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FROM WEYL GROUPS TO SEMISIMPLE GROUPS

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ABSTRACT. In this paper we show, using ideas from the theory of total positivity, how a number of properties of a semisimple group over the complex numbers can be presented purely in terms of the Weyl group. We also describe some new connections of the theory of canonical bases with total positivity.

1. In this (partly expository) paper we show (using ideas from the theory of total positivity) that many concepts related to a semisimple group G over $\mathbf{C}$ of simply laced type can be presented purely in terms of the Weyl group. This paper contains also a few new results. For example, we give a new characterization of the involution ϕ studied in [L97] in connection with the totally positive flag manifold. In Section 12 we show that the canonical basis [L90] of a finite dimensional irreducible representation of G can be indexed by a set which appears in the theory of total positivity (and whose definition involves the $\mathbf{Z}$-version of ϕ).

In A3 we show that the totally positive flag manifold has something close to a base point (a closed subset of dimension equal to the rank of G).

In A4 we describe a new (conjectural) connection of the theory of canonical bases with total positivity.

2. We first define the Weyl group following Coxeter. (For simplicity we restrict ourselves to the simply laced case.) Consider a finite connected graph with set of vertices I' and with edges denoted by $i - -j$ such that there exists a function $h : I' \to \mathbf{Z}_{>0}$ with the following properties:

(1) for any $i \in I'$ we have $h(i) = (1/2) \sum_{j \in I'; i--j} h(j)$ (harmonicity),

(2) $h(i) = 1$ for some $i \in I'$.

Let I be the graph obtained from I' by removing one $i \in I'$ such that $h(i) = 1$. Coxeter has shown that the resulting graphs are exactly those that appear in the classification of (simply laced) simple Lie algebras.

Here is an example of the graph I' with the harmonic function h:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & 3 & & \end{matrix}$$

The edges are pairs of numbers written next to each other. The graph I (said to be of type E_8) with the restriction of h is

$$\begin{matrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & 3 & & \end{matrix}$$

In the rest of this paper the graph I is fixed. Let E be the $\mathbf{Q}$-vector space with basis $\{\check{\alpha}_i; i \in I\}$. For $i \in I$ we define an automorphism $s_i : E \to E$ by $\check{\alpha}_j \mapsto \check{\alpha}_j - a_{ij}\check{\alpha}_i$

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where a_{ij} is 2 if $i=j$, is -1 if $i--j$, and is 0 if $i\neq j$ don't form an edge of I. Let W be the group of automorphisms of E generated by $\{s_i; i\in I\}$. This is the Weyl group. It is finite. For $w\in W$ we can write w as a product of s_i; the minimum number of factors in such a product is denoted by $|w|$. For example $|1|=0$; at the other extreme there is a unique $w\in W$ for which $|w|$ is maximum; we denote it by w_0 and we set $\nu=|w_0|$. Let $i\mapsto i^!$ be the involution of I such that $w_0s_iw_0=s_{i^!}$ for all $i\in I$.

3. It is known that to our graph (or to W) corresponds a simply connected semisimple algebraic group G over $\mathbf{C}$. Now G has two important (unipotent) subgroups, U^+,U^-. (For example to the graph with $I=\{i,j\}$ and with $i--j$ corresponds the algebraic group $SL_3(\mathbf{C})$ and U^+,U^- is the group of upper triangular or lower triangular matrices with 1 on diagonal; in this case, W is the symmetric group in 3 letters.) We would like to show how to construct G from W. We will first try to construct U^+,U^- from W. A similar method applies to the full G but this case will be only sketched.

4. Let $U_{\geq 0}$ be the semigroup with generators $\{i^a; i\in I, a\in\mathbf{R}_{>0}\}$ and relations (similar to those of a Coxeter group):

$i^ai^b=i^{a+b}$ for $i\in I$, a,b in $\mathbf{R}_{>0}$;
$i^aj^bi^c=j^{bc/(a+c)}i^{a+c}j^{ab/(a+c)}$ if $a_{ij}=-1$, a,b,c in $\mathbf{R}_{>0}$;
$i^aj^b=j^bi^a$ if $a_{ij}=0$, a,b in $\mathbf{R}_{>0}$.

There is a unique semigroup anti-automorphism $\Psi: U_{\geq 0}\to U_{\geq 0}$ such that $\Psi(i^a)=i^a$ for all $i\in I, a\in\mathbf{R}_{>0}$. We have $\Psi^2=1$.

Let $\mathcal{I}$ be the set of all sequences $\mathbf{i}=(i_1,\dots,i_\nu)$ in I such that $w_0=s_{i_1}\dots s_{i_\nu}$. For $\mathbf{i}\in\mathcal{I}$ we define $\kappa_{\mathbf{i}}:\mathbf{R}^\nu_{>0}\to U_{\geq 0}$ by

$$\mathbf{c}=(c_1,\dots,c_\nu)\mapsto \mathbf{i}^{\mathbf{c}}:=i_1^{c_1}i_2^{c_2}\dots i_\nu^{c_\nu}.$$

One can show that this map is injective and its image is independent of the choice of $\mathbf{i}$. We denote this image by $U_{>0}$. It is closed under multiplication in $U_{\geq 0}$ and is stable under Ψ.

Let $\mathbf{N}(Z_1,\dots,Z_\nu)$ be the set of rational functions (coefficients in $\mathbf{Q}$) in the indeterminates $Z_1,\dots,Z_\nu$ which are of the form $P(Z_1,\dots,Z_\nu)/P'(Z_1,\dots,Z_\nu)$ where P and P' are (nonempty) sums of monomials in $Z_1,\dots,Z_\nu$.

If $\mathbf{i}\in\mathcal{I},\mathbf{i}'\in\mathcal{I}$, then by [L94], $\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'}:\mathbf{R}^\nu_{>0}\to\mathbf{R}^\nu_{>0}$ is of the form

(a) $(z_1,\dots,z_\nu)\mapsto(\pi_1(z_1,\dots,z_\nu),\dots,\pi_\nu(z_1,\dots,z_\nu))$

where

(b) each $\pi_1,\dots,\pi_\nu$ belongs to $\mathbf{N}(Z_1,\dots,Z_\nu)$.

