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DUALS AND ADMISSIBILITY IN NATURAL CHARACTERISTIC

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Abstract. In this article we introduce a derived smooth duality functor $R\underline{\mathrm{Hom}}(-,k)$ on the unbounded derived category $D(G)$ of smooth k-representations of a p-adic Lie group G. Here k is a field of characteristic p. Using this functor we relate various subcategories of admissible complexes in $D(G)$.

1. Introduction

Let G be a p-adic Lie group of dimension d, and let k be a field of characteristic p. We denote by $\mathrm{Mod}(G)$ the abelian category of smooth G-representations in k-vector spaces.

In this paper we endow the unbounded derived category $D(G) = D(\mathrm{Mod}(G))$ with a tensor product $\otimes_k$ plus internal hom functor $R\underline{\mathrm{Hom}}$, and begin exploring the resulting closed symmetric monoidal category. The duality functor $R\underline{\mathrm{Hom}}(-,k)$ is of particular interest to us. It gives a derived approach to the higher smooth duality functors S^j introduced by Kohlhaase in [Koh], realizing them as cohomological functors $h^j(R\underline{\mathrm{Hom}}(-,k)) = \underline{\mathrm{Ext}}^j(-,k)$.

Our first result (Proposition 2.7) shows that the functors S^j are compatible with duals on the Hecke side. If H_U denotes the Hecke algebra of a torsion free open pro-p subgroup $U \subseteq G$, we give an H_U-equivariant spectral sequence with E_2-page $H^i(U, S^j(V))$ converging to the twisted dual Hecke modules $H^{d-i-j}(U,V)^\vee(\chi_G)$. Here the character $\chi_G : G \to k^\times$ turns out to coincide with the duality character in [Koh]. This is a non-trivial fact and we give a proof. In particular $\chi_G = 1$ if G is an open subgroup of the $\mathfrak{F}$-points of a connected reductive group over a p-adic field $\mathfrak{F}$.

Motivated by [DGA], which gives a differential graded version of the Hecke algebra $H_U^\bullet$ along with an equivalence between $D(G)$ and the derived category $D(H_U^\bullet)$ of differential graded modules over $H_U^\bullet$, we turn to studying the functor $R\underline{\mathrm{Hom}}(-,k)$ in the derived setting.

We first observe that $R\underline{\mathrm{Hom}}(-,k)$ is involutive on the subcategory $D_{adm}(G)$ of complexes $V^\bullet$ with admissible cohomology representations $h^i(V^\bullet)$ for all $i \in \mathbb{Z}$. We then introduce a possibly larger subcategory

$$D(G)^a \supseteq D_{adm}(G)$$

consisting of globally admissible complexes, by which we mean $H^i(U, V^\bullet)$ is finite-dimensional for all $i \in \mathbb{Z}$. As we show in Theorem 4.5, a complex $V^\bullet$ belongs to $D(G)^a$ precisely when the natural biduality morphism

$$\eta_{V^\bullet} : V^\bullet \longrightarrow R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(V^\bullet,k),k)$$

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is a quasi-isomorphism. As a result, the notion of being globally admissible is independent of the choice of U. Finally we show that a globally admissible $V^\bullet$ satisfying various boundedness conditions actually lies in the subcategory $D_{adm}(G)$. For instance, Corollary 4.12 tells us $D^b_{adm}(G)$ contains exactly those complexes $V^\bullet$ whose total cohomology $H^*(U, V^\bullet)$ is finite-dimensional.

To orient the reader we point out that $D(G)^a$ is equivalent to the category $D_{fin}(H^\bullet_U)$ of differential graded $H^\bullet_U$-modules with finite-dimensional cohomology spaces in each degree. We have work in progress aiming at an intrinsic description of the duality functor on $D(H^\bullet_U)$ corresponding to $R\underline{\mathrm{Hom}}(-, k)$.

2. Higher smooth duality

For any compact open subgroup $K \subseteq G$ we have the completed group ring $\Omega(K)$ of K over k. This is a noetherian ring (cf. [pLG, Theorem 33.4]). We let $\mathrm{Mod}(\Omega(K))$ denote the abelian category of left $\Omega(K)$-modules. However $\Omega(K)$ is also a pseudocompact ring (cf. [pLG, IV §19]). We therefore also have the abelian category $\mathrm{Mod}_{pc}(\Omega(K))$ of pseudocompact left $\Omega(K)$-modules together with the obvious forgetful functor $\mathrm{Mod}_{pc}(\Omega(K)) \to \mathrm{Mod}(\Omega(K))$. Both categories have enough projective objects. Any finitely generated $\Omega(K)$-module M is pseudocompact in a natural way. Moreover, such an M is projective in $\mathrm{Mod}_{pc}(\Omega(K))$ if and only if it is projective in $\mathrm{Mod}(\Omega(K))$. This leads to the natural isomorphism

$$\mathrm{Ext}^*_{\mathrm{Mod}_{pc}(\Omega(K))}(M, N) \cong \mathrm{Ext}^*_{\mathrm{Mod}(\Omega(K))}(M, N) \tag{1}$$

for any finitely generated module M in $\mathrm{Mod}(\Omega(K))$ and any pseudocompact module N in $\mathrm{Mod}_{pc}(\Omega(K))$.

Pontrjagin duality gives rise to the equivalence of categories

$$\mathrm{Mod}(K)^{op} \xrightarrow{\simeq} \mathrm{Mod}_{pc}(\Omega(K)),$$
$$V \longmapsto V^\vee := \mathrm{Hom}_k(V, k),$$

where, of course, in order to make $V^\vee$ a left module we use the inversion map $g \mapsto g^{-1}$ on K. See [Koh, Th. 1.5] for instance. In particular, we have the natural isomorphisms

$$\mathrm{Ext}^*_{\mathrm{Mod}(K)}(V_1, V_2) \cong \mathrm{Ext}^*_{\mathrm{Mod}_{pc}(\Omega(K))}(V_2^\vee, V_1^\vee) \ .$$

If we apply this with the trivial K-representation $V_2 := k$ and use (1) we obtain the natural isomorphism

$$\mathrm{Ext}^*_{\mathrm{Mod}(K)}(V, k) \cong \mathrm{Ext}^*_{\mathrm{Mod}(\Omega(K))}(k, V^\vee) \tag{2}$$

for V in $\mathrm{Mod}(K)$.

If $K' \subseteq K$ is another open subgroup then in (2) we have on both sides the obvious restriction maps. On the left-hand side this follows from the fact that the restriction functor $\mathrm{Mod}(K) \to \mathrm{Mod}(K')$ preserves injective objects (as follows from Frobenius reciprocity and the exactness of compact induction $\mathrm{ind}^K_{K'}$). On the right-hand side the functor $\mathrm{Mod}(\Omega(K)) \to \mathrm{Mod}(\Omega(K'))$ preserves projective objects since $\Omega(K)$ is free over $\Omega(K')$.

Hence we may pass to the inductive limit

$$\varinjlim_K \mathrm{Ext}^*_{\mathrm{Mod}(K)}(V, k) \cong \varinjlim_K \mathrm{Ext}^*_{\mathrm{Mod}(\Omega(K))}(k, V^\vee) \ . \tag{3}$$

Note that, for V in $\mathrm{Mod}(G)$, the right-hand side is Kohlhaase's higher smooth dual functors

$$S^*(V) := \varinjlim_K \mathrm{Ext}^*_{\mathrm{Mod}(\Omega(K))}(k, V^\vee)$$

in [Koh]. We use the left-hand side to understand these as derived functors. For any V_1, V_2 in $\mathrm{Mod}(G)$ we introduce

$$\underline{\mathrm{Hom}}(V_1, V_2) := \{f \in \mathrm{Hom}_k(V_1, V_2) : f \text{ is } K\text{-equivariant for some compact open subgroup } K \subseteq G\}.$$

Via the G-action defined by ${}^g f := gf(g^{-1}-)$, for $g \in G$, this is again an object in $\mathrm{Mod}(G)$. Since the functors

$$\underline{\mathrm{Hom}}(V_1, -) : \mathrm{Mod}(G) \longrightarrow \mathrm{Mod}(G), \quad V_2 \longmapsto \underline{\mathrm{Hom}}(V_1, V_2)$$

are left exact we have the corresponding right derived functors

$$\underline{\mathrm{Ext}}^i(V_1, V_2) \qquad \text{for } i \geq 0.$$

It is well-known (and easy to show) that $\mathrm{Mod}(G)$ has enough injectives. On the contrary $\mathrm{Mod}(G)$ does not in general have enough projectives.

Lemma 2.1.

(i) *If V_2 is injective in $\mathrm{Mod}(G)$ then $\underline{\mathrm{Hom}}(V_1, V_2)$ is $H^0(U, -)$-acyclic for any compact open subgroup $U \subseteq G$.*

(ii) $\underline{\mathrm{Ext}}^*(V_1, V_2) = \varinjlim_K \mathrm{Ext}^*_{\mathrm{Mod}(K)}(V_1, V_2)$.

Proof. By definition $\underline{\mathrm{Hom}}(V_1, V_2) = \varinjlim_K \mathrm{Hom}_{\mathrm{Mod}(K)}(V_1, V_2)$. Note that any injective object in $\mathrm{Mod}(G)$ remains injective when viewed in $\mathrm{Mod}(U)$, as explained right after (2). Therefore the lemma follows from Proposition 2.2 in the appendix by Verdier in [CG]. □

We see that, in particular, we can rewrite Kohlhaase's functors as the derived functors

$$S^*(V) = \underline{\mathrm{Ext}}^*(V, k) \ .$$

We first note that $S^j(V) = 0$ in the range $j > d$. More generally we have the following.

