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GENERALIZED WHITTAKER FUNCTIONS AND JACQUET MODULES

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Abstract. Let F be a non-Archimedean local field and G be (the F-points of) a connected reductive group defined over F. Fix U_0 to be the unipotent radical of a minimal parabolic subgroup P_0 of G, and $\psi : U_0 \to \mathbb{C}^\times$ be a non-degenerate character of U_0. Let $P = MU \supseteq P_0$ be a standard parabolic subgroup of G so that the restriction ψ_M of ψ to $M \cap U_0$ is non-degenerate. We denote by $\mathcal{W}(G, \psi)$ the space of smooth ψ-Whittaker functions on G and by $\mathcal{W}_c(G, \psi)$ its G-stable subspace consisting of functions with compact support modulo U_0. In this situation Bushnell and Henniart identified $\mathcal{W}_c(M, \psi_M^{-1})$ to the Jacquet module of $\mathcal{W}_c(G, \psi^{-1})$ with respect to P^- (Bushnell and Henniart [Amer. J. Math. 125 (2003), pp. 513–547]). On the other hand Delorme defined a constant term map from $\mathcal{W}(G, \psi)$ to $\mathcal{W}(M, \psi_M)$ which descends to the Jacquet module of $\mathcal{W}(G, \psi)$ with respect to P (Delorme [Trans. Amer. Math. Soc. 362 (2010), pp. 933–955]). We show (as we surprisingly could not find a proof of this statement in the literature) that the descent of Delorme's constant term map is the dual map of the isomorphism of Bushnell and Henniart, in particular the constant term map is surjective. We also show that the constant term map coincides on admissible submodules of $\mathcal{W}(G, \psi)$ with the inflation of the "germ map" defined by Lapid and Mao [Represent. Theory 13 (2009), pp. 63–81] following earlier works of Casselman and Shalika [Compositio Math. 41 (1980), pp. 207–231]. From these results we derive a simple proof of a slight generalization of a theorem of Delorme and Sakellaridis–Venkatesh ([Ast'erisque 396 (2017), pp. viii+360] for quasi-split G) on irreducible discrete series with a generalized Whittaker model to the setting of admissible representations with a central character under the split component of G, and similar statements in the cuspidal case (also generalizing a result of Delorme) and in the tempered case. We also show that the germ map of Lapid and Mao is injective, answering one of their questions. Finally using a result of Vignéras [*Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 773–801] and recent results of Dat, Helm, Kurinczuk, and Moss [*Finiteness for hecke algebras of p-adic groups*, arXiv:2203.04929, 2022], we show in the context ℓ-adic representations that the asymptotic expansion of Lapid and Mao can be chosen to be integral for functions in integral G-submodules of $\mathcal{W}(\pi, \psi)$ of finite length.

1. Introduction

Whittaker functions and their various generalizations play an important role in the representation theory of p-adic and real groups, in particular due to the fact that they appear naturally as a local analogue of Fourier coefficients of automorphic forms. In this paper F is a non-Archimedean local field and G be (the F-points of) a connected reductive group defined over F. We fix U_0 the unipotent radical

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of a minimal parabolic subgroup P_0 of G, and $\psi : U_0 \to \mathbb{C}^\times$ be a non-degenerate character of U_0. Let $P = MU \supseteq P_0$ be a standard parabolic subgroup of G so that the restriction ψ_M of ψ to $M \cap U_0$ is non-degenerate. We denote by $\mathcal{W}(G, \psi)$ the space of smooth ψ-Whittaker functions on G and by $\mathcal{W}_c(G, \psi)$ its G-stable subspace consisting of functions with compact support modulo U_0. In this situation Bushnell and Henniart identified $\mathcal{W}_c(M, \psi_M^{-1})$ to the Jacquet module of $\mathcal{W}_c(G, \psi^{-1})$ with respect to P^- the parabolic subgroup of G such that $P \cap P^- = M$ ([1, 2.2 Theorem], see Section 3.3). On the other hand Delorme defined a constant term map from $\mathcal{W}(G, \psi)$ to $\mathcal{W}(M, \psi_M)$ which descends to the Jacquet module of $\mathcal{W}(G, \psi)$ with respect to P ([6, Definition 3.12], see Section 3.2). Both maps have simple characterizations as shown by the authors of [1] and [6] repsectively, and this allows us to compare them without too much effort. In fact we prove in Theorem 4.1 (as we could not find a proof of this statement in the literature, though it might be known to experts) that the descent of Delorme's constant term map is the dual map of the isomorphism of Bushnell and Henniart, in particular the constant term map is surjective and its descent identifies the Jacquet module of $\mathcal{W}(G, \psi)$ with respect to P to $\mathcal{W}(M, \psi_M)$. We also show in Theorem 6.1 that the constant term map coincides on admissible submodules of $\mathcal{W}(G, \psi)$ with the inflation of the "germ map" defined by Lapid and Mao in [9] following earlier works of Casselman and Shalika from [3]. From these results we derive in Theorem 8.2 a very quick proof of a generalization of a theorem of Delorme [7, Théorème 9] and Sakellaridis–Venkatesh ([11, Corollary 6.3.5] where G is assumed to be quasi-split) on irreducible discrete series with a generalized Whittaker model to the setting of admissible representations with a central character under the split component of G. We get similar statements in the cuspidal and tempered case, generalizing [7, Théorème 10] in the cuspidal case. Theorem 6.1 also shows that the germ map of Lapid and Mao is injective, answering [9, Question 3.2]. After restating the asymptotic expansion of Lapid and Mao for functions in admissible submodules of $\mathcal{W}(\pi, \psi)$ (Theorem 7.1), we finally show in Theorem 9.8 in the context of ℓ-adic representations that the asymptotic expansion in question can be chosen to be integral for integral G-submodules of $\mathcal{W}(\pi, \psi)$ of finite length. This last result makes crucial use of results of Vignéras from [13] and of the very recent work [4] (see Section 9.1).

2. Notations

2.1. **Reductive groups.** Here F is a non-Archimedean local field with residual characteristic p and normalized absolute value $| \,|$. We denote by G the F-points of a connected reductive group defined over F. We fix a minimal parabolic subgroup P_0 of G, denote by U_0 its unipotent radical, and fix a Levi subgroup M_0 of P_0. We denote by A_0 the F-split component of (the center of) M_0, the group M_0/A_0 is compact. The letter P will always denote a standard parabolic subgroup of G (i.e. $P_0 \subseteq P$), we denote by M its standard Levi subgroup (i.e. M is the Levi component of P containing A_0) and by U its unipotent radical. We will denote by A_M the split component of M, and by A_M^1 the maximal compact (open) subgroup of A_M. We denote by P^- the parabolic subgroup opposite to P with respect to A_0, so that $P \cap P^- = M$ and we denote by U^- its unipotent radical, so that $P^- = MU^-$. We denote by $\Delta(A_M, P)_G$ or most of the time by $\Delta(A_M, P)$ (resp. $\Delta(A_M, P^-)$) the set of simple roots of A_M for its action on the Lie algebra of P (resp. P^-). We set

$\Delta_0 = \Delta(A_0, P_0)$, and when P is a standard parabolic subgroup of G, we set

$$\Delta_0^P = \Delta(A_0, M \cap P_0)_M \subseteq \Delta_0.$$

For $\epsilon > 0$ we set

$$A_M^-(P, \epsilon) = \{a \in A_M,\ |\alpha(a)| \leqslant \epsilon, \alpha \in \Delta(A_M, P)\}$$

and

$$A_M^-(P^-, \epsilon) = \{a \in A_M,\ |\alpha(a)| \leqslant \epsilon, \alpha \in \Delta(A_M, P^-)\}$$

so that

$$a \in A_M^-(P, \epsilon) \Longleftrightarrow a^{-1} \in A_M^-(P^-, \epsilon).$$

Note that $A_M^-(P, 1) \subseteq A_0^-(P_0, 1)$. For $0 < \epsilon \leqslant 1$ we set

$$A_{0,P}^-(\epsilon) = \{a \in A_0,\ |\alpha(a)| \leqslant \epsilon \text{ for } \alpha \in \Delta_0 - \Delta_0^P \text{ and } \epsilon < |\alpha(a)| \leqslant 1 \text{ for } \alpha \in \Delta_0^P\}.$$

We denote by K_0 a maximal compact open subgroup in good position with respect to (P_0, P_0^-) (see [10, V.2.1 and V.5.2] where the terminology is "K_0 is adapted to A_0", in particular the Iwasawa decomposition $G = P_0 K$ holds) with its family $\Omega_{\mathrm{I}}(G)$ of open subgroups K having an Iwahori decomposition with respect to (P, P^-) for any standard parabolic subgroup P of G, and such that $K \cap M$ has the same property with respect to standard parabolic subgroups of M (containing $P_0 \cap M$). It is known that $\Omega_{\mathrm{I}}(G)$ form a basis of neighborhoods of the identity in G. For $K \in \Omega_{\mathrm{I}}(G)$ and $P = MU$ a standard parabolic subgroup of G we set $K_U = K \cap U$, $K_{U^-} = K \cap U^-$ and $K_M = K \cap M$ so that $K = K_U K_M K_{U^-}$. For any $a \in A_M^-(P, 1)$ we have $a^{-1} K_{U^-} a \subseteq K_{U^-}$ and $a K_{U^-} a^{-1} \subseteq K_{U^-}$.