It follows that $\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'}$ can be regarded as a birational equivalence $\mathbf{C}^\nu-->\mathbf{C}^\nu$. Let $O[\mathbf{C}^\nu]$ be the algebra of regular functions $\mathbf{C}^\nu\to\mathbf{C}$ and let $O(\mathbf{C}^\nu)$ be the quotient field of this algebra. Now $\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'}$ induces a field isomorphism $O(\mathbf{C}^\nu)\to O(\mathbf{C}^\nu)$ denoted by $(\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'})_*$ (it is given by sending an element of $O(\mathbf{C}^\nu)$ to its composition with $\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'}$). Let $\underline{O}[U]$ be the set of all $(f_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}$ where $f_{\mathbf{i}}\in O(\mathbf{C}^\nu)$ satisfy:

$(\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'})_*(f_{\mathbf{i}'})=f_{\mathbf{i}}$ for any $\mathbf{i},\mathbf{i}'$ in $\mathcal{I}$;
$f_{\mathbf{i}}\in O[\mathbf{C}^\nu]$ for any $\mathbf{i}\in\mathcal{I}$.

This is a commutative algebra in an obvious way. The following result was conjectured in [L19, §6] and proved in [FL21].

(c) $\underline{O}[U]$ *is the algebra of regular functions* $O[U]$ *on a unipotent algebraic group* U *over* $\mathbf{C}$.

Note that any element of $U_{>0}$ gives rise (via evaluation) to an algebra homomorphism $\underline{O}[U] \to \mathbf{C}$; thus $U_{>0}$ can be regarded as a subset of U. The multiplication on U extends that on $U_{>0}$ and this defines it uniquely (by the requirement that it is regular). Now U is the same as U^+ in Section 3.

5. Let $O[U]_{\geq 0}$ be the set of all $(f_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}$ in $\underline{O}[U]$ such that for any $\mathbf{i} \in \mathcal{I}$, the function $\mathbf{C}^\nu \to \mathbf{C}$ given by $(c_1, \dots, c_\nu) \mapsto f_{\mathbf{i}}(c_1, \dots, c_\nu)$ is a polynomial in $(c_1, \dots, c_\nu)$ with coefficients in $\mathbf{R}_{\geq 0}$. Note that $O[U]_{\geq 0}$ is closed under addition, under multiplication and under scalar multiplication by elements in $\mathbf{R}_{\geq 0}$ (but not under substraction).

One can also define $O[U]'_{\geq 0}$ as the subset of $O[U]$ consisting of all $\mathbf{R}_{\geq 0}$-linear combinations of the elements in the dual canonical basis [L90] (at parameter 1) of $O[U]$; from the positivity properties of the canonical basis one can deduce that $O[U]'_{\geq 0} \subset O[U]_{\geq 0}$. It would be interesting to prove the reverse inclusion. (See A4 in the Appendix for an attempt in this direction.)

6. The semigroup $\mathfrak{G}(\mathbf{R}_{>0})$ defined in [L19, 2.10] by generators $i^a, \underline{i}^a, -i^a$ $(i \in I, a \in \mathbf{R}_{>0})$ and certain relations will be denoted here by $G_{\geq 0}$. We write $G_{>0}$ for the subset of $G_{\geq 0}$ which in [L19, 2.19] is denoted by $\mathfrak{G}(\mathbf{R}_{>0})_{w_0,-w_0}$; this is a subsemigroup of $G_{\geq 0}$. Let $M = 2\nu + |I|$. In [L19, 2.13(b)] a family of bijections $\theta_{\mathbf{h}} : \mathbf{R}^M_{>0} \to G_{>0}$ is described. Here $\mathbf{h}$ runs over a certain set of sequences with M terms; we will take $\mathbf{h}$ to be a sequence of a special kind, that is either:

- the first ν terms form a sequence in $\mathcal{I}$; the last ν terms form a sequence in $\mathcal{I}$ (with the sign $-$ attached) and the middle $|I|$ terms form a list of the elements of I (underlined), or

- the first ν terms form a sequence in $\mathcal{I}$ (with the sign $-$ attached); the last ν terms form a sequence in $\mathcal{I}$ and the middle $|I|$ terms form a list of the elements of I (underlined).

These sequences form a finite set $\mathbf{H}$. The compositions $\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'} : \mathbf{R}^M_{>0} \to \mathbf{R}^M_{>0}$ (with $\mathbf{h}, \mathbf{h}'$ in $\mathbf{H}$) satisfy a property similar to 4(a), (b). It follows that $\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'} : \mathbf{R}^M_{>0} \to \mathbf{R}^M_{>0}$ can be regarded as a birational equivalence $\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu - - > \mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu$. Let $O[\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu]$ be the algebra of regular functions $\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu \to \mathbf{C}$ and let $O(\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu)$ be the quotient field of this algebra. Now $\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'}$ induces a field isomorphism $O(\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu) \to O(\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu)$ denoted by $(\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'})_*$ (it is given by sending an element of $O(\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu)$ to its composition with $\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'}$). Let $\underline{O}[G]$ be the set of all $(f_{\mathbf{h}})_{\mathbf{h}\in\mathbf{H}}$ where $f_{\mathbf{h}} \in O(\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu)$ satisfy:

$(\theta_{\mathbf{h}}^{-1}\theta_{\mathbf{h}'})_*(f_{\mathbf{h}'}) = f_{\mathbf{h}}$ for any $\mathbf{h}, \mathbf{h}'$ in $\mathbf{H}$;
$f_{\mathbf{h}} \in O[\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu]$ for any $\mathbf{h} \in \mathbf{H}$.

This is a commutative algebra in an obvious way. The following result was conjectured in [L19, §6] and proved in [FL21].

