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CHARACTER SHEAVES FOR CLASSICAL SYMMETRIC PAIRS

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Abstract. We establish a Springer theory for classical symmetric pairs. We give an explicit description of character sheaves in this setting. In particular we determine the cuspidal character sheaves.

Contents

1. Introduction

In this paper we work out a theory analogous to the generalized Springer correspondence of [L1] in the context of symmetric pairs. We concentrate on classical symmetric pairs, but our methods are general and extend to other isogeny classes and exceptional types with some modifications.

Let G be a connected complex reductive algebraic group and $\theta : G \to G$ an involution. Let $K = G^\theta$ be the subgroup of fixed points of θ. The pair (G, K) is called a symmetric pair. It is called a *split* symmetric pair if there exists a maximal torus T of G that is θ-split, i.e., $\theta(t) = t^{-1}$ for all $t \in T$. Let $\mathfrak{g} = \operatorname{Lie} G$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the decomposition into θ-eigenspaces so that $d\theta|_{\mathfrak{g}_i} = (-1)^i$. Let $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}$ and let $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}_1$ be the nilpotent cone in $\mathfrak{g}_1$. We write $\operatorname{Char}_K(\mathfrak{g}_1)$ for the set of irreducible $\mathbb{C}^*$-conic K-equivariant perverse sheaves $\mathcal{F}$ on $\mathfrak{g}_1$ whose singular support is nilpotent, i.e., for $\mathcal{F}$ such that $\operatorname{SS}(\mathcal{F}) \subset \mathfrak{g}_1 \times \mathcal{N}_1$ where $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$ are identified via a K-invariant non-degenerate bilinear form on $\mathfrak{g}_1$. We call the sheaves in $\operatorname{Char}_K(\mathfrak{g}_1)$ character sheaves for the symmetric pair (G, K).

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The classical character sheaves of [L1] on a Lie algebra $\mathfrak{g}$ can be viewed as a special case of character sheaves for symmetric pairs, where one considers the symmetric pair $(G \times G, G)$ with θ switching the factors in $G \times G$.

In this paper we concentrate on the classical symmetric pairs, that is, when G is GL_n, Sp_{2n} or SO_n. We give a complete description of the set $\mathrm{Char}_K(\mathfrak{g}_1)$ for all classical symmetric pairs (G, K). Special cases have been considered before. The case (GL_{2n}, Sp_{2n}) was considered by Grinberg in [G3], Henderson in [H], and Lusztig in [L2]. The case $(GL_n, GL_p \times GL_q)$ was considered by Lusztig in [L2] where he treated GL_n in the case of arbitrary finite order semi-simple inner automorphisms. In these instances the Springer theory closely resembles the classical situation. In [CVX] we considered the case (SL_n, SO_n) where phenomena quite different from the classical case already occur. We treat the inner involutions of SL_n in a companion paper [VX1] as it requires an expansion of the techniques used here. The symmetric pairs associated to spin groups are treated in [X].

We write $\mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)$ for character sheaves whose support is all of $\mathfrak{g}_1$; we call these full support character sheaves. We show that all full support character sheaves arise from the nearby cycle construction in [GVX1], which, in turn, is based on ideas in [G1, G2]. Sheaves in $\mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)$ are IC-sheaves of certain K-equivariant local systems on $\mathfrak{g}_1^{rs}$, the regular semisimple locus of $\mathfrak{g}_1$. The equivariant fundamental group $\pi_1^K(\mathfrak{g}_1^{rs})$ is an extended braid group, see (2.3). In Section 3.2 we construct certain Hecke algebras, following [GVX1], and show how their simple modules give us full support character sheaves, see Proposition 5.2 and Corollary 5.5.

At the other extreme from full support character sheaves there are nilpotent support character sheaves, i.e., those supported on closures of nilpotent K-orbits in $\mathcal{N}_1$. We write $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ for these character sheaves. We show such sheaves occur only when the involution θ is inner. It turns out that the supports of these character sheaves are exactly the Richardson orbits attached to θ-stable Borel subgroups. From a geometric point of view these are the nilpotent K-orbits whose closures are images of conormal bundles of closed K-orbits on the flag manifold G/B under the moment map. They can be viewed as singular supports of discrete series representations. In this context these nilpotent orbits were studied and classified by Trapa in [T1]. We make use of his classification in our determination of $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ in Section 4. A crucial ingredient in determining $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ is Proposition 4.6. In that proposition we associate a specific θ-stable Borel subgroup to each Richardson orbit. We show that all nilpotent support character sheaves arise via parabolic induction from these specific Borels. The choice of these Borels also allows us to study parabolic induction in sufficient detail to prove our main result.

Following standard terminology, we call a character sheaf *cuspidal* if it does not appear as a direct summand (up to shift) in parabolic induction of character sheaves in $\mathrm{Char}_{L^\theta}(\mathfrak{l}_1)$ from θ-stable Levi subgroups L contained in proper θ-stable parabolic subgroups of G. We write $\mathrm{Char}^{\mathrm{cusp}}_K(\mathfrak{g}_1)$ for the set of cuspidal character sheaves. In [L1] Lusztig worked out the cuspidal character sheaves in the classical case, which have nilpotent supports.

As a corollary of our explicit description of character sheaves, we classify cuspidal character sheaves for classical symmetric pairs as follows.

Theorem 1.1. *(Corollary 6.7) The cuspidal character sheaves consist of*

(1) *The constant sheaf* $\mathbb{C}_{\mathfrak{g}_1}$ *for the pair* $(GL_2, GL_1 \times GL_1)$,

(2) *Full support character sheaves for split symmetric pairs. When* (G, K) *is a split symmetric pair, we have*

$$\mathrm{Char}_K^{cusp}(\mathfrak{g}_1) = \mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1) = \left\{ \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\rho) \,|\, \rho \in \Theta_{(G,K)} \right\}.$$

For an explicit description of this set, see Corollary 5.5.

We note that the skyscraper sheaf at the origin for the pair (GL_1, GL_1) is also cuspidal. The full support cuspidal sheaves in (2) are obtained via the nearby cycle construction from simple modules of Hecke algebras of [GVX1] as explained in §3.2. These Hecke algebras are not in general associated to Weyl groups of subgroups of G. However, as we will explain in §3.3, they are associated to Weyl groups of subgroups of the dual group $\check{G}$ and so one can think of them as arising from endoscopic groups.

Remark 1.2. For other isogeny classes such as special linear groups and spin groups, there are further cuspidal character sheaves of full support, or cuspidal character sheaves which are not of full support at various central characters. In particular, we have cuspidal character sheaves for special linear groups on quasi-split symmetric pairs, see [VX1]. For spin groups there are cuspidals for symmetric pairs which are not quasi-split, see [X].

We determine all character sheaves in Theorems 6.2, 6.3, 6.4. For each nilpotent K-orbit $\mathcal{O}$ on $\mathcal{N}_1$ we construct in §3.1 the corresponding dual stratum $\check{\mathcal{O}} \subset \mathfrak{g}_1$. The character sheaves are locally constant along the strata $\check{\mathcal{O}}$. For the zero orbit the dual stratum is $\mathfrak{g}_1^{rs}$. In §6.1 we determine explicitly the nilpotent orbits $\mathcal{O}$ such that $\check{\mathcal{O}}$ supports a character sheaf and we explicitly describe, analogously to the $\mathfrak{g}_1^{rs}$ case, the equivariant fundamental groups $\pi_1^K(\check{\mathcal{O}})$ of these $\check{\mathcal{O}}$ in terms of extended braid groups, see (3.4). We write down explicitly the local systems on the $\check{\mathcal{O}}$ in terms of certain Hecke algebras whose IC-sheaves are character sheaves. To prove the theorems we make a detailed study of parabolic induction of character sheaves from θ-stable Levi subgroups contained in proper θ-stable parabolic subgroups. To show that the nearby cycle construction and parabolic induction give a complete set of the character sheaves we rely on a counting argument, as in the classical case treated in [L1].

We work with varieties over complex numbers and with sheaves with complex coefficients. However, we can use any field of characteristic zero as coefficients, although sometimes we require the field to contain roots of unity. In [GVX1] we work over the integers. Thus, it seems reasonable that one can develop a modular theory along the lines of this paper. Furthermore, our results should hold for ℓ-adic sheaves in the finite characteristic setting, but at the moment [GVX1] is written in the classical topology only.

In [LY], the authors have studied the decomposition of the derived category $D_K(\mathcal{N}_1)$ into blocks using spiral induction where they treat the general case of finite order semi-simple automorphisms. Their work can be viewed as another generalization of the Springer theory. We expect that the theory presented in this paper can be extended to the case of finite order automorphism θ, see partial results in [GVX2, VX2], although we do not expect to obtain as detailed results as those presented in this paper. Graded Lie algebras are intimately connected to p-adic groups via the Moy-Prasad filtration (see, for example, [RY]). Our theory is also directly related to the affine Springer fibers studied by Oblomkov and Yun [OY].

Thus, we expect our theory presented here, and its generalization to the higher order cases, to have applications in p-adic groups as well as geometry of affine Springer fibers. We will connect our results to the work of Lusztig and Yun, and address these applications in future work.

The paper is organized as follows. In Section 2 we recall the preliminaries on symmetric pairs, nilpotent orbits, restricted roots, little Weyl groups, and set up the notation. In particular, we give an explicit description of the classical symmetric pairs we work with and explicitly describe the associated Lie theoretic data. In Section 3 we describe the general strategy to determine the set $\mathrm{Char}_K(\mathfrak{g}_1)$. In Section 4 we describe the nilpotent support character sheaves. In Section 5 we apply the nearby cycle construction in [GVX1] and describe the full support character sheaves. In Section 6 we state the main theorems, Theorems 6.2, 6.3, and 6.4, where the character sheaves are determined. In Section 7 we prove the main theorems by combining the results in the previous sections, parabolic induction and counting arguments. In Appendix A we discuss the dual strata in the classical situations. Appendix B by Dennis Stanton contains proofs of combinatorial formulas which are crucial for the proofs in this paper.

2. Preliminaries

Throughout the paper we work with algebraic groups over $\mathbb{C}$ and with sheaves with complex coefficients. For perverse sheaves we use the conventions of [BBD]. If $\mathcal{F}$ is a perverse sheaf up to a shift we often write $\mathcal{F}[-]$ for the corresponding perverse sheaf.

2.1. **Character sheaves.** Let G be a connected reductive algebraic group over $\mathbb{C}$ and $\theta : G \to G$ an involution. Let $K = G^\theta$. A torus $A \subset G$ is called θ-split, if $\theta(t) = t^{-1}$ for all $t \in A$. The symmetric pair (G, K) is called split if there exists a maximal torus A of G which is θ-split. Note that the involution θ is inner if and only if $\operatorname{rank} K = \operatorname{rank} G$.

Let $\mathfrak{g}$ be the Lie algebra of G. The involution θ gives rise to a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $d\theta|_{\mathfrak{g}_i} = (-1)^i$. Let $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}$ and let $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}_1$. The group K acts on $\mathcal{N}_1$ with finitely many orbits [KR]. We identify $\mathfrak{g}_1$ and $\mathfrak{g}_1^*$ via a K-invariant non-degenerate bilinear form on $\mathfrak{g}_1$, which can be obtained from a G-invariant and θ-invariant non-degenerate bilinear form on $\mathfrak{g}$.

Let $\mathcal{A}_K(\mathfrak{g}_1)$ denote the set of irreducible K-equivariant perverse sheaves on $\mathcal{N}_1$. That is,

$$\mathcal{A}_K(\mathfrak{g}_1) = \{\mathrm{IC}(\mathcal{O}, \mathcal{E}) \mid \mathcal{O} \subset \mathcal{N}_1 \text{ is an } K\text{-orbit and } \mathcal{E} \text{ is an irreducible } K\text{-equivariant local system on } \mathcal{O} \text{ (up to isomorphism)}\}.$$

Recall the set $\mathrm{Char}_K(\mathfrak{g}_1)$ of character sheaves, that is, the set of irreducible $\mathbb{C}^*$-conic K-equivariant perverse sheaves on $\mathfrak{g}_1$ whose singular support is nilpotent. Consider the Fourier transform $\mathfrak{F} : P_K(\mathfrak{g}_1) \to P_K(\mathfrak{g}_1)$. By definition the functor $\mathfrak{F}$ gives us a bijection

$$\mathfrak{F} : \mathcal{A}_K(\mathfrak{g}_1) \xrightarrow{\sim} \mathrm{Char}_K(\mathfrak{g}_1)\,. \tag{2.1}$$

This implies, in particular, that the set $\mathrm{Char}_K(\mathfrak{g}_1)$ is finite. Note that Lusztig calls the sheaves in $\mathcal{A}_K(\mathfrak{g}_1)$ orbital complexes and the sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)$ anti-orbital complexes.

There are two important extreme cases of character sheaves. The character sheaves supported on all of $\mathfrak{g}_1$ are called full support character sheaves, and we write $\mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)$ for them. The character sheaves supported on nilpotent K-orbits, are called nilpotent support character sheaves and we write $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ for them.

2.2. **Restricted roots and the little Weyl group.** Let (G,K) be a symmetric pair. We fix a Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}_1$, i.e., a maximal abelian subspace consisting of semisimple elements. We write A for the maximal θ-split torus of G with Lie algebra $\mathfrak{a}$. Let $T \subset Z_G(\mathfrak{a})$ be a maximal torus. Then $T \supset A$ and T is θ-stable. Let $C = T^\theta$ and $\mathfrak{c} = \mathrm{Lie}\, C \subset \mathfrak{g}_0$. We have $\mathrm{Lie}\, T = \mathfrak{t} = \mathfrak{a} \oplus \mathfrak{c}$.

Let $\Phi \subset X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ be the root system of $(\mathfrak{g}, T)$. For each $\alpha \in \Phi$, let $\check{\alpha} \in X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the corresponding coroot. We recall the following notions

$$\begin{aligned}
&\alpha \in \Phi \ \text{ is real if } \alpha|_{\mathfrak{c}} = 0 \iff \theta\alpha = -\alpha,\\
&\alpha \in \Phi \ \text{ is imaginary if } \alpha|_{\mathfrak{a}} = 0 \iff \theta\alpha = \alpha,\\
&\alpha \in \Phi \ \text{ is complex otherwise.}
\end{aligned}$$

Let $\Phi^{\mathrm{R}} \subset \Phi$ (resp. $\Phi^{Im} \subset \Phi$, $\Phi^{\mathrm{C}x} \subset \Phi$) be the set of real (resp. imaginary, complex) roots. Then Φ^{R} and Φ^{Im} are subroot systems of Φ. Furthermore, let $\rho^{\mathrm{R}} = \frac{1}{2}\sum_{\alpha\in\Phi^{\mathrm{R},+}} \alpha$, $\rho^{\mathrm{I}m} = \frac{1}{2}\sum_{\alpha\in\Phi^{\mathrm{I}m,+}} \alpha$ (with respect to some positive systems of Φ^{R} and Φ^{Im} respectively) and let

$$\Phi^{\mathrm{C}} = \{\alpha \in \Phi^{\mathrm{C}x} \mid (\alpha, \rho^{\mathrm{R}}) = (\alpha, \rho^{\mathrm{I}m}) = 0\}.$$

Then Φ^{C} is also a subroot system of Φ.

Let $W_{\mathfrak{a}} := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the little Weyl group of the pair (G,K). We have (see [V, Proposition 4.16])

$$W_{\mathfrak{a}} \cong W^{\mathrm{R}} \rtimes (W^{\mathrm{C}})^\theta$$

where W^{C} (resp. W^{R}) is the Weyl group of the root system Φ^{C} (resp. Φ^{R}) and $(W^{\mathrm{C}})^\theta \subset W^{\mathrm{C}}$ is the subgroup that consists of the elements in W^{C} commuting with θ. It is well-known that $W_{\mathfrak{a}}$ is also the Weyl group of the restricted root system, denoted by Σ, i.e., the (not necessarily reduced) root system formed by the restrictions of $\alpha \in \Phi$ to $\mathfrak{a}$. We have

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\bar{\alpha}\in\Sigma} \mathfrak{g}_{\bar{\alpha}},$$

where $\mathfrak{g}_{\bar{\alpha}}$ denotes the root space corresponding to the restricted root $\bar{\alpha}$. Note that if $\bar{\alpha}$ is the restriction of a complex root α, then the root space $\mathfrak{g}_{\bar{\alpha}}$ is at least two dimensional, since α and $-\theta\alpha$ restrict to the same restricted root $\bar{\alpha} = (\alpha - \theta\alpha)/2$.

Let $s \in W_{\mathfrak{a}}$ be a reflection, i.e., $s = s_{\bar{\alpha}}$ for some $\bar{\alpha} \in \Sigma$. We define

$$\delta(s) := \frac{1}{2} \sum_{\substack{\bar{\alpha}\in\Sigma \\ s_{\bar{\alpha}}=s}} \dim \mathfrak{g}_{\bar{\alpha}}. \tag{2.2}$$

If $\delta(s) = 1$, we write α_s for the unique positive real root in Φ^{R} such that $s_{\bar{\alpha}_s} = s$.

2.3. **The equivariant fundamental group.** We say that an element $x \in \mathfrak{g}_1$ is regular in $\mathfrak{g}_1$ if $\dim Z_K(x) \leq \dim Z_K(y)$ for all $y \in \mathfrak{g}_1$. Let $\mathfrak{g}_1^{rs}$ denote the set of regular semisimple elements of $\mathfrak{g}_1$ and set $\mathfrak{a}^{rs} = \mathfrak{a} \cap \mathfrak{g}_1^{rs}$. We write $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ for the equivariant fundamental group with base point $a \in \mathfrak{a}^{rs}$.

Consider the adjoint quotient map

$$f : \mathfrak{g}_1 \to \mathfrak{g}_1/\!/K \cong \mathfrak{a}/W_{\mathfrak{a}}.$$

As in [GVX1, §2.3] it gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I & \longrightarrow & \widetilde{B}_{W_{\mathfrak{a}}} := \pi_1^K(\mathfrak{g}_1^{rs}, a) & \xrightarrow{\tilde{q}} & B_{W_{\mathfrak{a}}} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} & & \\ 1 & \longrightarrow & I & \longrightarrow & \widetilde{W}_{\mathfrak{a}} := N_K(\mathfrak{a})/Z_K(\mathfrak{a})^0 & \xrightarrow{q} & W_{\mathfrak{a}} & \longrightarrow & 1\,, \end{array} \tag{2.3}$$

where

$$I = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0$$

is a 2-group and $B_{W_{\mathfrak{a}}} = \pi_1(\mathfrak{a}^{rs}/W_{\mathfrak{a}}, \bar{a})$ is the braid group associated to $W_{\mathfrak{a}}$, $\bar{a} = f(a)$.

In [GVX1, §2.3], we have constructed an explicit splitting of the top exact sequence and thus we write

$$\widetilde{B}_{W_{\mathfrak{a}}} \simeq I \rtimes B_{W_{\mathfrak{a}}}, \tag{2.4}$$

and we note that the braid group $B_{W_{\mathfrak{a}}}$ acts on I through the quotient $p : B_{W_{\mathfrak{a}}} \to W_{\mathfrak{a}}$. Such a splitting is not unique. The example in §3.7 illustrates how the splitting affects the labelling of character sheaves.

2.4. **The classical symmetric pairs.** In this section we summarize the detailed structure of classical symmetric pairs we will use to classify character sheaves. The classical symmetric pairs are (see for example [He])

(AI) $(GL_n, O_n))$ (AII) (GL_{2n}, Sp_{2n}) (AIII) $(GL_n, GL_p \times GL_q)$ $p + q = n$

(BDI) $(SO_N, S(O_p \times O_q))$ $p + q = N$ (DIII) $(SO_N, GL_{N/2})$

(CI) (Sp_{2n}, GL_n) (CII) $(Sp_{2n}, Sp_{2p} \times Sp_{2q})$ $p + q = n$.

The results for type (AI) can be derived from those in [CVX] where we considered the case $(G, K) = (SL_n, SO_n)$. We will include these results as remarks throughout the paper. Type (AII) was studied in [G3], [H], and [L2]. Thus we will focus on the remaining cases. We will make use of the following concrete descriptions of (G, K).

In type A, let $V = V^+ \oplus V^-$ be a $\mathbb{C}$-vector space of dimension n and let $G = GL_V$. In type BD (resp. C), let $V = V^+ \oplus V^-$ be a $\mathbb{C}$-vector space of dimension N (resp. $2n$) equipped with a non-degenerate symmetric bilinear (resp. symplectic) form $(\ ,\ )$ and let $G = SO_{V,(\ ,\ )}$ (resp. $G = Sp_{V,(\ ,\ )}$).

(AIII) $\dim V^+ = p$, $\dim V^- = q$, $K = GL_{V^+} \times GL_{V^-}$.

(BDI) (resp. (CII)) $(V^+, V^-) = 0$, $\dim V^+ = p$ (resp. $2p$), $\dim V^- = q$ (resp. $2q$), $K = S(O_{V^+} \times O_{V^-})$ (resp. $K = Sp_{V^+} \times Sp_{V^-}$).

(DIII) (CI) $(\ ,\ )|_{V^+} = (\ ,\ )|_{V^-} = 0$, $\dim V^+ = \dim V^- = n$, $K = G \cap (GL_{V^+} \times GL_{V^-}) \cong GL_n$.

We write $r = \dim \mathfrak{a}$. Then $r = \min(p, q)$ in type AIII, BDI, CII, $r = n$ in type CI, and $r = [n/2]$ in type DIII. Note that the split pairs are types AI, CI, and BDI with $r = [N/2]$.

We describe the various root systems and the little Weyl groups, for the convenience of the reader. As usual, we write $\Phi = \{\pm(\epsilon_i - \epsilon_j) \,|\, 1 \leq i < j \leq n\}$ in type A_{n-1}, $\Phi = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \leq i < j \leq n;\ \pm\epsilon_i$ (resp. $\pm 2\epsilon_i$), $1 \leq i \leq n\}$ in type B_n (resp. C_n), and $\Phi = \{\pm(\epsilon_i \pm \epsilon_j) \,|\, 1 \leq i < j \leq n\}$ in type D_n. Let

$$W_n \text{ denote the Weyl group of type } B_n \text{ (or } C_n)$$
$$\text{and } W'_n \text{ denote the Weyl group of type } D_n.$$

We use the convention that $W_0 = W'_0 = \{1\}$. In what follows we indicate the dimension of the restricted root spaces $\mathfrak{g}_{\bar{\alpha}}$ by the number within the parentheses after each restricted root $\bar{\alpha}$ in Σ.

(AIII) $\Phi^{\mathrm{R}} = \{\pm(\epsilon_{2i-1} - \epsilon_{2i}),\ 1 \leq i \leq r\}$,
$\Phi^{\mathrm{C}} = \{\pm(\epsilon_{2i-1} - \epsilon_{2j-1}),\ \pm(\epsilon_{2i} - \epsilon_{2j}),\ 1 \leq i < j \leq r\}$,
$\Sigma = \{\pm(\epsilon_i \pm \epsilon_j)\,(2),\ 1 \leq i < j \leq r,\ \pm 2\epsilon_i\,(1),\ \pm\epsilon_i\,(2n-4r),\ 1 \leq i \leq r\}$

(BI) $\Phi^{\mathrm{R}} = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \leq i < j \leq r,\ \pm\epsilon_i,\ 1 \leq i \leq r\},\ \Phi^{\mathrm{C}} = \emptyset$,
$\Sigma = \{\pm(\epsilon_i \pm \epsilon_j)\,(1),\ 1 \leq i < j \leq r,\ \pm\epsilon_i\,(|p-q|),\ 1 \leq i \leq r\}$

(DI) $\Phi^{\mathrm{R}} = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \leq i < j \leq r\}$,
$\Phi^{\mathrm{C}} = \{\pm(\epsilon_r \pm \epsilon_n)\}\ (r < n),\ \Phi^{\mathrm{C}} = \emptyset\ (r = n)$,
$\Sigma = \{\pm(\epsilon_i \pm \epsilon_j)\,(1), 1 \leq i < j \leq r,\ \pm\epsilon_i\,(|p-q|), 1 \leq i \leq r\}$

(CI) $\Phi^{\mathrm{R}} = \Sigma = \{\pm(\epsilon_i \pm \epsilon_j),\ 1 \leq i < j \leq r;\ \pm 2\epsilon_i,\ 1 \leq i \leq r\}$,

(CII) $\Phi^{\mathrm{R}} = \{\pm(\epsilon_{2i-1} + \epsilon_{2i}),\ 1 \leq i \leq r\}$,
$\Phi^{\mathrm{C}} = \{\pm(\epsilon_{2i-1} - \epsilon_{2j-1}),\ \pm(\epsilon_{2i} - \epsilon_{2j}),\ 1 \leq i < j \leq r\}$,
$\Sigma = \{\pm(\epsilon_i \pm \epsilon_j)\,(4),\ 1 \leq i < j \leq r,\ \pm 2\epsilon_i\,(3),\ \pm\epsilon_i\,(4n-8r),\ 1 \leq i \leq r\}$.

(DIII) $\Phi^{\mathrm{R}} = \{\pm(\epsilon_{2i-1} - \epsilon_{2i}),\ 1 \leq i \leq r\}$,
$\Phi^{\mathrm{C}} = \{\pm(\epsilon_{2i-1} - \epsilon_{2j-1}),\ \pm(\epsilon_{2i} - \epsilon_{2j}),\ 1 \leq i < j \leq r\}$,
$\Sigma = \{\pm(\epsilon_i \pm \epsilon_j)\,(4),\ 1 \leq i < j \leq r,\ \pm 2\epsilon_i\,(1),\ \pm\epsilon_i\,(4)\ (n \text{ odd}),\ 1 \leq i \leq r\}$.

We have $W_{\mathfrak{a}} \cong W_r$ except in type DI when $r = n$. In the latter case $W_{\mathfrak{a}} \cong W'_r$.

2.5. **Nilpotent orbits and component groups of the centralizers.** In this section we recall the classification of nilpotent K-orbits on $\mathcal{N}_1$ and the description of the components groups $A_K(x) = Z_K(x)/Z_K(x)^0$, $x \in \mathcal{N}_1$. Using the Sekiguchi-Kostant correspondence, this is reduced to the same question for real nilpotent orbits, see, for example, [CM] and [SS]. For completeness and to fix notations, we recall the results here.

A signed Young diagram is a Young diagram with each box labeled $+$ or $-$ so that signs alternate across rows. Two signed Young diagrams are regarded equivalent if and only if one can be obtained from the other by interchanging rows of equal length. A signed Young diagram is said to have signature (p, q) if there are p boxes labeled $+$ and q boxes labeled $-$.

From now on, we write a signed Young diagram as

$$\lambda = (\lambda_1)_+^{p_1}(\lambda_1)_-^{q_1}(\lambda_2)_+^{p_2}(\lambda_2)_-^{q_2}\cdots(\lambda_s)_+^{p_s}(\lambda_s)_-^{q_s}, \tag{2.5a}$$

where $\lambda = (\lambda_1)^{p_1+q_1}(\lambda_2)^{p_2+q_2}\cdots(\lambda_s)^{p_s+q_s}$ is the corresponding partition, $\lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$, for $i = 1, \ldots, s$, $p_i + q_i > 0$ is the multiplicity of λ_i in λ, and $p_i \geq 0$ (resp. $q_i \geq 0$) is the number of rows of length λ_i that begins with sign $+$

(resp. $-$). For later use, it is more convenient to use numbers rather than signs. Thus we will sometimes replace the subscript $+$ by 0 and $-$ by 1 and write the signed Young diagram in (2.5a) as

$$\lambda=(\lambda_1)_0^{p_1}(\lambda_1)_1^{q_1}(\lambda_2)_0^{p_2}(\lambda_2)_1^{q_2}\cdots(\lambda_s)_0^{p_s}(\lambda_s)_1^{q_s}. \tag{2.5b}$$

Given a signed Young diagram λ, we write $\mathcal{O}_\lambda$ for the corresponding nilpotent orbit in $\mathcal{N}_1$.

