal components). A component $V(\nu) \subset V(\lambda) \otimes V(\mu)$ is called δ -maximal if $V(\nu + k\delta) \not\subset V(\lambda) \otimes V(\mu)$ for any $k > 0$.

Let $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ be the derived subalgebra of \mathfrak{g} . By the Lie bracket given in (1), we have $\mathfrak{g}' = \mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$. In particular, the \mathfrak{g}' action on $V(\lambda)$ cannot detect the weight δ . Nevertheless, it is known that the restriction of $V(\lambda)$ to a \mathfrak{g}' -module remains irreducible (cf. [Ku3, Lemma 2.1.4]). This allows us to make Definition 2.3.

Definition 2.3. Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal component. We denote by W^ν the subspace

$$W^\nu := \sum_{k \geq 0} V(\nu - k\delta)^{\oplus m_k} \subset V(\lambda) \otimes V(\mu),$$

where $m_k := m_{\lambda, \mu}^{\nu - k\delta}$ is the multiplicity of $V(\nu - k\delta)$ in $V(\lambda) \otimes V(\mu)$.

That is, W^ν is the ν -isotypic component of $V(\lambda) \otimes V(\mu)$ with respect to the \mathfrak{g}' -action. In the next section, we make use of the Virasoro algebra to more closely examine the structure of such W^ν .

3. VIRASORO ALGEBRA AND GODDARD–KENT–OLIVE CONSTRUCTION

In this section, we recall the basics of the Virasoro algebra and its representation theory. In particular, we give an overview of the Goddard–Kent–Olive (GKO) construction of the Virasoro algebra and its action on tensor products of affine Lie algebra representations. We follow the exposition of [KRR].

Definition 3.1. The *Virasoro algebra* Vir is a Lie algebra over \mathbb{C} with basis $\{c, L_k | k \in \mathbb{Z}\}$ with commutation relations

$$(2) \quad [L_k, L_j] = (k - j)L_{k+j} + \frac{1}{12}(k^3 - k)\delta_{k,-j}c, \quad [Vir, c] = 0.$$

We set $Vir_0 := \mathbb{C}L_0 \oplus \mathbb{C}c$, and the dual space $Vir_0^* := \mathbb{C}h \oplus \mathbb{C}z$, where $\{h, z\}$ is the basis dual to that of $\{L_0, c\}$. Similar to the case of Kac–Moody representations, we have the notion of highest weight representations of Vir :

Definition 3.2. A representation V of Vir is defined to be a *highest weight representation with highest weight* $\lambda \in Vir_0^*$ if there is a vector $v \in V$ such that

$$X.v = \lambda(X)v \quad \forall X \in Vir_0, \quad L_k.v = 0 \quad \forall k \geq 1, \quad V = U \left(\bigoplus_{k < 0} \mathbb{C}L_k \right).v.$$

The structure of highest weight Vir representations is in many ways parallel to that of Kac–Moody representations. Denote by V_μ the μ -weight space of V . Then, by the defining relations (2) for Vir , we have $L_{-k} : V_\mu \rightarrow V_{\mu+kh}$. Further, if V has highest weight λ , then for any $v \in V$ (not necessarily highest weight vector) by equation (2) we again have $c.v = \lambda(c)v$. We refer to the value $\lambda(c)$ as the *central charge* of the representation V .

Definition 3.3. A Vir representation V is called *unitarizable* if there is a positive-definite Hermitian form $\langle \cdot | \cdot \rangle$ on V satisfying $\langle L_k.v | w \rangle = \langle v | L_{-k}.w \rangle$ and $\langle cv | w \rangle = \langle v | cw \rangle$ for all $k \in \mathbb{Z}$ and $v, w \in V$.

The existence of a positive-definite form on a highest weight representation V allows us to investigate the weight spaces of V . More specifically, the following Lemma 3.4 from [BrKu] determines when the weight spaces $\lambda+kh$ of an irreducible, highest weight representation with highest weight λ are nontrivial.

Lemma 3.4. *Let V be a unitarizable, highest weight (irreducible) representation of Vir with highest weight λ . Then, we have the following:*

- (1) *If $\lambda(L_0) \neq 0$, then $V_{\lambda+kh} \neq 0$ for any $k \in \mathbb{Z}_{\geq 0}$.*
- (2) *If $\lambda(L_0) = 0$ and $\lambda(c) \neq 0$, then $V_{\lambda+kh} \neq 0$ for all $k \geq 2$, and $V_{\lambda+h} = 0$.*
- (3) *If $\lambda(L_0) = \lambda(c) = 0$, then V is one-dimensional.*

Proof. Let v be the highest weight vector of V . Then, we have (for all $k > 0$):

$$0 \leq \langle L_{-k}v | L_{-k}v \rangle = \langle L_k L_{-k}v | v \rangle = (2k\lambda(L_0) + \frac{1}{12}(k^3 - k)\lambda(c))\langle v | v \rangle.$$

Thus, $\lambda(L_0)$ and $\lambda(c)$ both must be nonnegative numbers. So, if $\lambda(L_0) \neq 0$, we have $L_{-k}v \neq 0$ for any $k \geq 0$. If $\lambda(L_0) = 0$, then we have $(k^3 - k)\lambda(c) \geq 0$; if $\lambda(c) \neq 0$,

this is nonzero for $k > 1$ and zero for $k = 1$, so that $L_{-1}v = 0$ and $L_{-k}v \neq 0$ for $k > 1$. Finally, if $\lambda(L_0) = \lambda(c) = 0$, then we get that $L_{-k}v = 0$ for all $k > 0$, so V must be one-dimensional. \square

The Goddard–Kent–Olive construction. The similarity between representations of Kac–Moody algebras and the Virasoro algebra is not coincidental. A foundational result linking the two theories is given by the *Sugawara construction*, which embeds $Vir \hookrightarrow \hat{U}(\mathfrak{g})$ in a certain completion $\hat{U}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of an affine Lie algebra \mathfrak{g} such that an integrable highest weight (irreducible) representation $V(\lambda)$ of \mathfrak{g} becomes a unitarizable representation of Vir . We make use of a related construction, known as the *Goddard–Kent–Olive (for short GKO) construction*, to produce an action of Vir on a tensor product $V(\lambda) \otimes V(\mu)$ of \mathfrak{g} -modules. At its core, the GKO construction is “relative” and relies on the diagonal inclusion $\hat{\mathfrak{g}} \hookrightarrow \hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$, but we will not give complete details of the construction here; see Lecture 10 of [KRR] for a more in-depth treatment. Proposition 3.5 is the primary computational tool we need from this construction.

Proposition 3.5 ([KRR, Proposition 10.3]). *Let \mathfrak{g} be an affine Lie algebra and $\lambda, \mu \in \mathcal{P}^+$ be weights with levels l, m respectively. Then,*

- (1) $V(\lambda) \otimes V(\mu)$ is a unitarizable Vir representation with nonnegative central charge

$$(\dim \hat{\mathfrak{g}}) \left(\frac{l}{l+h^\vee} + \frac{m}{m+h^\vee} - \frac{l+m}{l+m+h^\vee} \right),$$

where h^\vee is the dual Coxeter number of \mathfrak{g} ([Kac, §6.1]).

- (2) L_0 acts on $V(\lambda) \otimes V(\mu)$ by

$$\frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l+h^\vee} + \frac{(\mu|\mu + 2\rho)}{m+h^\vee} - \frac{\Omega}{l+m+h^\vee} \right),$$

where Ω is the Casimir operator of \mathfrak{g} ([Ku3, §1.5]) and $(\cdot|\cdot)$ is the normalized form on \mathfrak{h}^* as in [Ku3, Lemma 13.1.8].

- (3) For all k , $[L_k, \mathfrak{g}'] = 0$; i.e., the L_k are intertwining operators for the representation of \mathfrak{g}' on $V(\lambda) \otimes V(\mu)$.

Remark 3.6. As the Casimir Ω acts on the \mathfrak{g} -module $V(\nu)$ by $(\nu|\nu + 2\rho)$ (cf., [Ku3, 2.1.16]), we can easily compute the action of L_0 on any component $V(\nu)$ of $V(\lambda) \otimes V(\mu)$; it will act via a scalar depending only on λ, μ, ν .

Now, let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal component as in Definition 2.2 and consider the subspace W^ν as in Definition 2.3. By Proposition 3.5(3), we can conclude the following immediate corollary.

Corollary 3.7. W^ν is a unitarizable Vir representation.

In general, W^ν is not an irreducible Vir representation. However, by Remark 3.6, L_0 acts on each $V(\nu - k\delta)$ as a scalar, so that each of these summands corresponds to a single Vir weight space.

Lemma 3.8. For any $k \in \mathbb{Z}$, $L_{-k} : V(\nu) \rightarrow V(\nu - k\delta)$, corresponding to shifting the Vir weight by kh , where ν is not necessarily a δ -maximal weight.

Proof. As all of W^ν has the same nonnegative central charge, we need only compute the change in the weight with respect to h . By Proposition 3.5 and Remark 3.6, we have that L_0 acts on $V(\nu - k\delta)$ via

$$\begin{aligned} & \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu - k\delta|\nu - k\delta + 2\rho)}{l + m + h^\vee} \right) \\ &= \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu|\nu + 2\rho) - 2k(\nu|\delta) - 2k(\rho|\delta)}{l + m + h^\vee} \right) \\ &= \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu|\nu + 2\rho) - 2k(l + m) - 2k(h^\vee)}{l + m + h^\vee} \right) \\ &= \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu|\nu + 2\rho)}{l + m + h^\vee} \right) + k. \end{aligned}$$

Now, as L_0 acts on $V(\nu)$ by $\frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu|\nu + 2\rho)}{l + m + h^\vee} \right)$ and we have that L_{-k} adds kh to a weight space, we get the desired result. \square

Therefore, combining Lemmas 3.8 and 3.4 we get the following key proposition, which allows us to determine if the multiplicities $m_k = m_{\lambda, \mu}^{\nu - k\delta}$ as in Definition 2.3 are nonzero.

Proposition 3.9. *Let $\lambda, \mu \in \mathcal{P}^+$ be of positive levels l, m , respectively. Let $V(\nu) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal component, and consider the *Vir* subrepresentation W^ν . Then, we have the following:*

- (1) *If $\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\nu|\nu + 2\rho)}{l + m + h^\vee} \neq 0$, we have $m_k \geq 1$ for all $k \geq 0$.*
- (2) *Else, we have $m_1 = 0$ and $m_k \geq 1$ for $k = 0, k \geq 2$.*

Proof. Choose a nonzero highest weight vector $v_\nu \in V(\nu) \subset W^\nu$. By Lemma 3.8, we have $L_k(v_\nu) = 0$ for all $k > 0$, as there are no components of the form $V(\nu + k\delta)$ in W^ν . Thus, v_ν is a highest weight vector for the *Vir* action, and generates a highest weight, unitarizable (irreducible) *Vir*-submodule of W^ν . Applying Lemma 3.4, we can determine if $L_{-k}(v_\nu)$ is nonzero by computing the L_0 -action on v_ν (done in the proof of Lemma 3.8). But $L_{-k}(v_\nu) \in V(\nu - k\delta)$ by Lemma 3.8 and if nonzero it is the highest weight vector of $V(\nu - k\delta)$ under the $U(\mathfrak{g}')$ action, since $[L_{-k}, \mathfrak{g}'] = 0$. Therefore, Lemma 3.4 gives the result. (Observe that c acts by a nonzero scalar on $V(\lambda) \otimes V(\mu)$ by Proposition 3.5(1).) \square

We conclude this section with two examples; the first is an alternate approach to [Kac, Exercise 12.16], and the second is an interpretation of root components for affine Lie algebras with respect to imaginary roots.

Example 3.10. Consider $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, and let $\lambda = \mu = \Lambda_0$. Then, of course $V(2\Lambda_0) \subset V(\Lambda_0) \otimes V(\Lambda_0)$ is a δ -maximal component. By Proposition 3.5 and Remark 3.6, the L_0 action on $V(2\Lambda_0)$ is given by the scalar:

$$\frac{1}{2} \left(\frac{(\Lambda_0|\Lambda_0 + 2\rho)}{3} + \frac{(\Lambda_0|\Lambda_0 + 2\rho)}{3} - \frac{(2\Lambda_0|2\Lambda_0 + 2\rho)}{4} \right).$$

But we have $(\Lambda_0|\Lambda_0) = (\Lambda_0|\rho) = 0$ (cf., [Kac, §6.2]), so that L_0 acts by 0 on $V(2\Lambda_0)$. Thus, by Proposition 3.9, we have $V(2\Lambda_0 - \delta) \not\subset V(\Lambda_0) \otimes V(\Lambda_0)$. But $V(2\Lambda_0 - k\delta) \subset V(\Lambda_0) \otimes V(\Lambda_0)$ for any $k > 1$; this agrees with Exercise 12.16 of [Kac].

Example 3.11. For any affine Lie algebra \mathfrak{g} , let $\lambda = \mu = \rho$. Then, of course, $V(2\rho) \subset V(\rho) \otimes V(\rho)$ is δ -maximal. We then get, since ρ is of level h^\vee , that L_0 acts on $V(2\rho)$ via

$$\frac{1}{2} \left(\frac{2(\rho|\rho + 2\rho)}{2h^\vee} - \frac{(2\rho|2\rho + 2\rho)}{3h^\vee} \right) = \frac{(\rho|\rho)}{6h^\vee} > 0.$$

Thus, by Proposition 3.9, we have $V(2\rho - k\delta) \subset V(\rho) \otimes V(\rho)$ for any $k \geq 0$.

4. SOME GENERAL RESULTS ON TENSOR PRODUCT DECOMPOSITION

Proposition 4.1 is crucial to our work on root components in the case of affine Lie algebras. For the semisimple Lie algebras, it is due to Kostant [Kos].

Proposition 4.1. *Let \mathfrak{g} be a symmetrizable Kac-Moody algebra, and fix $\lambda, \mu, \nu \in \mathcal{P}^+$. Then, the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$ is given by:*

$$m'_{\lambda, \mu} = \dim\{v \in V(\mu)_{\nu-\lambda} : e_i^{\lambda(\alpha_i^\vee)+1}.v = 0 \text{ for all simple roots } \alpha_i\},$$

where e_i is a root vector corresponding to the root α_i .

Proof. Define an increasing filtration $\{F_n(\lambda)\}_{n \geq 0}$ of $V(\lambda)$ by

$$F_n(\lambda) := \sum_{\beta \in Q^+ : |\beta| \leq n} V(\lambda)_{\lambda-\beta},$$

where $Q^+ := \oplus_i \mathbb{Z}_{\geq 0} \alpha_i$, for any $\beta = \sum n_i \alpha_i$, we define $|\beta| := \sum n_i$ and $V(\lambda)_\gamma$ denotes the γ -weight space. Then, clearly, each $F_n(\lambda)$ is \mathfrak{b} -stable. We also set

$$V(\lambda)^\vee := \oplus_{\beta \in Q^+} (V(\lambda)_{\lambda-\beta})^*.$$

Let \mathbb{C}_ν denote the one-dimensional \mathfrak{b} -module with the Cartan subalgebra \mathfrak{h} acting via the weight ν and of course the commutator $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ acts trivially. Now,

$$\begin{aligned} \text{Hom}_{\mathfrak{g}}(V(\nu), V(\lambda) \otimes V(\mu)) &\simeq \text{Hom}_{\mathfrak{b}}(\mathbb{C}_\nu, V(\lambda) \otimes V(\mu)) \\ &\simeq \varinjlim_n \text{Hom}_{\mathfrak{b}}(\mathbb{C}_\nu, F_n(\lambda) \otimes V(\mu)) \\ &\simeq \varinjlim_n \text{Hom}_{\mathfrak{b}}(F_n(\lambda)^* \otimes \mathbb{C}_\nu, V(\mu)), \\ &\quad \text{since } F_n(\lambda) \text{ is finite-dimensional} \\ (3) \quad &\simeq \text{Hom}_{\mathfrak{b}}(V(\lambda)^\vee \otimes \mathbb{C}_\nu, V(\mu)), \end{aligned}$$

where the last isomorphism is induced from the surjection $V(\lambda)^\vee \rightarrow F_n(\lambda)^*$ obtained from dualizing the embedding $F_n(\lambda) \hookrightarrow V(\lambda)$. It is indeed an isomorphism since any \mathfrak{b} -module homomorphism $f : V(\lambda)^\vee \otimes \mathbb{C}_\nu \rightarrow V(\mu)$ clearly descends to a homomorphism $\bar{f} : F_N(\lambda)^* \otimes \mathbb{C}_\nu \rightarrow V(\mu)$ for some large N because the weights of $V(\lambda)^\vee \otimes \mathbb{C}_\nu$ lie in $-\lambda + \nu + Q^+$ whereas the weights of $V(\mu)$ lie in $\mu - Q^+$.