(c) $\underline{O}[G]$ *is the algebra* $O[G]$ *of regular functions on a semisimple simply connected algebraic group* G *over* $\mathbf{C}$.

Note that any element of $G_{>0}$ gives rise (via evaluation) to an algebra homomorphism $\underline{O}[G] \to \mathbf{C}$; thus $G_{>0}$ can be regarded as a subset of G. The multiplication on G extends that on $G_{>0}$ and this defines it uniquely (by the requirement that it is regular). Now this G is the same as G of Section 3.

Let $O[G]_{\geq 0}$ be the set of all $(f_{\mathbf{h}})_{\mathbf{h}\in\mathbf{H}}$ in $\underline{O}[G]$ such that for any $\mathbf{h} \in \mathbf{H}$, the function $\mathbf{C}^\nu \times (\mathbf{C}^*)^{|I|} \times \mathbf{C}^\nu \to \mathbf{C}$ given by $(c_1, \dots, c_M) \mapsto f_{\mathbf{h}}(c_1, \dots, c_M)$ is an $\mathbf{R}_{\geq 0}$-linear combination of functions $(c_1, \dots, c_M) \mapsto c_1^{k_1} \dots c_M^{k_M}$ where $k_1, \dots, k_M$ are integers of which the first ν and the last ν are ≥ 0. Note that $O[G]_{\geq 0}$ is closed under addition, under multiplication and under scalar multiplication by elements in $\mathbf{R}_{\geq 0}$ (but not under substraction).

One can also define $O[G]'_{\geq 0} \subset O[G]$ as the subset of $O[G]$ consisting of all $\mathbf{R}_{\geq 0}$-linear combinations of the elements in the dual canonical basis (at parameter 1) of $O[G]$; it would be again interesting to compare it with $O[G]_{\geq 0}$. (See A5 in the Appendix for an attempt in this direction.)

One can define similarly the subset $O[G/U^-]_{\geq 0}$ of the coordinate algebra $O[G/U^-]$ of G/U^- in terms of the description of the algebra $O[G/U^-]$ conjectured in [L19, §6] and proved in [FL21]. Recall that

$$O[G/U^-] = \oplus_{\lambda \in \mathbf{N}^I} O[G/U^-]^\lambda$$

where $O[G/U^-]^\lambda$ are the irreducible finite dimensional representations of G. We define

$$O[G/U^-]^\lambda_{\geq 0} = O[G/U^-]^\lambda \cap O[G/U^-]_{\geq 0}.$$

Then $O[G/U^-]^\lambda_{\geq 0}$ is a subset of $O[G/U^-]^\lambda$ closed under addition, under scalar multiplication by elements in $\mathbf{R}_{\geq 0}$ and under the action of $G_{\geq 0}$. It would be again interesting to compare this subset with the subset of $O[G/U^-]^\lambda$ consisting of all $\mathbf{R}_{\geq 0}$-linear combinations of the elements in the dual canonical basis (at parameter 1) of $O[G/U^-]^\lambda$.

7. Now let K be a semifield, that is a set with two operations: $+, \times$ such that K is an abelian group with respect to $\times$, an abelian semigroup with respect to $+$ and such that the distributivity law $(a + b)c = ac + bc$ is satisfied. Here are three examples of semifields.

(i) $K = \mathbf{R}_{>0}$ with the usual $+, \times$;

(ii) $K = \mathbf{Z}$ with the semifield structure in which the sum of a, b is $\min(a, b)$ and the product of a, b is $a + b$;

(iii) $K = \{1\}$ with $1 + 1 = 1, 1 \times 1 = 1$.

If $\mathbf{i} \in \mathcal{I}, \mathbf{i}' \in \mathcal{I}$ and if we take $(z_1, \dots, z_\nu) \in K^\nu$, then (in view of 4(b)), the right hand side of 4(a) makes sense as an element of K^ν, so that 4(a) defines a map $(\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'})_K : K^\nu \to K^\nu$ (which is inverse to $(\kappa_{\mathbf{i}'}^{-1}\kappa_{\mathbf{i}})_K$ hence is a bijection).

Let U_K be the set of all $(\xi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}$ where $\xi_{\mathbf{i}} \in K^\nu$ satisfy:

$(\kappa_{\mathbf{i}}^{-1}\kappa_{\mathbf{i}'})_K(\xi_{\mathbf{i}'}) = \xi_{\mathbf{i}}$ for any $\mathbf{i}, \mathbf{i}'$ in $\mathcal{I}$.

Note that if $\mathbf{i} \in \mathcal{I}$, then $\mathbf{i}$ defines a bijection $K^\nu \xrightarrow{\sim} U_K$ whose inverse is $(\xi_{\mathbf{i}'})_{\mathbf{i}'\in\mathcal{I}} \mapsto \xi_{\mathbf{i}}$. We denote this bijection by $\mathbf{c} \mapsto \mathbf{i}^{\mathbf{c}}$. We have $U_{\mathbf{R}_{>0}} = U_{>0}$. From the definitions we see that the semigroup structure on $U_{>0}$ induces a semigroup structure on U_K.

There is a well defined involution $\Psi_K : U_K \to U_K$ given by $(\xi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}} \mapsto (\xi'_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}$ where for $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{I}$ we have $\xi'_{\mathbf{i}} = (c_1, \dots, c_\nu)$ whenever $\xi_{i_\nu,\dots,i_1} = (c_\nu, \dots, c_1)$. This is an anti-automorphism of U_K; when $K = \mathbf{R}_{>0}$, it coincides with the restriction of $\Psi : U_{\geq 0} \to U_{\geq 0}$ (in Section 4) to $U_{>0}$.