Lemma 2.2. $\underline{\mathrm{Ext}}^i(V_1, V_2) = 0$ *for any $i > d$.*

Proof. By [Bru, Theorem 4.1] the global dimension of $\Omega(K)$ as a pseudocompact ring is equal to the cohomological dimension of K. By Lazard (cf. [CG, I-47]) the latter is equal to d provided K is pro-p and torsion free. Since G contains arbitrarily small open pro-p subgroups without torsion we conclude from Lemma 2.1(ii) combined with the isomorphism $\mathrm{Ext}^*_{\mathrm{Mod}(K)}(V_1, V_2) \cong \mathrm{Ext}^*_{\mathrm{Mod}_{pc}(\Omega(K))}(V_2^\vee, V_1^\vee)$ that indeed $\underline{\mathrm{Ext}}^i(V_1, V_2) = 0$ for any $i > d$. □

Proposition 2.3. *For any compact open subgroup $U \subseteq G$ we have the E_2-spectral sequence*

$$H^i(U, \underline{\mathrm{Ext}}^j(V_1, V_2)) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathrm{Mod}(U)}(V_1, V_2) \ .$$

In particular,

$$H^i(U, S^j(V)) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathrm{Mod}(U)}(V, k) \ .$$

Proof. This is the composed functor spectral sequence which exists by Lemma 2.1(i). □

The above spectral sequence has an additional equivariance property which we now describe. We fix a compact open subgroup $U \subseteq G$ and consider the compact induction $\mathbf{X}_U := \mathrm{ind}_U^G(k)$ in $\mathrm{Mod}(G)$. We then have the endomorphism ring $H_U := \mathrm{End}_{\mathrm{Mod}(G)}(\mathbf{X}_U)^{op}$ so that $\mathbf{X}_U$ becomes a right H_U-module. Frobenius reciprocity gives a natural isomorphism of functors $H^0(U,-) \cong \mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}_U,-)$ on $\mathrm{Mod}(G)$. By using injective resolutions it extends to a natural isomorphism of cohomological functors

$$H^*(U,-) \cong \mathrm{Ext}^*_{\mathrm{Mod}(G)}(\mathbf{X}_U,-) \ .$$

Through its right action on $\mathbf{X}_U$ the right-hand side becomes a left H_U-module. In this way $H^*(U,-)$ is equipped with a left H_U-module structure. In particular, $\mathrm{Hom}_{\mathrm{Mod}(U)}(V_1,V_2) = H^0(U,\underline{\mathrm{Hom}}(V_1,V_2)) \cong \mathrm{Hom}_{\mathrm{Mod}(G)}(\mathbf{X}_U,\underline{\mathrm{Hom}}(V_1,V_2))$ carries a left H_U-module structure which is functorial in V_1 and V_2. By derivation we obtain a functorial left H_U-module structure on $\mathrm{Ext}^*_{\mathrm{Mod}(U)}(V_1,V_2)$. Up to isomorphism the latter H_U-module is independent of the choice of injective resolution of V_2.

Lemma 2.4. *The spectral sequence in Proposition* 2.3 *is* H_U*-equivariant.*

Proof. This is straightforward from the way the composed functor spectral sequence is constructed. □

We now suppose in addition that U is pro-p and torsion free. Then U is a Poincaré group of dimension d ([CG] I-47 Ex. (3)). A straightforward variant of the appendix by Verdier in [CG] (or Tate's Appendix 1 in the 1997 English translation) therefore gives the following: In $\mathrm{Mod}(U)$ we have the dualizing object

$$\hat{I} := \varinjlim_{K \subseteq U, \mathrm{cores}} \mathrm{Hom}_k(H^d(K,k),k) \ ,$$

which actually is isomorphic to the trivial representation k in $\mathrm{Mod}(U)$, together with an isomorphism

$$\mathrm{Hom}_k(H^i(U,V),k) \cong \mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) \cong \mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,k) \qquad \text{for any } i \geq 0 \tag{4}$$

which is natural in V in $\mathrm{Mod}(U)$; this latter isomorphism is induced by the Yoneda product

$$\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) \times H^i(U,V) \longrightarrow H^d(U,\hat{I})$$

(Definition 4.5, Proposition 3.1.5, and first displayed formula on p. V-20). In the following we will keep writing $\hat{I}$ and view it as a trivial G-representation. From now on we assume that V comes from a given G-representation (by restriction to U) and we will see that then all terms in the above Yoneda pairing carry a natural left H_U-action.

(A) From the proof of Proposition 8.4.i in [OS] we know a formula for the H_U action on $H^*(U,V)$. Viewing H_U as the convolution algebra of U-bi-invariant functions with compact support on G we denote by $\tau_h \in H_U$, for $h \in G$, the characteristic function of the double coset UhU in G. The diagram

$$\begin{array}{ccc} H^*(U,V) & \xrightarrow{\tau_h \cdot} & H^*(U,V) \\ \downarrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \mathrm{cores}} \\ H^*(U\cap h^{-1}Uh,V) & \xrightarrow{h_*} & H^*(U\cap hUh^{-1},V) \end{array}$$

is commutative.

(B) By [CG] I Proposition 18 the same $\hat{I}$ is also a dualizing object in $\mathrm{Mod}(U')$ for any open subgroup $U' \subseteq U$.

(C) As introduced above, we have a natural left H_U-action on $\mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,\hat{I})$. To give an explicit formula we let V' be any other object in $\mathrm{Mod}(G)$ and we first recall that, for any open subgroup $U' \subseteq U$ and any $h \in G$, we have the following natural maps:
- The restriction map $\mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,V') \xrightarrow{\mathrm{res}} \mathrm{Ext}^*_{\mathrm{Mod}(U')}(V,V')$ which derives the obvious forgetful map on homomorphisms. (Recall that restriction $\mathrm{Mod}(U) \to \mathrm{Mod}(U')$ preserves injective objects.)
- The corestriction map $\mathrm{Ext}^*_{\mathrm{Mod}(U')}(V,V') \xrightarrow{\mathrm{cores}} \mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,V')$ which derives the map which sends a U'-equivariant homomorphism $f : V \to V'$ to the U-equivariant homomorphism $\sum_{g\in U/U'} gf(g^{-1}-) : V \to V'$.
- The conjugation map $\mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,V') \xrightarrow{h_*} \mathrm{Ext}^*_{\mathrm{Mod}(hUh^{-1})}(V,V')$ which derives the map which sends a U-equivariant homomorphism $f : V \to V'$ to the hUh^{-1}-equivariant homomorphism $hf(h^{-1}-) : V \to V'$.

(D) As for (A) it is straightforward to verify that, for any $h \in G$, the diagram

$$\begin{array}{ccc} \mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,V') & \xrightarrow{\tau_h \cdot} & \mathrm{Ext}^*_{\mathrm{Mod}(U)}(V,V') \\ \downarrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \mathrm{cores}} \\ \mathrm{Ext}^*_{\mathrm{Mod}(U\cap h^{-1}Uh)}(V,V') & \xrightarrow{h_*} & \mathrm{Ext}^*_{\mathrm{Mod}(U\cap hUh^{-1})}(V,V') \end{array}$$

is commutative.

(E) It is easily checked that the map

$$\begin{aligned} H_U &\longrightarrow H_U, \\ \tau &\longmapsto \tau(-^{-1}) \end{aligned}$$

is an anti-involution of the k-algebra H_U, again viewed as a convolution algebra as in part (A) It sends τ_h to $\tau_{h^{-1}}$.

Lemma 2.5. *For any $0 \leq i \leq d$ and any $h \in G$ the diagram of Yoneda pairings*

$$\begin{array}{ccccc}
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) & \times & H^i(U,V) & \longrightarrow & H^d(U,\hat{I}) \\
\Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\uparrow{\scriptstyle \mathrm{cores}} & & \Big\uparrow{\scriptstyle \mathrm{cores}} \\
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U\cap h^{-1}Uh)}(V,\hat{I}) & \times & H^i(U\cap h^{-1}Uh,V) & \longrightarrow & H^d(U\cap h^{-1}Uh,\hat{I}) \\
\Big\downarrow{\scriptstyle h_*} & & \Big\uparrow{\scriptstyle h_*^{-1}} & & \Big\downarrow{\scriptstyle h_*} \\
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U\cap hUh^{-1})}(V,\hat{I}) & \times & H^i(U\cap hUh^{-1},V) & \longrightarrow & H^d(U\cap hUh^{-1},\hat{I}) \\
\Big\downarrow{\scriptstyle \mathrm{cores}} & & \Big\uparrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{cores}} \\
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) & \times & H^i(U,V) & \longrightarrow & H^d(U,\hat{I})
\end{array}$$

is commutative.

Proof. We fix injective resolutions $V \xrightarrow{\simeq} \mathcal{J}^\bullet$ and $\hat{I} \xrightarrow{\simeq} \mathcal{I}^\bullet$ in $\mathrm{Mod}(G)$, which as noted earlier remain injective resolutions after restriction to any given open subgroup of G.

