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THE INTEGRAL GEOMETRIC SATAKE EQUIVALENCE IN MIXED CHARACTERISTIC

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ABSTRACT. Let k be an algebraically closed field of characteristic p. Denote by $W(k)$ the ring of Witt vectors of k. Let F denote a totally ramified finite extension of $W(k)[1/p]$ and $\mathcal{O}$ its ring of integers. For a connected reductive group scheme G over $\mathcal{O}$, we study the category $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$ of L^+G-equivariant perverse sheaves in Λ-coefficient on the Witt vector affine Grassmannian Gr_G where $\Lambda = \mathbb{Z}_\ell$ and $\mathbb{F}_\ell$ ($\ell \neq p$), and prove that it is equivalent as a tensor category to the category of finitely generated Λ-representations of the Langlands dual group of G.

CONTENTS

1. INTRODUCTION

The geometric Satake equivalence establishes an equivalence between two symmetric monoidal categories which are of great importance in algebraic geometry, number theory, and representation theory. The first category is $\mathrm{Rep}_\Lambda(\hat{G})$, the category of finitely generated $\hat{G}$-modules over Λ, where $\hat{G}$ is the Langlands dual group of a connected reductive group G; and the second category is $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$, the category of Λ-coefficient L^+G-equivariant perverse sheaves on the affine Grassmannian Gr_G of G. This equivalence may be regarded as a categorification of the classical Satake isomorphism for connected reductive groups.

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In both equal characteristic (cf. [9], [7], [2], [11]) and mixed characteristic setting with $\bar{\mathbb{Q}}_\ell$-coefficient (cf. [17]), the geometric Satake equivalence has found many significant applications. For example, the equal characteristic geometric Satake equivalence is used in Lafforgue's proof of the "automorphic to Galois" direction of the Langlands correspondence over global function fields [8]. Another noticeable example is a recent work of Xiao-Zhu [15], in which they use the $\bar{\mathbb{Q}}_\ell$-coefficient geometric Satake equivalence in mixed characteristic to prove the "generic" cases of Tate conjecture for the mod p fibers of many Shimura varieties. In the present paper, we prove the integral coefficient geometric Satake equivalence in mixed characteristic setting.

1.1. **Main result.** Let k be an algebraically closed field of characteristic $p > 0$, and denote by $W(k)$ its ring of Witt vectors. Let F denote a totally ramified finite extension of $W(k)[1/p]$ and $\mathcal{O}$ the ring of integers of F. For a connected reductive group scheme G over $\mathcal{O}$, denote by Gr_G the Witt vector affine Grassmannian of G. In this paper, we consider the category $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$ of L^+G-equivariant perverse sheaves in Λ-coefficient on the affine Grassmannian Gr_G for $\Lambda = \mathbb{F}_\ell$ and $\mathbb{Z}_\ell$, where ℓ is a prime number different from p. We call this category the Satake category and sometimes write it as $\mathrm{Sat}_{G,\Lambda}$ for simplicity. The convolution product of sheaves equips the Satake category with a monoidal structure. Let $\hat{G}_\Lambda$ denote the Langlands dual group of G, i.e. the canonical pinned split reductive group scheme over Λ whose root datum is dual to that of G. Our main theorem is the geometric Satake equivalence in the current setting.

Theorem 1.1. *There is an equivalence of monoidal categories between*

$$\mathrm{P}_{L^+G}(Gr_G, \Lambda)$$

and the category of representations of the Langlands dual group $\hat{G}_\Lambda$ on finitely generated Λ-modules.

We mention that Scholze has announced a global version of Theorem 1.1 as part of his work on the local Langlands conjecture for p-adic groups using his beautiful theory of diamonds, cf. [6].

In the equal characteristic case, the Beilinson-Drinfeld Grassmannians play a crucial role in establishing the geometric Satake equivalence. In fact, they can be used to construct the monoidal structure of the hypercohomology functor

$$\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda$$

and the commutativity constraint in the Satake category by interpreting the convolution product as fusion product. In mixed characteristic, Scholze's theory of diamonds allows him to construct an analogue of the Beilinson-Drinfeld Grassmannian and prove the geometric Satake equivalence in this setting in a similar way as in [11]. We pursue a different strategy to construct the geometric Satake equivalence which makes use of some ideas in [17]. However, our situation is different from *loc.cit* and new difficulties arise. One of the most significant differences, the Satake category in $\bar{\mathbb{Q}}_\ell$-coefficient is semisimple, while, in our case, the semisimplicity of the Satake category fails. In addition, the monoidal structure of the hypercohomology functor was constructed by studying the equivariant cohomology of (convolutions of) irreducible objects in the Satake category in [17]. Nevertheless, in our situation, the equivariant cohomology may have torsion. Thus the method in *loc.cit* does not apply to our case directly. To deal with these difficulties, we give a new approach to construct the monoidal structure of the hypercohomology functor and the commutativity constraint in the Satake category. We briefly discuss our strategy as follows.

The first key ingredient of the proof is Proposition 1.2 (cf. Proposition 6.4).

Proposition 1.2. *The hypercohomology functor* $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda$ *is a monoidal functor.*

We study the $\mathbb{G}_m$-action (in fact, we consider the action of the perfection of the group scheme $\mathbb{G}_m$) on the convolution Grassmannian $Gr_G \tilde{\times} Gr_G$. Applying the Mirković-Vilonen theory for mixed characteristic affine Grassmannians established in [17] and Braden's hyperbolic localization functor [5], we can decompose the hypercohomology functor $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G \tilde{\times} Gr_G, \Lambda) \to \mathrm{Mod}(\Lambda)$ into a direct sum of compactly supported cohomologies. Each direct summand can be further realized as the tensor product of two compactly supported cohomologies on Gr_G by the Künneth formula. Putting these together completes the proof of Proposition 1.2. In particular, the monoidal structure constructed by our approach is compatible with that obtained in [17].

We further notice that as in the cases discussed in [11] and [17], the hypercohomology functor is representable by projective objects when restricted to certain full subcategories of the Satake category. In addition, these projective objects are isomorphic to the projective objects studied in [17] after base change to $\bar{\mathbb{Q}}_\ell$. This, together with Proposition 1.2, allows us to directly construct a Λ-algebra $B(\Lambda)$ as in [11]. The compatibility of the monoidal structure of H^* and the projective objects constructed in our case with those obtained in [17] enable us to inherit a commutative multiplication map of $B(\Lambda)$ from that of $B(\bar{\mathbb{Q}}_\ell)$, where the later comes from the commutativity constraint of $\mathrm{Sat}_{G,\bar{\mathbb{Q}}_\ell}$ constructed in *loc.cit.* In other words, the Λ-algebra $B(\Lambda)$ admits the structure of a commutative Hopf algebra with an antipode.

The general Tannakian construction (cf. [11]) yields an equivalence of tensor categories

$$\mathrm{P}_{L^+G}(Gr_G, \Lambda) \simeq \mathrm{Rep}_\Lambda(\widetilde{G}_\Lambda),$$

where $\widetilde{G}_\Lambda := \operatorname{Spec} B(\Lambda)$ is an affine flat group scheme and $\mathrm{Rep}_\Lambda(\widetilde{G}_\Lambda)$ denotes the category of $\widetilde{G}_\Lambda$-modules which are finitely generated over Λ. We give two approaches identifying $\widetilde{G}_\Lambda$ with $\hat{G}_\Lambda$ and conclude the proof of the theorem by a result of Prasad-Yu [13] on quasi-reductive group schemes (cf. Theorem 8.2).

1.2. **Organization of the paper.** We briefly discuss the organization of this paper. In §2, we give a quick review of the construction of affine Grassmannians in mixed characteristic. Section 3 is devoted to the construction of the Satake category and its monoidal structure. We study the hyperbolic localization functors on the affine Grassmannian and define the weight functors in §4. In §5, we prove that the weight functors are representable and study the structure of the projective objects. In §6, we prove that the hypercohomology functor H^* can be endowed with a monoidal structure. We apply a generalized Tannakian formalism in §7. In §8, we identify the group scheme constructed in §7 with the Langlands dual group and conclude the main theorem of this paper.

1.3. **Notations.** We fix an algebraically closed field k of characteristic $p > 0$. For any k-algebra R, its ring of Witt vectors is denoted by

$$W(R) = \{(r_0, r_1, \cdots) \mid r_i \in R\}.$$

We denote by $W_h(R)$ the ring of truncated Witt vectors of length h. For any perfect k-algebra R, we know that $W_h(R) = W(R)/p^h W(R)$.

Let $\mathcal{O}_0 = W(k)$, and $F_0 = W(k)[1/p]$. We denote by F a totally ramified finite extension of F_0 and $\mathcal{O}$ the ring of integers of F. We also fix a uniformizer ϖ in $\mathcal{O}$. For any k-algebra R, we define ring of Witt vectors in R with coefficient in $\mathcal{O}$ as

$$W_{\mathcal{O}}(R) := W(R)\hat{\otimes}_{W(k)}\mathcal{O} := \varprojlim_n W_{\mathcal{O},n}(R), \text{ and } W_{\mathcal{O},n}(R) = W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

We define the formal unit disk and formal punctured unit disk to be

$$D_{F,R} := \operatorname{Spec} W_{\mathcal{O}}(R), \text{ and } D^{\times}_{F,R} := \operatorname{Spec} W_{\mathcal{O}}(R)[1/p]$$

respectively. When F is clear from the context, we omit it from the subscripts in $D_{F,R}$ and $D^{\times}_{F,R}$. We will assume G to be a smooth affine group scheme over $\mathcal{O}$ with connected geometric fibers. Also, we denote by $\mathcal{E}_0$ the trivial G-torsor.

In the case G is a split reductive group, we will choose a Borel subgroup $B \subset G$ over $\mathcal{O}$ and a split maximal torus $T \subset B$. For the choice of (B,T) we denote by $U \subset B$ the unipotent radical of B. We also let $\overline{T} \subset \overline{B} \subset \overline{G}$ be the fibres of $T \subset B \subset G$ at $\mathcal{O}/\varpi$, respectively. We write $\Delta(G,T)$ for the root system of G with respect to T. We denote by $\Delta_+(G,B,T)$ the subset of positive roots determined by B and $\Delta_s(G,B,T)$ the subset of simple roots, and $2\rho^\vee$ the sum of all positive coroots. Similarly, we let $\Delta^\vee(G,T)$ denote the coroot system of G respect to T. We write $\Delta^\vee_+(G,B,T)$ for the subset of positive coroots determined by B and $\Delta^\vee_s(G,B,T)$ the subset of simple coroot.

Let $\mathbb{X}_\bullet$ and $\mathbb{X}^\bullet$ denote the coweight lattice and the weight lattice of T, respectively. Let $\mathbb{X}_\bullet^+$ denote the semi-group of dominant coweights with respect to the chosen Borel. Let $2\rho \in \mathbb{X}^\bullet$ be the sum of all positive roots. Define the partial order "$\leq$" on $\mathbb{X}_\bullet$ such that $\lambda \leq \mu$, if and only if $\mu - \lambda$ equals a non-negative integral linear combination of positive coroots. For any $\mu \in \mathbb{X}_\bullet$, denote ϖ^μ by the image of μ under the composition of maps

$$\mathbb{G}_m \to T \subset G.$$

The Langlands dual group of G is denoted by $\hat{G}$.

We will write $\mathbb{G}_m^{p^{-\infty}}$, the perfection of the group scheme $\mathbb{G}_m$, simply as $\mathbb{G}_m$ for convenience. We let $\bigwedge$ be $\mathbb{Z}_\ell$ on $\mathbb{F}_\ell$ for $\ell \neq p$ unless otherwise stated.

2. Mixed characteristic affine Grassmannians

In this section, we review the construction of affine Grassmannians in mixed characteristic and summarize their geometric properties which will be used later following [17]. Most properties appearing in this section have analogies in the equal characteristic setting, and we refer to [11] for a detailed discussion.

Let $\mathcal{X}$ be a finite type $\mathcal{O}$-scheme. We consider the following two presheaves on the category of perfect k-algebras defined as follows

$$L_p^+\mathcal{X}(R) := \mathcal{X}(W_{\mathcal{O}}(R)), \text{ and } L_p^h\mathcal{X}(R) := X(W_{\mathcal{O},h}(R)),$$

which are represented by schemes over k. Their perfections are denoted by

$$L^+\mathcal{X} := (L_p^+\mathcal{X})^{p^{-\infty}}, \text{ and } L^h\mathcal{X} := (L_p^h\mathcal{X})^{p^{-\infty}}$$

respectively, and we call them p-adic *jet spaces*.

Let X be an affine scheme over F. We define the p-adic *loop space* LX of X as the perfect space by assigning a perfect k-algebra R to the set

$$LX(R) = X(W_{\mathcal{O}}(R)[1/p]).$$

Now, let $\mathcal{X} = G$ be a smooth affine group scheme over $\mathcal{O}$. We write $G^{(0)} = G$ and define the h-th congruence group scheme of G over $\mathcal{O}$, denoted by $G^{(h)}$, as the dilatation of $G^{(h-1)}$ along the unit. The group $L^+G^{(h)}$ can be identified with $\ker(L^+G \to L^hG)$ via the natural map $G^{(h)} \to G$. Then L^+G acts on LG by multiplication on the right. We define the *affine Grassmannian* Gr_G of G to be the perfect space

$$Gr_G := [LG/L^+G]$$

on the category of perfect k-algebras.

In the work of Bhatt-Scholze [3], the functor Gr_G is proved to be representable by an inductive limit of perfections of projective varieties.

We recall Proposition 2.1 in [17] for later use.

Proposition 2.1. *Let $\rho : G \to GL_n$ be a linear representation such that GL_n/G is quasi-affine. Then ρ induces a locally closed embedding $Gr_G \to Gr_{GL_n}$. If in addition GL_n/G is affine, then $Gr_G \to Gr_{GL_n}$ is in fact a closed embedding.*

Explicitly, the affine Grassmannian Gr_G can be described as assigning a perfect k-algebra R the set of pairs (P, ϕ), where P is an L^+G-torsor over $\operatorname{Spec} R$ and $\phi : P \to LG$ is an L^+G-equivariant morphism. It is clear from the definition that $LG \to Gr_G$ is an L^+G-torsor and L^+G naturally acts on Gr_G. Then we can form the twisted product which we also call the *convolution product* in the current setting

$$Gr_G \tilde{\times} Gr_G := LG \times^{L^+G} Gr_G := [LG \times Gr_G / L^+G],$$

where L^+G acts on $LG \times Gr_G$ anti-diagonally as $g^+ \cdot ([g_1], [g_2]) := ([g_1(g^+)^{-1}], [g^+ g_2])$.

As in the equal characteristic case, the affine Grassmannians can be interpreted as the moduli stack of G-torsors on the formal unit disk with trivialization away from the origin. More precisely, for each perfect k-algebra R,

$$Gr_G(R) = \left\{ (\mathcal{E}, \beta) \middle| \begin{array}{l} \mathcal{E} \to D_R \text{ is a } G\text{-torsor, and} \\ \beta : \mathcal{E} \mid_{D_R^\times} \simeq \mathcal{E}_0 \mid_{D_R^\times} \end{array} \right\}.$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two G-torsors over D_R, and let $\beta : \mathcal{E}_1 \mid_{D_R^\times} \simeq \mathcal{E}_2 \mid_{D_R^\times}$ be an isomorphism. For any $x \in \operatorname{Spec} R$, one can define the relative position $\operatorname{Inv}(\beta_x)$ of β_x, the base change of β to x, as an element in $\mathbb{X}_\bullet^+$ as in [17]. For any $\mu \in \mathbb{X}_\bullet^+$, define

$$\begin{aligned}(\operatorname{Spec} R)_\mu &:= \{x \in \operatorname{Spec} R \mid \operatorname{Inv}(\beta_x) = \mu\} \\ &\subset (\operatorname{Spec} R)_{\leq\mu} := \{x \in \operatorname{Spec} R \mid \operatorname{Inv}(\beta_x) \leq \mu\}.\end{aligned}$$

It is well known that $(\operatorname{Spec} R)_{\leq\mu}$ is a closed subset of $\operatorname{Spec} R$, and $(\operatorname{Spec} R)_\mu \subset (\operatorname{Spec} R)_{\leq\mu}$ is an open subset (cf. [17, Lemma 1.22]). The affine Grassmannian admits a stratification of nice subspaces indexed by $\mathbb{X}_\bullet^+$.