8. For $p = (p_i) \in K^I$ we define a bijection $S_p : U_K \to U_K$ by

$$(\xi_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}} \mapsto (\xi'_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}}$$

where $\xi'_{\mathbf{i}}$ is obtained by multiplying $\xi_{\mathbf{i}}$ by $(p_{i_1}, p_{i_2}, \dots, p_{i_\nu})$ (component by component in K) where $\mathbf{i} = (i_1, \dots, i_\nu)$. See [L19, 4.3(a)]. From the definitions we have $S_p\Psi_K = \Psi_K S_p$.

We define an element $u(1) \in U_K$ as follows. Let $\mathbf{i} \in \mathcal{I}$. For $k \in [1, \nu]$ we have

$$s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\check{\alpha}_{i_k}) = \sum_{i\in I} r_{i,k}\check{\alpha}_i$$

where $r_{i,k} \in \mathbf{N}$. Let $r'_k = \sum_{i\in I} r_{i,k} \in \mathbf{Z}_{>0}$. Let $\mathbf{c} = (r'_1, r'_2, \dots, r'_\nu) \in K^\nu$. As in [L94, 11.2] we see that $\mathbf{i}^{\mathbf{c}} \in U_K$ is independent of the choice of $\mathbf{i}$; we denote it by $u(1)$. We define $q = (q_i) \in K^I$ by $q_i = \sum_{k\in[1,\nu]} r_{i,k} \in \mathbf{Z}_{>0} \subset K$; this is also independent of the choice of $\mathbf{i}$.

We define an imbedding $K^I \to U_K$ by $p \mapsto u(p) := S_p(u(1))$.

9. For $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in K^\nu$ and $c \in K$ we set ${}_c\mathbf{c} = (cc_1, c_2, \dots, c_\nu) \in K^\nu$. For any $i \in I, c \in K$ there is a unique bijection $T_{i,c} : U_K \to U_K$ such that $T_{i,c}(\mathbf{i}^{\mathbf{c}}) = \mathbf{i}^{{}_c\mathbf{c}}$ for some/any $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{I}$ such that $i_1 = i$ and any $\mathbf{c} \in K^\nu$ (see [L97, 2.3] for the case $K = \mathbf{Z}$ and [L19, 2.16] for a general K).

We regard U_K as the set of vertices of a graph in which $u \ne u'$ are joined if $u' = T_{i,c}(u)$ for some $i \in I, c \in K$. We have the following result.

(a) *If $K = \mathbf{R}_{>0}$, this graph is connected.*

The proof is given in the Appendix, see A2. (An analogous result in which $\mathbf{R}_{>0}$ is replaced by $\mathbf{Z}$ in 7(ii) appears in [L97, 2.8].)

10. We have the following result. (An analogous result in which $\mathbf{R}_{>0}$ is replaced by $\mathbf{Z}$ in 7(ii) appears in [L97, 2.9].)

(a) *There is a unique bijection $\phi : U_{>0} \to U_{>0}$ such that*
(i) $T_{i,c}\phi = \phi T_{i^!,c^{-1}}$ for all $i \in I, c \in \mathbf{R}_{>0}$,
(ii) $\phi(u(1)) = u(q^{-1})$.

The existence is proved in the Appendix, see A1. The uniqueness of ϕ follows from 9(a).

For example, if $I = \{i\}$, then ϕ is given by $i^c \mapsto i^{c^{-1}}$; if $I = \{i, j\}$ with $i - -j$, then ϕ is given by

$$i^a j^b i^c \mapsto i^{a/c(a+c)} j^{(a+c)/ab} i^{1/(a+c)} = j^{c/ab} i^{1/c} j^{1/b}.$$

11. The results in this subsection are based on the identification of ϕ in Section 10 with the bijection with the same name in [L97] (see the proof in A1). For $p \in \mathbf{R}^I_{>0}$ we have $S_p\phi = \phi S_{p^{-1}}$. (See [L19, 4.3(d)].) It follows that $\phi(u(p)) = u(q^{-1}p^{-1})$. We have $\phi^2 = 1$ (see [L19, 4.1]).

From [L97, 3.4] it follows that if $\mathbf{i} \in \mathcal{I}, \mathbf{i}' \in \mathcal{I}$, then $\kappa_{\mathbf{i}}^{-1}\phi\kappa_{\mathbf{i}'} : \mathbf{R}^\nu_{>0} \to \mathbf{R}^\nu_{>0}$ is of the form

$$(z_1, \dots, z_\nu) \mapsto (\rho_1(z_1, \dots, z_\nu)), \dots, \rho_\nu(z_1, \dots, z_\nu))$$

where each $\rho_1, \dots, \rho_\nu$ belongs to $\mathbf{N}(Z_1, \dots, Z_\nu)$. It follows that if K is a semifield, this map gives rise to a map $K^\nu \to K^\nu$ which can be viewed as a bijection $U_K \to U_K$ (denoted by ϕ_K) which does not depend on the choice of $\mathbf{i}, \mathbf{i}'$.

We have $\phi_K^2 = 1$. Since for $p \in K^I$ we have $S_p\phi_K = \phi_K S_{p^{-1}}$, we see that $(S_p\phi_K)^2 = 1$.

We set $\phi'_K = \Psi_K \phi_K \Psi_K : U_K \to U_K$, ($\Psi_K$ as in Section 7). For $p \in K^I$ we have $S_p \phi'_K = \phi'_K S_{p^{-1}}$, hence $(S_p \phi'_K)^2 = 1$.

12. We now assume that $K = \mathbf{Z}$ is as in 7(ii). For $i \in I$ we define $z_i : U_{\mathbf{Z}} \to \mathbf{Z}$ by $z_i((\xi_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}}) = c_\nu$ where $\xi_{\mathbf{i}} = (c_1, \dots, c_{\nu-1}, c_\nu)$ is defined in terms of $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{I}$ such that $i_\nu = i$. (It is easy to see that z_i is well defined.)

