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REGULAR FUNCTIONS ON THE K-NILPOTENT CONE

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Abstract. Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and let $G_{\mathbb{R}}$ be a real form of G with maximal compact subgroup $K_{\mathbb{R}}$. Associated to $G_{\mathbb{R}}$ is a $K \times \mathbb{C}^{\times}$-invariant subvariety $\mathcal{N}_{\theta}$ of the (usual) nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$. In this article, we will derive a formula for the ring of regular functions $\mathbb{C}[\mathcal{N}_{\theta}]$ as a representation of $K \times \mathbb{C}^{\times}$.

Some motivation comes from Hodge theory. In [*Hodge theory and unitary representations of reductive Lie groups, Frontiers of Mathematical Sciences*, Int. Press, Somerville, MA, 2011, pp. 397–420], Schmid and Vilonen use ideas from Saito's theory of mixed Hodge modules to define canonical good filtrations on many Harish-Chandra modules (including all standard and irreducible Harish-Chandra modules). Using these filtrations, they formulate a conjectural description of the unitary dual. If $G_{\mathbb{R}}$ is split, and X is the spherical principal series representation of infinitesimal character 0, then conjecturally $\mathrm{gr}(X) \simeq \mathbb{C}[\mathcal{N}_{\theta}]$ as representations of $K \times \mathbb{C}^{\times}$. So a formula for $\mathbb{C}[\mathcal{N}_{\theta}]$ is an essential ingredient for computing Hodge filtrations.

1. Introduction

Let G be a complex connected reductive algebraic group and let $G_{\mathbb{R}}$ be a real form of G. Choose a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and let $\theta : G \to G$ be the corresponding Cartan involution. Let K be the group of θ-fixed points (i.e. the complexification of $K_{\mathbb{R}}$). Note that K is a complex reductive algebraic group (it is often disconnected). Write $\mathfrak{g}$,$\mathfrak{k}$ for the Lie algebras and let $\mathfrak{p} = \mathfrak{g}^{-d\theta}$. There is a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{1.0.1}$$

Using (1.0.1), we can identify $\mathfrak{p}^*$ with $(\mathfrak{g}/\mathfrak{k})^*$, a subspace of $\mathfrak{g}^*$.

Let $G \times \mathbb{C}^{\times}$ act on $\mathfrak{g}^*$ by the usual formula

$$(g, z) \cdot \zeta = z \operatorname{Ad}^*(g)\zeta, \qquad g \in \widetilde{G},\ z \in \mathbb{C}^{\times},\ \zeta \in \mathfrak{g}^*.$$

Note that $\mathfrak{p}^*$ is stable under $K \times \mathbb{C}^{\times} \subset G \times \mathbb{C}^{\times}$. Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}^*$. Recall that $G \times \mathbb{C}^{\times}$ acts on $\mathcal{N}$ (with finitely many orbits). The *K-nilpotent cone* is the $K \times \mathbb{C}^{\times}$-invariant subvariety

$$\mathcal{N}_{\theta} = \mathcal{N} \cap \mathfrak{p}^* \subset \mathfrak{p}^*.$$

This subvariety (and the $K \times \mathbb{C}^{\times}$-action on it) is closely related to the representation theory of $G_{\mathbb{R}}$, see [16]. The main result of this paper is an explicit description of the ring of regular functions $\mathbb{C}[\mathcal{N}_{\theta}]$ as a representation of $K \times \mathbb{C}^{\times}$ (in the case when $G_{\mathbb{R}}$ is split modulo center). Some motivation for this problem will be discussed at the end of Section 2.

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Since K is (in general) a disconnected group, its irreducible representations cannot be easily parameterized using the theory of highest weights. In [14], Vogan gives a parameterization of a very different flavor (making essential use of the fact that K is a symmetric subgroup of G). We will recall some of the details in Section 2.

There are several well-known results regarding the structure of $\mathbb{C}[\mathcal{N}_\theta]$ as a K-representation. In [7], it is shown that for $G_\mathbb{R}$ quasi-split, $\mathbb{C}[\mathcal{N}_\theta]$ is isomorphic as a K-representation to the induced representation $\mathrm{Ind}_M^K \mathrm{triv}$, where M is centralizer in K of a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Equivalently, $\mathbb{C}[\mathcal{N}_\theta]$ is isomorphic as a K-representation to a spherical principal series representation of $G_\mathbb{R}$. Importantly, these results do not provide any information about the $\mathbb{C}^\times$-action on $\mathbb{C}[\mathcal{N}_\theta]$. To understand this structure, we must adopt a different approach. First, we relate the ring of regular functions $\mathbb{C}[\mathcal{N}_\theta]$ (regarded as a representation of $K \times \mathbb{C}^\times$) to the ring of regular functions $\mathbb{C}[\mathcal{N}]$ (regarded as a representation of $G \times \mathbb{C}^\times$). For $G_\mathbb{R}$ split modulo center, we prove the following formula in Corollary 6.0.2

$$\mathbb{C}[\mathcal{N}_\theta]|_{K\times\mathbb{C}^\times} = \mathbb{C}[\mathcal{N}]|_{K\times\mathbb{C}^\times} \otimes [\wedge(\mathfrak{k})]. \tag{1.0.2}$$

Here $[\wedge(\mathfrak{k})]$ denotes the signed graded exterior algebra associated to $\mathfrak{k}$ (this formula takes place in the Grothendieck group of admissible representations of $K\times\mathbb{C}^\times$). The proof of this result is not purely formal—we make essential use of the fact that $\mathcal{N}$ is Cohen-Macaulay and that $\mathcal{N}_\theta \subset \mathcal{N}$ is a complete intersection of codimension $\dim(\mathfrak{k})$ (for the latter assertion, we use that $G_\mathbb{R}$ is split modulo center). The structure of $\mathbb{C}[\mathcal{N}]$ as a $G\times\mathbb{C}^\times$-representation is well-known (it can be computed using Lusztig's q-analog of Kostant's partition function, see [10]). We then 'restrict' this description to $K \times \mathbb{C}^\times$ to obtain a formula for $\mathbb{C}[\mathcal{N}_\theta]$. The final result is Theorem 7.0.2.

Along the way, we introduce a restriction map on equivariant K-theory

$$K^G(\mathcal{N}) \to K^K(\mathcal{N}_\theta)$$

(this map is defined and studied in Section 3). In fact, this map arises as the 'associated graded' of a map from representations of G (regarded as a real group) to representations of $G_\mathbb{R}$ (see Remark 3.0.8 for more details). We will pursue this point in future work.

1.1. **Notation.** Let R be an algebraic group. An algebraic R-representation V is *admissible* if every irreducible R-representation appears in V with finite multiplicity. Consider the abelian categories

$$\begin{aligned} \mathrm{Rep}(R) &= \text{algebraic representations of } R \\ \mathrm{Rep}_a(R) &= \text{admissible algebraic representations of } R \\ \mathrm{Rep}_f(R) &= \text{finite-dimensional algebraic representations of } R. \end{aligned}$$

There are obvious embeddings $\mathrm{Rep}_f(R) \subset \mathrm{Rep}_a(R) \subset \mathrm{Rep}(R)$.