2.2. **Haar measures.** If X is a totally compact locally disconnected space, we denote by $\mathcal{C}_c^\infty(X)$ the space of (complex valued) compactly supported locally constant functions on X. If moreover X is a group, we denote by $\mathcal{C}^\infty(X)$ the space of smooth functions on X (i.e. functions which are fixed on the right by a compact open subgroup of X) and $\mathcal{C}_c^\infty(X) \subseteq \mathcal{C}^\infty(X)$. We fix a right Haar measure dh on any closed subgroup H of G normalized by the condition $\mu(H \cap K_0) = 1$. We denote by $\delta_H : H \to \mathbb{R}_{>0}$ the modulus character of H, which is determined by the fact that $\delta_H dh$ is a left Haar measure on H. If $P' \subseteq P$ are two standard parabolic subgroups of G then $\delta_{P'} \equiv \delta_P$ on A_M. Such choices fix a unique right invariant measure on $U \backslash G$ such that

$$\int_G f(g) dg = \int_{U \backslash G} \int_U f(ug) du dg$$

for any $f \in \mathcal{C}_c^\infty(G)$. The following integration formulas are valid for any $f \in \mathcal{C}_c^\infty(U \backslash G)$ and we will use them freely:

$$\int_{U \backslash G} f(g) dg = \int_M \int_{K_0} f(mk) \delta_P(m)^{-1} dk dm$$

and

$$\int_{U \backslash G} f(g) dg = \int_M \int_{U^-} f(mu^-) \delta_P(m)^{-1} dm du^-.$$

2.3. **Smooth representations.** We say that (π, V) is a smooth (complex) representation of G if every $v \in V$ is fixed by some $K \in \Omega_{\mathrm{I}}(G)$. We say that it is moreover admissible if V^K has finite dimension for all (equivalently some) $K \in \Omega_{\mathrm{I}}(G)$. If π is a smooth representation of G, we denote by $\pi^\vee$ its smooth dual. We denote by ψ a non-degenerate character of U_0 [1, 1.2. Definition], and if $P = MU \supseteq P_0$ we denote by ψ_M its restriction to $U_0 \cap M$ which is non-degenerate as well by [1, 2.2. Proposition]. We denote by

$$(\rho_{G,\psi}^\vee, \mathrm{Ind}_{U_0}^G(\psi)) \text{ or } (\rho_{G,\psi}^\vee, \mathcal{W}(G,\psi))$$

the representation by right translation of G on the space of smooth (i.e. fixed on the right by some $K \in \Omega_{\mathrm{I}}(G)$) functions from $W : G \to \mathbb{C}$ satisfying $W(ug) = \psi(u)W(g)$ for $u \in U_0,\ g \in G$. We denote by

$$(\rho_{G,\psi}, \mathrm{ind}_{U_0}^G(\psi^{-1})) \text{ or } (\rho_{G,\psi}, \mathcal{W}_c(G,\psi^{-1}))$$

the restriction of $\rho_{G,\psi^{-1}}^\vee$ to the G-invariant subspace of functions with compact support modulo U_0. This strange notation makes sense: the duality

$$\langle W', W\rangle_G = \int_{U_0\backslash G} W'(g)W(g)dg$$

for $W' \in \mathrm{ind}_{U_0}^G(\psi^{-1})$ and $W \in \mathrm{Ind}_{U_0}^G(\psi)$ identifies $\rho_{G,\psi}^\vee$ to $(\rho_{G,\psi})^\vee$. We will now omit the subscript ψ in $\rho_{G,\psi}$ and most of the time also omit the subscript G in ρ_G. We say that ξ in the algebraic dual V^* of a smooth representation (π, V) of G is a ψ-Whittaker functional if

$$\xi(\pi(u_0)v) = \psi(u_0)\xi(v)$$

for $u_0 \in U_0,\ v \in V$. For example the map $\delta_e : \mathcal{W}(G,\psi) \to \mathbb{C}$ defined by $W \mapsto W(e)$ (where e is the neutral element of G) is a ψ-Whittaker functional on $\mathcal{W}(G,\psi)$ which restricts non-trivially to any non-zero G-submodule of $\mathcal{W}(G,\psi)$.

3. Preliminary results of Bushnell–Henniart and Delorme

3.1. **Delorme's application of the second adjunction theorem.** We recall here the second adjointness theorem for smooth representations due to J. Bernstein (which generalizes that of W. Casselman for admissible representations). We then explain a consequence of it due to P. Delorme. For (π, V) a smooth (complex) representation of G and $P = MU$ a standard parabolic subgroup of G, we denote by $(r_P(\pi), r_P(V))$ its non-normalized Jacquet module and by $(J_P(\pi), J_P(V))$ its normalized Jacquet module (the second being the smooth M-module obtained by twisting the first by $\delta_P^{-1/2}$). We recall that Bernstein proved (following Jacquet and Casselman in the admissible case) that if (π, V) is a smooth representation of G and if $K \in \Omega_{\mathrm{I}}(G)$, then the map r_P sends V^K to $r_P(V)^{K\cap M}$ surjectively (see [10, VI.9.1 Théorème]). More precisely we have Bernstein's second adjointness theorem as stated in [6, Lemma 2.1] (in [6, Lemma 2.1] the quantity ϵ_K below is taken uniformly with respect to all smooth representations, but we don't need to know that this can be done).

Theorem 3.1. *Let P be a standard parabolic subgroup of G and $K \in \Omega_{\mathrm{I}}(G)$. Let (π, V) be a smooth representation of G, then $r_P(\pi)^\vee \simeq r_{P_-}(\pi^\vee)$ for any smooth representation π of G. Moreover one can take the M-invariant duality $\langle\ ,\ \rangle_P$*

between $r_{P_-}(\pi^\vee)$ and $r_P(\pi)$ identifying $r_{P_-}(\pi^\vee)$ to $r_P(\pi)^\vee$ to satisfy the following property: there exists $0 < \epsilon_K < 1$ such that for $(v, v^\vee) \in V^K \times (V^\vee)^K$ then

$$\langle r_P(\pi)(a) r_P(v), r_{P^-}(v^\vee)\rangle_P = \langle \pi(a)v, v^\vee\rangle \tag{1}$$

for all $a \in A_M^-(P, \epsilon_K)$. The duality $\langle\ ,\ \rangle_P$ is uniquely characterized by $\langle\ ,\ \rangle$ and the existence of ϵ_K such that (1) holds for all $a \in A_M^-(P, \epsilon_K)$ for every $K \in \Omega_{\mathrm{I}}(G)$. If moreover π is admissible, then $\langle\ ,\ \rangle_P$ is already uniquely characterized by $\langle\ ,\ \rangle$ and the fact that for $(v, v^\vee) \in V \times V^\vee$, there exists $0 < \epsilon < 1$ (depending on $(v, v^\vee)$) such that (1) holds for all $a \in A_M^-(P, \epsilon)$.

Remark 3.2. The fact that $\langle\ ,\ \rangle$ together with (1) characterize the duality $\langle\ ,\ \rangle_P$ follows from the following facts: for $K \in \Omega_{\mathrm{I}}(G)$ one has $r_P(V^K) = r_P(V)^{K\cap M}$, $r_{P^-}((V^\vee)^K) = r_{P^-}(V^\vee)^{K\cap M}$ and $r_P(\pi)(a)$ is a linear bijection of $r_P(V)^{K\cap M}$ for any $a \in A_M^-(P, \epsilon_K)$ (one suffices). Together they imply that $\langle\ ,\ \rangle_P$ is uniquely determined on $r_P(V)^{K\cap M} \times r_{P^-}(V^\vee)^{K\cap M}$ for all $K \in \Omega_{\mathrm{I}}(G)$, hence on $r_P(V) \times r_{P^-}(V^\vee)$. When π is admissible then the weaker assumption gives the existence of an ϵ_K by considering the minimal value of the positive numbers ϵ obtained for the vectors in a basis of $V^K \times (V^\vee)^K$, hence is enough to characterize the duality $\langle\ ,\ \rangle_P$.

It will be convenient to reformulate Theorem 3.1 as follows.

Corollary 3.3. *Let P be a standard parabolic subgroup of G and let (π, V) be a smooth representation of G, then for all $K \in \Omega_{\mathrm{I}}(G)$ there exists $\epsilon_K > 0$ such that*

$$\langle r_{P^-}(v), r_P(\pi^\vee(a)) r_P(v^\vee)\rangle_{P^-} = \langle v, \pi^\vee(a)v^\vee\rangle \tag{2}$$

for all $(v, v^\vee) \in V^K \times (V^\vee)^K$ and $a \in A_M^-(P, \epsilon_K)$. The duality $\langle\ ,\ \rangle_{P^-}$ is uniquely characterized by $\langle\ ,\ \rangle$ and (2) for all $K \in \Omega_{\mathrm{I}}(G)$. When π is admissible the duality $\langle\ ,\ \rangle_{P^-}$ is already uniquely characterized by $\langle\ ,\ \rangle$ and the fact that for $(v, v^\vee) \in V \times V^\vee$, there exists $0 < \epsilon < 1$ (depending on $(v, v^\vee)$) such that (2) holds for all $a \in A_M^-(P, \epsilon)$.

Delorme proves in [6, Theorem 3.4] a more general form of the following result.