Let $U_{\mathbf{N}}$ be the set of all $(\xi_{\mathbf{i}})_{\mathbf{i} \in \mathcal{I}} \in U_{\mathbf{Z}}$ such that for any $\mathbf{i} \in \mathcal{I}$, $\xi_{\mathbf{i}}$ is not only in $\mathbf{Z}^\nu$ but actually in $\mathbf{N}^\nu$. (Note that, if $\xi_{\mathbf{i}} \in \mathbf{N}^\nu$ for some $\mathbf{i}$, then $\xi_{\mathbf{i}} \in \mathbf{N}^\nu$ for all $\mathbf{i}$, see [L19, 2.14].)

Let $\lambda = (\lambda_i)_{i \in I} \in \mathbf{N}^I$. Now $S_\lambda : U_{\mathbf{Z}} \to U_{\mathbf{Z}}$ is defined as in Section 8 since $\lambda \in \mathbf{Z}^I$. It is known that λ indexes a finite dimensional irreducible representation V_λ of G (in Section 3) with a canonical basis [L90] in natural bijection with

$$U_{\mathbf{N},\lambda} := \{x \in U_{\mathbf{N}}; z_i(x) \le \lambda_i \text{ for all } i \in I\}.$$

Let

$$U'_{\mathbf{N},\lambda} = \{x \in U_{\mathbf{N}}; S_\lambda \phi_{\mathbf{Z}}(x) \in U_{\mathbf{N}}\}.$$

We shall prove the following result which appears as a conjecture in [L19, 8.2(b)].

(a) *We have* $U_{\mathbf{N},\lambda} = U'_{\mathbf{N},\lambda}$.

We shall write ϕ instead of $\phi_{\mathbf{Z}}$. From [L97, 4.9] (see also the errata in [L17, #130]) we have

(b) $S_\lambda \phi : U_{\mathbf{N},\lambda} \xrightarrow{\sim} U_{\mathbf{N},\lambda}$.

Since $U_{\mathbf{N},\lambda} \subset U_{\mathbf{N}}$, it follows that $U_{\mathbf{N},\lambda} \subset U'_{\mathbf{N},\lambda}$. Now let $x \in U'_{\mathbf{N},\lambda}$. We set $\tilde{x} = S_\lambda \phi(x) \in U_{\mathbf{N}}$. Define $\mu = (\mu_i) \in \mathbf{N}^I$ by $\mu_i = z_i(\tilde{x})$. We have $\tilde{x} \in U_{\mathbf{N},\mu}$. By (b) (for μ instead of λ) we have $\tilde{x} = S_\mu \phi(y)$ for some $y \in U_{\mathbf{N},\mu}$. Now $S_\lambda \phi(x) = S_\mu \phi(y)$ hence $\phi(S_{-\lambda}(x)) = \phi(S_{-\mu}(y))$ so that $S_{-\lambda}(x) = S_{-\mu}(y)$ and $x = S_{\lambda-\mu}(y)$. We have $z_i(x) = \lambda_i - \mu_i + z_i(y)$. Since $y \in U_{\mathbf{N},\mu}$, we have $z_i(y) \le \mu_i$, hence $\lambda_i - \mu_i + z_i(y) \le \lambda_i$ and $z_i(x) \le \lambda_i$. We see that $x \in U_{\mathbf{N},\lambda}$. This proves (a).

Let

$$U''_{\mathbf{N},\lambda} = \{x \in U_{\mathbf{N}}; S_\lambda \phi'_{\mathbf{Z}}(x) \in U_{\mathbf{N}}\}.$$

We have a bijection $U'_{\mathbf{N},\lambda} \to U''_{\mathbf{N},\lambda}$ given by $x \mapsto \Psi_{\mathbf{Z}}(x)$. (Note that $\Psi_{\mathbf{Z}}$ preserves $U_{\mathbf{N}}$.) Thus $U'_{\mathbf{N},\lambda}$ can be identified via $\Psi_{\mathbf{Z}}$ with $U''_{\mathbf{N},\lambda}$.

In particular we have

(c) $\dim(V_\lambda) = \sharp(U'_{\mathbf{N},\lambda}) = \sharp(U''_{\mathbf{N},\lambda})$.

Earlier formulas for $\dim(V_\lambda)$ were given by Weyl (as a quotient of two positive integers) and by Kostant (as a difference of two integers); in both of these formulas the result was not obviously a positive integer. On the other hand, last expression in (c) is either a positive integer or ∞. Combining any two of these three formulas shows that the result is a positive integer.

Appendix

A1. In this subsection we give a proof of the existence part of 10(a). Let G (over $\mathbf{C}$) be as in Section 3. Let $\mathfrak{g}$ be the Lie algebra of G. We assume given a maximal torus T of G and a pair B^+, B^- of opposed Borel subgroups of G containing T, with unipotent radicals U^+, U^-. For $i \in I$ we consider homomorphisms $x_i : \mathbf{C} \to U^+, y_i : \mathbf{C} \to U^-$ such that $(T, B^+, B^-, x_i, y_i; i \in I)$ is a pinning for G. Define e_i, f_i in $\mathfrak{g}$ by $\exp(ae_i) = x_i(a), \exp(af_i) = y_i(a)$ for all $a \in \mathbf{C}$. Let $h_i = [e_i, f_i]$. There is a unique semigroup imbedding $U_{\ge 0} \to U^+$, $u \mapsto u^+$, given by $i^c \mapsto x_i(c)$ for any $i \in I$

and any $c \in \mathbf{R}_{>0}$. There is a unique semigroup imbedding $U_{\ge 0} \to U^-$, $u \mapsto u^-$, given by $i^c \mapsto y_i(c)$ for any $i \in I$ and any $c \in \mathbf{R}_{>0}$.