Now let $\widetilde{R} = R \times \mathbb{C}^\times$. Then $\mathrm{Rep}(\widetilde{R})$, $\mathrm{Rep}_a(\widetilde{R})$, and $\mathrm{Rep}_f(\widetilde{R})$ can be defined as above. We will also consider the category

$$\mathrm{Rep}_{aa}(\widetilde{R}) = \text{algebraic representations of } \widetilde{R} \text{ which are admissible as representations of both } R \text{ and } \mathbb{C}^\times.$$

There are obvious embeddings $\mathrm{Rep}_f(\widetilde{R}) \subset \mathrm{Rep}_{aa}(\widetilde{R}) \subset \mathrm{Rep}_a(\widetilde{R}) \subset \mathrm{Rep}(\widetilde{R})$. We will denote the Grothendieck groups by $K(R), K_a(R), K_f(R), K_{aa}(\widetilde{R})$, and so on.

Write $\mathrm{Irr}(R)$ for the set of (equivalence classes of) irreducible representations of R. Then

$$\mathrm{Irr}(\widetilde{R}) = \{\tau q^n \mid \tau \in \mathrm{Irr}(R),\ n \in \mathbb{Z}\},$$

where q^n denotes the degree-n character of $\mathbb{C}^\times$ and τq^n is shorthand for the irreducible $\widetilde{R}$-representation $\tau \otimes q^n$. The group $K_f(R)$ (resp. $K_f(\widetilde{R})$) can be identified with (finite) integer combinations of $\tau \in \mathrm{Irr}(R)$ (resp. $\tau q^n \in \mathrm{Irr}(\widetilde{R})$). The group $K_a(R)$ (resp. $K_a(\widetilde{R})$) can be identified with *formal* integer combinations of $\tau \in \mathrm{Irr}(R)$ (resp. $\tau q^n \in \mathrm{Irr}(\widetilde{R})$). Finally, $K_{aa}(\widetilde{R})$ can be identified with formal integer combinations of τq^n such that for each $\tau \in \mathrm{Irr}(R)$ and for each $n \in \mathbb{Z}$ only finitely many τq^n appear with nonzero multiplicity. The tensor product of representations turns $K_f(R)$ into a commutative ring, $K_a(R)$ into a $K_f(R)$-module, and $K_{aa}(\widetilde{R})$ into a $K_f(\widetilde{R})$-module.

If X is a scheme equipped with an algebraic R-action, we write $\mathrm{Coh}^R(X)$ for the category of (strongly) R-equivariant coherent sheaves on X and $K^R(X)$ for its Grothendieck group.

2. Irreducible representations of K

In this section, we will recall a parameterization of $\mathrm{Irr}(K)$ due to Vogan [14].

Suppose H is a θ-stable maximal torus in G. Write $\Delta = \Delta(G, H)$ for the roots of H on $\mathfrak{g}$. Since H is θ-stable, θ acts on Δ by $\alpha \mapsto \alpha \circ \theta$. A root $\alpha \in \Delta$ is imaginary (resp. real, resp. complex) if $\theta\alpha = \alpha$ (resp. $\theta\alpha = -\alpha$, resp. $\theta\alpha \notin \{\pm\alpha\}$). If α is imaginary, then θ acts on the root space $\mathfrak{g}_\theta$ by $\pm\mathrm{Id}$. We say that α is compact (resp. noncompact) if $\theta|_{\mathfrak{g}_\alpha} = \mathrm{Id}$ (resp. $\theta|_{\mathfrak{g}_\alpha} = -\mathrm{Id}$). So Δ is partitioned

$$\Delta = \Delta_{i\mathbb{R}} \sqcup \Delta_{\mathbb{R}} \sqcup \Delta_{\mathbb{C}}$$

into imaginary, real, and complex roots, and $\Delta_{i\mathbb{R}}$ is partitioned

$$\Delta_{i\mathbb{R}} = \Delta_c \sqcup \Delta_n$$

into compact and noncompact roots. Note that $\Delta_{i\mathbb{R}}$ is a root system. If we choose a positive system $\Phi^+ \subset \Delta_{i\mathbb{R}}$, we can define the element $\rho_{i\mathbb{R}} = \frac{1}{2}\sum \Phi^+ \in \mathfrak{h}^*$.

Definition 2.0.1 (Sec 6, [1])**.** A continued Langlands parameter for (G, K) is a triple (H, γ, Φ^+) where

(1) H is a θ-stable maximal torus in G.
(2) γ is a formal sum $\gamma_0 + \rho_{i\mathbb{R}}$, where γ_0 is a one-dimensional $(\mathfrak{h}, H^\theta)$-module (we define $d\gamma = d\gamma_0 + \rho_{i\mathbb{R}}$).
(3) Φ^+ is a positive system for $\Delta_{i\mathbb{R}}$.

There is a K-action on the set of continued Langlands parameters. Two parameters are *equivalent* if they are conjugate under K. A continued Langlands parameter is *standard* if

(4) For every $\alpha \in \Phi^+$, $\langle d\gamma, \alpha^\vee \rangle \geq 0$.

A standard Langlands parameter is *nonzero* if

(5) For every $\alpha \in \Phi^+$ which is simple and compact, $\langle d\gamma, \alpha^\vee \rangle \neq 0$.

A nonzero Langlands parameter is *final* if

(6) If $\alpha \in \Delta_{\mathbb{R}}$ and $\langle d\gamma, \alpha^\vee \rangle = 0$, then α does not satisfy the Speh-Vogan parity condition ([12]).

Write $\mathcal{P}_L(G, K)$ for the set of equivalence classes of final Langlands parameters.

To the equivalence class of a continued Langlands parameter Γ, one can associate a virtual Harish-Chandra module $[I(\Gamma)]$. If Γ is standard, then $[I(\Gamma)]$ is represented by a distinguished $(\mathfrak{g}, K)$-module $I(\Gamma)$. If Γ is nonzero, then $I(\Gamma) \neq 0$. If Γ is final, there is a unique irreducible quotient $I(\Gamma) \twoheadrightarrow J(\Gamma)$. The following is a version of the Langlands classification.

Theorem 2.0.2 (Thm 6.1, [1])**.** *The map $\Gamma \mapsto J(\Gamma)$ defines a bijection between $\mathcal{P}_L(G, K)$ and isomorphism classes of irreducible $(\mathfrak{g}, K)$-modules.*

For any continued Langlands parameter Γ, the infinitesimal character of the virtual $(\mathfrak{g}, K)$-module $[I(\Gamma)]$ corresponds, under the Harish-Chandra isomorphism, to the W-orbit of $d\gamma$. If Γ is final, then $I(\Gamma)$ is tempered if and only if the restriction of $d\gamma$ to $\mathfrak{h}^{-d\theta}$ lies in the imaginary span of Δ (i.e. $d\gamma|_{\mathfrak{h}^{-d\theta}}$ is imaginary). In this case, $I(\Gamma)$ is irreducible, i.e. $I(\Gamma) = J(\Gamma)$.

Definition 2.0.3. A final Langlands parameter (H, γ, Φ^+) is *tempered with real infinitesimal character* if $d\gamma|_{\mathfrak{h}^{-d\theta}} = 0$. Write $\mathcal{P}_L^{t,\mathbb{R}}(G, K)$ for the set of equivalence classes of such parameters.