Theorem 3.4. *Let $P = MU$ be a standard parabolic subgroup of G. Let (π, V) be a smooth representation of G and let $\xi \in V^*$ a ψ-Whittaker functional. Then there is a unique ψ_M-Whittaker functional $r_{P^-}(\xi)$ in $r_{P^-}(V)^*$ such that for any $K \in \Omega_{\mathrm{I}}(G)$, there exists $0 < \epsilon_K < 1$ such that*

$$\langle r_P(\pi)(a) r_P(v), r_{P^-}(\xi)\rangle = \langle \pi(a)v, \xi\rangle \tag{3}$$

for all $a \in A_M^-(P, \epsilon_K)$ and $v \in V^K$. When π is admissible, the map $r_{P^-}(\xi)$ is already uniquely determined by the existence for any $v \in V$, of $0 < \epsilon < 1$ such that (3) holds for all $a \in A_M^-(P, \epsilon)$.

Remark 3.5. Uniqueness of $r_{P^-}(\xi)$ again follows from the facts that $r_P(V^K) = V^{K\cap M}$ and that $r_P(\pi)(a)$ is a linear bijection of $r_P(V)^{K\cap M}$ for any $a \in A_M^-(P, \epsilon_K)$. In the admissible case the weaker assumption characterizes the duality by the same argument as in Remark 3.2.

From now on, for every $K \in \Omega_{\mathrm{I}}(G)$, we fix $0 < \epsilon_K < 1$ such that the statement of Corollary 3.3 holds for $V = \mathcal{W}(G, \psi)$ with this choice of ϵ_K, and that of Theorem 3.4 also holds with this choice of ϵ_K. (We recall once again, though we shall not use it, that by [6, Lemma 2.1] there is a choice of ϵ_K which holds for all smooth representations together.)

3.2. **The constant term map.** Thanks to Theorem 3.4 Delorme defined in [6, Definition 3.12] a constant term map from $\mathcal{W}(G,\psi)$ to $\mathcal{W}(M,\psi_M)$.

Definition 3.6. For $P = MU$ a standard parabolic subgroup of G and $W \in \mathcal{W}(G,\psi)$, we set

$$W_P(m) = \delta_P^{-1/2}(m)\langle r_P(\rho^\vee(m)W), r_{P^-}(\delta_e)\rangle$$

for $m \in M$.

The map W_P belongs to $\mathcal{W}(M,\psi_M)$ and $D_P : W \mapsto W_P$ is a P-module homomorphism from $(\mathcal{W}(G,\psi), \delta_P^{-1/2}\rho^\vee)$ to $\mathcal{W}(M,\psi_M)$ where the action of U (resp. M) on $\mathcal{W}(M,\psi_M)$ is trivial (resp. given by $\rho_M^\vee$). In other words D_P induces an M-module homomorphism

$$\overline{D}_P : J_P(\mathcal{W}(G,\psi)) \to \mathcal{W}(M,\psi).$$

We call W_P the constant term of W along P. Delorme then shows in [6, Proposition 3.14] that the constant term map D_P is characterized as follows. We recall his proof in view of Corollary 3.9.

Proposition 3.7. *The constant term map $D_P : \mathcal{W}(G,\psi) \to \mathcal{W}(M,\psi_M)$ is characterized by the property that it is the unique P-module homomorphism from $(\delta_P^{-1/2}\rho_G^\vee, \mathcal{W}(G,\psi))$ to $\mathcal{W}(M,\psi_M)$ for U acting trivially and M by $\rho_M^\vee$ on $\mathcal{W}(M,\psi_M)$, such that for any $K \in \Omega_{\mathrm{I}}(G)$ we have*

$$\delta_P(a)^{1/2}W_P(a) = W(a) \tag{4}$$

for all $a \in A_M^-(P,\epsilon_K)$ and $W \in \mathcal{W}(G,\psi)^K$.

Proof. Suppose that $D' : \mathcal{W}(G,\psi) \to \mathcal{W}(M,\psi_M)$ also has the properties which we expect to characterize D_P. Then the map $W \mapsto D'(W)(e)$ factors through a map ψ_M-Whittaker functional $\eta \in r_P(V)^*$:

$$D'(W)(e) = \langle r_P(W), \eta\rangle$$

for all $W \in \mathcal{W}(G,\psi)$. Then for fixed $K \in \Omega_{\mathrm{I}}(G)$ and $W \in \mathcal{W}(G,\psi)^K$, Equation (4) for $D'(W)$ tells that

$$\langle\rho^\vee(a)W, \delta_e\rangle = W(a) = \delta_P(a)^{1/2}D'(W)(a) = (\delta_P(a)^{1/2}\rho_M^\vee(a)D'(W))(e)$$

for $a \in A_M^-(P,\epsilon_K)$. However by the M-equivariance property of D'

$$(\delta_P(a)^{1/2}\rho_M^\vee(a)D'(W))(e) = D'(\rho^\vee(a)W)(e) = \langle r_P(\rho^\vee(a)W), \eta\rangle.$$

Hence by Theorem 3.4 we have $\eta = j_{P^-}(\delta_e)$, so

$$D'(W)(e) = D_P(W)(e)$$

for all $W \in \mathcal{W}(G,\psi)$. As both maps D' and D_P have the same equivariance property under M, we deduce that $D'(W) = D_P(W)$ for all $W \in \mathcal{W}(G,\psi)$. $\square$

Remark 3.8. Of course in the statement of Proposition 3.7 we can take K varying in any subset of $\Omega_{\mathrm{I}}(G)$ which is still a basis of neighborhoods of the identity in G (because we can do so in the statements of Theorem 3.1, Corollary 3.3 and Theorem 3.4), which we will do in the proof of Theorem 4.1.

We end this section by giving a version of Proposition 3.7 for admissible submodules of $\mathcal{W}(G,\psi)$.

Corollary 3.9. *Let V be an admissible submodule of $\mathcal{W}(G,\psi)$. The retriction to V of the constant term map D_P is characterized by the property that it is the unique P-module homomorphism from $(\delta_P^{-1/2}\rho_G^\vee, V)$ to $\mathcal{W}(M,\psi_M)$ for U acting trivially and M by $\rho_M^\vee$ on $\mathcal{W}(M,\psi_M)$, such that for any $W \in V$, there is $\epsilon > 0$ such that*

$$\delta_P(a)^{1/2} W_P(a) = W(a) \tag{5}$$

for $a \in A_M^-(P,\epsilon)$.

Proof. The proof of Proposition 3.7 goes through with V instead of the full representation $\mathcal{W}(G,\psi)$, except that we appeal to the admissible version of Theorem 3.4 at the end of it. □

3.3. **An isomorphism of Bushnell–Henniart.** We recall from the introduction that $\mathcal{W}_c(G,\psi)$ is the G-stable subspace of $\mathcal{W}(G,\psi)$ consisting of functions compactly supported modulo U_0, and from Section 2.3 that we denote by ρ the action of G on $\mathcal{W}_c(G,\psi)$ by right translation. In [1], for $P = MU$ a standard parabolic subgroup of G, Bushnell and Henniart identify the Jacquet module of $\mathcal{W}_c(G,\psi)$ with respect to P^- to $\mathcal{W}_c(M,\psi_M)$. We recall their result [1, 2.2 Theorem], which gives a characterization of the map $W \mapsto J_{P^-}(W)$ via their identification.

Theorem 3.10. *Let $P = MU$ be a standard parabolic subgroup of G, and turn $\mathcal{W}_c(M,\psi_M)$ into a P^--module by making U^- act trivially on $\mathcal{W}_c(M,\psi_M)$ and M act by ρ_M. There is a unique P^--module homomorphism map*

$$BH_{P^-} : (\mathcal{W}_c(G,\psi^{-1}), \delta_P^{1/2}\rho) \to \mathcal{W}_c(M,\psi_M^{-1})$$

such that if U_c^- is a compact open subgroup of U^- and $W \in \mathcal{W}_c(G,\psi^{-1})^{U_c^-}$ has support in PU_c^-, then

$$BH_{P^-}(W) = \mathrm{vol}(U_c^-)\delta_P^{-1/2} W_{|M}.$$

The map BH_{P^-} is surjective and induces a M-module isomorphism

$$\overline{BH}_{P^-} : J_{P^-}(W) \mapsto BH_{P^-}(W)$$

between $J_{P^-}(\mathcal{W}_c(G,\psi^{-1}))$ and $\mathcal{W}_c(M,\psi_M^{-1})$.

4. Surjectivity of the constant term map

We now have the map $BH_{P^-} : \mathcal{W}_c(G,\psi) \to \mathcal{W}_c(M,\psi_M)$ which induces an M-module isomorphism $\overline{BH}_{P^-} : J_{P^-}(\rho) \simeq \rho_M$ and the constant term map $D_P : \mathcal{W}(G,\psi) \to \mathcal{W}(M,\psi_M)$ which induces an M-module homomorphism $\overline{D}_P : J_P(\rho^\vee) \to \rho_M^\vee$. Denote by

$$\overline{BH}_{P^-}^\vee : \rho_M^\vee \simeq (J_{P^-}\rho)^\vee$$

the M-module isomorphism dual to $\overline{BH}_{P^-}$, and by

$$\overline{BH}^P : \rho_M^\vee \simeq J_P(\rho^\vee)$$

the isomorphism obtained by composing $\overline{BH}_{P^-}^\vee$ with the isomorphism of the second adjunction theorem (Theorem 3.1). Finally we set

$$\overline{BH}_P := (\overline{BH}^P)^{-1} : J_P(\rho^\vee) \simeq \rho_M^\vee$$

and write

$$BH_P : \delta_P^{-1/2}\rho^\vee \twoheadrightarrow \rho_M^\vee$$

the surjection obtained by inflating $\overline{BH}_P$. We will now show the equality

$$D_P = BH_P,$$

which in particular implies that D_P is surjective.