The upper rectangle: Let $\beta^\bullet : \mathcal{J}^\bullet \to \mathcal{I}^\bullet[d-i]$ be a U-equivariant and $\alpha^\bullet : k \to \mathcal{J}^\bullet[i]$ a $U \cap h^{-1}Uh$-equivariant homomorphism of complexes representing classes $[\beta^\bullet] \in \mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I})$ and $[\alpha^\bullet] \in H^i(U\cap h^{-1}Uh, V)$, respectively. Then $\beta^\bullet$ also represents $\mathrm{res}[\beta^\bullet]$ whereas $\mathrm{cores}[\alpha^\bullet]$ is represented by $\sum_{g\in U/U\cap h^{-1}Uh} {}^g\alpha^\bullet$. We compute

$$\begin{aligned}
[\beta^\bullet[i]] \circ \mathrm{cores}[\alpha^\bullet] &= [\beta^\bullet[i] \circ \sum_{g\in U/U\cap h^{-1}Uh} {}^g\alpha^\bullet] = [\beta^\bullet[i] \circ \sum_{g\in U/U\cap h^{-1}Uh} g\alpha^\bullet(g^{-1}-)] \\
&= [\sum_{g\in U/U\cap h^{-1}Uh} g(\beta^\bullet[i]\circ\alpha^\bullet)(g^{-1}-)] = [\sum_{g\in U/U\cap h^{-1}Uh} {}^g(\beta^\bullet[i]\circ\alpha^\bullet)] \\
&= \mathrm{cores}(\mathrm{res}[\beta^\bullet[i]] \circ [\alpha^\bullet]) \ .
\end{aligned}$$

The middle rectangle: Let $\beta^\bullet : \mathcal{J}^\bullet \to \mathcal{I}^\bullet[d-i]$ be a $U\cap h^{-1}Uh$-equivariant and $\alpha^\bullet : k \to \mathcal{J}^\bullet[i]$ a $U\cap hUh^{-1}$-equivariant homomorphism of complexes representing classes $[\beta^\bullet] \in \mathrm{Ext}^{d-i}_{\mathrm{Mod}(U\cap h^{-1}Uh)}(V,\hat{I})$ and $[\alpha^\bullet] \in H^i(U\cap hUh^{-1},V)$, respectively. Then $h_*[\beta^\bullet]$ and $h_*^{-1}[\alpha^\bullet]$ are represented by ${}^h\beta^\bullet$ and ${}^{h^{-1}}\alpha^\bullet$. We compute

$$\begin{aligned}
h_*[\beta^\bullet[i]] \circ [\alpha^\bullet] &= [{}^h\beta^\bullet[i] \circ \alpha^\bullet] = [h\beta^\bullet[i](h^{-1}\alpha^\bullet(-))] = [h\beta^\bullet[i](h^{-1}\alpha^\bullet(hh^{-1}-))] \\
&= [h\beta^\bullet[i]({}^{h^{-1}}\alpha^\bullet(h^{-1}-))] = [h(\beta^\bullet[i] \circ {}^{h^{-1}}\alpha^\bullet)(h^{-1}-)] \\
&= [{}^h(\beta^\bullet[i] \circ {}^{h^{-1}}\alpha^\bullet)] = h_*([\beta^\bullet[i]] \circ h_*^{-1}[\alpha^\bullet]) \ .
\end{aligned}$$

The lower rectangle: This is entirely analogous to the computation for the upper rectangle. $\square$

By [CG] I-50(4) the two corestriction maps in the rightmost column of the diagram in Lemma 2.5 are isomorphisms between one-dimensional vector spaces. The composition

$$H^d(U,\hat{I}) \xleftarrow{\sim} H^d(U\cap h^{-1}Uh,\hat{I}) \xrightarrow{h_*} H^d(U\cap hUh^{-1},\hat{I}) \xrightarrow{\sim} H^d(U,\hat{I})$$

is therefore multiplication by a scalar $\chi_G(h) \in k^\times$, which happens to be independent of U.

Lemma 2.6. *The map $\chi_G : G \to k^\times$ is a character which is independent of U and trivial on any pro-p subgroup of G.*

Proof. We first show the independence of U. Suppose U' is another open torsion free pro-p subgroup of G and consider subgroups $U'' \subset U \cap U'$. Again by [CG] I-50(4) corestriction gives isomorphisms

$$H^d(U,k) \xleftarrow{\sim} H^d(U'',k) \xrightarrow{\sim} H^d(U',k).$$

This gives a canonical isomorphism between the dualizing objects $\hat{I}_U \simeq \hat{I}_{U'}$ which in turn gives a canonical isomorphism $H^d(U,\hat{I}_U) \simeq H^d(U',\hat{I}_{U'})$. Altogether this shows $\chi_G(h) \in k^\times$ is independent of U, and it is obviously trivial on U.

Suppose that we have checked the multiplicativity of χ_G already and let U_0 be any pro-p subgroup of G. Note that, as a p-adic Lie group, G always has an open torsion free pro-p subgroup; see [pLG, Theorem 27.1] for instance which even shows the existence of a p-valuable subgroup. Hence $\chi_G|U_0$ factorizes through a finite quotient which is a p-group. Since any finite subgroup of $k^\times$ has order prime to p it follows that χ_G is trivial on U_0. To establish multiplicativity let $g, h \in G$. Since conjugation commutes with corestriction we have the following three commutative diagrams, which together show our claim:

$$\begin{array}{ccc}
H^d(U \cap gUg^{-1}, \hat{I}) & \xleftarrow[\cong]{\text{cores}} & H^d(U \cap gUg^{-1} \cap ghU(gh)^{-1}, \hat{I}) \\
\uparrow{\scriptstyle g_*} & & \uparrow{\scriptstyle g_*} \\
H^d(U \cap g^{-1}Ug, \hat{I}) & \xleftarrow[\cong]{\text{cores}} & H^d(U \cap g^{-1}Ug \cap hUh^{-1}, \hat{I}),
\end{array}$$

$$\begin{array}{ccc}
H^d(U \cap hUh^{-1}, \hat{I}) & \xleftarrow[\cong]{\text{cores}} & H^d(U \cap g^{-1}Ug \cap hUh^{-1}, \hat{I}) \\
\uparrow{\scriptstyle h_*} & & \uparrow{\scriptstyle h_*} \\
H^d(U \cap h^{-1}Uh, \hat{I}) & \xleftarrow[\cong]{\text{cores}} & H^d(U \cap (gh)^{-1}Ugh \cap h^{-1}Uh, \hat{I}),
\end{array}$$

and

$$\begin{array}{ccc}
H^d(U \cap gUg^{-1} \cap ghU(gh)^{-1}, \hat{I}) & \xrightarrow[\cong]{\text{cores}} & H^d(U \cap ghU(gh)^{-1}, \hat{I}) \\
\uparrow{\scriptstyle g_*} & & \uparrow{\scriptstyle g_*} \\
H^d(U \cap g^{-1}Ug \cap hUh^{-1}, \hat{I}) & \xrightarrow[\cong]{\text{cores}} & H^d(g^{-1}Ug \cap hUh^{-1}, \hat{I}) \\
\uparrow{\scriptstyle h_*} & & \uparrow{\scriptstyle h_*} \\
H^d(U \cap (gh)^{-1}Ugh \cap h^{-1}Uh, \hat{I}) & \xrightarrow[\cong]{\text{cores}} & H^d(U \cap (gh)^{-1}Ugh, \hat{I}).
\end{array}$$

$$(gh)_* : H^d(U \cap (gh)^{-1}Ugh, \hat{I}) \longrightarrow H^d(U \cap ghU(gh)^{-1}, \hat{I})$$

□

The map

$$\begin{aligned} H_U &\longrightarrow H_U, \\ \tau &\longmapsto \chi_G \tau \ \text{(pointwise product of functions)} \end{aligned}$$

is an algebra homomorphism. Pulling back an H_U-module M along this homomorphism defines the twisted H_U-module $M(\chi_G)$. More explicitly $\tau_h \in H_U$ acts on $m \in M(\chi_G)$ by the rule $\tau_h \star m = \chi_G(h)\tau_h(m)$.

Also note that we may use the anti-involution in (E) to make the k-linear dual $M^\vee := \mathrm{Hom}_k(M,k)$ of a left H_U-module M again into a left H_U-module. More explicitly $(\tau_h f)(m) = f(\tau_{h^{-1}} m)$ for $f \in M^\vee$ and $h \in G$.

Using (A) and (C) we may rewrite the diagram in Lemma 2.5 as the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) & \times & H^i(U,V) & \longrightarrow & k \\
\Big\downarrow{\scriptstyle \tau_h\cdot} & & \Big\uparrow{\scriptstyle \tau_{h^{-1}}\cdot} & & \Big\downarrow{\scriptstyle \chi_G(h)\cdot} \\
\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,\hat{I}) & \times & H^i(U,V) & \longrightarrow & k.
\end{array}$$

Then this says that the duality isomorphism (4) in fact is an isomorphism of H_U-modules

$$\mathrm{Ext}^{d-i}_{\mathrm{Mod}(U)}(V,k) \xrightarrow{\cong} H^i(U,V)^\vee(\chi_G)\ . \tag{5}$$

Altogether this yields the following spectral sequence alluded to in Section 1.

Proposition 2.7. *For any compact open subgroup $U \subseteq G$ which is pro-p and torsion free and any V in* $\mathrm{Mod}(G)$ *we have an H_U-equivariant E_2-spectral sequence*

$$H^i(U, S^j(V)) \Longrightarrow H^{d-i-j}(U,V)^\vee(\chi_G)\ .$$

Proof. The spectral sequence arises by combining the second spectral sequence in Proposition 2.3 (observe Lemma 2.4) with the duality isomorphism (5). □

Remark 2.8. Suppose that $G = \mathbf{G}(\mathfrak{F})$ where $\mathfrak{F}/\mathbb{Q}_p$ is a finite extension and $\mathbf{G}$ is a connected reductive $\mathfrak{F}$-split group over $\mathfrak{F}$. Assuming that a pro-p Iwahori subgroup U of G is torsion free it is shown in [OS] Proposition 7.16 that $\chi_G = 1$. Under additional assumptions this was proved before in [Koz]. In the preprint [KS21] Koziol and Schwein give an alternate proof of the triviality of the orientation character χ_G via Moy-Prasad groups (still assuming pro-p Iwahori is torsion free). We extend this result in Lemma 2.10.