Corollary 2.0.4. *The map $\Gamma \mapsto J(\Gamma)$ defines a bijection between $\mathcal{P}_L^{t,\mathbb{R}}(G, K)$ and isomorphism classes of irreducible tempered $(\mathfrak{g}, K)$-modules with real infinitesimal character.*

Suppose μ is an irreducible representation of K. Choose a maximal torus $T \subset K$ and a positive system Φ_c^+ for $\Delta(K, T)$. Let $\rho_c = \frac{1}{2}\sum \Phi_c^+ \in \mathfrak{t}^*$. Let $\lambda \in \mathfrak{t}^*$ be a highest weight of μ (if K is disconnected, λ need not be unique). The K-norm of μ is defined by the formula

$$|\mu|^2 := \langle \lambda + 2\rho_c, \lambda + 2\rho_c \rangle.$$

It is not hard to see that $|\mu|$ is independent of Φ_c^+ and λ.

We say that μ is a *lowest K-type* in a $(\mathfrak{g}, K)$-module X if μ appears in X with nonzero multiplicity and $|\mu|$ is minimal among all irreducible representations of K with this property.

Theorem 2.0.5 (Thm 11.9, [14])**.** *The following are true:*

(i) *If $\Gamma \in \mathcal{P}_L^{t,\mathbb{R}}(G, K)$, then $I(\Gamma)$ contains a unique lowest K-type $\mu(\Gamma)$.*

(ii) *The map $\Gamma \mapsto \mu(\Gamma)$ defines a bijection between $\mathcal{P}_L^{t,\mathbb{R}}(G, K)$ and* $\mathrm{Irr}(K)$.

(iii) *There is a total order on $\mathcal{P}_L^{t,\mathbb{R}}(G, K)$ such that the (infinite) square matrix $m(\Gamma, \Gamma')$ defined by the formula*

$$[I(\Gamma)] = \sum_{\Gamma' \in \mathcal{P}_L^{t,\mathbb{R}}} m(\Gamma, \Gamma')\mu(\Gamma')$$

is upper triangular with 1's along the diagonal.

(iv) *In particular, this matrix $m(\Gamma, \Gamma')$ is invertible. Write $M(\Gamma, \Gamma')$ for its inverse (which is also upper triangular).*

(v) *The entries of the matrices $m(\Gamma, \Gamma')$ and $M(\Gamma, \Gamma')$ can be computed by an algorithm.*

The algorithm in (v) is described in [14].

Now suppose that $G_{\mathbb{R}}$ is split modulo center. This means that there is a maximal torus H_s in G such that $\Delta(G, H_s) = \Delta_{\mathbb{R}}(G, H_s)$ (and so $\Delta_{i\mathbb{R}}(G, H_s) = \emptyset$). Let Γ_0 be the parameter

$$\Gamma_0 := (H_s, 0, \emptyset) \in \mathcal{P}_L^{t,\mathbb{R}}(G, K).$$

By a result of Kostant ([6]) there is an identity in $K_a(K)$

$$\mathbb{C}[\mathcal{N}_\theta]|_K = I(\Gamma_0)|_K.$$

So by Theorem 2.0.5, we can write $\mathbb{C}[\mathcal{N}_\theta]$ as a formal integer sum of the irreducible K-representations $\mu(\Gamma)$ (this idea has been implemented in the atlas software). This is quite useful information, but it does *not* give us the grading on $\mathbb{C}[\mathcal{N}_\theta]$, which is part of what we're after.

In [11], Schmid and Vilonen define canonical good filtrations on all Harish-Chandra modules of 'functorial origin', including all standard and irreducible Harish-Chandra modules. These canonical good filtrations (and their associated gradeds) should be closely related to questions of unitarity.

The associated graded of a Harish-Chandra module with respect to a good filtration can be regarded as a representation of $\widetilde{K}$ (in fact, as a class in $K_{aa}(\widetilde{K})$). It is conjectured in [11] that

$$\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}} = \operatorname{gr} I(\Gamma_0)|_{\widetilde{K}}.$$

It is also suggested that the Hodge filtration on an *arbitrary* standard module (of an arbitrary group) can be reduced to this case (via cohomological induction and a deformation argument). So computing the class $\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}}$ in the case when $G_{\mathbb{R}}$ is split is central to the program of computing Hodge fitlrations. In Theorem 7.0.2, we will give a formula for $\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}}$ in terms of the classes $I(\Gamma)q^n$.

3. A restriction map $K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta)$

In this section, we will define a restriction map

$$K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta).$$

Since $\mathcal{N}$ and $\mathcal{N}_\theta$ are singular, we cannot proceed directly. Instead, we follow the standard approach outlined (for example) in [3, Sec 5.3]: we first regard $\mathcal{N}$ (resp. $\mathcal{N}_\theta$) as a subvariety of $\mathfrak{g}^*$ (resp. $\mathfrak{p}^*$) and then apply the restriction map $K^{\widetilde{G}}(\mathfrak{g}^*) \to K^{\widetilde{K}}(\mathfrak{p}^*)$ (defined in the usual way, as an alternating sum of Tor functors).

Our first proposition describes the relationship between $K^{\widetilde{G}}(\mathcal{N})$, $K^{\widetilde{G}}(\mathfrak{g}^*)$ $K_{aa}(\widetilde{G})$, and $K_f(\widetilde{G})$.

Proposition 3.0.1. *The following are true:*

(i) *If* $\mathcal{E} \in \operatorname{Coh}^{\widetilde{G}}(\mathcal{N})$, *then* $\Gamma(\mathcal{N}, \mathcal{E})|_{\widetilde{G}} \in \operatorname{Rep}_{aa}(\widetilde{G})$. *This defines an exact functor* $\operatorname{Coh}^{\widetilde{G}}(\mathcal{N}) \to \operatorname{Rep}_{aa}(\widetilde{G})$, *and hence a group homomorphism*

$$\Gamma(\bullet)|_{\widetilde{G}} : K^{\widetilde{G}}(\mathcal{N}) \to K_{aa}(\widetilde{G}).$$

(ii) *The homomorphism in (i) is injective.*

(iii) *If* $\mathcal{E} \in \operatorname{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$, *then* $\Gamma(\mathfrak{g}^*, \mathcal{E})|_{\widetilde{G}} \in \operatorname{Rep}_a(\widetilde{G})$. *This defines an exact functor* $\operatorname{Coh}^{\widetilde{G}}(\mathfrak{g}^*) \to \operatorname{Rep}_a(\widetilde{G})$, *and hence a group homomorphism*

$$\Gamma(\bullet)|_{\widetilde{G}} : K^{\widetilde{G}}(\mathfrak{g}^*) \to K_a(\widetilde{G}).$$

(iv) *Restriction along* $\{0\} \subset \mathfrak{g}^*$ *induces an exact functor* $\operatorname{Coh}^{\widetilde{G}}(\mathfrak{g}^*) \to \operatorname{Coh}^{\widetilde{G}}(\{0\}) \simeq \operatorname{Rep}_f(\widetilde{G})$, *which in turn induces a group isomorphism*

$$|_{\{0\}} : K^{\widetilde{G}}(\mathfrak{g}^*) \xrightarrow{\sim} K_f(\widetilde{G}).$$

(v) *The direct image along the closed embedding $j : \mathcal{N} \hookrightarrow \mathfrak{g}^*$ induces an exact functor* $\mathrm{Coh}^{\widetilde{G}}(\mathcal{N}) \to \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$*, and therefore a group homomorphism*

$$j_* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{G}}(\mathfrak{g}^*).$$

(vi) *If* $V \in \mathrm{Rep}_f(\widetilde{G})$*, then* $V \otimes \mathbb{C}[\mathfrak{g}^*] \in \mathrm{Rep}_a(\widetilde{G})$*. This defines an exact functor* $\mathrm{Rep}_f(\widetilde{G}) \to \mathrm{Rep}_a(\widetilde{G})$*, and hence a group homomorphism*

$$\phi_{\mathfrak{g}^*} : K_f(\widetilde{G}) \to K_a(\widetilde{G}).$$

(vii) *The following diagram commutes:*

$$\begin{tikzcd}
K_a(\widetilde{G}) & K_f(\widetilde{G}) \arrow[l, "\phi_{\mathfrak{g}^*}"'] \\
K_{aa}(\widetilde{G}) \arrow[u] & \\
K^{\widetilde{G}}(\mathcal{N}) \arrow[u, "\Gamma(\bullet)|_{\widetilde{G}}"] \arrow[r, "j_*"] & K^{\widetilde{G}}(\mathfrak{g}^*) \arrow[uul, "\Gamma(\bullet)|_{\widetilde{G}}"'] \arrow[uu, "|_{\{0\}}"']
\end{tikzcd}.$$

(viii) *The homomorphism in (v) is injective.*

Proof.