Definition 2.2. For each $\mu \in \mathbb{X}_\bullet^+$, we define functors on the opposite category of perfect k-algebras

(1) the (spherical) *Schubert variety*

$$Gr_{\leq\mu}(R) := \{(\mathcal{E}, \beta) \in Gr_G(R) \mid (\operatorname{Spec} R)_{\leq\mu} = \operatorname{Spec} R\},$$

(2) the (spherical) *Schubert cell*

$$Gr_\mu(R) := \{(\mathcal{E},\beta) \in Gr_G(R) \mid (\operatorname{Spec} R)_\mu = \operatorname{Spec} R\},$$

for each perfect k-algebra R.

We recall the following basic properties of Schubert cells and Schubert varieties (cf. [17, Proposition 1.23]).

Proposition 2.3.

(1) *Let $\mu \in \mathbb{X}_\bullet^+$, and $\varpi^\mu \in Gr_G$ be the corresponding point in the affine Grassmannian. Then the map*

$$i_\mu : L^+G/(L^+G \cap \varpi^\mu L^+G\varpi^{-\mu}) \longrightarrow [LG/L^+G], \text{ such that } g \longmapsto g\varpi^\mu$$

induces an isomorphism

$$L^+G/(L^+G \cap \varpi^\mu L^+G\varpi^{-\mu}) \simeq Gr_\mu.$$

(2) *Gr_μ is the perfection of a quasi-projective smooth variety of dimension $(2\rho,\mu)$.*

(3) *$Gr_{\leq\mu}$ is the Zariski closure of Gr_μ in Gr_G, and therefore is perfectly proper of dimension $(2\rho,\mu)$.*

The convolution Grassmannian $Gr_G\tilde{\times}Gr_G$ admits a moduli interpretation as follows

$$Gr_G\tilde{\times}Gr_G(R) = \left\{ (\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2) \middle| \begin{array}{l} \mathcal{E}_1,\mathcal{E}_2 \text{ are } G-\text{torsors on } D_R, \text{and} \\ \beta_1 : \mathcal{E}_1 \mid_{D_R^\times} \simeq \mathcal{E}_0 \mid_{D_R^\times}, \beta_2 : \mathcal{E}_2 \mid_{D_R^*} \simeq \mathcal{E}_1 \mid_{D_R^\times} \end{array} \right\}.$$

Via this interpretation, we define the convolution morphism as in the equal characteristic case

$$m : Gr_G\tilde{\times}Gr_G \longrightarrow Gr_G,$$

such that

$$(\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2) \longmapsto (\mathcal{E}_2,\beta_1\beta_2).$$

Note that there is also the natural projection morphism

$$pr_1 : Gr_G\tilde{\times}Gr_G \longrightarrow Gr_G,$$

such that

$$(\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2) \longmapsto (\mathcal{E}_1,\beta_1).$$

It is clear to see that $(pr_1, m) : Gr_G\tilde{\times}Gr_G \simeq Gr_G \times Gr_G$ is an isomorphism.

One can define the i-fold convolution Grassmannian $Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G$ in a similar manner as follows

$$Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G(R) := \left\{ (\mathcal{E}_k,\beta_k)_{k=1}^i \middle| \begin{array}{l} \mathcal{E}_k \text{ is a } G\text{-torsor over } D_R, \text{and} \\ \beta_k : \mathcal{E}_k \mid_{D_R^\times} \simeq \mathcal{E}_{k-1} \mid_{D_R^\times} \end{array} \right\}.$$

For $i = 1, 2, \cdots n$, we also define the i-fold convolution morphism

$$m_i : Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G \longrightarrow Gr_G$$

such that

$$(\mathcal{E}_k,\beta_k)_{k=1}^i \longmapsto (\mathcal{E}_i, \beta_1\beta_2\cdots\beta_i : \mathcal{E}_i \mid_{D_R^\times} \simeq \mathcal{E}_0 \mid_{D_R^\times}).$$

As for the 2-fold convolution Grassmannian, we have an isomorphism

$$(m_1, m_2, \cdots, m_n) : Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G \simeq Gr_G \times \cdots \times Gr_G.$$

Given a sequence of dominant coweights $\mu_\bullet = (\mu_1, \cdots, \mu_n)$ of G, we define the following closed subspace of $Gr_G \tilde{\times} \cdots \tilde{\times} Gr_G$,

$$Gr_{\leq \mu_\bullet} := Gr_{\leq \mu_1} \tilde{\times} \cdots \tilde{\times} Gr_{\leq \mu_n}.$$

For a perfect k-algebra R, $Gr_{\leq \mu_\bullet}(R)$ classifies isomorphism classes of modifications of G-torsors over D_R

$$\mathcal{E}_n \overset{\beta_1}{\dashrightarrow} \mathcal{E}_{n-1} \overset{\beta_{n-1}}{\dashrightarrow} \cdots \overset{\beta_1}{\dashrightarrow} \mathcal{E}_0,$$

where $\mathrm{Inv}(\beta_{i,x}) \leq \mu_i$ for any $x \in \operatorname{Spec} R$.

As in [17], we let $|\mu_\bullet| := \sum \mu_i$. Then the convolution map induces the following morphism

$$m : Gr_{\leq \mu_\bullet} \longrightarrow Gr_{\leq |\mu_\bullet|},$$

such that

$$(\mathcal{E}_i, \beta_i)_i \longmapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n).$$

Replacing $Gr_{\leq \mu_i}$ by Gr_{μ_i}, we can similarly define $Gr_{\mu_\bullet} := Gr_{\mu_1} \tilde{\times} \cdots \tilde{\times} Gr_{\mu_n}$. By Proposition 2.3, we have

$$Gr_{\leq \mu_\bullet} = \cup_{\mu'_\bullet \leq \mu_\bullet} Gr_{\mu'_\bullet}, \tag{2.1}$$

where $\mu'_\bullet \leq \mu_\bullet$ means $\mu'_i \leq \mu_i$ for each i. This gives a stratification of $Gr_{\mu_{\leq \bullet}}$.

3. The Satake category

In this section, we first define the Satake category $\mathrm{Sat}_{G,\Lambda}$ as the category of L^+G-equivariant Λ-coefficient perverse sheaves on Gr_G. We then define the convolution map which enables us to equip the Satake category with a monoidal structure.

Recall that Gr_G can be written as a limit of L^+G-invariant closed spaces

$$Gr_G = \lim_{\to \mu} Gr_{\leq \mu}.$$

For each $\mu \in \mathbb{X}_\bullet^+$, the Schubert variety is perfectly proper, and the action of L^+G on $Gr_{\leq \mu}$ factors through some perfectly of finite type quotient L^hG. Therefore, it makes sense to define the category of L^+G-equivariant perverse sheaves on $Gr_{\leq \mu}$ as in [17, §2.1.1], which we denote by $\mathrm{P}_{L^+G}(Gr_{\leq \mu}, \Lambda)$. Then we define the *Satake category* as

$$\mathrm{P}_{L^+G}(Gr_G, \Lambda) := \lim_{\to \mu} \mathrm{P}_{L^+G}(Gr_{\leq \mu}, \Lambda).$$

We denote by IC_μ for each $\mu \in \mathbb{X}_\bullet^+$ the *intersection cohomology sheaf* on $Gr_{\leq \mu}$. Its restriction to each open strata Gr_μ is constant and in particular, $\mathrm{IC}_\mu \mid_{Gr_\mu} \simeq \underline{\Lambda}[(2\rho, \mu)]$.

With the above preparation, we can define the monoidal structure in $\mathrm{Sat}_{G,\Lambda}$ by Lusztig's convolution of sheaves as in the equal characteristic counterpart. Consider the following diagram

$$Gr_G \times Gr_G \xleftarrow{p} LG \times Gr_G \xrightarrow{q} Gr_G \tilde{\times} Gr_G \xrightarrow{m} Gr_G,$$

where p and q are projection maps. We define for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{L^+G}(Gr_G, \Lambda)$,

$$\mathcal{A} \star \mathcal{A}_2 := Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2),$$

where $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \in \mathrm{P}_{L^+G}(Gr_G \tilde{\times} Gr_G, \Lambda)$ is the unique sheaf such that

$$q^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2) \simeq p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2)).$$

Unlike the construction in $\mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$, we emphasize that taking the 0-th perverse cohomology ${}^p\mathrm{H}(\bullet)$ in the above definition is necessary. This is because when we work with $\mathbb{Z}_\ell$-sheaves, the external tensor product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ may not be perverse. In fact, $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is perverse if one of $\mathrm{H}^*(\mathcal{A}_i)$ is a flat $\mathbb{Z}_\ell$-module. For more details, we refer to [11, Lemma 4.1] for a detailed explanation.

Proposition 3.1 is mentioned as a "miraculous theorem" of the Satake category in equal characteristic (cf. [2]).

Proposition 3.1. *For any $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{L^+G}(Gr_G, \Lambda)$, the convolution product $\mathcal{A}_1 \star \mathcal{A}_2$ is perverse.*

Proof. Note by [17, Proposition 2.3] that the convolution morphism m is a stratified semi-small morphism with respect to the stratification (2.1). Then the proposition follows from the argument of [11, Lemma 4.3]. □

We can also define the n-fold convolution product in $\mathrm{Sat}_{G,\Lambda}$

$$\mathcal{A}_1 \star \cdots \star \mathcal{A}_n := Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n),$$

where $\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n$ is defined in a similar way as $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$. By considering the following isomorphism:

$${}^p\,\mathrm{H}^0(\mathcal{A}_1 \boxtimes ({}^p\,\mathrm{H}^0(\mathcal{A}_2 \boxtimes \mathcal{A}_3)) \simeq {}^p\,\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2 \boxtimes \mathcal{A}_3) \simeq {}^p\,\mathrm{H}^0({}^p\,\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \boxtimes \mathcal{A}_3),$$

we conclude that the convolution product is associative:

$$(\mathcal{A}_1 \star \mathcal{A}_2) \star \mathcal{A}_3 \simeq Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \tilde{\boxtimes} \mathcal{A}_3) \simeq \mathcal{A}_1 \star (\mathcal{A}_2 \star \mathcal{A}_3).$$

Thus, the category $(\mathrm{Sat}_{G,\Lambda}, \star)$ is a monoidal category.

4. Semi-infinite orbits and weight functors

In this section, we review the construction and geometry of semi-infinite orbits of Gr_G. By studying a $\mathbb{G}_m$-action on the affine Grassmannian Gr_G, we realize the semi-infinite orbits as the attracting loci of the $\mathbb{G}_m$-action in the sense of [5]. We also define the weight functors and relate them to the hyperbolic localization functors and study their properties.

We follow notions introduced in §1.3. Since $U\backslash G$ is quasi-affine, Proposition 2.1 shows that $i : Gr_U \hookrightarrow Gr_G$ is a locally closed embedding. For any $\lambda \in \mathbb{X}_\bullet$, define the *semi-infinite orbit* to be the locally closed subspace

$$S_\lambda := \varpi^\lambda i(Gr_U)$$

in Gr_G. On the level of k-points, S_λ equals the orbit of ϖ^λ in Gr_G under the LU-action. By the Iwasawa decomposition for p-adic groups, we know that

$$Gr_G = \cup_{\lambda \in \mathbb{X}_\bullet} S_\lambda.$$

Similarly, consider the opposite Borel B^- and let U^- be its unipotent radical. We also define the *opposite semi-infinite orbits*

$$S_\lambda^- := \varpi^\lambda i(Gr_{U^-}).$$

On the level of k-points, S_λ^- is the LU^--orbit of ϖ^λ in Gr_G.

Recall the following closure relations as in [17, Proposition 2.5] (the equal characteristic analogue of this statement is proved in [11, Proposition 3.1]).

Proposition 4.1. *Let $\lambda \in \mathbb{X}_\bullet$, then $S_{\leq\lambda} := \overline{S_\lambda} = \cup_{\lambda' \leq \lambda} S_{\lambda'}$ and $S_{\leq\lambda}^- := \overline{S_\lambda^-} = \cup_{\lambda' \leq \lambda} S_{\lambda'}^-$.*

Base change the L^+G-torsor $LG \to Gr_G$ along $S_\mu \hookrightarrow Gr_G$, we obtain an L^+U-torsor $LU \to S_\mu$. This allows us to construct the convolution of semi-infinite orbits $S_{\mu_1}\tilde{\times}S_{\mu_2}\tilde{\times}\cdots\tilde{\times}S_{\mu_n}$. Let $\mu_\bullet=(\mu_1,\cdots,\mu_n)$ be a sequence of (not necessarily dominant) coweights of G. We define

$$S_{\mu_\bullet} := S_{\mu_1}\tilde{\times}S_{\mu_2}\tilde{\times}\cdots\tilde{\times}S_{\mu_n} \subset Gr_G\tilde{\times}Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G.$$

The morphism

$$m : S_{\mu_\bullet} \longrightarrow S_{\mu_1}\times S_{\mu_1+\mu_2}\times\cdots\times S_{|\mu_\bullet|}$$

given by

$$\begin{aligned}&(\varpi^{\mu_1}x_1,\varpi^{\mu_2}x_2,\cdots\varpi^{\mu_1 n}x_n)\\ &\qquad\longmapsto(\varpi^{\mu_1}x_1,\varpi^{\mu_1+\mu_2}(\varpi^{-\mu_2}x_1\varpi^{\mu_2}x_2),\cdots,\varpi^{|\mu_\bullet|}(\varpi^{-|\mu_\bullet|+\mu_1}x_1\cdots\varpi^{\mu_n}x_n))\end{aligned}$$

is an isomorphism. The morphism m fits into the following commutative diagram

$$\begin{tikzcd} S_{\leq\mu_\bullet} \arrow[r, "m"] \arrow[d, hook] & S_{\mu_1}\times S_{\mu_1+\mu_2}\times\cdots\times S_{|\mu_n|} \arrow[d, hook] \\ Gr_G\tilde{\times}Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G \arrow[r, "(m_1,\ldots,m_n)"] & Gr_G\times Gr_G\times\cdots\times Gr_G. \end{tikzcd}$$

We also note that there is a canonical isomorphism

$$(S_{\nu_1}\cap Gr_{\leq\mu_1})\tilde{\times}(S_{\nu_2}\cap Gr_{\leq\mu_2})\tilde{\times}\cdots\tilde{\times}(S_{\nu_n}\cap Gr_{\leq\mu_n})\cong S_{\nu_\bullet}\cap Gr_{\leq\mu_\bullet}. \tag{4.1}$$

Similar to the equal characteristic situation (cf. [11] (3.16), (3.17)), the semi-infinite orbits may be interpreted as the attracting loci of certain torus action which we describe here.

Recall the notation $2\rho^\vee$ as in §1.3 and regard it as a cocharacter of G. The projection map $L_p^+\mathbb{G}_m \to \mathbb{G}_m$ admits a unique section $\mathbb{G}_m \to L_p^+\mathbb{G}_m$ which identifies $\mathbb{G}_m$ as the maximal torus of $L_p^+\mathbb{G}_m$. This section allows us to define a cocharacter

$$\mathbb{G}_m \longrightarrow L^+\mathbb{G}_m \overset{L^+(2\rho^\vee)}{\longrightarrow} L^+T \subset L^+G.$$

Then the $\mathbb{G}_m$-action on Gr_G is induced by the action of L^+G on Gr_G. Under this action by $\mathbb{G}_m$, the set of fixed points are precisely $R := \{\varpi^\lambda \mid \lambda\in\mathbb{X}_\bullet\}$. The attracting loci of this action are semi-infinite orbits i.e.

$$S_\lambda = \{g\in Gr_G \mid \lim_{t\to 0} L^+(2\rho^\vee(t))\cdot(g)=\varpi^\lambda \text{ for } t\in\mathbb{G}_m\}.$$

The repelling loci are the opposite semi-infinite orbits i.e.

$$S_\lambda^- = \{g\in Gr_G \mid \lim_{t\to \infty} L^+(2\rho^\vee(t))\cdot(g)=\varpi^\lambda \text{ for } t\in\mathbb{G}_m\}.$$

Recall that if X is a scheme and $i : Y \hookrightarrow X$ is an inclusion of a locally closed subscheme, then for any $\mathcal{F}\in D_c^b(X,\Lambda)$, the local cohomology group is defined as $\mathrm{H}^k_Y(\mathcal{F}) := \mathrm{H}^k(Y, i^!\mathcal{F})$.