The spectral sequence in Proposition 2.7 was obtained by different means in [Ko, Theorem 1.3] in the generality of a p-adic reductive group G and a torsion free pro-p Iwahori subgroup U.

We will show that χ_G in fact coincides with the duality character introduced by Kohlhaase in [Koh] after Definition 3.12 and which we temporarily denote by χ_G^{Koh}.

Proposition 2.9. *We have $\chi_G = \chi_G^{\mathrm{Koh}}$.*

Proof. The character χ_G^{Koh} describes the G-action on a certain one dimensional k-vector space $E^d(k)$ the original definition of which we do not need. Instead we use [Koh] Proposition 3.2 which says that, for any compact open subgroup $G_0 \subseteq G$, there is a natural G_0-equivariant isomorphism $\ell_{G,G_0} : E^d(k) \xrightarrow{\cong} \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_0))}(k, \Omega(G_0))$ such that:

(1) For any $g \in G$ the diagram

$$\begin{array}{ccc}
E^d(k) & \xrightarrow{\ell_{G,G_0}} & \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_0))}(k, \Omega(G_0)) \\
\downarrow{\scriptstyle \chi_G^{\mathrm{Koh}}(g)\cdot} & & \downarrow{\scriptstyle g_*} \\
E^d(k) & \xrightarrow{\ell_{G,gG_0g^{-1}}} & \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(gG_0g^{-1}))}(k, \Omega(gG_0g^{-1}))
\end{array}$$

is commutative, where g_* is the conjugation isomorphism (compare with the argument in the third paragraph of the proof of [Koh] Proposition 3.13).

(2) For any open subgroup $G_1 \subseteq G_0$ the diagram

$$\begin{array}{ccc}
 & & \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_0))}(k, \Omega(G_0)) \\
 & \nearrow^{\ell_{G,G_0}} & \\
E^d(k) & & \downarrow{\scriptstyle \ell_{G_0,G_1}} \\
 & \searrow_{\ell_{G,G_1}} & \\
 & & \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_1))}(k, \Omega(G_1))
\end{array}$$

is commutative. Moreover ℓ_{G_0,G_1} is the composite of the restriction map

$$\mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_0))}(k, \Omega(G_0)) \xrightarrow{\mathrm{res}} \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_1))}(k, \Omega(G_0))$$

and the map

$$\mathrm{Ext}^d(k, j^\vee_{G_1.G_0}) : \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_1))}(k, \Omega(G_0)) \to \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_1))}(k, \Omega(G_1))$$

which is induced by the Pontrjagin dual $j^\vee_{G_1,G_0}$ of the extension by zero map $j_{G_1,G_0} : C^\infty(G_1, k) \to C^\infty(G_0, k)$.

The Pontrjagin dual of $C^\infty(G_0, k)$ being $\Omega(G_0)$ we have, using (2), the isomorphism

$$P_{G_0} : \mathrm{Ext}^d_{\mathrm{Mod}(\Omega(G_0))}(k, \Omega(G_0)) \xrightarrow{\cong} \mathrm{Ext}^d_{\mathrm{Mod}(G_0)}(C^\infty(G_0, k), k) \ .$$

Combining it with the above two diagrams we arrive at the commutative diagrams

(6)
$$\begin{array}{ccc}
E^d(k) & \xrightarrow[\cong]{P_{G_0} \circ \ell_{G,G_0}} & \mathrm{Ext}^d_{\mathrm{Mod}(G_0)}(C^\infty(G_0, k), k) \\
\downarrow{\scriptstyle \chi_G^{\mathrm{Koh}}(g)\cdot} & & \downarrow{\scriptstyle g_*} \\
E^d(k) & \xrightarrow[\cong]{P_{gG_0g^{-1}} \circ \ell_{G,gG_0g^{-1}}} & \mathrm{Ext}^d_{\mathrm{Mod}(gG_0g^{-1})}(C^\infty(gG_0g^{-1}, k), k)
\end{array}$$

and

(7)
$$\begin{array}{ccccc}
 & & \mathrm{Ext}^d_{\mathrm{Mod}(G_0)}(C^\infty(G_0, k), k) & & \\
 & \nearrow^{P_{G_0} \circ \ell_{G,G_0}}_{\cong} & & \searrow^{\mathrm{res}} & \\
E^d(k) & & & & \mathrm{Ext}^d_{\mathrm{Mod}(G_1)}(C^\infty(G_0, k), k) \\
 & \searrow^{\cong}_{P_{G_1} \circ \ell_{G,G_1}} & & \swarrow_{\mathrm{Ext}^d(j_{G_1,G_0}, k)} & \\
 & & \mathrm{Ext}^d_{\mathrm{Mod}(G_1)}(C^\infty(G_1, k), k). & &
\end{array}$$

On the other hand, taking now $G_0 = U$ we note that the duality isomorphism (4) for $V = C^\infty(U,k)$ and $i = 0$ is given by

$$\begin{aligned} \mathrm{Ext}^d_{\mathrm{Mod}(U)}(C^\infty(U,k),k) &\xrightarrow{\cong} \mathrm{Hom}_k(\mathrm{Hom}_{\mathrm{Mod}(U)}(k, C^\infty(U,k)), H^d(U,k)), \\ e &\longmapsto \big[\phi \mapsto \phi^*(e)\big] \ . \end{aligned}$$

Let $\mathrm{con}_U : k \to C^\infty(U,k)$ denote the map which sends $1 \in k$ to the constant function with value 1 on U. Then the above isomorphism is equivalent to the isomorphism

$$\begin{aligned} \mathrm{Ext}^d_{\mathrm{Mod}(U)}(C^\infty(U,k),k) &\xrightarrow{\cong} H^d(U,k), \\ e &\longmapsto \mathrm{con}_U^*(e) \ . \end{aligned}$$

The first isomorphism being natural in conjugation by $g \in G$ and this conjugation sending con_U to $\mathrm{con}_{gUg^{-1}}$ we see that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^d_{\mathrm{Mod}(U)}(C^\infty(U,k),k) & \xrightarrow{\mathrm{con}_U^*} & H^d(U,k) \\ \downarrow g_* & & \downarrow g_* \\ \mathrm{Ext}^d_{\mathrm{Mod}(gUg^{-1})}(C^\infty(gUg^{-1},k),k) & \xrightarrow{\mathrm{con}_{gUg^{-1}}^*} & H^d(gUg^{-1},k). \end{array} \tag{8}$$

Furthermore, if $U' \subseteq U$ is any open subgroup, then we have the commutative diagram of duality pairings

$$\begin{array}{ccccc} \mathrm{Ext}^d_{\mathrm{Mod}(U)}(C^\infty(U,k),k) & \times & H^0(U, C^\infty(U,k)) & \longrightarrow & H^d(U,k) \\ \downarrow \mathrm{res} & & \uparrow \mathrm{cores} & & \uparrow \mathrm{cores} \\ \mathrm{Ext}^d_{\mathrm{Mod}(U')}(C^\infty(U,k),k) & \times & H^0(U', C^\infty(U,k)) & \longrightarrow & H^d(U',k) \\ \downarrow \mathrm{Ext}^d(j_{U',U},k) & & \uparrow H^0(U',j_{U',U}) & & \| \\ \mathrm{Ext}^d_{\mathrm{Mod}(U')}(C^\infty(U',k),k) & \times & H^0(U', C^\infty(U',k)) & \longrightarrow & H^d(U',k). \end{array}$$

Here the top, resp. bottom, rectangle is commutative by the top rectangle in Lemma 2.5, resp. the functoriality of the Yoneda pairing. Note that the middle column maps $\mathrm{con}_{U'}$ to con_U. Hence we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^d_{\mathrm{Mod}(U)}(C^\infty(U,k),k) & \xrightarrow{\mathrm{con}_U^*} & H^d(U,k) \\ \downarrow \mathrm{Ext}^d(j_{U',U},k)\circ\mathrm{res} & & \uparrow \mathrm{cores} \\ \mathrm{Ext}^d_{\mathrm{Mod}(U')}(C^\infty(U',k),k) & \xrightarrow{\mathrm{con}_{U'}^*} & H^d(U',k). \end{array}$$

By combining it with the diagram (7) we deduce the left-hand triangle of the commutative diagram

$$\begin{array}{ccccc} & \overset{\mathrm{con}_U^* \circ P_U \circ \ell_{G,U}}{\nearrow \cong} & H^d(U,k) & \overset{\cong}{\searrow} & \\ E^d(k) & & \uparrow \mathrm{cores} & & k \\ & \underset{\mathrm{con}_{U'}^* \circ P_{U'} \circ \ell_{G,U'}}{\searrow \cong} & H^d(U',k), & \underset{\cong}{\nearrow} & \end{array}$$

where the right-hand oblique arrows are our standard identifications. This means that the isomorphism $\mathrm{con}_U^* \circ P_U \circ \ell_{G,U} : E^d(k) \xrightarrow{\cong} k$ does not depend on the subgroup U. With this information we consider the commutative diagram

$$\begin{array}{ccccc} E^d(k) & \xrightarrow[\cong]{\mathrm{con}_U^* \circ P_U \circ \ell_{G,U}} & H^d(U,k) & \xrightarrow{\cong} & k \\ {\scriptstyle \chi_G^{\mathrm{Koh}}(g)\cdot}\downarrow & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle \chi_G(g)\cdot} \\ E^d(k) & \xrightarrow[\cong]{\mathrm{con}_{gUg^{-1}}^* \circ P_{gUg^{-1}} \circ \ell_{G,gUg^{-1}}} & H^d(gUg^{-1},k) & \xrightarrow{\cong} & k \end{array}$$

whose left-hand rectangle arises by combining (6) and (8). Since the horizontal arrows coincide we conclude that $\chi_G^{\mathrm{Koh}}(g) = \chi_G(g)$. $\square$

One immediately infers the triviality of χ_G for open subgroups of p-adic reductive groups:

Lemma 2.10. *Suppose that* $\mathbf{G}$ *is a connected reductive group over a finite extension* $\mathfrak{F}$ *of* $\mathbb{Q}_p$*; if* G *is an open subgroup of* $\mathbf{G}(\mathfrak{F})$ *then* $\chi_G = 1$.