We recall that $(\rho^\vee, \mathcal{W}(G,\psi))$ is identified to the dual of $(\rho, \mathcal{W}_c(G,\psi^{-1}))$, thanks to the duality

$$\langle W', W\rangle_G = \int_{U_0\backslash G} W'(g)W(g)dg$$

for $(W', W) \in \mathcal{W}_c(G,\psi^{-1}) \times \mathcal{W}(G,\psi)$, and that $(\rho_M^\vee, \mathcal{W}(M,\psi_M))$ is identified to the dual of $(\rho_M, \mathcal{W}_c(M,\psi_M^{-1}))$, thanks to the duality

$$\langle W'_M, W_M\rangle_M = \int_{M\cap U_0\backslash M} W'_M(m)W_M(m)dm$$

for $(W'_M, W_M) \in \mathcal{W}_c(M,\psi_M^{-1}) \times \mathcal{W}(M,\psi_M)$. In particular through our various identifications we have for any $(W', W) \in \mathcal{W}_c(G,\psi^{-1}) \times \mathcal{W}(G,\psi)$ with respective image $(\overline{W'}, \overline{W}) \in J_{P^-}(\mathcal{W}_c(G,\psi^{-1}) \times J_P(\mathcal{W}(G,\psi^{-1}))$:

$$\langle \overline{W'}, \overline{W}\rangle_{P^-} = \langle \overline{BH}_{P^-}(\overline{W'}), \overline{BH}_P(\overline{W})\rangle_M = \langle BH_{P^-}(W'), BH_P(W)\rangle_M. \tag{6}$$

On the other hand Corollary 3.3 tells us that given $K \in \Omega_{\mathrm{I}}(G)$, for K-invariant W' and W one has

$$\langle \overline{W'}, J_P(\rho^\vee(a))\overline{W}\rangle_{P^-} = \delta_P(a)^{-1/2}\langle W', \rho^\vee(a)W\rangle_G \tag{7}$$

for all $a \in A_M^-(P,\epsilon_K)$. So take $K \in \Omega_{\mathrm{I}}(G)$ small enough such that $K_{U_0} \subseteq \mathrm{Ker}(\psi)$, take $W \in \mathcal{W}(G,\psi)^K$, and define $W_0 \in \mathcal{W}_c(G,\psi^{-1})^K$ to be function supported on $U_0K = U_0K_MK_{U^-}$ equal to 1 on K. Putting Equations (6) and (7) together we obtain for $a \in A_M^-(P, \epsilon_K)$

$$\begin{aligned}
&\delta_P^{1/2}(a)\langle BH_{P^-}(W_0), \rho_M^\vee(a)BH_P(W)\rangle_M \\
&\quad = \langle BH_{P^-}(W_0), BH_P(\rho^\vee(a)W)\rangle_M \\
&\quad = \langle W_0, \rho^\vee(a)W\rangle_G = \int_{U_0\backslash G} W_0(g)W(ga)dg \\
&\quad = \int_{U_0\cap M\backslash M}\int_{U^-} W_0(mu^-)W(mu^-a)\delta_P^{-1}(m)dmdu^- \\
&\quad = \int_{U_0\cap M\backslash M}\int_{U^-} W_0(mu^-)W(ma(a^{-1}u^-a))\delta_P^{-1}(m)dmdu^- \\
&\quad = \mathrm{vol}(K_{U^-})\int_{U_0\cap M\backslash M} W_0(m)W(ma)\delta_P^{-1}(m)dm \\
&\quad = \mathrm{vol}(K_{U^-})\int_{M\cap U_0^-}\int_{A_0} W_0(a_0v_0^-)W(a_0v_0^-a)\delta_{M\cap P_0}^{-1}(a_0)\delta_P^{-1}(a_0)da_0dv_0^- \\
&\quad = \mathrm{vol}(K_{U^-})\int_{M\cap U_0^-}\int_{A_0} W_0(a_0v_0^-)W(a_0a(a^{-1}v_0^-a))\delta_{M\cap P_0}^{-1}(a_0)\delta_P^{-1}(a_0)da_0dv_0^- \\
&\quad = \mathrm{vol}(K_{U^-})\,\mathrm{vol}(K_M\cap U_0^-)\int_{A_0} W_0(a_0)W(a_0a)\delta_{M\cap P_0}^{-1}(a_0)\delta_P^{-1}(a_0)da_0 \\
&\quad = \mathrm{vol}(K_{U^-})\,\mathrm{vol}(K_M\cap U_0^-)\int_{K\cap A_0} W(aa_0)\delta_{M\cap P_0}^{-1}(a_0)\delta_P^{-1}(a_0)da_0 \\
&\quad = \mathrm{vol}(K_{U^-})\,\mathrm{vol}(K_M\cap U_0^-)\,\mathrm{vol}(K\cap A_0)W(a).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\delta_P^{1/2}(a)\langle BH_{P^-}(W_0), \rho_M^\vee(a)BH_P(W)\rangle_M \\
&\quad = \delta_P^{1/2}(a)\,\mathrm{vol}(K_{U^-})\int_{U_0\cap M\backslash M} W_0(m)BH_P(W)(ma)\delta_P^{-1/2}(m)dm \\
&\quad = \delta_P^{1/2}(a)\,\mathrm{vol}(K_{U^-})\,\mathrm{vol}(K_M\cap U_0^-)\,\mathrm{vol}(K\cap A_0)BH_P(W)(a)
\end{aligned}$$

because $BH_P(W)$ is right $K\cap M$-invariant as W is right K-invariant. From the above discussion we obtain the equality

$$W(a) = \delta_P^{1/2}(a)BH_P(W)(a)$$

for all $W \in \mathcal{W}(G,\psi)^K$ and $a \in A_M^-(P,\epsilon_K)$. As moreover both BH_P and D_P have the same equivariance property with respect to P by definition of BH_P, we arrive at the following conclusion thanks to Proposition 3.7:

Theorem 4.1. *The maps BH_P and D_P from $\mathcal{W}(G,\psi)$ to $\mathcal{W}(M,\psi_M)$ are equal and in particular D_P is surjective.*

5. Further notations and preliminaries

Let P be a standard parabolic subgroup of G. For $\alpha \in \Delta_0$, we set $F_\alpha = F - \{0\}$ if $\alpha \in \Delta_0^P$ and $F_\alpha = F$ otherwise. We set

$$\mathcal{M}_P^G = \prod_{\alpha\in\Delta_0} F_\alpha$$

and recall that

$$\mathcal{C}_c^\infty(\mathcal{M}_P^G) \simeq \bigotimes_{\alpha\in\Delta_0} \mathcal{C}_c^\infty(F_\alpha).$$

We denote by $\mathfrak{r}_G : A_0 \to \prod_{\alpha\in\Delta_0} F - \{0\} \subseteq \mathcal{M}_P$ the map defined by $\mathfrak{r}_G(t)_\alpha = \alpha(t)$.

Because the restriction map from the lattice $X^*(M)$ of algebraic characters of M to that $X^*(A_M)$ of algebraic characters of A_M is injective with finite co-kernel, any $|\chi|$ for $\chi \in X^*(A_M)$ extends uniquely to a positive character of M still denoted by $|\chi|$. For $\epsilon > 0$ we set

$$M^-(P,\epsilon) = \{m \in M,\ |\alpha|(m) \leqslant \epsilon,\ \alpha \in \Delta(A_M, P)\}.$$

Whenever V is a $\mathbb{C}$-vector space affording a smooth representation π of A_M, we denote by $V_{A_M,\mathrm{fin}}$ its subspace of A_M-finite vectors, i.e. the space of vectors v in V such that $\mathbb{C}[A_M].v$ is finite dimensional. For example when $V = \mathcal{C}^\infty(A_M)$ with $\pi := \rho$ the action by translation, we set

$$\mathcal{F}(A_M) := \mathcal{C}^\infty(A_M)_{A_M,\mathrm{fin}}.$$

Note that

$$V_{A_M,\mathrm{fin}} = \bigoplus_{\chi\in\widehat{A_M}} V_{(\chi)}$$

where

$$V_{(\chi)} = \{v \in V,\ \exists n \in \mathbb{N} - \{0\}, (\pi(a) - \chi(a)\,\mathrm{Id})^n v = 0\ \forall a \in A_M\}.$$

The space $V_{(\chi)}$ is an A_M-submodule of $V_{A_M,\mathrm{fin}}$ which we call the characteristic subspace associated to χ, and which contains the eigenspace

$$V_\chi = \{v \in V,\ (\pi(a) - \chi(a)\,\mathrm{Id})v = 0\ \forall a \in A_M\}.$$

We denote by $\widehat{A_M}$ the group of smooth characters from A_M to $\mathbb{C}^*$ and set

$$\mathcal{E}(A_M, V) = \{\chi \in \widehat{A_M},\ V_{(\chi)} \neq \{0\}\} = \{\chi \in \widehat{A_M},\ V_\chi \neq \{0\}\}.$$

When V is an admissible representation of M then $V = V_{A_M,\mathrm{fin}}$. We recall from [2, Section 4] that if V is an admissible representation of G, then $J_P(V)$ is an admissible representation of M for $P = MU$ any standard parabolic subgroup of G.