By [L97, 3.3] there exists a unique bijection $\phi : U_{>0} \to U_{>0}$ such that

$$(\phi(u)^-)^{-1}B^+\phi(u)^- = (u^+)^{-1}B^-u^+$$

for all $u \in U_{>0}$. As stated in [L19, 4.2(a)], this ϕ satisfies 10(a)(i). (This follows immediately from [L97, Lemma 3.6].) To verify that this ϕ satisfies 10(a)(ii) it is enough to show that

$$(\phi(u(1))^-)^{-1}B^+\phi(u(1))^- = (u(q^{-1})^-)^{-1}B^+u(q^{-1})^-$$

or equivalently that

$$(u(1)^+)^{-1}B^-u(1)^+ = (u(q^{-1})^-)^{-1}B^+u(q^{-1})^-.$$

According to the conjecture [L94, 11.4(a)] (with all $p_i = 1$), proved in [FL97], we have $u(1)^+ = \exp(\sum_{i\in I} q_ie_i)$ (with q_i as in Section 8). The same proof applied with e_i, f_i replaced by $q_i^{-1}e_i, q_if_i$ shows that $u(q^{-1})^+ = \exp(\sum_{i\in I} e_i)$ hence $u(q^{-1})^- = \exp(\sum_{i\in I} f_i)$. Thus we are reduced to proving

$$^{\exp(-\sum_{i\in I} q_ie_i)}B^- = {}^{\exp(-\sum_{i\in I} f_i)}B^+$$

or, setting $\omega = \sum_{i\in I} q_ie_i, \omega' = \sum_{i\in I} f_i$, that

(a) $$\exp(\omega')\exp(-\omega)B^-\exp(\omega)\exp(-\omega') = B^+.$$

Here for $g \in G, B \in \mathcal{B}$ we write ^gB instead of gBg^{-1}. We have $[\omega, \omega'] = \sum_{i\in I} q_ih_i$. We show that $[[\omega, \omega'], \omega] = 2\omega$, $[[\omega, \omega'], \omega'] = -2\omega'$ or equivalently that

$$\sum_{i,j} q_iq_ja_{ij}e_j = 2\sum_{i\in I} q_ie_i, -\sum_{i,j} q_ia_{ij}f_j = -2\sum_{i\in I} f_i$$

or equivalently that $\sum_i q_ia_{ij} = 2$. This follows from the definition of q_i. We see that $\omega, \omega', [\omega, \omega']$ is an sl_2-triple. Hence the subgroup of G with Lie algebra $\mathbf{C}\omega \oplus \mathbf{C}\omega' \oplus \mathbf{C}[\omega, \omega']$ is isomorphic to $SL_2(\mathbf{C})$ or $PGL_2(\mathbf{C})$. By a property of $SL_2(\mathbf{C})$, we see that $Ad(\exp(\omega')\exp(-\omega)\exp(\omega'))$ maps the line spanned by ω' to the line spanned by ω. Since ω is regular nilpotent in $Lie(B^+)$, we see that B^+ is the unique Borel subgroup whose Lie algebra contains ω; similarly B^- is the unique Borel subgroup whose Lie algebra contains ω'. It follows that conjugation by $\exp(\omega')\exp(-\omega)\exp(\omega')$ takes B^- to B^+. Since $\exp(\omega') \in B^-$, it follows that (a) holds. This completes the proof of the existence part of 10(a).

A2. In this subsection we give a proof of 9(a). We use notation in A1. Let $\mathcal{B}$ be the variety of Borel subgroups of G. If $B \in \mathcal{B}, B' \in \mathcal{B}$ we denote by $pos(B, B')$ the relative position of B, B' (an element of W, the Weyl group of G). Let $G(\mathbf{R})$ be the group of real points of G defined by the pinning and let $\mathcal{B}(\mathbf{R})$ be the orbit of B^+ (or B^-) under the adjoint action of $G(\mathbf{R})$. If $i \in I$, an i-circle in $\mathcal{B}(\mathbf{R})$ is a subset L of $\mathcal{B}(\mathbf{R})$ other than a point such that for any $B \ne B'$ in L we have $pos(B, B') = s_i$. Let

$$\mathcal{B}_{>0} = \{^{u^+}B^-; u \in U_{>0}\} = \{^{u^-}B^+; u \in U_{>0}\}.$$

(The last equality follows from [L94].) This is an open subset of $\mathcal{B}(\mathbf{R})$.

Let $u \ne u'$ in $U_{>0}$ and $i \in I$ be such that

(a) $pos(^{u^-}B^+, {}^{u'^-}B^+) = s_i$.

We show that

(b) $\Psi(u') = T_{i,a}\Psi(u)$ for some $a \in \mathbf{R}_{>0}$, (Ψ as in Section 4).

Now $(u^{-1}u')^-$ is in the intersection of U^- with the parabolic subgroup generated by B^+ and by $y_i(\mathbf{C})$ hence is in $y_i(\mathbf{C})$. It follows that $(u')^- = u^- y_i(c)$ for some $c \in \mathbf{C}$ (which is necessarily in $\mathbf{R} - \{0\}$). If $c > 0$ we deduce $\Psi(u') = i^c\Psi(u)$, from which (b) follows immediately. If $c < 0$ then $u^- = (u')^- y_i(-c)$ with $-c > 0$ which implies $\Psi(u) = i^{-c}\Psi(u')$ and again (b) follows. (Conversely, it is easy to show that if (b) holds then (a) holds.) Note also that if $c > 0$ then $\{{}^{u^- y_i(c')}B^+; 0 \le c' \le c\}$ is contained in $\mathcal{B}_{>0}$ and in an i-circle and it contains both ${}^{u^-}B^+, {}^{u'^-}B^+$; if $c < 0$ then $\{{}^{u'^- y_i(c')}B^+; 0 \le c' \le -c\}$ is contained in $\mathcal{B}_{>0}$ and in an i-circle and it contains both ${}^{u^-}B^+, {}^{u'^-}B^+$. We see that the intersection of any i-circle in $\mathcal{B}(\mathbf{R})$ with $\mathcal{B}_{>0}$ is either empty or connected. (If it is nonempty, this intersection is called a half i-circle.)

We define a graph structure on $\mathcal{B}_{>0}$ in which $B \ne B'$ in $\mathcal{B}_{>0}$ are joined if for some $i \in I$, we have $pos(B, B') = s_i$ or equivalently if B, B' belong to the same half i-circle. We have the following result.