Proof. Proposition 2.9 together with [Koh, Corollary 5.2] shows the assertion in the case $\mathfrak{F} = \mathbb{Q}_p$. In general let $\mathbf{G}'$ denote the Weil restriction of $\mathbf{G}$ to $\mathbb{Q}_p$. It is shown in [Oes] App. 3 that $\mathbf{G}'$ again is a connected linear algebraic group with the property that $\mathbf{G}(\mathfrak{F}) = \mathbf{G}'(\mathbf{Q}_p)$ as p-adic Lie groups. Since our field extension is separable it follows from loc. cit. A.3.4 that with $\mathbf{G}$ also $\mathbf{G}'$ is reductive. This reduces the general case to the case $\mathfrak{F} = \mathbb{Q}_p$. $\square$

3. Derived smooth duality

We begin by recalling some general nonsense about the adjunction between tensor product and Hom-functor which for three k-vector spaces V_1, V_2, and V_3 is given by the linear isomorphism

$$\begin{aligned} (9) \qquad \mathrm{Hom}_k(V_1 \otimes_k V_2, V_3) &\xrightarrow{\cong} \mathrm{Hom}_k(V_1, \mathrm{Hom}_k(V_2, V_3)), \\ A &\longmapsto \lambda_A(v_1)(v_2) := A(v_1 \otimes v_2) \ . \end{aligned}$$

Suppose that all three vector spaces carry a left G-action. Then $\mathrm{Hom}_k(V_1 \otimes_k V_2, V_3)$ and $\mathrm{Hom}_k(V_1, \mathrm{Hom}_k(V_2, V_3))$ are equipped with the $G \times G \times G$-action defined by

$$^{(g_1,g_2,g_3)}A(v_1 \otimes v_2) := g_3 A(g_1^{-1} v_1 \otimes g_2^{-1} v_2)$$

and

$$^{(g_1,g_2,g_3)}\lambda(v_1)(v_2) := g_3(\lambda(g_1^{-1} v_1)(g_2^{-1} v_2)),$$

respectively. The above adjunction is equivariant for these two actions. If we restrict to the diagonal G-action, and take G-invariants, then the above adjunction induces the adjunction isomorphism

$$\mathrm{Hom}_{k[G]}(V_1 \otimes_k V_2, V_3) \xrightarrow{\cong} \mathrm{Hom}_{k[G]}(V_1, \mathrm{Hom}_k(V_2, V_3)) \ .$$

If the G-action on the V_i is smooth then this also can be written as an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}(G)}(V_1 \otimes_k V_2, V_3) \cong \mathrm{Hom}_{\mathrm{Mod}(G)}(V_1, \underline{\mathrm{Hom}}(V_2, V_3)) \ . \tag{10}$$

Let $D(G)$ denote the unbounded derived category of $\mathrm{Mod}(G)$. The tensor product functor

$$\begin{aligned} \mathrm{Mod}(G) \times \mathrm{Mod}(G) &\longrightarrow \mathrm{Mod}(G), \\ (V_1, V_2) &\longmapsto V_1 \otimes_k V_2 \ , \end{aligned}$$

where the G-action on the tensor product is the diagonal one, is exact in both variables. Therefore it extends directly (i.e., without derivation) to the functor

$$\begin{aligned} D(G) \times D(G) &\longrightarrow D(G), \\ (V_1^\bullet, V_2^\bullet) &\longmapsto \mathrm{tot}_\oplus(V_1^\bullet \otimes_k V_2^\bullet) \ , \end{aligned}$$

which we usually denote simply by $V_1^\bullet \otimes_k V_2^\bullet$.[1] On the other hand, since $\mathrm{Mod}(G)$ is a Grothendieck category, we have for any V_0 in $\mathrm{Mod}(G)$ the total derived functor

$$R\underline{\mathrm{Hom}}(V_0, -) : D(G) \longrightarrow D(G)$$

such that $R^j\underline{\mathrm{Hom}}(V_0, V) = \underline{\mathrm{Ext}}^j(V_0, V)$ for any V in $\mathrm{Mod}(G)$ and $j \geq 0$. We want to extend this to a bifunctor $D(G)^{op} \times D(G) \to D(G)$. First we recall that $\mathrm{Mod}(G)$ has arbitrary direct products (but which are not exact); we will denote these by $\prod^\infty$ to avoid confusion with the cartesian direct product. Hence, for any two complexes $V_1^\bullet$ and $V_2^\bullet$ in $\mathrm{Mod}(G)$ we may define the complex

$$\underline{\mathrm{Hom}}^\bullet(V_1^\bullet, V_2^\bullet) := \prod_{j \in \mathbb{Z}}{}^{\infty} \underline{\mathrm{Hom}}(V_1^j, V_2^{j+\bullet})$$

in $\mathrm{Mod}(G)$ in the usual way. By construction we have that

$$\begin{aligned} \underline{\mathrm{Hom}}^\bullet(V_1^\bullet, V_2^\bullet) &= \varinjlim_K \big(\prod_{j \in \mathbb{Z}} \underline{\mathrm{Hom}}(V_1^j, V_2^{j+\bullet})\big)^K = \varinjlim_K \prod_{j \in \mathbb{Z}} \underline{\mathrm{Hom}}(V_1^j, V_2^{j+\bullet})^K \\ &= \varinjlim_K \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}(K)}(V_1^j, V_2^{j+\bullet}) \qquad (11) \\ &= \varinjlim_K \mathrm{Hom}^\bullet_{\mathrm{Mod}(K)}(V_1^\bullet, V_2^\bullet) \end{aligned}$$

is the inductive limit over all compact open subgroups $K \subseteq G$ of the usual Hom-complexes for the abelian categories $\mathrm{Mod}(K)$.

The adjunction (10) shows that the assumptions of [KS] Theorem 14.4.8 are satisfied (with $\mathcal{P}_i = \mathcal{C}_i = \mathrm{Mod}(G)$, G the tensor product functor, and $F_1 = F_2 = \underline{\mathrm{Hom}}$). Hence we obtain the following result.

[1]This uses the fact that for any two complexes of vector spaces one of which is acyclic their tensor product is acyclic as well. Indeed $h^*(V_1^\bullet \otimes_k V_2^\bullet) \simeq h^*(V_1^\bullet) \otimes_k h^*(V_2^\bullet)$ by the Künneth formula. Recall that k is a field.

Proposition 3.1. *The total derived functor* $R\underline{\mathrm{Hom}}(-,-) : D(G)^{op} \times D(G) \longrightarrow D(G)$ *exists and can be computed by* $R\underline{\mathrm{Hom}}(V_1^\bullet, V_2^\bullet) = \underline{\mathrm{Hom}}^\bullet(V_1^\bullet, J^\bullet)$ *where* $V_2^\bullet \xrightarrow{\simeq} J^\bullet$ *is a homotopically injective resolution. Moreover, there are the natural adjunctions*

$$\mathrm{Hom}_{D(G)}(V_1^\bullet \otimes_k V_2^\bullet, V_3^\bullet) = \mathrm{Hom}_{D(G)}(V_1^\bullet, R\underline{\mathrm{Hom}}(V_2^\bullet, V_3^\bullet))$$

and

$$R\,\mathrm{Hom}_{\mathrm{Mod}(G)}(V_1^\bullet \otimes_k V_2^\bullet, V_3^\bullet) = R\,\mathrm{Hom}_{\mathrm{Mod}(G)}(V_1^\bullet, R\underline{\mathrm{Hom}}(V_2^\bullet, V_3^\bullet))$$

for any $V_i^\bullet$ *in* $D(G)$.

Remark 3.2. For future reference we mention that the local version of the above adjunction also holds. That is

$$R\underline{\mathrm{Hom}}(V_1^\bullet \otimes_k V_2^\bullet, V_3^\bullet) = R\underline{\mathrm{Hom}}(V_1^\bullet, R\underline{\mathrm{Hom}}(V_2^\bullet, V_3^\bullet))$$

for all $V_i^\bullet$. To see this pick a homotopically injective resolution $V_3^\bullet \xrightarrow{\simeq} J^\bullet$ in $\mathrm{Mod}(G)$. Note that $J^\bullet$ remains homotopically injective upon restriction to any compact open subgroup $K \subseteq G$ (by Frobenius reciprocity and exactness of ind_K^G). Furthermore $\underline{\mathrm{Hom}}^\bullet(V_2^\bullet, J^\bullet)$ is homotopically injective by adjunction and the previous footnote. By Proposition 3.1 for K we have

$$\mathrm{Hom}^\bullet_{\mathrm{Mod}(K)}(V_1^\bullet \otimes_k V_2^\bullet, J^\bullet) = \mathrm{Hom}^\bullet_{\mathrm{Mod}(K)}(V_1^\bullet, \underline{\mathrm{Hom}}^\bullet(V_2^\bullet, J^\bullet)).$$

Taking the limit over K and invoking the description (11) gives the result.