(i) This is [2, (5.4f)] (it is an immediate consequence of the following facts: $\widetilde{G}$ is reductive, $\mathcal{N}$ is affine, and $\widetilde{G}$ acts on $\mathcal{N}$ with finitely many orbits).

(ii) This is [2, Cor 7.4].

(iii) Since $\mathfrak{g}^*$ is affine, the functor $\Gamma(\bullet)|_{\widetilde{G}} : \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*) \to \mathrm{Rep}(\widetilde{G})$ is exact. It suffices to show that its image is contained in $\mathrm{Rep}_a(\widetilde{G})$. Let $\mathcal{E} \in \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$. Since $\mathfrak{g}^*$ is smooth, there is a $\widetilde{G}$-equivariant vector bundle $\mathcal{V}$ on $\mathfrak{g}^*$ and a surjection $\mathcal{V} \twoheadrightarrow \mathcal{E}$ in $\mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$. Since $\Gamma(\bullet)|_{\widetilde{G}}$ is exact, we get a surjection $\Gamma(\mathfrak{g}^*, \mathcal{V})|_{\widetilde{G}} \twoheadrightarrow \Gamma(\mathfrak{g}^*, \mathcal{E})|_{\widetilde{G}}$ in $\mathrm{Rep}(\widetilde{G})$. So it suffices to show that $\Gamma(\mathfrak{g}^*, \mathcal{V})|_{\widetilde{G}} \in \mathrm{Rep}_a(\widetilde{G})$. Let $V = \mathcal{V}|_{\{0\}}$. Then $V \in \mathrm{Rep}_f(\widetilde{G})$ and

$$\mathcal{V} \simeq V \otimes \mathcal{O}_{\mathfrak{g}^*}$$

in $\mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$. So $\Gamma(\mathfrak{g}^*, \mathcal{V})|_{\widetilde{G}} = V \otimes \mathbb{C}[\mathfrak{g}^*]|_{\widetilde{G}}$. Note that each graded component of $\mathbb{C}[\mathfrak{g}^*]$, and hence of $V \otimes \mathbb{C}[\mathfrak{g}^*]$, is finite-dimensional. So $V \otimes \mathbb{C}[\mathfrak{g}^*]|_{\widetilde{G}} \in \mathrm{Rep}_a(\widetilde{G})$, as required.

(iv) This is a well-known fact from equivariant K-theory, see [13, Thm 4.1].

(v) This follows from the fact that j is closed, and hence affine.

(vi) See the proof of (iii).

(vii) It suffices to show that the top triangle is commutative (the commutativity of the bottom triangle is obvious). If $\mathcal{V} \simeq V \otimes \mathcal{O}_{\mathfrak{g}^*} \in \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$ is a vector bundle, then

$$\Gamma(\mathfrak{g}^*, \mathcal{V})|_{\widetilde{G}} \simeq V \otimes \mathbb{C}[\mathfrak{g}^*]|_{\widetilde{G}} \simeq \mathcal{V}|_{\{0\}} \otimes \mathbb{C}[\mathfrak{g}^*]|_{\widetilde{G}}.$$

But since $\mathfrak{g}^*$ is smooth, $K^{\widetilde{G}}(\mathfrak{g}^*)$ is spanned by vector bundles. So the upper triangle is commutative.

(viii) By (vii), the map

$$K^{\widetilde{G}}(\mathcal{N}) \overset{\Gamma(\bullet)|_{\widetilde{G}}}{\to} K_{aa}(\widetilde{G}) \subset K_a(\widetilde{G}) \tag{3.0.1}$$

coincides with the composition

$$K^{\widetilde{G}}(\mathcal{N}) \overset{j_*}{\to} K^{\widetilde{G}}(\mathfrak{g}^*) \overset{\Gamma(\bullet)|_{\widetilde{G}}}{\to} K_a(\widetilde{G}).$$

By (ii), (3.0.1) is injective. Hence, $j_* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{G}}(\mathfrak{g}^*)$ must be injective as well.

□

Remark 3.0.2. We note that (ii) of Proposition 3.0.1 (and hence (viii), which is a consequence) is a very deep assertion—the proof of (ii) in [2] makes essential use of the Langlands classification.

Remark 3.0.3. It is worth considering what happens if we forget about the $\mathbb{C}^\times$-actions. Arguing exactly as in Proposition 3.0.1, we get a commutative diagram

$$\begin{tikzcd}
K(G) & & K_f(G) \arrow[ll, "\phi_{\mathfrak{g}^*}"'] \\
K_a(G) \arrow[u] & & \\
K^G(\mathcal{N}) \arrow[u, "\Gamma(\bullet)|_G"] \arrow[rr, "j_*"] & & K^G(\mathfrak{g}^*) \arrow[uu, "|_{\{0\}}"'] \arrow[uull, "\Gamma(\bullet)|_G"']
\end{tikzcd}$$

and $\Gamma : K^G(\mathcal{N}) \to K_a(G)$ is injective. However, $K(G) = 0$ (indeed, every algebraic G-representation V satisfies $V \oplus V^\infty \simeq V^\infty$, and therefore has image 0 in $K(G)$). So we cannot deduce that $j_* : K^G(\mathcal{N}) \to K^G(\mathfrak{g}^*)$ is injective (and in fact, it is not: the skyscraper sheaf at $\{0\}$ (with trivial G-action) lies in the kernel of the map $j_* : K^G(\mathcal{N}) \to K^G(\mathfrak{g}^*)$. So, the $\mathbb{C}^\times$-actions are essential for the proposition above.

If we replace $\widetilde{G}$ with $\widetilde{K}$, $\mathfrak{g}^*$ with $\mathfrak{p}^*$, and so on, we can prove a result which is completely analogous to Proposition 3.0.1 (the only change in the proof is that in (iii) we use [2, Cor 10.9] instead of [2, Cor 7.4]).

