REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 27, Pages 80–114 (April 26, 2023)
https://doi.org/10.1090/ert/627

ON ENDOMORPHISM ALGEBRAS OF GELFAND-GRAEV REPRESENTATIONS

TZU-JAN LI

ABSTRACT. For a connected reductive group G defined over $\mathbb{F}_q$ and equipped with the induced Frobenius endomorphism F, we study the relation among the following three $\mathbb{Z}$-algebras: (i) the $\mathbb{Z}$-model E_G of endomorphism algebras of Gelfand-Graev representations of G^F; (ii) the Grothendieck group K_{G^*} of the category of representations of G^{*F^*} over $\overline{\mathbb{F}_q}$ (Deligne-Lusztig dual side); (iii) the ring $\mathsf{B}_{G^\vee}$ of the scheme $(T^\vee/\!\!/W)^{F^\vee}$ over $\mathbb{Z}$ (Langlands dual side). The comparison between (i) and (iii) is motivated by recent advances in the local Langlands program.

Contents

Introduction and main results

0.1. **The problem.** In this article, we would like to study the relation among three algebras E_G, K_{G^*} and $\mathsf{B}_{G^\vee}$ coming from group representation theory and from invariant theory. Let us now describe these algebras.

Let p be a prime number, let $q = p^r$ for some $r \in \mathbb{N}^*$, and fix an algebraic closure $\overline{\mathbb{F}_q}$ of the finite field $\mathbb{F}_q$. Let G be a connected reductive group defined over $\mathbb{F}_q$, and let $F : G \longrightarrow G$ be the Frobenius endomorphism associated to the $\mathbb{F}_q$-structure of G. We write $G(\overline{\mathbb{F}_q})$ simply as G, so $G^F = \{x \in G : F(x) = x\}$ is a finite group. Fix an F-stable Borel subgroup B of G, fix an F-stable maximal torus T of G included in B, and let $W = N_G(T)/T$ be the Weyl group of (G,T); then W is F-stable. Let (G^*, T^*, F^*) be a Deligne-Lusztig dual (defined over $\overline{\mathbb{F}_q}$; see §0.7) of (G,T,F), and let $(G^\vee, T^\vee)$ be a Langlands dual (defined and split over $\mathbb{Z}$; see §3.1) of (G,T). The endomorphism F induces an endomorphism $F^\vee$ on the character

Received by the editors December 10, 2021, and, in revised form, June 7, 2022, and June 22, 2022.

2020 *Mathematics Subject Classification.* Primary 20C33.

This project received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754362.

group $X(T^\vee) = \mathrm{Hom}_{\mathrm{alg}}(T^\vee, \mathbb{G}_m)$ (§3.2). In addition, let $\overline{\mathbb{Q}}$ be the field of algebraic numbers, and let $\overline{\mathbb{Z}}$ be the ring of algebraic integers. With this setup:

- E_G is the $\mathbb{Z}$-model of endomorphism algebras of Gelfand-Graev representations of G^F (see §§1.3–1.5).
- K_{G^*} is the Grothendieck group of the category of finite-dimensional representations of G^{*F^*} over $\overline{\mathbb{F}_q}$ (§2.2); it is a $\mathbb{Z}$-algebra whose multiplication comes from the tensor product.
- $\mathsf{B}_{G^\vee}$ is the ring of functions of the affine $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$ (§3.3); more precisely, $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ where I is the ideal of $\mathbb{Z}[X(T^\vee)]^W$ generated by the set $\{F^\vee f - f : f \in \mathbb{Z}[X(T^\vee)]^W\}$.

Our goal is to compare these three algebras over rational and integral coefficients.