Corollary 3.3. $(D(G), \otimes_k, k, R\underline{\mathrm{Hom}})$ *is a closed symmetric monoidal category.*

For $V_2 = k$ viewed as complex concentrated in degree zero we, in particular, obtain the total derived duality functor

$$R\underline{\mathrm{Hom}}(-,k) : D(G)^{op} \longrightarrow D(G)$$

such that $R^j\underline{\mathrm{Hom}}(V,k) = \underline{\mathrm{Ext}}^j(V,k) = S^j(V)$ for any V in $\mathrm{Mod}(G)$ and any $j \geq 0$. In order to see in which way k is a dualizing object for $\mathrm{Mod}(G)$ we have to introduce two finiteness conditions.

First we make the following observation.

Lemma 3.4. *The functor* $R\underline{\mathrm{Hom}}(-,k)$ *is way-out in both directions, and in particular respects* $D^b(G)$.

Proof. We refer to [Har, p. 68] for what it means to be way-out, but the actual definition is not important here. By [Har, Proposition I.7.6] $R\,\mathrm{Hom}_{\mathrm{Mod}(K)}(-,k)$ is way-out (in both directions) if and only if there is an n_0 such that $\mathrm{Ext}^i_{\mathrm{Mod}(K)}(V,k) = 0$ for all $V \in \mathrm{Mod}(K)$ and $i > n_0$. By (the proof of) Lemma 2.2 we may take $n_0 = d$ when K is sufficiently small. Finally by (11) we conclude that $R\underline{\mathrm{Hom}}(-,k)$ itself is way-out. □

Remark 3.5. In general the trivial G-representation k does not have finite injective dimension in $\mathrm{Mod}(G)$. Nevertheless, as the previous proof shows, we have

$$R\underline{\mathrm{Hom}}(V,k) \in D^{[0,d]}(G)$$

for all V in $\mathrm{Mod}(G)$.

Next we recall that a representation V in $\mathrm{Mod}(G)$ is called admissible if, for any open subgroup $K \subseteq G$, the vector space of K-fixed vectors V^K is finite dimensional. In fact, it suffices to check the defining condition for a single compact open subgroup K (apply the Nakayama lemma to the dual $\Omega(K)$-module $V^\vee$ or see [Koh] Lemma 1.7). The full subcategory $\mathrm{Mod}_{adm}(G)$ of admissible representations in $\mathrm{Mod}(G)$ is a Serre subcategory (cf. [Em1] Proposition 2.2.13). Hence we have the strictly full triangulated subcategories $D^b_{adm}(G) \subseteq D^b(G)$ and $D_{adm}(G) \subseteq D(G)$ of those complexes whose cohomology representations are admissible.

Lemma 3.6. *The derived duality functor $R\underline{\mathrm{Hom}}(-,k)$ respects both subcategories $D^b_{adm}(G)$ and $D_{adm}(G)$.*

Proof. It is shown in [Koh] Corollary 3.15 that for an admissible representation V in $\mathrm{Mod}(G)$ the representations $S^j(V)$ are admissible as well. Hence for an admissible V the complex $R\underline{\mathrm{Hom}}(V,k)$ lies in $D^b_{adm}(G)$. On the other hand we have observed already that our functor is way-out in both directions in the sense of [Har] §7. Therefore our assertion follows from loc. cit. Proposition I.7.3. $\square$

Let $V^\bullet$ be any complex in $\mathrm{Mod}(G)$ and fix an injective resolution $k \xrightarrow{\simeq} \mathcal{J}^\bullet$. We construct a natural transformation

$$\eta_{V^\bullet} : V^\bullet \longrightarrow \underline{\mathrm{Hom}}^\bullet(\underline{\mathrm{Hom}}^\bullet(V^\bullet, \mathcal{J}^\bullet), \mathcal{J}^\bullet) \tag{12}$$

as follows. Inserting the definitions we have to produce, for any $\ell \in \mathbb{Z}$, a natural G-equivariant map

$$\eta_{V^\ell} : V^\ell \longrightarrow \prod_{j\in\mathbb{Z}}^{\infty} \underline{\mathrm{Hom}}(\prod_{i\in\mathbb{Z}}^{\infty} \underline{\mathrm{Hom}}(V^i, \mathcal{J}^{i+j}), \mathcal{J}^{j+\ell})$$

compatible with the differentials. It is straightforward to check that the maps

$$\eta_{V^\ell}(v)_j((f_{i,j})_i) := (-1)^{\ell j} f_{\ell,j}(v)$$

have these properties.

Proposition 3.7. *If the complex $V^\bullet$ has admissible cohomology then the natural transformation $\eta_{V^\bullet}$ is a quasi-isomorphism.*

Proof. Since we have a natural transformation between way-out functors the lemma on way-out-functors ([Har] Prop. I.7.1(iii)) tells us that we need to establish the assertion only in the case where our complex is a single admissible representation (viewed as a complex concentrated in degree zero). In fact, by loc. cit. Prop. I.7.1(iv) we can go one step further. Suppose given a class $\mathcal{P}$ of admissible representations such that every admissible representation is embeddable into a finite direct sum of representations in this class. Then it suffices to check the assertion for representations in $\mathcal{P}$. We cannot apply this directly, though. First let us fix a compact open subgroup K in G. Then we observe:

- Any admissible G-representation V is also admissible as a K-representation;
- $k \xrightarrow{\simeq} \mathcal{J}^\bullet$ is also an injective resolution in $\mathrm{Mod}(K)$;
- the natural transformation η_V remains the same if constructed for V considered only as a K-representation.

This means that, for the purposes of our proof, we may assume that our group G is compact. Let $C^\infty(G,k)$ denote, as before, the vector space of k-valued locally constant functions on G. Equipped with the left translation action it is an admissible smooth G-representation. We have $C^\infty(G,k)^\vee = \Omega(G)$. Let V be any admissible representation in $\mathrm{Mod}(G)$. Then $V^\vee$ is a finitely generated (pseudocompact) $\Omega(G)$-module ([Koh] Proposition 1.9(i)). Hence we find a surjection $\Omega(G)^m \twoheadrightarrow V^\vee$ in $\mathrm{Mod}_{pc}(\Omega(G))$ for some integer $m \geq 0$. It is the dual of an injective map $V \hookrightarrow C^\infty(G,k)^m$ in $\mathrm{Mod}(G)$. Therefore we can take the single object $C^\infty(G,k)$ for the class $\mathcal{P}$. By [Koh] Proposition 3.13 we have, for any integer j, that

$$R^j\underline{\mathrm{Hom}}(C^\infty(G,k),k) = S^j(C^\infty(G,k)) \cong \begin{cases} \chi_G \otimes_k C^\infty(G,k) & \text{for } j = d, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_G : G \to k^\times$ is Kohlhaase's duality character. Hence $R\underline{\mathrm{Hom}}(C^\infty(G,k),k) \simeq (\chi_G \otimes_k C^\infty(G,k))[-d]$ and then $R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(C^\infty(G,k),k),k) \simeq C^\infty(G,k)$. One checks from the proof in loc. cit. that the latter quasi-isomorphism is induced by the natural transformation $\eta_{C^\infty(G,k)}$. $\square$

In other words:

Corollary 3.8. *On $D_{adm}(G)$ the functor $R\underline{\mathrm{Hom}}(-,k)$ is involutive.*

Next we extend the involutivity of $R\underline{\mathrm{Hom}}(-,k)$ to a potentially larger category.

4. Globally admissible complexes

In this section we will generalize some of the results in Section 3 to a subcategory of $D(G)$ which is potentially larger than $D_{adm}(G)$. The possible drawback is that the defining condition for this subcategory is a "global" finiteness condition.

We let Vec denote the abelian category of k-vector spaces and $D(k)$ its unbounded derived category. In the following we fix an open subgroup $U \subseteq G$ which is pro-p and torsion free. As recalled in the proof of Lemma 2.2 the functor

$$\begin{aligned} \mathrm{Mod}(G) &\longrightarrow \mathrm{Vec}, \\ V &\longmapsto V^U = H^0(U,V) \end{aligned}$$

has finite cohomological dimension d. Hence its total derived functor $RH^0(U,-) : D(G) \longrightarrow D(k)$ exists (cf. [Har] Corollary I.5.3)). It is given by composing

$$R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k,-) : D(U) \longrightarrow D(k)$$

with the restriction functor forget : $D(G) \longrightarrow D(U)$.

On the other hand the functor $\mathrm{Hom}_k(-,k)$ on Vec of taking the k-linear dual is exact and therefore passes directly to a functor form $D(k)^{op}$ to $D(k)$ which, for simplicity, we also denote by $\mathrm{Hom}_k(-,k)$.