(c) *This graph on $\mathcal{B}_{>0}$ is connected.*

Since (a) implies (b), we see that to prove 9(a) it is enough to prove (c).

We first verify the following statement.

(d) Let $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{I}$. If $B \in \mathcal{B}_{>0}$, then for any $B' \in \mathcal{B}_{>0}$ sufficiently close to B^+ there is a unique sequence $B' = B_0, B_1, \dots, B_\nu = B$ in $\mathcal{B}_{>0}$ such that $pos(B_0, B_1) = s_{i_1}, pos(B_1, B_2) = s_{i_2}, \dots, pos(B_{\nu-1}, B_\nu) = s_{i_\nu}$.

We have $B = {}^{u^-}B^+$ where $u \in U_{>0}$.

If $u' \in U_{>0}$ is such that u'^- is sufficiently close to 1 then:

(i) $pos(B, {}^{(u')^-}B^+) = w_0$ and
(ii) $(u'^{-1}u)^- \in U^-_{>0}$.

Indeed, for u'^- close to 1, (ii) holds since $u^- \in U^-_{>0}$ and $U^-_{>0}$ is open in the group of real points of U^-. Also, we have $pos(B, B^+) = w_0$ so that for u'^- close to 1, (i) holds (we use that ${}^{(u')^-}B^+$ is close to B^+ hence is contained in the open subset $\{B_1 \in \mathcal{B}; pos(B, B_1) = w_0\}$ of $\mathcal{B}$).

We write $(u'^{-1}u)^- = y_{i_1}(c_1)\dots y_{i_\nu}(c_\nu)$ where $c_1, \dots, c_\nu$ are in $\mathbf{R}_{>0}$. We set

$B_0 = {}^{(u')^-}B^+$, $B_1 = {}^{(u')^- y_{i_1}(c_1)}B^+$, $B_2 = {}^{(u')^- y_{i_1}(c_1) y_{i_2}(c_2)}B^+, \dots,$
$B_\nu = {}^{(u')^- y_{i_1}(c_1) \dots y_{i_\nu}(c_\nu)}B^+$.

For $k = 0, 1, 2, \dots, \nu$ we have $u'y_{i_1}(c_1)\dots y_{i_k}(c_k) \in U^-_{>0}U^-_{\ge 0} \subset U^-_{>0}$ hence B_0, $B_1, \dots, B_\nu$ are in $\mathcal{B}_{>0}$. Note that $B_\nu = {}^{u^-}B^+ = B$. We have

$pos(B_0, B_1) = pos(B^+, {}^{y_{i_1}(c_1)}B^+) = s_1$, $pos(B_1, B_2) = pos(B^+, {}^{y_{i_2}(c_2)}B^+) = s_2$, $\dots,$
$pos(B_{\nu-1}, B_\nu) = pos(B^+, {}^{y_{i_\nu}(c_\nu)}B^+) = s_\nu$.

This proves the existence in (d). The uniqueness is obvious.

We now prove (c). Let $B \ne \tilde{B}$ in $\mathcal{B}_{>0}$. If $B' \in \mathcal{B}_{>0}$ is sufficiently close to B^+ then by (d), B can be joined with B' through a sequence of edges of our graph and $\tilde{B}$ can be joined with B' through a sequence of edges of our graph. Thus $B, \tilde{B}$ are in the same connected component of our graph. This completes the proof of (c) hence that of 9(a).

A3. Let $p \in \mathbf{R}^I_{>0}$. From the equality $\phi(u(p)) = u(q^{-1}p^{-1})$ (see Section 11) and the equality $(\phi(u)^-)^{-1}B^+\phi(u)^- = (u^+)^{-1}B^-u^+$ for $u \in U_{>0}$ we deduce

$$(u(p)^+)^{-1}B^-u(p)^+ = (u(q^{-1}p^{-1})^-)^{-1}B^+u(q^{-1}p^{-1})^-.$$

Applying the antiautomorphism of G which keeps each $x_i(a), y_i(a)$ fixed (hence keeps $u(p)^+, u(p)^-$ fixed), we deduce

$$u(p)^+B^-(u(p)^+)^{-1} = u(q^{-1}p^{-1})^-B^+(u(q^{-1}p^{-1})^-)^{-1}.$$

It follows that

$$\{{}^{u(p)^-}B^+; p \in \mathbf{R}^I_{>0}\} = \{{}^{u(p)^+}B^-; p \in \mathbf{R}^I_{>0}\}.$$

The two sides of this equality form a (closed) subset $\mathcal{B}^\bullet_{>0}$ of $\mathcal{B}_{>0}$ which is a single orbit under a (free) $\mathbf{R}^I_{>0}$-action on $\mathcal{B}_{>0}$. This subset is the closest we can come to having a base point of $\mathcal{B}_{>0}$.

A4. In this subsection we assume that $I = \{i, j\}$, $i - -j$. In this case the vector space $\underline{O}[U]$ in Section 4 consists of all pairs $[\Pi; \Pi']$ where Π and Π' are polynomials in the indeterminates a, b, c with coefficients in $\mathbf{C}$ and we have

$$\Pi(bc/(a+c), a+c, ab/(a+c)) = \Pi'(a, b, c).$$

The following are examples of pairs in $\underline{O}[U]$:

$$f_{i,j,k} = [a^ib^jc^k; ((bc/(a+c))^i(a+c)^j(ab/(a+c))^k]$$

$$\text{(a)} \qquad = [a^ib^jc^k; \sum_{h',h'' \text{ in } \mathbf{N}; h'+h''=j-i-k} \binom{j-i-k}{h'} a^{h'+k}b^{i+k}c^{h''+i}]$$

with i, j, k in $\mathbf{N}$, $j \geq i + k$,

$$f'_{i,j,k} = [((bc/(a+c))^i(a+c)^j(ab/(a+c))^k; a^ib^jc^k]$$

$$\text{(b)} \qquad = [\sum_{h',h'' \text{ in } \mathbf{N}; h'+h''=j-i-k} \binom{j-i-k}{h'} a^{h'+k}b^{i+k}c^{h''+i}; a^ib^jc^k]$$

with i, j, k in $\mathbf{N}$, $j \geq i + k$.