0.2. **Analysis over $\overline{\mathbb{Q}}$.** In the case of $\overline{\mathbb{Q}}$-coefficients, Curtis-Deligne-Lusztig's theory (§1.2–§1.4) and Brauer theory (§2.3) give us two $\overline{\mathbb{Q}}$-algebra isomorphisms:

$$\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \quad \text{and} \quad \overline{\mathbb{Q}}\mathsf{K}_{G^*} \simeq \overline{\mathbb{Q}}^{G_{p'}^{*F^*}/\sim}. \tag{0.2.1}$$

In (0.2.1), the first isomorphism only depends on a choice of identifications $(\mathbb{Q}/\mathbb{Z})_{p'} \simeq \overline{\mathbb{F}_q}^\times \hookrightarrow \overline{\mathbb{Q}}^\times$, and the second one only depends on a choice of embedding $\overline{\mathbb{F}_q}^\times \hookrightarrow \overline{\mathbb{Q}}^\times$. Via (0.2.1), the natural maps of finite sets

$$(G_{p'}^{*F^*}/\sim) =\!=\!= (G_{\mathrm{ss}}^{*F^*}/\sim) \xrightarrow[\text{Lemma 3.6}]{\text{Lemma 3.1}} (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \tag{0.2.2}$$

induce the following $\overline{\mathbb{Q}}$-algebra homomorphisms:

$$\overline{\mathbb{Q}}\mathsf{K}_{G^*} \xleftrightarrow{\sim} \overline{\mathbb{Q}}\mathsf{E}_G \longleftarrow (\overline{\mathbb{Q}}\mathsf{B}_{G^\vee})_{\mathrm{red}} \overset{\S 3.5}{=\!=\!=} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadleftarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}. \tag{0.2.3}$$

(For a ring A, we denote by A_{red} the associated reduced ring.)

It is then natural to ask: when are the maps in (0.2.3) all isomorphisms? Could we change the $\overline{\mathbb{Q}}$-coefficients in (0.2.3) by Λ-coefficients with Λ a subring of $\overline{\mathbb{Q}}$, such that we still have analogous Λ-algebra homomorphisms?

In [BoKe] (see also Lemma 1.8–Theorem 1.9), Bonnafé and Kessar have proved a "saturatedness property" for the Curtis homomorphism, where they used symmetrizing forms to descend an equality of algebras over rational coefficients into an equality of algebras over integral coefficients. This idea will be the starting point of our present work.

0.3. **Main results.** Keep the notation and assumptions introduced so far.

(a) [see Theorem 2.3–Corollary 2.4] The $\overline{\mathbb{Q}}$-algebra isomorphism $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in (0.2.3) descends to a $\mathbb{Z}[\frac{1}{p|W|}]$-algebra isomorphism

$$\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{E}_G \simeq \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}.$$

(b) [see Proposition 3.4 and Theorem 3.9] $\mathsf{B}_{G^\vee,\mathrm{red}}$ is a free $\mathbb{Z}$-module of rank

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{B}_{G^\vee,\mathrm{red}} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |(G_{\mathrm{ss}}^*/\sim)^{F^*}|; \tag{0.3.1}$$

moreover, if G_{der}^* (or equivalently $G_{\mathrm{der}}^\vee$) is simply-connected, then $\mathsf{B}_{G^\vee}$ is a reduced ring (so $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}}$), and the above rank (0.3.1) is also equal to $|G_{\mathrm{ss}}^{*F^*}/\sim|$.

(c) [see Proposition 3.12–Theorem 3.13] The formal character isomorphism

$$\mathrm{ch}:\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(T^\vee)]^W$$

induces an inclusion of rings $\mathsf{B}_{G^\vee,\mathrm{red}} \hookrightarrow \mathsf{K}_{G^*}$ which is compatible with the identifications in (0.2.3); moreover, if G^*_{der} is simply-connected, then this inclusion of rings becomes a ring isomorphism:

$$\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}} \simeq \mathsf{K}_{G^*}.$$