Let $i : \mathfrak{p}^* \hookrightarrow \mathfrak{g}^*$ be the inclusion. The restriction functor $i^* : \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*) \to \mathrm{Coh}^{\widetilde{K}}(\mathfrak{p}^*)$ is not exact in general (if $\mathcal{E} \in \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$, then $i^*\mathcal{E}$ corresponds to the $\mathbb{C}[\mathfrak{p}^*]$-module $\mathbb{C}[\mathfrak{p}^*] \otimes_{\mathbb{C}[\mathfrak{g}^*]} \mathcal{E}$. The functor $\mathbb{C}[\mathfrak{p}^*] \otimes_{\mathbb{C}[\mathfrak{g}^*]} (\bullet)$ is only right exact). Write $L_n i^*$ for its higher derived functors (if $\mathcal{E} \in \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$, then $L_n i^* \mathcal{E}$ corresponds to the $\mathbb{C}[\mathfrak{p}^*]$-module $\mathrm{Tor}_n^{\mathbb{C}[\mathfrak{g}^*]}(\mathbb{C}[\mathfrak{p}^*], \mathcal{E})$). Since $\mathfrak{g}^*$ is smooth, $L_n i^* \mathcal{E} = 0$ for n very large (see e.g. [3, Prop 5.1.28]). So we can define a homomorphism

$$i^* : K^{\widetilde{G}}(\mathfrak{g}^*) \to K^{\widetilde{K}}(\mathfrak{p}^*), \qquad i^*[\mathcal{E}] = \sum_{n=0}^{\infty} (-1)^n [L_n i^* \mathcal{E}].$$

If $[\mathcal{E}]$ is supported in $\mathcal{N}$, then $i^*[\mathcal{E}]$ is supported in $\mathcal{N}_\theta$. So $i^* : K^{\widetilde{G}}(\mathfrak{g}^*) \to K^{\widetilde{K}}(\mathfrak{p}^*)$ restricts to a (unique) homomorphism $i^* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta)$

$$\begin{tikzcd} K^{\widetilde{G}}(\mathcal{N}) \arrow[r, hook, "j_*"] \arrow[d, "i^*"] & K^{\widetilde{G}}(\mathfrak{g}^*) \arrow[d, "i^*"] \\ K^{\widetilde{K}}(\mathcal{N}_\theta) \arrow[r, hook, "j_*"] & K^{\widetilde{K}}(\mathfrak{p}^*) \end{tikzcd} \tag{3.0.2}$$

We can compute these restriction maps in terms of $\widetilde{K}$-representations. The key ingredient is a graded Koszul identity in $K_a(\widetilde{K})$. Consider the class

$$[\wedge(\mathfrak{k})] := \sum_{n=0}^{\infty} (-1)^n [\wedge^n(\mathfrak{k})] \in K_f(\widetilde{K}).$$

Here, as usual, we put $\mathfrak{k}$ in degree 1.

Lemma 3.0.4. *There is an identity in $K_a(\widetilde{K})$*

$$\mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] = \mathrm{triv}.$$

Proof. Consider the Koszul resolution of the trivial $\mathbb{C}[\mathfrak{k}^*]$-module

$$0 \to \mathbb{C}[\mathfrak{k}^*] \otimes \wedge^{\dim(\mathfrak{k})}(\mathfrak{k}) \to \cdots \to \mathbb{C}[\mathfrak{k}^*] \otimes \wedge^1(\mathfrak{k}) \to \mathbb{C}[\mathfrak{k}^*] \otimes \wedge^0(\mathfrak{k}) \to \mathbb{C} \to 0. \tag{3.0.3}$$

We can regard each term as a representation of $\widetilde{K}$, and it is easy to check that the differentials are $\widetilde{K}$-equivariant. Restricting to $\widetilde{K}$ we get an exact sequence in $\mathrm{Rep}_a(\widetilde{K})$

$$0 \to \mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes \wedge^{\dim(\mathfrak{k})}(\mathfrak{k}) \to \cdots \to \mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes \wedge^1(\mathfrak{k}) \to \mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes \wedge^0(\mathfrak{k}) \to \mathrm{triv} \to 0.$$

Now the required identity follows from the Euler-Poincare principle

$$\mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] = \sum_n (-1)^n \mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes [\wedge^n(\mathfrak{k})] = \mathrm{triv}.$$

$\square$

If $V \in \mathrm{Rep}_a(\widetilde{G})$, then $V|_{\widetilde{K}} \in \mathrm{Rep}_a(\widetilde{K})$. This defines an exact functor $\mathrm{Rep}_a(\widetilde{G}) \to \mathrm{Rep}_a(\widetilde{K})$, and hence a group homomorphism

$$|_{\widetilde{K}} : K_a(\widetilde{G}) \to K_a(\widetilde{K}).$$

Tensoring with the class $[\wedge(\mathfrak{k})] \in K_f(\widetilde{K})$, we obtain a further homomorphism

$$r : K_a(\widetilde{G}) \to K_a(\widetilde{K}), \qquad r[V] = [V]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})].$$

Lemma 3.0.5. *The following diagram is commutative*

$$\begin{tikzcd} K_a(\widetilde{G}) \arrow[d, "r"] & K_f(\widetilde{G}) \arrow[l, "\phi_{\mathfrak{g}^*}"'] \arrow[d, "|_{\widetilde{K}}"] \\ K_a(\widetilde{K}) & K_f(\widetilde{K}) \arrow[l, "\phi_{\mathfrak{p}^*}"] \end{tikzcd}.$$

Proof. Let $V \in K_f(\widetilde{G})$. Then by Lemma 3.0.4 we have

$$\begin{aligned}\phi_{\mathfrak{g}^*}(V)|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] &= (V \otimes \mathbb{C}[\mathfrak{g}^*])|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] \\ &= V|_{\widetilde{K}} \otimes \mathbb{C}[\mathfrak{p}^*]|_{\widetilde{K}} \otimes \mathbb{C}[\mathfrak{k}^*]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] \\ &= V|_{\widetilde{K}} \otimes \mathbb{C}[\mathfrak{p}^*]|_{\widetilde{K}} \\ &= \phi_{\mathfrak{p}^*}(V|_{\widetilde{K}})\end{aligned}$$

as desired. $\square$

Lemma 3.0.6. *The following diagram is commutative*

$$\begin{tikzcd} K^{\widetilde{G}}(\mathfrak{g}^*) \arrow[r, "|_{\{0\}}"] \arrow[d, "i^*"] & K_f(\widetilde{G}) \arrow[d, "|_{\widetilde{K}}"] \\ K^{\widetilde{K}}(\mathfrak{p}^*) \arrow[r, "|_{\{0\}}"] & K_f(\widetilde{K}) \end{tikzcd}.$$

Proof. If $\mathcal{V} \simeq V \otimes \mathcal{O}_{\mathfrak{g}^*} \in \mathrm{Coh}^{\widetilde{G}}(\mathfrak{g}^*)$ is a vector bundle, then $L_n i^* \mathcal{V} = 0$ for $n > 0$ ($\mathcal{V}$ corresponds to a flat, and hence projective, $\mathbb{C}[\mathfrak{g}^*]$-module, so all higher Tor groups vanish). Consequently

$$\begin{aligned}(i^*[\mathcal{V}])|_{\{0\}} &= [i^*\mathcal{V}]|_{\{0\}} \\ &= (V|_{\widetilde{K}} \otimes \mathcal{O}_{\mathfrak{p}^*})|_{\{0\}} \\ &= V|_{\widetilde{K}} \\ &= ([\mathcal{V}]|_{\{0\}})|_{\widetilde{K}}.\end{aligned}$$

So the diagram commutes on vector bundles. But since $\mathfrak{g}^*$ is smooth, $K^{\widetilde{G}}(\mathfrak{g}^*)$ is spanned by vector bundles. This completes the proof. $\square$

The next result gives a method for computing $i^* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta)$ on the level of $\widetilde{K}$-representations.