Proposition 4.2. *For any $\mathcal{F}\in \mathrm{P}_{L^+G}(Gr_G,\Lambda)$, there is an isomorphism*

$$\mathrm{H}^k_c(S_\mu,\mathcal{F}) \simeq \mathrm{H}^k_{S^-_\mu}(\mathcal{F}),$$

and both sides vanish if $k\neq(2\rho,\mu)$.

Proof. The proof is similar to the equal characteristic case (cf. [11, Theorem 3.5]) as the dimension estimation of the intersections of the semi-infinite orbits with Schubert varieties is established in [17, Corollary 2.8]. Since $\mathcal{F}$ is perverse, for any $\nu \in \mathbb{X}_\bullet^+$, we know that $\mathcal{F}\mid_{Gr_\nu} \in D^{\leq -\dim(Gr_\nu)} = D^{\leq -(2\rho,\nu)}$. By [17, Corollary 2.8], we know that $\mathrm{H}_c^k(S_\mu \cap Gr_{\leq\nu}, \mathcal{F}) = 0$ if $k > 2\dim(S_\mu \cap Gr_{\leq\nu}) = (2\rho, \mu+\nu)$. Filtering Gr_G by $Gr_{\leq\mu}$, we apply a dévissage argument and conclude that

$$\mathrm{H}_c^k(S_\mu, \mathcal{F}) = 0 \text{ if } k > (2\rho, \mu).$$

An analogous argument proves that

$$\mathrm{H}^k_{S_\mu^-}(\mathcal{F}) = 0$$

if $k < (2\rho, \mu)$.

Now by regarding S_μ and S_μ^- as the attracting and repelling loci of the $\mathbb{G}_m$-action, we apply the hyperbolic localization as in [5] and obtain

$$\mathrm{H}_c^k(S_\mu, \mathcal{F}) \simeq \mathrm{H}^k_{S_\mu^-}(\mathcal{F}).$$

The proposition is thus proved. □

Let Mod_Λ denote the category of finitely generated Λ-modules and $\mathrm{Mod}_\Lambda(\mathbb{X}_\bullet)$ denote the category of $\mathbb{X}_\bullet$-graded finitely generated Λ-modules.

Definition 4.3. For any $\mu \in \mathbb{X}_\bullet$, we define

(1) the *weight functor*

$$\mathrm{CT}_\mu : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda(\mathbb{X}_\bullet),$$

by

$$\mathrm{CT}_\mu(\mathcal{F}) := \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F}),$$

(2) the *total weight functor*

$$\mathrm{CT} := \bigoplus_\mu \mathrm{CT}_\mu : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda(\mathbb{X}_\bullet),$$

by

$$\mathrm{CT}(\mathcal{F}) := \bigoplus_\mu \mathrm{CT}_\mu(\mathcal{F}) := \bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F}).$$

We denote by F the forgetful functor from $\mathrm{Mod}_\Lambda(\mathbb{X}_\bullet)$ to Mod_Λ.

Proposition 4.4. *There is a canonical isomorphism of functors*

$$\mathrm{H}^*(Gr_G, \bullet) \cong F \circ \mathrm{CT} : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda.$$

In addition, both functors are exact and faithful.

Proof. By the definition of the semi-infinite orbits and the Iwasawa decomposition, we obtain two stratifications of Gr_G by $\{S_\mu \mid \mu \in \mathbb{X}_\bullet\}$ and $\{S_\mu^- \mid \mu \in \mathbb{X}_\bullet\}$, respectively. The first stratification induces a spectral sequence with E_1-terms $\mathrm{H}_c^k(S_\mu, \mathcal{F})$ and abutment $\mathrm{H}^*(Gr_G, \mathcal{F})$. This spectral sequence degenerates on the E_1-page by Proposition 4.2. Thus, there is a filtration of $\mathrm{H}^*(Gr_G, \mathcal{F})$ indexed by $(\mathbb{X}_\bullet, \leq)$ defined as

$$\mathrm{Fil}_{\geq\mu} \mathrm{H}^*(Gr_G, \mathcal{F}) := \ker(\mathrm{H}^*(Gr_G, \mathcal{F}) \longrightarrow \mathrm{H}^*(S_{<\mu}, \mathcal{F})),$$

where $S_{<\mu} := \cup_{\mu'<\mu} S_{\mu'}$. Direct computation yields that the associated graded of the above filtration is $\bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F})$.

Consider the second stratification of Gr_G. It also induces a filtration of $\mathrm{H}^*(Gr_G, \mathcal{F})$ as

$$\mathrm{Fil}'_{<\mu} \mathrm{H}^k(Gr_G, \mathcal{F}) := \mathrm{Im}(\mathrm{H}^*_{S^-_{\leq\mu}}(\mathcal{F}) \longrightarrow \mathrm{H}^*(Gr_G, \mathcal{F})),$$

where $S^-_{<\mu} := \cup_{\mu'<\mu} S^-_{\mu'}$.

Now, by Proposition 4.2, the two filtrations are complementary to each other and together define the decomposition $\mathrm{H}^*(Gr_G, \bullet) \simeq \bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \bullet)$.

Next, we prove that the total weight functor CT is exact. To do so, it suffices to show that the weight functor CT_μ is exact for each $\mu \in \mathbb{X}_\bullet$. Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence in $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$. It is given by a distinguished triangle

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \xrightarrow{+1}$$

in $D^b_c(Gr_G, \Lambda)$. We thus have a long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}^k_c(S_\mu, \mathcal{F}_1) \longrightarrow \mathrm{H}^k_c(S_\mu, \mathcal{F}_2) \longrightarrow \mathrm{H}^k_c(S_\mu, \mathcal{F}_3) \longrightarrow \mathrm{H}^{k+1}_c(S_\mu, \mathcal{F}_1) \longrightarrow \cdots .$$

Then Proposition 4.2 gives the desired exact sequence

$$0 \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_1) \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_2) \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_3) \longrightarrow 0.$$

We conclude the proof by showing that CT is faithful. Since CT is exact, it suffices to prove that CT maps non-zero objects to non-zero objects. Let $\mathcal{F} \in \mathrm{Sat}_{G,\Lambda}$ be a non-zero object. Then $\mathrm{supp}(\mathcal{F})$ is a finite union of Schubert cells Gr_ν. Choose ν to be maximal for this property. Then $\mathcal{F} \mid_{Gr_\nu} \simeq \underline{\Lambda}^{\oplus n}[(2\rho, \nu)]$ for some positive integer n and it follows that $\mathrm{CT}_\nu(\mathcal{F}) \neq 0$. Thus the functor H^* is faithful. □

Remark 4.5. The weight functor is in fact independent of the choice of the maximal torus T. The proof for this is analogous to the equal characteristic case (cf. [11, Theorem 3.6]), and we omit it here.

We note that the analogue of [17, Corollary 2.9] also holds in our setting. In particular, $\mathrm{H}^*(\mathrm{IC}_\mu)$ is a free Λ-module for any $\mu \in \mathbb{X}^+_\bullet$.

We end this section by proving a weaker statement of [11, Proposition 2.1] which will be used in the process of identification of group schemes in §8.

Lemma 4.6. *There is a natural equivalence of tensor categories*

$$\alpha : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \cong \mathrm{P}_{L^+(G/Z)}(Gr_G, \Lambda),$$

where Z is the center of G.

Proof. We first note that the category $\mathrm{P}_{L^+(G/Z)}(Gr_G, \Lambda)$ can be identified as a full subcategory of $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$. Let $X \subset Gr_G$ be a finite union of L^+G-orbits. Since L^+Z acts on Gr_G trivially, the action of L^+G on Gr_G factors through the quotient $L^+(G/Z)$. In other words, the following diagram commutes

$$\begin{tikzcd} L^+G \times X \arrow[r, "a_1"] \arrow[d, "q"] & X \\ L^+(G/Z) \times X \arrow[ur, "a_2"] & \end{tikzcd}$$

where a_1 and a_2 are the action maps and q is the natural projection map. In addition, the following diagram is clearly commutative.

$$\begin{tikzcd} L^+G \times X \arrow[r, "p_1"] \arrow[d, "q"] & X \\ L^+(G/Z) \times X \arrow[ur, "p_2"] & \end{tikzcd}$$

It follows that if $\mathcal{F} \in \mathrm{P}_{L^+(G/Z)}(Gr_G, \Lambda)$, then $\mathcal{F}$ is automatically L^+G-equivariant.

Thus it suffices to prove the reverse direction. We prove by induction on the number of L^+G-orbits in X following the idea in the proof of [11, Proposition A.1]. First, we assume that X contains exactly one L^+G-orbit. Write $X = Gr_\mu$ for some $\mu \in \mathbb{X}_\bullet^+$. Recall Proposition 2.2.3.(1) and [17, 1.4.4]. There is a natural projection with fibres isomorphic to the perfection of affine spaces

$$\pi_\mu : Gr_\mu \simeq L^+G/(L^+G \cap \varpi^\mu L^+G\varpi^{-\mu}) \longrightarrow (\overline{G}/\overline{P}_\mu)^{p^{-\infty}}$$

$$(gt^\mu \bmod L^+G) \longmapsto (\bar{g} \bmod \overline{P}_\mu^{p^{-\infty}}),$$

where P_μ denotes the parabolic subgroup of G generated by T and the root subgroups U_α of G corresponding to those roots α satisfying $\langle \alpha, \mu \rangle \le 0$, and $\overline{P}_\mu$ denotes the fibre of P_μ at $\mathcal{O}/\varpi$. Assume that L^+G acts on $(\overline{G}/\overline{P}_\mu)^{p^{-\infty}}$ by a finite type quotient L^nG. Since the stabilizer of this action of L^+G is connected, we have a canonical equivalence of categories (cf. [17, A.3.4]) $\mathrm{P}_{L^+G}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda) \simeq \mathrm{P}_{L^nG}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda)$. Finally, we note that $\mathrm{P}_{L^nG}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda)$ is equivalent to $\mathrm{Mod}_\Lambda(\mathbb{B}L^nG)$, we conclude that $\mathrm{P}_{L^+G}(Gr_\mu, \Lambda) \simeq \mathrm{Mod}_\Lambda(\mathbb{B}L^nG)$. A completely similar argument implies that $\mathrm{P}_{L^+(G/Z)}(Gr_\mu, \Lambda) \simeq \mathrm{Mod}_\Lambda(\mathbb{B}L^nG)$ which concludes the proof in the case $X = Gr_\mu$.

Now we treat the general X. The machinery we will use here is the gluing construction for perverse sheaves established in [10], [14]. Assume that L^+G and $L^+(G/Z)$ act on X through finite type quotient L^mG and $L^m(G/Z)$ respectively. Choose $\mu \in \mathbb{X}_\bullet^+$ such that $Gr_\mu \subset X$ is a closed subspace, and let $U := X \backslash Gr_\mu$. By induction hypothesis, we know that $\mathrm{P}_{L^+G}(U, \Lambda)$ is equivalent to $\mathrm{P}_{L^+(G/Z)}(U, \Lambda)$. Denote by $i : Gr_\mu \hookrightarrow X$ and $j : U \hookrightarrow X$ the closed and open embeddings, respectively. Let $\widetilde{Gr}_\mu := L^m(G/Z) \times Gr_\mu$, $\widetilde{X} := L^m(G/Z) \times X$, and $\widetilde{U} := L^m(G/Z) \times U$. Denote by $\tilde{j} : \widetilde{U} \hookrightarrow \widetilde{X}$ the open embedding. The stratification on X induces a stratification on $\widetilde{X}$ which has strata equal to products of $L^m(G/Z)$ with strata in X. Restricting to $\widetilde{U}$, we get a stratification of $\widetilde{U}$. Considering the action of L^mG on $\widetilde{X}$ and $\widetilde{U}$ by left multiplication on the second factor, we can define categories $\mathrm{P}_{L^+G}(\widetilde{X}, \Lambda)$ and $\mathrm{P}_{L^+G}(\widetilde{U}, \Lambda)$. Define the functor

$$\widetilde{\mathrm{CT}}_\mu : \mathrm{P}_{L^mG}(\widetilde{X}, \Lambda) \longrightarrow \mathrm{Loc}_\Lambda(L^m(G/Z))$$

$$\widetilde{\mathrm{CT}}_\mu(\mathcal{F}) := \mathcal{H}^{(2\rho,\mu)+\dim L^m(G/Z)}(\pi_! \tilde{i}^*(\mathcal{F}))$$

for any $\mathcal{F} \in \mathrm{P}_{L^+G}(\widetilde{X}, \Lambda)$, where $\tilde{i} : L^m(G/Z) \times (S_\mu \cap X) \hookrightarrow \widetilde{X}$ is the locally closed embedding, $\pi : L^m(G/Z) \times (S_\mu \cap X) \to L^m(G/Z)$ is the natural projection, and $\mathrm{Loc}_\Lambda(L^m(G/Z))$ denotes the category of Λ-local systems on $L^m(G/Z)$. A completely similar argument as in Proposition 4.2 shows that $\widetilde{\mathrm{CT}}_\mu$ is an exact functor. Let $\widetilde{F}_1 := \widetilde{\mathrm{CT}}_\mu \circ {}^p\tilde{j}_!, \widetilde{F}_2 := \widetilde{\mathrm{CT}}_\mu \circ {}^pj_* : \mathrm{P}_{L^+G}(\widetilde{U}, \Lambda) \to \mathrm{Loc}_\Lambda(L^m(G/Z))$. Finally let $\widetilde{T} := \widetilde{\mathrm{CT}}_\mu({}^p\tilde{j}_! \to {}^p\tilde{j}_*)$. Then as in [11, Appendix A], we get an equivalence of abelian categories

$$\widetilde{E} : \mathrm{P}_{L^+G}(\widetilde{X}, \Lambda) \simeq \mathcal{C}(\widetilde{F}_1, \widetilde{F}_2, \widetilde{T}),$$

where the second category in the above is defined in *loc.cit.* The same argument in [11, Proposition A.1] applies here and gives

$$\widetilde{E}(a_2^*\mathcal{F}) \simeq \widetilde{E}(p_2^*\mathcal{F}).$$

Then we deduce an isomorphism $a_2^*\mathcal{F} \simeq p_2^*\mathcal{F}$ and the lemma is thus proved. □

5. Representability of weight functors and the structure of representing objects

In §4, we construct the weight functors and the total weight functor

$$\mathrm{CT}_\mu, \mathrm{CT} : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda .$$

We will prove in this section that both functors are (pro)representable, so that we can apply the (generalized) Deligne and Milne's Tannakian formalism as in [11, §11]. In the following, we will recall the induction functor (cf. [11]) to explicitly construct the representing object of each weight functor and use the representability of the total weight functor to prove that the Satake category has enough projective objects. At the end of this section, we give a few propositions of the representing objects which will be used to apply the (generalized) Tannakian formalism.

Let $Z \subset Gr_G$ be a closed subspace which is a union of finitely many L^+G-orbits. Choose $n \in \mathbb{Z}$ large enough so that L^+G acts on Z via the quotient L^nG. Let $\nu \in \mathbb{X}_\bullet$. As in [11, §9], we consider the following commutative diagram

$$\begin{tikzcd}
S_\nu^- \cap Z \arrow[d, hook, "i"] & L^nG \times (S_\nu^- \cap Z) \arrow[l] \arrow[r, "\tilde{a}"] \arrow[d, hook] & Z \arrow[d, equal] \\
Z & L^nG \times Z \arrow[l, "p"] \arrow[r, "a"] & Z
\end{tikzcd}$$

where i is the locally closed embedding, a and $\tilde{a}$ are the action maps, and p is the projection map. Then we define

$$P_Z(\nu, \Lambda) := {}^p\mathrm{H}^0(a_! p^! i_! \underline{\Lambda}_{S_\nu^- \cap Z}[-(2\rho, \nu)]).$$

Most results appear in this section can be proved similar to the equal characteristic case. Thus, we will briefly discuss the main ideas in the proofs and refer readers to relevant references for details.