From the definition of the integrable highest weight module (cf. [Ku3, Definition 2.1.5 and Corollary 2.2.6]) (rather its analogue for the integrable lowest module $V(\lambda)^\vee$), we get that $V(\lambda)^\vee$ is obtained from the Verma module for the negative Borel $M_{\mathfrak{b}^-}(-\lambda)$ (with lowest weight $-\lambda$) by taking its quotient by the submodule generated by $\{e_i^{\lambda(\alpha_i^\vee)+1}.v_{-\lambda}\}$, where e_i runs over all the (positive) simple root vectors and $v_{-\lambda}$ is the lowest weight vector of $M_{\mathfrak{b}^-}(-\lambda)$. In particular, the map

$$U(\mathfrak{n}) \rightarrow V(\lambda)^\vee, \quad a \mapsto a \cdot v_{-\lambda},$$

is surjective with kernel the left ideal generated by $\{e_i^{\lambda(\alpha_i^\vee)+1}\}_i$. From this and the above isomorphism (3), we get the proposition. \square

The above result also follows from the work of Kashiwara [Kas1], [Kas2] on the existence of crystal base and global base for quantum group representations.

We can deduce from Proposition 4.1 the following well-known fact, which we refer to as *additivity in the tensor decomposition*. Following [Ku1, Corollary 1.5], we give a purely algebraic proof, but remark that this can also be deduced from the Kac-Moody analogue of the Borel-Weil theorem.

Corollary 4.2. *Let $\lambda, \lambda', \mu, \mu', \nu \in \mathcal{P}^+$. Then, $m_{\lambda, \mu}^\nu \leq m_{\lambda+\lambda', \mu+\mu'}^{\nu+\lambda'+\mu'}$.*

Proof. Define a map $\xi : V(\mu) \rightarrow V(\mu + \mu')$ via $\xi(v) = \pi(v \otimes v_{\mu'})$, where $v_{\mu'}$ is a nonzero highest weight vector of $V(\mu')$ and $\pi : V(\mu) \otimes V(\mu') \rightarrow V(\mu + \mu')$ is the projection onto the Cartan component. For any nonzero $v \in V(\mu)$, $v_\mu \otimes v_{\mu'} \in U(\mathfrak{g}) \cdot (v \otimes v_{\mu'})$, so the map ξ is clearly injective. Now, by Proposition 4.1, choose a vector $v \in V(\mu)_{\nu-\lambda}$ satisfying $e_i^{\lambda(\alpha_i^\vee)+1} \cdot v = 0$ for all the simple roots α_i . Then, $\xi(v) \in V(\mu + \mu')_{\nu+\mu'-\lambda}$ and satisfies $e_i^{\lambda(\alpha_i^\vee)+1} \cdot (\xi(v)) = 0$. Thus, by Proposition 4.1, we get $m_{\lambda, \mu}^\nu \leq m_{\lambda+\mu', \mu+\mu'}^{\nu+\mu'}$. Repeating the argument a second time for λ and λ' gives the result. \square

5. EXISTENCE OF ROOT COMPONENTS FOR IMAGINARY ROOTS

Now, let $\beta = k\delta$ be a positive imaginary root. By the regularity condition (P2) for root components as in Section 1, if (λ, μ, β) is a Wahl triple, then necessarily λ and μ are both regular dominant weights. Combining Example 3.11 and additivity in the tensor decomposition (Corollary 4.2), we get the first example of root components via Corollary 5.1.

Corollary 5.1. *Let $\beta = k\delta$ be a positive imaginary root, and (λ, μ, β) a Wahl triple. Then, $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$.*

Remark 5.2. In fact, the stronger statement that $V(\lambda + \mu - k\delta) \subset V(\lambda) \otimes V(\mu)$ is true if **at least one** of λ or μ is regular dominant follows immediately from the same approach as in Example 3.11.

6. EXISTENCE OF ROOT COMPONENTS FOR $\beta = \gamma + k\delta$, FOR $\gamma \in \mathring{\Phi}^+$

Fix for the remainder of this section a simple untwisted affine Lie algebra \mathfrak{g} and a Wahl triple $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi_{Re}^+$. Without loss of generality, assume that $\lambda(d) = \mu(d) = 0$. Let l, m be the levels of λ and μ , respectively. Then, by Proposition 3.5(2), we know precisely how $L_0 \in Vir$ acts on $V(\lambda) \otimes V(\mu)$ via the GKO construction. The goal of this section is to understand how this action interacts with potential root components $V(\lambda + \mu - \beta)$. As a first step, we consider the Cartan component $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$ in Lemma 6.1.

Lemma 6.1. *For $\lambda, \mu \in \mathcal{P}^+$ with levels l, m , respectively, we have*

$$\frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\lambda + \mu|\lambda + \mu + 2\rho)}{l + m + h^\vee} \right) \geq 0.$$

Proof. Let $v := v_\lambda \otimes v_\mu \in V(\lambda) \otimes V(\mu)$, where v_λ, v_μ are nonzero highest weight vectors in $V(\lambda), V(\mu)$, respectively. Then, v is the highest weight vector of the

Cartan component $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$. Now, consider the Virasoro action $Vir.v$ coming from the GKO construction. As $V(\lambda + \mu)$ is a δ -maximal component of $V(\lambda) \otimes V(\mu)$, by Lemma 3.8, $L_k.v = 0$ for all $k \geq 1$. So, consider the highest weight, unitarizable (irreducible) Virasoro submodule of $V(\lambda) \otimes V(\mu)$ generated by v . By Proposition 3.5, we know that the action of L_0 on v is given precisely by the scalar value above. Further, by the proof of Lemma 3.4, we know that this value must be nonnegative. \square

Now, we can obtain from this the following key proposition pertaining to Wahl triples.

Proposition 6.2. *Let $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi_{Re}^+$ be a Wahl triple. Then,*

$$\frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\lambda + \mu - \beta|\lambda + \mu - \beta + 2\rho)}{l + m + h^\vee} \right) > 0.$$

Proof. We use the bilinearity of the form $(\cdot|\cdot)$ to expand the third term as

$$\frac{(\lambda + \mu - \beta|\lambda + \mu - \beta + 2\rho)}{l + m + h^\vee} = \frac{(\lambda + \mu|\lambda + \mu + 2\rho)}{l + m + h^\vee} - \frac{2(\lambda + \mu|\beta) + 2(\rho|\beta) - (\beta|\beta)}{l + m + h^\vee}.$$

In total, we therefore have

$$\begin{aligned} & \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\lambda + \mu - \beta|\lambda + \mu - \beta + 2\rho)}{l + m + h^\vee} \right) \\ &= \frac{1}{2} \left(\frac{(\lambda|\lambda + 2\rho)}{l + h^\vee} + \frac{(\mu|\mu + 2\rho)}{m + h^\vee} - \frac{(\lambda + \mu|\lambda + \mu + 2\rho)}{l + m + h^\vee} \right) \\ & \quad + \frac{1}{2} \left(\frac{2(\lambda + \mu|\beta) + 2(\rho|\beta) - (\beta|\beta)}{l + m + h^\vee} \right). \end{aligned}$$

By Lemma 6.1, we know that the first term is nonnegative. Further, we know that $2(\rho|\beta) - (\beta|\beta) \geq 0$, as $1 \leq \rho(\beta^\vee) = \frac{2(\rho|\beta)}{(\beta|\beta)}$. Finally, by the property (P1) of the Wahl triple, we have $(\lambda + \mu|\beta) > 0$ (since $\lambda + \mu - \beta \in \mathcal{P}^+$, we get $(\lambda + \mu - \beta|\beta) \geq 0$ and hence $(\lambda + \mu|\beta) > 0$), so that in total the expression as a whole is strictly positive. \square

As a consequence of Propositions 6.2 and 3.9, we get Corollary 6.3.

Corollary 6.3. *Let $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi_{Re}^+$ be a Wahl triple and let $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal root component. In particular, λ, μ have strictly positive levels. Then, for any $k \geq 0$, we have $V(\lambda + \mu - \beta - k\delta) \subset V(\lambda) \otimes V(\mu)$.*

Proof. By Propositions 6.2, 3.5 and Remark 3.6, L_0 acts on $V(\lambda + \mu - \beta)$ via a scalar > 0 . Thus, by Proposition 3.9,

$$V(\lambda + \mu - \beta - k\delta) \subset V(\lambda) \otimes V(\mu), \text{ for all } k \geq 0.$$

This proves the corollary. \square

Note that while Corollary 6.3 allows us to conclude the existence of infinitely many root components from the existence of its associated δ -maximal component, the action of the Virasoro does not *a priori* say anything about the appearance of the δ -maximal root components in the tensor product decomposition. Nevertheless, to conclude the proof of Theorem I of Section 1, our work is reduced by Corollary 6.3 to showing the existence of these δ -maximal root components.

Recall that, for any $\beta \in \Phi_{Re}^+$, we can write $\beta = \gamma + k\delta$ for some $\gamma \in \mathring{\Phi}$ and $k \in \mathbb{Z}_{\geq 0}$. This can further be separated into two cases: If $\gamma \in \mathring{\Phi}^+$, then we can take any $k \geq 0$. If $\gamma \in \mathring{\Phi}^-$, then necessarily $k \geq 1$. Therefore, when considering the δ -maximal root components $V(\lambda + \mu - \beta)$, it suffices to consider the two possibilities: $V(\lambda + \mu - \gamma)$ and $V(\lambda + \mu - (-\gamma + \delta))$, where $\gamma \in \mathring{\Phi}^+$. The first of these cases follows immediately from Kumar’s result in the finite case [Kul].

Proposition 6.4. *Let (λ, μ, γ) be a Wahl triple with $\gamma \in \mathring{\Phi}^+$. Then, $m_{\lambda, \mu}^{\lambda + \mu - \gamma} \neq 0$.*

Proof. Let $\mathring{V}(\lambda) \subset V(\lambda)$ denote the $\mathring{\mathfrak{g}}$ -submodule generated by the highest weight vector, and similarly for $\mathring{V}(\mu)$. Then, by [Kul, Theorem 1.1], we have

$$\mathring{V}(\lambda + \mu - \gamma) \subset \mathring{V}(\lambda) \otimes \mathring{V}(\mu).$$

Denote by $v_\gamma \in V(\lambda) \otimes V(\mu)$ the highest weight vector for $\mathring{\mathfrak{g}}$ that generates $\mathring{V}(\lambda + \mu - \gamma)$. We claim that this is in fact a highest weight vector for \mathfrak{g} . Indeed, all that remains to check is that $e_0.v_\gamma = 0$. By the assumption that $\lambda(d) = \mu(d) = 0$, we have that no vector in $V(\lambda) \otimes V(\mu)$ can have weight ν with $\nu(d) \geq 1$. However, since $\gamma \in \mathring{\Phi}^+$, if $e_0.v_\gamma \neq 0$, we have $(\lambda + \mu - \gamma + \alpha_0)(d) = \alpha_0(d) = 1$, a contradiction. Thus, $e_0.v_\gamma = 0$ and generates a \mathfrak{g} -submodule $V(\lambda + \mu - \gamma) \subset V(\lambda) \otimes V(\mu)$. \square

From this, via Corollary 6.3, we have Corollary 6.5.

Corollary 6.5. *Let (λ, μ, β) be a Wahl triple with $\beta = \gamma + k\delta$ for some $\gamma \in \mathring{\Phi}^+$ and $k \geq 0$. Then, $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$.*

7. EXISTENCE OF ROOT COMPONENTS FOR $\beta = \gamma + k\delta$, FOR $\gamma \in \mathring{\Phi}^-$

We now show the existence of the δ -maximal root components of the form $V(\lambda + \mu - (-\gamma + \delta))$ for $\gamma \in \mathring{\Phi}^+$. The results and arguments of this section parallel those in [Kul]; we reproduce some of the original arguments therein here for completeness.

Recall from Proposition 4.1 that

$$m_{\lambda, \mu}^{\lambda + \mu - \beta} = \dim\{v \in V(\mu)_{\mu - \beta} : e_i^{\lambda(\alpha_i^\vee) + 1}.v = 0 \text{ for all the simple roots } \alpha_i\}.$$

Our goal is to explicitly construct such a vector $v \in V(\mu)_{\mu - \beta}$ for $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi_{Re}^+$ a Wahl triple. To do so, we begin with the following preparatory lemma.

Lemma 7.1. *Let $\beta \in \Phi_{Re}^+$, with $\beta = \gamma + k\delta$ for some $\gamma \in \mathring{\Phi}$ and $k \in \mathbb{Z}_{\geq 0}$, and suppose $\beta - 2\alpha_i$ is a root for some $0 \leq i \leq \ell$. Then, neither of $\beta - 2\alpha_j$ or $\beta - 2\alpha_j - \alpha_i$ is a root or zero for $j \neq i$.*

Proof. If $\beta \in \mathring{\Phi}^+$ (that is, if $k = 0$), then the lemma holds by explicit knowledge of the finite root systems; see [Bou81] for example. Else, first suppose $\beta - 2\alpha_0$ is a root. Then, we have $\beta - 2(\delta - \theta) = (\gamma + 2\theta) + (k - 2)\delta$ is a root, where $\theta \in \mathring{\Phi}^+$ is the highest root of the underlying semisimple Lie algebra. Thus, $\beta - 2\alpha_0$ is a root if and only if $\gamma + 2\theta$ is a root in $\mathring{\Phi}$, so that $\gamma = -\theta$. In this case, neither of $\beta - 2\alpha_j$ or $\beta - 2\alpha_j - \alpha_0$ is a root for any $j \neq 0$, as the first would mean that $-\theta - 2\alpha_j \in \mathring{\Phi}$, and the second would mean that $-2\alpha_j \in \mathring{\Phi}$, a clear contradiction. Also, clearly, $\beta - 2\alpha_j$ or $\beta - 2\alpha_j - \alpha_0$ cannot be zero.

Now, suppose $\beta - 2\alpha_i$ is a root for $i \neq 0$. Then, we have $(\gamma - 2\alpha_i) + k\delta$ is a root, so that $\gamma - 2\alpha_i \in \mathring{\Phi}$. By the finite case, we get that $\beta - 2\alpha_j$ and $\beta - 2\alpha_j - \alpha_i$ are not roots or zero for $j \neq 0, i$, as this would have to hold for γ . (For γ positive,

we have seen this above and for γ negative, we can similarly see using the explicit knowledge of the root system as in [Bou81].) As above, $\beta - 2\alpha_0 \in \Phi$ would imply that $\gamma = -\theta$, contradicting $\beta - 2\alpha_i$ is a root. Finally, $\beta - 2\alpha_0 - \alpha_i \in \Phi$ would imply that $\gamma - \alpha_i + 2\theta \leq \theta$, so that $\gamma = -\theta$ or $\gamma = -\theta + \alpha_i$; in each of these cases, $\gamma - 2\alpha_i$ is not a root, contradicting $\beta - 2\alpha_i$ being a root. Further, it is easy to see that $\beta - 2\alpha_j - \alpha_i$ cannot be zero for any $j \neq i$. \square

We next introduce the following set of indices, which will play a role in the construction of sufficient Wahl triples associated to a positive root β .