In (a), the inversion of $|W|$ is mainly due to the usage of saturatedness technique (see the proofs of Theorems 1.9 and 2.3), while we expect that such an inversion is unnecessary. Through recent collaboration with J. Shotton, we can replace this inversion of $|W|$ by only the inversion of bad primes for the root system of G, but it will take more work to know if we really have the expected isomorphism $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}$ for all G. The inversion of p, on the other hand, arises from the usage of an idempotent e_ψ of U^F (with U the unipotent radical of B) in the identification $\Lambda\mathsf{E}_G = e_\psi \Lambda G^F e_\psi$ (with Λ a suitable ring), where we need to invert $|U^F|$ (a power of p); see §1.2 and §1.4. For now, I don't know if we can remove the inversion of p and reach the isomorphism $\mathsf{E}_G \simeq \mathsf{K}_{G^*}$.

For the reducedness of $\mathsf{B}_{G^\vee}$ in (b), the simple-connectedness of G^*_{der} is imposed mainly to fit the induction technique on heights (see Theorem 3.9 and its proof); without this condition on G^*_{der}, we still expect that $\mathsf{B}_{G^\vee}$ is reduced, but little is known for now (see §3.4 for partial progress on this). The reducedness of $\mathsf{B}_{G^\vee}$ is a necessary condition of the isomorphism $\mathsf{B}_{G^\vee} \simeq \mathsf{K}_{G^*}$ in (c) (as K_{G^*} is reduced), but even if $\mathsf{B}_{G^\vee}$ is reduced, the isomorphism $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}} \simeq \mathsf{K}_{G^*}$ may still not hold without the simple-connectedness of G^*_{der}, as the inequality $\dim \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} = |(G^*_{\mathrm{ss}}/\sim)^{F^*}| \leq |G^{*F^*}_{\mathrm{ss}}/\sim| = \dim \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ may not be equality. For example, when $G^* = \mathrm{PGL}_2(\overline{\mathbb{F}_q})$ with q odd, $\mathsf{B}_{G^\vee}$ is reduced and $\dim \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = q$, whereas $\dim \overline{\mathbb{Q}}\mathsf{K}_{G^*} = q+1$, so $\mathsf{B}_{G^\vee} \not\simeq \mathsf{K}_{G^*}$.

0.4. **Motivation.** The problem of comparison between E_G and $\mathsf{B}_{G^\vee}$ is a finite group analogue of the "local Langlands correspondence in families" (LLIF) conjecture in [DHKM], which asserts the existence of an isomorphism from the ring of functions of the moduli stack of Langlands parameters of a p-adic reductive group to the endomorphism ring of its Whittaker space. Note that the LLIF conjecture is implied by the Fargues-Scholze conjecture in [FaSc, Conj. I.10.2].

In the case of a reductive group $\mathbf{G}$ defined and split over a p-adic field F whose residue field is $\mathbb{F}_q$, let O_F be the ring of integers of F and write $G = \mathbf{G}(\overline{\mathbb{F}_q})$; then a special case of the LLIF conjecture predicts a ring isomorphism

$$\mathscr{O}_{\mathscr{Z}^1_{\mathrm{tame}}/\!\!/\mathbf{G}^\vee} \simeq \mathrm{End}(\text{c-Ind}\,^{\mathbf{G}(F)}_{\mathbf{G}(O_F)}\Gamma_G)$$

which is compatible with the classical local Langlands conjectures; in the above,

$$\mathscr{Z}^1_{\mathrm{tame}} = \{(s,\mathcal{F}) \in \mathbf{G}^\vee \times \mathbf{G}^\vee : \mathcal{F}s\mathcal{F}^{-1} = s^q\}$$

is the moduli space of tame Langlands parameters for $\mathbf{G}^\vee$, $\mathscr{O}_{\mathscr{Z}^1_{\mathrm{tame}}/\!\!/\mathbf{G}^\vee}$ is the ring of functions of $\mathscr{Z}^1_{\mathrm{tame}} /\!\!/ \mathbf{G}^\vee$, Γ_G is a Gelfand-Graev representation of G^F, and c-Ind $^{\mathbf{G}(F)}_{\mathbf{G}(O_F)}\Gamma_G$ is (isomorphic to) the depth-zero part of the corresponding Whittaker space. The first projection map $\mathbf{G}^\vee \times \mathbf{G}^\vee \longrightarrow \mathbf{G}^\vee$ induces a morphism