Corollary 3.0.7. *The following diagram is commutative*

$$\begin{tikzcd} & K_a(\widetilde{G}) \arrow[dd, "r"] & & K_f(\widetilde{G}) \arrow[ll, hook', "\phi_{\mathfrak{g}^*}"'] \arrow[dd, "|_{\widetilde{K}}"] \\ K^{\widetilde{G}}(\mathcal{N}) \arrow[ur, hook, "\Gamma"] \arrow[rr, hook, "j_*"] \arrow[dd, "i^*"'] & & K^{\widetilde{G}}(\mathfrak{g}^*) \arrow[ur, "|_{\{0\}}"'] \arrow[dd, "i^*"] & \\ & K_a(\widetilde{K}) & & K_f(\widetilde{K}) \arrow[ll, hook', "\phi_{\mathfrak{p}^*}"] \\ K^{\widetilde{K}}(\mathcal{N}_\theta) \arrow[ur, hook, "\Gamma"] \arrow[rr, hook, "j_*"] & & K^{\widetilde{K}}(\mathfrak{p}^*) \arrow[ur, "|_{\{0\}}"'] & \end{tikzcd}.$$

In particular, for every $[\mathcal{E}] \in K^{\widetilde{G}}(\mathcal{N})$, the restriction $i^*[\mathcal{E}]$ is uniquely determined by the following identity in $K_a(\widetilde{K})$

$$\Gamma(i^*[\mathcal{E}])|_{\widetilde{K}} = \Gamma([\mathcal{E}])|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})]. \tag{3.0.4}$$

Proof. The front face is commutative by the definition of $i^* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta)$, see (3.0.2). The right face is commutative by Lemma 3.0.6. The back face is commutative by Lemma 3.0.5. The top face is commutative by Proposition 3.0.1. The bottom face is commutative by its analog for $\widetilde{K}$. The commutativity of the left face follows as a formal consequence of the commutativity of the others. □

Remark 3.0.8. There are surjective 'forgetful' maps

$$K^{\widetilde{G}}(\mathcal{N}) \twoheadrightarrow K^G(\mathcal{N}), \qquad K^{\widetilde{K}}(\mathcal{N}_\theta) \twoheadrightarrow K^K(\mathcal{N}_\theta).$$

It is not hard to show that the restriction map $i^* : K^{\widetilde{G}}(\mathcal{N}) \to K^{\widetilde{K}}(\mathcal{N}_\theta)$ descends to a (necessarily unique) homomorphism

$$i^* : K^G(\mathcal{N}) \to K^K(\mathcal{N}_\theta).$$

Since we will not use this fact, we will not prove it here. Write $K\mathcal{M}(G_\mathbb{R})$ for the Grothendieck group of finite-length admissible $G_\mathbb{R}$-representations. Similarly, write $K\mathcal{M}(G)$ (here G is regarded as a real reductive group by restriction of scalars). There are 'associated graded' maps

$$\mathrm{gr} : K\mathcal{M}(G) \to K^G(\mathcal{N}), \qquad \mathrm{gr} : K\mathcal{M}(G_\mathbb{R}) \to K^K(\mathcal{N}_\theta)$$

(see [16] for definitions). An intriguing question, which we will not pursue in this paper, is whether there is a natural homomorphism $K\mathcal{M}(G) \stackrel{?}{\to} K\mathcal{M}(G_\mathbb{R})$ such that the following diagram commutes

$$\begin{array}{ccc} K\mathcal{M}(G) & \xrightarrow{?} & K\mathcal{M}(G_\mathbb{R}) \\ \big\downarrow \mathrm{gr} & & \big\downarrow \mathrm{gr} \\ K^G(\mathcal{N}) & \xrightarrow{i^*} & K^K(\mathcal{N}_\theta) \end{array} .$$

4. Regular functions on $\mathcal{N}$

In this section, we will recall a (well-known) formula for $\mathbb{C}[\mathcal{N}]$ as a representation of $\widetilde{G}$. Choose a maximal torus $H \subset G$ and a Borel subgroup $B \subset G$ containing H. Let $\Lambda \subset \mathfrak{h}^*$ denote the weight lattice, $\Phi^+ \subset \Lambda$ the positive roots, and $\Lambda^+ \subset \Lambda$ the dominant weights. Then

$$\mathrm{Irr}(G) = \{\tau_\lambda \mid \lambda \in \Lambda^+\},$$

where τ_λ is the irreducible representation of G with highest weight λ. Recall *Kostant's partition function*

$$\mathcal{P} : \Lambda \to \mathbb{Z}, \qquad \mathcal{P}(\lambda) = \#\{m : \Phi^+ \to \mathbb{Z} \mid \lambda = \sum_{\alpha \in \Phi^+} m(\alpha)\alpha\}.$$

Define

$$\mathcal{M} : \Lambda^+ \times \Lambda \to \mathbb{Z}, \qquad \mathcal{M}(\lambda, \mu) = \sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}(w(\lambda + \rho) - (\mu + \rho))$$

where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the length function and $\rho = \frac{1}{2}\sum \Phi^+$. For each $\lambda \in \Lambda^+$, there is an identity in $K_a(H)$

$$\tau_\lambda|_H = \sum_{\mu \in \Lambda} \mathcal{M}(\lambda, \mu) e^\mu. \tag{4.0.1}$$

This is a version of the Weyl character formula.

Lusztig has introduced q-analogs of both $\mathcal{P}$ and $\mathcal{M}$ ([8]). The q-analog of $\mathcal{P}$ is defined by the formula

$$\mathcal{P}_q : \Lambda \to \mathbb{Z}[q], \qquad \mathcal{P}_q(\lambda) = \sum_{n=0}^{\infty} \mathcal{P}_q^n(\lambda) q^n$$

where

$$\mathcal{P}_q^n(\lambda) = \#\{m : \Phi^+ \to \mathbb{Z} \mid \lambda = \sum_{\alpha \in \Phi^+} m(\alpha)\alpha \text{ and } n = \sum_{\alpha \in \Phi^+} m(\alpha)\}.$$

The q-analog of $\mathcal{M}$ is

$$\mathcal{M}_q : \Lambda^+ \times \Lambda \to \mathbb{Z}[q], \qquad \mathcal{M}_q(\lambda, \mu) = \sum_{w \in W} (-1)^{\epsilon(w)} \mathcal{P}_q(w(\lambda + \rho) - (\mu + \rho)).$$

The following can be extracted from [10].

Proposition 4.0.1. *There is an identity in* $K_a(\widetilde{G})$

$$\mathbb{C}[\mathcal{N}]|_{\widetilde{G}} = \sum_{\lambda \in \Lambda^+} \tau_\lambda \mathcal{M}_q(\lambda, 0).$$

Sketch of proof. Consider the Springer resolution $\eta : T^*(G/B) \to \mathcal{N}$. Since $T^*(G/B)$ is symplectic, there is an identification (of $\widetilde{G}$-equivariant sheaves)

$$\mathcal{O}_{T^*(G/B)} \simeq \omega_{T^*(G/B)}$$

where $\omega_{T^*(G/B)}$ is the canonical sheaf on $T^*(G/B)$. So by the theorem of Grauert and Riemenschneider ([4])

$$R^i \eta_* \mathcal{O}_{T^*(G/B)} = 0, \qquad \forall i > 0.$$

On the other hand, $R^0 \eta_* \mathcal{O}_{T^*(G/B)} \simeq \mathcal{O}_{\mathcal{N}}$. Using the Leray spectral sequence (and the fact that $\mathcal{N}$ is affine), we get an identity in $K_a(\widetilde{G})$

$$\mathbb{C}[\mathcal{N}]|_{\widetilde{G}} = \sum_i (-1)^i H^i(T^*(G/B), \mathcal{O}_{T^*(G/B)}). \tag{4.0.2}$$