Definition 7.2. For $\beta \in \Phi^+$, define $F_\beta := \{0 \leq i \leq \ell : \beta - \alpha_i \notin \Phi \sqcup \{0\}\}$.

Proposition 7.3. Let (λ, μ, β) be a Wahl triple with β a real root, and suppose that $\beta - 2\alpha_i \notin \Phi^+$ for any $0 \leq i \leq \ell$, or else $\mu(\beta^\vee) = 1$. Then, $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$.

Thus, for simply-laced \mathfrak{g} (i.e., $\mathring{\mathfrak{g}}$ is simply-laced), for any Wahl triple,

$$(4) \quad V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu).$$

Proof. Let v_μ denote a nonzero highest weight vector of $V(\mu)$, and choose a nonzero root vector $X_{-\beta} \in \mathfrak{g}_{-\beta}$. Set $v := X_{-\beta}.v_\mu$. For any $0 \neq X_\beta \in \mathfrak{g}_\beta$, we have (using [Ku3, Theorem 1.5.4]),

$$X_\beta.v = X_\beta(X_{-\beta}.v_\mu) = [X_\beta, X_{-\beta}]v_\mu = (X_\beta|X_{-\beta})\nu^{-1}(\beta).v_\mu = (X_\beta|X_{-\beta})\mu(\nu^{-1}(\beta))v_\mu,$$

where $\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*$ is the map induced by $(\cdot|\cdot)$. This is nonzero as $(X_\beta|X_{-\beta}) \neq 0$ and by the regularity condition (P2), $\mu(\nu^{-1}(\beta)) > 0$. Then, $v \in V(\mu)_{\mu-\beta}$ is a nonzero vector.

By the dominance of λ we have $\lambda(\alpha_i^\vee) + 1 \geq 1$. Then, if $i \in F_\beta$, we get

$$e_i.v = e_i(X_{-\beta}.v_\mu) = [e_i, X_{-\beta}].v_\mu = 0,$$

as by definition $\alpha_i - \beta \notin \Phi \cup \{0\}$. Else, if $i \notin F_\beta$, we have by the regularity condition (P2) that $\lambda(\alpha_i^\vee) + 1 \geq 2$, and

$$e_i^2.v = ((\text{ad}_{e_i})^2(X_{-\beta})).v_\mu = 0$$

by the assumption that $\beta - 2\alpha_i \notin \Phi^+$. Thus, for all i , $e_i^{\lambda(\alpha_i^\vee)+1}.v = 0$. This proves the proposition in the case $\beta - 2\alpha_i \notin \Phi^+$ for any i by using Proposition 4.1.

For the proof in the case $\mu(\beta^\vee) = 1$, observe first that for any weight ν of $V(\mu)$, $(\mu|\mu) \geq (\nu|\nu)$ (cf. [Kac, Proposition 11.4 (a)]). Denoting $n := \lambda(\alpha_i^\vee) + 1$, we have:

$$\begin{aligned} (\mu - \beta + n\alpha_i|\mu - \beta + n\alpha_i) &= (\mu|\mu) + n^2(\alpha_i|\alpha_i) + 2n(\mu - \beta|\alpha_i), \text{ since } s_\beta\mu = \mu - \beta \\ &= (\mu|\mu) + n^2(\alpha_i|\alpha_i) + n(\alpha_i|\alpha_i)(\mu - \beta|\alpha_i^\vee) \\ &= (\mu|\mu) + n(\alpha_i|\alpha_i)((\lambda + \mu - \beta|\alpha_i^\vee) + 1) \\ &> (\mu|\mu), \text{ since } \lambda + \mu - \beta \in \mathcal{P}^+ \text{ by (P1)}. \end{aligned}$$

Thus, $V(\mu)_{\mu-\beta+n\alpha_i} = 0$.

To prove equation (4), by Corollaries 5.1, 6.3 and 6.5, it suffices to assume that $\beta = \delta - \gamma$, for a positive root $\gamma \in \mathring{\Phi}^+$. Any such root β does satisfy the condition that $\beta - 2\alpha_i \notin \Phi^+$ for any simple root α_i . Hence the first part of the proposition proves (4). \square

Remark 7.4. This proof holds more generally for any symmetrizable Kac-Moody algebra \mathfrak{g} and triple (λ, μ, β) satisfying (P1) and (P2) under the condition on β as in Proposition 7.3.

Next, we consider triples (λ, μ, β) where β is a real root with $\beta - 2\alpha_i \in \Phi^+$ for some $0 \leq i \leq \ell$. By Lemma 7.1, this i is unique. For technical reasons, we assume that $\mathfrak{g} \neq G_2^{(1)}$; this will be handled separately.

Proposition 7.5. *Let $\mathfrak{g} \neq G_2^{(1)}$, and let (λ, μ, β) be a Wahl triple with β a real root. Suppose α_i is the unique simple root such that $\beta - 2\alpha_i \in \Phi^+$. In particular, $\mu(\beta^\vee) \geq 1$ by the condition (P2). Then, $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ assuming at least one of the following:*

- (1) $\lambda \in \mathcal{P}^+$ is **regular dominant**, or
- (2) $F_\beta = F_{\beta - \alpha_i}$.

Proof. Again, let $v_\mu \in V(\mu)$ denote a nonzero highest weight vector, $X_{-\beta} \in \mathfrak{g}_{-\beta}$ a nonzero root vector, and define $X_{-\beta + \alpha_i} := [e_i, X_{-\beta}]$, $X_{-\beta + 2\alpha_i} := [e_i, X_{-\beta + \alpha_i}]$. Since $\mathfrak{g} \neq G_2^{(1)}$, we have that $|\beta(\alpha_i^\vee)| \leq 2$, so that $\beta - 3\alpha_i, \beta + \alpha_i \notin \Phi$. Set

$$v := (X_{-\beta + \alpha_i} f_i - 2\mu(\alpha_i^\vee) X_{-\beta}) \cdot v_\mu \in V(\mu)_{\mu - \beta}.$$

Then, by a similar computation as in the proof of Proposition 7.3 (making use of the condition (P2) since $i \notin F_\beta$), we have $X_\beta \cdot v \neq 0$ for any $0 \neq X_\beta \in \mathfrak{g}_\beta$, so that $v \neq 0$. (Since $\beta + \alpha_i \notin \Phi$, $\beta(\alpha_i^\vee) \geq 0$.) Now, for any $j \neq i$, we have

$$\begin{aligned} e_j^2 \cdot v &= e_j^2(X_{-\beta + \alpha_i} f_i \cdot v_\mu) - 2\mu(\alpha_i^\vee) e_j^2(X_{-\beta} \cdot v_\mu) \\ &= ((\text{ad}_{e_j})^2(X_{-\beta + \alpha_i}) f_i) \cdot v_\mu - 2\mu(\alpha_i^\vee) ((\text{ad}_{e_j})^2(X_{-\beta})) \cdot v_\mu \\ &= 0 \end{aligned}$$

as $(\text{ad}_{e_j})^2(X_{-\beta + \alpha_i}) = 0$ and $(\text{ad}_{e_j})^2(X_{-\beta}) = 0$ by Lemma 7.1. Further, we have

$$\begin{aligned} e_i \cdot v &= X_{-\beta + 2\alpha_i} f_i v_\mu + \mu(\alpha_i^\vee) X_{-\beta + \alpha_i} v_\mu - 2\mu(\alpha_i^\vee) X_{-\beta + \alpha_i} v_\mu \\ &= X_{-\beta + 2\alpha_i} f_i v_\mu - \mu(\alpha_i^\vee) X_{-\beta + \alpha_i} v_\mu, \\ \implies e_i^2 \cdot v &= \mu(\alpha_i^\vee) X_{-\beta + 2\alpha_i} v_\mu - \mu(\alpha_i^\vee) X_{-\beta + 2\alpha_i} v_\mu \\ &= 0. \end{aligned}$$

Therefore $e_j^2 \cdot v = 0$ for all j .

If $\lambda \in \mathcal{P}^+$ is regular dominant, then $\lambda(\alpha_j^\vee) + 1 \geq 2$ for all j . So, by Proposition 4.1 we get $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$.

Else, for any j such that $\lambda(\alpha_j^\vee) = 0$, by condition (P2) we have $j \in F_\beta$; note that $i \notin F_\beta$. Thus, if $F_\beta = F_{\beta - \alpha_i}$, we have

$$e_j \cdot v = [e_j, X_{-\beta + \alpha_i}] f_i \cdot v_\mu - 2\mu(\alpha_i^\vee) [e_j, X_{-\beta}] \cdot v_\mu = 0,$$

so, again by Proposition 4.1, we have $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$. □

To end this section, recall the set of indices F_β for $\beta \in \Phi^+$ from Definition 7.2. The following dominant weights will play a crucial role for the remaining constructions.

Definition 7.6. Given $\beta \in \Phi_{Re}^+$, define $\rho_\beta := \sum_{i \notin F_\beta} \Lambda_i$.

By construction, the weight ρ_β satisfies condition (P2) for the root β and is the *minimal* dominant weight with $\rho_\beta(d) = 0$ that can do so. Therefore, we can conclude Lemma 7.7 by the additivity property of the tensor decomposition Corollary 4.2.

Lemma 7.7. *If $2\rho_\beta - \beta \in \mathcal{P}^+$ is such that $m_{\rho_\beta, \rho_\beta}^{2\rho_\beta - \beta} \neq 0$, then $m_{\lambda, \mu}^{\lambda + \mu - \beta} \neq 0$ for any Wahl triple (λ, μ, β) .*

Further, it is easy to see that $2\rho_\beta - \beta \in \mathcal{P}^+$ for any \mathfrak{g} except possibly for G_2 -type \mathfrak{g} .

8. EXCEPTIONAL MAXIMAL COMPONENTS

In light of Corollaries 5.1, 6.3 and 6.5, all that remains to show to complete the proof of Theorem I is the existence of root components $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ for $\beta = -\gamma + \delta$, where $\gamma \in \mathring{\Phi}^+$. We can reduce further, by making use of Proposition 7.3 and Lemma 7.1, to the case that $\beta - 2\alpha_i \in \mathring{\Phi}^+$ for some unique i (in particular, \mathfrak{g} is not simply-laced). We can exclude the case $i = 0$, via the following easy lemma.

Lemma 8.1. *Let (λ, μ, β) be a Wahl triple such that $\beta = -\gamma + k\delta$ for some $\gamma \in \mathring{\Phi}^+$ and $k \geq 1$ and such that $\beta - 2\alpha_0 \in \mathring{\Phi}$. Then, $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$.*

Proof. Direct computation shows that

$$\beta - 2\alpha_0 = -\gamma + k\delta - 2\alpha_0 = 2\theta - \gamma + (k - 2)\delta.$$

This is a root if and only if $2\theta - \gamma \in \mathring{\Phi}$, so necessarily $\gamma = \theta$. But then we have $\beta = \alpha_0 + (k - 1)\delta$. Thus, we can reduce to the δ -maximal component associated to $\beta = \alpha_0$; the existence of $V(\lambda + \mu - \alpha_0)$ in $V(\lambda) \otimes V(\mu)$ is clear by taking any linear combination: $c(f_0 \cdot v_\lambda) \otimes v_\mu + dv_\lambda \otimes f_0 \cdot v_\mu$, for $c, d \in \mathbb{C}$. □

The remaining options for such $\beta = -\gamma + \delta$ (where $\gamma \in \mathring{\Phi}^+$), with $i \neq 0$, correspond to roots $\gamma \in \mathring{\Phi}^+$ such that $\gamma + 2\alpha_i \in \mathring{\Phi}^+$. We collect these possibilities, organized by the type \mathfrak{g} , $\gamma \in \mathring{\Phi}^+$, and the associated unique simple root denoted $\alpha_i(\gamma)$, in Table 1.

TABLE 1. Exceptional components

\mathfrak{g}	γ	$\alpha_i(\gamma)$
$B_\ell^{(1)}$	$\sum_{i \leq k < \ell} \alpha_k, 1 \leq i < \ell$	α_ℓ
$C_\ell^{(1)}$	$(2 \sum_{i \leq k < \ell} \alpha_k) + \alpha_\ell, 2 \leq i \leq \ell$	α_{i-1}
$F_4^{(1)}$	α_2	α_3
$F_4^{(1)}$	$\alpha_1 + \alpha_2$	α_3
$F_4^{(1)}$	$\alpha_2 + 2\alpha_3$	α_4
$F_4^{(1)}$	$\alpha_1 + \alpha_2 + 2\alpha_3$	α_4
$F_4^{(1)}$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$	α_4
$F_4^{(1)}$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	α_3
$G_2^{(1)}$	α_2	α_1
$G_2^{(1)}$	$\alpha_1 + \alpha_2$	α_1

We refer to these roots β and their associated root components as “exceptional,” since root behavior of these does not appear in the semisimple setting and the methods to demonstrate their existence in the tensor decomposition do not uniformly fit into the general approach of [Ku1]. Instead, we will primarily make use of the PRV components to construct these final δ -maximal components. *Throughout the*

remainder of this section, we deal with Wahl triples $(\lambda, \mu, \beta = \delta - \gamma)$ for each specified γ in Table 1. We first recall the following results:

Theorem 8.2 ([Ku2, Theorem 3.7], [M2, Corollaire 3] (PRV components)). *Let \mathfrak{g} be any symmetrizable Kac-Moody Lie algebra. For any $\lambda, \mu \in \mathcal{P}^+$ and the Weyl group elements $v, w \in W$ such that $\eta := v\lambda + w\mu \in \mathcal{P}^+$, we have $V(\eta) \subset V(\lambda) \otimes V(\mu)$.*

8.1. $B_\ell^{(1)}$. For $\gamma = \sum_{i \leq k < \ell} \alpha_k$, $1 \leq i < \ell$, $\beta = -\gamma + \delta$, we have α_ℓ as the unique simple root such that $\beta - 2\alpha_\ell \in \Phi^+$. Setting $[\ell] := \{0, 1, 2, \dots, \ell\}$, we have:

$$F_\beta = \begin{cases} [\ell] \setminus \{i - 1, \ell\} & i \neq 2, \\ [\ell] \setminus \{0, 1, \ell\} & i = 2. \end{cases}$$

Comparing this to $F_{\beta - \alpha_\ell}$, we see that $F_\beta = F_{\beta - \alpha_\ell}$ in all the cases; thus we can apply Proposition 7.5.

8.2. $C_\ell^{(1)}$. For any $\gamma = \gamma_i = (2 \sum_{i \leq k < \ell} \alpha_k) + \alpha_\ell$ ($2 \leq i \leq \ell$) in Table 1, $F_{\beta_i} = [\ell] \setminus \{i - 1\}$, where $\beta_i := \delta - \gamma_i$. In this case $\rho_{\beta_i}(\beta_i^\vee) = 1$. Thus, Proposition 7.3 and Lemma 7.7 take care of this case.

8.3. $F_4^{(1)}$.

- $\gamma = \alpha_2$: Here, $\beta = -\alpha_2 + \delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $F_\beta = \{0, 2, 4\}$, and we can take $\lambda = \mu = \rho_\beta = \Lambda_1 + \Lambda_3$. We use the following variant of the construction from [Ku1]: define $v \in V(\rho_\beta)_{\rho_\beta - \beta}$ by

$$v := (X_{-\beta + \alpha_3} f_3 - 2X_{-\beta} - X_{-\beta + \alpha_3 + \alpha_4} [f_3, f_4]) v_{\Lambda_1 + \Lambda_3}.$$

Then, one can check directly as previously that $X_\beta.v \neq 0$, so that $v \neq 0$, and that $e_0.v = e_2.v = e_4.v = e_1^2.v = e_3^2.v = 0$, as required to apply Proposition 4.1.