Proposition 5.1. *The restriction of the weight functor* CT_ν *to* $\mathrm{P}_{L^+G}(Z, \Lambda)$ *is represented by the projective object* $P_Z(\nu, \Lambda)$ *in* $\mathrm{P}_{L^+G}(Z, \Lambda)$.

Proof. The proof uses the idea in [11, Proposition 9.1] and we sketch it here.

For any $\mathcal{F} \in \mathrm{P}_{L^+G}(Z, \Lambda)$, by adjunction we have

$$\mathrm{CT}_\nu(\mathcal{F}) = \mathrm{Ext}^0_{D_{L^nG}(Z,\Lambda)}(a_! p^! i_! \underline{\Lambda}_{S_\nu^- \cap Z}[-(2\rho, \nu)], \mathcal{F}).$$

Then it suffices to prove that $a_! p^! i_! \underline{\Lambda}_{S_\nu^- \cap Z}[-(2\rho, \nu)] \in {}^pD^{\leq 0}(Z, \Lambda)$. This can be shown by Proposition 4.2.

□

Corollary 5.2. *The category* $\mathrm{P}_{L^+G}(Z, \Lambda)$ *has enough projectives.*

Proof. For any $\mathcal{F} \in \mathrm{P}_{L^+G}(Z, \Lambda)$, choose a finitely generated Λ-projective covers $P_\nu \to \mathrm{CT}(\mathcal{F})$. Then in fact, $\oplus_{\nu \in \mathbb{X}}(P_\nu \otimes_\Lambda P_Z(\nu, \Lambda)) \to \mathcal{F}$ is a projective cover. For details, see [11, Corollary 9.2]. $\square$

Let $P_Z(\Lambda) := \oplus_{\nu \in \mathbb{X}} P_Z(\nu, \Lambda)$. We have the following properties of projective objects.

Proposition 5.3.

(1) *Let $Y \subset Z$ be a closed subset which is a union of L^+G-orbits. Then*

$$P_Y(\Lambda) = {}^p\,\mathrm{H}^0(P_Z(\Lambda)\,|_Y),$$

and there is a canonical surjective morphism

$$p_Y^Z : P_Z(\Lambda) \longrightarrow P_Y(\Lambda).$$

(2) *For each L^+G-orbit Gr_λ, denote by $j_\lambda : Gr_\lambda \hookrightarrow Gr_G$ the open embedding. The projective object $P_Z(\Lambda)$ has a filtration with associated graded*

$$gr(P_Z(\Lambda)) \simeq \bigoplus_{Gr_\lambda \subset Z} \mathrm{CT}({}^p j_{\lambda,*}\underline{\Lambda}_{Gr_\lambda}[(2\rho,\lambda)])^* \otimes {}^p j_{\lambda,!}\underline{\Lambda}_{Gr_\lambda}[(2\rho,\lambda)].$$

In particular, $\mathrm{H}^*(P_Z(\Lambda))$ is a free Λ-module of finite rank.

(3) *For $\Lambda = \bar{\mathbb{Q}}_\ell$ and $\mathbb{F}_\ell$, there is a canonical isomorphism*

$$P_Z(\Lambda) \cong P_Z(\mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell} \Lambda.$$

Proof. The proof is similar to that of [11, Proposition.10.1], and we sketch it here.

Let $i : Y \hookrightarrow Z$ be a closed embedding of L^+G-orbits. Then $\mathrm{Hom}(P_Z(\Lambda), i_*\mathcal{F}) = \mathrm{Hom}(P_Z(\Lambda)|_Y, \mathcal{F})$ for any $\mathcal{F} \in \mathrm{P}_{L^+G}(Y, \Lambda)$. Since $P_Z(\Lambda)|_Y \in {}^pD^{\leq 0}(Y, \Lambda)$, we have $P_Y(\Lambda) = {}^p\,\mathrm{H}^0(P_Z(\Lambda)|_Y)$. The isomorphism of functors $\mathrm{Hom}(P_Z(\Lambda), i_*(\bullet)) \simeq \mathrm{Hom}(P_Z(\Lambda), i_*(\bullet))$ induces the desired canonical surjective morphism in (1).

We apply induction on the number of L^+G-orbits in Z to prove part (2). Let $Gr_\lambda \subset Z$ is open and $Y := Z \backslash Gr_\lambda$ be the closed complement. By (1), we have a short exact sequence

$$0 \to K \to P_Z(\Lambda) \to P_Y(\Lambda) \to 0, \tag{5.1}$$

where K denotes the kernel. Let M be a finitely generated Λ-module. Apply the functor $\mathrm{RHom}(\bullet, {}^p j_{\lambda,*}M)$ to (5.1), we get by adjunction an isomorphism

$$\mathrm{Hom}(K|_{Gr_\lambda}, \underline{M}_{Gr_\lambda}[(2\rho,\lambda)] \simeq \mathrm{CT}({}^p j_{\lambda,*}M[(2\rho,\lambda)]). \tag{5.2}$$

The study of standard and co-standard sheaves (cf. [11, §8]) implies that the functor

$$F : \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda, \quad M \mapsto \mathrm{CT}({}^p j_{\lambda,*}M[(2\rho,\lambda)])$$

is represented by the free Λ-module $\mathrm{CT}({}^p j_{\lambda,*}\Lambda[(2\rho,\lambda)])^*$. Then (5.2) implies that

$$K|_{Gr_\lambda} \simeq \mathrm{CT}({}^p j_{\lambda,*}\Lambda[(2\rho,\lambda)])^* \otimes \underline{\Lambda}_{Gr_\lambda}[(2\rho,\lambda)]. \tag{5.3}$$

By adjunction, (5.3) gives rise to the following exact sequence

$$0 \to K' \to \mathrm{CT}({}^p j_{\lambda,*}\Lambda)^* \otimes {}^p j_{\lambda,!}\Lambda[(2\rho,\lambda)] \to K \to C \to 0, \tag{5.4}$$

where K' and C denote the kernel and cokernel. Applying the functor $\mathrm{RHom}(\bullet, C)$ to (5.1), we conclude that $C = 0$ since it is supported on Y. Let $\Lambda = \mathbb{Z}_\ell$, [11, Proposition 8.2] implies that $K' = 0$. Thus, we deduce from (5.4) that

$$K \simeq \mathrm{CT}({}^p j_{\lambda,*}\Lambda)^* \otimes {}^p j_{\lambda,!}\Lambda[(2\rho,\lambda)]. \tag{5.5}$$

Then (5.1), (5.5), together with the induction hypothesis prove (b) for $\Lambda = \mathbb{Z}_\ell$.

Since $\Lambda \otimes_{\mathbb{Z}_\ell} {}^p j_{\lambda,!}\mathbb{Z}_\ell[(2\rho,\lambda)] \simeq {}^p j_{\lambda,!}\Lambda[(2\rho,\lambda)]$ (cf. [11, Proposition 8.1.(a)]), our proof for (b) in the case of $\Lambda = \mathbb{Z}_\ell$ shows that $\Lambda \otimes_{\mathbb{Z}_\ell} P_Z(\mathbb{Z}_\ell)$ is perverse. By adjunction, we have (cf. [11, (10.9)])

$$\mathrm{Hom}(\Lambda \otimes_{\mathbb{Z}_\ell} {}^p j_{\lambda,!}\mathbb{Z}_\ell[(2\rho,\lambda)], \bullet) \simeq \mathrm{CT}(\bullet) : P_{L^+G}(Z,\Lambda) \to \mathrm{Mod}_\Lambda .$$

We thus have $\Lambda \otimes_{\mathbb{Z}_\ell} P_Z(\mathbb{Z}_\ell) \simeq P_Z(\Lambda)$. Statement (c) and statement (b) for $\Lambda = \mathbb{Z}_\ell$ imply the general case of (b). $\square$

For the rest of this section, we set $\Lambda = \mathbb{Z}_\ell$.

Proposition 5.4. *Let $\mathcal{F} \in \mathrm{P}_{L^+G}(Z,\Lambda)$ be a projective object. Then $\mathrm{H}^*(\mathcal{F})$ is a projective Λ-module. In particular, $\mathrm{H}^*(\mathcal{F})$ is torsion-free.*

Proof. Since $\mathrm{Hom}(P_Z(\Lambda), \bullet)$ is exact and faithful, the object $P_Z(\Lambda)$ is a projective generator of $\mathrm{P}_{L^+G}(Z,\Lambda)$. Then each object in the Satake category admits a resolution by direct sums of $P_Z(\Lambda)$. Choose such a resolution for $\mathcal{F}$

$$P_Z(\Lambda)^{\oplus m} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{5.6}$$

In this way, $\mathcal{F}$ can be realized as a direct summand of $P_Z(\Lambda)^{\oplus m}$. By Proposition 5.3(2), we notice that $\mathrm{H}^*(P_Z(\Lambda)^{\oplus m})$ is a finitely generated free Λ-module. Finally, by the exactness of the global cohomology functor $\mathrm{H}^*(\bullet)$, we conclude that $\mathrm{H}^*(\mathcal{F})$ is a direct summand of $\mathrm{H}^*(P_Z(\Lambda)^{\oplus m})$ and is thus a projective Λ-module. $\square$

Remark 5.5. Proposition 5.4 becomes immediate once the geometric Satake equivalence is established.

6. The monoidal structure of H^*

In this section, we study the $\mathbb{G}_m$-action on $Gr_G \tilde{\times} Gr_G$ and apply the hyperbolic localization theorem to prove that the hypercohomology functor $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G,\Lambda) \to \mathrm{Mod}(\Lambda)$ is a monoidal functor. Then we study the relation between the total weight functor CT and the hypercohomology functor H^*. At the end of this section, we prove that the monoidal structure on H^* we constructed is compatible with the one constructed in [17].

We recall the action of $\mathbb{G}_m$ on Gr_G defined in §4, and let $\mathbb{G}_m$ act on $Gr_G \times Gr_G$ diagonally. Then,

$$R \times R := \{(g_1,g_2) \in Gr_G \times Gr_G \mid L^+(2\rho^\vee(t)) \cdot (g_1,g_2) = (g_1,g_2)\},$$

$$S_{\mu_1} \times S_{\mu_2} = \{(g_1,g_2) \in Gr_G \times Gr_G \mid \lim_{t\to 0} L^+(2\rho^\vee(t)) \cdot (g_1,g_2) = (\varpi^{\mu_1},\varpi^{\mu_2})\},$$

and

$$S^-_{\mu_1} \times S^-_{\mu_2} = \{(g_1,g_2) \in Gr_G \times Gr_G \mid \lim_{t\to \infty} L^+(2\rho^\vee(t)) \cdot (g_1,g_2) = (\varpi^{\mu_1},\varpi^{\mu_2})\}$$

are the stable, attracting, and repelling loci of the $\mathbb{G}_m$-action, respectively. Recall that there is an isomorphism $(m_1,m_2) : Gr_G \tilde{\times} Gr_G \simeq Gr_G \times Gr_G$ (cf. §2). Then the diagonal action of $\mathbb{G}_m$ on $Gr_G \times Gr_G$ induces an action on $Gr_G \tilde{\times} Gr_G$. Explicitly, write $(g_1 \tilde{\times} g_2) = (m_1,m_2)^{-1}(g_1, g_1 g_2) \in Gr_G \tilde{\times} Gr_G$, then $t(g_1 \tilde{\times} g_2) := (tg_1 \tilde{\times} g_1^{-1} g_2)$ for any $t \in \mathbb{G}_m, (g_1\, g_2) \in Gr_G$. The stable, attracting, and repelling loci of the $\mathbb{G}_m$-action on $Gr_G \tilde{\times} Gr_G$ are

$$R \tilde{\times} R = \{(\varpi^{\mu_1} \tilde{\times} \varpi^{\mu_2-\mu_1}) \mid \mu_1,\mu_2 \in \mathbb{X}^+_\bullet\},$$

$$S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}=\{(g_1\tilde{\times}g_2)\in Gr_G\tilde{\times}Gr_G\mid \lim_{t\to 0}L^+(2\rho^\vee(t))\cdot(g_1\tilde{\times}g_2)=(\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})\},$$

and

$$\begin{aligned}S^-_{\mu_1}\tilde{\times}S^-_{\mu_2-\mu_1}&=\{((g_1\tilde{\times}g_2)\in Gr_G\tilde{\times}Gr_G\mid \lim_{t\to\infty}L^+(2\rho^\vee(t))\cdot(g_1\tilde{\times}g_2)\\&=(\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})\},\end{aligned}$$

respectively.

Lemma 6.1. *For any $\mathcal{F},\mathcal{G}\in\mathrm{Sat}_{G,\Lambda}$, we have the following isomorphisms*

$$\begin{aligned}\mathrm{H}^*_{S^-_{\mu_1}\tilde{\times}S^-_{\mu_2-\mu_1}}(Gr_G\tilde{\times}Gr_G,\mathcal{F}\tilde{\boxtimes}\mathcal{G})&\simeq\mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\mathcal{F}\tilde{\boxtimes}\mathcal{G})\\&\simeq\mathrm{H}^*_c(S_{\mu_1},\mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1},\mathcal{G}).\end{aligned}\tag{6.1}$$

In addition, the above cohomology groups vanish outside degree $(2\rho,\mu_2)$.

Proof. By our discussion on the $\mathbb{G}_m$-action on $Gr_G\tilde{\times}Gr_G$ above, the first isomorphism can be obtained by applying Braden's hyperbolic localization theorem [5]. Therefore, we are left to prove the second isomorphism and the vanishing property of the cohomology. We first establish a canonical isomorphism

$$\mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\mathcal{F}\tilde{\boxtimes}\mathcal{G})\cong\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1},{}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})).$$

The idea of constructing this isomorphism is completely similar to the one that appears in [17, Corollary 2.17], and we sketch it here.

Assume LU acts on S_{μ_1} via the quotient L^nU for some positive integer n. Denote by $S^{(n)}_{\mu_1}$ the pushout of the L^+U-torsor $LU\to S_{\mu_1}$ along $L^+U\to L^nU$. Then $\pi: S^{(n)}_{\mu_1}\to S_{\mu_1}$ is an L^nU-torsor. Denote by $\pi^*\mathcal{F}$ the pullback of $\mathcal{F}$ along π. Then we have the following projection morphisms

$$S_{\mu_1}\times S_{\mu_2-\mu_1}\xleftarrow{\pi\times\mathrm{id}}S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1}\xrightarrow{q}S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}.$$

Since L^nU is isomorphic to the perfection of an affine space of dimension $n\dim U$, we have the following canonical isomorphisms

$$\begin{aligned}&\mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\mathcal{F}\tilde{\boxtimes}\mathcal{G})\\&\quad\cong\mathrm{H}^*_c(S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1},q^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}))\\&\quad\cong\mathrm{H}^*_c(S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1},(\pi\times\mathrm{id})^*({}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})))\\&\quad\cong\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1},{}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})).\end{aligned}$$

Next, we prove that there is a natural isomorphism

$$\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1},{}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G}))\cong\mathrm{H}^*_c(S_{\mu_1},\mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1},\mathcal{G}).\tag{6.2}$$

Assume that $\mathcal{G}$ is a projective object in the Satake category. Then by Proposition 5.4 and discussion in §2, we have ${}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})=\mathcal{F}\boxtimes\mathcal{G}$ and (6.2) thus holds. Now we come back to the general situation. Since $\mathcal{F}\boxtimes\mathcal{G}\in{}^pD^{\leq 0}(Gr_G\times Gr_G,\Lambda)$, there is a natural morphism $\mathcal{F}\boxtimes\mathcal{G}\to{}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})$. It then induces a map

$$\begin{aligned}&\mathrm{H}^*_c(S_{\mu_1},\mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1},\mathcal{G})\\&\quad\simeq\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1},\mathcal{F}\boxtimes\mathcal{G})\to\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1},{}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})).\end{aligned}$$

By Corollary 5.2, we can find a projective resolution $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ for $\mathcal{F}$. Since the functor ${}^p\mathrm{H}^0(\bullet \boxtimes \mathcal{G})$ is right exact, we get the following exact sequence

$$ {}^p\mathrm{H}^0(\mathcal{F}_2 \boxtimes \mathcal{G}) \longrightarrow {}^p\mathrm{H}^0(\mathcal{F}_1 \boxtimes \mathcal{G}) \longrightarrow {}^p\mathrm{H}^0(\mathcal{F} \boxtimes \mathcal{G}) \longrightarrow 0. \tag{6.3}$$

Applying Proposition 4.2 to $G \times G$, we get from the previous discussion the long exact sequence

$$\begin{aligned} \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_2) \otimes \mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G}) &\to \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_1) \otimes \mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G}) \\ &\to \mathrm{H}_c^*(S_{\mu_1} \times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F} \boxtimes \mathcal{G})) \to 0. \end{aligned}$$

Comparing the above exact sequence with the one obtained from tensoring the following exact sequence

$$\mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_2) \longrightarrow \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_1) \longrightarrow \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}) \longrightarrow 0$$

with $\mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G})$, we complete the proof of (6.2).