(AIII) The orbits are parametrized by signed Young diagrams with signature (p,q). We have $A_K(x)=1$ for all $x\in\mathcal{N}_1$.

(BDI) The orbits are parametrized by signed Young diagrams of signature (p,q) such that, in terms of (2.5a), $p_i=q_i$ when λ_i is even, except that each signed Young diagram with all λ_i even corresponds to two orbits. The latter case can only happen when $(G,K)=(SO_{4m},S(O_{2m}\times O_{2m}))$; we write $\mathcal{O}_\lambda^{\mathrm{I}}$ and $\mathcal{O}_\lambda^{\mathrm{II}}$ for those two orbits. We have

$$\begin{gathered}A_K(x)=(\mathbb{Z}/2\mathbb{Z})^{r_\lambda},\\ r_\lambda=|\{i\in[1,s]\,|\,\lambda_i\text{ odd and }p_i>0\}|+|\{i\in[1,s]\,|\,\lambda_i\text{ odd and }q_i>0\}|-1,\\ \text{if at least one part of }\lambda\text{ is odd};\quad r_\lambda=0\text{ otherwise.}\end{gathered} \tag{2.6}$$

(DIII) The orbits are parametrized by signed Young diagrams of signature (n,n) such that, in terms of (2.5a), $p_i=q_i$ if λ_i is odd, and both p_i and q_i are even if λ_i is even. We have that $A_K(x)=1,$ for all $x\in\mathcal{N}_1$.

(CI) The orbits are parametrized by signed Young diagrams of signature (n,n) such that, in terms of (2.5a), $p_i=q_i$ if λ_i is odd. We have

$$\begin{gathered}A_K(x)=(\mathbb{Z}/2\mathbb{Z})^{r_\lambda},\\ r_\lambda=|\{i\in[1,s]\,|\,\lambda_i\text{ even and }p_i>0\}|+|\{i\in[1,s]\,|\,\lambda_i\text{ even and }q_i>0\}|.\end{gathered} \tag{2.7}$$

(CII) The orbits are parametrized by signed Young diagrams of signature $(2p,2q)$ such that, in terms of (2.5a), $p_i=q_i$ if λ_i is even, and both p_i and q_i are even if λ_i is odd. We have that $A_K(x)=1,$ for all $x\in\mathcal{N}_1$.

2.6. **Weyl groups and Hecke algebras.** In our setting Hecke algebras with (unequal) parameters ±1 arise. In this section we recall some results about their simple modules and the generating functions of the numbers of simple modules.

Let $\mathcal{H}_{W_n,c_0,c_1}$ denote the Hecke algebra generated by T_{s_i}, $i\in[1,n]$, with relations

$$\begin{gathered}T_{s_i}T_{s_{i+1}}T_{s_i}=T_{s_{i+1}}T_{s_i}T_{s_{i+1}},\ i\in[1,n-2],\\ T_{s_{n-1}}T_{s_n}T_{s_{n-1}}T_{s_n}=T_{s_n}T_{s_{n-1}}T_{s_n}T_{s_{n-1}},\ T_{s_i}T_{s_j}=T_{s_j}T_{s_i},\ |i-j|>1;\\ (T_{s_i}-c_0)(T_{s_i}+1)=0,\ i\in[1,n-1],\ (T_{s_n}-c_1)(T_{s_n}+1)=0.\end{gathered}$$

When $c_0=1$ and $c_1=-1$, the group algebra $\mathbb{C}[S_n]$ is naturally a subalgebra of $\mathcal{H}_{W_n,1,-1}$. By [DJ, §5.4], there is a natural bijection between the set of simple modules of $\mathbb{C}[S_n]$ and the set of simple modules of $\mathcal{H}_{W_n,1,-1}$ as follows: each simple module of $\mathbb{C}[S_n]$ naturally extends to a simple module of $\mathcal{H}_{W_n,1,-1}$ by letting T_{s_n} act by -1. Thus, the simple modules of $\mathcal{H}_{W_n,1,-1}$ are parametrized by $\mathcal{P}(n)$, the set of partitions of n. We write

$$\operatorname{Irr}\mathcal{H}_{W_n,1,-1}=\{L_\beta\,|\,\beta\in\mathcal{P}(n)\}. \tag{2.8}$$

When $c_0=c_1=1$, the set of simple $\mathcal{H}_{W_n,1,1}=\mathbb{C}[W_n]$-modules is parametrized by $\mathcal{P}_2(n)$, the set of bi-partitions of n, i.e., pairs of partitions (μ,ν) with $|\mu|+|\nu|=n$.

We write

$$\operatorname{Irr}\mathcal{H}_{W_n,1,1}=\{L_\rho\,|\,\rho\in\mathcal{P}_2(n)\}.$$

Thus, the numbers of simple modules of $\mathcal{H}_{W_n,1,-1}$ and $\mathcal{H}_{W_n,1,1}$ are given by the following generating functions

$$\sum_{n\geq 0}|\operatorname{Irr}\mathcal{H}_{W_n,1,-1}|x^n=\prod_{s\geq 1}\frac{1}{(1-x^s)},\quad \sum_{n\geq 0}|\operatorname{Irr}\mathcal{H}_{W_n,1,1}|x^n=\prod_{s\geq 1}\frac{1}{(1-x^s)^2}.$$

For $\mathcal{H}_{W_k,-1,-1}$ and $\mathcal{H}_{W_k,-1,1}$ the combinatorial description of the simple modules is more involved. However, by [AM], we have the following generating functions

$$\begin{aligned}\sum_{k}|\operatorname{Irr}\mathcal{H}_{W_k,-1,-1}|x^k&=\prod_{s\geq 1}(1+x^{2s})(1+x^s),\\ \sum_{k}|\operatorname{Irr}\mathcal{H}_{W_k,-1,1}|x^k&=\prod_{s\geq 1}(1+x^{2s-1})(1+x^s).\end{aligned}\tag{2.9}$$

Let $\mathcal{H}_{W_n',-1}$ denote the Hecke algebra of W_n' with parameter -1. According to [Ge],

$$|\operatorname{Irr}\mathcal{H}_{W_n',-1}|=\frac{1}{2}|\operatorname{Irr}\mathcal{H}_{W_n,-1,1}|,\ n\geq 1.\tag{2.10}$$

3. General strategy

In this section we describe our general strategy to determine character sheaves for symmetric pairs. We will carry out this strategy for the classical symmetric pairs in the subsequent sections. We refer the readers to this section for notational conventions.

3.1. Dual strata for symmetric spaces and supports of character sheaves.

In this section we extend the discussion in Appendix A to symmetric pairs.

For each nilpotent K-orbit $\mathcal{O}$ in $\mathcal{N}_1$ we consider its conormal bundle

$$\Lambda_{\mathcal{O}}=T^*_{\mathcal{O}}\mathfrak{g}_1=\{(x,y)\in\mathfrak{g}_1\times\mathfrak{g}_1\mid x\in\mathcal{O}\ \ [x,y]=0\}\,.$$

Consider the projection $\widetilde{\mathcal{O}}$ of $\Lambda_{\mathcal{O}}$ to the second coordinate:

$$\widetilde{\mathcal{O}}=\{y\in\mathfrak{g}_1\mid \text{there exist an }\ x\in\mathcal{O}\ \text{ with }\ [x,y]=0\}\,.$$

We construct an open (dense) subset $\check{\mathcal{O}}$ of $\widetilde{\mathcal{O}}$ such that the projection $\Lambda_{\mathcal{O}}\to\widetilde{\mathcal{O}}$ has constant maximum rank above $\check{\mathcal{O}}$. Since the Fourier transform preserves the singular support, the $\check{\mathcal{O}}$ are subvarieties of $\mathfrak{g}_1$ with the following property:

For any $\mathcal{F}\in\mathrm{P}_K(\mathcal{N}_1)$ the Fourier transform $\mathfrak{F}(\mathcal{F})$ is smooth along all the $\check{\mathcal{O}}$.

Moreover, for each $\mathrm{IC}(\mathcal{O},\mathcal{E})\in\mathrm{P}_K(\mathcal{N}_1)$,

$$\operatorname{Supp}\mathfrak{F}(\mathrm{IC}(\mathcal{O},\mathcal{E}))=\overline{\check{\mathcal{O}'}},\ \text{for some }\mathcal{O}'\subset\bar{\mathcal{O}}.\tag{3.1}$$

As in Appendix A we consider the adjoint quotient map $\mathfrak{g}_1\xrightarrow{f}\mathfrak{g}_1/\!/K\cong\mathfrak{a}/W_{\mathfrak{a}}$. We have:

$$\begin{array}{ccccc}\widetilde{\mathcal{O}} & \xrightarrow{\tilde f} & f(\widetilde{\mathcal{O}}) & \xrightarrow{\sim} & \mathfrak{a}^\phi/W_{\mathfrak{a}^\phi}\\ \downarrow & & \downarrow & & \downarrow\\ \mathfrak{g}_1 & \xrightarrow{f} & \mathfrak{g}_1/\!/K & \xrightarrow{\sim} & \mathfrak{a}/W_{\mathfrak{a}}\end{array}$$

where the vertical arrows are inclusions and the upper righthand corner is constructed as follows. Consider an element $e \in \mathcal{O}$ and a normal $\mathfrak{sl}_2$-triple $\phi = (e, f, h)$ such that $f \in \mathfrak{g}_1$ and $h \in \mathfrak{g}_0$. Recall that we have

$$\mathfrak{g}^e = \mathfrak{g}^\phi \oplus \mathfrak{u}^e,$$

where $\mathfrak{g}^\phi = \mathfrak{g}^e \cap \mathfrak{g}^h$, $\mathfrak{u} = \oplus_{i \geq 1} \mathfrak{g}(i)$, see (A.1). Thus

$$\mathfrak{g}_1^e = \mathfrak{g}_1^\phi \oplus (\mathfrak{u}^e \cap \mathfrak{g}_1).$$

Let $\mathfrak{a}^\phi \subset \mathfrak{g}_1^\phi$ be a Cartan subspace such that every semisimple element in $\mathfrak{g}_1^\phi$ is K^ϕ-conjugate to some element in $\mathfrak{a}^\phi$, where $K^\phi = G^\phi \cap K = Z_K(e) \cap Z_K(f) \cap Z_K(h)$. We choose $\mathfrak{a}^\phi$ such that it lies in $\mathfrak{a}$. Let $W_{\mathfrak{a}^\phi}$ be the little Weyl group $N_{K^\phi}(\mathfrak{a}^\phi)/Z_{K^\phi}(\mathfrak{a}^\phi)$. The same argument as in Appendix A shows that $f(\widetilde{\mathcal{O}}) \cong \mathfrak{a}^\phi / W_{\mathfrak{a}^\phi}$ and we write

$$\tilde{f} : \tilde{\mathcal{O}} \to \mathfrak{a}^\phi / W_{\mathfrak{a}^\phi}.$$

Note also that, analogously to (A.2) we have

$$N_K(\mathfrak{a}^\phi)/Z_K(\mathfrak{a}^\phi) = N_{K^\phi}(\mathfrak{a}^\phi)/Z_{K^\phi}(\mathfrak{a}^\phi) = W_{\mathfrak{a}^\phi}\,. \tag{3.2}$$

For an element $x \in \mathfrak{g}_1$, we write $x = x_s + x_n$ for the Jordan decomposition of x into semisimple part x_s and nilpotent part x_n. Then x_s, $x_n \in \mathfrak{g}_1$ and $[x_s, x_n] = 0$. Let $(\mathfrak{a}^\phi)^{rs}$ denote the regular semisimple locus of $\mathfrak{a}^\phi$ defined with respect to the symmetric pair (G^ϕ, K^ϕ). Proceeding again as in Appendix A we have the following:

Lemma 3.1. *If $x = x_s + x_n \in \mathfrak{g}_1^e$ and $x_s \in (\mathfrak{a}^\phi)^{rs}$, then $x_n \in \bar{\mathcal{O}}$.*

We now define

$$\check{\mathcal{O}} = \{ y \in \widetilde{\mathcal{O}} \mid y = y_s + y_n,\ \ \tilde{f}(y) \in (\mathfrak{a}^\phi)^{rs}/W_{\mathfrak{a}^\phi},\ \ y_n \in \mathcal{O} \}\,. \tag{3.3}$$

This establishes a correspondence:

$$\mathcal{O} \leftrightarrow \check{\mathcal{O}}.$$

Repeating the arguments in Appendix A we obtain the following description of the equivariant fundamental groups of $\check{\mathcal{O}}$:

$$\begin{tikzcd} 1 \arrow[r] & Z_{K^\phi}(\mathfrak{a}^\phi)/Z_{K^\phi}(\mathfrak{a}^\phi)^0 \arrow[r] \arrow[d, equal] & \pi_1^{K^\phi}((\mathfrak{g}_1^\phi)^{rs}) \arrow[r, "\tilde{q}"] \arrow[d, equal] & B_{W_{\mathfrak{a}^\phi}} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & Z_K(\mathfrak{a}^\phi + e)/Z_K(\mathfrak{a}^\phi + e)^0 \arrow[r] & \pi_1^K(\check{\mathcal{O}}) \arrow[r, "\tilde{q}"] & B_{W_{\mathfrak{a}^\phi}} \arrow[r] & 1\,, \end{tikzcd} \tag{3.4}$$

where $B_{W_{\mathfrak{a}^\phi}} = \pi_1((\mathfrak{a}^\phi)^{rs}/W_{\mathfrak{a}^\phi})$ is the braid group associated to $W_{\mathfrak{a}^\phi}$. Recall that, as in Appendix A, we use the terminology "braid group" even when $W_{\mathfrak{a}^\phi}$ is not a Coxeter group.

In view of (3.1), each character sheaf is supported on some $\overline{\check{\mathcal{O}}}$. For each symmetric pair considered here, we will describe explicitly the set of nilpotent orbits $\mathcal{O}$ for which the corresponding $\overline{\check{\mathcal{O}}}$ supports a character sheaf. We also write down the representations of $\pi_1^K(\check{\mathcal{O}}) = \pi_1^{K^\phi}((\mathfrak{g}_1^\phi)^{rs})$ whose IC-sheaves are character sheaves. When $\check{\mathcal{O}}$ supports a character sheaf, $W_{\mathfrak{a}^\phi}$ turns out to be a Coxeter group and the rows in (3.4) can be split as in (2.3) following [GVX1, Section 4.4]. We will later construct such explicit splittings.

3.2. **Full support character sheaves.** In this section we recall the main construction from [GVX1] and explain how it will be used to construct full support character sheaves. All such character sheaves are of the form $\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{L})$, where $\mathcal{L}$ is an irreducible K-equivariant local system on $\mathfrak{g}_1^{rs}$.

Recall the notation from §2.3. Let $X_{\bar{a}}=f^{-1}(\bar{a})$, the fiber of the adjoint quotient map $f:\mathfrak{g}_1\to\mathfrak{a}/W_{\mathfrak{a}}$ at $\bar{a}\in\mathfrak{a}^{rs}/W_{\mathfrak{a}}$. Let $\hat{I}$ denote the set of irreducible characters of the 2-group I. Consider a character $\chi\in\hat{I}$ and note that the equivariant fundamental group of $X_{\bar{a}}$ is given by

$$\pi_1^K(X_{\bar{a}},a)\cong I=Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0\,,$$

where $f(a)=\bar{a}$. Therefore, the character χ gives rise to a rank one K-equivariant local system $\mathcal{L}_\chi$ on $X_{\bar{a}}$. We base change f to the family

$$f_{\bar{a}}:\mathcal{Z}_{\bar{a}}=\{(x,c)\in\mathfrak{g}_1\times\mathbb{C}\,|\,f(x)=c\,\bar{a}\}\to\mathbb{C}$$

where the $\mathbb{C}$-action on $\mathfrak{a}/W_{\mathfrak{a}}$ is induced by the action on $\mathfrak{a}$ so that $f(ca)=cf(a)$. By construction this family is K-equivariant. We define the nearby cycle sheaf associated to $\chi\in\hat{I}$ as

$$P_\chi=\psi_{f_{\bar{a}}}\mathcal{L}_\chi[-]\in\mathrm{Perv}_K(\mathcal{N}_1). \tag{3.5}$$

As the group K is not necessarily connected, a character of the component group $K/K^0=I/I^0$ enters the description of the Fourier transform $\mathfrak{F}P_\chi$, where we have written

$$I^0=Z_{K^0}(\mathfrak{a})/Z_{K^0}(\mathfrak{a})^0,$$

see [GVX1, §3.4]. It was denoted by τ there but we denote it by ι in this paper, i.e.,

$$\iota:I\to I/I^0=K/K^0\to\{\pm1\}$$

is the character determined by the action of $M_{\mathbb{R}}/(M_{\mathbb{R}}\cap K_{\mathbb{R}}^0)\cong K/K^0$ on $\wedge^{\mathrm{top}}(\mathfrak{k}_{\mathbb{R}}/\mathfrak{m}_{\mathbb{R}})$, where $K_{\mathbb{R}}$ is the compact form of K, $M_{\mathbb{R}}=Z_{K_{\mathbb{R}}}(\mathfrak{a})$, $\mathfrak{k}_{\mathbb{R}}=\mathrm{Lie}\,K_{\mathbb{R}}$ and $\mathfrak{m}_{\mathbb{R}}=\mathrm{Lie}\,M_{\mathbb{R}}$. In particular, if $K=K^0$, then ι is the trivial character. Recall that the $W_{\mathfrak{a}}$ action on $\hat{I}$ leaves ι fixed. Let $\mathbb{C}_\iota$ denote the rank one K-equivariant local system on $\mathfrak{g}_1^{rs}$ given by the representation of $\pi_1^K(\mathfrak{g}_1^{rs})=I\rtimes B_{W_{\mathfrak{a}}}$ where I acts via the character ι and $B_{W_{\mathfrak{a}}}$ acts trivially. We have ([GVX1, Theorem 3.6])

$$\mathfrak{F}P_\chi=\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{M}_\chi\otimes\mathbb{C}_\iota) \tag{3.6}$$

where $\mathcal{M}_\chi$ is the K-equivariant local system on $\mathfrak{g}_1^{rs}$ given by the following $\widetilde{B}_{W_{\mathfrak{a}}}=\pi_1^K(\mathfrak{g}_1^{rs})$ representation

$$M_\chi\ =\ \mathbb{C}[\widetilde{B}_{W_{\mathfrak{a}}}]\otimes_{\mathbb{C}[\widetilde{B}^{\chi,0}_{W_{\mathfrak{a}}}]}(\mathbb{C}_\chi\otimes\mathcal{H}_{W^0_{\mathfrak{a},\chi}})\,. \tag{3.7}$$

To explain the notations in the above formula, recall the map $p:B_{W_{\mathfrak{a}}}\to W_{\mathfrak{a}}$ in (2.3). Let

$$W_{\mathfrak{a},\chi}=\mathrm{Stab}_{W_{\mathfrak{a}}}(\chi)\subset W_{\mathfrak{a}},\quad B^\chi_{W_{\mathfrak{a}}}=p^{-1}(W_{\mathfrak{a},\chi})\subset B_{W_{\mathfrak{a}}}.$$

Recall the Coxeter group $W^0_{\mathfrak{a},\chi}\subset W_{\mathfrak{a},\chi}$ defined by (see [GVX1, §3.3])

$$\begin{aligned}W^0_{\mathfrak{a},\chi}\ =\ &\text{the subgroup of } W_{\mathfrak{a}} \text{ generated by reflections } s\in W_{\mathfrak{a}} \text{ such that,}\\ &\text{either } \delta(s)>1,\ \text{or } \delta(s)=1 \text{ and } \chi(\check{\alpha}_s(-1))=1,\end{aligned}$$

where $\delta(s)$ is defined in (2.2) and $\check{\alpha}_s(-1)$ is viewed as an element in I via the natural projection $Z_K(\mathfrak{a}) \to I$. We show in §3.4 that the quotient $W_{\mathfrak{a},\chi}/W^0_{\mathfrak{a},\chi}$ is a 2-group, see (3.18). We set

$$B^{\chi,0}_{W_\mathfrak{a}} \;=\; p^{-1}(W^0_{\mathfrak{a},\chi}) \subset B_{W_\mathfrak{a}} \quad \text{and} \quad \widetilde{B}^{\chi,0}_{W_\mathfrak{a}} = I \rtimes B^{\chi,0}_{W_\mathfrak{a}} \subset \widetilde{B}_{W_\mathfrak{a}} \text{ (see (2.4))}.$$

Let $\mathcal{H}_{W^0_{\mathfrak{a},\chi}}$ be the Hecke algebra associated to the Coxeter group $W^0_{\mathfrak{a},\chi}$ defined as follows. We choose a set of simple reflections $s_{\bar{\alpha}_1},\ldots,s_{\bar{\alpha}_\ell}$ that generate $W^0_{\mathfrak{a},\chi}$. We write T_i for the generators of $\mathcal{H}_{W^0_{\mathfrak{a},\chi}}$ associated to $s_{\bar{\alpha}_i}$. Then the Hecke algebra $\mathcal{H}_{W^0_{\mathfrak{a},\chi}}$ is generated by the T_i subject to the braid relations plus the relations

$$(T_i-1)(T_i+q_i) \;=\; 0\,, \quad q_i = (-1)^{\delta(s_{\bar{\alpha}_i})}\,.$$

Now $\mathbb{C}_\chi \otimes \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$ is the $\mathbb{C}[\widetilde{B}^{\chi,0}_{W_\mathfrak{a}}]$-module where I acts via the character χ and $B^{\chi,0}_{W_\mathfrak{a}}$ acts via the composition of maps $\mathbb{C}[B^{\chi,0}_{W_\mathfrak{a}}] \twoheadrightarrow \mathbb{C}[B_{W^0_{\mathfrak{a},\chi}}] \twoheadrightarrow \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$. Here $B_{W^0_{\mathfrak{a},\chi}}$ is the braid group associated to the Coxeter group $W^0_{\mathfrak{a},\chi}$. The first map is induced by the map $B^{\chi,0}_{W_\mathfrak{a}} = \pi_1(\mathfrak{a}^{rs}/W^0_{\mathfrak{a},\chi}) \twoheadrightarrow B_{W^0_{\mathfrak{a},\chi}} = \pi_1(\mathfrak{a}^{rs}_\chi/W^0_{\mathfrak{a},\chi})$, which in turn is induced by the inclusion $\mathfrak{a}^{rs}_\chi \subset \mathfrak{a}^{rs}$, where $\mathfrak{a}^{rs}_\chi$ is the root hyperplane arrangement corresponding to $W^0_{\mathfrak{a},\chi}$. The second map is given by the natural projection map $\mathbb{C}[B_{W^0_{\mathfrak{a},\chi}}] \twoheadrightarrow \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$.

Let us write

$$\begin{aligned}\Theta_{(G,K)} = \{&\text{irreducible representations of } \pi_1^K(\mathfrak{g}_1^{rs}) \text{ that appear} \\ &\text{as composition factors of } M_\chi \otimes \mathbb{C}_\iota,\ \chi \in \hat{I}\}.\end{aligned} \tag{3.8}$$

For each $\rho \in \Theta_{(G,K)}$, let $\mathcal{L}_\rho$ denote the corresponding K-equivariant local system on $\mathfrak{g}_1^{rs}$. It follows from (3.5) and (3.6) that

$$\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{L}_\rho) \in \mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1),\ \rho \in \Theta_{(G,K)}. \tag{3.9}$$

We will show that for the symmetric pairs considered in this paper,

$$\big\{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{L}_\rho)\,|\,\rho \in \Theta_{(G,K)}\big\} = \mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)\,. \tag{3.10}$$

To determine the set $\Theta_{(G,K)}$, it suffices to determine the composition factors of M_χ. Let us note that

$$M_\chi \;=\; \mathbb{C}[\widetilde{B}_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[\widetilde{B}^\chi_{W_\mathfrak{a}}]} \otimes (\mathbb{C}_\chi \otimes (\mathbb{C}[B^\chi_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[B^{\chi,0}_{W_\mathfrak{a}}]} \mathcal{H}_{W^0_{\mathfrak{a},\chi}}))\,.$$

Making use of the semidirect product decomposition (2.4) we conclude that the irreducible representations of $\widetilde{B}_{W_\mathfrak{a}}$ appearing as composition factors of M_χ are of the form $\mathbb{C}[\widetilde{B}_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[\widetilde{B}^\chi_{W_\mathfrak{a}}]} \otimes (\mathbb{C}_\chi \otimes \rho)$, where ρ is an irreducible representation of $B^\chi_{W_\mathfrak{a}}$ which appears as a composition factor in $\mathbb{C}[B^\chi_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[B^{\chi,0}_{W_\mathfrak{a}}]} \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$. Since $B^\chi_{W_\mathfrak{a}}/B^{\chi,0}_{W_\mathfrak{a}}$ is a 2-group, it suffices to study the decomposition $\mathbb{C}[B^\chi_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[B^{\chi,0}_{W_\mathfrak{a}}]} \tau$ using Clifford theory, where $\tau \in \operatorname{Irr} \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$. In particular, we see that all irreducible representations of $\widetilde{B}_{W_\mathfrak{a}}$ which appear as composition factors of $M_\chi \otimes \mathbb{C}_\iota$ can be obtained as quotients of $M_\chi \otimes \mathbb{C}_\iota$.

For $\tau \in \operatorname{Irr} \mathcal{H}_{W^0_{\mathfrak{a},\chi}}$, we write

$$V_{\tau,\chi} = \Big(\mathbb{C}[\widetilde{B}_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[\widetilde{B}^{\chi,0}_{W_\mathfrak{a}}]} \otimes (\mathbb{C}_\chi \otimes \tau)\Big) \otimes \mathbb{C}_\iota. \tag{3.11}$$

When $W^0_{\mathfrak{a},\chi} = W_{\mathfrak{a},\chi}$, $V_{\tau,\chi}$ is an irreducible representation of $\widetilde{B}_{W_{\mathfrak{a}}}$. When $W^0_{\mathfrak{a},\chi} \neq W_{\mathfrak{a},\chi}$, we write

(3.12) $\quad V^{\delta}_{\tau,\chi}$ for the non-isomorphic irreducible summands of $V_{\tau,\chi}$

as representations of $\widetilde{B}_{W_{\mathfrak{a}}}$.

3.3. **The nearby cycle construction in the split case and endoscopy.** The groups $W_{\mathfrak{a},\chi}$ and $W^0_{\mathfrak{a},\chi}$ are not in general Weyl groups of subgroups of G. We show in this section that they are Weyl groups of the dual group $\check{G}$ in the case of split symmetric pairs. We explain at the end of the section how the case of non-trivial characters χ can be viewed as endoscopy.