- $\gamma = \alpha_1 + \alpha_2$: We have $\beta = -\gamma + \delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$, $F_\beta = \{1, 2, 4\}$, and we can take $\lambda = \mu = \rho_\beta = \Lambda_0 + \Lambda_3$. Set $w = s_3 s_2 s_4 s_3 s_2 s_4 s_3$, $v = s_1 s_0$. Then,

$$v\rho_\beta = \rho_\beta - \alpha_0 - \alpha_1, \quad w\rho_\beta = \rho_\beta - 2\alpha_2 - 4\alpha_3 - 2\alpha_4.$$

Thus, we have $V(w\rho_\beta + v\rho_\beta) = V(2\rho_\beta - \beta) \subset V(\rho_\beta) \otimes V(\rho_\beta)$ as a PRV component.

- $\gamma = \alpha_2 + 2\alpha_3$: In this case, we have $\beta = -\gamma + \delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, and α_4 is the unique simple root such that $\beta - 2\alpha_4 \in \Phi^+$. Then, $F_\beta = \{0, 2, 3\} = F_{\beta - \alpha_4}$ and we can apply Proposition 7.5.
- $\gamma = \alpha_1 + \alpha_2 + 2\alpha_3$: We have $\beta = -\gamma + \delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$, and α_4 is the unique simple root such that $\beta - 2\alpha_4 \in \Phi^+$. As in the previous case, we compute $F_\beta = \{1, 3\} = F_{\beta - \alpha_4}$, so that we can apply Proposition 7.5.
- $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3$: For this case, we get $\beta = -\gamma + \delta = \alpha_0 + \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$. Set $w = s_4 s_3 s_1 s_2 s_3 s_4$, $v = s_0$, and use $\lambda = \mu = \rho_\beta = \Lambda_0 + \Lambda_4$. Then,

$$v\rho_\beta = \Lambda_0 + \Lambda_4 - \alpha_0, \quad w\rho_\beta = \rho_\beta - \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4,$$

so that we get $V(w\rho_\beta + v\rho_\beta) = V(2\rho_\beta - \beta) \subset V(\rho_\beta) \otimes V(\rho_\beta)$ as a PRV component.

- $\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$: In this final case, we have $\beta = -\gamma + \delta = \alpha_0 + \alpha_1 + \alpha_2 + 2\alpha_3$, and α_3 is the unique simple root such that $\beta - 2\alpha_3 \in \Phi^+$. Then, we see that $F_\beta = \{1, 2, 4\} = F_{\beta - 2\alpha_3}$, so we can apply Proposition 7.5.

8.4. $G_2^{(1)}$.

- $\gamma = \alpha_2$: We have $\beta = -\alpha_2 + \delta = \alpha_0 + 3\alpha_1 + \alpha_2$. Then, the minimal dominant $\lambda, \mu \in \mathcal{P}^+$ satisfying the necessary conditions (P1) and (P2) are $\lambda = \Lambda_0 + \Lambda_1, \mu = \Lambda_0 + 2\Lambda_1$. Setting $w = s_1s_2s_1, v = s_0$, we have

$$v\mu = \Lambda_0 + 2\Lambda_1 - \alpha_0, \quad w\lambda = s_1s_2s_1(\Lambda_0 + \Lambda_1) = \lambda - 3\alpha_1 - \alpha_2,$$

so that we have $V(w\lambda + v\mu) = V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ as a PRV component.

- $\gamma = \alpha_1 + \alpha_2$: We have $\beta = -\alpha_1 - \alpha_2 + \delta = \alpha_0 + 2\alpha_1 + \alpha_2, F_\beta = \{2\}$. Then, we have $\rho_\beta = \Lambda_0 + \Lambda_1$, and $2\rho_\beta - \beta \in \mathcal{P}^+$. Setting $w = s_2s_1$ and $v = s_0s_1$, we have $w\rho_\beta = \Lambda_0 + \Lambda_1 - \alpha_1 - \alpha_2$ and $v\rho_\beta = \Lambda_0 + \Lambda_1 - \alpha_0 - \alpha_1$, so that we have $V(w\rho_\beta + v\rho_\beta) = V(2\rho_\beta - \beta) \subset V(\rho_\beta) \otimes V(\rho_\beta)$ as a PRV component.

9. MAIN THEOREM ON ROOT COMPONENTS

Combining Corollaries 5.1, 6.3 and 6.5, Lemma 7.7 and Proposition 7.3 and the results from Section 8, we get the following main result of our paper (cited as Theorem I in Section 1):

Recall that by a Wahl triple we mean a triple $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi^+$ such that

- (P1) $\lambda + \mu - \beta \in \mathcal{P}^+$, and
- (P2) If $\lambda(\alpha_i^\vee) = 0$ or $\mu(\alpha_i^\vee) = 0$, then $\beta - \alpha_i \notin \Phi \sqcup \{0\}$.

Theorem 9.1. *For any affine Lie algebra \mathfrak{g} and Wahl triple $(\lambda, \mu, \beta) \in (\mathcal{P}^+)^2 \times \Phi^+$,*

$$V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu).$$

10. STUDY OF THE COHOMOLOGY $H^*(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu))$

Let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra with its standard Borel subalgebra \mathfrak{b} and standard Cartan subalgebra \mathfrak{h} [Ku3, §1.1.2]. Let $\Phi \subset \mathfrak{h}^*$ be the set of roots and Φ^+ (resp. Φ^-) be the subset of positive (resp. negative) roots and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset \Phi^+$ be the subset of simple roots with α_i^\vee the corresponding simple coroots. Let \mathcal{P}^+ be the set of dominant integral weights. The integrable, highest weight (irreducible) \mathfrak{g} -modules are parameterized by \mathcal{P}^+ . For $\lambda \in \mathcal{P}^+$, let $V(\lambda)$ be the corresponding integrable, highest weight \mathfrak{g} -module. For any subset $S \subset \{1, \dots, \ell\}$, let $\mathfrak{p} = \mathfrak{p}_S := \mathfrak{b} \oplus (\oplus_{\alpha \in \Phi_S^+} \mathfrak{g}_{-\alpha})$ be the corresponding standard parabolic subalgebra, where \mathfrak{g}_α is the root space corresponding to the root α and $\Phi_S^+ := \Phi^+ \cap (\oplus_{k \in S} \mathbb{Z}_{\geq 0} \alpha_k)$. Then,

$$\mathfrak{u} := \oplus_{\alpha \in \Phi^+(S)} \mathfrak{g}_\alpha$$

is the nil-radical of \mathfrak{p} , where $\Phi^+(S) := \Phi^+ \setminus \Phi_S^+$. Let $\mathcal{G}, \mathcal{B}, \mathcal{P} = \mathcal{P}_S, \mathcal{U}, \mathcal{H}$ be the corresponding ‘maximal’ groups associated to the Lie algebras $\mathfrak{g}, \mathfrak{b}, \mathfrak{p}, \mathfrak{u}, \mathfrak{h}$ respectively (cf. [Ku3, Section 6.1]). Let W be the Weyl group of \mathfrak{g} and let $W'_\mathcal{P}$ be the set of smallest length coset representatives in $W/W_\mathcal{P}$, where $W_\mathcal{P}$ is the subgroup of W generated by the simple reflections $\{s_k\}_{k \in S}$.

For any finite dimensional pro-algebraic \mathcal{P} -module M , by $\mathcal{L}(M)$ we mean the corresponding homogeneous vector bundle (locally free sheaf) on $\mathcal{X}_\mathcal{P} := \mathcal{G}/\mathcal{P}$ associated to the principal \mathcal{P} -bundle: $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{P}$ by the representation M of \mathcal{P} (cf.

[Ku3, Corollary 8.2.5]). For any integral weight $\lambda \in \mathfrak{h}^*$, such that $\lambda(\alpha_k^\vee) = 0$ for all $k \in S$, the one dimensional \mathcal{H} -module \mathbb{C}_λ (given by the character λ) admits a unique \mathcal{P} -module structure (extending the \mathcal{H} -module structure); in particular, we have the line bundle $\mathcal{L}(\mathbb{C}_\lambda)$ on $\mathcal{X}_{\mathcal{P}}$. We abbreviate the line bundle $\mathcal{L}(\mathbb{C}_{-\lambda})$ by $\mathcal{L}(\lambda)$ and its restriction to the Schubert variety $X_w^{\mathcal{P}} := \overline{\mathcal{B}w\mathcal{P}/\mathcal{P}}$ by $\mathcal{L}_w(\lambda)$ (for any $w \in W'_{\mathcal{P}}$). Given two line bundles $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$, we can form their external tensor product to get the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ on $\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$.

A dominant integral weight μ is called S -regular if $\mu(\alpha_k^\vee) = 0$ if and only if $k \in S$. The set of such weights is denoted by \mathcal{P}_S^{+o} . With this definition, we have the following:

Proposition 10.1. *For any $\mu \in \mathcal{P}_S^{+o}$ and $w \in W'_{\mathcal{P}}$ such that $X_w^{\mathcal{P}}$ is \mathcal{P} -stable under the left multiplication (i.e., $w^{-1}\alpha_k \in \Phi_S^+ \sqcup \Phi^-$ for all the simple roots α_k with $k \in S$),*

- (a) $H^p(X_w^{\mathcal{P}}, \mathcal{I}_e \otimes \mathcal{L}_w(\mu)) = 0$, for all $p > 0$,
- (b) $H^p(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu)) = 0$, for all $p > 0$, and
- (c) $H^0(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu))$ (resp. $H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu))$) is canonically isomorphic (as \mathcal{P} -modules) with $(V_w(\mu)/(\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu))^*$ (resp. $(\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu)^*$), where \mathcal{I}_e denotes the ideal sheaf of the base point $e = \mathcal{P}/\mathcal{P} \subset X_w^{\mathcal{P}}$, \mathcal{O}_w denotes the structure sheaf of $X_w^{\mathcal{P}}$, $v_\mu \neq 0 \in [V(\mu)]_\mu$, $V_w(\mu) \subset V(\mu)$ is the Demazure module (which is the \mathfrak{b} -submodule of $V(\mu)$ generated by the extremal weight vector $v_{w\mu}$) and $\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu := \mathbb{C}v_\mu \oplus T_e(X_w^{\mathcal{P}}) \cdot v_\mu$ (here $T_e(X_w^{\mathcal{P}})$ is the Zariski tangent space of $X_w^{\mathcal{P}}$ at e).

(Observe that, since $\mu(\alpha_k^\vee) = 0$ for $k \in S$, $\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu \subset V(\mu)$ is stable under \mathcal{P} , and also the point e being \mathcal{P} -fixed, $H^0(X_w^{\mathcal{P}}, \mathcal{I}_e^k \otimes \mathcal{L}_w(\mu))$ and $H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^k) \otimes \mathcal{L}_w(\mu))$ have canonical \mathcal{P} -module structures for any $k \geq 1$.)

Proof. (a) Consider the sheaf exact sequence:

$$(5) \quad 0 \rightarrow \mathcal{I}_e \otimes \mathcal{L}_w(\mu) \rightarrow \mathcal{L}_w(\mu) \rightarrow (\mathcal{O}_w/\mathcal{I}_e) \otimes \mathcal{L}_w(\mu) \rightarrow 0.$$

Clearly, $H^p(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e) \otimes \mathcal{L}_w(\mu)) = H^p(e, \mathcal{L}(\mu)|_e) = 0$, for all $p > 0$. Also, $H^p(X_w^{\mathcal{P}}, \mathcal{L}_w(\mu))=0$, for $p>0$, and the restriction map: $H^0(X_w^{\mathcal{P}}, \mathcal{L}_w(\mu)) \rightarrow H^0(e, \mathcal{L}_w(\mu)|_e)$ is surjective (cf. [Ku3, Theorem 8.2.2]). So, the long exact cohomology sequence associated to the sheaf sequence (5) gives (a).

(b) Similarly, consider the sheaf exact sequence:

$$(6) \quad 0 \rightarrow \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu) \rightarrow \mathcal{L}_w(\mu) \rightarrow (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu) \rightarrow 0.$$

We write a part of the corresponding cohomology exact sequence:

$$(7) \quad \begin{aligned} 0 \rightarrow H^0(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu)) \rightarrow H^0(X_w^{\mathcal{P}}, \mathcal{L}_w(\mu)) \xrightarrow{\pi} H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu)) \\ \rightarrow H^1(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu)) \rightarrow 0. \end{aligned}$$

The sheaf exact sequence:

$$(8) \quad 0 \rightarrow (\mathcal{I}_e/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu) \rightarrow (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu) \rightarrow (\mathcal{O}_w/\mathcal{I}_e) \otimes \mathcal{L}_w(\mu) \rightarrow 0$$

gives:

$$(9) \quad 0 \rightarrow T_e(X_w^{\mathcal{P}})^* \otimes \mathbb{C}_\mu^* \rightarrow H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu)) \rightarrow \mathbb{C}_\mu^* \rightarrow 0.$$

Now, we assert that the map π is surjective: To prove this, observe first that the map

$$\theta : \mathfrak{u}^- \rightarrow V(\mu), \quad X \mapsto X \cdot v_\mu$$

is injective, where \mathfrak{u}^- is the opposite nil-radical of \mathfrak{p} : For any root $\beta \in \Phi^+(S)$, pick root vectors $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$. Then,

$$X_\beta Y_\beta \cdot v_\mu = (X_\beta | Y_\beta)(\mu | \beta) v_\mu, \quad \text{by [Ku3, Theorem 1.5.4].}$$

From the nondegeneracy of $(\cdot | \cdot)$ over \mathfrak{g} and the weight consideration, we get the injectivity of θ . (Here we have used the assumption that $\mu \in \mathcal{P}_S^{+o}$.) Since π is a \mathfrak{p} and hence \mathfrak{h} -module map, from the exact sequence (9) and the following identification with the dual Demazure module:

$$H^0(X_w^{\mathcal{P}}, \mathcal{L}_w(\mu)) \simeq V_w(\mu)^* \quad (\text{cf. [Ku3, Corollary 8.1.26]}),$$

to prove the surjectivity of π , it suffices to show that $\theta_w : T_e(X_w^{\mathcal{P}}) \rightarrow V_w(\mu)$, $X \mapsto X \cdot v_\mu$, is injective, where we identify $T_e(X_w^{\mathcal{P}}) \hookrightarrow T_e(\mathcal{X}_{\mathcal{P}}) \simeq \mathfrak{u}^-$. But the injectivity of θ_w clearly follows from the injectivity of θ .

So, the exact sequence (7) (and the surjectivity of π) establishes the vanishing of $H^1(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu))$. Vanishing of $H^p(X_w^{\mathcal{P}}, \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu))$, for $p > 1$, follows by considering the cohomology exact sequence associated to the sheaf exact sequence:

$$(10) \quad 0 \rightarrow \mathcal{I}_e^2 \otimes \mathcal{L}_w(\mu) \rightarrow \mathcal{I}_e \otimes \mathcal{L}_w(\mu) \rightarrow (\mathcal{I}_e / \mathcal{I}_e^2) \otimes \mathcal{L}_w(\mu) \rightarrow 0,$$

and the (a)-part of the proposition. So, (b) follows.

(c) It is easy to see, from the description of the map π , that

$$(11) \quad \text{Ker } \pi = \{f \in V_w(\mu)^* : f|_{(\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu)} = 0\} \simeq \left(V_w(\mu) / (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu) \right)^*.$$

So, using the exact sequences (7) and (9), (c) follows. □

From Proposition 10.1 together with [Ku3, Theorem 8.2.2] we immediately get:

Corollary 10.2. *Under the assumptions of Proposition 10.1, $H^p(X_w^{\mathcal{P}}, (\mathcal{O}_w / \mathcal{I}_e^k) \otimes \mathcal{L}_w(\mu)) = 0$, for all $p > 0$ and $k = 1, 2$.*

11. SURJECTIVITY OF THE GAUSSIAN MAP

We continue to follow the notation and assumptions from Section 10. For any $w \in W'_{\mathcal{P}}$ such that the Schubert variety $X_w^{\mathcal{P}}$ is \mathcal{P} -stable, set

$$\hat{\mathcal{X}}_w^{\mathcal{P}} := \mathcal{G} \times^{\mathcal{P}} X_w^{\mathcal{P}}.$$

Consider the isomorphism:

$$\delta : \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}} \simeq \mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \quad [g, x] \mapsto (g\mathcal{P}, gx), \quad \text{for } g \in \mathcal{G} \text{ and } x \in \mathcal{X}_{\mathcal{P}}.$$

We have an embedding

$$\hat{\mathcal{X}}_w^{\mathcal{P}} \hookrightarrow \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}}.$$

In particular, we can restrict the line bundle $\mathcal{L}(\lambda \boxtimes \mu)$ via the above isomorphism δ to $\hat{\mathcal{X}}_w^{\mathcal{P}}$ to get the line bundle denoted $\mathcal{L}_w(\lambda \boxtimes \mu)$.