Finally, consider Proposition 4.2 together with (6.2), and we conclude the proof of the lemma. □

The previous lemma motivates us to study the analogue of the total weight functor

$$\begin{aligned} \mathrm{CT}' : &\mathrm{P}_{L^+G}(Gr_G, \Lambda) \times \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda(\mathbb{X}_\bullet). \\ &(\mathcal{F}, \mathcal{G}) \mapsto \bigoplus_{\mu_1,\, \mu_2 \in \mathbb{X}} \mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G})'' \end{aligned}$$

Recall that we denote $F : \mathrm{Mod}_\Lambda(\mathbb{X}_\bullet) \to \mathrm{Mod}_\Lambda$ to be the forgetful functor.

Proposition 6.2. *There is a canonical isomorphism*

(6.4)

$$\mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong F \circ \mathrm{CT}'(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \times \mathrm{P}_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda,$$

for all $\mathcal{F}, G \in P_{L^+G}(Gr_G, \Lambda)$.

Proof. The convolution Grassmannian $Gr_G \tilde{\times} Gr_G$ admits a stratification by the convolution of semi-infinite orbits

$$\{S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1} \mid \mu_1, \mu_2 \in \mathbb{X}_\bullet\}.$$

For any $\mathcal{F}, \mathcal{G} \in \mathrm{P}_{L^+G}(Gr_G, \Lambda)$, there is a spectral sequence with E_1-terms $\mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$ and abutment $\mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$. By the Lemma 6.1, this spectral sequence degenerates on the E_1 page. Hence, there exists a filtration

$$\mathrm{Fil}_{\geq \mu_1, \mu_2} \mathrm{H}^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) := \ker(\mathrm{H}^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \to \mathrm{H}_c^*(S_{<\mu_1, <\mu_2}, \mathcal{F} \tilde{\boxtimes} \mathcal{G})),$$

where $S_{<\mu_1, <\mu_2} := \cup_{\nu_1 < \mu_1, \nu_1+\nu_2 < \mu_2} S_{\nu_1} \tilde{\times} S_{\nu_2-\nu_1}$. It is clear that the associated graded of this filtration is $\oplus_{\mu_1, \mu_2 \in \mathbb{X}_\bullet} \mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$.

Similarly, consider the stratification $\{S^-_{\mu_1} \tilde{\times} S^-_{\mu_2-\mu_1} \mid, \mu_1, \mu_2 \in \mathbb{X}_\bullet\}$ of $Gr_G \tilde{\times} Gr_G$. It also induces a filtration

$$\mathrm{Fil}'_{<\mu_1, \mu_2} \mathrm{H}^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) := \mathrm{Im}(\mathrm{H}^*_{T_{<\mu_1, <\mu_2}}(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \to \mathrm{H}^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}))$$

on $\mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$ where $T_{<\mu_1, <\mu_2} := \cup_{\nu_1 < \mu_1, \nu_1+\nu_2 < \mu_2} T_{\nu_1} \tilde{\times} T_{\nu_2-\nu_1}$. The two filtrations are complementary to each other by Lemma 6.1 and the proposition is proved. □

Proposition 6.3. *Under the canonical isomorphism*

$$\mathrm{H}^*(Gr_G, \mathcal{F} \star \mathcal{G}) \cong \mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G}),$$

the weight functor decomposition of the hypercohomology functor obtained in Proposition 4.2 *and the analogous decomposition given by Proposition* 6.2 *are compatible. More precisely, for any $\mathcal{F}, \mathcal{G} \in \mathrm{P}_{L^+G}(Gr_G, \Lambda)$ and any $\mu_2 \in \mathbb{X}_\bullet$, we have the following isomorphism*

$$\mathrm{H}_c^*(S_{\mu_2}, \mathcal{F} \star \mathcal{G}) \simeq \bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}), \tag{6.5}$$

which identifies both sides as direct summands of the direct sum decomposition of $\mathrm{H}^*(Gr_G, \mathcal{F} \star \mathcal{G})$ and $\mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$, respectively.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} m^{-1}(S_{\mu_2}) \arrow[r, hook, "\tilde{f}^+"] \arrow[d, "m_1"] & Gr_G \tilde{\times} Gr_G \arrow[d, "m"] \\ S_{\mu_2} \arrow[r, hook, "f^+"] & Gr_G. \end{tikzcd}$$

Here, f and $\tilde{f}^+$ are the natural locally closed embeddings. The morphism m_1 is the convolution morphism m restricted to $m^{-1}(S_{\mu_2})$.

Consider the $\mathbb{G}_m$-equivariant isomorphism $(pr_1, m) : Gr_G \tilde{\times} Gr_G \simeq Gr_G \times Gr_G$. The preimage of S_{μ_2} along m can be described as

$$(pr_1, m) : m^{-1}(S_{\mu_2}) \simeq Gr_G \times S_{\mu_2}.$$

As before, the diagonal action of $\mathbb{G}_m$ on $Gr_G \times S_{\mu_2}$ induces a $\mathbb{G}_m$- action on $m^{-1}(S_{\mu_2})$ with invariant loci $\{(\varpi^{\mu_1}, \varpi^{\mu_2}) \mid \mu_1 \in \mathbb{X}_\bullet\}$. Via the isomorphism $(pr_1, m)^{-1}$, the attracting and repelling loci for $(\varpi^{\mu_1} \tilde{\times} \varpi^{\mu_2 - \mu_1})$ in $m^{-1}(S_{\mu_2})$ are

$$S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1},$$

and

$$T_{\mu_1, \mu_2} := (pr_1, m)^{-1}(S_{\mu_1}^- \times \{\varpi^{\mu_2}\}),$$

respectively. Applying the hyperbolic localization theorem to $m^{-1}(S_{\mu_2})$, we have the following isomorphism

$$\mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \simeq \mathrm{H}^*_{T_{\mu_1, \mu_2}}(\mathcal{F} \tilde{\boxtimes} \mathcal{G}). \tag{6.6}$$

By Lemma 6.1, the above cohomology groups concentrate in a single degree.

Filtering the space $m^{-1}(S_{\mu_2})$ by $\{S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1} \mid \mu_1 \in \mathbb{X}_\bullet\}$, we get a spectral sequence with E_1-terms $\mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$. As noticed in Lemma 6.1, this spectral sequence degenerates on the E_1-page. Then, there exists a filtration

$$\mathrm{Fil}_{\mu_1, \mu_2'} := \ker(\mathrm{H}^*(m^{-1}(S_{\mu_2}), \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \to \mathrm{H}^*(\cup_{\mu_1' < \mu_1} S_{\mu_1'} \tilde{\times} S_{\mu_2 - \mu_1'}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}))$$

with associated graded

$$\bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2 - \mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}).$$

Similarly, filtering $m^{-1}(S_{\mu_2})$ by $\{T_{\mu_1,\mu_2} \mid \mu_1 \in \mathbb{X}_\bullet\}$, we get an induced spectral sequence with E_1-terms $\mathrm{H}^*_{T_{\mu_1,\mu_2}}(\mathcal{F}\tilde{\boxtimes}\mathcal{G})$. This spectral sequence also degenerates on the E_1-page and there is an induced filtration

$$\mathrm{Fil}'_{\mu_1,\mu_2} := \mathrm{Im}(\mathrm{H}^*_{T_{<\mu_1,\mu_2}}(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) \to \mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G})),$$

where $T_{<\mu_1,\mu_2} := \cup_{\mu_1'<\mu_1} T_{\mu_1',\mu_2}$. The two filtrations are complementary to each other by (6.6) and together define the decomposition

$$\mathrm{H}^*_c(S_{\mu_2}, \mathcal{F}\star\mathcal{G}) \simeq \bigoplus_{\mu_1} \mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}).$$

□

Proposition 6.4. *The hypercohomology functor* $\mathrm{H}^*(Gr_G, \bullet) : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda$ *is a monoidal functor. In addition, the obtained monoidal structure is compatible with the weight functor decomposition established in Proposition* 4.2

Proof. Recall for $\mathcal{F}, \mathcal{G} \in \mathrm{P}_{L^+G}(Gr_G, \Lambda)$, the convolution product $\mathcal{F}\star\mathcal{G}$ is defined as $\mathcal{F}\star\mathcal{G} = Rm_!(\mathcal{F}\tilde{\boxtimes}\mathcal{G})$. Then by Lemma 6.1 and Proposition 6.2, there are canonical isomorphisms

$$\begin{aligned}
&\mathrm{H}^*(Gr_G, \mathcal{F}\star\mathcal{G})\\
&\quad\cong \mathrm{H}^*(Gr_G\tilde{\times}Gr_G, \mathcal{F}\tilde{\boxtimes}\mathcal{G})\\
&\quad\cong \bigoplus_{\mu_1,\mu_2} \mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G})\\
&\quad\cong \bigoplus_{\mu_1,\mu_2} \Big(\mathrm{H}^*_c(S_{\mu_1}, \mathcal{F}) \otimes \mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G})\Big)\\
&\quad\cong \Big(\bigoplus_{\mu_1} \mathrm{H}^*_c(S_{\mu_1}, \mathcal{F})\Big) \otimes \Big(\bigoplus_{\mu_2} \mathrm{H}^*_c(S_{\mu_2}, \mathcal{G})\Big)\\
&\quad\cong \mathrm{H}^*(\mathcal{F}) \otimes \mathrm{H}^*(\mathcal{G}).
\end{aligned}$$

Note that by Proposition 4.2, we have the decomposition of the total weight functor into direct sum of weight functors $\mathrm{H}^*(Gr_G, \mathcal{F}\star\mathcal{G}) \simeq \oplus_\lambda \mathrm{H}^*_c(S_\lambda, \mathcal{F}\star\mathcal{G})$. Proposition 6.3 then shows that the monoidal structure obtained above is compatible with the weight functor decomposition. Finally, we need to show that the monoidal structure of H^* is compatible with the associativity constraint. This can be proved by considering the $\mathbb{G}_m$- action on $Gr_G\tilde{\times}Gr_G\tilde{\times}Gr_G$ induced by the diagonal action of $\mathbb{G}_m$ on $Gr_G \times Gr_G \times Gr_G$ via the isomorphism

$$(m_1, m_2, m_3)^{-1} : Gr_G \times Gr_G \times Gr_G \simeq Gr_G\tilde{\times}Gr_G\tilde{\times}Gr_G.$$

Note that in this case we can still split the intersection $(S_{\nu_1}\tilde{\times}S_{\nu_2}\tilde{\times}S_{\nu_3}) \cap (Gr_{\leq\mu_1}\tilde{\times}Gr_{\leq\mu_2}\tilde{\times}Gr_{\leq\mu_3})$ by (4.1). This allows us to apply the hyperbolic localization theorem and a similar spectral sequence argument as before. We obtain the desired compatibility property and the proposition is thus proved. □

With the monoidal structure of H^* established above, we are now ready to prove the following results.

Proposition 6.5. *For any* $\mathcal{F} \in \mathrm{Sat}_{G,\Lambda}$*, the functors* $(\bullet)\star\mathcal{F}$ *and* $\mathcal{F}\star(\bullet)$ *are both right exact. If in addition* $\mathcal{F}$ *is a projective object, then these functors are exact.*

Proof. Let

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0 \tag{6.7}$$

be an exact sequence in $\mathrm{Sat}_{G,\Lambda}$. By Proposition 4.2, taking global cohomology gives an exact sequence

$$\mathrm{H}^*(\mathcal{G}') \longrightarrow \mathrm{H}^*(\mathcal{G}) \longrightarrow \mathrm{H}^*(\mathcal{G}'') \longrightarrow 0. \tag{6.8}$$

Tensoring (6.8) with $\mathrm{H}^*(\mathcal{F})$ gives the exact sequence

$$\mathrm{H}^*(\mathcal{G}') \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}) \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}'') \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow 0. \tag{6.9}$$

By Proposition 6.3, (6.9) is canonically isomorphic to the following sequence

$$\mathrm{H}^*(\mathcal{G}' \star \mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G} \star \mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}'' \star \mathcal{F}) \longrightarrow 0. \tag{6.10}$$

Notice that by Proposition 4.2, the global cohomology functor $\mathrm{H}^*(\bullet)$ is faithful, then the exactness of (6.9) implies that the sequence

$$\mathcal{G}' \star \mathcal{F} \longrightarrow \mathcal{G} \star \mathcal{F} \longrightarrow \mathcal{G}'' \star \mathcal{F} \longrightarrow 0$$

is also exact. The right exactness for $\mathcal{F} \star (\bullet)$ can be proved similarly.

Now, assume $\mathcal{F}$ to be a projective object in the Satake category. By Proposition 5.4, we know that the functors $\mathrm{H}^*(\bullet) \otimes \mathrm{H}^*(\mathcal{F})$ and $\mathrm{H}^*(\mathcal{F}) \otimes \mathrm{H}^*(\bullet)$ are both exact. Then arguing as before and using the monoidal structure and the faithfulness of the functor $\mathrm{H}^*(\bullet)$, we conclude the proof. □

Again, Proposition 6.5 becomes obvious once Theorem 1.1 is proved.

We conclude the discussion on the monoidal structure of H^* by identifying it with the one constructed in [17]. For this purpose, we briefly recall the construction in *loc.cit.*

Let $\mathcal{F}, \mathcal{G} \in \mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$. Assume that L^+G acts on $\mathrm{supp}(\mathcal{G})$ via the quotient $L^+G \to L^mG$. Define $\mathrm{supp}(\mathcal{F}) \tilde{\times} \mathrm{supp}(\mathcal{G}) := \mathrm{supp}(\mathcal{F})^{(m)} \times^{L^mG} \mathrm{supp}(\mathcal{G})$ and denote by π the projection morphism $\mathrm{supp}(\mathcal{F})^{(m)} \to \mathrm{supp}(\mathcal{F})$. Then we have an $L^+G \times L^mG$-equivariant projection morphism

$$p : \mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}) \longrightarrow \mathrm{supp}(\mathcal{F}) \tilde{\times} \mathrm{supp}(\mathcal{G}),$$

the L^+G action on $\mathrm{supp}(\mathcal{F})^{(m)}$ is by multiplication on the left and the L^mG action on $\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G})$ is the one used in constructing the twisted product $\mathrm{supp}(\mathcal{F}) \tilde{\times} \mathrm{supp}(\mathcal{G})$. Then p induces a canonical isomorphism of the L^+G-equivariant cohomology (cf. [17, A.3.5])

$$\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}) \tilde{\times} \mathrm{supp}(\mathcal{G}), \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong \mathrm{H}^*_{L^+G \times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}), \pi^*\mathcal{F} \boxtimes \mathcal{G}). \tag{6.11}$$

By the equivariant Künneth formula (cf. [16, A.1.15]), there is a canonical isomorphism

$$\begin{aligned} &\mathrm{H}^*_{L^+G \times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}), \pi^*\mathcal{F} \boxtimes \mathcal{G}) \\ &\quad \cong \mathrm{H}^*_{L^+G \times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)}, \mathcal{F}) \otimes \mathrm{H}^*_{L^+G \times L^mG}(\mathrm{supp}(\mathcal{G}), \mathcal{G}). \end{aligned} \tag{6.12}$$

Combine (6.11) with (6.12), and we conclude a canonical isomorphism

$$\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}) \tilde{\times} \mathrm{supp}(\mathcal{G}), \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \cong \mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}), \mathcal{F}) \otimes \mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{G}), \mathcal{G}). \tag{6.13}$$

We denote by $\overline{G}_{\bar{\mathbb{Q}}_\ell}$ the base change of $\overline{G}$ to $\bar{\mathbb{Q}}_\ell$. Let $R_{\overline{G},\ell} := \mathrm{Sym}(\mathfrak{g}_{\bar{\mathbb{Q}}_\ell}(-1))^{G_{\bar{\mathbb{Q}}_\ell}}$ denote the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}_{\bar{\mathbb{Q}}_\ell}(-1)$. Then

(6.13) induces an isomorphism of $R_{\overline{G},\ell}$-bimodules. In addition, the two $R_{\overline{G},\ell}$-module structures coincide ([17, Lemma 2.19]) and the base change of (6.13) along the augmentation map $R_{\overline{G},\ell} \to \bar{\mathbb{Q}}_\ell$, the canonical isomorphism

$$\mathrm{H}^*_{L^+G}(\mathcal{F}) \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell \cong \mathrm{H}^*(\mathcal{F}) \tag{6.14}$$

gives the monoidal structure of H^* in the $\bar{\mathbb{Q}}_\ell$-case ([17, Proposition 2.20]).