In this section let (G, K) be a split symmetric pair. Then $A = T$, $W_{\mathfrak{a}} = W = N_G(T)/Z_G(T)$, and $I = Z_K(\mathfrak{a})$. In particular, $\Phi^{\mathrm{R}} = \Phi$ and $\delta(s) = 1$ for all reflections s. Thus by definition

$$W^0_{\mathfrak{a},\chi} = \langle s_{\bar{\alpha}}, \alpha \in \Phi,\, \chi(\check{\alpha}(-1)) = 1\rangle.$$

We first discuss the interpretation of $W^0_{\mathfrak{a},\chi} \subset W_{\mathfrak{a},\chi} = \mathrm{Stab}_{W_{\mathfrak{a}}}(\chi)$ as a Weyl group on the dual side.

Recall that we have a surjection $A[2] \to I$ of order two elements of A to I ([GVX1, §2.3 eq(2.13)]). Thus, we can regard the character $\chi \in \hat{I}$ as a character of $A[2]$.

Let $\check{G}$ be the dual group and $\check{A} \subset \check{G}$ the dual torus. It is standard, and not difficult to see, that we can identify $\mathrm{Hom}(A[2], \mathbb{G}_m) \cong \check{A}[2]$, the order 2 elements in $\check{A}$. Thus, χ regarded as an element in $\check{A}[2]$, gives rise to an involution on $\check{G}$ which we denote by $\check{\theta}_\chi$, i.e. $\check{\theta}_\chi = \mathrm{Int}\,\chi$. Let

$$\check{K}(\chi) = \check{G}^{\check{\theta}_\chi}, \quad \check{K}(\chi)^0 = (\check{G}^{\check{\theta}_\chi})^0.$$

We show that

$$W_{\mathfrak{a},\chi} \;=\; W(\check{K}(\chi), \check{A}) \tag{3.13a}$$

$$W^0_{\mathfrak{a},\chi} \;=\; W(\check{K}(\chi)^0, \check{A}). \tag{3.13b}$$

The root system of $(\check{G}, \check{A})$ can be identified with $\check{\Phi} = \{\check{\alpha} \,|\, \alpha \in \Phi\}$. With respect to the $\check{\theta}_\chi$-stable torus $\check{A}$, all roots in $\check{\Phi}$ are imaginary since $\check{\theta}_\chi|_{\check{A}} = 1$. They split into compact imaginary and non-compact imaginary according to the value of $\chi(\check{\alpha}(-1))$. So, we set

$$\check{\Phi}_{\mathrm{ci}} = \{\check{\alpha} \mid \alpha \in \Phi,\ \chi(\check{\alpha}(-1)) = 1\} \qquad \check{\Phi}_{\mathrm{nci}} = \{\check{\alpha} \mid \alpha \in \Phi,\ \chi(\check{\alpha}(-1)) = -1\}\,.$$

By construction we have

$$W_{\mathfrak{a},\chi} \;=\; \{w \in W \mid w : \check{\Phi}_{\mathrm{ci}} \to \check{\Phi}_{\mathrm{ci}},\ w : \check{\Phi}_{\mathrm{nci}} \to \check{\Phi}_{\mathrm{nci}}\} \;=\; \{w \in W \mid w : \check{\Phi}_{\mathrm{ci}} \to \check{\Phi}_{\mathrm{ci}}\}\,.$$

Equation (3.13a) can be seen as follows. First,

$$W(\check{K}(\chi), \check{A}) = N_{\check{K}(\chi)}(\check{A})/Z_{\check{K}(\chi)}(\check{A}) = N_{\check{K}(\chi)}(\check{A})/\check{A} \subset N_{\check{G}}(\check{A})/\check{A} = W\,.$$

Let $n_w \in N_{\check{G}}(\check{A})$ denote a representative of $w \in W$. Now, $n_w \in \check{K}(\chi)$ if and only if $\chi n_w \chi^{-1} = n_w$ if and only if $n_w^{-1}\chi n_w = n_w$, i.e., if and only if $w\chi = \chi$. Equation (3.13b) follows from the fact that the roots of $(\check{K}(\chi)^0, \check{A})$ are the $\check{\alpha}$'s such that $\chi(\check{\alpha}(-1)) = 1$.

Note that the Weyl group $W(\check{K}(\chi), \check{A})$ is not necessarily a Coxeter group as $\check{K}(\chi)$ might not be connected. It is connected if $\check{G}$ is simply connected and then G

is adjoint. The quotient $W(\check{K}(\chi),\check{A})/W(\check{K}(\chi)^0,\check{A}) = \check{K}(\chi)/\check{K}(\chi)^0$ is a 2-group, called the R-group.

The above results imply that

$$W_{\mathfrak{a},\chi}/W^0_{\mathfrak{a},\chi} = \check{K}(\chi)/\check{K}(\chi)^0 \text{ is a 2-group.} \tag{3.14}$$

Moreover,

$$\text{if } G \text{ is adjoint, then } W_{\mathfrak{a},\chi} = W^0_{\mathfrak{a},\chi}. \tag{3.15}$$

Cheng-Chiang Tsai pointed out to us that the previous discussion can be viewed from the endoscopic point of view as follows. Note that in the split case for the trivial character $\chi = 1$, in (3.7) the Hecke algebra $\mathcal{H}_{W^0_{\mathfrak{a},\chi}} = \mathcal{H}_{W,-1}$, i.e., the Hecke algebra of W with parameter -1.

For any $\chi \in \hat{I}$ we obtained the symmetric pair $(\check{G},\check{K}(\chi))$. By construction, the maximal torus $\check{A}$ of $\check{G}$ is also a maximal torus of $\check{K}(\chi)$. Let $G(\chi)$ be the dual group of $\check{K}(\chi)^0$ and consider the split symmetric pair of $G(\chi)$. By construction the split maximal torus $A(\chi)$ of $G(\chi)$ is canonically isomorphic to A and $W(G(\chi),A) = W(\check{K}(\chi),\check{A}) = W^0_{\mathfrak{a},\chi}$. We have

$$\mathcal{H}_{W^0_{\mathfrak{a},\chi}} = \mathcal{H}_{W^0_{\mathfrak{a},\chi},-1}\,. \tag{3.16}$$

The Hecke algebra on the right hand side is the one corresponding the trivial character for the split pair of $G(\chi)$. We can thus interpret the the character sheaves associated to non-trivial characters χ as arising from the endoscopic groups $G(\chi)$.

3.4. **The groups $W_{\mathfrak{a},\chi}$ and $W^0_{\mathfrak{a},\chi}$ for a general symmetric pair.** In this section we discuss the groups $W_{\mathfrak{a},\chi}$ and $W^0_{\mathfrak{a},\chi}$ for a general symmetric pair. Recall from [GVX1, Corollary 5.2] that

$$W_{\mathfrak{a},\chi} = W^{\mathrm{R}}_\chi \rtimes (W^{\mathrm{C}})^\theta, \quad W^{\mathrm{R}}_\chi = W^{\mathrm{R}} \cap W_{\mathfrak{a},\chi}.$$

To determine W^{R}_χ, let us consider

$$G_{\mathrm{s}} = Z_G(\mathfrak{c})/C^0.$$

Here $C = T^\theta$ and $\mathfrak{a} = \operatorname{Lie} C$, where $T \subset Z_G(\mathfrak{a})$ is a maximal torus. The involution θ induces an involution θ_{s} on G_{s} and we write $K_{\mathrm{s}} = G_{\mathrm{s}}^{\theta_{\mathrm{s}}}$. Clearly $A = T/C^0$ is a maximal torus of G_{s} and it is θ_{s}-split. Thus the symmetric pair $(G_{\mathrm{s}},K_{\mathrm{s}})$ is a split pair with $W(G_{\mathrm{s}},A) = W^{\mathrm{R}}$ and the root system of (G_{s},A) can be identified with Φ^{R}. Furthermore, we have

$$I \;=\; Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0 \;=\; C/C^0 \;=\; Z_{K_{\mathrm{s}}}(\mathfrak{a})/Z_{K_{\mathrm{s}}}(\mathfrak{a})^0\,.$$

Let $W_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})}} = W^{\mathrm{R}}$ be the little Weyl group of the split symmetric pair $(G_{\mathrm{s}},K_{\mathrm{s}})$. We write $W_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})},\chi}$ for $\operatorname{Stab}_{W_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})}}}(\chi)$ and $W^0_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})},\chi}$ for the corresponding subgroup of $W_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})},\chi}$. Define $W^{\mathrm{R},0}_\chi$ to be the subgroup of W^{R} generated by those reflecitions s_α, $\alpha \in \Phi^{\mathrm{R}}$, such that $\chi(\check{\alpha}(-1)) = 1$. Then we have

$$W^{\mathrm{R}}_\chi = W_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})},\chi}, \qquad W^{\mathrm{R},0}_\chi = W^0_{\mathfrak{a}_{(G_{\mathrm{s}},K_{\mathrm{s}})},\chi}. \tag{3.17}$$

Note that $(W^{\mathrm{C}})^\theta \ltimes (W^R_\chi)^0 \subset W^0_{\mathfrak{a},\chi} \subset (W^{\mathrm{C}})^\theta \ltimes W^{\mathrm{R}}_\chi = W_{\mathfrak{a},\chi}$. Applying (3.14) and (3.15) to the split pair $(G_{\mathrm{s}},K_{\mathrm{s}})$ we see that

$$W_{\mathfrak{a},\chi}/W^0_{\mathfrak{a},\chi} \subset W^{\mathrm{R}}_\chi/(W^{\mathrm{R}}_\chi)^0 \text{ is a 2-group.} \tag{3.18}$$

Moreover,

$$\text{if } G_{\mathrm{s}} \text{ is adjoint, then } W_{\mathfrak{a},\chi} = W^0_{\mathfrak{a},\chi} = (W^{\mathrm{C}})^\theta \ltimes (W^R_\chi)^0. \tag{3.19}$$

3.5. **Nilpotent support character sheaves.** In this section we classify nilpotent support character sheaves. They exist in the cases we consider only when θ is inner and one expects that it also holds for exceptional groups.

Note that if the Fourier transform of $\mathrm{IC}(\mathcal{O},\mathcal{E})$ has nilpotent support then the support of the Fourier transform is also $\bar{\mathcal{O}}$. Moreover $\mathcal{O} = \check{\mathcal{O}}$ and so $\mathcal{O}$ is self dual. This can be seen as follows. Assume that $\mathfrak{F}\,\mathrm{IC}(\mathcal{O},\mathcal{E}) = \mathrm{IC}(\mathcal{O}',\mathcal{E}')$. By (3.1), $\bar{\mathcal{O}}' = \bar{\check{\mathcal{O}}}''$, where $\mathcal{O}'' \subset \bar{\mathcal{O}}$. Now if $\check{\mathcal{O}}''$ is nilpotent, then $\check{\mathcal{O}}'' = \mathcal{O}''$. We conclude that $\mathcal{O}' = \mathcal{O}'' \subset \bar{\mathcal{O}}$. Similarly, we have $\mathcal{O} \subset \bar{\mathcal{O}}'$. Thus $\mathcal{O}' = \mathcal{O}$. In particular, we see that if the Fourier transform of $\mathrm{IC}(\mathcal{O},\mathcal{E})$ has nilpotent support, then $\mathcal{O}$ is distinguished. That is, $\mathcal{O}$ consists of distinguished nilpotent elements, i.e., $e \in \mathcal{N}_1$ such that $\mathfrak{g}_1^e := Z_{\mathfrak{g}_1}(e)$ consists of nilpotent elements.

Here is a general construction of nilpotent support character sheaves. Let X denote the flag variety of G. Consider a closed K-orbit Q on X and its conormal bundle $T^*_Q X$. Let $B \subset G$ be a point on Q, i.e., a θ-stable Borel subgroup. Let $\mathfrak{b} = \operatorname{Lie} B$ and let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$. We write $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i$, $\mathfrak{n}_i = \mathfrak{n} \cap \mathfrak{g}_i$, $i = 0, 1$ and $B_K = B \cap K = B^\theta$. We have a moment map

$$\mu : T^*_Q X = K \times^{B_K} \mathfrak{n}_1 \to \mathcal{N}_1.$$

Since θ is inner, we have that (see, for example, [L2, §3.2])

$$\mathfrak{b}_1 = \mathfrak{n}_1. \tag{3.20}$$

Using the functoriality of Fourier transform and (3.20), we see that

$$\mathfrak{F}(\mu_* \mathbb{C}_{T^*_Q X}[-]) \cong \mu_* \mathbb{C}_{T^*_Q X}[-].$$

It follows that all direct summands of $\mu_* \mathbb{C}_{T^*_Q X}[-]$ are nilpotent support character sheaves (up to shift).

To describe the character sheaves that arise in this way, we first note that the image of $\mu(T^*_Q X)$ is the closure of the unique K-orbit $\mathcal{O}$ in $\mathcal{N}_1$ such that $\mathcal{O} \cap \mathfrak{b}_1 = \mathcal{O} \cap \mathfrak{n}_1$ is dense in $\mathfrak{b}_1 = \mathfrak{n}_1$; we denote this orbit by $\mathcal{O}_B$. We say that the orbit $\mathcal{O}_B$ is the Richardson orbit attached to (the K-orbit of) the θ-stable Borel subgroup B. Let us now write

$$\pi_B : K \times^{B_K} \mathfrak{n}_1 \to \bar{\mathcal{O}}_B \qquad \text{and} \qquad \mathring{\pi}_B : K \times^{B_K} \mathfrak{n}_1^r \to \mathcal{O}_B, \quad \mathfrak{n}_1^r = \mathfrak{n}_1 \cap \mathcal{O}_B$$

for the moment map μ and its restriction to $\mu^{-1}(\mathcal{O}_B)$. Let $\underline{\mathcal{N}_1} = \mathcal{N}_1/K$ denote the set of K-orbits on $\mathcal{N}_1$. We write

$$\underline{\mathcal{N}_1^0} = \{\text{Richardson orbits attached to } \theta\text{-stable Borel subgroups}\} \subset \underline{\mathcal{N}_1}. \tag{3.21}$$

Let $A_K(\mathcal{O}) := A_K(x)$, $x \in \mathcal{O}$. We write $\widehat{A}_K(\mathcal{O})$ for the set of irreducible characters of $A_K(\mathcal{O})$.

For an orbit $\mathcal{O} \in \underline{\mathcal{N}_1^0}$, let $\Pi^B_{\mathcal{O}} \subset \widehat{A}_K(\mathcal{O})$ denote the set of irreducible characters of $A_K(\mathcal{O})$ which appear in the permutation representations of $A_K(\mathcal{O})$ on the set of irreducible components of $\pi_B^{-1}(x)$, where $x \in \mathcal{O}$ and B is a θ-stable Borel subgroup with $\mathcal{O}_B = \mathcal{O}$. Given $\phi \in \widehat{A}_K(\mathcal{O})$, we write $\mathcal{E}_\phi$ for the irreducible K-equivariant local system on $\mathcal{O}$ corresponding to ϕ. The $\mathrm{IC}(\mathcal{O},\mathcal{E}_\phi)$ are nilpotent support character sheaves. We see this as follows. Let d be the fiber dimension of the fibration $\mathring{\pi}_B$. Decomposing $R^{2d}(\mathring{\pi}_B)_*(\mathbb{C})$ into direct summands

$$R^{2d}(\mathring{\pi}_B)_*(\mathbb{C}) \;\cong\; \bigoplus_{\phi\in\Pi^B_{\mathcal{O}_B}} \mathcal{E}_\phi\,.$$

We will show that all nilpotent support character sheaves can be obtained in this way:

$$\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1) = \Big\{\mathrm{IC}(\mathcal{O},\mathcal{E}_\phi)\,|\,\mathcal{O}\in\underline{\mathcal{N}_1^0},\ \phi\in\Pi_{\mathcal{O}}\Big\} \tag{3.22}$$

where $\Pi_{\mathcal{O}}=\cup_{B:\mathcal{O}_B=\mathcal{O}}\Pi^B_{\mathcal{O}}$ and the union runs through all θ-stable Borel subgroups (up to K-conjugacy) with $\mathcal{O}_B=\mathcal{O}$.

It turns out that the nilpotent support character sheaves can be determined as follows. Let $\mathcal{O}\in\underline{\mathcal{N}_1^0}$. Note that two θ-stable Borel subgroups that are not conjugate under K can give rise to the same Richardson orbit. In this case, the sets of irreducible components of $\pi^{-1}_{B_1}(x)$ and $\pi^{-1}_{B_2}(x)$, $x\in\mathcal{O}$ can be different. However, we will show that we can find a θ-stable Borel subgroup B such that $\mathcal{O}_B=\mathcal{O}$ and such that $\Pi_{\mathcal{O}}=\Pi^B_{\mathcal{O}}$. Thus, to determine $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$, for any $\mathcal{O}\in\underline{\mathcal{N}_1^0}$ we only need to consider that specific θ-stable Borel in (3.22).

Remark 3.2. Following Lusztig [L3], we say that an orbit $\mathcal{O}\subset\mathcal{N}_1$ is $\mathfrak{F}$-thin, if $\mathfrak{F}(\mathrm{IC}(\mathcal{O},\mathbb{C}))$ has nilpotent support. It follows that (in the cases considered here) an orbit $\mathcal{O}\subset\mathcal{N}_1$ is $\mathfrak{F}$-thin, if and only if, $\mathcal{O}$ is a Richardson orbit attached to some θ-stable Borel subgroup. This is consistent with the speculation in [L3] that $\mathfrak{F}$-thin orbits exist only if θ is inner.

3.6. **Induced character sheaves.** In order to produce all character sheaves, we consider parabolic induction from θ-stable Levi subgroups. We recall the notion of induction functors in our setting (see [H, L2]). Let L be a θ-stable Levi subgroup contained in a θ-stable parabolic subgroup $P\subset G$. We write

$$\mathfrak{l}=\mathrm{Lie}\,L,\ \mathfrak{p}=\mathrm{Lie}\,P,\ L_K=L\cap K,\ P_K=P\cap K,\ \mathfrak{l}_1=\mathfrak{l}\cap\mathfrak{g}_1,\ \mathfrak{p}_1=\mathfrak{p}\cap\mathfrak{g}_1.$$

The parabolic induction functor $\mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}:D_{L_K}(\mathfrak{l}_1)\to D_K(\mathfrak{g}_1)$ is defined as follows. Consider the diagram

$$\mathfrak{l}_1 \xleftarrow{\ \mathrm{pr}\ } \mathfrak{p}_1 \xleftarrow{\ p_1\ } K\times\mathfrak{p}_1 \xrightarrow{\ p_2\ } K\times^{P_K}\mathfrak{p}_1 \xrightarrow{\ \pi\ } \mathfrak{g}_1\ , \tag{3.23}$$

where p_1 and p_2 are natural projection maps and $\pi:(k,x)\mapsto\mathrm{Ad}(k)(x)$. The maps in (3.23) are $K\times P_K$-equivariant, where K acts trivially on $\mathfrak{l}_1$ and $\mathfrak{p}_1$, by left multiplication on the K-factor on $K\times\mathfrak{p}_1$ and on $K\times^{P_K}\mathfrak{p}_1$, and by adjoint action on $\mathfrak{g}_1$, and P_K acts on $\mathfrak{l}_1$ by $a.l=\mathrm{pr}(\mathrm{Ad}\,a(l))$, by adjoint action on $\mathfrak{p}_1$, by $a.(k,p)=(ka^{-1},\mathrm{Ad}\,a(p))$ on $K\times\mathfrak{p}_1$, trivially on $K\times^{P_K}\mathfrak{p}_1$ and $\mathfrak{g}_1$.

Let $\mathcal{T}$ be a complex in $D_{L_K}(\mathfrak{l}_1)$. Then $(\mathrm{pr}\circ p_1)^*\mathcal{T}\cong p_2^*\mathcal{T}'$ for a well-defined complex $\mathcal{T}'$ in $D_K(K\times^{P_K}\mathfrak{p}_1)$. Set $\mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}\mathcal{T}=\pi_!\mathcal{T}'[\dim P-\dim L]$. Note that as the map π is proper the induction functor takes semisimple objects to semisimple objects. Moreover, the Fourier transform commutes with induction:

$$\mathfrak{F}(\mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}\mathcal{T})\cong\mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}(\mathfrak{F}(\mathcal{T})). \tag{3.24}$$

In our setting we will define a family of θ-stable parabolic subgroups P and study the character sheaves obtained by inducing full support character sheaves from the corresponding θ-stable Levi subgroups L.

3.7. **An example.** In this section we consider the special case (SL_2, SO_2), or equivalently (Sp_2, GL_1), to illustrate how the description of the character sheaves depends on the choice of the splitting of the top row of (2.3). The choice of a splitting comes up also in (6.1). It affects the labelling of the local systems in the way we illustrate here.

We have $\mathfrak{g}_1 = \mathbb{C}^2$ and $K = \mathbb{C}^*$. The $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $c \cdot (x, y) = (cx, c^{-1}y)$. The nilpotent cone is given by $\mathcal{N}_1 = \{(x, y) \mid xy = 0\}$. There are three nilpotent orbits, the zero orbit $\mathcal{O}_0$, the two regular nilpotent orbits $\mathcal{O}_1^{\mathrm{I}}$ and $\mathcal{O}_1^{\mathrm{II}}$, corresponding to the coordinate axes. The orbits $\mathcal{O}_1^{\mathrm{I}}$ and $\mathcal{O}_1^{\mathrm{II}}$ are self dual and $\check{\mathcal{O}}_0 = \mathfrak{g}_1^{rs}$. In this case diagram (2.3) becomes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & (\mathbb{Z} \oplus \mathbb{Z})/2\mathbb{Z} & \xrightarrow{\tilde{q}} & \mathbb{Z} & \longrightarrow & 1 \\ & & \Big\| & & \big\downarrow{\scriptstyle \tilde{p}} & & \big\downarrow{\scriptstyle p} & & \\ 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{q} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1\,. \end{array}$$

There are two character sheaves with full support where the $I = \mathbb{Z}/2\mathbb{Z}$ acts non-trivially: the Fourier transforms of the IC-sheaves associated to non-trivial local systems on the orbits $\mathcal{O}_1^{\mathrm{I}}$ and $\mathcal{O}_1^{\mathrm{II}}$. Once we choose a splitting of $\tilde{q}$ one of those local system corresponds to the trivial representation of $B_1 = \mathbb{Z}$ and the other to the non-trivial one.

4. Nilpotent support character sheaves

In this section we determine all nilpotent support character sheaves. They only arise for the symmetric pairs (G, K) in §2.4 for which θ is an inner involution. We use the notation of that section. We assume that θ is inner in this section. Recall our strategy from §3.5. At the beginning of this section we will concentrate on specifying for each $\mathcal{O} \in \underline{\mathcal{N}_1^0}$ (see (3.21)) a special θ-stable Borel B such that we only need to consider

$$\mathring{\pi}_B : K \times^{B_K} \mathfrak{n}_1^r \to \mathcal{O}_B = \mathcal{O}$$

to obtain all character sheaves supported on $\bar{\mathcal{O}}$.

The set $\underline{\mathcal{N}_1^0}$ of Richardson orbits has been determined in each case by Trapa in [T1]. We recall his result:

Proposition 4.1 ([T1])**.** *Suppose that θ is inner. Let $\mathcal{O}_\lambda$ be a K-orbit in $\mathcal{N}_1$ corresponding to a signed Young diagram λ of the form* (2.5b)*. Then $\mathcal{O}_\lambda \in \underline{\mathcal{N}_1^0}$ if and only if*

- (AIII) *for each i, either $p_i = 0$ or $q_i = 0$.*
- (CI) *for λ_i odd, $p_i = q_i = 0$, and for λ_i even, either $p_i = 0$ or $q_i = 0$.*
- (CII) *for λ_i even, $p_i = q_i \leq 1$, and for λ_i odd, either $p_i = 0$ or $q_i = 0$.*
- (DIII) *for λ_i odd, $p_i = q_i \leq 1$, and for λ_i even, either $p_i = 0$ or $q_i = 0$.*
- (BDI) *the following two conditions hold*
  1. *for λ_i even, $p_i = q_i = 0$, and for λ_i odd, either $p_i = 0$ or $q_i = 0$. Namely the signed Young diagram λ is of the form $(2\mu_1+1)_{\epsilon_1}(2\mu_2+1)_{\epsilon_2}\cdots(2\mu_k+1)_{\epsilon_k}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0$, $\epsilon_i \in \{0, 1\}$ and $\epsilon_i = \epsilon_j$ if $\mu_i = \mu_j$.*
  2. (a) *if N is odd, $\epsilon_1 \equiv q \bmod 2$ and $\epsilon_{2i} + \mu_{2i} \equiv \epsilon_{2i+1} + \mu_{2i+1} \bmod 2$, $i \geq 1$;*
     (b) *if N is even, $\epsilon_{2i-1} + \mu_{2i-1} \equiv \epsilon_{2i} + \mu_{2i} \mod 2$, $i \geq 1$.*

Let us write

$$\mathrm{SYD}^0_{(G,K)}$$

for the set of signed Young diagrams corresponding to the orbits in $\underline{\mathcal{N}_1^0}$.

Theorem 4.2. *Assume that* (G,K) *is* $(Sp_{2n}, Sp_{2p} \times Sp_{2q})$, (Sp_{2n}, GL_n), *or* (SO_{2n}, GL_n). *The set of nilpotent support character sheaves is*

$$\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1) = \{\mathrm{IC}(\mathcal{O}, \mathbb{C}) \mid \mathcal{O} \in \underline{\mathcal{N}_1^0}\}.$$

Remark 4.3. The theorem holds also for the symmetric pair $(GL_n, GL_p \times GL_q)$. This is a special case of [L2].

Suppose that $(G,K) = (SO_N, S(O_p \times O_q))$, where either p or q is even. Let $\mathcal{O} \in \underline{\mathcal{N}_1^0}$. We define a set of characters $\Pi_{\mathcal{O}} \subset \widehat{A}_K(\mathcal{O})$ and we show that it is exactly the set of characters $\Pi_{\mathcal{O}} \subset \widehat{A}_K(\mathcal{O})$ defined in §3.5. Proposition 4.1 implies that $\mathcal{O} = \mathcal{O}_\mu$, where

$$\mu = (2\mu_1+1)^{m_1}_{\epsilon_1}(2\mu_2+1)^{m_2}_{\epsilon_2}\cdots(2\mu_s+1)^{m_s}_{\epsilon_s}, \tag{4.1}$$

$\mu_1 > \mu_2 > \cdots > \mu_s \geq 0$, $m_i > 0$ and $\epsilon_i \in \{0,1\}$, $1 \leq i \leq s$. From §2.5 we conclude

$$A_K(\mathcal{O}) \cong S(O_{m_1} \times \cdots \times O_{m_s})/S(O_{m_1} \times \cdots \times O_{m_s})^0 \cong (\mathbb{Z}/2\mathbb{Z})^{s-1}. \tag{4.2}$$

For $1 \leq i \leq s-1$, let

$$\begin{gathered}\delta_i \text{ correspond to the non-trivial element in} \\ S(O_{m_i} \times O_{m_{i+1}})/(SO_{m_i} \times SO_{m_{i+1}}) \cong \mathbb{Z}/2\mathbb{Z}.\end{gathered} \tag{4.3}$$

Then $A_K(\mathcal{O})$ is generated by δ_i, $1 \leq i \leq s-1$.

We define $\Omega_{\mathcal{O}} = \Omega_\mu \subset \{1, \ldots, s\}$ to be the set of $j \in [1,s]$ such that

(1) $\sum_{a=j}^s m_a$ is even,
(2) if $j \geq 2$, then either $\mu_{j-1} \geq \mu_j + 2$ or $\epsilon_{j-1} = \epsilon_j$.