If $p : T^*(G/B) \to G/B$ is the projection, then $p_* \mathcal{O}_{T^*(G/B)}$ is identified (as a $\widetilde{G}$-equivariant sheaf) with (the sheaf of local sections of) the $\widetilde{G}$-equivariant vector bundle $G \times_B S(\mathfrak{g}/\mathfrak{b})$. Since p is affine, the direct image functor p_* preserves cohomology, i.e.

$$H^i(T^*(G/B), \mathcal{O}_{T^*(G/B)}) \simeq H^i(G/B, p_* \mathcal{O}_{T^*(G/B)}), \qquad \forall i \geq 0. \tag{4.0.3}$$

Combining (4.0.2) and (4.0.3), we get a further identity in $K_a(\widetilde{G})$

$$\mathbb{C}[\mathcal{N}]|_{\widetilde{G}} = \sum_i (-1)^i H^i(G/B, G \times_B S(\mathfrak{g}/\mathfrak{b})).$$

The right hand side can be computed using Borel-Weil-Bott. The result follows. □

5. Some commutative algebra

Using the results of Section 3, we get a well-defined class $i^*[\mathcal{O}_{\mathcal{N}}] \in K^{\widetilde{K}}(\mathcal{N}_\theta)$, which can be regarded as the restriction of $\mathcal{O}_{\mathcal{N}}$ to $\mathfrak{p}^*$. However, it is not at all clear that $i^*[\mathcal{O}_{\mathcal{N}}] = [\mathcal{O}_{\mathcal{N}_\theta}]$. For groups split modulo center, we will see that this equality always holds, but the proof will require some commutative algebra.

Lemma 5.0.1. *Let R be a Noetherian ring and let M be a finitely-generated R-module. Suppose $x_1, \dots, x_m \in R$ is an R-regular sequence which is also M-regular. Consider the ideal $I = (x_1, \dots, x_m) \subset R$. Then*

$$\mathrm{Tor}_n^R(R/I, M) = 0, \qquad n > 0.$$

Proof. Since $x_1, \dots, x_m$ is R-regular, the Koszul complex $K(x_1, \dots, x_m; R)$ is a resolution of R/I. So $\mathrm{Tor}_\bullet^R(R/I, M)$ is the homology of $K(x_1, \dots, x_m; M) := K(x_1, \dots, x_m; R) \otimes_R M$

$$H_n(K(x_1, \dots, x_m; M)) \simeq \mathrm{Tor}_n^R(R/I, M), \qquad \forall n.$$

But since $x_1, \dots, x_m$ is an M-regular sequence, the complex $K(x_1, \dots, x_m; M)$ is acyclic, see [9, Thm 16.5(i)]. So $\mathrm{Tor}_n^R(R/I, M) = 0$ for $n > 0$. $\square$

Lemma 5.0.2 (Thm 17.4, [9])**.** *Suppose A is Cohen-Macaulay, and let $I = (x_1, \dots, x_n)$
$\subset A$ be an ideal. If*

$$\dim(A/I) = \dim(A) - n$$

then $x_1, \dots, x_n$ is an A-regular sequence.

Proposition 5.0.3. *Suppose X is a smooth Noetherian scheme and let Y, Z be closed subschemes of X. Write $i : Y \hookrightarrow X$ and $j : Z \hookrightarrow X$ for the inclusions and form the Cartesian diagram of schemes*

$$\begin{tikzcd} Z \arrow[r, hook, "j"] & X \\ Z \cap Y \arrow[u, hook] \arrow[r, hook] & Y \arrow[u, hook, "i"'] \end{tikzcd} .$$

Assume

(i) *Y is smooth.*
(ii) *Z is Cohen-Macaulay.*
(iii) *$\dim(Z \cap Y) = \dim(Z) + \dim(Y) - \dim(X)$.*

Then

$$L_n i^*(j_* \mathcal{O}_Z) = 0, \qquad n > 0. \tag{5.0.1}$$

Proof. The statement is local in X, so we can assume all schemes are affine. Let $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(R/I)$ and $Z = \mathrm{Spec}(A)$, so that $Z \cap Y = \mathrm{Spec}(A/I)$. Since Y is smooth, we can find an R-regular sequence $x_1, \dots, x_m \in R$ such that $I = (x_1, \dots, x_m)$, where $m = \dim(X) - \dim(Y)$. Now by Lemma 5.0.2, $(x_1, \dots, x_m)$ is an A-regular sequence. So by Lemma 5.0.1 (applied to the R-module $M = A$)

$$\mathrm{Tor}_n^R(R/I, A) = 0, \qquad n > 0.$$

This is equivalent to (5.0.1). $\square$

6. Regular functions on $\mathcal{N}_\theta$

First, we will assume that $G_{\mathbb{R}}$ is split. This means that G contains a maximal torus $H_s \subset G$ on which θ acts by inversion. Our first lemma shows that the codimension of $\mathcal{N}_\theta \subset \mathcal{N}$ equals the codimension of $\mathfrak{p}^* \subset \mathfrak{g}^*$.

Lemma 6.0.1. $\dim(\mathcal{N}_\theta) = \dim(\mathcal{N}) + \dim(\mathfrak{p}) - \dim(\mathfrak{g})$.

Proof. By [7, Prop 9]

$$\dim(\mathcal{N}_\theta) = \dim(\mathfrak{p}) - \dim(\mathfrak{h}_s). \tag{6.0.1}$$

By the Iwasawa decomposition

$$\dim(\mathfrak{g}) = \dim(\mathfrak{k}) + \dim(\mathfrak{h}_s) + \dim(\mathfrak{n}). \tag{6.0.2}$$

Finally

$$\dim(\mathcal{N}) = 2\dim(\mathfrak{n}) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_s). \tag{6.0.3}$$

Combining (6.0.1), (6.0.2), and (6.0.3) proves the lemma. □

Corollary 6.0.2. *There is an equality in $K^{\widetilde{K}}(\mathcal{N}_\theta)$*

$$[\mathcal{O}_{\mathcal{N}_\theta}] = i^*[\mathcal{O}_{\mathcal{N}}] \tag{6.0.4}$$

and hence an equality in $K_{aa}(\widetilde{K})$

$$\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}} = \mathbb{C}[\mathcal{N}]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})]. \tag{6.0.5}$$

Proof. For (6.0.4), we will apply Proposition 5.0.3. So let $X = \mathfrak{g}^*$, $Z = \mathcal{N}$, and $Y = \mathfrak{p}^*$. Clearly, X and Y are smooth. By [5, Thm 0.1], $\mathcal{N}$ is a complete intersection, and therefore Cohen-Macaulay. Condition (iii) of Proposition 5.0.3 is the content of Lemma 6.0.1. So by Proposition 5.0.3, we have

$$L_n i^*(j_*\mathcal{O}_{\mathcal{N}}) = 0, \qquad n > 0$$

and therefore

$$i^*[\mathcal{O}_{\mathcal{N}}] = \sum_n (-1)^n [L_n i^* j_* \mathcal{O}_{\mathcal{N}}] = [i^* j_* \mathcal{O}_{\mathcal{N}}] = [\mathcal{O}_{\mathcal{N}_\theta}].$$

This proves (6.0.4). Now (6.0.5) follows from (6.0.4) and Corollary 3.0.7. □

Corollary 6.0.2 can be easily extended to the case when $G_{\mathbb{R}}$ is split modulo center.