Then to identify the monoidal structures, it suffices to prove Proposition 6.6.

Proposition 6.6. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{P}_{L^+G}(Gr_G, \mathbb{Z}_\ell)$ be two projective objects. We denote $\mathcal{F}\otimes\bar{\mathbb{Q}}_\ell$ and $\mathcal{G}\otimes\bar{\mathbb{Q}}_\ell$ by $\mathcal{F}'$ and $\mathcal{G}'$, respectively. Then the following diagram commutes*
(6.15)

$$\begin{tikzcd}
\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}')\tilde{\times}\,\mathrm{supp}(\mathcal{G}'), \mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell \arrow[r, "(6.14)"] \arrow[d, "(6.13)"] & \mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \arrow[d, "\alpha"] \\
(\mathrm{H}^*_{L^+G}(\mathcal{F}') \otimes_{R_{\overline{G},\ell}} \mathrm{H}^*_{L^+G}(\mathcal{G}')) \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell \arrow[d, "\cong"] & \bigoplus_{\mu_1,\mu_2} \mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \arrow[d, "\beta"] \\
(\mathrm{H}^*_{L^+G}(\mathcal{F}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell) \otimes_{\bar{\mathbb{Q}}_\ell} (\mathrm{H}^*_{L^+G}(\mathcal{G}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell) \arrow[r, "\simeq"] & (\bigoplus_{\mu_1} \mathrm{H}^*_c(S_{\mu_1}, \mathcal{F}')) \otimes_{\bar{\mathbb{Q}}_\ell} (\bigoplus_{\mu_2} \mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G}'))
\end{tikzcd}$$

where the morphisms α and β are the base change of isomorphisms (6.4) *and* (6.1) *to $\bar{\mathbb{Q}}_\ell$, respectively.*

Proof. Consider the filtrations

$$\mathrm{Fil}_{\geq \mu_1,\mu_2} \mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'), \mathrm{Fil}_{\geq\mu} \mathrm{H}^*(\mathcal{F}'), \text{ and } \mathrm{Fil}_{\geq\mu} \mathrm{H}^*(\mathcal{G}')$$

defined as in Proposition 6.2 and Proposition 4.2. To prove the proposition, it suffices to prove that these filtrations respect (6.13). Then taking the Verdier dual (note now $\mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$, $\mathrm{H}^*(\mathcal{F}')$, and $\mathrm{H}^*(\mathcal{G}')$ are all $\bar{\mathbb{Q}}_\ell$-vector spaces) implies that the complementary filtrations $\mathrm{Fil}'_{<\mu_1,\mu_2}$ and $\mathrm{Fil}'_{<\mu}$ also respect (6.13). This will provide the commutativity of (6.15).

The approach we will use is similar to the one given in [16, Proposition 5.3.14], and we sketch it here. Although the semi-infinite orbit S_μ does not admit an L^+G-action, it is stable under the action of the constant torus $T \subset L^+T \subset L^+G$. Then so is the convolution product of semi-infinite orbits $S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}$. Stratifying $Gr_G\tilde{\times}Gr_G$ by $\{S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1} \mid \mu_1,\mu_2 \in \mathbb{X}_\bullet\}$, we get a spectral sequence with E_1-terms $\mathrm{H}^*_{T,c}(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$ which abuts to $\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$. By [17, Proposition 2.7], the spectral sequence degenerates on the E_1-page and the filtration $\mathrm{Fil}_{\geq\mu_1,\mu_2}$ thus lifts to a new filtration of H^*_T

$$\mathrm{Fil}_{\geq\mu_1,\mu_2} \mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') := \ker(\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \to \mathrm{H}^*_T(S_{<\mu_1}\tilde{\times}S_{<\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')).$$

Using a similar argument as in the proof of Proposition 6.2, the associated graded of this filtration equals $\bigoplus_{\mu_1,\mu_2} \mathrm{H}^*_{T,c}(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$. Note that all the terms in this filtration and the associated graded are in fact free $R_{\overline{T},\ell}$-modules, then base change to $\bar{\mathbb{Q}}_\ell$ along the augmentation map $R_{\overline{T},\ell} \to \bar{\mathbb{Q}}_\ell$ recovers our original filtration $\mathrm{Fil}_{\geq\mu_1,\mu_2}$. Similarly, we can define the filtrations $\mathrm{Fil}_{\geq\mu} \mathrm{H}^*_T(\mathcal{F}')$ and $\mathrm{Fil}_{\geq\mu} \mathrm{H}^*_T(\mathcal{G}')$ which recover the original filtrations $\mathrm{Fil}_{\geq\mu} \mathrm{H}^*(\mathcal{F}')$ and $\mathrm{Fil}_{\geq\mu} \mathrm{H}^*(\mathcal{G}')$ in the same way.

Since

$$\mathrm{H}^*_T(\bullet) \simeq \mathrm{H}^*_{L^+G}(\bullet) \otimes_{R_{\overline{G},\ell}} R_{\overline{T},\ell} : \mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell) \longrightarrow \mathrm{Vect}_{\bar{\mathbb{Q}}_\ell},$$

then (6.13) induces a monoidal structure on the T-equivariant cohomology

$$\mathrm{H}^*_T(\mathcal{F}' \star G') \simeq \mathrm{H}^*_T(\mathcal{F}') \otimes \mathrm{H}^*_T(\mathcal{G}'). \tag{6.16}$$

Then we are left to show that (6.16) is compatible with the filtrations $\mathrm{Fil}_{\geq \mu_1,\mu_2}$ and $\mathrm{Fil}_{\geq\mu}$. It suffices to check the compatibility with filtrations $\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*_T$ and $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T$ over the generic point of $\operatorname{Spec} R_{\overline{T},\ell}$. Denote

$$\mathrm{H}_\lambda := \mathrm{H}^*_T \otimes_{R_{\overline{T},\ell}} Q,$$

where Q is the fraction filed of $R_{\overline{T},\ell}$. By the equivariant localization theorem, we have isomorphisms

$$\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \simeq \bigoplus_{\mu_1,\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}' \mid_{(\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})})$$

and

$$\mathrm{H}^*_\lambda(S_{<\mu_1,<\mu_2}\,\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \simeq \bigoplus_{\nu_1<\mu_1,\nu_2<\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}' \mid_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})}).$$

Then it follows that

$$\begin{aligned}&\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')\\ &\quad := \mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \otimes_{R_{\overline{T},\ell}} Q \simeq \bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}' \mid_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})}).\end{aligned}$$

Applying the equivariant localization theorem again gives isomorphisms

$$\mathrm{H}_\lambda(\mathcal{F}') \simeq \bigoplus_{\mu} \mathrm{H}_\lambda(\mathcal{F}' \mid_{\varpi^\mu})$$

and

$$\mathrm{H}_\lambda(S_{<\mu},\mathcal{F}') \simeq \bigoplus_{\nu<\mu} \mathrm{H}_\lambda(\mathcal{F}' \mid_{\varpi^\nu}).$$

Similarly, we get a filtration $\mathrm{Fil}_{\geq\mu}\mathrm{H}_\lambda(\mathcal{F}') \simeq \oplus_{\nu\geq\mu}\mathrm{H}_\lambda(\mathcal{F}' \mid_{\varpi^\nu})$ induced by $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T(\mathcal{F})$.

Notice that as for $\mathrm{H}^*_{L^+G}(\bullet)$, the monoidal structure (6.16) is defined via the composition of the following isomorphisms

$$\begin{aligned}&\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')\\ &\simeq \bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}' \mid_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})})\\ &\simeq \bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}' \mid_{\varpi^{\nu_1}}) \otimes \mathrm{H}_\lambda(\mathcal{G}' \mid_{\varpi^{\nu_2-\nu_1}})\\ &\simeq \bigoplus_{\nu_1\geq\mu_1} \mathrm{H}_\lambda(\mathcal{F}' \mid_{\varpi^{\nu_1}}) \otimes \bigoplus_{\nu_2\geq\mu_2-\mu_1} \mathrm{H}_\lambda(\mathcal{G}' \mid_{\varpi^{\nu_2}})\\ &\simeq \mathrm{Fil}_{\geq\mu_1}\mathrm{H}_\lambda(\mathcal{F}') \bigotimes \mathrm{Fil}_{\geq\mu_2-\mu_1}\mathrm{H}_\lambda(\mathcal{G}')\end{aligned}$$

where the second isomorphism is obtained by an analogue of (6.13) for T-equivariant cohomology and the equivariant Künneth formula. Note that the monoidal structure of the total weight functor CT is compatible with that of the hypercohomology functor H^* by Proposition 6.3. We thus conclude the proof. $\square$

7. Tannakian construction

The monoidal structure of the hypercohomology functor H^* allows us to perform a generalized Tannakian formalism which we discuss below.

Let $Z \subset Gr_G$ denote a closed subspace consisting of a finite union of L^+G-orbits. Then any $\mathcal{F} \in \mathrm{P}_{L^+G}(Z, \Lambda)$ admits a presentation

$$P_1 \longrightarrow P_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where P_1 and P_0 are finite direct sums of $P_Z(\Lambda)$.

Write $A_Z(\Lambda)$ for $\mathrm{End}_{\mathrm{P}_{L^+G}(Z,\Lambda)}(P_Z(\Lambda))^{op}$. By Proposition 5.3(2), $A_Z(\Lambda)$ is a finite free Λ-module, and any finitely generated $A_Z(\Lambda)$-module is also finitely presented. Now we recall the following version of Gabriel and Mitchell's theorem as formulated in [1, Theorem 9.1].

Theorem 7.1. *Let $\mathcal{C}$ be an abelian category. Let P be a projective object and write $A = \mathrm{End}_{\mathcal{C}}(P)^{op}$. Denote $\mathcal{M}$ to be the full subcategory of $\mathcal{C}$ consisting of objects M which admits a presentation*

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where P_1 and P_0 are finite direct sums of P. Let $\mathcal{M}'_A$ be the category of finitely presented right A-modules. Then

(1) *there is an equivalence of abelian categories $\mathcal{M} \simeq \mathcal{M}'_A$ induced by the functor $\mathrm{Hom}_{\mathcal{C}}(P, \bullet)$,*
(2) *there is a canonical isomorphism between the endomorphism ring of the functor $\mathrm{Hom}_{\mathcal{C}}(P, \bullet)$ and A^{op}.*

Theorem 7.1 and the discussion before it enable us to deduce an equivalence of abelian categories

$$E_Z : \mathrm{P}_{L^+G}(Z, \Lambda) \simeq \mathcal{M}'_{A_Z(\Lambda)}.$$

Let $i : Y \hookrightarrow Z$ be an inclusion of closed subsets consisting of L^+G-orbits, then we have the functor $i_* : \mathrm{P}_{L^+G}(Y, \Lambda) \to \mathrm{P}_{L^+G}(Z, \Lambda)$. In addition, i_* induces a functor $(i^Z_Y)^* : \mathcal{M}'_{A_Y(\Lambda)} \to \mathcal{M}'_{A_Z(\Lambda)}$ which in turn gives a ring homomorphism $i^Z_Y : A_Z(\Lambda) \to A_Y(\Lambda)$. Note for any $a \in A_Z(\Lambda)$ and $\mathcal{F} \in \mathrm{P}_{L^+G}(Y, \Lambda)$, we have isomorphisms

$$\begin{aligned} & a \cdot E_Z(i_*\mathcal{F}) \\ & \simeq a \cdot \mathrm{Hom}(P_Z(\Lambda), i_*\mathcal{F}) \\ & \simeq a \cdot (i^Z_Y)^*(\mathrm{Hom}(P_Z(\Lambda), i_*\mathcal{F}) \\ & \simeq i^Z_Y(a) \cdot \mathrm{Hom}(P_Y(\Lambda), \mathcal{F}) \\ & \simeq i^Z_Y(a) \cdot E_Y(\mathcal{F}). \end{aligned}$$

Define $B_Z(\Lambda) := \mathrm{Hom}(A_Z(\Lambda), \Lambda)$. Since $A_Z(\Lambda)$ is a finite free Λ-module, then so is $B_Z(\Lambda)$ and we have the following canonical equivalence of abelian categories

$$\mathcal{M}'_{A_Z(\Lambda)} \cong \mathrm{Comod}_{B_Z(\Lambda)}.$$

The dual map of i^Z_Y gives a map $\iota^Y_Z : B_Y(\Lambda) \to B_Z(\Lambda)$. Let $B(\Lambda) = \varinjlim B_Z(\Lambda)$, we conclude that $\mathrm{Sat}_{G,\Lambda} \simeq \mathrm{Comod}_{B(\Lambda)}$ as abelian categories. Moreover, by Proposition 5.3(3) we know that

$$B(\Lambda) \simeq B(\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \Lambda \tag{7.1}$$

for $\Lambda = \bar{\mathbb{Q}}_\ell$ and $\mathbb{F}_\ell$.