We set $l_\mu = l_{\mathcal{O}} := |\Omega_{\mathcal{O}}|$. Suppose that $\Omega_{\mathcal{O}} = \{j_1, \ldots, j_l\}$, $l = l_{\mathcal{O}}$, and we write $j_{l+1} = s+1$. Note that $j_1 = 1$ if and only if N is even. Thus $l_{\mathcal{O}} \geq 1$ when N is even. The subset $\Pi_{\mathcal{O}} \subset \widehat{A}_K(\mathcal{O})$ is defined as follows

$$\Pi_{\mathcal{O}} = \{\chi \in \widehat{A}_K(\mathcal{O}) \mid \chi(\delta_r) = 1 \text{ if } r+1 \notin \Omega_{\mathcal{O}}\}. \tag{4.4}$$

In other words $\chi(\delta_r)$ is allowed to take the value -1 precisely when $r+1 \in \Omega_{\mathcal{O}}$. In particular, $|\Pi_{\mathcal{O}}| = 2^{l_{\mathcal{O}}}$ if N is odd and $|\Pi_{\mathcal{O}}| = 2^{l_{\mathcal{O}}-1}$ if N is even as in the latter case $j_1 = 1$.

Theorem 4.4. *Assume that* $(G,K) = (SO_N, S(O_p \times O_q))$, *where either* p *or* q *is even. The set of nilpotent support character sheaves is*

$$\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1) = \{\mathrm{IC}(\mathcal{O}, \mathcal{E}_\phi) \mid \mathcal{O} \in \underline{\mathcal{N}_1^0},\ \phi \in \Pi_{\mathcal{O}}\},$$

where $\Pi_{\mathcal{O}}$ *is defined in* (4.4).

The fact that the sheaves in Theorems 4.2 and 4.4 belong to $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ follows from the discussion in §3.5, Propositions 4.5 and 4.6 below. The fact that these are all of $\mathrm{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)$ will follow once we show that we have constructed sufficiently many character sheaves.

4.1. **Conjugacy classes of θ-stable Borel subgroups.** The K-conjugacy classes of θ-stable Borel subgroups of G are precisely the closed K-orbits on the flag variety $X = G/B$. Let T be a maximal torus of K which is also a maximal torus in G as θ is inner. The closed K-orbits are given by K-orbits of the fixed point set X^T. The Weyl group $W(G,T)$ acts transitively on X^T and in this manner we get an identification

$$\{\text{closed } K\text{-orbits on } X = G/B\} \longleftrightarrow W(G,T)/W(K,T)\,.$$

Recall our concrete description of (G,K) and the subspaces V^+, V^- of V (see §2.4). We denote by V_i (resp. V_i^+, V_i^-) a subspace of V (resp. V^+, V^-) of dimension i. Let $n = [N/2]$ if $G = SO_N$. Let $\tilde{K} = O_p \times O_q$ in type DI and $\tilde{K} = K$ otherwise. It is easy to see that the $\tilde{K}$-conjugacy classes of θ-stable Borel subgroups in G are parametrized by *ordered* sequences $a_1, \ldots, a_n$, $a_i \in \{0,1\}$, $1 \le i \le n$, such that

$$\sum_{i=1}^n a_i = q \text{ (AIII) (CII)}, \qquad \sum_{i=1}^n a_i = [\frac{q}{2}] \text{ (BDI)}.$$

Let $B_{(a_i)}$ denote a θ-stable Borel subgroup of G corresponding to such a sequence (a_i). Concretely $B_{(a_i)}$ can be taken as the subgroup of G which stabilizes a flag of the following form

$$\begin{aligned} 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \qquad & \text{(AIII)},\\ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset V_n^\perp \subset \cdots \subset V_1^\perp \subset V \qquad & \text{others},\end{aligned}$$

where $V_i = V^+_{i-\sum_{j=1}^i a_j} \oplus V^-_{\sum_{j=1}^i a_j}$, $i = 1, \ldots, n$.

Let $\mathcal{O}_{(a_i)} = \mathcal{O}_{B_{(a_i)}}$. In the case of type DI, each $\tilde{K}$-orbit of θ-stable Borel subgroups in G decomposes into two K-orbits; we denote them by $B^\omega_{(a_i)}$, $\omega = \mathrm{I}, \mathrm{II}$. Note that $\mathcal{O}_{B^I_{(a_i)}} = \mathcal{O}_{B^{\mathrm{II}}_{(a_i)}}$.

4.2. **Springer fibers.** Let B be a θ-stable Borel subgroup of G. Recall the map $\pi_B : K \times^{B_K} \mathfrak{n}_1 \to \bar{\mathcal{O}}_B$. Let $x \in \mathcal{O}_B$. Recall that $K \times^{B_K} \mathfrak{n}_1 = T^*_Q X$ and then $\pi_B^{-1}(\mathcal{O}_B)$ is an open subset of $T^*_Q X$. Let $x \in \mathcal{O}_B$. Applying generic smoothness to the map $\pi_B^{-1}(\mathcal{O}_B) \to \mathcal{O}_B$ we see that the fiber $\pi_B^{-1}(x)$ is a smooth subvariety of the Springer fiber $\mathcal{B}_x$. In general $\pi_B^{-1}(x)$ is not connected. However, as the map $\pi_B^{-1}(\mathcal{O}_B) \to \mathcal{O}_B$ is K-equivariant and $\pi_B^{-1}(\mathcal{O}_B)$ is K-connected we conclude that the components of $\pi_B^{-1}(x)$ form a single orbit under $A_K(x)$. For details, see, for example, [BZ, Proposition 2] or [CNT, Proposition 2.12].

Proposition 4.5 is proved in [BZ, Corollary 33] for type CI, and in [T2] for types AIII, CII, and DIII.

Proposition 4.5 ([BZ], [T2])**.** *Assume that (G,K) is not of type BDI. Then the fibers $\pi_B^{-1}(x)$ are irreducible for $x \in \mathcal{O}_B$.*

As was pointed out earlier, it follows from this proposition and the discussion in §3.5 that the IC sheaves in Theorem 4.2 are nilpotent support character sheaves.

In the remainder of this section we study the pairs $(G,K) = (SO_N, S(O_p \times O_q))$.

Let $\mathcal{O} = \mathcal{O}_\mu \in \underline{\mathcal{N}_1^0}$, where μ is as in (4.1). Recall the set $\Omega_{\mathcal{O}} = \Omega_\mu = \{j_1, \ldots, j_l\}$, $l = l_{\mathcal{O}}$, defined at the beginning of the section. Let

$$\sigma_k = \sum_{a=j_k}^{j_{k+1}-1} m_a \text{ for } 1 \le k \le l.$$

The σ_k are even. Let $x \in \mathcal{O}$. Let $\{x, y, h\}$ be a normal $\mathfrak{sl}_2$-triple. Then $h \in \mathfrak{g}_0$. In particular, $hV^+ \subset V^+$ and $hV^- \subset V^-$. Let

$$V = \oplus_{i\geq 0} V(i)_+^{\oplus p_i} \oplus_{i\geq 0} V(i)_-^{\oplus q_i}$$

be the decomposition of V into irreducible $\mathfrak{sl}_2$-modules under $\{x, y, h\}$, where $V(i)_+$ (resp. $V(i)_-$) denotes an irreducible $\mathfrak{sl}_2$-module of highest weight i such that the lowest weight vector lies in V^+ (resp. V^-). Let $V(i) = V(i)_+$ or $V(i)_-$. We set

$$W_k = \bigoplus_{a=j_k}^{j_{k+1}-1} V(2\mu_a)_0^{\oplus m_a},\ 1 \leq k \leq l,$$

where $V(2\mu_a)_0$ is the 0-weight space of $V(2\mu_a)$. We have $\dim W_k = \sigma_k$ and $(,)|_{W_k}$ is non-degenerate.

For a subspace W of V let

$$\mathrm{IGr}(k, W) \ = \ \{V_k \subset W \mid \dim V_k = k,\, V_k \text{ is isotropic in } V\}\,.$$

Proposition 4.6. *There exists a θ-stable Borel subgroup $B \subset G$ such that $\mathcal{O} = \mathcal{O}_B$ and such that for $x \in \mathcal{O}$ we have*

$$\mathrm{H}^{top}(\tilde{\pi}_B^{-1}(x)) \cong \bigotimes_{k=1}^{l} \mathrm{H}^{top}(\mathrm{IGr}(\sigma_k/2, W_k))\,,$$

where $\tilde{\pi}_B : \tilde{K} \times^{B_K} \mathfrak{n}_1 \to \bar{\mathcal{O}}_B$. This isomorphism is compatible with the action of $A_K(x)$.

The group $A_K(x)$ from (4.2) acts in the natural way on $\mathrm{H}^{top}(\mathrm{IGr}(\sigma_k/2, W_k)) = \mathbb{C} \oplus \mathbb{C}$. More precisely, $\delta_i \in A_K(x)$ (see (4.3)) acts trivially if $i+1 \notin \Omega_{\mathcal{O}}$, and δ_{j_k-1}, $j_k \in \Omega_{\mathcal{O}}$, $k = 1, \ldots, l$, permutes the two copies of $\mathbb{C}$ in $\mathrm{H}^{top}(\mathrm{IGr}(\sigma_j/2, W_j))$ for $j = k-1, k$, while acting trivially on $\mathrm{H}^{top}(\mathrm{IGr}(\sigma_j/2, W_j))$ for $j \notin \{k-1, k\}$. In particular, the action of $A_K(x)$ on the set of irreducible (or equivalently, connected) components of $\tilde{\pi}_B^{-1}(x)$ is transitive when N is odd but has two orbits when N is even (recall $j_1 = 1$). In the N even case we only consider one of the orbits. Taking into account the discussion from §3.5 and the explicit description of the action of $A_K(x)$ on $\mathrm{H}^{top}(\mathrm{IGr}(\sigma_k/2, W_k))$ we conclude that the IC sheaves in Theorem 4.4 are nilpotent support character sheaves.

Proof of Proposition 4.6. Recall $n = [N/2]$ and the $\tilde{K}$-orbits of θ-stable Borel subgroups are parametrized by ordered sequences $a_1, \ldots, a_n$, where $a_i \in \{0, 1\}$ and $\sum_{1\leq i\leq n} a_i = [q/2]$. We define the special θ-stable Borel by induction as follows. Assume that $1 \leq k \leq s$ is such that $\epsilon_1 = \ldots = \epsilon_k = \epsilon$ and $\epsilon_k \neq \epsilon_{k+1}$ (if $k < s$). Note that if $\mu_k = 0$, then $k = s$. Let

$$m_\mu = \sum_{a=1}^{k} m_a \text{ (resp. } \sum_{a=1}^{k-1} m_a + [\frac{m_k}{2}]) \text{ if } \mu_k \neq 0 \text{ (resp. } \mu_k = 0). \tag{4.5}$$

We define

$$a_1 = \ldots = a_{m_\mu} = \epsilon.$$

We write $m = m_\mu$ and without loss of generality we assume that $\epsilon = 0$. Let

$$V_m = \ker x \cap \mathrm{Im}\, x^{2\mu_k} \text{ if } \mu_k \geq 1, \text{ and } V_m \in \mathrm{IGr}(m, \ker x) \text{ if } \mu_k = 0.$$

We note that $V_m \subset V^\epsilon$ (here $V^0 = V^+$ and $V^1 = V^-$). Let $V' = (V_m)^\perp/V_m$. Then $(,)$ induces a non-degenerate bilinear form $(,)'$ on V'. Moreover, θ induces an involution θ' on $SO_{V',(,)'}$ and we obtain a symmetric pair $(G',K') = (SO_{N-2m}, S(O_{p-2m}\times O_q))$.

Consider the map $x' : V_m^\perp/V_m \to V_m^\perp/V_m$ induced by $x \in \mathcal{O}_\mu$. Then $x' \in \mathcal{O}' = \mathcal{O}_{\mu'}$ where

$$(4.6)\quad \mu' = \begin{cases} (2\mu_1-1)_{\epsilon+1}^{m_1}(2\mu_2-1)_{\epsilon+1}^{m_2}\cdots(2\mu_k-1)_{\epsilon+1}^{m_k}(2\mu_{k+1}+1)_{\epsilon+1}^{m_{k+1}}\cdots & \text{if } \mu_k \geq 1\\ (2\mu_1-1)_{\epsilon+1}^{m_1}(2\mu_2-1)_{\epsilon+1}^{m_2}\cdots(2\mu_{k-1}-1)_{\epsilon+1}^{m_{k-1}}1_\epsilon^{m_k-2[m_k/2]} & \text{if } \mu_k = 0.\end{cases}$$

Note that in passing from μ to μ' only the first $\sum_{i=1}^k m_i$ terms in the partition change. Let m'_μ be defined for μ' as in (4.5). We define

$$a_{m_\mu+1} = \ldots = a_{m_\mu+m_{\mu'}} \equiv \epsilon+1.$$

Continuing in this manner we produce a sequence $a_1,\ldots,a_n$.

Let B' be a θ-stable Borel subgroup defined by the sequence $a_{m_\mu+1},\ldots,a_n$ for the symmetric pair $(G',K') = (SO_{N-2m}, S(O_{p-2m}\times O_q))$. Let us write $\tilde\pi_B : \tilde K \times^{B_K} \mathfrak{n}_1 \to \bar{\mathcal{O}}_B$ and $\tilde\pi'_{B'} : \tilde K' \times^{B'_{K'}} \mathfrak{n}'_1 \to \bar{\mathcal{O}}_{B'}$. By the induction hypothesis, we can assume that $\mathcal{O}_{B'} = \mathcal{O}_{\mu'}$, where μ' is as in (4.6). Then $\mathcal{O}_B = \mathcal{O}_\mu$ by [T1, Proposition 7.1]. We write B for the θ-stable Borel subgroup defined by the sequence $a_1,\ldots,a_n$.

We now determine the components of $\tilde\pi_B^{-1}(x)$ by induction. We first show that

$$(4.7)\quad \Omega_{\mathcal{O}} = \Omega_{\mathcal{O}'} \sqcup \{k\} \text{ (resp. } \Omega_{\mathcal{O}'}) \text{ if } \mu_k = 0 \text{ and } m_k \text{ is even (resp. otherwise).}$$

Thus

$$(4.8)\quad l_{\mathcal{O}'} = l_{\mathcal{O}} - 1 \text{ (resp. } l_{\mathcal{O}'}) \text{ if } \mu_k = 0 \text{ and } m_k \text{ is even (resp. otherwise).}$$

Since $\epsilon_{k-1} = \epsilon_k$, whether $k \in \Omega_{\mathcal{O}}$ depends only on the fact whether $\sum_{a=k}^s m_a$ is even. Consider first the case $\mu_k = 0$. Then $k = s$ and, if m_k is even (resp. odd) then $k \in \Omega_{\mathcal{O}}$ (resp. $k \notin \Omega_{\mathcal{O}}$) and $k \notin \Omega_{\mathcal{O}'}$. Thus we are reduced to the case $\mu_k \geq 1$ and it suffices to check that $k+1 \in \Omega_{\mathcal{O}}$ if and only if $k+1 \in \Omega'_{\mathcal{O}}$. We have the following cases

(1) $\sum_{a=k+1}^s m_a$ is odd, then $k+1 \notin \Omega_{\mathcal{O}}$ and $k+1 \notin \Omega_{\mathcal{O}'}$;
(2) $\sum_{a=k+1}^s m_a$ is even and $\mu_k \geq \mu_{k+1}+2$. Thus $k+1 \in \Omega_{\mathcal{O}}$ and $k+1 \in \Omega_{\mathcal{O}'}$ because at step $k+1$ there is no sign change.
(3) $\sum_{a=k+1}^s m_a$ is even and $\mu_k = \mu_{k+1}+1$. Then, as $\epsilon \neq \epsilon_{k+1}$ we see that $k+1 \notin \Omega_{\mathcal{O}}$. Also, $k+1 \notin \Omega_{\mathcal{O}'}$ as in this case $\Omega_{\mathcal{O}'} \subset \{1,\ldots,s\} - \{k+1\}$.

This proves (4.7) and thus (4.8).

Let us write $l = l_{\mathcal{O}}$, $l' = l_{\mathcal{O}'}$ and let

$$W'_b = \bigoplus_{a=j_b}^{j_{b+1}-1} V(2\mu'_a)_0^{\oplus m_a},\ 1 \leq b \leq l',$$

where, we recall, $\mu'_a = \mu_a - 1$ for $1 \leq a \leq k$. It is now easy to verify that

$$(4.9)\quad \dim\tilde\pi_B^{-1}(x) = \dim(\tilde\pi'_{B'})^{-1}(x') + \frac{m^2-m}{2} + \begin{cases} 0 & \text{if } \mu_k \geq 1,\\ \frac{m_k(m_k-2)}{8} & \text{if } \mu_k = 0 \text{ and } m_k \text{ even},\\ \frac{m_k^2-1}{8} & \text{if } \mu_k = 0 \text{ and } m_k \text{ odd}.\end{cases}$$

Note that if $\mathcal{F} = (U_i) \in \tilde\pi_B^{-1}(x)$, then $U_m \subset \ker x$ as $U_m \subset V^+$. Consider the map

$$p : \tilde\pi_B^{-1}(x) \to \mathrm{IGr}(m,\ker x),\ \mathcal{F} = (U_i) \mapsto U_m.$$

Suppose that $\mu_k \geq 1$. By construction $p^{-1}(V_m) \cong \mathcal{F}(V_m) \times (\tilde{\pi}'_{B'})^{-1}(x')$, where $\mathcal{F}(V_m)$ is the complete flag variety of $V_m = \ker x \cap \operatorname{Im} x^{2\mu_k}$. By (4.9), we then conclude that $p^{-1}(V_m) = \tilde{\pi}_B^{-1}(x)$ and hence $\tilde{\pi}_B^{-1}(x) \cong \mathcal{F}(V_m) \times (\tilde{\pi}'_{B'})^{-1}(x')$. Therefore

$$\mathrm{H}^{top}(\tilde{\pi}_B^{-1}(x)) \cong \mathrm{H}^{top}((\tilde{\pi}'_{B'})^{-1}(x')) .$$

Let us now consider the case when $\mu_k = 0$. We first note that

$$\mathrm{IGr}(m, \ker x) \cong \mathrm{IGr}([m_k/2], V(0)_0^{\oplus m_k}) ,$$

where $\dim V(0)_0^{\oplus m_k} = m_k$ and $(,)|_{V(0)_0^{\oplus m_k}}$ is non-degenerate. For each $U_m \in \mathrm{IGr}(m, \ker x)$, the fiber $p^{-1}(U_m) \cong \mathcal{F}(U_m) \times (\tilde{\pi}'_{B'})^{-1}(x')$. Therefore we conclude that

$$\mathrm{H}^{top}(\tilde{\pi}_B^{-1}(x)) \cong \mathrm{H}^{top}(\mathrm{IGr}([m_k/2], m_k)) \otimes \mathrm{H}^{top}((\tilde{\pi}'_{B'})^{-1}(x')) .$$

We have

$$\mathrm{H}^{top}(\mathrm{IGr}([m_k/2], m_k)) = \mathbb{C} \text{ (resp. } \mathbb{C} \oplus \mathbb{C}) \text{ if } m_k \text{ is odd (resp. even).}$$

We note that $\mathrm{IGr}(\sigma_b/2, W_b) \cong \mathrm{IGr}(\sigma_{b'}/2, W'_b)$, $1 \leq b \leq l'$ and we can assume by induction that the theorem holds for $\mathrm{H}^{top}((\tilde{\pi}'_{B'})^{-1}(x'))$. The discussion above implies that $\mathrm{H}^{top}(\tilde{\pi}_B^{-1}(x)) \cong \mathrm{H}^{top}((\tilde{\pi}'_{B'})^{-1}(x'))$ except when $\mu_k = 0$ and m_k is even. In that case $\sigma_l = m_k$ and $W_l = V(0)_0^{\oplus m_k}$, and thus

$$\mathrm{H}^{top}(\tilde{\pi}_B^{-1}(x)) \cong \mathrm{H}^{top}(\mathrm{IGr}(\sigma_l/2, W_l)) \otimes \mathrm{H}^{top}((\tilde{\pi}'_{B'})^{-1}(x')) .$$

This concludes the induction step. □

5. Nearby cycle sheaves

In this section, we determine the Fourier transform $\mathfrak{F}(P_\chi)$ of the nearby cycle sheaves P_χ (see §3.2) explicitly for the symmetric pairs (G, K) in §2.4. Recall that $r = \dim \mathfrak{a}$.

5.1. **The group I and the Weyl group action.** In this section we study the groups $I = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0$ and the action of $W_\mathfrak{a}$ on the set $\hat{I}$ of its irreducible characters. We will make use of the discussion in §2.4 and §3.4. Note that $I^0 = Z_{K^0}(\mathfrak{a})/Z_{K^0}(\mathfrak{a})^0$ is generated by $\check{\alpha}(-1)$, $\alpha \in \Phi^{\mathrm{R}}$, see, for example [K, Theorem 7.55].

Lemma 5.1. *We have that*

(1) $I = 1$, *if (G, K) is of type* AIII, CII *or* DIII.
(2) $I = (\mathbb{Z}/2\mathbb{Z})^r$, $I^0 = (\mathbb{Z}/2\mathbb{Z})^{r-1}$, *if (G, K) is of type* BDI, *and $I = I^0 = (\mathbb{Z}/2\mathbb{Z})^r$, if (G, K) is of type* CI. *Moreover, the action of $W_\mathfrak{a}$ on $\hat{I}$ has $r+1$ orbits.*

Proof. The claim (1) can be checked easily. Note that $I/I^0 = K/K^0$. Thus if G is simply connected, then $K = K^0$ and $I = I^0$. The claim (2) is readily checked using the following description of I and I^0.

Suppose that (G, K) is of type BDI. Then $K/K^0 = \mathbb{Z}/2\mathbb{Z}$. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq r-1$, $\alpha_r = \epsilon_r$ (N odd), $\alpha_r = \epsilon_{r-1} + \epsilon_r$ (N even), be a set of simple real roots. Then $I^0 = \langle \gamma_i = \check{\alpha}_i(-1), 1 \leq i \leq r-1 \rangle$ and $I = \langle \gamma_i = \check{\alpha}_i(-1), 1 \leq i \leq r-1, \gamma_r = \epsilon_r(-1) \rangle$.

Suppose that (G, K) is of type CI. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \leq i \leq r-1, \alpha_r = 2\epsilon_r$, be a set of simple real roots. Then $I = \langle \gamma_i = \check{\alpha}_i(-1), 1 \leq i \leq r \rangle$. □

In the remainder of this section we assume that (G,K) is of type BDI or CI. We fix a set of representatives for the $W_{\mathfrak{a}}$-orbits in $\hat{I}$ as follows,

$$\chi_0 = 1,\ \chi_m(\gamma_i) = 1 \text{ if } i \neq m \text{ and } \chi_m(\gamma_m) = -1,\ 1 \leq m \leq r, \tag{5.1}$$

where $\gamma_i \in I$ are defined in the proof of Lemma 5.1. We describe the subgroups $W_{\mathfrak{a},\chi_m}$ and $W^0_{\mathfrak{a},\chi_m}$.

Suppose that (G,K) is of type BI. Then G_{s} is semisimple and adjoint. One checks that

$$\begin{gathered}\{\alpha \in \Phi^{\mathrm{R}} \mid \chi_m(\check{\alpha}(-1)) = 1\}\\ = \{\pm(\epsilon_i \pm \epsilon_j), 1 \leq i < j \leq m,\ m < i < j \leq r;\ \pm\epsilon_i, 1 \leq i \leq r\}.\end{gathered}$$

Thus (see (3.19))

$$W_{\mathfrak{a},\chi_m} = W^0_{\mathfrak{a},\chi_m} = \langle s_1, \ldots, s_{m-1}, \tau_m\rangle \times \langle s_{m+1}, \ldots, s_r\rangle \cong W_m \times W_{r-m},$$

where $s_i = s_{\epsilon_i - \epsilon_{i-1}}$, $1 \leq i \leq m-1$, $s_r = s_{\epsilon_r}$, and $\tau_m = s_m \ldots s_{r-1} s_r s_{r-1} \ldots s_m = s_{\epsilon_m}$.

Suppose that (G,K) is of type DI. Let $s_i = s_{\epsilon_i - \epsilon_{i-1}}$, $1 \leq i \leq r-1$, $s_r = s_{\epsilon_{r-1}+\epsilon_r}$, and $\tau_m = (s_m s_{m+1} \ldots s_{r-2} s_r s_{r-1} \ldots s_{m+1} s_m) = s_{\epsilon_m} s_{\epsilon_n}$. Assume that $r = n$. Then

$$\begin{aligned}W_{\mathfrak{a},\chi_m} &= \langle s_{m+1}, \ldots, s_n\rangle \rtimes \langle s_1, \ldots, s_{m-1}, \tau_m\rangle\\ &\cong W'_{n-m} \rtimes W_m,\ 1 \leq m \leq n-1\\ W^0_{\mathfrak{a},\chi_m} &= \langle s_1, \ldots, s_{m-1}, \tau_m s_{m-1} \tau_m\rangle \times \langle s_{m+1}, \ldots, s_n\rangle\\ &\cong W'_m \times W'_{n-m},\ 1 \leq m \leq n-1,\\ W_{\mathfrak{a},\chi_m} &= W^0_{\mathfrak{a},\chi_m} = W'_n,\ m = 0 \text{ or } n.\end{aligned} \tag{5.2}$$

Assume that $r < n$. We have $G_{\mathrm{s}} \cong SO_{2r}$. Using (3.17) and (5.2), we see that

$$W^{\mathrm{R}}_{\chi_m} = \langle s_1, \ldots, s_{m-1}, \tau_m, s_{m+1}, \ldots, s_r\rangle.$$

Write $(W^{\mathrm{C}})^\theta = \{1, t\} = \mathbb{Z}/2\mathbb{Z}$, where $t = s_{\epsilon_r} s_{\epsilon_n}$. Then $W_{\mathfrak{a}} = \langle s_1, \ldots, s_{r-1}, t\rangle = W_r$. Thus

$$\begin{gathered}W_{\mathfrak{a},\chi_m} = W^{\mathrm{R}}_{\chi_m} \rtimes (W^{\mathrm{C}})^\theta = \langle s_1, \ldots, s_{m-1}, \tau_m, s_{m+1}, \ldots, s_r, t\rangle\\ = \langle s_1, \ldots, s_{m-1}, s_m s_{m+1} \cdots s_{r-2} s_{r-1} t s_{r-1} \cdots s_{m+1} s_m\rangle \times \langle s_{m+1}, \ldots, s_{r-1}, t\rangle\\ \cong W_m \times W_{r-m}.\end{gathered}$$

On the other hand, one checks that $\{\alpha \in \Phi^{\mathrm{R}} \mid \chi_m(\check{\alpha}(-1)) = 1\} = \{\pm(\epsilon_i \pm \epsilon_j), 1 \leq i < j \leq m,\ m+1 \leq i < j \leq r\}$. Thus

$$W^0_{\mathfrak{a},\chi_m} = \langle s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_{r-1},\ s_{\epsilon_i}, i = 1, \ldots, r\rangle = W_{\mathfrak{a},\chi_m} \cong W_m \times W_{r-m}.$$

Suppose that (G,K) is of type CI. Let $s_i = s_{\epsilon_i - \epsilon_{i+1}}$, $1 \leq i \leq r-1$, $s_r = s_{\epsilon_r}$, and $\tau_m = s_m s_{m+1} \ldots s_{r-1} s_r s_{r-1} \ldots s_m = s_{\epsilon_m}$. We have

$$\begin{aligned}W_{\mathfrak{a},\chi_m} &= \langle s_1, \ldots, s_{m-1}, \tau_m\rangle \times \langle s_{m+1}, \ldots, s_r\rangle \cong W_m \times W_{r-m},\\ W^0_{\mathfrak{a},\chi_m} &= \langle s_1, \ldots, s_{m-1}, \tau_m s_{m-1} \tau_m = s_{\epsilon_{m-1}+\epsilon_m}\rangle \times \langle s_{m+1}, \ldots, s_r\rangle\\ &\cong W'_m \times W_{r-m}.\end{aligned} \tag{5.3}$$

5.2. **Nearby cycle sheaves and full support character sheaves.** Using the discussions in §2.4, §3.2, and §5.1, we explicitly describe the Fourier transforms of the nearby cycle sheaves.