Lemma 11.1. *For $\lambda, \mu \in \mathcal{P}_S^{+o}, k = 1, 2, w \in W'_P$ as above, and any $p \geq 0$, there are canonical isomorphisms:*

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e^k \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \simeq H^p(\mathcal{X}_P, \mathcal{L}(\lambda) \otimes \mathcal{L}(H^0(X_w^{\mathcal{P}}, \mathcal{I}_e^k \otimes \mathcal{L}_w(\mu)))),$$

and

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^k) \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \simeq H^p(\mathcal{X}_P, \mathcal{L}(\lambda) \otimes \mathcal{L}(H^0(X_w^{\mathcal{P}}, (\mathcal{O}_w/\mathcal{I}_e^k) \otimes \mathcal{L}_w(\mu)))),$$

where $\hat{\mathcal{I}}_e$ denotes the ideal sheaf of $\hat{\mathcal{X}}_w^{\mathcal{P}}$ in $\hat{\mathcal{X}}_w^{\mathcal{P}}$ and $\hat{\mathcal{O}}_w$ is the structure sheaf of $\hat{\mathcal{X}}_w^{\mathcal{P}}$.

Further,

$$(12) \quad H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

(We will see later that $H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0$, for all $p > 0$.)

Proof. Consider the fibration

$$\sigma : \hat{\mathcal{X}}_w^{\mathcal{P}} \rightarrow \mathcal{X}_P, [g, x] \mapsto gP$$

and the associated Leray spectral sequence. Then, it is easy to see that $R^p\sigma_*(\hat{\mathcal{I}}_e^k \otimes \mathcal{L}_w(\lambda \boxtimes \mu))$ is canonically isomorphic with $\mathcal{L}(\lambda) \otimes \mathcal{L}(H^p(X_w^{\mathcal{P}}, \mathcal{I}_e^k \otimes \mathcal{L}_w(\mu)))$ for any $p \geq 0$. (A similar statement is true for $\hat{\mathcal{I}}_e^k$ replaced by $\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^k$.) In particular, by Proposition 10.1, Corollary 10.2 and the (degenerate) Leray spectral sequence, first part of the lemma follows.

To prove the vanishing (12), use the cohomology exact sequence associated to the sheaf exact sequence:

$$0 \rightarrow \hat{\mathcal{I}}_e \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e) \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow 0,$$

and [Ku2, Theorem 2.7] (also see [M2, Lemmas 21, 22]). □

Lemma 11.2 is used in proving our crucial Proposition 11.4.

Lemma 11.2. *Let \mathfrak{g} be an affine Kac-Moody Lie algebra (cf. Section 2). Take $\lambda, \mu \in \mathcal{P}_S^{+o}$ and $\theta \in \mathcal{P}^+$. Then, if $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee)$ is nonzero, then $\theta = \lambda + \mu - \beta$, for some $\beta \in \Phi^+(S) \cup \{0\}$.*

If β is zero, then the above space is one dimensional.

If such a β is a real root, then the above space is nonzero if and only if $S \subset F_\beta$ (cf. Definition 7.2), in which case it is one dimensional.

If β is an imaginary root, then this space is of dimension $(\dim \mathfrak{g}_\beta) - |S|$.

Proof. It is easy to see that, as a \mathfrak{p} -module,

$$\mathfrak{g} \cdot v_\mu \simeq (\mathfrak{g}/\text{Ann } v_\mu) \otimes \mathbb{C}_\mu,$$

where $\text{Ann } v_\mu$ is the annihilator of v_μ in \mathfrak{g} and the right side is given the tensor product \mathfrak{p} -module structure. Hence, using the nondegenerate invariant form $(\cdot | \cdot)$ on \mathfrak{g} , we get:

$$(13) \quad (\mathfrak{g} \cdot v_\mu)^\vee \simeq (\mathfrak{u} \oplus (\text{Ker } \mu)^\perp) \otimes \mathbb{C}_\mu^*,$$

where $\text{Ker } \mu := \{h \in \mathfrak{h} : \mu(h) = 0\}$, $(\text{Ker } \mu)^\perp := \{h \in \mathfrak{h} : (h | \text{Ker } \mu) = 0\}$ and $\mathfrak{u} \oplus (\text{Ker } \mu)^\perp$ is a \mathfrak{p} -module under the adjoint representation. By the presentation of $V(\theta)^\vee$ as a $U(\mathfrak{b})$ -module as in the proof of Proposition 4.1, we get:

$$\begin{aligned} \text{Hom}_{\mathfrak{b}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) &\simeq \text{Hom}_{\mathfrak{b}}\left(V(\theta)^\vee, \mathbb{C}_{(\lambda+\mu)}^* \otimes (\mathfrak{u} \oplus (\text{Ker } \mu)^\perp)\right), \text{ by (13)} \\ &\simeq \{v \in [\mathfrak{u} \oplus (\text{Ker } \mu)^\perp]_{\lambda+\mu-\theta} : (\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}v = 0\}, \text{ for all the simple roots } \alpha_i, \end{aligned}$$

where e_i is a root vector corresponding to the simple root α_i . It is easy to see, from the above description, that

$$(14) \quad \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) \simeq \{v \in [\mathfrak{u} \oplus (\text{Ker } \mu)^\perp]_{\lambda+\mu-\theta} : (\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}v = 0, \\ \text{for all the simple roots } \alpha_i \text{ and } (\text{ad } f_k)v = 0 \text{ for all } k \in S\},$$

where f_k is a root vector corresponding to the negative simple root $-\alpha_k$. Clearly, the following conditions (1) and (2) are both satisfied if the right side of (14) is nonzero:

- (1) $\lambda + \mu - \theta = \beta$, for some $\beta \in \Phi^+(S) \cup \{0\}$, and
- (2) If $\beta \neq 0$ is real, $F_\beta \supset S$.

To prove (2), observe that if $[f_k, \mathfrak{g}_\beta] = 0$ then $\beta - \alpha_k$ is not a root of \mathfrak{g} . Conversely, for any dominant θ which satisfies the conditions (1) and (2) as above, the right side of (14) (and hence $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee)$) is one dimensional if either $\beta = 0$ or β is a real root:

If $\beta = 0$, $(\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}((\text{Ker } \mu)^\perp) = 0$ for all $i \notin S$, since $\lambda, \mu \in \mathcal{P}_S^{+o}$. Moreover, $\text{ad } e_k((\text{Ker } \mu)^\perp) = \text{ad } f_k((\text{Ker } \mu)^\perp) = 0$ for all $k \in S$ (since $\mu(\alpha_k^\vee) = 0$).

So, assume now that $\beta \in \Phi^+(S)$ is a real root and pick $0 \neq X_\beta \in \mathfrak{g}_\beta$. By the above condition (2), $(\text{ad } f_k)X_\beta = 0$, for $k \in S$. So, by the $sl(2)$ -module theory, $\beta(\alpha_k^\vee) \leq 0$ and $(\text{ad } e_k)^{1-\beta(\alpha_k^\vee)}X_\beta = 0$. In particular, $(\text{ad } e_k)^{1+\theta(\alpha_k^\vee)}X_\beta = 0$. For $i \notin S$, we again claim that $(\text{ad } e_i)^{\theta(\alpha_i^\vee)+1}X_\beta = 0$, since $\beta + (\theta(\alpha_i^\vee) + 1)\alpha_i \notin \Phi^+$: We have

$$(\beta + (\theta(\alpha_i^\vee) + 1)\alpha_i)(\alpha_i^\vee) = (\lambda + \mu + \theta)(\alpha_i^\vee) + 2 \geq 4,$$

since $\lambda, \mu \in \mathcal{P}_S^{+o}$ and $\theta \in \mathcal{P}^+$. But, this contradicts [Bou81, Page 278, Fact 6]. (Here we used the assumption on \mathfrak{g} being an affine Kac-Moody Lie algebra.) This completes the proof of the lemma in the case β is either zero or a real root.

Assume now that $\beta = \lambda + \mu - \theta$ is an imaginary root $p\delta, p > 0$. For any $h \in \mathfrak{h}$ take $v(h) = h \otimes t^p \in \mathfrak{g}_{p\delta}$. Then, for any $k \in S$,

$$(\text{ad } e_k)v(h) = (\text{ad } f_k)v(h) = 0 \text{ if and only if } \alpha_k(h) = 0,$$

where α_0 is interpreted as the negative of the highest root. Moreover, for $i \notin S$,

$$(15) \quad (\text{ad } e_i)^{(\lambda+\mu-p\delta)(\alpha_i^\vee)+1}v(h) = 0, \text{ since } \lambda, \mu \in \mathcal{P}_S^{+o} \text{ and } p\delta(\alpha_i^\vee) = 0.$$

Thus, by equation (14), for $\theta = \lambda + \mu - p\delta$,

$$(16) \quad \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) \simeq \{h \otimes t^p : \alpha_k(h) = 0 \ \forall k \in S\}.$$

In particular, it has dimension $\dim \mathfrak{g}_{p\delta} - |S|$. □

Lemma 11.3. *Let the notation and assumptions be as in Lemma 11.2 and let M be a \mathfrak{p} -module quotient $\pi : R \twoheadrightarrow M$, where $R := \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee$. Then, for any $\theta \in \mathcal{P}^+$, the canonical map*

$$(17) \quad \hat{\pi}_M : \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, R) \rightarrow \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$$

is surjective.

Proof. By the proof of Lemma 11.2, both the sides of (17) are zero unless $\theta = \lambda + \mu - \beta$, for some $\beta \in \Phi^+(S) \cup \{0\}$. We identify $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, R)$ with the right side of (14) and a similar identification for $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$. To prove the lemma, we consider the three cases separately:

Case I ($\beta = 0$). In this case $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, R) \simeq (\text{Ker } \mu)^\perp$ (cf. Proof of Lemma 11.2) and thus the lemma follows easily in this case.

Case II (β is a real root). In this case, following the proof of Lemma 11.2,

$$\begin{aligned} \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, R) &\simeq \mathfrak{g}_\beta, \text{ if } F_\beta \supset S \\ &= 0, \text{ otherwise.} \end{aligned}$$

Identifying R with $\mathfrak{u} \oplus (\text{Ker } \mu)^\perp$ via equation (13), it suffices to prove that if $0 \neq \pi(\mathfrak{g}_\beta) \subset \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$, then $F_\beta \supset S$. Since $0 \neq \pi(\mathfrak{g}_\beta) \subset \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$, we get that, for all $k \in S$, $[f_k, \pi(\mathfrak{g}_\beta)] = 0$, which implies $\beta(\alpha_k^\vee) \leq 0$. Assume, if possible, that $F_\beta \not\supset S$, i.e., for some $k \in S, \beta - \alpha_k \in \Phi^+$ (it cannot be zero since $\beta \in \Phi^+(S)$). Then, $[e_k, \mathfrak{g}_{\beta-\alpha_k}]$ is nonzero and hence, \mathfrak{g}_β being one dimensional, $[e_k, \mathfrak{g}_{\beta-\alpha_k}] = \mathfrak{g}_\beta$ giving rise to $[e_k, \pi(\mathfrak{g}_{\beta-\alpha_k})] \neq 0$. From this, by the sl_2 -module theory, we easily see that $[f_k, \pi(\mathfrak{g}_\beta)] \neq 0$. A contradiction (to equation (14)) since $\pi(\mathfrak{g}_\beta) \subset \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$. Thus, $F_\beta \supset S$. From this the lemma follows in this case.

Case III (β is an imaginary root $p\delta$ for $p > 0$). Take $h \otimes t^p$ (for $h \in \mathfrak{h}$) such that $\pi(h \otimes t^p) \subset \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, M)$. Define the subset $S' \subset S$ by

$$S' := \{k \in S : \pi(\alpha_k^\vee \otimes t^p) = 0\},$$

where we set $\alpha_0^\vee = -\theta^\vee$ for the highest root θ of \mathfrak{g} . Consider the decomposition

$$(18) \quad \mathfrak{h} = (\oplus_{k \in S'} \mathbb{C}\alpha_k^\vee) \oplus \mathfrak{h}(S'),$$

where $\mathfrak{h}(S') := \{x \in \mathfrak{h} : \alpha_k(x) = 0 \forall k \in S'\}$. Let h' be the projection of h in the $\mathfrak{h}(S')$ -factor (under the above decomposition). Then, from the definition of S' , we see that $\pi(h \otimes t^p) = \pi(h' \otimes t^p)$. Moreover, it is easy to see that $h' \otimes t^p \in \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, R)$, i.e.,

$$(\text{ad } e_k)(h' \otimes t^p) = (\text{ad } f_k)(h' \otimes t^p) = 0, \text{ for all } k \in S$$

(cf. equation (15)). (To show that $\alpha_k(h') = 0$ for $k \in S \setminus S'$, use the nonvanishing $\pi(\alpha_k^\vee \otimes t^p) \neq 0$ and $e_k \cdot \pi(h' \otimes t^p) = 0$.)

□

We prove the following crucial result towards the proof of Theorem 11.7. We follow the notation as in Lemma 11.1.

Proposition 11.4. *Let \mathfrak{g} be an affine Kac-Moody Lie algebra (cf. Section 2). Then, for any $\lambda, \mu \in \mathcal{P}_S^{+o}$ (where S is an arbitrary subset of the simple roots of \mathfrak{g}) and any $w \in W'_P$ such that the Schubert variety X_w^P is \mathcal{P} -stable, consider the following two conditions:*

- (a) $H^1(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0$.
- (b) For all the real roots $\beta \in \Phi^+(S)$, satisfying $F_\beta \supset S$ and $\lambda + \mu - \beta \in \mathcal{P}^+$ (in particular, (λ, μ, β) is a Wahl triple), there exists an

$$f_\beta \in \text{Hom}_{\mathfrak{b}}(\mathbb{C}_{\lambda+\mu-\beta} \otimes V(\lambda)^\vee, V(\mu))$$

such that

$$(19) \quad X_\beta(f_\beta(\mathbb{C}_{\lambda+\mu-\beta} \otimes v_\lambda^*)) \neq 0, \text{ for } X_\beta \neq 0 \in \mathfrak{g}_\beta, \text{ where } v_\lambda^* \neq 0 \in [V(\lambda)^\vee]_{-\lambda}.$$

Then, the condition (b) implies the condition (a).

Proof. Consider the sheaf exact sequence over $\hat{\mathcal{X}}_w^{\mathcal{P}}$:

$$(20) \quad 0 \rightarrow \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow 0.$$

From the cohomology exact sequence associated to the above sheaf exact sequence and the vanishing of $H^1(\hat{\mathcal{X}}_w^{\mathcal{P}}, \mathcal{L}_w(\lambda \boxtimes \mu))$ (cf. [Ku2, Theorem 2.7]), we get that (a) is equivalent to the surjectivity of the canonical map

$$\tau_w : H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, \mathcal{L}_w(\lambda \boxtimes \mu)) \rightarrow H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu)).$$

By Lemma 11.1 and Proposition 10.1 we get:

$$H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \simeq H^0(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}((\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*)).$$

Further, by [Ku3, Theorem 8.2.2]

$$H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, \mathcal{L}_w(\lambda \boxtimes \mu)) \simeq H^0(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}(V_w(\mu)^*)).$$

Transporting the map τ_w under these identifications, we get the map

$$\tilde{\tau}_w : H^0(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}(V_w(\mu)^*)) \rightarrow H^0(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}((\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*)),$$

which is induced from the canonical restriction: $\mathcal{L}(V_w(\mu)^*) \rightarrow \mathcal{L}((\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*)$.