Theorem 4.1. *The diagram*

$$\begin{array}{ccc} D(G)^{op} & \xrightarrow{R\underline{\mathrm{Hom}}(-,\hat{I})} & D(G) \\ \downarrow{\scriptstyle \text{forget}} & & \downarrow{\scriptstyle \text{forget}} \\ D(U)^{op} & \xrightarrow{R\underline{\mathrm{Hom}}(-,\hat{I})} & D(U) \\ \downarrow{\scriptstyle R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k,-)} & & \downarrow{\scriptstyle R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k,-)} \\ D(k)^{op} & \xrightarrow{\mathrm{Hom}_k(-,k)[-d]} & D(k) \end{array}$$

is commutative (up to a natural isomorphism). More precisely, there is a natural isomorphism of functors

$$RH^0(U, R\underline{\mathrm{Hom}}(-, \hat{I})) \xrightarrow{\sim} \mathrm{Hom}_k(RH^0(U,-), k)[-d].$$

Proof. The upper rectangle is commutative since restriction from G to U preserves homotopically injective resolutions. For the lower triangle we first observe that the second adjunction formula in Proposition 3.1 tells us that the composed functor $RH^0(U, R\underline{\mathrm{Hom}}(-, \hat{I}))$ is naturally isomorphic to the functor $R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(-, \hat{I})$. Hence it remains to exhibit a natural isomorphism

$$R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(-, \hat{I}) \longrightarrow \mathrm{Hom}_k(RH^0(U,-), k)[-d].$$

For this we start with the Yoneda pairing

$$R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(V^\bullet, \hat{I}) \times R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, V^\bullet) \longrightarrow R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, \hat{I}) \ .$$

By our assumption on the group U the natural homomorphism

$$\tau^{\leq d} R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, \hat{I}) \xrightarrow{\cong} R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, \hat{I})$$

is an isomorphism and the upper truncation $\tau^{\leq d} R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, \hat{I})$ at degree d (cf. [Har] p. 69/70) maps to its cohomology $H^d(U, \hat{I})[-d] \cong k[-d]$ in degree d. (The latter identification is given by the trace map ϱ in Verdier's appendix to [CG].)

The Yoneda pairing therefore induces a pairing

$$R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(V^\bullet, \hat{I}) \times R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, V^\bullet) \longrightarrow k[-d]$$

and hence a natural homomorphism

$$R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(V^\bullet, \hat{I}) \longrightarrow \mathrm{Hom}_k(R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k, V^\bullet), k[-d]) \ .$$

To show that it is an isomorphism we need to check that the map induced on cohomology

$$\mathrm{Ext}^*_{\mathrm{Mod}(U)}(V^\bullet, \hat{I}) \longrightarrow \mathrm{Hom}_k(H^{d-*}(U, V^\bullet), k) \tag{13}$$

is bijective. If $V^\bullet$ is a single representation in degree zero then we have seen this already in (4). By Example 1 on p. 68 in [Har] the functor $RH^0(U,-)$ and hence also the functor $\mathrm{Hom}_k(R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(k,-), k[-d])$ are way-out in both directions. Similarly, by Lemma 2.2 and [Har] Proposition I.7.6 the functor $R\,\mathrm{Hom}_{\mathrm{Mod}(U)}(-, k)$ is way-out in both directions as well. Hence it follows from [Har] Proposition I.7.1(iii) that (13) is always bijective. □

Definition 4.2. A complex $V^\bullet$ in $D(G)$ is globally admissible if its cohomology groups $H^i(U, V^\bullet)$, for any $i \in \mathbb{Z}$, are finite dimensional vector spaces. Let $D(G)^a \subseteq D(G)$ denote the strictly full triangulated subcategory of all globally admissible complexes.

We will see only later in Corollary 4.6 that Definition 4.2, indeed, does not depend on the choice of U. To rephrase Definition 4.2 let $D_{fin}(k) \subseteq D(k)$ denote the strictly full triangulated subcategory of all objects all of whose cohomology vector spaces are finite dimensional. Then $D(G)^a$ is the full preimage in $D(G)$ of $D_{fin}(k)$ under the functor $RH^0(U,-)$.

Corollary 4.3. *The duality functor $R\underline{\mathrm{Hom}}(-k)$ respects the subcategory $D(G)^a$.*

Proof. This is immediate from Theorem 4.1 since the functor $\mathrm{Hom}_k(-,k)$ on $D(k)$ respects the subcategory $D_{fin}(k)$. □

In (12) we introduced the biduality morphism $\eta_{V^\bullet} : V^\bullet \to R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(V^\bullet, k), k)$. Our further analysis of it will be based upon the following general observation.

Lemma 4.4. *A homomorphism $V_1^\bullet \to V_2^\bullet$ in $D(G)$ is an isomorphism if and only if the induced map $H^i(U, V_1^\bullet) \to H^i(U, V_2^\bullet)$, for any $i \in \mathbb{Z}$, is bijective.*

Proof. This is an immediate consequence of the equivalence H between $D(G)$ and the derived category of a certain differential graded algebra in [DGA] Theorem 9. By construction the functor H has the property that $h^*(H(-)) = H^*(U, -)$. □

Theorem 4.5. *The biduality morphism $\eta_{V^\bullet}$, for any $V^\bullet$ in $D(G)$, is an isomorphism if and only if $V^\bullet$ lies in $D(G)^a$.*

Proof. According to Lemma 4.4 we have to check that the maps

$$H^i(U, \eta_{V^\bullet}) : H^i(U, V^\bullet) \to H^i(U, R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(V^\bullet, k), k))$$

are bijective for any $i \in \mathbb{Z}$ if and only if $V^\bullet$ lies in $D(G)^a$. By Proposition 4.1 we have natural isomorphisms

$$\xi^i_{V^\bullet} : H^i(U, R\underline{\mathrm{Hom}}(V^\bullet, \hat{I})) \xrightarrow{\cong} \mathrm{Hom}_k(H^{d-i}(U, V^\bullet), k) \ .$$

For the remainder of this proof we fix an isomorphism $\hat{I} \simeq k$. The trace map $\varrho : H^d(U, \hat{I}) \to k$ then yields an isomorphism $H^d(U, k) \simeq k$. We will just write k instead of $\hat{I}$ in what follows.

We now claim that the diagram

$$\begin{array}{ccc}
H^i(U, V^\bullet) & \xrightarrow{H^i(U, \eta_{V^\bullet})} & H^i(U, R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(V^\bullet, k), k)) \\
\Big\downarrow b & & \cong \Big\downarrow \xi^i_{R\underline{\mathrm{Hom}}(V^\bullet, k)} \\
\mathrm{Hom}_k(\mathrm{Hom}_k(H^i(U, V^\bullet), k), k) & \xrightarrow[\cong]{\mathrm{Hom}_k(\xi^{d-i}_{V^\bullet}, k)} & \mathrm{Hom}_k(H^{d-i}(U, R\underline{\mathrm{Hom}}(V^\bullet, k)), k),
\end{array}$$

where b denotes the natural map from a k-vector space into its double dual, is commutative up to the sign $(-1)^{i(d-i)}$. This immediately shows that $H^i(U, \eta_{V^\bullet})$ is bijective if and only if b is bijective which, of course, is the case if and only if the vector space $H^i(U, V^\bullet)$ is finite dimensional.

To establish this claim we compute $R\underline{\mathrm{Hom}}(-, k)$ by using an injective resolution $\mathcal{J}^\bullet$ of k in $\mathrm{Mod}(G)$ and hence in $\mathrm{Mod}(U)$. Then $R\underline{\mathrm{Hom}}(V^\bullet, k) = \underline{\mathrm{Hom}}^\bullet(V^\bullet, \mathcal{J}^\bullet)$ by Proposition 3.1. Moreover the adjunction property (10) implies that $\underline{\mathrm{Hom}}^\bullet(V^\bullet, \mathcal{J}^\bullet)$ always is homotopically injective. Finally we may also assume that $V^\bullet$ is homotopically injective. Our diagram therefore becomes

$$\begin{array}{ccc}
h^i((V^\bullet)^U) & \xrightarrow{H^i(U, \eta_{V^\bullet})} & \mathrm{Hom}_{K(U)}(\prod^\infty_{r \in \mathbb{Z}} \underline{\mathrm{Hom}}(V^r, \mathcal{J}^{r+\bullet}), \mathcal{J}^\bullet[i]) \\
\Big\downarrow b & & \cong \Big\downarrow \xi^i_{R\underline{\mathrm{Hom}}(V^\bullet, k)} \\
\mathrm{Hom}_k(\mathrm{Hom}_k(h^i((V^\bullet)^U), k), k) & \xrightarrow[\cong]{\mathrm{Hom}_k(\xi^{d-i}_{V^\bullet}, k)} & \mathrm{Hom}_k(\mathrm{Hom}_{K(U)}(V^\bullet, \mathcal{J}^\bullet[d-i]), k),
\end{array}$$

where $K(U)$ denotes as usual the unbounded homotopy category of complexes in $\mathrm{Mod}(U)$. We first recall that, under our identification $h^d((\mathcal{J}^\bullet)^U) = H^d(U, k) \cong k$, the map $\xi^i_{V^\bullet}$ is explicitly given by

$$\xi^i_{V^\bullet} : \mathrm{Hom}_{K(U)}(V^\bullet, \mathcal{J}^\bullet[i]) \longrightarrow \mathrm{Hom}_k(h^{d-i}((V^\bullet)^U), k),$$
$$[\epsilon^\bullet] \longmapsto \Big[[\delta_{d-i}] \longmapsto [\epsilon^{d-i}(\delta_{d-i})]\Big] .$$