Let χ_0 denote the trivial character of I. In types BDI and CI, let $\chi_m \in \hat{I}$, $0 \leq m \leq r$, be the characters defined in (5.1). Let

$$\widetilde{B}_{W_k} = (\mathbb{Z}/2\mathbb{Z})^k \rtimes B_{W_k},\ \widetilde{B}_{W_k'} = (\mathbb{Z}/2\mathbb{Z})^k \rtimes B_{W_k'}. \tag{5.4}$$

Let $\mathcal{H}_{W_r,c_0,c_1}$ (resp. $\mathcal{H}_{W_n',-1}$) be the Hecke algebra defined in §2.6. We write $\mathcal{H}_{W_r,c_0,c_1}$ also for the representation of B_{W_r} (resp. $B_{W_n'}$) arising from the regular representation of $\mathcal{H}_{W_r,c_0,c_1}$ (resp. $\mathcal{H}_{W_n',-1}$) via the natural surjective map $\mathbb{C}[B_{W_r}] \to \mathcal{H}_{W_r,c_0,c_1}$ (resp. $\mathbb{C}[B_{W_n'}] \to \mathcal{H}_{W_n',-1}$). We set $\mathcal{H}_{W_0,c_0,c_1} = \mathcal{H}_{W_0',-1} = \mathbb{C}$.

Proposition 5.2.

(i) *Suppose that (G,K) is of type* AIII, CII *or* DIII. *We have*

$$\mathfrak{F}(P_{\chi_0}) = \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{H}_{W_r,1,-1}).$$

(ii) *Suppose that (G,K) is of type* BCDI. *We have*

$$\mathfrak{F}(P_{\chi_m}) = \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_{\chi_m} \otimes \mathbb{C}_\iota),$$

where $\mathcal{M}_{\chi_m}$ is given by the following representation M_{χ_m} of $\pi_1^K(\mathfrak{g}_1^{rs})$

$$M_{\chi_m} \cong \mathbb{C}[\widetilde{B}_{W_r}] \otimes_{\mathbb{C}[\widetilde{B}_{W_r}^{\chi_m}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_m,-1,-1} \otimes \mathcal{H}_{W_{r-m},-1,-1}) \quad \textit{(BI)},$$

$$M_{\chi_m} \cong \mathbb{C}[\widetilde{B}_{W_n}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n}^{\chi_m}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_m,-1,1} \otimes \mathcal{H}_{W_{n-m},-1,-1}) \quad \textit{(CI)},$$

$$M_{\chi_m} \cong \mathbb{C}[\widetilde{B}_{W_n'}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n'}^{\chi_m,0}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_m',-1} \otimes \mathcal{H}_{W_{n-m}',-1}) \quad \textit{(DI) } p = q,$$

$$M_{\chi_m} \cong \mathbb{C}[\widetilde{B}_{W_r}] \otimes_{\mathbb{C}[\widetilde{B}_{W_r}^{\chi_m}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_m,-1,1} \otimes \mathcal{H}_{W_{r-m},-1,1}) \quad \textit{(DI) } p \neq q,$$

and $\iota = 1$ unless $G = SO_{2n+1}$ in which case ι is the nontrivial character of $K/K^0 = \mathbb{Z}/2\mathbb{Z}$.

In the case of type CI, the isomorphism in the above proposition is proved in the same way as in [GVX1, §7.6], noting that $W_{\mathfrak{a},\chi_m}$ is a Coxeter group. In particular, the parameter for the simple reflection τ_m of $W_{\mathfrak{a},\chi_m}$ in (5.3) is 1 since $\chi(\check{\alpha}(-1)) = -1$ for $\alpha = \epsilon_k$.

In the case of split type DI, recall from (5.2) that when $1 \leq m \leq n-1$, $W_{\mathfrak{a},\chi_m} = W^0_{\mathfrak{a},\chi_m} \rtimes \langle \tau_m \rangle$ and $W^0_{\mathfrak{a},\chi_m} = \langle s_1, \ldots, s_{m-1}, \tau_m s_{m-1} \tau_m \rangle \times \langle s_{m+1}, \ldots, s_n \rangle$, where $\tau_m^2 = 1$, and τ_m acts on the generators of $W^0_{\mathfrak{a},\chi_m}$ as follows: τ_m fixes s_i for $i < m-1$ and $m+1 \leq i < n-1$, τ_m interchanges $\tau_m s_{m-1} \tau_m$ (resp. s_{n-1}) with s_{m-1} (resp. s_n). Thus when $1 \leq m \leq n-1$, for each irreducible representation ρ of $B_{W_n'}^{\chi_m,0}$, the $\mathbb{C}[B_{W_n'}^{\chi_m}] \otimes_{\mathbb{C}[B_{W_n'}^{\chi_m,0}]} \rho$ decomposes into the direct sum of two non-isomorphic irreducible representations of $B_{W_n'}^{\chi_m}$.

Remark 5.3. For the symmetric pair (GL_n, O_n), we have $I \cong (\mathbb{Z}/2\mathbb{Z})^n$, $W_{\mathfrak{a}} = S_n$, and we can choose a set $\{\chi_m,\ m = 0, \ldots, n\}$ of representatives of $W_{\mathfrak{a}}$-orbits on $\hat{I}$ such that $W_{\mathfrak{a},\chi_m} \cong S_m \times S_{n-m}$. Let $\mathcal{H}_{S_m,-1}$ denote the Hecke algebra of S_m with parameter -1. We have

$$\mathfrak{F}(P_{\chi_m}) = \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_{\chi_m} \otimes \mathbb{C}_\iota),\ 0 \leq m \leq n,$$

where $\mathcal{M}_{\chi_m}$ is given by the following representation M_{χ_m} of $\pi_1^K(\mathfrak{g}_1^{rs})$

$$M_{\chi_m} \cong \mathbb{C}[\widetilde{B}_{S_n}] \otimes_{\mathbb{C}[\widetilde{B}_{S_n}^{\chi_m}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{S_m,-1} \otimes \mathcal{H}_{S_{n-m},-1}).$$

Remark 5.4. For the symmetric pair (GL_{2n}, Sp_{2n}), the restricted root system is of type A_{n-1} with each restricted root space of dimension 4. We have $W_{\mathfrak{a}} = S_n$ and $I = 1$. Let χ_0 denote the trivial character of $I = 1$. We have

$$\mathfrak{F}(P_{\chi_0}) = \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathbb{C}[S_n]).$$

Recall the set $\Theta_{(G,K)}$ defined in (3.8). Applying Proposition 5.2, the discussions in §3.2 and the above discussion, we conclude that the sheaves in the following corollary are full support character sheaves. As before, the fact that these are all of them will follow once we construct all character sheaves.

Corollary 5.5. *Let (G,K) be a classical symmetric pair. All full support character sheaves arise from the nearby cycle constructions, i.e.,*

$$\left\{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\rho) \mid \rho \in \Theta_{(G,K)}\right\} = \mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1)\,.$$

Moreover, we have

$$\begin{aligned}
&\Theta_{(G,K)} = \{V_\tau \mid \tau \in \mathrm{Irr}\,\mathcal{H}_{W_r,1,-1}\}\ \text{(AIII) (CII) (DIII)}\,,\\
&\Theta_{(GL_{2n},Sp_{2n})} = \{V_\tau \mid \tau \in \mathrm{Irr}\,S_n\}\,,\\
&\Theta_{(GL_n,O_n)} = \{V_{\rho_1\boxtimes\rho_2,\chi_k} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{S_k,-1}, \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{S_{n-k},-1}, k \in [0,n]\},\\
&\Theta_{(SO_{2n+1},S(O_p\times O_q))} = \{V_{\rho_1\boxtimes\rho_2,\chi_k} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W_k,-1,-1},\\
&\qquad\qquad\qquad\qquad \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W_{r-k},-1,-1}, k \in [0,r]\},\\
&\Theta_{(Sp_{2n},GL_n)} = \{V_{\rho_1\boxtimes\rho_2,\chi_k} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W_k,-1,1},\\
&\qquad\qquad\qquad\qquad \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W_{n-k},-1,-1}, k \in [0,n]\},\\
&\Theta_{(SO_{2n},S(O_p\times O_q))} = \{V_{\rho_1\boxtimes\rho_2,\chi_k} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W_k,-1,1},\\
&\qquad\qquad\qquad\qquad \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W_{r-k},-1,1}, k \in [0,r]\}\ (p \neq q),\\
&\Theta_{(SO_{2n},S(O_n\times O_n))} = \{V_{\rho_1\boxtimes\rho_2,\chi_k} \mid, \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W'_k,-1},\\
&\qquad\qquad\qquad\qquad \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W'_{n-k},-1}, k = 0,n\}\\
&\cup \{V^\delta_{\rho_1\boxtimes\rho_2,\chi_k} \mid \delta = I,II,\ \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W'_k,-1}, \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W'_{n-k},-1}, k \in [1,n-1]\},
\end{aligned}$$

where $V_{\rho,\chi}$ and $V^\delta_{\rho,\chi}$ are defined as in (3.11) *and* (3.12).

Let us write

$$\Theta_n^A = \Theta_{(GL_n,O_n)}\,, \tag{5.5}$$

$$\Theta_n^B = \Theta_{(SO_{2n+2t-1},S(O_{n+2t-1}\times O_n))},\ \Theta_n^D = \Theta_{(SO_{2n+2t},S(O_{n+2t}\times O_n))}, \tag{5.6}$$
$$\Theta_n^C = \Theta_{(Sp_{2n},GL_n)},$$

$$\Theta_n^{D,0} = \Theta_{(SO_{2n},S(O_n\times O_n))}, \tag{5.7}$$

where $t > 0$. It follows that

$$|\Theta_n^B| = \sum_{k=0}^{q} d(k)d(n-k), \quad |\Theta_n^C| = \sum_{k=0}^{n} d(k)e(n-k), \tag{5.8}$$

$$|\Theta_n^D| = \sum_{k=0}^{n} e(k)e(n-k), \quad |\Theta_n^{D,0}| = \frac{1}{2}\sum_{k=0}^{n} e(k)e(n-k), \tag{5.9}$$

where we have used (2.10) and $d(k) = |\operatorname{Irr}\mathcal{H}_{W_k,-1,-1}|$, $e(k) = |\operatorname{Irr}\mathcal{H}_{W_k,-1,1}|$, see §2.6.

6. Character sheaves

In this section we give a description of the set $\mathrm{Char}_K(\mathfrak{g}_1)$ of character sheaves for the symmetric pairs (G,K) in §2.4. We use the notation from that section.

6.1. **Supports of the character sheaves.** To describe the supports of character sheaves, we define a set $\underline{\mathcal{N}_1^{\mathrm{cs}}}$ of nilpotent orbits such that for $\mathcal{O} \in \underline{\mathcal{N}_1^{\mathrm{cs}}}$ the corresponding $\check{\mathcal{O}}$ supports a character sheaf. Conversely, all supports of character sheaves are of this form. The $\underline{\mathcal{N}_1^{\mathrm{cs}}}$ consists of the following orbits:

$$
\begin{array}{lll}
\text{(AIII)} & \mathcal{O}_{k,\mu} = \mathcal{O}_{1_+^k 1_-^k \sqcup \mu} & 0 \leq k \leq r,\ \mu \in \mathrm{SYD}^0_{(GL_{n-2k}, GL_{p-k}\times GL_{q-k})},\\
\text{(BDI)} & \mathcal{O}_{m,k,\mu} = \mathcal{O}_{1_+^m 1_-^m 2_+^k 2_-^k \sqcup \mu} & m \equiv r \mod 2 \text{ if } N \text{ is even},\ 0 \leq m+2k \leq r,\\
 & & \mu \in \mathrm{SYD}^0_{(SO_{N-2m-4k}, S(O_{p-m-2k}\times O_{q-m-2k}))},\\
\text{(CI)} & \mathcal{O}_{m,k,\mu} = \mathcal{O}_{1_+^m 1_-^m 2_+^k 2_-^k \sqcup \mu} & 0 \leq m+2k \leq n,\ \mu \in \mathrm{SYD}^0_{(Sp_{2n-2m-4k}, GL_{n-m-2k})}\\
\text{(CII)} & \mathcal{O}_{k,\mu} = \mathcal{O}_{1_+^{2k} 1_-^{2k} \sqcup \mu} & 0 \leq k \leq r,\ \mu \in \mathrm{SYD}^0_{(Sp_{2n-4k}, Sp_{2p-2k}\times Sp_{2q-2k})}\\
\text{(DIII)} & \mathcal{O}_{k,\mu} = \mathcal{O}_{1_+^{2k} 1_-^{2k} \sqcup \mu} & 0 \leq k \leq [n/2],\ \mu \in \mathrm{SYD}^0_{(SO_{2n-4k}, GL_{n-2k})}\,.
\end{array}
$$

Here $\lambda \sqcup \mu$ denotes the signed Young diagram obtained by joining λ and μ together, i.e., the rows of $\lambda \sqcup \mu$ are the rows of λ and μ rearranged according to the lengths of the rows. Note that $\underline{\mathcal{N}_1^0} \subset \underline{\mathcal{N}_1^{\mathrm{cs}}}$. When $G = K$, the set $\mathrm{SYD}^0_{(G,K)}$ consists of the signed Young diagram which corresponds to the zero orbit. Also note that when $(G,K) = (SO_{4k}, S(O_{2k}\times O_{2k}))$, we have written $\mathcal{O}_{0,k,\emptyset}$ for both $\mathcal{O}^{\mathrm{I}}_{0,k,\emptyset}$ and $\mathcal{O}^{\mathrm{II}}_{0,k,\emptyset}$.

Remark 6.1. For the symmetric pair (SL_N, SO_N) considered in [CVX], the $\check{\mathcal{O}}$'s that support character sheaves are given by the nilpotent orbits $\mathcal{O}_{2^k 1^{N-2k}}$, where as usual, there are two orbits $\mathcal{O}^{\mathrm{I}}_{2^{N/2}}$ and $\mathcal{O}^{\mathrm{I}}_{2^{N/2}}$ when N is even.

Using (3.4), one readily checks that the equivariant fundamental groups $\pi_1^K(\check{\mathcal{O}})$ are as follows

$$\text{(6.1a)}\quad \text{(AIII) (CII) (DIII)}\quad \pi_1^K(\check{\mathcal{O}}_{k,\mu}) = B_{W_k};$$

$$\text{(6.1b)}\quad \text{(BDI)}\quad \pi_1^K(\check{\mathcal{O}}_{m,k,\mu}) = \begin{cases} \widetilde{B}_{W'_m} \times \widetilde{B}_{W_k} \text{ if } \mu = \emptyset \\ \widetilde{B}_{W_m} \times \widetilde{B}_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu} \text{ if } \mu \neq \emptyset \end{cases}$$

$$\text{(6.1c)}\quad \text{(CI)}\quad \pi_1^K(\check{\mathcal{O}}_{m,k,\mu}) = \widetilde{B}_{W_m} \times \widetilde{B}_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu};$$

where r_μ are defined in (2.6) and (2.7). Here we use the convention that $B_{W_0} = B_{W'_0} = \widetilde{B}_{W_0} = \widetilde{B}_{W'_0} = \{1\}$.

To illustrate the idea, let us explain the calculation in the case of type BDI. Also, for later use, the argument below fixes the splitting of $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$ into a product. We assume that $\mu = (\mu_1)^{m_1} \cdots (\mu_s)^{m_s} 1^{m_0}$, where each μ_i is odd and

$\mu_1 > \cdots > \mu_s > 1$. We then have

$$G^\phi = \{(g_1, \ldots, g_s, g_0, h) \in \prod_{i=1}^{s} O_{m_i} \times O_{2m+m_0} \times Sp_{2k} \mid \det(g_1 \cdots g_s g_0) = 1\}$$

$$K^\phi = \{(g_1, \ldots, g_s, h_1, h_2, h_0) \in \prod_{i=1}^{s} O_{m_i} \times O_m \times O_{m+m_0} \times GL_k \mid \det(g_1 \cdots g_s h_1 h_2) = 1\}.$$

Assume that $\mu \neq \emptyset$, i.e., that s and m_0 are not both zero. Then $Z_{K^\phi}(\mathfrak{a}^\phi)/Z_{K^\phi}(\mathfrak{a}^\phi)^0 = I_1 \times I_2 \times I_3$, $I_1 = (\mathbb{Z}/2\mathbb{Z})^m$, $I_2 = (\mathbb{Z}/2\mathbb{Z})^k$, $I_3 = (\mathbb{Z}/2\mathbb{Z})^{s-\delta_{m_0,0}} = (\mathbb{Z}/2\mathbb{Z})^{r_\mu} \cong S(\prod_{i=1}^s O_{m_i} \times O_{m_0})/S(\prod_{i=1}^s O_{m_i} \times O_{m_0})^0$ and $B_{W_{\mathfrak{a}^\phi}} = B_{W_m} \times B_{W_k}$. Moreover the action of $B_{W_m} \times B_{W_k}$ on $I_1 \times I_2 \times I_3$ factors through the action of B_{W_m} on I_1 and the action of B_{W_k} on I_2.

6.2. **Explicit description of character sheaves.** In this section we state our main theorems which give an explicit description of character sheaves. We begin by writing down representations of the fundamental groups in (6.1). We use the notation from §2.6. Let $\tau \in \mathcal{P}(k)$. We continue to write L_τ for the B_{W_k}-representation obtained by pulling back the simple module L_τ of $\mathcal{H}_{W_k,1,-1}$ (see (2.8)) via the surjective map $\mathbb{C}[B_{W_k}] \to \mathcal{H}_{W_k,1,-1}$.

Recall the sets $\Theta_n^{B,C,D}$ (resp. $\Theta_n^{D,0}$) of irreducible representations of $\widetilde{B}_{W_n}$ (resp. $\widetilde{B}_{W_n'}$) defined in (5.6) (resp. (5.7)).

(AIII, CII, DIII) For each $\tau \in \mathcal{P}(k)$, let $\mathcal{T}_\tau$ denote the irreducible K-equivariant local system on $\check{\mathcal{O}}_{k,\mu}$ corresponding to the $\pi_1^K(\check{\mathcal{O}}_{k,\mu})$-representation where B_{W_k} acts via L_τ, i.e. $\mathcal{T}_\tau = L_\tau$. Here $\mathcal{T}_\tau = \mathbb{C}$ is the trivial local system if $k = 0$.

(CI) For $\rho \in \Theta_m^C$ and $\tau \in \mathcal{P}(k)$, let $\mathcal{T}_{\rho,\tau}$ denote the irreducible K-equivariant local system on $\check{\mathcal{O}}_{m,k,\mu}$ corresponding to the representation of $\pi_1^k(\check{\mathcal{O}}_{m,k,\mu})$, where $\widetilde{B}_{W_m}$ acts via ρ, $\widetilde{B}_{W_k}$ acts via the B_{W_k}-representation L_τ, and $(\mathbb{Z}/2\mathbb{Z})^{r_\mu}$ acts trivially, i.e., $\mathcal{T}_{\rho,\tau} = L_\rho \boxtimes L_\tau \boxtimes 1$.

(BI (resp. DI)) Suppose that $\mu \neq \emptyset$. For each $\rho \in \Theta_m^B$ (resp. Θ_m^D), $\tau \in \mathcal{P}(k)$, and $\phi \in \Pi_{\mathcal{O}_\mu}$ (see (4.4)), let $\mathcal{T}_{\rho,\tau,\phi}$ denote the irreducible K-equivariant local system on $\check{\mathcal{O}}_{m,k,\mu}$ given by the representation of $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$, where $\widetilde{B}_{W_m}$ acts via ρ, $\widetilde{B}_{W_k}$ acts via the B_{W_k}-representation L_τ, and $(\mathbb{Z}/2\mathbb{Z})^{r_\mu}$ acts via ϕ, i.e., $\mathcal{T}_{\rho,\tau,\phi} = L_\rho \boxtimes L_\tau \boxtimes \phi$.

Suppose that $\mu = \emptyset$. For each $\rho \in \Theta_m^{D,0}$ and $\tau \in \mathcal{P}(k)$, let $\mathcal{T}_{\rho,\tau}$ denote the irreducible K-equivariant local system on $\check{\mathcal{O}}_{m,k,\emptyset}$ corresponding to the representation $L_\rho \boxtimes L_\tau$ of $\pi_1^K(\check{\mathcal{O}}_{m,k,\emptyset})$, where $\widetilde{B}_{W_m'}$ acts via ρ and $\widetilde{B}_{W_k}$ acts via the B_{W_k}-representation L_τ, i.e., $\mathcal{T}_{\rho,\tau} = L_\rho \boxtimes L_\tau$.

Note that $\mathcal{T}_{\rho,\tau} = \mathbb{C}$ is the trivial local system and $\mathcal{T}_{\rho,\tau,\phi} = \mathcal{E}_\phi$ if $m = k = 0$.

Theorem 6.2. *Suppose that* (G, K) *is of type* AIII, CII, *or* DIII. *We have*

$$\mathrm{Char}_K(\mathfrak{g}_1) = \left\{ \mathrm{IC}(\check{\mathcal{O}}_{k,\mu}, \mathcal{T}_\tau) \mid \tau \in \mathcal{P}(k) \right\}.$$

Theorem 6.3. *Suppose that* (G, K) *is of type* CI. *We have*

$$\mathrm{Char}_K(\mathfrak{g}_1) = \left\{ \mathrm{IC}(\check{\mathcal{O}}_{m,k,\mu}, \mathcal{T}_{\rho,\tau}) \mid \rho \in \Theta_m^C,\ \tau \in \mathcal{P}(k) \right\}.$$

Theorem 6.4.

(i) *Suppose that* (G,K) *is of type* BI. *We have*

$$\mathrm{Char}_K(\mathfrak{g}_1)=\Big\{\mathrm{IC}(\check{\mathcal{O}}_{m,k,\mu},\mathcal{T}_{\rho,\tau,\phi})\,|\,\rho\in\Theta^B_m,\tau\in\mathcal{P}(k),\ \phi\in\Pi_{\mathcal{O}_\mu}\Big\}.$$

(ii) *Suppose that* (G,K) *is of type* DI. *Then*

$$\begin{aligned}\mathrm{Char}_K(\mathfrak{g}_1) &= \Big\{\mathrm{IC}(\check{\mathcal{O}}_{m,k,\mu},\mathcal{T}_{\rho,\tau,\phi})\,|\,\mu\neq\emptyset,\ \rho\in\Theta^D_m,\ \omega=\mathrm{I},\mathrm{II},\ \tau\in\mathcal{P}(k),\ \phi\in\Pi_{\mathcal{O}_\mu}\Big\}\\ &\sqcup\ \Big\{\mathrm{IC}(\check{\mathcal{O}}_{m,k,\emptyset},\mathcal{T}_{\rho,\tau})\,|\,m\geq 1,\ \rho\in\Theta^{D,0}_m,\ \tau\in\mathcal{P}(k)\Big\}\ \ (\textit{if } p=q)\\ &\sqcup\ \Big\{\mathrm{IC}(\check{\mathcal{O}}^\omega_{0,n/2,\emptyset},\mathcal{T}_{\tau})\,|\,\omega=\mathrm{I},\mathrm{II},\ \tau\in\mathcal{P}(n/2)\Big\}\ \ (\textit{if } p=q \textit{ even}).\end{aligned}$$

Remark 6.5. For the symmetric pair (GL_{2n},Sp_{2n}), the nilpotent K-orbits in $\mathcal{N}_1$ are labelled by partitions of n. Moreover, $A_K(x)=1$ for $x\in\mathcal{N}_1$. We have (see Remark 5.4)

$$\mathrm{Char}_K(\mathfrak{g}_1)=\{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{L}_\tau)\,|\,\tau\in\mathcal{P}(n)\},$$

where $\mathcal{L}_\tau$ is the irreducible K-equivariant local system on $\mathfrak{g}_1^{rs}$ given by the irreducible representation of $\pi_1^K(\mathfrak{g}_1^{rs})=B_{S_n}$, where B_{S_n} acts through S_n via the irreducible representation corresponding to $\tau\in\mathcal{P}(n)$. This recovers the corresponding result in [G2, H, L2].

Remark 6.6. For the symmetric pair (GL_n,O_n), analysing as in [CVX], we have

$$\mathrm{Char}_K(\mathfrak{g}_1)=\{\mathrm{IC}(\check{\mathcal{O}}_{2^k1^{n-2k}},\mathcal{T}_{\tau,\rho})\,|\,\tau\in\mathcal{P}(k),\rho\in\Theta^A_{n-2k}\},$$

where Θ^A_m is the set of irreducible representations of $\widetilde{B}_{S_m}$ defined in (5.5) and $\mathcal{T}_{\tau,\rho}$ is the irreducible K-equivariant local system on $\check{\mathcal{O}}_{2^k1^{n-2k}}$ given by the irreducible representation $L_\tau\boxtimes\rho$ of $\pi_1^K(\check{\mathcal{O}}_{2^k1^{n-2k}})$ via the surjective map $\pi_1^K(\check{\mathcal{O}}_{2^k1^{n-2k}})\to B_{S_k}\times\widetilde{B}_{S_{n-2k}}$, where B_{S_k} acts through S_k on the irreducible representation L_τ corresponding to $\tau\in\mathcal{P}(k)$ and $\widetilde{B}_{S_{n-2k}}$ acts on ρ.

As mentioned earlier, Theorems 4.2 and 4.4, and Corollary 5.5 follow from the above theorems whose proofs are given in the next two sections. We also obtain the following corollary.

Corollary 6.7. *Theorem* 1.1 *from the introduction holds.*

Proof. We first note that all full support character sheaves for split symmetric pairs are cuspidal. Since Fourier transform commutes with parabolic induction, this follows from the fact that $\mathrm{Supp}\,\mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}A\subset K.\mathfrak{p}_1\subsetneq\mathfrak{g}_1$ for any θ-stable Levi subgroup L contained in a *proper* θ-stable parabolic subgroup P when the symmetric pair is split.