6. The germ map

Here we recall ideas of Casselman and Shalika [3] further developed by Lapid and Mao in [9]. One says that two functions f and f' in $\mathcal{C}^\infty(A_M)$ (resp. $\mathcal{C}^\infty(M)$) have the same germ if there is $\epsilon > 0$ such that $f_{|A_M^-(P,\epsilon)} = f'_{|A_M^-(P,\epsilon)}$ (resp. $f_{|M^-(P,\epsilon)} = f'_{|M^-(P,\epsilon)}$) and we write $f \sim f'$. This defines an equivalence relation on $\mathcal{C}^\infty(A_M)$ (resp. $\mathcal{C}^\infty(M)$) and we denote by $[f]$ the class of f and call it its germ. We set $\mathcal{G}(A_M) = [\mathcal{C}^\infty(A_M)]$ (resp. $\mathcal{G}(M) = [\mathcal{C}^\infty(M)]$). The map $f \mapsto [f]$ from $\mathcal{C}^\infty(A_M)$ to $\mathcal{G}(A_M)$ (resp. from $\mathcal{C}^\infty(M)$ to $\mathcal{G}(M)$) is a smooth A_M-bimodules (resp. M-bimodules) homomorphism. We note that the paper [9] is valid for F of positive characteristic as well (the characteristic zero assumption plays no role in the paper). By [9, Lemma 2.9], the map $f \mapsto [f]$ induces an isomorphism Γ_M between $\mathcal{C}^\infty(A_M)_{A_M,\mathrm{fin}}$ and $\mathcal{G}(A_M)_{A_M,\mathrm{fin}}$. From this one deduces as in the proof of [9, Corollary 2.11] that $f \mapsto [f]$ induces an isomorphism

$$\iota_M : \mathcal{C}^\infty(M)_{A_M,\mathrm{fin}} \simeq \mathcal{G}(M)_{A_M,\mathrm{fin}}.$$

It is then proved in the proof of [9, Theorem 3.1] that if $\pi \subseteq \mathcal{W}(G,\psi)$ is an admissible G-module (Lapid and Mao work under a finite length assumption which is unnecessary), then the map

$$W \mapsto [\delta_P^{-1/2} W_{|M}]$$

factors through a map

$$\kappa_P : J_P(\pi) \to \mathcal{G}(M).$$

Moreover as

$$J_P(\pi) = J_P(\pi)_{A_M,\mathrm{fin}}$$

because $J_P(\pi)$ is admissible, we deduce that

$$\kappa_P : J_P(\pi) \to \mathcal{G}(M)_{A_M,\mathrm{fin}}.$$

We then define

$$\Xi_P := \iota_M^{-1} \circ \kappa_P : J_P(\pi) \to \mathcal{C}^\infty(M)_{A_M,\mathrm{fin}}.$$

The authors of [9] note that in fact

$$\Xi_P : J_P(\pi) \to \mathcal{W}(M,\psi_M)_{A_M,\mathrm{fin}}.$$

Here we answer [9, Question 3.2].

Theorem 6.1. *Let $\pi \subseteq \mathcal{W}(G,\psi)$ be an admissible G-module, then the "germ map" Ξ_P coincides with $\overline{D_P} = \overline{BH}_P : J_P(W) \mapsto W_P$ on π. In particular it is injective, hence* $\mathrm{Ker}(\kappa_P) = \{0\}$.

Proof. Thanks to the characterization of D_P given in Corollary 3.9, it is sufficient to show that there exists $\epsilon > 0$ such that $\Xi_P(W) = \delta_P^{-1/2}(a)W(a)$ on $A^-(P,\epsilon)$. However by definition $\Xi_P(W)$ and $\delta_P^{-1/2}W_{|M}$ have the same germ, hence in particular they agree on $A_M^-(P,\epsilon) \subseteq M^-(P,\epsilon)$ for some $\epsilon > 0$. □

7. The asymptotic expansion of Whittaker functions

The following expansion is due to Lapid and Mao [9, Theorem 3.1]. We reproduce parts of their proof which holds verbatim in our more general context, as we shall need to use them later in the context of integral ℓ-adic representations.

Theorem 7.1. *Let (π, V) be an admissible $\mathbb{C}[G]$-submodule of $\mathcal{W}(G,\psi)$ and $W \in V$, then for each $P = MU \supseteq P_0$ and $\chi \in \mathcal{E}(A_M, J_P(\pi))$, there is a finite indexing set $I_{W,\chi}$ (where we choose $I_{W,\chi}$ disjoint from $I_{W,\chi'}$ if $\chi \neq \chi'$) such that one can write for $t \in A_0$*

$$W(t) = \sum_{P=MU\supseteq P_0} \delta_P(t)^{1/2} \sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))} \sum_{i\in I_{W,\chi}} f_i(t)\phi_i(\mathfrak{r}_G(t))$$

for some functions $f_i \in \mathcal{F}(A_0)_{(\chi)}$ and $\phi_i \in \mathcal{C}_c^\infty(\mathcal{M}_P^G)$.

Proof. We do an induction on the split rank r of G/A_G. If $r = 0$ then $\mathcal{C}_c^\infty(\mathcal{M}_G^G)$ is just the space of constant functions on $A_0 = A_G$. Take $W \in V$, because V is admissible W is A_G-finite and the expansion follows. If $r > 0$ we can always suppose that $W \in V_{(\mu)}$ for $\mu \in \mathcal{E}(A_G,\pi)$. For $\emptyset \neq J \subseteq \Delta_0$ we denote by $P_J = M_J U_J$ the standard parabolic subgroup of G such that $\Delta_0 - \Delta_0^{P_J} = J$. By induction we have for any subset $\emptyset \neq J \subseteq \Delta_0$:

$$W_{P_J}(t) = \sum_{P\subseteq P_J} \delta_{P\cap M_J}(t)^{1/2} \sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))} F_{J,\chi}(t)$$

where

$$F_{J,\chi}(t)=\sum_{k\in I_{W,J,\chi}} f_k(t)\phi_k(\mathfrak{r}_{M_J}(t))$$

for some finite indexing set $I_{W,J,\chi}$ (with all such sets disjoint from one another when J and χ vary) and $f_k\in\mathcal{F}(A_0)_{(\chi)}$, $\phi_k\in\mathcal{C}_c^\infty(\mathcal{M}_{P\cap M_J}^{M_J})$. Hence arguing as in the proof of [9, Theorem 3.1] there is $\epsilon>0$ such that for all subsets $\emptyset\neq J\subseteq\Delta_0$ one has

$$W(t)=\sum_{P\subseteq P_J}\delta_P(t)^{1/2}\sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))} F_{J,\chi}(t)$$

on $A_0^-(P_0,\epsilon)$. Setting

$$F_\chi(t)=\sum_{\emptyset\neq J\subseteq\Delta_0-\Delta_0^P}(-1)^{|J|-1}F_{J,\chi}(t)\prod_{\alpha\in J}\mathbf{1}_{]0,\epsilon[}(|\alpha(t)|)$$

we observe that

$$F_{J,\chi}(t)\prod_{\alpha\in J}\mathbf{1}_{]0,\epsilon[}(|\alpha(t)|)=\sum_{k\in I_{W,J,\chi}} f_k(t)\phi_k'(\mathfrak{r}_G(t))$$

where

$$\phi_k'(\mathfrak{r}_G(t))=\phi_k(\mathfrak{r}_{M_J}(t))\prod_{\alpha\in J}\mathbf{1}_{]0,\epsilon[}(|\alpha(t)|)\in\mathcal{C}_c^\infty(\mathcal{M}_P^G).$$

Then Lapid and Mao show that

$$W(t)=\sum_{P=MU\subsetneq G}\ \sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))}\delta_P^{1/2}(t)F_\chi(t)$$

for any $t\in A_0$ such that there exists $\alpha\in\Delta_0$ with $|\alpha(t)|<\epsilon$, whereas the sum on the right vanishes for t not satisfying this property. But then we claim that the function

$$\Phi(t)=W(t)\prod_{\alpha\in\Delta_0}\mathbf{1}_{[\epsilon,+\infty[}(|\alpha(t)|)$$

belongs to $\mathcal{F}(A_0)_{(\mu)}\mathcal{C}_c^\infty(\mathcal{M}_G^G)$ and the equality

$$W(t)=\sum_{P=MU\subsetneq G}\ \sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))}\delta_P^{1/2}(t)F_\chi(t)+\Phi(t)$$

will then end the proof of the theorem. Indeed using the equivariance property of W under U_0 we have $W(t)=0$ whenever $|\alpha(t)|$ is large enough for some $\alpha\in\Delta_0$, hence there is C such that

$$\Phi(t)=W(t)\prod_{\alpha\in\Delta_0}\mathbf{1}_{[\epsilon,C]}(|\alpha(t)|).$$

Now W is finite under A_G because V is admissible and the result follows. □

8. Applications

In this section we generalize to admissible representations of G on which A_G acts by a character, a theorem of Delorme [7, Proposition 13 and Théorème 9] and Sakellaridis–Venkatesh ([11, Corollary 6.3.5] for quasi-split groups) for irreducible representations, characterizing discrete series with a generalized Whittaker model. We also prove similar statements for cuspidal and tempered representations, generalizing [7, Théorème 10] in the cuspidal case.