$\mathcal{Z}^1_{\mathrm{tame}} /\!\!/ \mathbf{G}^\vee \longrightarrow (\mathbf{G}^\vee /\!\!/ \mathbf{G}^\vee)^{(\cdot)^q} \simeq (T^\vee /\!\!/ W)^{(\cdot)^q}$ and hence a ring homomorphism $\mathsf{B}_{G^\vee} \longrightarrow \mathcal{O}_{\mathcal{Z}^1_{\mathrm{tame}}/\!\!/\mathbf{G}^\vee}$. We may thus draw the following diagram:

$$\begin{tikzcd} \mathcal{O}_{\mathcal{Z}^1_{\mathrm{tame}}/\!\!/\mathbf{G}^\vee} \arrow[rr, leftrightarrow, "\mathrm{LLIF}", "\sim"'] & & \mathrm{End}(\text{c-Ind}\,^{\mathbf{G}(F)}_{\mathbf{G}(O_F)}\Gamma_G) \\ \mathsf{B}_{G^\vee} \arrow[u] \arrow[rr, dashed, no head, "?"] & & \mathsf{E}_G \arrow[u, hook] \end{tikzcd}$$

If the center of G is connected, for a sufficiently large integral extension $\Lambda \supset \mathbb{Z}[\frac{1}{p}]$, the ring $\Lambda\mathsf{B}_{G^\vee}$ (resp. $\Lambda\mathsf{E}_G$) should be thought of as the integral closure of the scalars Λ in $\Lambda.\mathcal{O}_{\mathcal{Z}^1_{\mathrm{tame}}/\!\!/\mathbf{G}^\vee}$ (resp. in $\Lambda.\mathrm{End}(\text{c-Ind}\,^{\mathbf{G}(F)}_{\mathbf{G}(O_F)}\Gamma_G)$), so the LLIF conjecture should give us a Λ-algebra isomorphism

$$\Lambda\mathsf{B}_{G^\vee} \simeq \Lambda\mathsf{E}_G. \tag{0.4.1}$$

In [Hel] and [HeMo], the LLIF has been proved for $\mathbf{G} = \mathrm{GL}_n(F)$, and, as a corollary, a Λ-algebra isomorphism (0.4.1) has been deduced for $G = \mathrm{GL}_n(\overline{\mathbb{F}_q})$ and for Λ being the ring of Witt vectors of $\overline{\mathbb{F}_\ell}$ (note that (0.4.1) was also used as an ingredient in their proof). However, as (0.4.1) is basically a result of finite groups, an argument without p-adic techniques (such as those used for the LLIF) is expected, and the present article provides such an argument under some additional hypotheses:

Theorem 0.1 (§0.3(a), (c) or Corollary 3.14)**.** *If G^*_{der} is simply-connected, then we have the Λ-algebra isomorphism* (0.4.1) *for* $\Lambda = \mathbb{Z}[\frac{1}{p|W|}]$.

0.5. **Plan of the article.** In §1, we review properties of endomorphism algebras of a modular Gelfand-Graev representation, construct the $\mathbb{Z}$-model E_G, and study Curtis homomorphisms; in §2, we introduce the Grothendieck group K_{G^*} on the Deligne-Lusztig dual side, review the Brauer theory, and then compare K_{G^*} with the algebra E_G. In §3, using methods from algebraic geometry, combinatorics of root datum and algebraic representation theory, we turn to study the algebra $\mathsf{B}_{G^\vee}$ on the Langlands-dual side and finally compare $\mathsf{B}_{G^\vee}$ with K_{G^*} and E_G.