By [M1, Proposition 12], for any finite dimensional pro-algebraic \mathcal{P} -module M , we have a canonical \mathcal{G} -module isomorphism:

$$H^0(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(M)) \simeq \bigoplus_{\theta \in \mathcal{P}^+} V(\theta)^* \otimes [V(\theta) \otimes M]^{\mathfrak{p}},$$

where \mathcal{G} acts trivially on $[V(\theta) \otimes M]^{\mathfrak{p}}$. So, the surjectivity of the map $\tilde{\tau}_w$ (and hence of τ_w) is equivalent to the surjectivity of the canonical restriction maps

$$\gamma_{\theta}^w : \text{Hom}_{\mathfrak{p}}(V(\theta)^{\vee}, \mathbb{C}_{\lambda}^* \otimes V_w(\mu)^*) \rightarrow \text{Hom}_{\mathfrak{p}}\left(V(\theta)^{\vee}, \mathbb{C}_{\lambda}^* \otimes (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*\right)$$

for all $\theta \in \mathcal{P}^+$.

For $\theta = \lambda + \mu$, the map γ_{θ}^w is clearly nonzero and hence surjective by Lemmas 11.2 and 11.3.

Making use of Lemmas 11.2 and 11.3 again, we can assume (for the surjectivity of γ_{θ}^w) that θ is of the form $\theta = \lambda + \mu - \beta$ for $\beta \in \Phi^+(S)$ and if β is a real root, then $F_{\beta} \supset S$.

We first analyze the surjectivity of

$$\gamma_{\theta} : \text{Hom}_{\mathfrak{p}}(V(\theta)^{\vee}, \mathbb{C}_{\lambda}^* \otimes V(\mu)^{\vee}) \rightarrow \text{Hom}_{\mathfrak{p}}(V(\theta)^{\vee}, \mathbb{C}_{\lambda}^* \otimes (\mathfrak{g} \cdot v_{\mu})^{\vee}).$$

Choose a positive definite Hermitian form $\{\cdot|\cdot\}$ on $V(\lambda)$ (and $V(\mu)$) satisfying

$$\{Xv|w\} = \{v|\sigma(X)w\}, \text{ for } v, w \in V(\lambda) \text{ and } X \in \mathfrak{g},$$

where σ is a conjugate-linear involution of \mathfrak{g} which takes \mathfrak{g}_{α} to $\mathfrak{g}_{-\alpha}$ for any root α (cf. [Ku3, Theorem 2.3.13]). Now, set the tensor product form (again denoted by $\{\cdot|\cdot\}$) on $V(\lambda) \otimes V(\mu)$.

Consider the following diagram (for $\theta = \lambda + \mu - \beta$):

$$\begin{array}{ccccc} \text{Hom}_{\mathfrak{g}}(V(\theta), V(\lambda) \otimes V(\mu)) & \xrightarrow[\simeq]{\gamma_2} & \text{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu), V(\theta)) & \xrightarrow[\simeq]{\gamma_3} & \text{Hom}_{\mathfrak{p}}(\mathbb{C}_{\lambda} \otimes V(\mu), V(\theta)) \\ \simeq \downarrow \gamma_1 & & & & \downarrow \hat{\gamma}_{\theta} \\ \text{Hom}_{\mathfrak{b}}(\mathbb{C}_{\theta} \otimes V(\lambda)^{\vee}, V(\mu)) & & & & \text{Hom}_{\mathfrak{p}}(\mathbb{C}_{\lambda} \otimes (\mathfrak{g} \cdot v_{\mu}), V(\theta)), \end{array}$$

where γ_1 and γ_3 are the canonical restriction maps, which are isomorphisms (as can be easily seen respectively from the isomorphism (3) and [Ku3, Identity 3.1.13.9]). We now describe the map γ_2 : For any nonzero $f \in \text{Hom}_{\mathfrak{g}}(V(\theta), V(\lambda) \otimes V(\mu))$, write

$V(\lambda) \otimes V(\mu) = \text{Image } f \oplus (\text{Image } f)^\perp$ and set $\gamma_2(f)$ as the projection on the first factor (identifying it with $V(\theta)$ under f). It is easy to see that γ_2 is a bijection (though nonlinear). Further, $\hat{\gamma}_\theta$ is the canonical restriction map.

We identify

$$\text{Hom}_{\mathfrak{p}}(\mathbb{C}_\lambda \otimes V(\mu), V(\theta)) \simeq \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes V(\mu)^\vee)$$

and

$$\text{Hom}_{\mathfrak{p}}(\mathbb{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu), V(\theta)) \simeq \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee).$$

We first take $\theta = \lambda + \mu - \beta$ for a real root $\beta \in \Phi^+(S)$ such that $S \subset F_\beta$. Following through the various isomorphisms in the above diagram, it can be seen that (denoting $\mathfrak{n}^+ = [\mathfrak{b}, \mathfrak{b}]$ and \mathfrak{n}^- is the opposite subalgebra to \mathfrak{n})

$$\begin{aligned} &\hat{\gamma}_\theta \text{ is nonzero (and hence surjective since the range has dimension 1)} \\ \Leftrightarrow &\text{there exists a } \mathfrak{b}\text{-morphism } f = f_\beta : \mathbb{C}_\theta \otimes V(\lambda)^\vee \rightarrow V(\mu) \\ &\text{such that } \{(\gamma_1^{-1}f)V(\theta)|\mathbb{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu)\} \neq 0 \\ \Leftrightarrow &\{U(\mathfrak{n}^-)((\gamma_1^{-1}f)\mathbb{C}_\theta)|\mathbb{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu)\} \neq 0 \\ \Leftrightarrow &\{(\gamma_1^{-1}f)\mathbb{C}_\theta|\mathbb{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu)\} \neq 0 \text{ by the invariance of} \\ &\{\cdot|\cdot\}; \text{ since } \mathbb{C}_\lambda \otimes (\mathfrak{g} \cdot v_\mu) \text{ is } U(\mathfrak{n})\text{-stable} \\ \Leftrightarrow &\{f(\mathbb{C}_\theta \otimes v_\lambda^*)|\mathfrak{g} \cdot v_\mu\} \neq 0 \\ \Leftrightarrow &\{f(\mathbb{C}_\theta \otimes v_\lambda^*)|X_{-\beta} \cdot v_\mu\} \neq 0, \text{ for } 0 \neq X_{-\beta} \in \mathfrak{g}_{-\beta} \\ \Leftrightarrow &\{X_\beta(f(\mathbb{C}_\theta \otimes v_\lambda^*))|v_\mu\} \neq 0, \text{ by the invariance of } \{\cdot|\cdot\} \\ \Leftrightarrow &X_\beta(f(\mathbb{C}_\theta \otimes v_\lambda^*)) \neq 0. \end{aligned}$$

For $\beta = 0$, $\hat{\gamma}_\theta$ is clearly surjective.

So, let us now take β to be an imaginary root $p\delta$, for $p > 0$:

Recall from the isomorphism (16) that, under the identification (13),

$$\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) \simeq \{h \otimes t^p : h \in \mathfrak{h} \text{ and } \alpha_k(h) = 0 \ \forall k \in S\}.$$

We next show that, under the restriction map

$$V(\mu)^\vee \rightarrow (\mathfrak{g} \cdot v_\mu)^\vee \simeq (\mathfrak{u} \oplus (\text{Ker } \mu)^\perp) \otimes \mathbb{C}_\mu^*,$$

for any $h \in \mathfrak{h}$, the element $x_h := (-\frac{h \otimes t^p}{p\mu(K)}) \cdot v_\mu^*$ goes to

$$(21) \quad x_h \mapsto (h \otimes t^p) \otimes v_\mu^*.$$

To see this, take any $h' \in \mathfrak{h}$. Then,

$$\begin{aligned} x_h((h' \otimes t^{-p}) \cdot v_\mu) &= v_\mu^* \left(\left[\frac{h \otimes t^p}{p\mu(K)}, h' \otimes t^{-p} \right] \cdot v_\mu \right) \\ &= (h|h'). \end{aligned}$$

Similarly,

$$\begin{aligned} ((h \otimes t^p) \otimes v_\mu^*)((h' \otimes t^{-p}) \cdot v_\mu) &= (h \otimes t^p|h' \otimes t^{-p}) \\ &= (h|h'). \end{aligned}$$

This proves (21). Now, for any $h \in \mathfrak{h}$ such that $\alpha_k(h) = 0$ for all $k \in S$, the element x_h satisfies

$$(22) \quad e_i^{\theta(\alpha_i^\vee)+1} \cdot x_h = 0 \ \forall i, \text{ and } f_k \cdot x_h = 0 \ \forall k \in S.$$

To prove this, observe that (in $V(\mu)^\vee$)

$$e_i^{\theta(\alpha_i^\vee)+1} \cdot (h \otimes t^p) \cdot v_\mu^* = \sum_{j=0}^{\theta(\alpha_i^\vee)+1} \binom{\theta(\alpha_i^\vee)+1}{j} ((\text{ad } e_i)^j (h \otimes t^p)) \cdot (e_i^{\theta(\alpha_i^\vee)+1-j} \cdot v_\mu^*),$$

cf. the proof of [Ku3, Lemma 1.3.3]

$$= 0, \text{ since } (\text{ad } e_i)^2 (h \otimes t^p) = 0 \text{ and } e_i^{\mu(\alpha_i^\vee)+1} \cdot v_\mu^* = 0,$$

by the presentation of $V(\mu)^\vee$ as a \mathfrak{b} -module given in the proof of Proposition 4.1. Further, for $k \in S$,

$$f_k \cdot (h \otimes t^p) \cdot v_\mu^* = [f_k, h \otimes t^p] \cdot v_\mu^* = 0, \text{ since } \alpha_k(h) = 0 \text{ by assumption.}$$

Thus, $v_\theta^* \mapsto x_h$ extends uniquely to a \mathfrak{p} -module morphism $V(\theta)^\vee \rightarrow \mathbb{C}_\lambda^* \otimes V(\mu)^\vee$. This proves the surjectivity of the restriction map

$$\gamma_\theta : \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes V(\mu)^\vee) \rightarrow \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee)$$

for $\theta = \lambda + \mu - p\delta$.

Consider the commutative diagram for any $\theta \in \mathcal{P}^+$ of the form $\theta = \lambda + \mu - \beta$, for $\beta \in \Phi^+(S) \cup \{0\}$ and $w \in W_{\mathcal{P}}'$ such that $X_w^{\mathcal{P}}$ is \mathcal{P} -stable:

$$\begin{CD} \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes V(\mu)^\vee) @>\gamma_\theta>> \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) \\ @VVV @VV\hat{\pi}_M V \\ \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes V_w(\mu)^*) @>\gamma_\theta^w>> \text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu)^*) \end{CD}$$

where the vertical maps are the canonical restriction maps and $M := \mathbb{C}_\lambda^* \otimes (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_\mu)^*$. By Lemma 11.3, $\hat{\pi}_M$ is surjective. Further, by the above proof, γ_θ is surjective for all $\theta = \lambda + \mu - \beta$ such that $\beta = 0$ or an imaginary root. Moreover, as proved above, for a real root $\beta \in \Phi^+(S)$ with $F_\beta \supset S$, γ_θ is surjective if and only if there exists a \mathfrak{b} -morphism $f_\beta : \mathbb{C}_\theta \otimes V(\lambda)^\vee \rightarrow V(\mu)$ such that $X_\beta(f_\beta(\mathbb{C}_\theta \otimes v_\lambda^*)) \neq 0$.

By Lemma 11.2, for any real root $\beta \in \Phi^+(S)$ satisfying $F_\beta \not\supset S$, we have $\text{Hom}_{\mathfrak{p}}(V(\theta)^\vee, \mathbb{C}_\lambda^* \otimes (\mathfrak{g} \cdot v_\mu)^\vee) = 0$. In particular, γ_θ^w is surjective for $\beta = 0$ or an imaginary root or a real root $\beta \in \Phi^+(S)$ such that $F_\beta \not\supset S$ and it is surjective for a real root $\beta \in \Phi^+(S)$ with $F_\beta \supset S$ if the condition (b) of the proposition is satisfied for β . Thus, $\tilde{\tau}_w$ (equivalently τ_w) is surjective if the condition (b) of the proposition is satisfied. This proves the proposition. \square

Lemma 11.5. *Let (λ, μ, β) be a Wahl triple for a real root β and let $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ be a δ -maximal root component. Observe that $\beta \in \mathring{\Phi}^+$ or $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$ (cf. Theorem 9.1). Then,*

- (1) *If $\beta \in \mathring{\Phi}^+$, the validity of condition (b) of Proposition 11.4 for $V(\lambda + \mu - \beta)$ (i.e., the validity of (19)) implies its validity for $V(\lambda + \mu - \beta - k\delta)$ for any $k \geq 0$.*

(2) If $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$, then we have (with notation as introduced below in the proof, see the identity (32))

$$(l + m + h^\vee)X_{-\gamma}(k + 1) \cdot v_\lambda^*(L_{-k}w) = l(k + 1)X_{-\gamma}(1) \cdot w_{\mu-\beta} - (\lambda|_{\mathring{\mathfrak{h}}}, \gamma|_{\mathring{\mathfrak{h}}})X_{-\gamma}(1) \cdot w_{\mu-\beta} + \sum_{\substack{\beta_i \in \mathring{\Phi}^+ \\ \beta_i + \gamma \in \mathring{\Phi}^+}} [X_{-\gamma}(1), X_{-\beta_i}]X_{\beta_i} \cdot w_{\mu-\beta} + A,$$

$$\text{where } A := \left(\sum_{\substack{\beta_i \in \mathring{\Phi}^+ \\ \beta_i - \gamma \in \mathring{\Phi}^+}} [X_{-\gamma} X_{\beta_i}]X_{-\beta_i}(1) \cdot w_{\mu-\beta} \right) + ((k + 1)m - (\mu|\gamma))X_{-\gamma}(1) \cdot w_{\mu-\beta}.$$

Proof. Let $w \in V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ be the highest weight vector. Then, $L_{-k}(w)$ is a highest weight vector in $V(\lambda) \otimes V(\mu)$ with weight $\lambda + \mu - \beta - k\delta$ (cf. Proposition 3.5(3) and Lemma 3.8). As before, let l, m be the central charges of $V(\lambda)$ and $V(\mu)$, respectively, and h^\vee the dual Coxeter number of \mathfrak{g} . Write

$$w = v_\lambda \otimes w_{\mu-\beta} + \sum_{q \in Q^+ \setminus \{0\}} v_{\lambda-q}^\bullet \otimes w_{\mu-\beta+q}^\bullet,$$

where $v_{\lambda-q}^\bullet$ is a basis of weight vectors of $V(\lambda)$ of weight $\lambda - q$, $w_{\mu-\beta+q}^\bullet \in V(\mu)_{\mu-\beta+q}$ (possibly zero), and $Q^+ := \bigoplus_{j=0}^\ell \mathbb{Z}_{\geq 0} \alpha_j$. Observe that $v_{\lambda-q}^\bullet$ occurs as many times (possibly zero) as the dimension of the weight space $V(\lambda)_{\lambda-q}$.

Denote by $v_\lambda^* : V(\lambda) \otimes V(\mu) \rightarrow V(\mu)$ the contraction via the highest weight vector v_λ . Then, by the proof of Proposition 4.1, we have that $v_\lambda^*(w)$, for the highest weight vector as above, satisfies the conditions $e_i^{\lambda(\alpha_i^\vee)+1} \cdot (v_\lambda^*(w)) = 0$ for all simple roots α_i .