We claim that all full support character sheaves for the pair $(GL_{2n},GL_n\times GL_n)$ can be obtained via parabolic induction from the constant sheaf $\mathbb{C}_{\mathfrak{l}_1}$ for the θ-stable Levi $L=GL_2^n$ and $L^\theta=(GL_1\times GL_1)^n$. This can be seen as follows. Recall that $G=GL_V$ and $K=GL_{V^+}\times GL_{V^-}$, where $V^+=\mathrm{span}\{e_1,\dots,e_n\}$ and $V^-=\mathrm{span}\{f_1,\dots,f_n\}$. Consider the θ-stable parabolic subgroup P that stabilizes the following flag $0\subset V_2\subset V_4\subset\cdots\subset V_{2n-2}\subset V_{2n}=V$, $V_{2m}=\mathrm{span}\{e_1,\dots,e_m\}\oplus\mathrm{span}\{f_1,\dots,f_m\}$. Let L be the natural θ-stable Levi subgroup.

Then $(L, L^\theta) \cong (GL_2^n, (GL_1 \times GL_1)^n)$. Consider the map $\pi : K \times^{P_K} \mathfrak{p}_1 \to \mathfrak{g}_1$. One checks readily that

$$\pi_*\mathbb{C}[-] = \bigoplus_{\tau \in \mathcal{P}(n)} \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\tau) \oplus \cdots .$$

Since $\pi_*\mathbb{C}[-] \cong \mathrm{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathbb{C}_{\mathfrak{l}_1}$, the claim follows.

One then checks that for the symmetric pairs $(Sp_{4n}, Sp_{2n} \times Sp_{2n})$ and (SO_{4n}, GL_{2n}), the full support character sheaves can be obtained via parabolic induction from the full support character sheaves for the θ-stable Levi subgroup $L = GL_{2n}$ with $L^\theta = GL_n \times GL_n$.

We can deduce from [H, §6] and [CVX, Corollary 4.8], respectively, that all character sheaves for the symmetric pair (GL_{2n}, Sp_{2n}), and all character sheaves which are not of full support for the symmetric pair (GL_n, O_n), are not cuspidal. To complete the proof we appeal to Theorems 6.2, 6.3, and 6.4. $\square$

7. Proof of Theorems 6.2–6.4

In this section we prove Theorems 6.2, 6.3, and 6.4 by producing new character sheaves via parabolic induction. Combining these character sheaves with nilpotent support ones and full support ones already constructed we have then shown that the sheaves listed in Theorems 6.2, 6.3, and 6.4 are indeed character sheaves. To prove that we have all of them we show that the number of the sheaves we have constructed coincides with the number of irreducible K-equivariant local systems on $\mathcal{N}_1$ (see (2.1)).

Let $\mathcal{O}_{m,k,\mu} \in \underline{\mathcal{N}_1^{\mathrm{cs}}}$, where we write $\mathcal{O}_{0,k,\mu} = \mathcal{O}_{k,\mu}$ in type AIII, CII, DIII. We begin by associating to each pair (G, K) a family of θ-stable parabolic subgroups $P_{m,k,\mu}$ together with their θ-stable Levi subgroups $L_{m,k,\mu}$ such that

$$K.(\mathfrak{p}_{m,k,\mu})_1 = \overline{\check{\mathcal{O}}}_{m,k,\mu},$$

where $(\mathfrak{p}_{m,k,\mu})_1 = \mathfrak{p}_{m,k,\mu} \cap \mathfrak{g}_1$, $\mathfrak{p}_{m,k,\mu} = \mathrm{Lie}\, P_{m,k,\mu}$. We then study the parabolic induction of full support character sheaves in $\mathrm{Char}^{\mathrm{f}}_{L^\theta_{m,k,\mu}}((\mathfrak{l}_{m,k,\mu})_1)$ (see §3.6).

7.1. A family of θ-stable parabolic subgroups of G.

Let $\mathcal{O}_{m,k,\mu} \in \underline{\mathcal{N}_1^{\mathrm{cs}}}$. Consider an *ordered* sequence $a_1, \ldots, a_{n-m-2k}$, with $a_i \in \{0, 1\}$, such that

$$\text{(AIII, CII)} \sum a_i = q - k, \quad \text{(BDI)} \sum a_i = [\frac{q-m-2k}{2}].$$

The ordered sequence (a_i) defines a $\tilde{K}'$-conjugacy class of θ-stable Borel subgroup(s) $B'_{(a_i)}$ for the symmetric pair (G', K')

$$\text{(AIII)}\ (GL_{n-2k}, GL_{p-k} \times GL_{q-k}) \quad \text{(BDI)}\ (SO_{N-2a}, S(O_{p-a} \times O_{q-a}))$$
$$\text{(CI)}\ (Sp_{2n-2m-4k}, GL_{n-m-2k}) \qquad \text{(CII)}\ (Sp_{2n-4k}, Sp_{2p-2k} \times Sp_{2q-2k})$$
$$\text{(DIII)}\ (SO_{2n-4k}, GL_{n-2k}).$$

We can and will choose a sequence (a_i) such that the corresponding Richardson orbit $\mathcal{O}_{B'_{(a_i)}}$ is $\mathcal{O}_\mu$. Moreover, in type BDI, we choose (a_i) such that the corresponding θ-stable Borel subgroup $B'_{(a_i)}$ is as in Proposition 4.6 for $\mathcal{O}_\mu$. We write $B'_\mu = B'_{(a_i)}$.

We fix such a sequence (a_i) and define the parabolic subgroup $P_{m,k,\mu} = P_{m,k,(a_i)}$ to be the subgroup of G that stabilizes a flag of the following form

$$\begin{aligned}
&\text{(AIII)} && 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-m-2k} \subset V \\
&\text{(BDI) (CI)} && 0 \subset V_1 \subset \cdots \subset V_{n-m-2k} \subset V_{n-m} \\
& && \subset V_{n-m}^{\perp} \subset V_{n-m-2k}^{\perp} \subset \cdots \subset V_1^{\perp} \subset V \\
&\text{(CII) (DIII)} && 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-m-2k} \subset V_{n-m-2k}^{\perp} \subset \cdots \subset V_1^{\perp} \subset V,
\end{aligned}$$

where $V_i = V^+_{i-c_i} \oplus V^-_{c_i}$, $1 \leq i \leq n-m-2k$, $V_{n-m} = V^+_{n-m-k-c_{n-m-2k}} \oplus V^-_{k+c_{n-m-2k}}$, $c_i = \sum_{j=1}^{i} a_j$. The parabolic subgroup $P_{m,k,\mu}$ is θ-stable. Let $L_{m,k,\mu} = L_{m,k,(a_i)}$ be the natural θ-stable Levi subgroup of $P_{m,k,\mu}$. We have

$$\begin{aligned}
&\text{(AIII)} && L_{k,\mu} \cong GL_{2k} \times GL_1^{n-2k},\ L^{\theta}_{k,\mu} \cong GL_k \times GL_k \times GL_1^{n-2k}; \\
&\text{(BDI)} && L_{m,k,\mu} \cong SO_{2m+m_0} \times GL_{2k} \times GL_1^{n-m-2k}, \\
& && L^{\theta}_{m,k,\mu} \cong S(O_{m+m_0} \times O_m) \times GL_k \times GL_k \times GL_1^{n-m-2k}; \\
&\text{(CI)} && L_{m,k,\mu} \cong Sp_{2m} \times GL_{2k} \times GL_1^{n-m-2k}, \\
& && L^{\theta}_{m,k,\mu} \cong GL_m \times GL_k \times GL_k \times GL_1^{n-m-2k}; \\
&\text{(CII)} && L_{m,k,\mu} \cong Sp_{4k} \times GL_1^{n-2k},\ L^{\theta}_{m,k,\mu} \cong Sp_{2k} \times Sp_{2k} \times GL_1^{n-2k}; \\
&\text{(DIII)} && L_{m,k,\mu} \cong SO_{4k} \times GL_1^{n-2k},\ L^{\theta}_{m,k,\mu} \cong GL_{2k} \times GL_k^{n-2k},
\end{aligned}$$

where $m_0 = 1$ (resp. 0) in type BI (resp. DI). One readily checks that $K.(\mathfrak{p}_{m,k,\mu})_1 = \overline{\check{\mathcal{O}}}_{m,k,\mu}$.

Note that in type DI, when $m = 0$, the $\tilde{K}$-conjugacy class of $P_{0,k,\mu}$ in G/P decomposes into two K-conjugacy classes; as before, we write $P^{\mathrm{I}}_{0,k,\mu}$ and $P^{\mathrm{I}}_{0,k,\mu}$ for the two θ-stable parabolic subgroups that are conjugate under $\tilde{K}$ but not conjugate under K. Moreover, when $\mu \neq \emptyset$, $K.(\mathfrak{p}^{\mathrm{I}}_{0,k,\mu})_1 = K.(\mathfrak{p}^{\mathrm{II}}_{0,k,\mu})_1$, and when $\mu = \emptyset$, let $K.(\mathfrak{p}^{\omega}_{0,n/2,\emptyset})_1 = \overline{\check{\mathcal{O}}^{\omega}}_{0,n/2,\emptyset}$, $\omega = \mathrm{I}, \mathrm{II}$.

For later use, let us describe the locus $\check{\mathcal{O}}_{m,k,\mu}$ in more detail.

(AIII) Let $x \in \check{\mathcal{O}}_{k,\mu}$. Then x has eigenvalues $\pm a_1, \ldots, \pm a_k$, each of multiplicity 1, and eigenvalue 0 of multiplicity $n-2k$. Moreover, there exist $v_i^+ \in V^+$, $v_i^- \in V^-$, $i = 1, \ldots, k$, such that $V = W_0 \oplus_{i=1}^{k} (W_i \oplus W_{-i})$, where $W_i = \operatorname{span}\{v_i^+ + v_i^-\}$, $W_{-i} = \operatorname{span}\{v_i^+ - v_i^-\}$, $x|W_i = a_i$, $x|_{W_{-i}} = -a_i$, $i = 1, \ldots, k$, and W_0 is the generalized eigenspace of x with eigenvalue 0. We obtain a symmetric pair $(GL_{W_0}, GL_{p-k} \times GL_{q-k})$ and $x|_{W_0} \in \mathcal{O}_\mu$.

(BDI) Let $x \in \check{\mathcal{O}}_{m,k,\mu}$. Then x has eigenvalues $\pm a_1, \ldots, \pm a_m$, each of multiplicity 1, eigenvalues $\pm b_1, \ldots, \pm b_k$, each of multiplicity 2, and eigenvalue 0 of multiplicity $N - 2m - 4k$. Moreover, we have an orthogonal decomposition $V = W_0 \oplus \oplus_{i=1}^{m} W_i \oplus \oplus_{j=1}^{k} U_j$, where W_0 is the generalized eigenspace of x with eigenvalue 0, $W_i = W_{a_i} \oplus W_{-a_i}$, $x|_{W_{a_i}} = a_i$, $x|_{W_{-a_i}} = -a_i$, and $U_j = U_{b_j} \oplus U_{-b_j}$ with $U_{\pm b_j}$ the generalized eigenspace of x with eigenvalue $\pm b_j$. Furthermore, there exist $v_i^+ \in V^+$, $v_i^- \in V^-$, $i = 1, \ldots, m$, such that $W_i = \operatorname{span}\{v_i^+ + v_i^-\}$, $W_{-i} = \operatorname{span}\{v_i^+ - v_i^-\}$, and $(v_i^+, v_i^+) \neq 0$, $(v_i^-, v_i^-) \neq 0$. Similarly, $U_{b_j} = \operatorname{span}\{w_j = w_j^+ + w_j^-, u_j = u_j^+ + u_j^-\}$, $U_{-b_j} = \operatorname{span}\{w_j^+ - w_j^-, u_j^+ - u_j^-\}$, where $xw_j = b_j w_j$, $xu_j = b_j u_j + w_j$, $(w_j^+, w_j^+) = (w_j^-, w_j^-) = 0$, $(w_j^+, u_j^+) \neq 0$ and $(w_j^-, u_j^-) \neq 0$. (The properties about the pairing of the basis elements can be deduced using $(xv, w) + (v, xw) = 0$ and the fact that

x takes V^+ to V^- and vice versa.) Since $(,)$ is nondegenerate on W_0, we get a symmetric pair $(SO_{N-2m-4k}, S(O_{p-m-2k} \times O_{q-m-2k}))$ and $x|_{W_0} \in \mathcal{O}_\mu$.

(CI) The description of $x \in \check{\mathcal{O}}_{m,k,\mu}$ is entirely similar to the case of type BD I except that we have $\langle v_i^+, v_i^- \rangle \neq 0$, $\langle w_j^+, w_j^- \rangle = 0$, $\langle w_j^+, u_j^- \rangle \neq 0$ and $\langle w_j^-, u_j^+ \rangle \neq 0$.

(CII) (DIII) Let $x \in \check{\mathcal{O}}_{k,\mu}$. Then x has eigenvalues $\pm a_1, \ldots, \pm a_k$, each of multiplicity 2, and eigenvalue 0 of multiplicity $2n-4k$. We have an orthogonal decomposition $V = W_0 \oplus \oplus_{i=1}^k W_i$, $W_i = W_{a_i} \oplus W_{-a_i}$, $x|_{W_{a_i}} = a_i$, $x|_{W_{-a_i}} = -a_i$, $\dim W_{a_i} = \dim W_{-a_i} = 2$, and $x|_{W_0} \in \mathcal{O}_\mu$.

7.2. **Equivariant fundamental groups.** In this section, we assume that either $m \neq 0$ or $k \neq 0$. In what follows, to simplify the notation, we omit the subscripts and write $L = L_{m,k,\mu}$, $P = P_{m,k,\mu}$, and $\bar{\check{\mathcal{O}}} = K.\mathfrak{p}_1$ etc. We describe the relations among the equivariant fundamental groups $\pi_1^{P_K}(\mathfrak{p}_1^r)$, $\pi_1^K(\check{\mathcal{O}})$ and $\pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$, where $P_K = P \cap K$, $\mathfrak{p}_1^r = \mathfrak{p}_1 \cap \check{\mathcal{O}}$, and $\mathfrak{l}_1^{rs}$ is the set of regular semisimple elements in $\mathfrak{l}_1$ with respect to the symmetric pair (L, L^θ).

Consider the natural projection map $\mathrm{pr} : \mathfrak{p}_1 \to \mathfrak{l}_1$. From the description of $\check{\mathcal{O}}_{m,k,\mu}$ at the end of the previous section, we see that $\mathrm{pr}(\mathfrak{p}_1^r) = \mathfrak{l}_1^{rs}$. Arguing as in [CVX, §3.2], we see that

$$\text{the canonical map } \Phi : \pi_1^{P_K}(\mathfrak{p}_1^r) \to \pi_1^{L^\theta}(\mathfrak{l}_1^{rs}) \text{ is surjective.} \tag{7.1}$$

We have

$$\pi_1^{L^\theta}(\mathfrak{l}_1^{rs}) \cong \begin{cases} B_{W_k} & \text{(AIII) (CII) (DIII)} \\ \widetilde{B}_{W_m} \times B_{W_k} \text{ (resp. } \widetilde{B}_{W'_m} \times B_{W_k}) & \text{(BI) (CI) (resp. DI).} \end{cases}$$

To determine $\pi_1^{P_K}(\mathfrak{p}_1^r)$ we consider the fibration

$$\mathring{\pi} : K \times^{P_K} \mathfrak{p}_1^r \to \check{\mathcal{O}}$$

which is the smooth part of the map $\pi : K \times^{P_K} \mathfrak{p}_1 \to \bar{\check{\mathcal{O}}}$. We first note that $\pi_1^{P_K}(\mathfrak{p}_1^r) = \pi_1^K(K \times^{P_K} \mathfrak{p}_1^r)$. Now, $\pi_1^K(\check{\mathcal{O}})$ acts on the set $\mathrm{Irr}(\pi^{-1}(x))$ of irreducible components of $\pi^{-1}(x)$, $x \in \check{\mathcal{O}}$, with $\pi_1^K(K \times^{P_K} \mathfrak{p}_1^r)$ as stabilizer. Thus we are reduced to study the action of $\pi_1^K(\check{\mathcal{O}})$ on the set $\mathrm{Irr}(\pi^{-1}(x))$. To understand this action we describe the fiber $\pi^{-1}(x)$, $x \in \check{\mathcal{O}}$.

For $x \in \check{\mathcal{O}} = \check{\mathcal{O}}_{m,k,\mu}$, let $x' = x|_{W_0}$, where W_0 is the generalized eigenspace of x with eigenvalue 0. Let (G', K') be the symmetric pair defined by W_0. Then $x' \in \mathcal{O}_\mu \subset \mathfrak{p}'_1$. As before let $\tilde{K}' = O_{p'} \times O_{q'}$ when (G', K') is of type DI and $\tilde{K}' = K'$ otherwise. Consider

$$\tilde{\pi}_{B'_\mu} : \tilde{K}' \times^{(B'_\mu)_{K'}} (\mathfrak{n}'_\mu)_1 \to \bar{\mathcal{O}}_\mu.$$

Lemma 7.1. *We have*

$$\pi^{-1}(x) \cong \tilde{\pi}_{B'_\mu}^{-1}(x')\,.$$

Proof. We prove the lemma in the case of type BDI. The other cases are entirely similar. We show that the map

$$\pi^{-1}(x) \to \tilde{\pi}_{B'_\mu}^{-1}(x') \qquad (V_1 \subset \cdots \subset V_{n-m-2k} \subset V_{n-m}) \mapsto (V_1 \subset \cdots \subset V_{n-m-2k})$$

is an isomorphism.

Suppose that $(V_1 \subset \cdots \subset V_{n-m-2k} \subset V_{n-m}) \in \pi^{-1}(x)$. We first show that $x|_{V_{n-m-2k}} = 0$, i.e., $V_{n-m-2k} \subset W_0$, the generalized eigenspace of x with eigenvalue

0. Since V_1 is x-stable and by definition either $V_1 \subset V^+$ or $V_1 \subset V^-$, $x|_{V_1} = 0$. Continuing by induction with x applying to V_i/V_{i-1}, which is one dimensional, we conclude that $x|_{V_{n-m-2k}} = 0$. Since $\dim W_0 = N - 2m - 4k$ and $n = [N/2]$, we see that V_{n-m-2k} is a maximal isotropic subspace in W_0. Since $x|_{W_0} = x'$, it follows that $(V_1 \subset \cdots \subset V_{n-m-2k}) \in \tilde{\pi}^{-1}_{B'_\mu}(x')$. It remains to show that V_{n-m} is uniquely determined by V_i, $i = 1, \ldots, n-m-2k$.

We make use of the description of $\check{\mathcal{O}}$ given in the end of §7.1. Recall the orthogonal decomposition $V = W_0 \oplus \oplus_{i=1}^m W_i \oplus \oplus_{j=1}^k U_j$ and the basis vectors defined there. We claim that $V_{n-m} = V_{n-m-2k} \oplus \oplus_{j=1}^k \operatorname{span}\{w_j^+, w_j^-\}$. This can be seen as follows. Consider the map induced by x on V_{n-m}/V_{n-m-2k} and the eigenvalues of x. Note that by definition, if a vector $v = v^+ + v^- \in V_{n-m}$, $v^\pm \in V^\pm$, then $v^+ \in V_{n-m}$ and $v^- \in V_{n-m}$. Thus we conclude that the eigenvalue can not be $\pm a_i$ as $(v_i^+, v_i^-) \neq 0$ and V_{n-m} is isotropic. Now there exists some j such that $w_j^+, w_j^- \in V_{n-m}$. Thus $V_{n-m} \subset \{w_j^+, w_j^-\}^\perp$. Continuing by induction, we see that the claim holds. □

Using the above lemma and Proposition 4.5 we conclude that

(7.2a) $\pi_1^{P_K}(\mathfrak{p}_1^r) = \pi_1^K(\check{\mathcal{O}})$ (DI) when $\mu = \emptyset$, or (AIII), (CI), (CII), (DIII).

It remains to consider the types BI and DI when $\mu \neq \emptyset$. We first observe that $K \times^{P_K} \mathfrak{p}_1^r$ is connected and hence the action of $\pi_1^K(\check{\mathcal{O}})$ on the set of connected components of $\pi^{-1}(x)$ is transitive. Also, by the previous lemma and Proposition 4.6 the cardinality of the set of connected components of the fiber $\pi^{-1}(x)$, and hence that of the quotient $\pi_1^K(\check{\mathcal{O}})/\pi_1^{P_K}(\mathfrak{p}_1^r)$, is 2^{l_μ}. In particular, the action of $\pi_1^K(\check{\mathcal{O}})$ factors through $A_{\tilde{K}'}(x')$. Using the specific splitting of $\pi_1^K(\check{\mathcal{O}})$ as a product in (6.1b), we can write

$$\pi_1^{P_K}(\mathfrak{p}_1^r) \cong \widetilde{B}_{W_m} \times \widetilde{B}_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu - l_\mu} \qquad \text{(BI)} \tag{7.2b}$$

$$\pi_1^{P_K}(\mathfrak{p}_1^r) \cong \widetilde{B}^0_{W_m} \times \widetilde{B}_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu - l_\mu + 1} \qquad \text{(DI) when } \mu \neq \emptyset, \tag{7.2c}$$

where $\widetilde{B}^0_{W_m} = (\mathbb{Z}/2\mathbb{Z})^m \rtimes B^0_{W_m} \subset (\mathbb{Z}/2\mathbb{Z})^m \rtimes B_{W_m}$ and $B^0_{W_m}$ is the subgroup of B_{W_m} defined as the kernel of the map $B_{W_m} \to \mathbb{Z}/2\mathbb{Z}$ where a word in the braid generators $\sigma_1, \ldots, \sigma_m$ (σ_m is the special one) is mapping to 0 (resp. 1) if the braid generator σ_m appears even (resp. odd) number of times. Moreover in type BI (resp. DI), the factor $(\mathbb{Z}/2\mathbb{Z})^{r_\mu - l_\mu}$ (resp. $(\mathbb{Z}/2\mathbb{Z})^{r_\mu - l_\mu + 1}$) is given by $\{a \in A_{K'}(\mathcal{O}_\mu) \mid \phi(a) = 1 \text{ for all } \phi \in \Pi_{\mathcal{O}_\mu}\}$.

Note that in type DI when $\mu \neq \emptyset$, the map Φ in (7.1) gives rise to a surjective map

$$\zeta : \widetilde{B}^0_{W_m} \twoheadrightarrow \widetilde{B}_{W'_m}. \tag{7.3}$$

Remark 7.2. It was previously known that such a surjective map $B^0_{W_m} \to B_{W'_m}$ exists, see, for example, a discussion in [BT].

7.3. **Induced nilpotent orbital complexes.** Let us consider the parabolic induction of full support character sheaves on $\mathfrak{l}_1$. For each irreducible representation ψ of $\pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$, we write ψ also for the irreducible representation of $\pi_1^{P_K}(\mathfrak{p}_1^r)$ obtained via pull-back from the surjective map $\Phi : \pi_1^{P_K}(\mathfrak{p}_1^r) \to \pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$ in (7.1). Then we have:

$$\operatorname{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1 \subset \mathfrak{p}_1} \operatorname{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\psi) \;=\; \pi_! \operatorname{IC}(K \times^{P_K} \mathfrak{p}_1^r, \mathcal{L}_\psi)[-]\,.$$

Furthermore, by the decomposition theorem, we have:

$$\pi_! \operatorname{IC}(K \times^{P_K} \mathfrak{p}_1^r, \mathcal{L}_\psi) \cong \operatorname{IC}(\check{\mathcal{O}}, R^{2d}\mathring{\pi}_!(\mathcal{L}_\psi))[-] \oplus \{\text{other terms}\}\,,$$

where d is the relative dimension of π. By construction we have

$$R^{2d}\mathring{\pi}_!(\mathcal{L}_\psi) \cong \mathbb{C}[\pi_1^K(\check{\mathcal{O}})] \otimes_{\mathbb{C}[\pi_1^{P_K}(\mathfrak{p}_1^r)]} \psi\,.$$

Consider the following decomposition

$$\mathbb{C}[\pi_1^K(\check{\mathcal{O}})] \otimes_{\mathbb{C}[\pi_1^{P_K}(\mathfrak{p}_1^r)]} \psi = \psi_1 \oplus \cdots \oplus \psi_k$$

into irreducible representations of $\pi_1^K(\check{\mathcal{O}})$. By the discussion above we have

$$\bigoplus_{i=1}^{k} \operatorname{IC}(\check{\mathcal{O}}, \mathcal{L}_{\psi_i}) \text{ appears as a direct summand of } \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \operatorname{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\psi) \text{ up to shift.} \tag{7.4}$$

Let us write $A[-] \subseteq_{\oplus} B$ to indicate that A appears as a direct summand of B up to shift.

For each $\tau \in \mathcal{P}(k)$, we continue to write L_τ for the B_{W_k}-representation obtained by pulling back $L_\tau \in \operatorname{Irr} \mathcal{H}_{W_k,1,-1}$ via the surjective map $\mathbb{C}[B_{W_k}] \to \mathcal{H}_{W_k,1,-1}$. The following claims follow from the description of $\pi_1^K(\check{\mathcal{O}})$ in (6.1a)–(6.1c), the description of $\pi_1^{P_K}(\mathfrak{p}_1^r)$ in (7.2a)–(7.2c) and (7.4).