Take any $\mu \in \mathbb{X}_\bullet^+$, and write $A_\mu(\Lambda)$ and $B_\mu(\Lambda)$ for $A_{Gr_{\leq\mu}}(\Lambda)$ and $B_{Gr_{\leq\mu}}(\Lambda)$, respectively. For any $\mu, \nu \in \mathbb{X}_\bullet^+$ such that $\mu \leq \nu$, use notations i_μ^ν and ι_ν^μ for $i_{Gr_{\leq\mu}}^{Gr_{\leq\nu}}$ and $\iota_{Gr_{\leq\nu}}^{Gr_{\leq\mu}}$, respectively. Also, we denote by $P_\theta(\Lambda)$ the projective object $P_{Gr_{\leq\theta}(\Lambda)}$ for any $\theta \in \mathbb{X}_\bullet^+$. Note the following canonoical isomorphism obtained by the monoidal structure of H^* established by Proposition 6.4.

$$
\begin{aligned}
&\mathrm{Hom}(P_{\mu+\nu}(\Lambda), P_\mu \star P_\nu)\\
&\simeq \mathrm{H}^*(P_\mu \star P_\nu)\\
&\simeq \mathrm{H}^*(P_\mu) \otimes \mathrm{H}^*(P_\nu)\\
&\simeq \mathrm{Hom}(P_\mu(\Lambda), P_\mu(\Lambda)) \otimes \mathrm{Hom}(P_\nu(\Lambda), P_\nu(\Lambda)).
\end{aligned}
$$

Then, the element $id_{P_\mu(\Lambda)} \otimes id_{P_\nu(\Lambda)} \in \mathrm{Hom}(P_\mu(\Lambda), P_\mu(\Lambda)) \otimes \mathrm{Hom}(P_\nu(\Lambda), P_\nu(\Lambda))$ gives rise to a morphism

$$f_{\mu,\nu} : P_{\mu+\nu}(\Lambda) \longrightarrow P_\mu(\Lambda) \star P_\nu(\Lambda).$$

Applying the functor H^* and dualizing, we get a morphism

$$g_{\mu,\nu} : B_\mu(\Lambda) \otimes B_\nu(\Lambda) \to B_{\mu+\nu}(\Lambda).$$

We check that the multiplication maps $g_{\bullet,\bullet}$ are compatible with the maps $\iota_\bullet^\bullet$ i.e. for any $\mu \leq \mu', \nu \leq \nu' \in \mathbb{X}_\bullet^+$, the following diagram commutes

$$
\begin{tikzcd}
B_\mu(\Lambda) \otimes B_\nu(\Lambda), \arrow[r, "g_{\mu,\nu}"] \arrow[d, "\iota^\mu_{\mu'} \otimes \iota^\nu_{\nu'}"] & B_{\mu+\nu}(\Lambda) \arrow[d, "\iota^{\mu+\nu}_{\mu'+\nu'}"] \\
B_{\mu'}(\Lambda) \otimes B_{\nu'}(\Lambda) \arrow[r, "g_{\mu',\nu'}"] & B_{\mu'+\nu'}(\Lambda).
\end{tikzcd}
\tag{7.2}
$$

By the constructions of g's and ι's, it suffices to check the commutativity for

$$
\begin{tikzcd}
P_{\mu'+\nu'}(\Lambda) \arrow[r, "f_{\mu',\nu'}"] \arrow[d, "p^{\mu'+\nu'}_{\mu+\nu}"] & P_{\mu'}(\Lambda) \star P_{\nu'}(\Lambda) \arrow[d, "p^{\mu'}_{\mu} \star p^{\nu'}_{\nu}"] \\
P_{\mu+\nu}(\Lambda) \arrow[r, "f_{\mu,\nu}"] & P_\mu(\Lambda) \star P_\nu(\Lambda).
\end{tikzcd}
\tag{7.3}
$$

Here, maps $p_\bullet^\bullet$ appearing in the above diagram are the maps $p_\bullet^\bullet$ in Proposition 5.3(1). The construction of f's implies that we are left to show that the following diagram commutes

$$
\begin{tikzcd}
\mathrm{H}^*(P_{\mu'}(\Lambda) \star P_{\nu'}(\Lambda)) \arrow[r, "\cong"] \arrow[d, "\mathrm{H}^*(p^{\mu'}_{\mu} \star p^{\nu}_{\nu'})"] & \mathrm{H}^*(P_{\mu'}(\Lambda)) \otimes \mathrm{H}^*(P_{\nu'}(\Lambda)) \arrow[d, "\mathrm{H}^*(p^{\mu'}_{\mu}) \otimes \mathrm{H}^*(p^{\nu'}_{\nu})"] \\
\mathrm{H}^*(P_\mu(\Lambda) \star P_\nu(\Lambda)) \arrow[r, "\cong"] & \mathrm{H}^*(P_\mu(\Lambda)) \otimes \mathrm{H}^*(P_\nu(\Lambda))
\end{tikzcd}
$$

The monoidal structure of H^* implies that the above diagram commutes and so is diagram (7.2). Taking direct limit, the morphisms $g_{\mu,\nu}$ give a multiplication map on $B(\Lambda)$ by the above discussion. Our observation at the end of §3 ensures that the multiplication on $B(\Lambda)$ is associative.

Note clearly that $B_0(\Lambda) = \Lambda$ and the canonical map $B_0(\Lambda) \to B(\Lambda)$ gives the unit map for $B(\Lambda)$. Now to endow $B(\Lambda)$ with a bialgebra structure in the sense of [4, §2], it suffices to prove that the multiplication on $B(\Lambda)$ is commutative and $B(\Lambda)$ admits an antipode. The later statement can be proved in a completely similar manner as in [1, Proposition 13.4] once the former statement is proved. Thus it suffices to construct the commutativity of the multiplication of $B(\Lambda)$, for $\Lambda = \mathbb{Z}_\ell$.

By the compatibility of morphisms $g_{\bullet,\bullet}$ with $\iota_\bullet^\bullet$, it suffices to prove for each $\mu \in \mathbb{X}_\bullet^+$ that the multiplication on $B_\mu(\Lambda)$ is commutative. For $\Lambda = \mathbb{Z}_\ell$, consider the following diagram

$$\begin{tikzcd} B_\mu(\Lambda) \otimes B_\mu(\Lambda) \arrow[r, "d"] \arrow[d, hook] & B_{2\mu}(\Lambda) \arrow[d, hook] \\ B_\mu(\bar{\mathbb{Q}}_\ell) \otimes B_\mu(\bar{\mathbb{Q}}_\ell) \arrow[r, "d'"] & B_{2\mu}(\bar{\mathbb{Q}}_\ell). \end{tikzcd} \tag{7.4}$$

The vertical arrows are inclusions by noting (7.1) and the fact that $B_\mu(\Lambda)$ is a finite free Λ-module. The map d is defined to map $b_1 \otimes b_2$ to $g_{\mu,\mu}(b_1 \otimes b_2) - g_{\mu,\mu}(b_2 \otimes b_1)$, and the map d' is defined similarly. By Proposition 6.6 and isomorphism (7.1), diagram (7.4) is commutative. The construction of the commutativity constraint in $\mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$ in [17] allows us to conclude that d is the zero map and the multiplication map in $B(\Lambda)$ is thus commutative. Thus, by a completely similar argument as in [4, Proposition 2.16], the category $\mathrm{Comod}_{B(\Lambda)}$ can be equipped with a commutativity constraint. This commutativity constraint then induces that of $\mathrm{Sat}_{G,\Lambda}$. Thus we have endowed $\mathrm{Sat}_{G,\Lambda}$ with a tensor category structure via an analogous argument as in [1, Proposition 13.4] which shows that $B(\Lambda)$ admits an antipode.

8. Identification of group schemes

With the work in previous sections, we have constructed the category $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$, and equipped it with

(1) the convolution product $\star$ and an associativity constraint,
(2) the hypercohomology functor $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda$ which is Λ-linear, exact, and faithful,
(3) a commutativity constraint which makes $\mathrm{Sat}_{G,\Lambda}$ a tensor category,
(4) a unit object IC_0,
(5) a bialgebra $B(\Lambda)$ such that $\mathrm{Sat}_{G,\Lambda}$ is equivalent to $\mathrm{Comod}_{B(\Lambda)}$ as tensor categories.

Note that by Proposition 5.3(2), $\mathrm{H}^*(P_Z(\mathbb{Z}_\ell))$ is a free $\mathbb{Z}_\ell$-module for any closed subspace $Z \subset Gr_G$ consisting of a finite union of L^+G-orbits. We also know that the representing object $P_Z(\mathbb{Z}_\ell)$ is compatible with base change by Proposition 5.3(3). Let $\widetilde{G}_\Lambda := \mathrm{Spec}(B(\Lambda))$, for $\Lambda = \mathbb{Z}_\ell, \mathbb{F}_\ell, \bar{\mathbb{F}}_\ell$. By our discussion in the previous section, we have the following generalized Tannakian construction similar to [11, Proposition 11.1]

Proposition 8.1. *The category of representations of the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ which are finitely generated over $\mathbb{Z}_\ell$ is equivalent to $\mathrm{P}_{L^+G}(Gr_G, \mathbb{Z}_\ell)$ as tensor categories. Furthermore, the coordinate ring of $\widetilde{G}_{\mathbb{Z}_\ell}$ is free over $\mathbb{Z}_\ell$ and $\widetilde{G}_{\mathbb{F}_\ell} = \mathrm{Spec}(\mathbb{F}_\ell) \times_{\mathbb{Z}_\ell} \widetilde{G}_{\mathbb{Z}_\ell}$.*

We are left to identify the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ with the Langlands dual group $\hat{G}_{\mathbb{Z}_\ell}$.

Note that split reductive group schemes over $\mathbb{Z}_\ell$ are uniquely determined by their root data. Then it suffices to prove the following for our purpose

(1) $\widetilde{G}_{\mathbb{Z}_\ell}$ is smooth over $\mathbb{Z}_\ell$,
(2) the group scheme $\widetilde{G}_{\overline{\mathbb{F}}_\ell}$ is reductive,
(3) the dual split torus $\hat{T}_{\mathbb{Z}_\ell}$ is a maximal torus of $\widetilde{G}_{\mathbb{Z}_\ell}$.

In §7, we showed that $B(\mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-modules. As a result, the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ is affine flat over $\mathbb{Z}_\ell$. Then the affineness of $\widetilde{G}_{\mathbb{Z}_\ell}$ together with the statements (1) and (2) in this paragraph amount to the definition of a reductive group over $\mathbb{Z}_\ell$. Recall in [13], a group scheme $\mathcal{G}$ over a discrete valuation ring R with uniformizer π, field of fractions K, and residue field κ is said to be *quasi-reductive* if

(1) $\mathcal{G}$ is affine flat over R,
(2) $\mathcal{G}_K := \mathcal{G} \otimes_R K$ is connected and smooth over K,
(3) $\mathcal{G}_\kappa := \mathcal{G} \otimes_R \kappa$ is of finite type over κ and the neutral component $(\mathcal{G}_{\bar{\kappa}})^\circ_{\mathrm{red}}$ of the reduced geometric fibre is a reductive group of dimension equals $\dim \mathcal{G}_K$.

We will make use of Theorem 8.2 for quasi-reductive group schemes proved in *loc.cit.*

Theorem 8.2. *Let $\mathcal{G}$ be a quasi-reductive group scheme over R. Then*

(1) *$\mathcal{G}$ is of finite type over R*
(2) *$\mathcal{G}_K$ is reductive*
(3) *$\mathcal{G}_\kappa$ is connected.*

In addition, if

(4) *the type of $\mathcal{G}_{\bar{K}}$ is of the same type as that of $(\mathcal{G}_{\bar{\kappa}})^\circ_{\mathrm{red}}$,*

then $\mathcal{G}$ is reductive.

As noted above, the requirement (1) of quasi-reductiveness is satisfied by $\widetilde{G}_{\mathbb{Z}_\ell}$. In addition, by [17], the group scheme $\widetilde{G}_{\mathbb{Q}_\ell}$ is connected reductive with root datum dual to that of G and condition (2) of quasi-reductiveness is met.

Lemma 8.3. *The group scheme $\widetilde{G}_{\bar{\mathbb{F}}_\ell}$ is connected.*

Proof. Note that the same proof as in [11, §12] and [1, Lemma 9.3] applies in our setting to show that the Satake category $\mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{F}}_\ell)$ has no object $\mathcal{F}$ such that the subcategory $\langle \mathcal{F} \rangle$, which is the strictly full subcategory of $\mathrm{P}_{L^+G}(Gr_G, \bar{\mathbb{F}}_\ell)$ whose objects are those isomorphic to a subquotient of $\mathcal{F}^{\star n}$ for some $n \in \mathbb{N}$, is stable under $\star$. This is equivalent to the fact that there does not exist an object $X \in \mathrm{Rep}_{\bar{\mathbb{F}}_\ell}(\widetilde{G}_{\bar{\mathbb{F}}_\ell})$ such that $\langle X \rangle$ is stable under $\bigotimes$ via Proposition 8.1. Then by [1, Corollary 2.11.2], we conclude our proof. $\square$

From now on, let $\kappa = \bar{\mathbb{F}}_\ell$. We have proved in Proposition 6.6 that the monoidal structure of H^* is compatible with the weight functor decomposition. In other words, we get a monoidal functor

$$\mathrm{CT} : \mathrm{Sat}_{G,\mathbb{Z}_\ell} \longrightarrow \mathrm{Mod}_{\mathbb{Z}_\ell}(\mathbb{X}_\bullet) \simeq \mathrm{Sat}_{T,\mathbb{Z}_\ell}.$$

Base change to κ, the same reasoning yields a monoidal functor

$$\mathrm{CT} : \mathrm{Sat}_{G,\kappa} \longrightarrow \mathrm{Mod}_\kappa(\mathbb{X}_\bullet) \simeq \mathrm{Sat}_{T,\kappa}.$$

Applying the construction in §7 to the above two Satake categories, we get a natural homomorphism $\hat{T} \to \widetilde{G}$. Note that by [17, Corollary 2.8] and Proposition 5.3(2), any $M \in \mathrm{Mod}_\kappa(\mathbb{X}_\bullet)$ can be realized as a subquotient of some projective object in $\mathrm{Sat}_{G,\kappa}$. It then follows from [4, Proposition 2.21(b)] that the homomorphism $\hat{T} \to \widetilde{G}$ is in fact a closed embedding, which realizes the dual torus $\hat{T}_\kappa$ as a subtorus of $\widetilde{G}_\kappa$. In addition, since $\widetilde{G}_{\mathbb{Z}_\ell}$ is flat, the same argument in [11, §12] applies to give the following dimension estimate

$$\dim G = \dim \widetilde{G}_{\mathbb{Q}_\ell} \geq \dim(\widetilde{G}_\kappa)_{\mathrm{red}}. \tag{8.1}$$

We can write $\widetilde{G}_\kappa = \varprojlim \widetilde{G}^*_\kappa$ where $\widetilde{G}^*_\kappa$ satisfies the following conditions

(1) $\widetilde{G}^*_\kappa$ is of finite type,
(2) the canonical map $\mathrm{Irr}_{\widetilde{G}^*_\kappa} \to \mathrm{Irr}_{\widetilde{G}_\kappa}$ is a bijection, where Irr denotes the set of irreducible representations.

In addition, we require that the transition morphisms are surjective. The first requirement may be satisfied since any group scheme is a projective limit of group schemes of finite type. To ensure that condition (2) can be satisfied, it is enough to choose $\widetilde{G}^*_\kappa$ sufficiently large so that the irreducible representations $L(\eta)$ associated to a finite set of generators η of the semigroup of dominant cocharacters $\mathbb{X}^+_\bullet$, are pullbacks of representations of $\widetilde{G}^*_\kappa$. For any $\mu, \nu \in \mathbb{X}^+_\bullet$, the sheaf $\mathrm{IC}_{\mu+\nu}$ supports on $Gr_{\leq \mu+\nu}$ and hence is a subquotient of $\mathrm{IC}_\mu \star \mathrm{IC}_\nu$. Thus all irreducible representations of $\widetilde{G}_\kappa$ come from $\widetilde{G}^*_\kappa$. By our choice of the finite type quotients, we have $(\widetilde{G}_\kappa)_{\mathrm{red}} = \varprojlim (\widetilde{G}^*_\kappa)_{\mathrm{red}}$. In addition, the composition of maps $\hat{T}_\kappa \to \widetilde{G}_\kappa \to \widetilde{G}^*_\kappa$ is a closed embedding.

We claim that

(8.2)

Each finite type quotient $\widetilde{G}^*_\kappa$ is connected, reductive, and isomorphic to $\hat{G}_\kappa$.

If (8.2) holds, then the arguments in [12] apply and yield $\widetilde{G}^*_\kappa = \hat{G}_\kappa$. Thus we deduce that condition (3) of the quasi-reductiveness and condition (4) in Theorem 8.2 are satisfied by $\widetilde{G}_\kappa$, and we complete the identification of group schemes by Theorem 8.2. Next, we prove (8.2) following the approach given in [11, §12].

Write H for the reductive quotient of $(\widetilde{G}^*_\kappa)_{\mathrm{red}}$, and we have that $\hat{T}_\kappa \to H$ is a closed embedding. Note that any irreducible representation of $(\widetilde{G}^*_\kappa)_{\mathrm{red}}$ is trivial on the unipotent radical. We then have:

$$\text{The canonical map } \mathrm{Irr}_H \to \mathrm{Irr}_{(\widetilde{G}^*_\kappa)_{\mathrm{red}}} \text{ is a bijection.} \tag{8.3}$$

We first note Lemma 8.4.

Lemma 8.4. *The subtorus $\hat{T}_\kappa$ is a maximal subtorus of H.*

Proof. Choose a maximal torus T_H for H and denote its Weyl group W_H. Then the irreducible representations of H are parametrized by $\mathbb{X}^\bullet(T_H)/W_H$. On the other hand, write the Weyl group for G by W_G, then Proposition 2.3 implies that $\mathbb{X}_\bullet(T_\kappa)/W_G$ parametrizes Schubert cells in Gr_{G_κ}. The IC-sheaf attached to each Schubert cell is an irreducible object in the Satake category, and thus gives rise to an irreducible representation of $\widetilde{G}_\kappa$. By our choice of $\widetilde{G}^*_\kappa$ and (8.3), we get a bijection $\mathbb{X}^\bullet(T_H)/W_H \simeq \mathbb{X}_\bullet(T)/W_G$. Hence, $T_H/W_H \simeq \hat{T}_\kappa/W_G$. Note that the Weyl group acts faithfully on the maximal torus, and we conclude that $\mathbb{X}^\bullet(T_H) = \mathbb{X}_\bullet(T)$ and $\hat{T}_\kappa$ is a maximal torus in H.