By [KRR, Proposition 10.3], for any $k > 0$, the GKO operator (cf. Proposition 3.5):

$$(l+m+h^\vee)L_{-k}(w) = m(L_{-k}^\mathfrak{g} \otimes Id)(w) + lv_\lambda \otimes (L_{-k}^\mathfrak{g} w_{\mu-\beta}) + l \sum_{q \neq 0} v_{\lambda-q}^\bullet \otimes (L_{-k}^\mathfrak{g} w_{\mu-\beta+q}^\bullet) - \sum_{j \in \mathbb{Z}, i} (u_i(-j) \cdot v_\lambda) \otimes (u^i(j-k) \cdot w_{\mu-\beta}) - \sum_{j \in \mathbb{Z}, i} \sum_{0 \neq q \leq \beta} (u_i(-j) \cdot v_{\lambda-q}^\bullet) \otimes (u^i(j-k) \cdot w_{\mu-\beta+q}^\bullet),$$

where $\{u_i\}$ is a basis of $\mathring{\mathfrak{g}}$, $\{u^i\}$ is its dual basis, and $L_{-k}^\mathfrak{g}$ is the Virasoro operator of \mathfrak{g} (cf. [KRR, Corollary 10.1]). Now, take a root basis $\{X_{\beta_i}\}$ of $\mathring{\mathfrak{n}}^+ := \bigoplus_{\alpha \in \mathring{\Phi}^+} \mathring{\mathfrak{g}}_\alpha$ and the dual basis $\{X_{-\beta_i}\}$ of $\mathring{\mathfrak{n}}^- := \bigoplus_{\alpha \in \mathring{\Phi}^+} \mathring{\mathfrak{g}}_{-\alpha}$, and a basis $\{h_j\}$ of $\mathring{\mathfrak{h}}$ and its dual basis $\{h^j\}$, with respect to the normalized form. That is, $(X_{\beta_i}, X_{-\beta_j}) = \delta_{ij}$, $(h_i, h^j) = \delta_{ij}$. Then, we have for $\beta \in \mathring{\Phi}^+$,

$$(23) \quad (l + m + h^\vee)v_\lambda^*(L_{-k}(w)) = l L_{-k}^\mathfrak{g}(w_{\mu-\beta}) - \sum_i v_\lambda^*(u_i \cdot v_\lambda)(u^i(-k) \cdot w_{\mu-\beta}) - \sum_{i, 0 \neq q \leq \beta} v_\lambda^*(u_i \cdot v_{\lambda-q}^\bullet)(u^i(-k) \cdot w_{\mu-\beta+q}^\bullet)$$

For $\beta = \delta - \gamma$ with $\gamma \in \check{\Phi}^+$, the above equation holds if we augment the right side with

$$- \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \gamma \leq \beta_i}} v_\lambda^*(X_{-\beta_i}(1) \cdot v_{\lambda-\delta+\beta_i}^\bullet)(X_{\beta_i}(-k-1) \cdot w_{\mu+\gamma-\beta_i}^\bullet).$$

Now, we consider the two cases separately. First, assume $\beta \in \check{\Phi}^+$. Then, for $X_\beta(k) := X_\beta \otimes t^k$, we have by equation (23):

$$\begin{aligned} (l+m+h^\vee)X_\beta(k) \cdot v_\lambda^*(L_{-k}(w)) &= lkX_\beta \cdot w_{\mu-\beta} + \sum_j \lambda(h_j)\beta(h^j)X_\beta \cdot w_{\mu-\beta} \\ &- \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta-\beta_i \in \check{\Phi}^+}} v_\lambda^*(X_{\beta_i} \cdot v_{\lambda-\beta_i}^\bullet)[X_\beta, X_{-\beta_i}] \cdot w_{\mu-\beta+\beta_i}^\bullet - v_\lambda^*(X_\beta \cdot v_{\lambda-\beta}^\bullet)((\mu, \beta) + km)w_\mu^\bullet, \end{aligned}$$

by [KRR, Corollary 10.1, Identity 10.13b]

$$\begin{aligned} (24) \quad &= lkX_\beta \cdot w_{\mu-\beta} + (\lambda|_{\mathfrak{h}}, \beta|_{\mathfrak{h}})X_\beta \cdot w_{\mu-\beta} - \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta-\beta_i \in \check{\Phi}^+}} v_\lambda^*(X_{\beta_i} \cdot v_{\lambda-\beta_i}^\bullet)[X_\beta, X_{-\beta_i}] \cdot w_{\mu-\beta+\beta_i}^\bullet \\ &- v_\lambda^*(X_\beta \cdot v_{\lambda-\beta}^\bullet)((\mu, \beta) + km)w_\mu^\bullet. \end{aligned}$$

For any root $\beta_i \in \check{\Phi}^+$, choose a basis of $V(\lambda)_{\lambda-\beta_i}$ consisting of $X_{-\beta_i} \cdot v_\lambda$ (which we will denote simply by $v_{\lambda-\beta_i}^\circ$) and any basis $v_{\lambda-\beta_i}^{\bullet'}$ of $V(\lambda)_{\lambda-\beta_i}$ annihilated by X_{β_i} . With such a basis, equation (24) becomes:

$$\begin{aligned} (25) \quad (l+m+h^\vee)X_\beta(k) \cdot v_\lambda^*(L_{-k}(w)) &= lkX_\beta \cdot w_{\mu-\beta} + (\lambda|_{\mathfrak{h}}, \beta|_{\mathfrak{h}})X_\beta \cdot w_{\mu-\beta} \\ &- \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta-\beta_i \in \check{\Phi}^+}} v_\lambda^*(X_{\beta_i}X_{-\beta_i} \cdot v_\lambda)[X_\beta, X_{-\beta_i}] \cdot w_{\mu-\beta+\beta_i}^\circ - v_\lambda^*(X_\beta X_{-\beta} \cdot v_\lambda)((\mu, \beta) + km)w_\mu^\circ. \end{aligned}$$

Since X_{β_i} annihilates w , we get

$$(26) \quad -X_{\beta_i} \cdot w_{\mu-\beta} = (\lambda, \beta_i)w_{\mu-\beta+\beta_i}^\circ.$$

Combining equations (25) and (26), we get taking $X_\beta \cdot w_{\mu-\beta} = w_\mu$ (which is possible by the assumption that condition (b) of Proposition 11.4 is valid for the δ -maximal component $V(\lambda + \mu - \beta)$):

$$\begin{aligned} (27) \quad (l+m+h^\vee)X_\beta(k) \cdot v_\lambda^*(L_{-k}(w)) &= lkw_\mu + (\lambda|_{\mathfrak{h}}, \beta|_{\mathfrak{h}})w_\mu \\ &+ \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta-\beta_i \in \check{\Phi}^+}} [X_\beta, X_{-\beta_i}]X_{\beta_i} \cdot w_{\mu-\beta} + ((\mu, \beta) + km)w_\mu. \end{aligned}$$

We next claim that for any $\beta_i \in \check{\Phi}^+$ such that $\beta - \beta_i \in \check{\Phi}^+$,

$$(28) \quad [X_\beta, X_{-\beta_i}]X_{\beta_i} + [X_\beta, X_{-(\beta-\beta_i)}]X_{\beta-\beta_i} = d_{\beta_i}X_\beta, \text{ for some } d_{\beta_i} \in \mathbb{Z}_{>0}.$$

To prove this, write

$$(29) \quad [X_\beta, X_{-\beta_i}] = cX_{\beta-\beta_i}, \text{ for some } c \neq 0.$$

Then,

$$\begin{aligned} ([X_\beta, X_{-(\beta-\beta_i)}], X_{-\beta_i}) &= -c(X_{-(\beta-\beta_i)}, X_{\beta-\beta_i}), \text{ by (29)} \\ &= -c. \end{aligned}$$

Thus,

$$(30) \quad [X_\beta, X_{-(\beta-\beta_i)}] = -cX_{\beta_i}.$$

Combining (29) and (30), we get

$$\begin{aligned} & [X_\beta, X_{-\beta_i}]X_{\beta_i} + [X_\beta, X_{-(\beta-\beta_i)}]X_{\beta-\beta_i} \\ &= c[X_{\beta-\beta_i}, X_{\beta_i}] \\ &= [[X_\beta, X_{-\beta_i}], X_{\beta_i}], \text{ by (29)} \\ &= d_{\beta_i}X_\beta, \text{ for some } d_{\beta_i} \in \mathbb{Z}_{>0} \text{ by } sl_2\text{-module theory.} \end{aligned}$$

This proves (28). Substituting (28) into (27), we get

$$(31) \quad (l + m + h^\vee)X_\beta(k) \cdot v_\lambda^*(L_{-k}(w)) = \left(lk + (\lambda_{|\mathfrak{h}^\circ}, \beta_{|\mathfrak{h}^\circ}) + (\mu, \beta) + km + \frac{1}{2} \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta - \beta_i \in \check{\Phi}^+}} d_{\beta_i} \right) w_\mu \neq 0,$$

since $\beta \in \check{\Phi}^+$. This proves the first part of the lemma.

Now, consider the case where $\beta = \delta - \gamma$ for some $\gamma \in \check{\Phi}^+$. Recall that θ is the highest root of \mathfrak{g} . By the analogue of equation (23) for β ,

$$\begin{aligned} & (l + m + h^\vee)X_{-\gamma}(k + 1) \cdot v_\lambda^*(L_{-k}(w)) \\ &= l(k + 1)X_{-\gamma}(1) \cdot w_{\mu-\beta} - \sum_j \lambda(h_j)\gamma(h^j)X_{-\gamma}(1) \cdot w_{\mu-\beta} \\ &\quad - \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta_i + \gamma \in \check{\Phi}^+}} v_\lambda^*(X_{\beta_i} \cdot v_{\lambda-\beta_i}^\bullet)[X_{-\gamma}, X_{-\beta_i}](1) \cdot w_{\mu-\beta+\beta_i} + A \end{aligned}$$

using the analogue of the equation (26) obtained from $X_{-\beta_i}(1) \cdot w = 0$.

$$\begin{aligned} &= l(k + 1)X_{-\gamma}(1) \cdot w_{\mu-\beta} - (\lambda_{|\mathfrak{h}^\circ}, \gamma_{|\mathfrak{h}^\circ})X_{-\gamma}(1) \cdot w_{\mu-\beta} \\ &\quad - \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta_i + \gamma \in \check{\Phi}^+}} v_\lambda^*(X_{\beta_i}X_{-\beta_i} \cdot v_\lambda)[X_{-\gamma}, X_{-\beta_i}](1) \cdot w_{\mu-\beta+\beta_i} + A \\ &= l(k + 1)X_{-\gamma}(1) \cdot w_{\mu-\beta} - (\lambda_{|\mathfrak{h}^\circ}, \gamma_{|\mathfrak{h}^\circ})X_{-\gamma}(1) \cdot w_{\mu-\beta} \\ &\quad + \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta_i + \gamma \in \check{\Phi}^+}} [X_{-\gamma}, X_{-\beta_i}](1)X_{\beta_i} \cdot w_{\mu-\beta} + A, \text{ using equation (26).} \end{aligned}$$

Finally, we get

$$(32) \quad \begin{aligned} & (l + m + h^\vee)X_{-\gamma}(k + 1) \cdot v_\lambda^*(L_{-k}w) = l(k + 1)X_{-\gamma}(1) \cdot w_{\mu-\beta} \\ & \quad - (\lambda_{|\mathfrak{h}^\circ}, \gamma_{|\mathfrak{h}^\circ})X_{-\gamma}(1) \cdot w_{\mu-\beta} + \sum_{\substack{\beta_i \in \check{\Phi}^+ \\ \beta_i + \gamma \in \check{\Phi}^+}} [X_{-\gamma}(1), X_{-\beta_i}]X_{\beta_i} \cdot w_{\mu-\beta} + A. \end{aligned}$$

□

Proposition 11.6. *Let (λ, μ, β) be a Wahl triple for a real root β . Then, the condition (b) of Proposition 11.4 (specifically, the identity (19)) is satisfied for some embedding $V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu)$ in the following cases:*

- (a) \mathfrak{g} is arbitrary but β is of the form $k\delta + \gamma$, for any $\gamma \in \mathring{\Phi}^+$ and $k \geq 0$.
- (b) \mathfrak{g} is arbitrary but β is of the form $(k + 1)\delta - \theta$, for any $k \geq 0$, where θ is the highest root of \mathfrak{g} .
- (c) \mathfrak{g} is simply-laced.
- (d) \mathfrak{g} is of type B_ℓ ($\ell \geq 2$).
- (e) \mathfrak{g} is of type C_ℓ ($\ell \geq 2$).
- (f) \mathfrak{g} is not of type G_2 and λ is regular dominant.

Proof. (a) By [Ku1, §2.7], the proposition is true for $\beta = \gamma$. Thus, (a) follows from Lemma 11.5.

(b) In this case $\beta = \alpha_0 + k\delta$ for $k \geq 0$. For $\beta = \alpha_0$, the proposition follows from the proof of Lemma 8.1. Now, to prove the proposition for $\beta = \alpha_0 + k\delta$, by the identity (32) of the proof of Lemma 11.5, we get that the right side of the identity (32) equals $((l + m)(k + 1) - (\lambda + \mu|\theta)) X_{-\theta}(1) \cdot w_{\mu - \alpha_0}$. Now, $X_{-\theta}(1) \cdot w_{\mu - \alpha_0} \neq 0$ since (λ, μ, α_0) is a Wahl triple; and $(l + m)(k + 1) - (\lambda + \mu|\theta) = (l + m)k + (\lambda + \mu)(\alpha_0^\vee) \neq 0$. This proves (b).

(c) Using (a) and (b), it suffices to prove the proposition for $\beta = (k + 1)\delta - \gamma$, for $k \geq 0$ and $\gamma \in \mathring{\Phi}^+ \setminus \{\theta\}$. In this case the proposition follows from the proof of Proposition 7.3.

(d) As in (c), it suffices to prove the proposition for $\beta = (k + 1)\delta - \gamma$, for $k \geq 0$ and $\gamma \in \mathring{\Phi}^+ \setminus \{\theta\}$. Observe first that if $\beta - 2\alpha_i \notin \mathring{\Phi}^+$ for any simple root α_i ($0 \leq i \leq \ell$), then by the proof of Proposition 7.3, the proposition follows. So, assume that $\beta - 2\alpha_{i_o} \in \mathring{\Phi}^+$ for some (and hence unique by Lemma 7.1) simple root α_{i_o} . If $i_o \neq 0$, then by Section 8.1, $F_\beta = F_{\beta - \alpha_{i_o}}$. (Observe that $F_\beta = F_{\beta + k\delta}$ for any $k \in \mathbb{Z}$.) Hence, the proposition in this case follows from the proof of Proposition 7.5. So, assume now that $i_o = 0$. In this case, by the proof of Lemma 8.1, $\beta = (k + 1)\delta - \theta$, which is already covered by (b). This completes the proof of the proposition in the case (d).

(e) As in (d), it suffices to prove the proposition for $\beta = (k + 1)\delta - \gamma$, for $k \geq 0$ and $\gamma \in \mathring{\Phi}^+ \setminus \{\theta\}$ such that $\beta - 2\alpha_j \in \mathring{\Phi}^+$ for some (and hence unique) simple root α_j ($1 \leq j \leq \ell$). Thus, $\beta = (k + 1)\delta - \gamma_i$, $k \geq 0$, where $\gamma = \gamma_i$ ($2 \leq i \leq \ell$) is as in Example of $C_\ell^{(1)}$ in Section 8. Further, using Lemma 7.7, we can assume that $\lambda = \mu = \rho_\beta = \Lambda_{i-1}$ (cf. Example of $C_\ell^{(1)}$ in Section 8). (Observe that if the proposition is valid for the Wahl triple (λ, μ, β) , then the proposition is valid for $(\lambda + \lambda', \mu + \mu', \beta)$ for any dominant weights $\lambda', \mu' \in \mathcal{P}^+$ by the proof of Corollary 4.2.)