(c) These pairs are distinct except for the equality $f_{i,j,k} = f'_{k,j,i}$ when $j = i + k$. We show that they are linearly independent. Indeed, $\underline{O}[U] = O[U]$ is a direct sum of weight spaces coming from a $(\mathbf{C}^*)^2$-action on U and indexed by pairs $(m, n) \in \mathbf{N}^2$. The pair $f_{i,j,k}$ is in the weight space indexed by $(i + k, j)$ and the pair $f'_{i,j,k}$ is in the weight space indexed by $(j, i + k)$. Hence the pairs (a), (b) in a given weight space are all of type (a) or all of type (b). But the pairs of type (a) are linearly independent (since their first component are clearly linearly independent) and the pairs of type (b) are linearly independent (since their second component are clearly linearly independent). Thus the pairs (a), (b) (with the identification (c)) are linearly independent. We can verify that the number of pairs (a), (b) in a given weight space is equal to the known dimension of that weight space. It follows that the pairs (a), (b) (with the identification (c)) form a basis of $\underline{O}[U] = O[U]$.

Let $O[U]''_{\geq 0}$ be the set of $\mathbf{R}_{\geq 0}$-linear combinations of pairs in (a), (b). Since each pair in (a), (b) belongs to $O[U]_{\geq 0}$, we have $O[U]''_{\geq 0} \subset O[U]_{\geq 0}$. Conversely consider a pair $[x; x'] \in O[U]_{\geq 0}$. Then the projections of $[x; x']$ to the various weight spaces of $\underline{O}[U]$ are also in $O[U]_{\geq 0}$. Hence to prove that $[x; x'] \in O[U]''_{\geq 0}$ we can assume that $[x; x']$ is in a weight space of $\underline{O}[U]$ (and in $O[U]_{\geq 0}$). Thus we have either

(i) $[x; x'] = \sum_{j \geq i+k} c_{i,j,k} f_{i,j,k}$ or

(ii) $[x;x'] = \sum_{j\ge i+k} c'_{i,j,k} f'_{i,j,k}$
with $c_{i,j,k}\in \mathbf{C}$, $c'_{i,j,k}\in\mathbf{C}$. If (i) holds then

$$x = \sum_{j\ge i+k} c_{i,j,k} a^i b^j c^k$$

and since $[x;x'] \in O[U]_{\ge 0}$ we must have $c_{i,j,k}\in\mathbf{R}_{\ge 0}$. If (ii) holds then

$$x' = \sum_{j\ge i+k} c'_{i,j,k} a^i b^j c^k$$

and since $[x;x'] \in O[U]_{\ge 0}$ we must have $c'_{i,j,k}\in\mathbf{R}_{\ge 0}$. We see than in either case we have $[x;x'] \in O[U]''_{\ge 0}$. Thus we have $O[U]_{\ge 0} = O[U]''_{\ge 0}$.

One can show that under the identification $\underline{O}[U] = O[U]$ in 4(c) the pairs (a), (b) form precisely the dual canonical basis [L90] (with parameter 1). It follows that in this case we have $O[U]_{\ge 0} = O[U]'_{\ge 0}$ (see Section 5). We see that (at least in this case) the dual canonical basis of $O[U]$ can be recovered (up to multiplication by scalars in $\mathbf{R}_{>0}$) without using the theory of quantum groups and without intersection cohomology: it is the only basis of $O[U]$ (up to multiplication by scalars in $\mathbf{R}_{>0}$) such that the set of $\mathbf{R}_{\ge 0}$-linear combination of its elements is exactly $O[U]_{\ge 0}$.

A5. In this subsection we assume that $I = \{i\}$. In this case the vector space $\underline{O}[G]$ in Section 6 consists of all pairs $[\Pi;\Pi']$ where Π and Π' are polynomials in the indeterminates a, b, b^{-1}, c with coefficients in $\mathbf{C}$ and we have

$$\Pi(a, b^{-1}, c) = \Pi'(c/(ac+b^2), b/(ac+b^2), a/(ac+b^2)).$$

The following are examples of pairs in $\underline{O}[G]$:

$$g_{i,j,k} = [a^i b^{-j} c^k; (c/(ac+b^2))^i (b/(ac+b^2))^j (a/(ac+b^2))^k]$$

$$\text{(a)} \qquad = [a^i b^{-j} c^k; \sum_{h',h'' \text{ in } \mathbf{N}; h'+h''=-i-j-k} \binom{-i-j-k}{h'} a^{k+h'} b^{j+2h''} c^{i+h'}]$$

with i,k in $\mathbf{N}$, $j\in\mathbf{Z}$, $-i-j-k\ge 0$,

$$g'_{i,j,k} = [(c/(ac+b^{-2}))^i (b^{-1}/(ac+b^{-2}))^j (a/(ac+b^{-2}))^k; a^i b^j c^k]$$

$$\text{(b)} \qquad = [\sum_{h',h'' \text{ in } \mathbf{N}; h'+h''=-i-j-k} a^k c^i b^{2i+2k+2j-j} (ab^2c+1)^{-i-j-k}; a^i b^j c^k]$$

with i,k in $\mathbf{N}$, $j\in\mathbf{Z}$, $-i-j-k\ge 0$.

(c) These pairs are distinct except for the equality $g_{i,j,k} = g'_{k,j,i}$ when $i+j+k = 0$.

One can verify that the pairs (a), (b) (with the identification (c)) form precisely the dual canonical basis (with parameter 1) of $\underline{O}[G] = O[G]$; these pairs are clearly in $O[G]_{\ge 0}$ and one can verify that $O[G]_{\ge 0} = O[G]'_{\ge 0}$.

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