0.6. **A graphical summary.** Let Λ be an integral domain such that $\mathbb{Z}[\frac{1}{p}] \subset \Lambda \subset \overline{\mathbb{Q}}$. The following diagram, which will eventually be shown to be commutative (see Lemma 1.6 and Proposition 3.12), summarizes the relations among the principal objects studied in this article.

Notations used in the diagram: $G^{*F^*}_{\mathrm{ss}}/\sim$ (resp. $G^{*F^*}_{p'}/\sim$) denotes the set of semisimple (resp. p-regular) conjugacy classes in G^{*F^*}; "Res" means the obvious restriction maps; $T_w := {}^gT = gTg^{-1}$ is the F-stable maximal torus of G whose G^F-conjugacy class corresponds to $w \in W$, so $g \in G$ and the image of $g^{-1}F(g) \in N_G(T)$ in W is w (recall that the G^F-conjugacy classes of F-stable tori in G are parametrized by F-conjugacy classes in W in this way); $\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w\in W}$ is the Curtis embedding which will be defined in §1.7.

$$
\begin{tikzcd}[column sep=small]
\prod_{w\in W}\overline{\mathbb{Q}}^{T_w^{*F^*}} \arrow[dd, leftrightarrow, "\wr"'] & & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[ll, hook', "\mathrm{Res}"'] \arrow[dd, leftrightarrow, "\wr"'] \arrow[dr, leftrightarrow, "\sim"] & & \overline{\mathbb{Q}}^{(T^\vee/\!\!/W)^{F^\vee}(\overline{\mathbb{Q}})} \arrow[ll, hook'] \arrow[dl, hook'] \arrow[dd, leftrightarrow, "\wr"'] & \\
 & & & \overline{\mathbb{Q}}^{G_{p'}^{*F^*}/\sim} \arrow[dd, leftrightarrow, "\wr"'] & & \\
\prod_{w\in W}\overline{\mathbb{Q}}T_w^F \arrow[dr, leftrightarrow, "\sim"] & & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[ll, hook', "\mathrm{Cur}^G"'] \arrow[dr, leftrightarrow, "\sim"] & & \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \arrow[ll, hook'] \arrow[dl, hook'] & \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \arrow[l, two heads] \\
 & \prod_{w\in W}\overline{\mathbb{Q}}\mathsf{K}_{T_w^*} & & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[ll, hook', "\mathrm{Res}"] & & \\
\prod_{w\in W}\Lambda T_w^F \arrow[uu, hook] \arrow[dr, leftrightarrow, "\sim"] & & \Lambda\mathsf{E}_G \arrow[uu, hook] \arrow[ll, hook', "\mathrm{Cur}^G"] \arrow[rr, dashed, "?"] \arrow[dr, dashed, "?"] & & \Lambda\mathsf{B}_{G^\vee,\mathrm{red}} \arrow[uu, hook] \arrow[dl, hook'] & \Lambda\mathsf{B}_{G^\vee} \arrow[uu] \arrow[l, two heads] \\
 & \prod_{w\in W}\Lambda\mathsf{K}_{T_w^*} \arrow[uu, hook] & & \Lambda\mathsf{K}_{G^*} \arrow[uu, hook] \arrow[ll, hook', "\mathrm{Res}"] & & \\
\prod_{w\in W}\mathbb{Z}T_w^F \arrow[uu, hook] \arrow[dr, leftrightarrow, "\sim"] & & & & \mathsf{B}_{G^\vee,\mathrm{red}} \arrow[uu, hook] \arrow[dl, hook'] & \mathsf{B}_{G^\vee} \arrow[uu] \arrow[l, two heads] \\
 & \prod_{w\in W}\mathsf{K}_{T_w^*} \arrow[uu, hook] & & \mathsf{K}_{G^*} \arrow[uu, hook] \arrow[ll, hook', "\mathrm{Res}"] & & \\
 & & & & \mathbb{Z}[X(T^\vee)]^W \arrow[uu, two heads] \arrow[r, equal] & \mathbb{Z}[X(T^\vee)]^W \arrow[uu, two heads] \\
 & & & \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[uu, hook, "\mathrm{Res}"] \arrow[ur, "\sim", "\mathrm{ch}"'] & &
\end{tikzcd}
$$

0.7. **Notation and convention.** The following notation and assumptions, as well as those introduced so far, will be used throughout the article.