For $\beta = \delta - \gamma_i$, the proposition follows from the proof of Proposition 7.3 since $\Lambda_{i-1}(\beta^\vee) = 1$. We now prove the proposition for $\beta = (k + 1)\delta - \gamma_i$ for $k \geq 1$ (and $\lambda = \mu = \Lambda_{i-1}$). We use the identity (32) of the proof of Lemma 11.5 to show that in this case $X_{-\gamma_i}(k + 1) \cdot v_\lambda^*(L_{-k}w) \neq 0$:

By the proof of Proposition 7.3, we can take $w_{\Lambda_{i-1} - (\delta - \gamma_i)} = X_{\gamma_i}(-1) \cdot w_{\Lambda_{i-1}}$. We freely use the notation from [Bou81, Planche III]. Then, for any $\beta_j \in \mathring{\Phi}^+$ such that $\beta_j + \gamma_i \in \mathring{\Phi}^+$, we get that β_j is precisely of the form

$\epsilon_j - \epsilon_i$ for $1 \leq j < i$ (and $\gamma_i = 2\epsilon_i$). Thus,

$$\begin{aligned}
 & [X_{-\gamma_i}(1), X_{-\beta_j}]X_{\beta_j}X_{\gamma_i}(-1) \cdot w_{\Lambda_{i-1}} \\
 &= [[X_{-\gamma_i}(1), X_{-\beta_j}], [X_{\beta_j}, X_{\gamma_i}(-1)]] \cdot w_{\Lambda_{i-1}} \\
 &= ([X_{-\gamma_i}(1), X_{-\beta_j}], [X_{\beta_j}, X_{\gamma_i}(-1)]) (\Lambda_{i-1}, \delta - \gamma_i - \beta_j)w_{\Lambda_{i-1}}, \\
 &\quad \text{by [Ku3, Theorem 1.5.4]} \\
 &= (X_{-\gamma_i}(1), [X_{-\beta_j}, [X_{\beta_j}, X_{\gamma_i}(-1)]]) (\Lambda_{i-1}, \delta - \gamma_i - \beta_j)w_{\Lambda_{i-1}} \\
 &= (X_{-\gamma_i}(1), X_{\gamma_i}(-1)) (\Lambda_{i-1}, \delta - \gamma_i - \beta_j)w_{\Lambda_{i-1}}, \\
 (33) \quad &= \frac{1}{2}w_{\Lambda_{i-1}}, \text{ since } (\Lambda_{i-1}, \delta - \gamma_i - \beta_j) = 1/2.
 \end{aligned}$$

Hence,

$$\sum_{1 \leq j < i} [X_{-\gamma_i}(1), X_{-\beta_j}]X_{\beta_j}X_{\gamma_i}(-1) \cdot w_{\Lambda_{i-1}} = \frac{(i-1)}{2}w_{\Lambda_{i-1}}.$$

Further,

$$\begin{aligned}
 X_{-\gamma_i}(1)X_{\gamma_i}(-1) \cdot w_{\Lambda_{i-1}} &= [X_{-\gamma_i}(1), X_{\gamma_i}(-1)] \cdot w_{\Lambda_{i-1}} = w_{\Lambda_{i-1}}, \\
 &\quad \text{since } l = m := \Lambda_{i-1}(K) = 1.
 \end{aligned}$$

So, the right side of the identity (32) of the proof of Lemma 11.5 becomes

$$\left(2(k+1) + \frac{(i-1)}{2} \right) w_{\Lambda_{i-1}} \neq 0.$$

This proves the proposition in this case.

- (f) Proof of the proposition in this case follows from the proofs of Propositions 7.3 and 7.5.

□

Following is the main theorem of this section:

Theorem 11.7. *Let \mathfrak{g} be an affine Kac-Moody Lie algebra and let $w \in W'_P$ be such that the Schubert variety X_w^P is \mathcal{P} -stable. Then, for any $\lambda, \mu \in \mathcal{P}_S^{+o}$ (where S is an arbitrary subset of the simple roots of \mathfrak{g}) such that the condition (b) of Proposition 11.4 is satisfied for all the Wahl triples (λ, μ, β) for any real root $\beta \in \Phi^+(S)$ (cf. Proposition 11.6),*

$$(34) \quad H^p(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

In particular, the canonical Gaussian map

$$(35) \quad H^0(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \rightarrow H^0(\hat{\mathcal{X}}_w^P, (\hat{\mathcal{I}}_e/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu))$$

is surjective.

In particular, (34) and (35) are valid for any simply-laced \mathfrak{g} and \mathfrak{g} of types B_ℓ, C_ℓ . Moreover, they also are valid for \mathfrak{g} of type F_4 in the case \mathcal{P} is the Borel subgroup \mathcal{B} .

Proof. By Proposition 11.4, we get the vanishing $H^1(\hat{\mathcal{X}}_w^P, \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) = 0$.

To prove the higher cohomology vanishing, consider the cohomology exact sequence, associated to the sheaf exact sequence over $\hat{\mathcal{X}}_w^P$:

$$(36) \quad 0 \rightarrow \hat{\mathcal{I}}_e^2 \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu) \rightarrow 0.$$

First of all by [Ku2, Theorem 2.7] (also see [M2, Lemmes 21]),

$$(37) \quad H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \mathcal{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

Further, by Lemma 11.1 and Proposition 10.1,

$$(38) \quad H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathcal{O}}_w/\hat{\mathcal{I}}_e^2) \otimes \mathcal{L}_w(\lambda \boxtimes \mu)) \simeq H^p(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}((\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*)).$$

Now, we prove the vanishing:

$$(39) \quad H^p(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(\lambda) \otimes \mathcal{L}((\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*)) = 0, \text{ for all } p > 0.$$

Take a \mathcal{P} -module filtration of the finite dimensional \mathcal{P} -module $M_w := \mathbb{C}_{\lambda}^* \otimes (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*$:

$$0 = F_0 \subset F_1 \subset \dots \subset F_d = M_w,$$

such that $V_j := F_j/F_{j-1}$ is an irreducible \mathcal{P} -module for each $1 \leq j \leq d$. By Engel's theorem, V_j is a trivial \mathfrak{u} -module, thus it can be considered as an irreducible module for the (finite dimensional) Levi algebra $\mathfrak{l} := \mathfrak{p}/\mathfrak{u}$. Let δ_j be the lowest weight of V_j as an \mathfrak{l} -module. Thus,

$$(40) \quad \delta_j(\alpha_k^{\vee}) \leq 0, \text{ for all } k \in S.$$

To prove the vanishing (39), it suffices to show that

$$(41) \quad H^p(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(V_j)) = 0, \text{ for all } p > 0.$$

From the classical Borel-Weil-Bott theorem for the finite dimensional reductive Lie algebra \mathfrak{l} and the Leray spectral sequence for the fibration $\mathcal{G}/\mathcal{B} \rightarrow \mathcal{G}/\mathcal{P}$, we get

$$(42) \quad H^p(\mathcal{X}_{\mathcal{P}}, \mathcal{L}(V_j)) \simeq H^p(\mathcal{X}_{\mathcal{B}}, \mathcal{L}(-\delta_j)), \text{ for all } p \geq 0.$$

Observe further that δ_j , in particular, is a weight of $\mathbb{C}_{\lambda}^* \otimes (\hat{T}_e(X_w^{\mathcal{P}}) \cdot v_{\mu})^*$ which is a quotient of $\mathbb{C}_{\lambda}^* \otimes (\mathfrak{g} \cdot v_{\mu})^{\vee}$. Thus, δ_j is of the form $\delta_j = -(\lambda + \mu - \beta)$, for some $\beta \in \Phi^+(S) \cup \{0\}$. Hence, for $i \notin S$, since $\lambda, \mu \in \mathcal{P}_S^{+o}$, by [Bou81, Page 278, item 6],

$$(43) \quad -\delta_j(\alpha_i^{\vee}) \geq -1.$$

Thus, combining equations (40), (43) and (42), the vanishing (41) follows from the affine analogue of the Borel-Weil-Bott theorem (cf. [Ku3, Corollary 8.3.12]). This proves the vanishing (39).

Thus, the cohomology exact sequence associated to the sheaf exact sequence (36) and the above two vanishing results (37) and (39) (using the identification (38)) prove (34) and (35) of the theorem.

The last 'in particular' statement of the theorem follows from Proposition 11.6. □

For any $w \in W_{\mathcal{P}}'$ such that $X_w^{\mathcal{P}}$ is \mathcal{P} -stable, we think of $\hat{\mathcal{X}}_w^{\mathcal{P}}$ as a sub-ind-variety of $\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$ via the isomorphism $\delta : \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}} \rightarrow \mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$ (as in the beginning of Section 11). From now on, we think of the sheaf $\hat{\mathcal{I}}_e^k = \hat{\mathcal{I}}_e(w)^k$ over $\hat{\mathcal{X}}_w^{\mathcal{P}}$ (for any $k \geq 1$) as a sheaf over $\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$ by considering $\delta_*(i_w)_*(\hat{\mathcal{I}}_e(w)^k)$, where $i_w : \mathcal{G} \times^{\mathcal{P}} X_w^{\mathcal{P}} \rightarrow \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}}$ is the canonical embedding. Let \mathcal{I}_D be the ideal sheaf of the diagonal $D \subset \mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$. Then, from the sheaf exact sequence:

$$0 \rightarrow \hat{\mathcal{I}}_e(w) \rightarrow \hat{\mathcal{O}}_w \rightarrow \hat{\mathcal{O}}_e \rightarrow 0,$$

we get that

$$\mathcal{I}_D \simeq \varprojlim_w \hat{\mathcal{I}}_e(w).$$

Define

$$\tilde{\mathcal{I}}_D^2 := \varprojlim_w \hat{\mathcal{I}}_e(w)^2.$$

Observe that $\mathcal{I}_D^2 \subset \tilde{\mathcal{I}}_D^2$ and the inclusion is strict in general.

As a corollary of Theorem 11.7, we get the following. This was conjectured in the finite case by Wahl [Wah] and proved in that case by Kumar [Ku1].

Corollary 11.8. *Under the notation and assumptions of Theorem 11.7, the canonical Gaussian map*

$$H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)) \rightarrow H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\mathcal{I}_D / \tilde{\mathcal{I}}_D^2) \otimes \mathcal{L}(\lambda \boxtimes \mu))$$

is surjective.

In particular, it is surjective for any simply-laced \mathfrak{g} as well as \mathfrak{g} of types B_ℓ, C_ℓ . In addition, it is surjective for \mathfrak{g} of type F_4 in the case \mathcal{P} is the Borel subgroup \mathcal{B} .

Proof. By Theorem 11.7, for any $w \in W'_\mathcal{P}$ such that $X_w^\mathcal{P}$ is \mathcal{P} -stable, the canonical map

$$\eta_w : H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \hat{\mathcal{I}}_e(w) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \rightarrow H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2) \otimes \mathcal{L}(\lambda \boxtimes \mu))$$

is surjective. Moreover, η_w is a \mathcal{G} -module morphism between \mathcal{G} -modules which are finite direct sums of $V(\theta)^*$ for some $\theta \in \mathcal{P}^+$ (cf. proof of Proposition 11.4). Let K_w be the kernel of η_w . Then, K_w being a finite direct sum of $V(\theta)^*$, it is easy to see that $\{K_w\}_w$ satisfies the Mittag-Leffler condition (cf. [H2, Chap. II, §9]). Thus, by [H2, Chap. II, Proposition 9.1], the canonical map

$$(44) \quad \varprojlim_w H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \hat{\mathcal{I}}_e(w) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \rightarrow \varprojlim_w H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2) \otimes \mathcal{L}(\lambda \boxtimes \mu))$$

is surjective. Further, by [H2, Chap. II, Proposition 9.2],

$$(45) \quad \varprojlim_w H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \hat{\mathcal{I}}_e(w) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \simeq H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu))$$

and

$$(46) \quad \begin{aligned} \varprojlim_w H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \\ \simeq H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\varprojlim_w \hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2) \otimes \mathcal{L}(\lambda \boxtimes \mu)). \end{aligned}$$

From the left exactness of \varprojlim in the category of sheaves,

$$(47) \quad j : \mathcal{I}_D / \tilde{\mathcal{I}}_D^2 \hookrightarrow \varprojlim_w \hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2.$$

Hence, by combining equations (44)–(47), we get that the composite

$$\begin{aligned} H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \mathcal{I}_D \otimes \mathcal{L}(\lambda \boxtimes \mu)) &\xrightarrow{\eta} H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \mathcal{I}_D / \tilde{\mathcal{I}}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu)) \\ &\xrightarrow{j_*} H^0(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, (\varprojlim_w \hat{\mathcal{I}}_e(w) / \hat{\mathcal{I}}_e(w)^2) \otimes \mathcal{L}(\lambda \boxtimes \mu)) \end{aligned}$$

is surjective and hence so is η (and j_* is an isomorphism). This proves the corollary. \square

Remark 11.9.

- (a) It is very likely that, under the assumptions of Corollary 11.8, $H^p(\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \mathcal{I}_D^2 \otimes \mathcal{L}(\lambda \boxtimes \mu)) = 0$ for all $p > 0$ (cf. [H1, Chap. I, Theorem 4.5]). Moreover, for $p = 1$, it is equivalent to the condition (b) of Proposition 11.4.
- (b) It is likely that Theorem 11.7 (and hence Corollary 11.8) is valid for any \mathfrak{g} and any $\lambda, \mu \in \mathcal{P}_S^{+o}$.

REFERENCES

- [Bou81] Nicolas Bourbaki, *Éléments de mathématique* (French), Masson, Paris, 1981. Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6]. MR647314
- [BrKu] Merrick Brown and Shrawan Kumar, *A study of saturated tensor cone for symmetrizable Kac-Moody algebras*, Math. Ann. **360** (2014), no. 3-4, 901–936, DOI 10.1007/s00208-014-1040-8. MR3273648
- [H1] Robin Hartshorne, *On the De Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 5–99. MR432647
- [H2] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157
- [Kac] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511626234. MR1104219
- [KRR] Victor G. Kac, Ashok K. Raina, and Natasha Rozhkovskaya, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, 2nd ed., Advanced Series in Mathematical Physics, vol. 29, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013, DOI 10.1142/8882. MR3185361
- [Kas1] M. Kashiwara, *On crystal bases of the Q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516, DOI 10.1215/S0012-7094-91-06321-0. MR1115118
- [Kas2] Masaki Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485, DOI 10.1215/S0012-7094-93-06920-7. MR1203234
- [Kos] Bertram Kostant, *A formula for the multiplicity of a weight*, Trans. Amer. Math. Soc. **93** (1959), 53–73, DOI 10.2307/1993422. MR109192
- [Ku1] Shrawan Kumar, *Proof of Wahl’s conjecture on surjectivity of the Gaussian map for flag varieties*, Amer. J. Math. **114** (1992), no. 6, 1201–1220, DOI 10.2307/2374759. MR1198300
- [Ku2] Shrawan Kumar, *Existence of certain components in the tensor product of two integrable highest weight modules for Kac-Moody algebras*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys., vol. 7, World Sci. Publ., Teaneck, NJ, 1989, pp. 25–38. MR1026944
- [Ku3] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston, Inc., Boston, MA, 2002, DOI 10.1007/978-1-4612-0105-2. MR1923198
- [Ku4] Shrawan Kumar, *Tensor product decomposition*, Proceedings of the International Congress of Mathematicians. Volume III, Hindustan Book Agency, New Delhi, 2010, pp. 1226–1261. MR2827839
- [M1] Olivier Mathieu, *Formules de caractères pour les algèbres de Kac-Moody générales* (French, with English summary), Astérisque **159-160** (1988), 267. MR980506
- [M2] Olivier Mathieu, *Construction d’un groupe de Kac-Moody et applications* (French), Compositio Math. **69** (1989), no. 1, 37–60. MR986812
- [Wah] Jonathan Wahl, *Gaussian maps and tensor products of irreducible representations*, Manuscripta Math. **73** (1991), no. 3, 229–259, DOI 10.1007/BF02567640. MR1132139

SCHOOL OF MATHEMATICS AND PHYSICS, THE UNIVERSITY OF QUEENSLAND, BRISBANE, QUEENSLAND 4072, AUSTRALIA

Email address: s.jeralds@uq.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NORTH CAROLINA 27599-3250

Email address: shrawan@email.unc.edu