Example 6.0.3. Let $G = \mathrm{SL}_2(\mathbb{C})$ and let $\theta(g) = (g^{-1})^t$ (this is the involution corresponding to split real form $\mathrm{SL}_2(\mathbb{R})$). Then $K = \mathrm{SO}_2(\mathbb{C})$. Write

$$\mathrm{Irr}(G) = \{\tau_m \mid m = 0, 1, 2, \dots\}, \qquad \mathrm{Irr}(K) = \{\chi_n \mid n \in \mathbb{Z}\}$$

(here τ_m is the irreducible with highest weight m and χ_n is the degree-n character of $\mathrm{SO}_2(\mathbb{C})$). We have the following branching rules

$$\tau_m|_K = \chi_{-2m} + \chi_{-2m+2} + \cdots + \chi_{2m}.$$

From Proposition 4.0.1, we deduce

$$\mathbb{C}[\mathcal{N}]|_{\widetilde{G}} = \sum_{m=0}^{\infty} \tau_{2m} q^m.$$

Also

$$[\wedge(\mathfrak{k})] = \chi_0 - \chi_0 q.$$

So by Corollary 6.0.2

$$\begin{aligned}\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}} &= \mathbb{C}[\mathcal{N}]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})] \\ &= (\sum_{m=0}^{\infty}(\chi_{-2m}+\cdots+\chi_{2m})q^m) \otimes (\chi_0 - \chi_0 q) \\ &= \sum_{m=0}^{\infty}((\chi_{-2m}+\cdots+\chi_{2m}) - (\chi_{-2m+2}+\cdots+\chi_{2m-2}))q^m \\ &= \sum_{m=0}^{\infty}(\chi_{2m}+\chi_{-2m})q^m.\end{aligned}$$

7. Branching to K

In this section, we will use Corollary 6.0.2 to compute $\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}}$ as a formal integer combination of classes of the form $I(\Gamma)q^n$.

Suppose (H, γ, Φ^+) is a continued Langlands parameter (see Definition 2.0.1) and let $\chi \in K_f(K)$. It is easy to compute the tensor product $[I(H,\gamma,\Phi^+)]|_K \otimes \chi$ as a representation of K.

Lemma 7.0.1 (Lem 12.13, [14]). *Choose a finite multiset $S_H(\chi)$ in $X^*(H)$ such that*

$$\chi|_{H^\theta} = \sum_{\mu \in S_H(\chi)} \mu|_{H^\theta}.$$

Then there is an identity in $K_a(K)$

$$[I(H,\gamma,\Phi^+)]|_K \otimes \chi = \sum_{\mu \in S_H(\chi)} [I(H,\gamma+\mu,\Phi^+)].$$

We will use Lemma 7.0.1 (together with Zuckerman's character formula for the trivial representation) to compute $\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}}$ in terms of the classes $I(\Gamma)q^n$. For an arbitrary class $\chi \in K_f(K)$, it may be difficult to find a multiset $S_H(\chi)$ as in Lemma 7.0.1. Fortunately, in our setting, χ is *not* arbitrary. For the problem at hand, we will need to compute $S_H(\chi)$ in the following two cases:

(1) χ is the restriction to K of an irreducible representation τ_λ of G.
(2) χ is the class $\wedge^n(\mathfrak{k})$.

First, suppose $\chi = \tau_\lambda|_K$. By the Weyl character formula (4.0.1), we have

$$\tau_\lambda|_H = \sum_{\mu \in X^*(H)} \mathcal{M}(\lambda,\mu)e^\mu.$$

So we can take

$$S_H(\tau_\lambda) = \{\mathcal{M}(\lambda,\mu)e^\mu \mid \mu \in X^*(H)\}$$

(the coefficients $\mathcal{M}(\lambda,\mu)$ above denote the multiset multiplicities. We will use similar notation below).

Next, suppose $\chi = \mathfrak{k}$. Choose a subset $\Delta'_{\mathbb{C}} \subset \Delta_{\mathbb{C}}$ such that for each $\alpha \in \Delta_{\mathbb{C}}$, exactly one of $\{\alpha, \theta\alpha\}$ appears in $\Delta'_{\mathbb{C}}$. Then there is a decomposition of $\mathfrak{k}$ into weight spaces for H^θ

$$\mathfrak{k} \simeq \bigoplus_{\alpha \in \Delta_c} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta'_{\mathbb{C}}} (1+d\theta)\mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta^+_{\mathbb{R}}} (1+d\theta)\mathfrak{g}_\alpha.$$

So we can take

$$S_H(\mathfrak{k}) = \{e^\alpha \mid \alpha \in \Delta_c\} \cup \{e^\alpha \mid \alpha \in \Delta'_{\mathbb{C}}\} \cup \{|\Delta^+_{\mathbb{R}}|e^0\}.$$

More generally

$$S_H(\wedge^n \mathfrak{k}) = \{\sum R \mid R \subseteq S_H(\mathfrak{k}),\ |R| = n\}.$$

Zuckerman's character formula for the trivial representation (see [15, Thm 9.4.16]) can be interpreted as an identity in $K_a(K)$

$$\mathrm{triv} = \sum_H \sum_{\Delta^+} (-1)^{\ell(\Delta^+)} [I(H, \rho_{i\mathbb{R}}, \Delta^+_{i\mathbb{R}})]|_K.$$

The outer sum runs over K-conjugacy classes of θ-stable maximal tori $H \subset G$ and the inner sum over $W(K, H^\theta)$-conjugacy classes of positive systems $\Delta^+ \subset \Delta(G, H)$. The integer $\ell(\Delta^+)$ is the codimension of the K-orbit on the flag variety containing the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ corresponding to the positive system Δ^+. Now assume that $G_{\mathbb{R}}$ is split modulo center. Using Corollary 6.0.2, we obtain an identity in $K_a(\widetilde{K})$

$$\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}} = \left(\sum_H \sum_{\Delta^+} (-1)^{\ell(\Delta^+)} [I(H, \rho_{i\mathbb{R}}, \Delta^+_{i\mathbb{R}})]|_K \right) \otimes \mathbb{C}[\mathcal{N}]|_{\widetilde{K}} \otimes [\wedge(\mathfrak{k})].$$

Using Proposition 4.0.1 and Lemma 7.0.1, we can rewrite the right hand side in terms of classes of the form $[I(\Gamma)]q^n$

Theorem 7.0.2. *Assume $G_{\mathbb{R}}$ is split modulo center. Then there is an identity in $K_a(\widetilde{K})$*

$$\mathbb{C}[\mathcal{N}_\theta]|_{\widetilde{K}}$$
$$= \sum_H \sum_{\Delta^+} \sum_{\lambda \in \Lambda^+} \sum_{\mu \in \Lambda} \sum_{R \subseteq S_H(\mathfrak{k})} (-1)^{\ell(\Delta^+)+|R|} \mathcal{M}(\lambda, \mu) \mathcal{M}_q(\lambda, 0) [I(H, \rho_{i\mathbb{R}} + \mu + |R|, \Delta^+_{i\mathbb{R}})] q^{|R|}.$$

Note that the terms on the right are not final. We can rewrite the sum in terms of final parameters using the Hecht-Schmid identities (this sort of thing is easy to do in atlas).

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