Root data [Sp]. Let $X(T) = \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m)$ (resp. $Y(T) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, T)$) be the character group (resp. cocharacter group) of T. Denote by $R \subset X(T)$ and $R^\vee \subset Y(T)$ (resp. $\Delta \subset X(T)$ and $\Delta^\vee \subset Y(T)$) the set of roots and the set of coroots (resp. the set of simple roots and the set of simple coroots) determined by (G,T) (resp. by (G,T,B)). Then $(X(T), Y(T), R, R^\vee)$ the root datum of (G,T).

Deligne-Lusztig dual [DiMi]. Fix a Deligne-Lusztig dual (G^*, T^*, F^*) of (G,T,F) (all choices are isomorphic), in the sense that (G^*, T^*, F^*) is defined over $\mathbb{F}_q$ and is obtained by assigning its character group (resp. cocharacter group, resp. set of roots, resp. set of coroots) as $Y(T)$ (resp. $X(T)$, resp. $R^\vee$, resp. R). In particular, we have the identifications $X(T^*) = Y(T)$ and $Y(T^*) = X(T)$, both of which are compatible with the Frobenius actions. Again, we write $G^*(\overline{\mathbb{F}_q})$ simply as G^*.

Let $G_{\mathrm{ss}}^{*F^*}$ be the set of semisimple elements of G^{*F^*}, and let $G_{\mathrm{ss}}^{*F^*}/\sim$ be the set of G^{*F^*}-conjugacy classes in $G_{\mathrm{ss}}^{*F^*}$; for each $x \in G_{\mathrm{ss}}^{*F^*}$, we shall denote by $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$ its G^{*F^*}-conjugacy class.

Finite group representation theory [Se1], [DeLu], [DiMi]. Let Λ be a commutative ring and let H be a finite group. The group ring $\Lambda[H]$ will often be written as ΛH; an element $f : H \longrightarrow \Lambda$ of ΛH is identified with the formal sum $\sum_{h\in H} f(h)h$. Let $\mathrm{Rep}_\Lambda(H)$ be the category of finite-dimensional representations of H over Λ, and let $\mathrm{Irr}_\Lambda(H)$ be the set of isomorphism classes of simple objects in $\mathrm{Rep}_\Lambda(H)$. For $V_1, V_2 \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(H)$ whose characters are χ_1, χ_2 respectively, their canonical pairings $\langle V_1, V_2\rangle_H := \dim_{\overline{\mathbb{Q}}} \mathrm{Hom}_{\overline{\mathbb{Q}}H}(V_1, V_2)$ and $\langle \chi_1, \chi_2\rangle_H :=$

$|H|^{-1}\sum_{h\in H}\chi_1(h^{-1})\chi_2(h)$ coincide. Ordinary induction functors like $\mathrm{Ind}_{U^F}^{G^F}$, ordinary restriction functors like $\mathrm{Res}_{T^{*F^*}}^{G^{*F^*}}$, and Deligne-Lusztig induction functors like R_T^G are defined in the usual way.

Dualities of tori [DiMi]. We fix a group isomorphism $\iota : \overline{\mathbb{F}_q}^\times \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$ (choices of roots of unity) and an injective group homomorphism $\jmath : (\mathbb{Q}/\mathbb{Z})_{p'} \hookrightarrow \overline{\mathbb{Q}}^\times$. Then $\kappa := \jmath\circ\iota : \overline{\mathbb{F}_q}^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ enables us to identify $\overline{\mathbb{F}_q}^\times$ as a subgroup of $\overline{\mathbb{Q}}^\times$. For an F-stable maximal torus S of G, we choose an F^*-stable maximal torus S^* of G^* which is dual to S (all choices of S^* are G^{*F^*}-conjugate). Then ι identifies $\mathrm{Irr}_{\overline{\mathbb{F}_q}}(S^F)\simeq S^{*F^*}$, and (ι,κ) identifies $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)\simeq S^{*F^*}$.