□

From now on, we write W_H for the Weyl group of H with respect to $\hat{T}_\kappa$. Recall that a (co)character of a reductive group is called regular if the cardinality of its orbit under the Weyl group action attains the maximum. Then 2ρ is a regular character in G with respect to T. By the proof of Lemma 8.4, it is a cocharacter in H with respect to $\hat{T}_\kappa$. In addition, the proof of Lemma 8.4 also shows that $W_H \cdot 2\rho = W_G \cdot 2\rho$ and thus the Weyl group orbit $W_H \cdot 2\rho$ has maximal cardinality and it follows that 2ρ is a regular cocharacter in H. Thus 2ρ fixes a Borel B_H which only depends on the Weyl chamber containing 2ρ. It also fixes a set of positive roots.

From the proof of Lemma 8.4, we deduce the followings

(8.4)

the (dominant) weights of $(H, B_H, \hat{T}_\kappa)$ coincide with (dominant) coweights of G.

(8.5)

W_H coincides with W_G together with their subsets of simple reflections identified.

To show (8.2), we hope to prove the following:

$$\Delta(H, B_H, \hat{T}_\kappa) = \Delta^\vee(G, B, T) \text{ and } \Delta^\vee(H, B_H, \hat{T}_\kappa) = \Delta(G, B, T). \tag{8.6}$$

We first prove a weaker version of (8.6).

Lemma 8.5. *Statement* (8.6) *holds if G is semisimple,*

Proof. Since G is assumed to be semisimple, then $\mathbb{Q} \cdot \mathbb{X}_\bullet^+(T) = \mathbb{Q} \cdot \Delta^\vee(G, B, T)$. Hence,

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s(G, B, T) = \{\alpha \in \mathbb{X}^\bullet(T) \mid \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathbb{X}_\bullet^+(T)\}. \tag{8.7}$$

On the other hand, it follows from (8.5) that W_H and W_G have the same cardinality. Together with (8.4), we conclude that H is also semisimple. Thus,

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa) = \{\alpha^\vee \in \mathbb{X}_\bullet(\hat{T}_\kappa) \mid \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathbb{X}_+^\bullet(\hat{T}_\kappa)\}. \tag{8.8}$$

Comparing (8.7) and (8.8), we have

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s(G, B, T) = \mathbb{Z}_{\geq 0} \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa).$$

Thus, $\Delta_s(G, B, T) = \Delta_s^\vee(H, B_H, \hat{T}_\kappa)$ and we conclude that

$$\Delta(G, B, T) = \Delta^\vee(H, B_H, \hat{T}_\kappa)$$

by noting (8.5). Finally, since for a semisimple reductive group, the coroots are uniquely determined by roots and vice versa, we also conclude that $\Delta^\vee(G, B, T) = \Delta(H, B_H, \hat{T}_\kappa)$. □

In fact, Lemma 8.5 may be proved following the idea of [1, §14][1] and [11, §12] and, we sketch this approach here.

[1]We note that the situation considered in [1, §14] is slightly different from ours. In the equal characteristic case, the group scheme $\widetilde{G}_\kappa$ is proved to be algebraic by directly exhibiting a tensor generator in the Satake category. Thus, there is no need to pass to finite type quotient $\widetilde{G}_\kappa^*$ as we do in this section.

Lemma 8.6. *We have the following inclusion of lattices*

$$\mathbb{Z} \cdot \Delta(H, \hat{T}_\kappa) \subseteq \mathbb{Z} \cdot \Delta^\vee(G, T) \tag{8.9}$$

for general G.

Proof. The proof is similar to that for [11, (12.21)] and we sketch it here. Note that the Satake category $\mathrm{Sat}_{G,\kappa}$ is equipped with a grading by $\pi_0(Gr_G) \simeq \pi_1(G) = \mathbb{X}_\bullet(T)/\mathbb{Z}\cdot\Delta^\vee(G, T)$ by [17, Proposition 1.21]. In addition, this grading is compatible with the tensor structure in $\mathrm{Sat}_{G,\kappa}$. Write Z for the center of G, then it can be identified with the group scheme

$$\mathrm{Hom}(\mathbb{X}_\bullet(T)/\mathbb{Z} \cdot \Delta^\vee(G, T), \mathbb{G}_{m,\kappa}). \tag{8.10}$$

Our previous observation implies that the forgetful functor

$$\mathrm{Sat}_{G,\kappa} \simeq \mathrm{Rep}_\kappa(\widetilde{G}_\kappa) \longrightarrow \mathrm{Rep}_\kappa(Z)$$

is compatible with the grading considered above. In this way, Z is realized as a central subgroup of $\widetilde{G}_\kappa$. Since $\hat{T}_\kappa \to H$ is a closed embedding, Z is also contained in the center of H. Finally, note that the center of H can be identified with the group scheme

$$\mathrm{Hom}(\mathbb{X}^\bullet(\hat{T}_\kappa)/\mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa), \mathbb{G}_{m,\kappa}). \tag{8.11}$$

Our discussion together with (8.10) and (8.11) completes the proof of the lemma. □

Lemma 8.7. *The set of dominant weights of $(H, \hat{T}_\kappa)$ is equal to $\mathbb{X}_\bullet^+(T) \subset \mathbb{X}_\bullet(T) = \mathbb{X}^\bullet(\hat{T}_\kappa)$.*

Proof. By our construction, we have a bijection between the set of irreducible representations of $\widetilde{G}_\kappa$ and that of $\widetilde{G}_\kappa^*$. Since irreducible representations restrict trivially to the unipotent radical, we get a bijection between the set of irreducible representations of $\widetilde{G}_\kappa$ and that of H. Thus, the dominant weights of $(H, \hat{T}_\kappa)$ equal that of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$.

Let $\lambda \in \mathbb{X}_\bullet(T)$ be a dominant weight of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$ and write the $L^{\widetilde{G}_\kappa}(\lambda)$ for the irreducible representation of $\widetilde{G}_\kappa$ associated to λ. Assume $\mu \in \mathbb{X}_\bullet^+(T)$ to be a dominant coweight of G such that the simple perverse sheaf corresponding to $L^{\widetilde{G}_\kappa}(\lambda)$ is IC_μ. Note that in the Grothendieck group of $\mathrm{Sat}_{G,\kappa}$, we have

$$[\mathrm{IC}_\mu] = \left[{}^p j_{\mu,*}\underline{\kappa}_{Gr_\mu}[(2\rho, \mu)]\right] + \sum_{\nu \in \mathbb{X}_\bullet^+(T), \nu < \mu} a_\nu^\mu \left[{}^p j_{\nu,!}\underline{\kappa}_{Gr_\nu}[(2\rho, \nu)]\right].$$

Then we conclude that $\lambda = \mu \in \mathbb{X}_\bullet^+(T)$.

On the other hand, if $\mu \in \mathbb{X}_\bullet^+(T)$, then the weights of the $\widetilde{G}_\kappa$-representation which correspond to ${}^p j_{\mu,!}\underline{\kappa}_{Gr_\mu}$ are independent of the coefficient κ by [11, Proposition 8.1]. Hence, they are weights of the irreducible $\hat{G}_{\bar{\mathbb{Q}}_\ell}$-representation of highest weight μ. Thus μ is a dominant weight of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$. □

Lemma 8.8. *The Weyl groups W_G and W_H coincide when considered as automorphism groups of $\mathbb{X}_\bullet(T)$, and their subsets of simple reflections S_G and S_H coincide.*

Proof. The proof of this lemma is completely similar to the proof of [1, Lemma 14.9] and we sketch it here. For any $\lambda \in \mathbb{X}_\bullet^+(T)$, we consider it as a dominant weight of $(H, \hat{T}_\kappa)$. Then the orbit $W_H \cdot \lambda$ is the set of extremal points of the convex polytope consisting of the convex hull of weights of the irreducible H-representation $L^H(\lambda)$. Since the set of irreducible representations of H are bijective to that of $\widetilde{G}_\kappa$, we conclude that

$$W_H \cdot \lambda = W_G \cdot \lambda. \tag{8.12}$$

Then for a regular $\lambda \in \mathbb{X}_\bullet^+(T)$, the orbit $S_G \cdot \lambda \subset W \cdot \lambda$ is the subset of $W_G \cdot \lambda$ consisting of elements μ such that the line segment connecting λ and μ is extremal in the convex hull of $W_G \cdot \lambda$. By (8.12), we have the same description for the orbit $S_H \cdot \lambda$. Thus,

$$S_G \cdot \lambda = S_H \cdot \lambda. \tag{8.13}$$

Choose an arbitrary $s_G \in S_G$. For any $\lambda \in \mathbb{X}_\bullet^+(T)$ regular, by (8.13) there exists $s_H \in S_H$ such that $s_G \cdot \lambda = s_H \cdot \lambda$. In addition, the direction of the line segment connecting λ with $s_G \cdot \lambda$ is determined by the line segment joining the coroot of G associated with s_G with the root of H associated with s_H. Thus for any other $\lambda' \in \mathbb{X}_\bullet^+(T)$ regular, we also have $s_G \cdot \lambda' = s_H \cdot \lambda'$. It follows that $s_G = s_H$ and thus $S_G = S_H$. Thus, we deduce that $W_G = W_H$. ☐

Lemma 8.9. *We have the following inclusion of lattices*

$$\mathbb{Z} \cdot \Delta(G, T) \subseteq \mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa).$$

Proof. We adapt the idea in the proof of [1, Lemma 14.10]. Firstly, we observe that Lemma 8.7 gives rise to the following equality

$$\mathbb{Q}_+ \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa) = \mathbb{Q}_+ \cdot \Delta_s(G, B, T). \tag{8.14}$$

This is because both sets consist of extremal rays of the rational convex polyhedral cone determined by $\{\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}^\bullet(T) \mid \text{ for any } \mu \in \mathbb{X}_\bullet^+(T), \langle \lambda, \mu \rangle \geq 0\}$. For $\mu \in \Delta_s(G, B, T)$, it follows from (8.14) that there exists $a \in \mathbb{Q}_+ \backslash \{0\}$ such that $a\mu \in \Delta_s^\vee(H, B_H, \hat{T}_\kappa)$. Lemma 8.8 then implies that

$$\mathrm{id} - \langle \mu^\vee, \bullet \rangle = \mathrm{id} - \langle (a\mu)^\vee, \bullet \rangle (a\mu)$$

as an automorphism of $\mathbb{X}^\bullet(T) = \mathbb{X}_\bullet(\hat{T}_\kappa)$. Thus, $(a\mu)^\vee = \frac{1}{a}\mu^\vee$. Note that Lemma 8.6 shows that $a\mu \in \mathbb{Z} \cdot \Delta^\vee(G, T)$. Thus, $\frac{1}{a} \in \mathbb{Z}$ and $\mu = \frac{1}{a}(a\mu) \in \mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa)$. ☐

The arguments above prepare us for a second proof of Lemma 8.5 as follows.

Proof. If G is in particular semisimple of adjoint type, then $\mathbb{Z} \cdot \Delta(G, T) = \mathbb{X}^*(T)$. Lemma 8.9 then implies that $\mathbb{Z} \cdot \Delta(G, T) = \mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa)$. Then the arguments in the proof of Lemma 8.9 imply that $\Delta_s(G, T) = \Delta_s^\vee(H, \hat{T}_\kappa)$. In addition, $\Delta_s^\vee(G, B, T) = \Delta_s(H, B_H, \hat{T}_\kappa)$, and the canonical bijections between the roots and coroots of H and G coincide. It then follows from Lemma 8.8 that $\Delta(H, B_H, \hat{T}_\kappa) = \Delta^\vee(G, T)$ and $\Delta^\vee(H, \hat{T}_\kappa) = \Delta(G, T)$. Thus, the root datum of H with respect to $\hat{T}_\kappa$ is dual to that of (G, T). Then the dimension estimate (8.1) concludes the proof of the lemma in the semisimple of adjoint type case.

Assume G is a general semisimple reductive group scheme. Recall notations in §1.3. We denote by G_{ad} the adjoint quotient of G and by T_{ad} the quotient of the maximal torus T. The construction in §7 goes through and we get the group scheme

$(\widetilde{G}_{\mathrm{ad}})_\kappa$. As noted in the proof of Lemma 8.6, the Satake category $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$ admits a grading by the finite group $\pi_1(G_{\mathrm{ad}})/\pi_1(G)$ which is compatible with the tensor structure of $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$. By Lemma 4.6, the category $\mathrm{Sat}_{G,\kappa}$ can be realized as a tensor subcategory of $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$ corresponding to the identity coset of $\pi_1(G)$. Thus, we have a surjective quotient

$$(\widetilde{G})_{\mathrm{ad},\kappa} \twoheadrightarrow \widetilde{G}_\kappa$$

with finite central kernel given by $\mathrm{Hom}(\pi_1(G_{\mathrm{ad}})/\pi_1(G), \mathbb{G}_{m,\kappa})$. Hence, $\widetilde{G}_\kappa$ is reductive and in particular semisimple. The result for G being semisimple of adjoint type applies here to complete the proof. □

Now, we complete the final step of identifying the group schemes.

Lemma 8.10. *Let G be a general connected reductive group, then the same result as in Lemma* 8.5 *holds.*

Proof. We sketch a proof similar to the arguments for [11, §12] and [1, Lemma 14.13]. Denote by $Z(G)$ the center of G and let $A = Z(G)^\circ$. Then A is a torus and G/A is semisimple. As in *loc.cit*, the exact sequence

$$1 \to A \to G \to G/A \longrightarrow 1$$

induces maps

$$Gr_A \xrightarrow{i} Gr_G \xrightarrow{\pi} Gr_{G/A}$$

which exhibit Gr_G as a trivial Gr_A-cover over $Gr_{G/A}$. This induces an exact sequence of functors

$$P_{L^+A}(Gr_A, \kappa) \xrightarrow{i_*} \mathrm{P}_{L^+G}(Gr_G, \kappa) \xrightarrow{\pi_*} P_{L^+G/A}(Gr_{G/A}, \kappa). \tag{8.15}$$

Note that $(Gr_A)_{\mathrm{red}}$ is a set of discrete points indexed by $\mathbb{X}_\bullet^+(A)$, then taking pushforward along i gives a fully faithful functor $i_* : P_{L^+A}(Gr_A, \kappa) \to \mathrm{P}_{L^+G}(Gr_G, \kappa)$. The functor π_* makes sense because of Lemma 4.6 and is essentially surjective.

Applying the Tannakian construction as in §7, we get flat affine group schemes $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$. Lemma 8.5 implies that $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$ are isomorphic to the dual groups of H and G/A respectively. The same arguments in [11, §12] and [1, §14] apply here to deduce that the sequence

$$1 \longrightarrow \widetilde{G/A}_\kappa \longrightarrow \widetilde{G}_\kappa \longrightarrow \widetilde{A}_\kappa \longrightarrow 1$$

induced by (8.15) is exact. Then $\widetilde{G}_\kappa$ is identified as the extension of smooth group schemes $\widetilde{A}_\kappa$ and $\widetilde{G/A}_\kappa$, and is thus also smooth. Moreover, the unipotent radical of $\widetilde{G}_\kappa$ has trivial image in the torus $\widetilde{A}_\kappa$. Hence it is included in $\widetilde{G/A}_\kappa$. Since the latter group is semisimple, it follows that $\widetilde{G}_\kappa$ is also reductive. Arguing as in [1, Lemma 14.14], we complete the proof of the lemma. □

Thus we identify the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ which arises from the general Tannakian construction with the Langlands dual group $\hat{G}_{\mathbb{Z}_\ell}$. We have our main theorem.

Theorem 8.11. *There is an equivalence of tensor categories between $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$ and the category of Λ-representations of the Langlands dual group $\hat{G}_\Lambda$ of G which are finitely generated over Λ for $\Lambda = \mathbb{F}_\ell$, and $\mathbb{Z}_\ell$ $(\ell \neq p)$.*

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