Lemma 8.1. *Let $\pi \subseteq \mathcal{W}(G,\psi)$ be an admissible G-submodule. Let $P = MU$ be a standard parabolic subgroup of G (possibly G itself) such that $\mathcal{E}(A_M, J_P(\pi)) \neq \emptyset$ and take $\chi \in \mathcal{E}(A_M, J_P(\pi))$. Then there is $W \in \pi$ and $\epsilon > 0$ such that $W(t) = \delta_P^{1/2}(t)\chi(t)$ on $A_M^-(P,\epsilon)$.*

Proof. By Theorem 4.1 and the discussion before it the map $W \mapsto W_P$ on $\mathcal{W}(G,\psi)$ identifies with the normalized Jacquet projection J_P, hence the same holds on any G-submodule of $\mathcal{W}(G,\psi)$, for example π. In particular the space W_P for $W \in \pi$ is the normalized Jacquet module of π with respect to P. Now take $W \in \pi$ such that $W_P \neq 0$ in the χ-eigenspace of $J_P(\pi)$. Up to translating W by an element of M, and normalizing it by a non-zero scalar in $\mathbb{C}^\times$, we can suppose that $W_P(1) = 1$, which in turn implies that $W_P(t) = (\rho(t)W_P)(1) = \chi(t)$ for all $t \in A_M$. On the other hand the asymptotic expansion of Theorem 7.1 gives the existence of $\epsilon > 0$ and distinct characters $\chi_1, \dots, \chi_r$ of A_M such that one has on $A_M^-(P,\epsilon)$ an expansion of the form $W(t) = \sum_{i=1}^r f_i(t)$ with $f_i \in \mathcal{F}(A_M)_{(\chi_i)}$. Hence according to Corollary 3.9 we have $W_P(t) = \sum_{i=1}^r \delta_P(t)^{-1/2} f_i(t)$ on $A_M^-(P,\epsilon)$ up to taking ϵ even smaller. We recall that as already noticed at the beginning of Section 6, there is a unique $f \in \mathcal{C}^\infty(A_M)_{A_M,\mathrm{fin}}$ such that $[f] = [(W_P)_{|A_M}]$, and this f has to be at the same time $\sum_{i=1}^r \delta_P^{-1/2} f_i$ and χ, so that only one f_i (say f_1) is non-zero and $\delta_P^{-1/2} f_1 = \chi$. This last equality gives the sought equality $W(t) = \delta_P(t)^{1/2}\chi(t)$ on $A_M^-(P,\epsilon)$, where $W \in \pi$ by assumption. □

Let (π, V) be an admissible representation of G. We recall that π is called cuspidal if all its Jacquet modules associated to proper standard parabolic subgroups are zero. If A_G acts on V by a unitary character, then we say that π is a discrete series if the module of every matrix coefficient of π belongs to $L^2(A_G\backslash G)$. Thanks to Casselman, this is known to be equivalent to the fact that for every proper standard parabolic subgroup $P = MU$ of G, every $\chi \in \mathcal{E}(A_M, J_P(V))$ satisfies $|\chi(t)| < 1$ for $t \in A_M^-(P,1) \cap (A_0^1 A_G)^c$, we say that χ is *positive* [2, Theorem 4.4.6]. Similarly still under the hypothesis that A_G acts on V by a unitary character, then we say that π is tempered if the module of every matrix coefficient of π belongs to $L^{2+r}(A_G\backslash G)$ for any $r > 0$. One shows as in [2, Theorem 4.4.6] that it is equivalent to the fact that for every proper standard parabolic subgroup $P = MU$ of G, every $\chi \in \mathcal{E}(A_M, J_P(V))$ satisfies $|\chi(t)| \leqslant 1$ for $t \in A_M^-(P,1)$, we say that χ is *non-negative*. In particular discrete series are tempered. If c is a character of A_G, we denote by $\mathcal{C}^\infty(A_G U_0\backslash G, c\otimes\psi)$ the set of smooth functions W from G to $\mathbb{C}$ which satisfy $W(au_0 g) = c(a)\psi(u_0)W(g)$ for $a \in A_G,\ u_0 \in U_0,\ g \in G$. We denote by $\mathcal{C}_c^\infty(A_G U_0\backslash G, c\otimes\psi)$ its G-submodule consisting of functions with support compact modulo $A_G U_0$. If c is moreover unitary, for $p > 0$ we denote by $L^p(A_G U_0\backslash G, c\otimes\psi)^\infty$ the G-submodule of $\mathcal{C}^\infty(A_G U_0\backslash G, c\otimes\psi)$ consisting of functions W such that $|W|^p$ is integrable on $A_G U_0\backslash G$.

Theorem 8.2. *Let π be an admissible submodule of $\mathcal{W}(G,\psi)$ such that A_G acts on π by a unitary character c, then π is cuspidal, resp. square integrable, resp. tempered if and only if $\pi \subseteq \mathcal{C}_c^\infty(A_G U_0\backslash G, c\otimes\psi)$, resp. $\pi \subseteq L^2(A_G U_0\backslash G, c\otimes\psi)^\infty$, resp. $\pi \subseteq L^{2+r}(A_G U_0\backslash G, c\otimes\psi)^\infty$ for all $r > 0$. In the cuspidal case the unitary assumption on c is superfluous.*

Proof. One direction of all statements easily follows from Theorem 7.1 and the Iwasawa decomposition $G = U_0 M_0 K_0$ together with the compactness of the quotient

M_0/A_0. We give the proof of the converse direction in the cuspidal and the tempered case, the square integrable case being similar to the tempered case. Suppose that $\pi \subseteq \mathcal{C}_c^\infty(A_GU_0\backslash G, c\otimes\psi)$ and let $P = MU$ be a proper standard parabolic subgroup of G. Suppose that $J_P(\pi) \neq \{0\}$ so that $\mathcal{E}(A_M, J_P(\pi)) \neq \emptyset$ and take χ inside it. The function W given by Lemma 8.1 can't belong to $\mathcal{C}_c^\infty(A_GU_0\backslash G, c\otimes\psi)$, hence $J_P(\pi)$ must be equal to $\{0\}$ and π is cuspidal. If $\pi \subseteq L^{2+r}(A_GU_0\backslash G, c\otimes\psi)^\infty$ for all $r > 0$ then

$$\int_{A_G\backslash A_0} |W(t)|^{2+r}\delta_{P_0}^{-1}(t)dt < +\infty$$

for all $W \in \pi$ thanks to the Iwasawa decomposition. In particular for all standard parabolic subgroups P, the function $F = W\delta_{P_0}^{-1/(2+r)}$ must satisfy that $|F|^{2+r}$ is summable on $A_GK_{A_0}\backslash A_{0,P}^-(\epsilon)$ for some $0 < \epsilon \leqslant 1$ and some compact open subgroup K_{A_0} of A_0. Now take $P = MU$ a proper standard parabolic subgroup of G such that $J_P(\pi) \neq \{0\}$ if it exists (if not π is cuspidal, hence tempered). Take $\chi \in \mathcal{E}(A_M, J_P(\pi))$ and W as in Lemma 8.1. First we notice that, thanks to the asymptotic expansion of Theorem 7.1, for ϵ small enough, the function F restricts to $A_{0,P}^-(\epsilon)$ as an A_M-finite function. Then by our choice of W, the character $\delta_P^{1/2}\chi$ is associated to $W_{|A_{0,P}^-(\epsilon)}$ in the sense of [2, Proposition 4.4.4], hence $\delta_P^{1/2}\chi\delta_{P_0}^{-1/(2+r)} = \delta_P^{r/2(2+r)}\chi$ (as δ_{P_0} coincides with δ_P on A_M) is associated to $F_{|A_{0,P}^-(\epsilon)}$. We conclude by [2, Proposition 4.4.4] that $\delta_P^{-r/2(2+r)}\chi$ must be positive for all $r > 0$, hence that χ must be non-negative. We conclude that π is tempered. ☐

9. Asymptotics of integral ℓ-adic Whittaker functions

In this last section we consider for $\ell \neq p$ a prime number, the field $\overline{\mathbb{Q}_\ell}$ (a fixed algebraic closure of $\mathbb{Q}_\ell$) instead of $\mathbb{C}$. We fix an isomorphism between $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$ which we use to transport smooth representations of any closed subgroup H of G over $\mathbb{C}$ to smooth representations of H over $\overline{\mathbb{Q}_\ell}$, this in particular applies to the modulus character δ_H. We denote by $\overline{\mathbb{Z}_\ell}$ the ring of integers of $\overline{\mathbb{Q}_\ell}$, and by $\mathbb{Q}_\ell^u$ the maximal unramified extension of $\mathbb{Q}_\ell$. We denote by $\mathbb{Q}_\ell(p^\infty)$ the algebraic extension of $\overline{\mathbb{Q}_\ell}$ obtained by adjoining all roots of unity of order a power of p, hence $\mathbb{Q}_\ell(p^\infty) \subseteq \mathbb{Q}_\ell^u$. If $E \subseteq \overline{\mathbb{Q}_\ell}$ is an extension of $\mathbb{Q}_\ell$ we denote by O_E its ring of integers ($O_E = E\cap\overline{\mathbb{Z}_\ell}$). We note that if E is contained in a finite extension of $\mathbb{Q}_\ell^u$, then O_E is principal.