(AIII) (CII) (DIII) For each $\tau \in \mathcal{P}(k)$, let $\mathcal{L}_\tau$ be the irreducible L^θ-equivairant local system on $\mathfrak{l}_1^{rs}$ given by L_τ. Then

$$\operatorname{IC}(\check{\mathcal{O}}, \mathcal{T}_\tau)[-] \subseteq_{\oplus} \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \operatorname{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\tau).$$

(CI) Let $\mathcal{L}_\rho \boxtimes \mathcal{L}_\tau$, $\rho \in \Theta_m^C$ and $\tau \in \mathcal{P}(k)$, be the irreducible L^θ-equivairant local system on $\mathfrak{l}_1^{rs}$ given by the irreducible representation $\rho \boxtimes L_\tau$ of $\pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$, where $\widetilde{B}_{W_m}$ acts via ρ and B_{W_k} acts via L_τ. Then

$$\operatorname{IC}(\check{\mathcal{O}}, \mathcal{T}_{\rho,\tau})[-] \subseteq_{\oplus} \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \operatorname{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\rho \boxtimes \mathcal{L}_\tau).$$

(BI) Let $\mathcal{L}_\rho \boxtimes \mathcal{L}_\tau$, $\rho \in \Theta_m^B$, and $\tau \in \mathcal{P}(k)$, be the irreducible L^θ-equivairant local system on $\mathfrak{l}_1^{rs}$ given by the irreducible representation $\rho \boxtimes L_\tau$ of $\pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$, where $\widetilde{B}_{W_m}$ acts via ρ and B_{W_k} acts via L_τ. Then

$$\bigoplus_{\phi \in \Pi_{\mathcal{O}_\mu}} \operatorname{IC}(\check{\mathcal{O}}, \mathcal{T}_{\rho,\tau,\phi})[-] \subseteq_{\oplus} \operatorname{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \operatorname{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\rho \boxtimes \mathcal{L}_\tau).$$

(DI) Let $\mathcal{L}_{\rho_0} \boxtimes \mathcal{L}_\tau$, $\rho_0 \in \Theta_m^{D,0}$ and $\tau \in \mathcal{P}(k)$, be the irreducible L^θ-equivairant local system on $\mathfrak{l}_1^{rs}$ given by the irreducible representation $\rho_0 \boxtimes L_\tau$ of $\pi_1^{L^\theta}(\mathfrak{l}_1^{rs})$, where $\widetilde{B}_{W_m}$ acts via ρ_0 and B_{W_k} acts via L_τ. Let ρ_0 also denote the representation of $\widetilde{B}_{W_m}^0$ obtained via the surjective map $\widetilde{B}_{W_m}^0 \to \widetilde{B}_{W_m'}$ (see (7.3)). Then, as representations of $\widetilde{B}_{W_m}$, $\mathbb{C}[\widetilde{B}_{W_m}] \otimes_{\mathbb{C}[\widetilde{B}_{W_m}^0]} \rho_0 = \rho_0^{\mathrm{I}} \oplus \rho_0^{\mathrm{II}}$, where ρ_0^{I} and ρ_0^{II} are non-isomorphic irreducible representations of $\widetilde{B}_{W_m}$. In view of Proposition 5.2, we conclude that

$\rho_0^{\mathrm{I}}, \rho_0^{\mathrm{II}} \in \Theta_m^D$. Moreover, $\Theta_m^D = \{\rho_0^{\mathrm{I}}, \rho_0^{\mathrm{II}} \mid \rho_0 \in \Theta_m^{D,0}\}$. Thus

$$\bigoplus_{\phi \in \Pi_{\mathcal{O}_\mu}} \bigoplus_{\omega=\mathrm{I},\mathrm{II}} \mathrm{IC}(\check{\mathcal{O}}, \mathcal{T}_{\rho_0^\omega,\tau,\phi})[-] \oplus \mathrm{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_{\rho_0} \boxtimes \mathcal{L}_\tau) \text{ when } \mu \neq \emptyset;$$

$$\mathrm{IC}(\check{\mathcal{O}}_{m,k,\emptyset}, \mathcal{T}_{\rho,\tau})[-] \oplus \mathrm{Ind}_{\mathfrak{l}_1 \subset \mathfrak{p}_1}^{\mathfrak{g}_1} \mathrm{IC}(\mathfrak{l}_1^{rs}, \mathcal{L}_\rho \boxtimes \mathcal{L}_\tau), \text{ when } \mu = \emptyset \text{ and } m > 0.$$

$$\mathrm{IC}(\check{\mathcal{O}}_{0,n/2,\emptyset}^\omega, \mathcal{T}_\tau)[-] \oplus \mathrm{Ind}_{\mathfrak{l}_1^\omega \subset \mathfrak{p}_1^\omega}^{\mathfrak{g}_1} \mathrm{IC}((\mathfrak{l}_1^\omega)^{rs}, \mathcal{L}_\tau), \ \omega = \mathrm{I}, \mathrm{II}, \text{ when } \mu = \emptyset \text{ and } m = 0.$$

Since Fourier transform commutes with parabolic induction (see (3.24)), in view of (3.9), we conclude that the IC sheaves in Theorems 6.2–6.4 are character sheaves. Since they are pairwise non-isomorphic, to prove Theorems 6.2–6.4, it remains to check that the number of the IC sheaves equals the number of character sheaves in each case.

7.4. **Proof of Theorem 6.2.** In each case, we prove the theorem by establishing an explicit bijection of the two sides.

7.4.1. Suppose that $(G, K) = (GL_n, GL_p \times GL_q)$. Given a signed Young diagram $\lambda = (\lambda_1)_+^{p_1}(\lambda_1)_-^{q_1} \cdots (\lambda_s)_+^{p_s}(\lambda_s)_-^{q_s}$ of signature (p, q), we associate it with the following data

(1) the partition $\alpha = (\lambda_1)^{l_1} \cdots (\lambda_s)^{l_s} \in \mathcal{P}(k)$, $l_i = \min\{p_i, q_i\}$, $k = \sum \lambda_i l_i$
(2) the signed Young diagram of signature $(p - k, q - k)$

$$\mu = (\lambda_1)_+^{p_1 - l_1}(\lambda_1)_-^{q_1 - l_1} \cdots (\lambda_s)_+^{p_s - l_s}(\lambda_s)_-^{q_s - l_s}.$$

The map

$$\mathrm{IC}(\mathcal{O}_\lambda, \mathbb{C}) \mapsto \mathrm{IC}(\check{\mathcal{O}}_{1_+^k 1_-^k \sqcup \mu}, \mathcal{T}_{\alpha^t}) \text{ for all } \lambda;$$

where α^t denotes the transpose partition of α, establishes the required bijection.

7.4.2. Suppose that $(G, K) = (Sp_{2n}, Sp_{2p} \times Sp_{2q})$, $q \leq n$. Let

$$\lambda = (2\lambda_1)_+^{a_1}(2\lambda_1)_-^{a_1} \cdots (2\lambda_s)_+^{a_s}(2\lambda_s)_-^{a_s}(2\mu_1+1)_+^{2p_1}(2\mu_1+1)_-^{2q_1} \cdots (2\mu_t+1)_+^{2p_t}(2\mu_t+1)_-^{2q_t}$$

be a signed Young diagram of signature $(2p, 2q)$. Let $l_i = \min\{p_i, q_i\}$ and $k = \sum 2[\frac{a_i}{2}]\lambda_i + \sum l_i(2\mu_i + 1)$. We associate to λ the following data

(1) the partition $\alpha = (2\lambda_1)^{[\frac{a_1}{2}]} \cdots (2\lambda_s)^{[\frac{a_s}{2}]}(2\mu_1 + 1)^{l_1} \cdots (2\mu_t + 1)^{l_t} \in \mathcal{P}(k)$,
(2) the signed Young diagram of signature $(2p - 2k, 2q - 2k)$

$$\mu = (2\lambda_1)_+^{a_1 - 2[\frac{a_1}{2}]}(2\lambda_1)_-^{a_1 - 2[\frac{a_1}{2}]} \cdots (2\lambda_s)_+^{a_s - 2[\frac{a_s}{2}]}(2\lambda_s)_-^{a_s - 2[\frac{a_s}{2}]}$$
$$(2\mu_1 + 1)_+^{2p_1 - 2l_1}(2\mu_1 + 1)_-^{2q_1 - 2l_1} \cdots (2\mu_t + 1)_+^{2p_t - 2l_t}(2\mu_t + 1)_-^{2q_t - 2l_t}.$$

The map

$$\mathrm{IC}(\mathcal{O}_\lambda, \mathbb{C}) \mapsto \mathrm{IC}(\check{\mathcal{O}}_{1_+^{2k} 1_-^{2k} \sqcup \mu}, \mathcal{T}_{\alpha^t})$$

establishes the required bijection.

7.4.3. Suppose that $(G,K)=(SO_{2n},GL_n)$. Let

$\lambda=(2\lambda_1+1)_+^{a_1}(2\lambda_1+1)_-^{a_1}\cdots(2\lambda_s+1)_+^{a_s}(2\lambda_s+1)_-^{a_s}(2\mu_1)_+^{2p_1}(2\mu_1)_-^{2q_1}\cdots(2\mu_t)_+^{2p_t}(2\mu_t)_-^{2q_t}$

be a signed Young diagram of signature (n,n). Let $l_i=\min\{p_i,q_i\}$ and $k=\sum[\frac{a_i}{2}](2\lambda_i+1)+\sum 2\mu_i l_i$. We associate to the above Young diagram the following data

(1) the partition $\alpha=(2\lambda_1+1)^{[\frac{a_1}{2}]}\cdots(2\lambda_s+1)^{[\frac{a_s}{2}]}(2\mu_1)^{l_1}\cdots(2\mu_t)^{l_t}\in\mathcal{P}(k)$,
(2) the signed Young diagram of signature $(n-2k,n-2k)$

$$\mu=(2\lambda_1+1)_+^{a_1-2[\frac{a_1}{2}]}(2\lambda_1+1)_-^{a_1-2[\frac{a_1}{2}]}\cdots(2\lambda_s+1)_+^{a_s-2[\frac{a_s}{2}]}(2\lambda_s+1)_-^{a_s-2[\frac{a_s}{2}]}$$
$$(2\mu_1)_+^{2p_1-2l_1}(2\mu_1)_-^{2q_1-2l_1}\cdots(2\mu_t)_+^{2p_t-2l_t}(2\mu_t+1)_-^{2q_t-2l_t}.$$

The map

$$\mathrm{IC}(\mathcal{O}_\lambda,\mathbb{C})\mapsto\mathrm{IC}(\check{\mathcal{O}}_{1_+^{2k}1_-^{2k}\sqcup\mu},\mathcal{T}_{\alpha^t})$$

establishes the required bijection.

Let us record that

$$|\mathcal{A}_{GL_n}((\mathfrak{so}_{2n})_1)|=\sum_{k=0}^{[\frac{n}{2}]}\mathbf{p}(k)\mathbf{p}(l,k)2^k\mathbf{do}(n-2l)=\text{Coefficient of }x^n\text{ in }\prod_{s\geq1}\frac{1+x^s}{(1-x^{2s})^2}$$

where $\mathbf{p}(k)$ denotes the number of partitions of k, $\mathbf{p}(l,k)$ denotes the number of partitions of l into (not necessarily distinct) parts of exactly k different sizes and $\mathbf{do}(l)$ denotes the number of partitions of l into distinct odd parts. Here we have used

$$\sum_k\mathbf{p}(n,k)\,2^k=\text{Coefficient of }x^n\text{ in }\prod_{s\geq1}\frac{1+x^s}{1-x^s}. \tag{7.5}$$

7.5. **Proof of Theorem 6.3.** Let $(G,K)=(Sp_{2n},GL_n)$.

Lemma 7.3. *We have*

$$|\mathcal{A}_{GL_n}((\mathfrak{sp}_{2n})_1)|=\textit{Coefficient of }x^n\textit{ in }\prod_{s\geq1}\frac{(1+x^s)^3}{(1-x^s)^2}. \tag{7.6}$$

Proof. Let us write $\mathcal{P}(C_n)$ for the set of partitions of $2n$ such that each odd part occurs with an even multiplicity. Given a partition

$$\lambda=(2\lambda_1)^{m_1}\cdots(2\lambda_s)^{m_s}(2\lambda_{s+1}+1)^{2m_{s+1}}\cdots(2\lambda_t+1)^{2m_t}\in\mathcal{P}(C_n),$$

using (2.7) we see that the number of orbital complexes $\mathrm{IC}(\mathcal{O},\mathcal{E})$ supported on the nilpotent orbits whose underlying Young diagrams have shape λ is $(\prod_{i=1}^s m_i)4^s:=M_\lambda$. Thus we have that

$$|\mathcal{A}_{GL_n}((\mathfrak{sp}_{2n})_1)|=\sum_{\lambda\in\mathcal{P}(C_n)}M_\lambda.$$

For a partition $\mu=(\mu_1)^{m_1}\cdots(\mu_s)^{m_s}$, we define $N_\mu=(\prod_{i=1}^s m_i)4^s$. Then we have

$$\sum_{\lambda\in\mathcal{P}(C_n)}M_\lambda=\sum_k\sum_{\mu\in\mathcal{P}(k)}N_\mu\cdot\mathbf{od}(n-k),$$

where $\mathbf{od}(l)$ denotes the number of the partitions of l into odd parts. Since the number of partition of n into distinct parts equals the number of partitions of n into odd parts, we have $\sum \mathbf{od}(l)x^l = \prod_{s\geq 1}(1+x^s)$. The lemma follows from

$$\sum_{\mu\in\mathcal{P}(n)} N_\mu = \text{Coefficient of } x^n \text{ in } \prod_{s\geq 1}\frac{(1+x^s)^2}{(1-x^s)^2}.$$

This can be seen using (7.5) and the following observation. We can rewrite a partition $(\mu_1)^{m_1}\cdots(\mu_s)^{m_s}$ into the union of two partitions $(\mu_1)^{m_1'}\cdots(\mu_s)^{m_s'}$ and $(\mu_1)^{m_1-m_1'}\cdots(\mu_s)^{m_s-m_s'}$, where $0\leq m_i'\leq m_i$. Thus we get

$$\left(1+\sum_{j=1}^{s}\prod_{1\leq i_1<i_2<\cdots<i_j\leq s}(m_{i_1}-1)\cdots(m_{i_j}-1)\right)4^s = \left(\prod_{i=1}^{s} m_i\right)4^s.$$

□

Let us write $b_C(n) = |\,\mathrm{SYD}^0_{(Sp_{2n},GL_n)}\,|$. Using Proposition 4.1 one readily checks that

$$b_C(n) = \sum_k \mathbf{p}(n,k)2^k = \text{coefficient of } x^n \text{ in } \prod_{s\geq 1}\frac{1+x^s}{1-x^s}.$$

Using (5.8) and (2.9), we see that the number of IC sheaves in Theorem 6.3 is

$$\sum_{m=0}^{n}|\Theta_m^C|\sum_{k=0}^{[\frac{n-m}{2}]}\mathbf{p}(k)\,b_C(n-m-2k) = \text{Coefficient of } x^n$$
$$\text{in } \prod_{s\geq 1}(1+x^s)^3\prod_{s\geq 1}\frac{1}{1-x^{2s}}\prod_{s\geq 1}\frac{1+x^s}{1-x^s} \overset{(7.6)}{=} |\,\mathrm{Char}_{GL_n}((\mathfrak{sp}_{2n})_1)|.$$

7.6. **Proof of Theorem 6.4.** Let $(G,K) = (SO_N, S(O_p\times O_q))$. We write $\mathrm{SYD}_{p,q}$ for the set of signed Young diagrams that parametrizes K-orbits in $\mathcal{N}_1$. Let us also write

$$A_{p,q} = |\mathcal{A}_{S(O_p\times O_q)}((\mathfrak{so}_{p+q})_1)|. \tag{7.7}$$

Since $A_K(\mathcal{O}_\lambda) = (\mathbb{Z}/2\mathbb{Z})^{r_\lambda}$ by (2.6), we have

$$A_{p,q} = \sum_{\lambda\in\mathrm{SYD}_{p,q}} 2^{r_\lambda} + \mathbf{p}(\tfrac{q}{2})\,\delta_{p,q},$$

where $\mathbf{p}(k)$ is the number of partitions of k and we set $\mathbf{p}(\frac{q}{2}) = 0$ if q is odd. We note that the second term in the above equation arises only when $p=q$ is even. In the latter case there are two nilpotent orbits corresponding to each partition with only even parts.

The formula in the following proposition is derived and proved by Dennis Stanton.

Proposition 7.4.

$$\sum_{p,q=0}^{\infty}2A_{p,q}u^pv^q - 3\sum_{q=0}^{\infty}\mathbf{p}(q)u^{2q}v^{2q} = \prod_{k=1}^{\infty}\frac{1}{1-u^{2k}v^{2k}}\prod_{m=0}^{\infty}\frac{(1+u^{m+1}v^m)(1+u^mv^{m+1})}{(1-u^{m+1}v^m)(1-u^mv^{m+1})}.$$

Proof. The proposition follows from Proposition B.2 in Appendix B, since by (7.7) and (B.1) we have

$$\begin{aligned} wt(p,q) &= 2A_{p,q} \text{ if } p+q \text{ is odd,} \\ wt(p,q) &= 2A_{p,q} - 3\mathbf{p}(\tfrac{q}{2})\,\delta_{p,q} \text{ if } p+q \text{ is even.} \end{aligned} \tag{7.8}$$

□

Suppose that either p or q is even. Let us now write

$$b_{p,q} = \sum_{\lambda\in \mathrm{SYD}^0_{(SO_{p+q},S(O_p\times O_q))}} |\Pi_{\mathcal{O}_\lambda}|$$

where $\Pi_{\mathcal{O}}$ is defined in (4.4). Namely, $b_{p,q}$ is the number of $\mathrm{IC}(\mathcal{O},\mathcal{E}_\phi)$'s such that $\mathcal{O}\in \underline{\mathcal{N}_1^0}$ and $\phi\in\Pi_{\mathcal{O}}$. Note that we have $b_{p,q}=b_{q,p}$.

Let us write $\mathcal{P}^{odd}(N)$ for the set of partitions of N into odd parts. We write a partition $\lambda\in\mathcal{P}^{odd}(N)$ as

$$\lambda = (2\mu_1+1)+(2\mu_2+1)+\cdots+(2\mu_s+1)$$

where $\mu_1\geq\mu_2\geq\cdots\geq\mu_s\geq 0$. Note that $s\equiv N \mod 2$. We set

$$wt_\lambda = 3^{\#\{1\leq j\leq (s-1)/2\,|\,\mu_{2j-1}=\mu_{2j}+1\}}\,4^{\#\{1\leq j\leq (s-1)/2\,|\,\mu_{2j-1}\geq\mu_{2j}+2\}} \text{ when } N \text{ is odd;}$$

$$wt_\lambda = 3^{\#\{1\leq j\leq s/2-1\,|\,\mu_{2j}=\mu_{2j+1}+1\}}\,4^{\#\{1\leq j\leq s/2-1\,|\,\mu_{2j}\geq\mu_{2j+1}+2\}} \text{ when } N \text{ is even.}$$

We then have

$$\sum_{p=0}^{2n+1} b_{p,2n+1-p} = 2\sum_{\lambda\in\mathcal{P}^{odd}(2n+1)} wt_\lambda,\qquad \sum_{p=0}^{n} b_{2p,2n-2p} = 2\sum_{\lambda\in\mathcal{P}^{odd}(2n)} wt_\lambda.$$

Proposition 7.5. *We have*

$$\sum_{p=0}^{2n+1} b_{p,2n+1-p} = \textit{Coefficient of } x^{2n+1} \textit{ in } 2x\prod_{k\geq1}(1+x^{4k})^2(1+x^{2k})^2 \tag{7.9a}$$

$$\sum_{p=0}^{n} b_{2p,2n-2p} = \textit{Coefficient of } x^{2n} \textit{ in } \frac{1}{2}\prod_{k\geq1}(1+x^{4k-2})^2(1+x^{2k})^2. \tag{7.9b}$$

Proof. See §B.2 in Appendix B. □

Let us further write $f_{m+1,m}=|\Theta^B_m|$ and $f_{m,m}=|\Theta^{D,0}_m|$. It follows from (2.9), (5.8) and (5.9) that

$$\sum_{m\geq0} f_{m+1,m}x^m = \prod_{k\geq1}(1+x^{2k})^2(1+x^k)^2 \tag{7.10a}$$

$$\sum_{m\geq0} f_{m,m}x^m = \frac{1}{2}\prod_{k\geq1}(1+x^{2k-1})^2(1+x^k)^2. \tag{7.10b}$$

Note that in the above we have used the notation that $b_{0,0}=1/2$ and $f_{0,0}=1/2$.

Let $A'_{p,q}$ denote the number of IC sheaves in Theorem 6.4. Recall $r=\min\{p,q\}$. We have

$$A'_{p,q} = \sum_{m=0}^{r} f_{m+1,m}\sum_{k=0}^{[\frac{r-m}{2}]}\mathbf{p}(k)\,b_{p-m-2k,q-m-2k},\quad \text{if } p+q\equiv 1 \mod 2, \tag{7.11a}$$

$$A'_{p,q} = 2\sum_{m=0}^{\frac{r-1}{2}} f_{2m+1,2m+1} \sum_{k=0}^{\frac{r-2m-1}{2}} \mathbf{p}(k)\, b_{p-2m-2k-1,q-2m-2k-1}, \quad \text{if } p \equiv q \equiv 1 \mod 2, \tag{7.11b}$$

$$A'_{p,q} = 2\sum_{m=0}^{\frac{q}{2}} f_{2m,2m} \sum_{k=0}^{\frac{q-2m}{2}} \mathbf{p}(k)\, b_{p-2m-2k,q-2m-2k} + \frac{3}{2}\mathbf{p}(\frac{q}{2})\delta_{p,q}, \quad \text{if } p \equiv q \equiv 0 \mod 2. \tag{7.11c}$$

To prove Theorem 6.4, it suffices to show that

$$\sum_{p=0}^{2n+1} A_{p,2n+1-p} = \sum_{p=0}^{2n+1} A'_{p,2n+1-p}, \qquad \sum_{p=0}^{2n} A_{p,2n-p} = \sum_{p=0}^{2n} A'_{p,2n-p} \tag{7.12}$$

since by construction $A'_{p,2n+1-p} \leq A_{p,2n+1-p}$ and $A'_{p,2n-p} \leq A_{p,2n-p}$. It then follows that

$$A_{p,q} = A'_{p,q}. \tag{7.13}$$

In what follows we prove (7.12). Setting $u = v = x$ in Proposition 7.4 we see that

$$\begin{aligned} \sum_{p=0}^{2n+1} A_{p,2n+1-p} = {} & \text{Coefficient of } x^{2n+1} \text{ in} \\ & \frac{1}{2}\prod_{k\geq 1}\frac{1}{1-x^{4k}}\prod_{m\geq 1}\frac{(1+x^{2m-1})^2}{(1-x^{2m-1})^2} \end{aligned} \tag{7.14a}$$

$$\begin{aligned} \sum_{p=0}^{2n} A_{p,2n-p} = {} & \text{Coefficient of } x^{2n} \text{ in} \\ & \frac{1}{2}\prod_{k\geq 1}\frac{1}{1-x^{4k}}\prod_{m\geq 1}\frac{(1+x^{2m-1})^2}{(1-x^{2m-1})^2} + \frac{3}{2}\prod_{k\geq 1}\frac{1}{1-x^{4k}}. \end{aligned} \tag{7.14b}$$

The equations (7.9a), (7.10a), and (7.11a) imply that

$$\begin{aligned} \sum_{p=0}^{2n+1} A'_{p,2n+1-p} = {} & \text{Coefficient of } x^{2n+1} \text{ in } \left(\prod_{k\geq 1}(1+x^{4k})^2(1+x^{2k})^2\right) \\ & \cdot\left(\prod_{k\geq 1}\frac{1}{1-x^{4k}}\right)\left(2x\prod_{m\geq 1}(1+x^{4m})^2(1+x^{2m})^2\right). \end{aligned} \tag{7.15a}$$

The equations (7.9b), (7.10b), (7.11b) and (7.11c) imply that

$$\begin{aligned} \sum_{p=0}^{2n} A'_{p,2n-p} = {} & \text{Coefficient of } x^{2n} \text{ in } 2\left(\frac{1}{2}\prod_{k\geq 1}(1+x^{4k-2})^2(1+x^{2k})^2\right) \\ & \cdot\left(\prod_{k\geq 1}\frac{1}{1-x^{4k}}\right)\cdot\left(\frac{1}{2}\prod_{m\geq 1}(1+x^{4m-2})^2(1+x^{2m})^2\right) + \frac{3}{2}\prod_{k\geq 1}\frac{1}{1-x^{4k}}. \end{aligned} \tag{7.15b}$$

Let us now write $F(x)=\prod_{m\geq 1}\frac{(1+x^{2m-1})^2}{(1-x^{2m-1})^2}$. Then (7.12) follows from (7.14a), (7.14b), (7.15a), (7.15b) and the following identities

(7.16a) $$F(x)-F(-x)=8\,x\prod_{k\geq 1}(1+x^{4k})^4(1+x^{2k})^4$$

(7.16b) $$F(x)+F(-x)=2\prod_{k\geq 1}(1+x^{4k-2})^4(1+x^{2k})^4.$$

The proof of equations (7.16a) and (7.16b) is given in §B.3 in Appendix B. This completes the proof of Theorem 6.4.

Corollary 7.6. *Let* $(G,K)=(SO_N,S(O_p\times O_q))$.

(1) *The number of character sheaves*

$$|\operatorname{Char}_K(\mathfrak{g}_1)|=\textit{coefficient of } x^q \textit{ in } \frac{1}{1+x^{p-q}}\prod_{s\geq 1}\frac{1+x^s}{(1-x^s)^3}+\frac{3\delta_{p,q}}{2}\prod_{s\geq 1}\frac{1}{1-x^{2s}}.$$

(2) *Suppose that either p or q is even. The number of nilpotent support character sheaves,*

$$|\operatorname{Char}_K^{\mathrm{n}}(\mathfrak{g}_1)| = \textit{coefficient of } x^q \textit{ in } \begin{cases}\displaystyle\frac{1}{1+x^{p-q}}\prod_{s\geq 1}\frac{(1+x^{2s-1})^2}{(1-x^{2s})^2} & \textit{if } N \textit{ is odd,}\\ \displaystyle\frac{1}{1+x^{p-q}}\prod_{s\geq 1}\frac{(1+x^{2s})^2}{(1-x^{2s})^2} & \textit{if } p\equiv q\equiv 0 \bmod 2.\end{cases}$$

Proof. We can assume that $p\geq q$. The case when $q<p$ follows readily. Part (1) follows from (7.8) and Corollary B.3 in Appendix B, as $|\operatorname{Char}_K(\mathfrak{g}_1)|=A_{p,q}$.

We prove part (2) in the case of p and q both even. The other case is entirely similar and simpler. Fix $2l=p-q\geq 0$. It follows from (7.11b), (7.11c) and (7.13) that

$$\sum_q A_{q+2l,q}\,x^q=2\left(\sum_m f_{m,m}x^m\right)\left(\sum_k \mathbf{p}(k)x^{2k}\right)\left(\sum_q b_{2q+2l,2q}x^{2q}\right)+\frac{3}{2}\delta_{l,0}\sum_q \mathbf{p}(q)x^{2q}.$$

By part (1), we have that

$$\sum_q A_{q+2l,q}\,x^q=\frac{1}{1+x^{2l}}\prod_{s\geq 1}\frac{1+x^s}{(1-x^s)^3}+\frac{3}{2}\delta_{l,0}\prod_{s\geq 1}\frac{1}{1-x^{2s}},$$

Thus using (7.10b), we obtain

$$\begin{aligned}\sum_q b_{2q+2l,2q}x^{2q} &= \frac{1}{1+x^{2l}}\prod_{s\geq 1}\frac{(1+x^s)(1-x^{2s})}{(1-x^s)^3(1+x^{2s-1})^2(1+x^s)^2}\\ &=\frac{1}{1+x^{2l}}\prod_{s\geq 1}\frac{(1+x^{2s})^2}{(1-x^{2s})^2}.\end{aligned}$$

$\square$

Appendix A. Dual strata

We will describe the dual strata of nilpotent orbits. Let G be a reductive algebraic group which for the purposes of this section can be assumed to be semisimple. We write $\mathfrak{g}$ for its Lie algebra and $\mathcal{N}$ for the nilpotent cone. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. For each nilpotent G-orbit $\mathcal{O}$ in $\mathcal{N}$ we consider its conormal bundle

$$\Lambda_{\mathcal{O}} = T^*_{\mathcal{O}}\mathfrak{g} = \{(x,y) \in \mathfrak{g} \times \mathfrak{g} \mid x \in \mathcal{O} \ \ [x,y] = 0\}\,.$$

Consider the projection $\widetilde{\mathcal{O}}$ of $\Lambda_{\mathcal{O}}$ to the second coordinate:

$$\widetilde{\mathcal{O}} = \{y \in \mathfrak{g} \mid \text{there exist an } \ x \in \mathcal{O} \ \text{ with } \ [x,y] = 0\}\,.$$

We construct an open (dense) subset $\check{\mathcal{O}}$ of $\widetilde{\mathcal{O}}$ such that the projection $\Lambda_{\mathcal{O}} \to \widetilde{\mathcal{O}}$ has constant maximum rank above $\check{\mathcal{O}}$. Thus the $\check{\mathcal{O}}$ are subvarieties of $\mathfrak{g}$ and they have the following property:

For any $\mathcal{F} \in \mathrm{P}_G(\mathcal{N})$ the Fourier transform $\mathfrak{F}(\mathcal{F})$ is smooth along all the $\check{\mathcal{O}}$.