Now let $[v_i] \in h^i((V^\bullet)^U)$. By definition of $\eta_{V^\bullet}$ its image under the top horizontal arrow in the above diagram is the homotopy class of the homomorphism of complexes

$$\prod_{r\in\mathbb{Z}}^{\infty} \underline{\mathrm{Hom}}(V^r, \mathcal{J}^{r+\bullet}) \longrightarrow \mathcal{J}^\bullet[i],$$
$$(f_{r,\bullet})_r \longmapsto (-1)^{i\bullet} f_{i,\bullet}(v_i)$$

induced by $\eta_{V^i}(v_i)_\bullet$. Under the right vertical arrow it is further mapped to the linear map

$$\mathrm{Hom}_{K(U)}(V^\bullet, \mathcal{J}^\bullet[d-i]) \longrightarrow k, \qquad (14)$$
$$[(f_{r,d-i})_r] \longmapsto (-1)^{i(d-i)}[f_{i,d-i}(v_i)].$$

But $[(f_{r,d-i})_r]$ corresponds under $\xi^{d-i}_{V^\bullet}$ to the linear map in $\mathrm{Hom}_k(h^i((V^\bullet)^U), k)$ sending $[\delta_i]$ to $[f_{i,d-i}(\delta_i)]$. Hence the preimage of (14) under the bottom horizontal map in the diagram is equal to $(-1)^{i(d-i)}b([v_i])$ as claimed. □

Corollary 4.6. *The subcategory $D(G)^a$ in $D(G)$ is independent of the choice of the subgroup $U \subseteq G$.*

What is the relation between the subcategories $D_{adm}(G)$ and $D(G)^a$? We had observed earlier that a representation V in $\mathrm{Mod}(G)$ is admissible if and only if the vector space $H^0(U, V)$ is finite dimensional. Moreover, by [Em2] Lemma 3.3.4, we have the following fact.

Lemma 4.7. *If V in $\mathrm{Mod}(G)$ is admissible then all the vector spaces $H^i(U, V)$, for $i \geq 0$, are finite dimensional.*

Lemma 4.7 says that, for an admissible V, the complex $RH^0(U, V)$ lies in $D_{fin}(k)$. By Example 1 on p. 68 in [Har] the functor $RH^0(U, -)$ is way-out in both directions. Therefore [Har] Proposition I.7.3(iii) implies that the functor $RH^0(U, -)$ maps $D_{adm}(G)$ to $D_{fin}(k)$. This proves the following.

Proposition 4.8. $D_{adm}(G) \subseteq D(G)^a$.

Alternatively this can be seen by combining Proposition 3.7 and Proposition 4.5.

On the full subcategories $D^\pm(G)$ of complexes bounded below or above we have stronger results.

Proposition 4.9.

(i) *A complex $V^\bullet$ in $D^+(G)$ lies in $D_{adm}(G)$ if and only if $H^i(U, V^\bullet)$ is finite dimensional for any $i \in \mathbb{Z}$. I.e., we have*

$$D^+(G) \cap D_{adm}(G) = D^+(G) \cap D(G)^a.$$

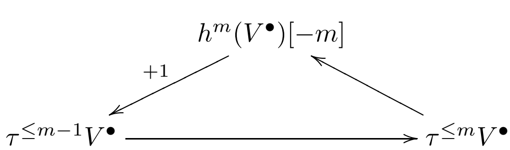
Similarly for $D^-(G)$.

(ii) *More generally, a globally admissible complex with <u>some</u> vanishing differential lies in the subcategory $D_{adm}(G)$.*

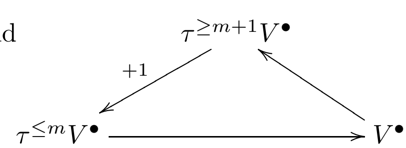
Proof. First of all, in part (i) it suffices to show the $D^+(G)$-version. For if $V^\bullet$ lies in $D^-(G)$ then its dual $R\underline{\mathrm{Hom}}(V^\bullet, k) = \underline{\mathrm{Hom}}^\bullet(V^\bullet, \mathcal{J}^\bullet)$ lies in $D^+(G)$. Furthermore $R\underline{\mathrm{Hom}}(V^\bullet, k)$ belongs to $D(G)^a$ if $V^\bullet$ does by Corollary 4.3. In that case, once we show the $D^+(G)$-version, we conclude that $R\underline{\mathrm{Hom}}(V^\bullet, k)$ is an object of $D_{adm}(G)$. However, by Lemma 3.6 the functor $R\underline{\mathrm{Hom}}(-, k)$ preserves $D_{adm}(G)$. Since the functor is involutive on $D(G)^a$ by Proposition 4.5 we conclude that $V^\bullet$ indeed belongs to $D_{adm}(G)$.

We proceed to show the $D^+(G)$-version in part (i). The direct implication holds true by Proposition 4.8. For the reverse implication we now assume that all the $H^i(U, V^\bullet)$ are finite dimensional, and $V^\bullet$ is bounded below.

Choose an integer m such that $h^j(V^\bullet) = 0$ for any $j < m$. In this situation it is a standard fact (cf. [KS] Exercise 13.3) that we have $H^0(U, h^m(V^\bullet)) = R^m H^0(U, V^\bullet) = H^m(U, V^\bullet)$. Hence $H^0(U, h^m(V^\bullet))$ is finite dimensional. As recalled before Lemma 4.7 this implies that $h^m(V^\bullet)$ is admissible. Moreover, Lemma 4.7 then says that $H^i(U, h^m(V^\bullet))$ is finite dimensional for any $i \in \mathbb{Z}$. We now use the distinguished triangles



and



in $D(G)$ (cf. [KS] Proposition 13.1.15(i)). Since $\tau^{\leq m-1}V^\bullet \simeq 0$ in $D(G)$ the left triangle implies that $H^i(U, \tau^{\leq m}V^\bullet) \cong H^{i-m}(U, h^m(V^\bullet))$ is finite dimensional for any $i \in \mathbb{Z}$. Using this as an input for the long exact cohomology sequence associated with the right triangle we conclude that $H^i(U, \tau^{\geq m+1}V^\bullet)$ is finite dimensional for any $i \in \mathbb{Z}$ as well. This proves the $n = 0$ case of the following statement P_n:

$$h^{m+n}(V^\bullet) \text{ is admissible and } \tau^{\geq m+n+1}V^\bullet \text{ is globally admissible.}$$

Proceeding inductively, to show $P_{n-1} \Rightarrow P_n$ for $n > 0$ we may repeat our initial reasoning for the complex $\tau^{\geq m+n}V^\bullet$. We obtain in particular that $h^j(V^\bullet)$ is admissible for any $j \in \mathbb{Z}$.

Finally part (ii) is a combination of the $D^\pm(G)$-versions. If the differential $V^n \to V^{n+1}$ vanishes one can decompose $V^\bullet$ as a sum of the two naive truncations $V^\bullet = \sigma^{\leq n}V^\bullet \oplus \sigma^{\geq n+1}V^\bullet$. If $V^\bullet$ is globally admissible so are the direct summands $\sigma^{\leq n}V^\bullet$ and $\sigma^{\geq n+1}V^\bullet$. Therefore they both lie in $D_{adm}(G)$ by part (i), which immediately implies $V^\bullet$ also lies in $D_{adm}(G)$ as claimed. □

Remark 4.10. One can relax the condition in part (ii) of Proposition 4.9 slightly. If $V^\bullet$ is *split somewhere*, meaning at some n there is a morphism $s : V^n \to V^{n-1}$ such that $dsd = d$, then the map $ds : V^n \to \ker(d)$ gives rise to a quasi-isomorphism $V^\bullet \to \tau^{\leq n}V^\bullet \oplus \tau^{\geq n+1}V^\bullet$. The direct sum is a complex with a vanishing differential at n. Applying (ii) shows that $V^\bullet$ lies in $D_{adm}(G)$ provided it is globally admissible. For the definition of a split complex we refer the reader to [Wei, Df. 1.4.1].

Unfortunately we do not have an example showing the inclusion in Proposition 4.8 could be strict for certain G.

Proposition 4.11. *For any $V^\bullet$ in $D(G)$ and any particular $i \in \mathbb{Z}$ we have*

$$H^i(U, V^\bullet) = 0 \Longrightarrow h^i(V^\bullet) = 0.$$

In particular, if $V^\bullet$ in $D(G)^a$ satisfies $H^i(U, V^\bullet) = 0$ for all $i \ll 0$ (resp. $i \gg 0$) then $V^\bullet$ belongs to $D^+_{adm}(G)$ (resp. $D^-_{adm}(G)$).

Proof. The proof of the first claim is almost literally the same argument as the one for the reverse implication in [DGA] Proposition 5, but for a single i. Now invoke Proposition 4.9. □

We finish with a characterization of $D^b_{adm}(G) = D^b(G) \cap D_{adm}(G)$.

Corollary 4.12. *The subcategory $D^b_{adm}(G)$ consists of all complexes $V^\bullet$ in $D(G)$ whose total cohomology $H^*(U, V^\bullet)$ is finite dimensional.*

Proof. This is an immediate consequence of Proposition 4.11, Lemma 4.7, and the hypercohomology spectral sequence. □

Remark 4.13. If G is compact then the natural functor

$$D^+(\mathrm{Mod}_{adm}(G)) \xrightarrow{\simeq} D^+_{adm}(G) := D^+(G) \cap D_{adm}(G)$$

is an equivalence. Similarly for $D^b_{adm}(G)$. This follows from [Em2] Proposition 2.1.9, and [Har, Proposition I.4.8] (which is also an easy consequence of [KS, Theorem 13.2.8]).

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