9.1. **A result of Vignéras on integral submodules of $\mathcal{W}(G,\psi)$.** Let (π, V) be an admissible representation of G, following [12, I. Definition 9.1] we say that a G-stable $\overline{\mathbb{Z}_\ell}$-submodule $\mathcal{L}$ of V is an admissible lattice if $\mathcal{L}^K$ is a $\overline{\mathbb{Z}_\ell}$-lattice in V^K (i.e. free of rank $\dim_{\overline{\mathbb{Q}_\ell}}(V^K)$) for all $K \in \Omega_{\mathrm{I}}(G)$. Let $E \subseteq \overline{\mathbb{Q}_\ell}$ be an algebraic extension of $\mathbb{Q}_\ell$. We say that a smooth (ℓ-adic) representation (π, V) is realizable over E if it has an E-structure (π_E, V_E): the E-vector space $V_E \subseteq V$ is G-stable and $V_E \otimes_E \overline{\mathbb{Q}_\ell} = V$. If E is contained in a finite extension of $\mathbb{Q}_\ell^u$ and (π, V) is admissible and has an E-structure, then (π, V) is integral if and only if there is a G-stable O_E-lattice $\mathcal{L}_E$ in V_E. Indeed it follows from [12, I.9.2] that if $\mathcal{L}_E$ is such a lattice, then $\mathcal{L} = \mathcal{L}_E \otimes_{O_E} \overline{\mathbb{Z}_\ell}$ is an admissible lattice in V, and conversely if $\mathcal{L}$ is an admissible lattice in V then $\mathcal{L}_E := \mathcal{L}\cap V_E$ is an admissible lattice in V_E as we now justify: for any $x \in V$ there is $m_x \in \mathbb{N}$ large enough such that $\ell^{m_x}x \in \mathcal{L}$ from which we deduce that for any $K \in \Omega_{\mathrm{I}}(G)$ the E-span of $\mathcal{L}_E^K$ is V_E^K, and moreover $\mathcal{L}_E$ being torsion free and E being principal we deduce that $\mathcal{L}_E^K$ is a free O_E-module. By [12, II.4.7] any irreducible representation π of G is realizable over a finite extension of $\mathbb{Q}_\ell$, and the proof of this fact extends to finite length representations, we add it here.

Proposition 9.1. *Let (π, V) be an ℓ-adic representation of finite length, then it is realizable over a finite extension of $\mathbb{Q}_\ell$.*

Proof. We suppose that $V \neq \{0\}$. Take $\{0\} = V_0 \subsetneq \cdots \subsetneq V_r = V$ a composition series of V of length r. Take $K \in \Omega_{\mathrm{I}}(G)$ small enough such that each $(V_{i+1}/V_i)^K$ is non-zero. Then one checks by induction on the length of V that $\pi(G).V^K = V$. Denote by $\mathcal{H}(G,K)$ the Hecke algebra of bi-K-invariant functions inside $\mathcal{C}_c^\infty(G)$. It is well-known that the category of smooth representations spanned over G by their K-invariants is equivalent to that of $\mathcal{H}(G,K)$-modules ([12, II. 3.12] for example). Now $\mathcal{H}(G,K)$ is finitely generated (see [12, II. 2.13]), hence the image of $\mathcal{H}$ of $\mathcal{H}(G,K)$ inside $\mathrm{End}_{\overline{\mathbb{Q}_\ell}}(V^K)$ is finitely generated. Take $B = (e_1, \dots, e_n)$ a basis of V^K, then $\mathrm{Mat}_B(\mathcal{H})$ is a finitely generated subalgebra of $\mathcal{M}_n(\overline{\mathbb{Q}_\ell})$, and hence is contained in $\mathcal{M}_n(E)$ for E a finite extension of $\mathbb{Q}_\ell$. Set $W_E = \mathrm{Vect}_E(e_1, \dots, e_n)$, then W_E is a finite length $\mathcal{H}(G,K)_E$-module (where $\mathcal{H}(G,K)_E$ consists of functions in $\mathcal{H}(G,K)$ with values in E). Then by [12, II. 3.12] there is a unique $E[G]$-module of finite length V_E such that $V_E^K = W_E$. One checks that V_E is an E-structure for V. □

We notice that the non-degenerate character $\psi : U_0 \to \overline{\mathbb{Q}_\ell}$ takes values in $O_{\mathbb{Q}_\ell(p^\infty)}$. Whenever R is a subring of $\overline{\mathbb{Q}_\ell}$ which contains $O_{\mathbb{Q}_\ell(p^\infty)}$, we denote by $\mathcal{W}(G,\psi)_R$ the R-submodule of $\mathcal{W}(G,\psi)$ of functions with values in R and set $\mathcal{W}_c(G,\psi^{-1})_R = \mathcal{W}_c(G,\psi^{-1}) \cap \mathcal{W}(G,\psi^{-1})_R$. Combining [13, Theorem IV.2.1, Corollary II.8.3, Proposition II.8.2] of Vignéras we obtain the following result.

Proposition 9.2. *Let E be an algebraic extension of $\mathbb{Q}_\ell(p^\infty)$ contained in a finite extension of $\mathbb{Q}_\ell^u$ and $(\pi_E, V_E) \subseteq (\mathcal{W}(G,\psi)_E, \rho^\vee)$ be an admissible representation of G which is integral. Then the $O_E[G]$-module*

$$\mathcal{L}_E := V_E \cap \mathcal{W}(G,\psi)_{O_E}$$

is an O_E-lattice in V_E. Moreover $(\pi_E^\vee, V_E^\vee)$ is a quotient of $(\mathcal{W}_c(G,\psi^{-1})_E, \rho)$ and if we denote by $s : W' \in \mathcal{W}_c(G,\psi^{-1})_E \mapsto \overline{W'} \in V_E^\vee$ the surjection such that the injection $i : V_E \subseteq \mathcal{W}(G,\psi)_E$ is dual to it, then $\mathcal{L}'_E := \overline{\mathcal{W}_c(G,\psi^{-1})_{O_E}} = s(\mathcal{W}_c(G,\psi^{-1})_{O_E})$ is an $O_E[G]$-lattice in $\pi_E^\vee$ which identifies $\mathcal{L}_E$ with the dual lattice of $\mathcal{L}'_E$:

$$\mathcal{L}_E = \{W \in V_E,\ \langle \mathcal{W}_c(G,\psi^{-1})_{O_E}, W\rangle \subseteq O_E\} = \{W \in V_E,\ \langle \mathcal{L}'_E, W\rangle \subseteq O_E\}.$$

Note that with the above notations, if E' is a finite extension of E, we have the inclusion $\mathcal{L}'_E \subseteq \mathcal{L}'_{E'}$ (in fact $\mathcal{L}'_{E'} = \mathcal{L}'_E \otimes_{O_E} O_{E'}$). Indeed set $s' : W' \in \mathcal{W}_c(G,\psi^{-1})_{E'} \mapsto \overline{W'} \in V_{E'}^\vee$ which we recall is by definition the surjection such that the injection $i' : V_{E'} \subseteq \mathcal{W}(G,\psi)_{E'}$ is dual to it, then s' restricts to $\mathcal{W}_c(G,\psi^{-1})_E$ as s, or equivalently $s' = s \otimes_E E'$, because i' also satisfies the relation $i' = i \otimes_E E'$. The claim follows.

As a corollary we obtain:

Corollary 9.3. *Let E be an algebraic extension of $\mathbb{Q}_\ell(p^\infty)$ contained in a finite extension of $\mathbb{Q}_\ell^u$ and $(\pi_E, V_E) \subseteq (\mathcal{W}(G,\psi)_E, \rho^\vee)$ be an admissible representation* *of G which is integral. Let E' be a finite extension of E and $V_{E'} = V_E \otimes_E E'$, clearly $(\pi_{E'}, V_{E'}) \subseteq (\mathcal{W}(G,\psi)_{E'}, \rho^\vee)$. Then $\mathcal{L}_{E'} := V_{E'} \cap \mathcal{W}(G,\psi)_{O_{E'}}$ is equal to $\mathcal{L}_E \otimes_{O_E} O_{E'}$ for $\mathcal{L}_E$ as in Proposition 9.2.*

Proof. The inclusion $\mathcal{L}_E \otimes_{O_E} O_{E'} \subseteq \mathcal{L}_{E'}$ is obvious. To prove the converse take $W \in \mathcal{L}_{E'}$, and in fact take $W \in \mathcal{L}_{E'}^K$ for $K \in \Omega_{\mathrm{I}}(G)$ small enough. Because $\mathcal{L}_E^K$ is a lattice in V_E^K any O_E-basis $(W_i)_i$ of $\mathcal{L}_E^K$ is an E'-basis of $V_{E'}^K$, hence $W = \sum_i a_i W_i$ for $a_i \in E'$. Because $\mathcal{L}_E^K = \{W \in V_E^K,\ \langle {\mathcal{L}'_E}^K, W\rangle \subseteq O_E\}$ is the dual lattice of ${\mathcal{L}'_E}^K$, the vectors $(\overline{W'_i})_i$ forming the dual basis of $(W_i)_i$ belong to ${\mathcal{L}'_E}^K$. Hence $a_i = \langle \overline{W'_i}, W\rangle \in O_{E'}$ because $\overline{W'_i} \in \mathcal{L}'_E \subseteq \mathcal{L}'_{E'}$ as observed before the corollary. ☐

The following result now follows.