This property follows from the fact that the Fourier transform preserves the singular support.

Before giving the general construction we make a few general comments. If $\mathcal{O} = \{0\}$ is the zero orbit then $\widetilde{\mathcal{O}} = \mathfrak{g}$ and $\check{\mathcal{O}} = \mathfrak{g}^{rs}$, the set of regular semisimple elements in $\mathfrak{g}$. The general description of $\check{\mathcal{O}}$ will be similar.

Let $\mathfrak{t}$ be a Cartan subspace of $\mathfrak{g}$ and W the Weyl group of G. Cconsider the adjoint quotient $\mathfrak{g} \xrightarrow{f} \mathfrak{g}/\!/G \cong \mathfrak{t}/W$. We have:

$$\begin{array}{ccccc}
\widetilde{\mathcal{O}} & \xrightarrow{\tilde{f}} & f(\widetilde{\mathcal{O}}) & \xrightarrow{\sim} & \mathfrak{t}^{\phi}/W^{\phi} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathfrak{g} & \xrightarrow{f} & \mathfrak{g}/\!/G & \xrightarrow{\sim} & \mathfrak{t}/W
\end{array}$$

where the vertical arrows are inclusions. We first analyze $f(\widetilde{\mathcal{O}})$ and explain the upper righthand corner. Let $\mathfrak{g}^{ss}$ denote the set of semisimple elements in $\mathfrak{g}$ and

$$\widetilde{\mathcal{O}}^{ss} \ = \ \{y \in \mathfrak{g}^{ss} \mid \text{there exist an } \ x \in \mathcal{O} \ \text{ with } \ [x,y] = 0\}$$

denote the semisimple locus of $\widetilde{\mathcal{O}}$. Then we have $f(\widetilde{\mathcal{O}}) = f(\widetilde{\mathcal{O}}^{ss})$ because we can regard $\mathfrak{g}/\!/G \cong \mathfrak{t}/W$ as consisting of the set of semisimple G-orbits in $\mathfrak{g}$. Consider an element $e \in \mathcal{O}$ and consider a corresponding $\mathfrak{sl}_2$-triple $\phi = (e,f,h)$. Then we have

$$\mathfrak{g}^e = \mathfrak{g}^{\phi} \oplus \mathfrak{u}^e,$$

where

$$\text{(A.1)} \qquad \mathfrak{g}^{\phi} = \mathfrak{g}^e \cap \mathfrak{g}^h, \ \ \mathfrak{u} = \oplus_{i \geq 1}\mathfrak{g}(i), \ \mathfrak{g}(i) = \{z \in \mathfrak{g} \,|\, [h,z] = iz\}.$$

Recall that $\mathfrak{g}^{\phi}$ is reductive and $\mathfrak{u}$ is nilpotent. In particular, the algebra $\mathfrak{g}^{\phi}$ is the centralizer of the $\mathfrak{sl}_2$ given by $\phi = (e,f,h)$ and we write G^{ϕ} for the corresponding centralizer group.

We observe that any element in $\widetilde{\mathcal{O}}^{ss}$ is G-conjugate to an element in $(\mathfrak{g}^e)^{ss}$. Thus, we see that:

$$f(\widetilde{\mathcal{O}}) = f(\widetilde{\mathcal{O}}^{ss}) = \widetilde{\mathcal{O}}^{ss}/G = (\mathfrak{g}^e)^{ss}/G^e\,.$$

Furthermore, any element in $(\mathfrak{g}^e)^{ss}$ is G^e-conjugate to an element in $(\mathfrak{g}^\phi)^{ss}$. As $G^e = G^\phi \ltimes U^e$ is a semidirect product of a reductive group and a unipotent group it is easy to see that if two elements in $(\mathfrak{g}^\phi)^{ss}$ are G^e-conjugate they are also G^ϕ-conjugate. Thus we conclude that $(\mathfrak{g}^e)^{ss}/G^e = (\mathfrak{g}^\phi)^{ss}/G^\phi$. Finally, putting this all together, we get:

$$f(\widetilde{\mathcal{O}}) = (\mathfrak{g}^e)^{ss}/G^e = (\mathfrak{g}^\phi)^{ss}/G^\phi = \mathfrak{g}^\phi /\!/ G^\phi\,.$$

Let $\mathfrak{t}^\phi \subset \mathfrak{g}^\phi$ be a maximal abelian subspace such that every semisimple element in $\mathfrak{g}^\phi$ is G^ϕ-conjugate to some element in $\mathfrak{t}^\phi$. We choose $\mathfrak{t}^\phi$ such that it lies in $\mathfrak{t}$ and adjust $\mathfrak{t}$ so that $h \in \mathfrak{t}$. Thus we have $\mathfrak{t}^\phi = \mathfrak{t}^e$. Writing Φ for the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and writing $e = \sum_{\alpha\in\Phi} a_\alpha x_\alpha$, $x_\alpha \in \mathfrak{g}_\alpha$, we have a concrete expression for $\mathfrak{t}^\phi$:

$$\mathfrak{t}^\phi \;=\; \bigcap_{\alpha\in\Phi_e} \{t \in \mathfrak{t} \mid \alpha(t) = 0\} \qquad \Phi_e = \{\alpha \in \Phi \mid a_\alpha \neq 0\} \ \text{ with } e = \sum_{\alpha\in\Phi} a_\alpha x_\alpha\,,$$

i.e., $\mathfrak{t}^\phi$ is given by the intersection of the root hyperplanes for those roots that occur in the expression of e. Write

$$W^\phi = N_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi).$$

Thus, we conclude that

$$f(\widetilde{\mathcal{O}}) = \mathfrak{g}^\phi /\!/ G^\phi = \mathfrak{t}^\phi/W^\phi.$$

Let us call the composition map $\tilde{f} : \widetilde{\mathcal{O}} \to \mathfrak{t}^\phi/W^\phi$. Note that the equality $\widetilde{\mathcal{O}}^{ss}/G = (\mathfrak{g}^\phi)^{ss}/G^\phi$, which follows from the arguments above, shows that

$$W^\phi = N_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi) = N_G(\mathfrak{t}^\phi)/Z_G(\mathfrak{t}^\phi) = N_G(L)/L\,, \tag{A.2}$$

where $L = Z_G(\mathfrak{t}^\phi)$. Note that W^ϕ is not necessarily a Coxeter group.

In what follows we write $x = x_s + x_n$ for the decomposition of an element in its semisimple and nilpotent parts and then $[x_s, x_n] = 0$. We now define

$$\check{\mathcal{O}} = \{y \in \widetilde{\mathcal{O}} \mid y = y_s + y_n,\ \ \tilde{f}(y) \in (\mathfrak{t}^\phi)^{rs}/W^\phi,\ \ y_n \in \mathcal{O}\}\,;$$

here the $(\mathfrak{t}^\phi)^{rs}$ stands for elements in $\mathfrak{t}^\phi$ that are regular semisimple in the ambient algebra $\mathfrak{g}^\phi$. By construction, the group G acts on $\check{\mathcal{O}}$. To show that $\check{\mathcal{O}}$ has the desired property we first claim:

Lemma A.1. *If $x = x_s + x_n \in \mathfrak{g}^e$ and $x_s \in (\mathfrak{t}^\phi)^{rs}$, then $x_n \in \bar{\mathcal{O}}$.*

Proof. First, recall that e is distinguished if all elements in $\mathfrak{g}^e$ are nilpotent. By [P, Theorem 1], if e is distinguished and $x \in \mathfrak{g}^e$, then $x \in \bar{\mathcal{O}}$.

Let now e be arbitrary. We have $x = x_s + x_n \in \mathfrak{g}^e$ and $x_s \in (\mathfrak{t}^\phi)^{rs}$. Then also $x_n \in \mathfrak{g}^e$. But, now $G^{x_s} = Z_G(\mathfrak{t}^\phi)$ because x_s is semisimple. As $[x_s, x_n] = 0$, $x_n \in Z_{\mathfrak{g}}(\mathfrak{t}^\phi)$. Consider the group $H = Z_G(\mathfrak{t}^\phi)/T^\phi$. Let us write $\bar{e}$ for e regarded as a nilpotent element in $\mathfrak{h} = \mathrm{Lie}(H)$. The element $\bar{e}$ is a distinguished nilpotent in $\mathfrak{h}$. To see this it suffices to show that if $y \in Z_{\mathfrak{g}}(\mathfrak{t}^\phi) \cap \mathfrak{g}^e$ is semisimple then $y \in \mathfrak{t}^\phi$. Since $\mathfrak{t}^\phi$ is a maximal toral subalgebra of $\mathfrak{g}^e$, $y \in \mathfrak{t}^\phi$. But, now $G^{x_s}/T^\phi = H$ and x_n can be viewed as an element in $\mathfrak{h}^{\bar{e}}$. As the H-orbit $\mathcal{O}_{\bar{e}}$ of $\bar{e}$ is distinguished we see that x_n lies in $\overline{\mathcal{O}_{\bar{e}}}$ and hence $x_n \in \bar{\mathcal{O}}$. □

We now observe that $\tilde{f}^{-1}((\mathfrak{t}^\phi)^{rs})$ is open dense in $\widetilde{\mathcal{O}}$ and it remains to show that $\check{\mathcal{O}}$ is dense in $\tilde{f}^{-1}((\mathfrak{t}^\phi)^{rs})$. Let us decompose an element $y \in \tilde{f}^{-1}((\mathfrak{t}^\phi)^{rs})$ into its semisimple and nilpotent parts $y = y_s + y_n$ and as $y \in \widetilde{\mathcal{O}}$ there exists an element

$x \in \mathcal{O}$ such that $[x, y] = 0$. Now, G acts on $\tilde{f}^{-1}((\mathfrak{t}^\phi)^{rs})$ and we can arrange the element x to be e. Thus, up to G-action $y = y_s + y_n \in \mathfrak{g}^e$. Thus, by the above lemma $y_n \in \bar{\mathcal{O}}$. But, $\check{\mathcal{O}}$ consists of such elements with $y_n \in \mathcal{O}$. Thus, $\check{\mathcal{O}}$ is open (and dense) in $\widetilde{\mathcal{O}}$. It is now easy to check that the projection $\Lambda_{\mathcal{O}} \to \widetilde{\mathcal{O}}$ has constant rank above $\check{\mathcal{O}}$ and hence $\check{\mathcal{O}}$ is a subvariety.

This establishes a correspondence:

$$\mathcal{O} \leftrightarrow \check{\mathcal{O}}$$

Let us now analyze the equivariant fundamental group $\pi_1^G(\check{\mathcal{O}})$. For an element $a \in (\mathfrak{t}^\phi)^{rs}$ let $a' = a + e$ and $X_{a'} = Ga'$, the G-orbit of a'. We have the following exact sequence:

$$1 \to \pi_1^G(X_{a'}) \to \pi_1^G(\check{\mathcal{O}}) \to B_{W^\phi} \to 1$$

where $B_{W^\phi} = \pi_1((\mathfrak{t}^\phi)^{rs}/W^\phi)$ denotes the braid group. We note that we use this terminology even when W^ϕ is not a Coxeter group. We have that

$$\pi_1^G(X_{a'}) \;=\; Z_G(a')/Z_G(a')^0 \,.$$

Now,

$$Z_G(a') = Z_G(e + \mathfrak{t}^\phi) = G^e \cap G^{\mathfrak{t}^\phi} = (G^\phi \cdot U^e) \cap G^{\mathfrak{t}^\phi} = Z_{G^\phi}(\mathfrak{t}^\phi) \cdot (U^e \cap U^{\mathfrak{t}^\phi})\,.$$

Thus:

$$\pi_1^G(X_{a'}) \;=\; Z_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi)^0 \,.$$

We conclude that we have the following exact sequences:

$$\begin{aligned} &1 \to Z_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi)^0 \to \pi_1^G(\check{\mathcal{O}}) \to B_{W^\phi} \to 1 \\ &1 \to Z_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi)^0 \to \pi_1^{G^\phi}((\mathfrak{g}^\phi)^{rs}) \to B_{W^\phi} \to 1\,. \end{aligned}$$

The natural map $\mathfrak{t}^\phi \to \mathfrak{t}^\phi + e$ induces an embedding

$$(\mathfrak{g}^\phi)^{rs} \subset \check{\mathcal{O}} \qquad gt \mapsto g(t+e) \qquad \text{for} \quad t \in (\mathfrak{t}^\phi)^{rs} \;\; g \in G^\phi \,.$$

This embedding identifies $\pi_1^G(\check{\mathcal{O}})$ and $\pi_1^{G^\phi}((\mathfrak{g}^\phi)^{rs})$ and the two exact sequences above:

$$\begin{tikzcd} 1 \arrow[r] & Z_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi)^0 \arrow[r] \arrow[d, equal] & \pi_1^{G^\phi}((\mathfrak{g}^\phi)^{rs}) \arrow[r, "\tilde{q}"] \arrow[d, equal] & B_{W^\phi} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & Z_{G^\phi}(\mathfrak{t}^\phi)/Z_{G^\phi}(\mathfrak{t}^\phi)^0 \arrow[r] & \pi_1^G(\check{\mathcal{O}}) \arrow[r, "\tilde{q}"] & B_{W^\phi} \arrow[r] & 1\,. \end{tikzcd}$$

Appendix B. The combinatorial formulas

Let us write

$$(A;q)_\infty = \prod_{s=0}^{\infty}(1 - Aq^s),\; (A;q)_n = \frac{(A;q)_\infty}{(Aq^n;q)_\infty} = \prod_{i=0}^{n-1}(1 - Aq^i),$$

$$(A, B; q)_n = (A;q)_n(B;q)_n, \text{etc.}$$

B.1. **Weighted type (p,q) allowable signed Young diagrams.** Recall from §7.6 that $\mathrm{SYD}_{p,q}$ denotes the set of signed Young diagrams that label the nilpotent K-orbits for the symmetric pair $(SO_N, S(O_p\times O_q))$. For each $\lambda\in\mathrm{SYD}_{p,q}$, we associate the weight $2^{r_\lambda+1}$ if λ has at least one odd part and weight 1 otherwise, where r_λ is defined in (2.6).

Definition B.1. Let $wt(p,q)$ denote the weighted sum of the type (p,q) allowable signed Young diagrams, i.e.,

$$wt(p,q)=\sum_{\lambda\in\mathrm{SYD}_{p,q}}2^{r_\lambda+1}-\mathbf{p}(\frac{q}{2})\delta_{p,q}, \tag{B.1}$$

where $\mathbf{p}(k)$ is the number of partitions of k and we set $\mathbf{p}(q/2)=0$ if q is odd.

Proposition B.2. *The generating function for $wt(p,q)$ is*

$$F(u,v)=\sum_{p,q=0}^{\infty}wt(p,q)u^pv^q=\prod_{k=1}^{\infty}\frac{1}{1-u^{2k}v^{2k}}\prod_{m=0}^{\infty}\frac{(1+u^{m+1}v^m)(1+u^mv^{m+1})}{(1-u^{m+1}v^m)(1-u^mv^{m+1})}.$$

Proof. An odd part of $2m+1$ can occur, with multiplicity $0,1,2,\ldots,r,\ldots$, which contributes

$$\begin{aligned}
&1+2(u^{m+1}v^m+u^mv^{m+1})+(2u^{2(m+1)}v^{2m}+4u^{2m+1}v^{2m+1}+2u^{2m}v^{2(m+1)})+\cdots\\
&\quad+(2u^{r(m+1)}v^{rm}+4u^{rm+r-1}v^{rm+1}+\cdots+4u^{rm+1}v^{rm+r-1}+2u^{rm}v^{r(m+1)})+\cdots\\
&=1+2\sum_{r=1}^{\infty}(u^mv^m)^r\left(\frac{u^{r+1}-v^{r+1}}{u-v}+uv\frac{u^{r-1}-v^{r-1}}{u-v}\right)\\
&=\frac{(1+u^{m+1}v^m)(1+u^mv^{m+1})}{(1-u^{m+1}v^m)(1-u^mv^{m+1})}.
\end{aligned}$$

An even part $2k$, since it has even multiplicity, clearly gives

$$\frac{1}{1-u^{2k}v^{2k}}.$$

□

Recall the q-Gauss theorem is

$$\sum_{m=0}^{\infty}\frac{(a;q)_m(b;q)_m}{(q;q)_m(c;q)_m}\left(\frac{c}{ab}\right)^m=\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}. \tag{B.2}$$

Corollary B.3. *Let $k\geq 0$. The generating function of the k^{th} diagonal is*

$$F_{kdiag}(t)=\sum_{p=0}^{\infty}wt(p+k,p)t^{2p}=\frac{2}{1+t^{2k}}\frac{\prod_{m=1}^{\infty}(1+t^{2m})}{\prod_{m=1}^{\infty}(1-t^{2m})^3}.$$

Proof. We use the q-binomial theorem with $q=uv$ for $F(u,v)=\sum_{p,q=0}^{\infty}wt(p,q)u^pv^q$:

$$\begin{aligned}
F(u,v)=&\frac{1}{(u^2v^2;u^2v^2)_\infty}\times\frac{(-u;uv)_\infty}{(u;uv)_\infty}\times\frac{(-v;uv)_\infty}{(v;uv)_\infty}\\
=&\frac{1}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-1;q)_m}{(q;q)_m}u^m\sum_{m=0}^{\infty}\frac{(-1;q)_m}{(q;q)_m}v^m.
\end{aligned}$$

So the k-th diagonal term is

$$\frac{1}{(q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-1;q)_{m+k}(-1;q)_m}{(q;q)_{m+k}(q;q)_m}q^m.$$

This sums by the q-Gauss theorem to

$$\begin{aligned}&=\frac{1}{(q^2;q^2)_\infty}\frac{(-1;q)_k}{(q;q)_k}\sum_{m=0}^{\infty}\frac{(-q^k;q)_m(-1;q)_m}{(q^{k+1};q)_m(q;q)_m}q^m\\ &=\frac{1}{(q^2;q^2)_\infty}\frac{(-1;q)_k}{(q;q)_k}\frac{(-q;q)_\infty(-q^{k+1},q)_\infty}{(q^{k+1};q)_\infty(q;q)_\infty}=\frac{2}{1+q^k}\frac{(-q;q)_\infty}{(q;q)_\infty^3},\end{aligned}$$

which is the given answer with $q=t^2$. □

B.2. **Proof of Proposition 7.5.** Recall that $\mathcal{P}^{odd}(N)$ denotes the set of partitions of N into odd parts. We write a partition $\lambda\in\mathcal{P}^{odd}(N)$ as

$$\lambda=(2\mu_1+1)+(2\mu_2+1)+\cdots+(2\mu_s+1)$$

where $\mu_1\geq\mu_2\geq\cdots\geq\mu_s\geq 0$. Note that $s\equiv N \mod 2$. We set

$$wt_\lambda=3^{\#\{1\leq j\leq(s-1)/2\,|\,\mu_{2j-1}=\mu_{2j}+1\}}\,4^{\#\{1\leq j\leq(s-1)/2\,|\,\mu_{2j-1}\geq\mu_{2j}+2\}}\text{ when } N \text{ is odd;}$$

$$wt_\lambda=3^{\#\{1\leq j\leq s/2-1\,|\,\mu_{2j}=\mu_{2j+1}+1\}}\,4^{\#\{1\leq j\leq s/2-1\,|\,\mu_{2j}\geq\mu_{2j+1}+2\}}\text{ when } N \text{ is even.}$$

Let us write

$$b_n=\sum_{\lambda\in\mathcal{P}^{odd}(2n+1)}wt_\lambda,\quad c_n=\sum_{\lambda\in\mathcal{P}^{odd}(2n)}wt_\lambda.$$

B.2.1. *Equation* (7.9a). The equation (7.9a) is equivalent to

$$\sum_{n=0}^{\infty}b_nq^{2n+1}=q(-q^4;q^4)_\infty^2(-q^2;q^2)_\infty^2.$$

This can be seen as follows. Suppose λ with odd parts has an odd number of parts, say $2k+1$. Consider the columns of λ, which have possible sizes $1,2,\cdots,2k+1$.

The part $2k+1$ occurs an odd number of times, the generating function is

$$\frac{q^{2k+1}}{1-q^{4k+2}}.$$

The part $2k$ occurs an even number of times, the generating function is

$$\frac{1}{1-q^{4k}}.$$

The part $2k-1$ occurs an even number of times, the weighted generating function is

$$1+3q^{4k-2}+\frac{4q^{8k-4}}{1-q^{4k-2}}.$$

This continues down to part size 1, to obtain the generating function

$$\sum_{n=0}^{\infty}b_nq^{2n+1}=$$

$$\sum_{k=0}^{\infty}\frac{q^{2k+1}}{1-q^{4k+2}}\frac{1}{(1-q^4)(1-q^8)\cdots(1-q^{4k})}\prod_{i=1,\text{odd}}^{2k-1}\left(1+3q^{2i}+\frac{4q^{4i}}{1-q^{2i}}\right).$$

Because

$$1+3x+\frac{4x^2}{1-x}=\frac{(1+x)^2}{1-x}$$

this may be written as

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^{2n+1} &= \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1-q^{4k+2}} \frac{(-q^2;q^4)_k(-q^2;q^4)_k}{(q^4;q^4)_k(q^2;q^4)_k} \\ &= \frac{q}{1-q^2}\sum_{k=0}^{\infty}\frac{(-q^2;q^4)_k(-q^2;q^4)_k}{(q^6;q^4)_k(q^4;q^4)_k}q^{2k}.\end{aligned}$$

Applying q-Gauss theorem (B.2) with $q\to q^4$ and $a=b=-q^2, c=q^6$, we obtain

$$\sum_{n=0}^{\infty} b_n q^{2n+1} = \frac{q}{1-q^2}\frac{(-q^4;q^4)_\infty(-q^4;q^4)_\infty}{(q^6;q^4)_\infty(q^2;q^4)_\infty} = q(-q^4;q^4)_\infty^2(-q^2;q^2)_\infty^2.$$

Here we have used the number of partitions of n into odd parts equals the number of partitions of n into distinct parts, i.e.,

$$(-q;q)_\infty = 1/(q;q^2)_\infty.$$

B.2.2. *Equation* (7.9b). The equation (7.9b) is equivalent to

$$\sum_{n=0}^{\infty} c_n q^{2n} = \frac{1}{4}(-q^2;q^4)_\infty^2(-q^2;q^2)_\infty.$$

Assume λ is a partition into odd parts with $2k$ parts. We argue as in §B.2.1, this time the even parts have weights. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} c_n q^{2n} &= \sum_{k=0}^{\infty}\frac{q^{2k}}{1-q^{4k}}\frac{1}{(1-q^2)(1-q^6)\cdots(1-q^{2(2k-1)})} \\ &\quad \prod_{i=1,\text{even}}^{2k-2}\left(1+3q^{2i}+\frac{4q^{4i}}{1-q^{2i}}\right) \\ &= \sum_{k=0}^{\infty}\frac{q^{2k}}{1-q^{4k}}\frac{(-q^4;q^4)_{k-1}(-q^4;q^4)_{k-1}}{(q^4;q^4)_{k-1}(q^2;q^4)_k} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1;q^4)_k(-1;q^4)_k}{(q^2;q^4)_k(q^4;q^4)_k}q^{2k} \\ &= \frac{1}{4}\frac{(-q^2;q^4)_\infty(-q^2;q^4)_\infty}{(q^2;q^4)_\infty(q^2;q^4)_\infty} = \frac{1}{4}(-q^2;q^4)_\infty^2(-q^2;q^2)_\infty.\end{aligned}$$

B.3. **Proof of equations** (7.16a) **and** (7.16b)**.**

Definition B.4. Let

$$F(q)=\frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2}.$$

The equations (7.16a) and (7.16b) are

Proposition B.5.

$$\begin{aligned} F(q)-F(-q) &= 8q(-q^4;q^4)_\infty^4(-q^2;q^2)_\infty^4 \\ \textit{and}\ \ F(q)+F(-q) &= 2(-q^2;q^4)_\infty^4(-q^2;q^2)_\infty^4.\end{aligned}$$

Recall Ramanujan's ${}_1\psi_1$ formula

Proposition B.6.

$$ {}_1\psi_1(a,b;q;x) =: \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax, q/ax, q, b/a; q)_\infty}{(x, b/ax, b, q/a; q)_\infty}. $$

for $|b/a| < |x| < 1$.

Letting $q \to q^2$, $a = -1, b = -q^2, x = q$ in Ramanujan's ${}_1\psi_1$ gives

$$ \sum_{k=-\infty}^{\infty} \frac{q^k}{1+q^{2k}} = \frac{1}{2}\ {}_1\psi_1(-1, -q^2; q^2; q) = \frac{1}{2} \frac{(-q, -q, q^2, q^2; q^2)_\infty}{(q, q, -q^2, -q^2; q^2)_\infty} $$

$$ = \frac{1}{2} F(q) \frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}. $$

We take the odd terms in the sum

$$ \sum_{k=-\infty, odd}^{\infty} \frac{q^k}{1+q^{2k}} = \frac{1}{4} \left(F(q) - F(-q)\right) \frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty} $$

and the even terms in the sum

$$ \sum_{k=-\infty, even}^{\infty} \frac{q^k}{1+q^{2k}} = \frac{1}{4} \left(F(q) + F(-q)\right) \frac{(q^2;q^2)^2_\infty}{(-q^2;q^2)^2_\infty}. $$

However

$$ \begin{aligned} \sum_{k=-\infty, odd}^{\infty} \frac{q^k}{1+q^{2k}} &= q \frac{1}{1+q^2}\ {}_1\psi_1(-q^2, -q^6; q^4; q^2) \\ &= q \frac{1}{1+q^2} \frac{(-q^4, -1, q^4, q^4; q^4)_\infty}{(q^2, q^2, -q^6, -q^2; q^4)_\infty} \\ &= 2q \frac{(-q^4, -q^4, q^4, q^4; q^4)_\infty}{(q^2, q^2, -q^2, -q^2; q^4)_\infty}, \\ \sum_{k=-\infty, even}^{\infty} \frac{q^k}{1+q^{2k}} &= \frac{1}{2}\ {}_1\psi_1(-1, -q^4; q^4; q^2) = \frac{1}{2} \frac{(-q^2, -q^2, q^4, q^4; q^4)_\infty}{(q^2, q^2, -q^4, -q^4; q^4)_\infty}. \end{aligned} $$

So

$$ \begin{aligned} \frac{1}{4}\left(F(q) - F(-q)\right) &= 2q \frac{(-q^4, -q^4, q^4, q^4; q^4)_\infty}{(q^2, q^2, -q^2, -q^2; q^4)_\infty} \frac{(-q^2;q^2)^2_\infty}{(q^2;q^2)^2_\infty} \\ &= 2q \frac{(-q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty} \frac{(-q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty} \\ &= 2q \frac{(-q^4;q^4)^4_\infty}{(q^2;q^4)^4_\infty} = 2q(-q^4;q^4)^4_\infty (-q^2;q^2)^4_\infty \\ \frac{1}{4}\left(F(q) + F(-q)\right) &= \frac{1}{2} \frac{(-q^2, -q^2, q^4, q^4; q^4)_\infty}{(q^2, q^2, -q^4, -q^4; q^4)_\infty} \frac{(-q^2;q^2)^2_\infty}{(q^2;q^2)^2_\infty} \\ &= \frac{1}{2}(-q^2;q^4)^4_\infty (-q^2;q^2)^4_\infty. \end{aligned} $$

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