Theorem 9.4. *Let $(\pi, V) \subseteq (W(G,\psi), \rho^\vee)$ be an ℓ-adic admissible representation realizable over an algebraic extension of $\mathbb{Q}_\ell$ contained in a finite extension of $\mathbb{Q}_\ell^u$. If π is integral, then the $\overline{\mathbb{Z}_\ell}[G]$-module $V \cap W(G,\psi)_{\overline{\mathbb{Z}_\ell}}$ is an admissible lattice in V.*

Proof. Call E the extension of the statement and set $L = \langle \mathbb{Q}_\ell(p^\infty), E\rangle$; it is an extension of $\mathbb{Q}_\ell(p^\infty)$ contained in a finite extension of $\mathbb{Q}_\ell^u$. Then because π has an E structure, it has an L-structure and Proposition 9.2 shows that $\mathcal{L}_L := V_L \cap W(G,\psi)_{O_L}$ is a G-stable O_L-lattice in V_L. The theorem will follow from the equality $\mathcal{L}_{\overline{\mathbb{Z}_\ell}} := V \cap W(G,\psi)_{\overline{\mathbb{Z}_\ell}} = \mathcal{L}_L \otimes_{O_L} \overline{\mathbb{Z}_\ell}$. The inclusion $\mathcal{L}_L \otimes_{O_L} \overline{\mathbb{Z}_\ell} \subseteq \mathcal{L}_{\overline{\mathbb{Z}_\ell}}$ is clear. Conversely take $W \in \mathcal{L}_{\overline{\mathbb{Z}_\ell}}$, because $\mathcal{L}_L$ contains an L-basis $(W_i)_i$ of V_L which must be a $\overline{\mathbb{Q}_\ell}$-basis of V, we can write $W = \sum_i a_i W_i$ for $a_i \in \overline{\mathbb{Q}_\ell}$. Set $L' = L((a_i)_i)$, it is a finite extension of L because all a_i are algebraic over $\mathbb{Q}_\ell$, hence over L, and W takes values in L' because all the W_i do. But W also takes values in $\overline{\mathbb{Z}_\ell}$ by definition, hence in $O_{L'}$, so $W \in \mathcal{L}_{L'}$. We deduce from Corollary 9.3 that $W \in \mathcal{L}_L \otimes_{O_L} O_{L'} \subseteq \mathcal{L}_L \otimes_{O_L} \overline{\mathbb{Z}_\ell}$ and the result follows. ☐

We recall that finite length representations of G are admissible (see [10, VI.6.3 Théorème]). From Proposition 9.1 we obtain the following immediate consequence of Theorem 9.4.

Corollary 9.5. *Let $(\pi, V) \subseteq (W(G,\psi), \rho^\vee)$ be an ℓ-adic representation of finite length. If π is integral, then the $\overline{\mathbb{Z}_\ell}[G]$-module $V \cap W(G,\psi)_{\overline{\mathbb{Z}_\ell}}$ is an admissible lattice in V.*

9.2. **The asymptotic expansion of integral Whittaker functions.** If π is smooth admissible $\overline{\mathbb{Q}_\ell}[G]$-module of G, we say that $\mathcal{E}(A_G,\pi)$ is integral if all its elements take values in $\overline{\mathbb{Z}_\ell}^\times$. The following lemma is straightforward from [12, I.9.3].

Lemma 9.6. *Let V be an integral admissible $\overline{\mathbb{Q}_\ell}[G]$-module with $\overline{\mathbb{Z}_\ell}[G]$-admissible lattice $\mathcal{L}$. Then $\mathcal{E}(A_G,\pi)$ is integral and $\mathcal{L} = \bigoplus_{\chi\in\mathcal{E}(A_G,\pi)} \mathcal{L}_{(\chi)}$, where $\mathcal{L}_{(\chi)} = \mathcal{L} \cap V_{(\chi)}$.*

For $P = MU$ a standard parabolic subgroup of G and $\chi \in \widehat{A_M}$ with values in $\overline{\mathbb{Z}_\ell}^\times$, we denote by $\mathcal{F}(A_M)_{(\chi),\overline{\mathbb{Z}_\ell}}$ the $\overline{\mathbb{Z}_\ell}$-module of functions in $\mathcal{F}(A_M)_{(\chi)}$ taking values in $\overline{\mathbb{Z}_\ell}$, and by $\mathcal{C}_c^\infty(\mathcal{M}_P^G)_{\overline{\mathbb{Z}_\ell}}$ that of functions in $\mathcal{C}_c^\infty(\mathcal{M}_P^G)$ taking values in $\overline{\mathbb{Z}_\ell}$. Whenever (π, V) is an integral finite length G-submodule of $(\mathcal{W}(G,\psi), \rho^\vee)$, we set

$$\mathcal{L}_\pi = V \cap \mathcal{W}(G,\psi)_{\overline{\mathbb{Z}_\ell}}.$$

In particular $\mathcal{L}_\pi$ is a lattice in V according to Corollary 9.5. First we prove the expected asymptotic expansion when G/A_G has split rank 0.

Lemma 9.7. *Suppose that G/A_G has split rank 0. Let V be an integral $\overline{\mathbb{Q}_\ell}[G]$-submodule of $\mathcal{W}(G,\psi)$ of finite length and $W \in \mathcal{L}_\pi$, then for each $\chi \in \mathcal{E}(A_G,\pi)$, there is a finite set $I_{W,\chi}$ such that one can write for $z \in A_G$*

$$W(z) = \sum_{\chi \in \mathcal{E}(A_G,\pi)} \sum_{i \in I_{W,\chi}} f_i(z)\phi_i(\mathfrak{r}_G(z))$$

for some functions $f_i \in \mathcal{F}(A_G)_{(\chi),\overline{\mathbb{Z}_\ell}}$ and $\phi_i \in \mathcal{C}_c^\infty(\mathcal{M}_G^G)_{\overline{\mathbb{Z}_\ell}}$ (here ϕ_i is just a constant in $\overline{\mathbb{Z}_\ell}$).

Proof. Note that the $\overline{\mathbb{Q}_\ell}[G]$-module $\pi' := \overline{\mathbb{Q}_\ell}[G].W$ has finite dimension because G/A_G is compact and A_G is commutative, hence we can consider a basis $(W_1,\ldots,W_r)$ of the $\overline{\mathbb{Z}_\ell}[G]$-lattice $\mathcal{L}_{\pi'}$. The action of A_G on $\mathcal{L}_{\pi'}$ in this basis has coefficients given by functions in $\mathcal{F}(A_G)_{\overline{\mathbb{Z}_\ell}}$. Writing W as a sum of the W_i's with integral coefficients, we see that $\pi(z)W$ is of the form $\sum_{i=1}^r h_i(z)W_i$ for $h_i \in \mathcal{F}(A_G)_{\overline{\mathbb{Z}_\ell}}$, hence evaluating $\pi(z)W$ at 1 we see that W is a linear combination with integral coefficients (the scalars $W_i(1) \in \overline{\mathbb{Z}_\ell}$) of the functions h_i, and we conclude by applying Lemma 9.6. □

It was expected for a long time that whenever π is admissible and integral, all its Jacquet modules are integral (see [5]). Though this fact sounds elementary it is far from being the case and Dat proved it (with sophisticated arguments) in [5] at least when F has characteristic zero and G is $\mathrm{GL}_n(F)$ or an inner form of it, or a classical group. However this statement has been very recently shown to hold in full generality in [4], as a consequence of (the very deep results of) [8] together with some finiteness results on the Galois side of the local Langlands correspondence established in [4]. This allows us to remove the hypothesis that the Jacquet modules of π are integral in the statement below, which is an integral version of Theorem 7.1.

Theorem 9.8. *Let π be an integral $\overline{\mathbb{Q}_\ell}[G]$-submodule of $W(G,\psi)$ of finite length with integral Jacquet modules. Then $\mathcal{E}(A_M, J_P(\pi))$ is integral for each $P = MU \supseteq P_0$. Moreover for each $P = MU \supseteq P_0$ and $\chi \in \mathcal{E}(A_M, J_P(\pi))$, there is a finite indexing set $I_{W,\chi}$ such that one can write for $t \in A_0$*

$$W(t) = \sum_{P=MU\supseteq P_0} \delta_P(t)^{1/2} \sum_{\chi\in\mathcal{E}(A_M,J_P(\pi))} \sum_{i\in I_{W,\chi}} f_i(t)\phi_i(\mathfrak{r}_G(t))$$

for some functions $f_i \in \mathcal{F}(A_0)_{(\chi),\overline{\mathbb{Z}_\ell}}$ and $\phi_i \in \mathcal{C}_c^\infty(\mathcal{M}_P^G)_{\overline{\mathbb{Z}_\ell}}$.

Proof. First for each $P = MU \supseteq P_0$, the Jacquet module $J_P(\pi)$ has finite length (see [10, VI.6.4]) and it is integral, thanks to [4, Corollary 1.5], in particular $\mathcal{E}(A_M, J_P(\pi))$ is integral. Moreover the map $\Xi_P = \overline{D}_P = \overline{BH}_P$ identifies $J_P(\pi)$ with a submodule of $\mathcal{W}(M,\psi)$ (Theorem 6.1 for example), in particular $\mathcal{L}_{J_P(\pi)}$ is a lattice in $J_P(\pi)$ thanks to Corollary 9.5. These observations, together with the fact that functions of the form $\mathbf{1}_{]0,\epsilon[}(|\alpha(\ )|)$, $\mathbf{1}_{[\epsilon,+\infty[}(|\alpha(\ )|)$ and $\mathbf{1}_{[\epsilon,C]}(|\alpha(\ )|)$ have integral values, are enough to apply the induction hypothesis going through the proof of Theorem 7.1 and to deduce the expected expansion. □

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