1. Endomorphism algebras of Gelfand-Graev representations

The Gelfand-Graev representation $\Gamma_{G,\psi}$.

1.1. **Regular linear characters.** [DLM, Sec. 2] Let U be the unipotent radical of the chosen F-stable Borel subgroup B of G, so that U itself is also F-stable. Denote by $U_\bullet$ the subgroup of U generated by the root subgroups of non-simple roots (roots in $R-\Delta$); note that $U_\bullet=[U,U]$. Let Δ/F be the set of F-orbits in Δ. For each $i\in\Delta/F$, let U_i be the product of root subgroups of (simple) roots in i. The quotient group $U/U_\bullet$ is canonically isomorphic to the product group $\prod_{i\in\Delta/F}U_i$, and this isomorphism is F-stable, so that $U^F/U_\bullet^F\simeq\prod_{i\in\Delta/F}U_i^F$ as abelian groups. Then a linear character ψ on U^F is called regular if ψ is trivial $(=1)$ on $U_\bullet^F$ and is non-trivial on every U_i^F $(i\in\Delta/F)$.

Let Ψ be the set of $\overline{\mathbb{Z}}^\times$-valued regular linear characters of U^F; the group T^F acts on Ψ by adjoint action (for $t\in T^F$ and $\psi\in\Psi$, ${}^t\psi := \psi({}^{t^{-1}}(\cdot)) = \psi(t^{-1}(\cdot)t)\in\Psi$). Let Z be the center of G, so the adjoint group of G is $G_{\mathrm{ad}}=G/Z$. Let $T_{\mathrm{ad}}=T/Z$ be the image of T in G_{ad}; the group T_{ad}^F also acts on Ψ by adjoint action (for $x\in T_{\mathrm{ad}}^F$ and $\psi\in\Psi$, choose a $t\in T$ such that $x=tZ$, and set ${}^x\psi := {}^t\psi$), and this T_{ad}^F-action on Ψ is regular (= free and transitive). Hence the first Galois cohomology group $H^1(F,Z)$, identified with T_{ad}^F/T^F, acts regularly on Ψ/T^F (the set of T^F-orbits in Ψ) by adjoint action.

1.2. **Gelfand-Graev representations.** [DiMi, Ch. 14] Let $\psi : U^F \longrightarrow \overline{\mathbb{Z}}^\times$ be a regular linear character (so $\psi\in\Psi$ in the notation of §1.1), and consider the primitive central idempotent

$$e_\psi := \frac{1}{|U^F|}\sum_{u\in U^F}\psi(u^{-1})u \in \overline{\mathbb{Z}}[\tfrac{1}{p}]U^F \tag{1.2.1}$$

of ψ ($|U^F|$ is a power of p). Let $(\overline{\mathbb{Z}}[\frac{1}{p}])_\psi$ be the $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$-module $\overline{\mathbb{Z}}[\frac{1}{p}]$ on which U^F acts by ψ; we have $(\overline{\mathbb{Z}}[\frac{1}{p}])_\psi\simeq\overline{\mathbb{Z}}[\frac{1}{p}]U^Fe_\psi=\overline{\mathbb{Z}}[\frac{1}{p}]e_\psi$ as $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$-modules. Then

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U^F}^{G^F}((\overline{\mathbb{Z}}[\tfrac{1}{p}])_\psi)\simeq\overline{\mathbb{Z}}[\tfrac{1}{p}]G^Fe_\psi\in\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F) \tag{1.2.2}$$

is called a Gelfand-Graev representation of G; the character of $\Gamma_{G,\psi}$ is $\mathrm{Ind}_{U^F}^{G^F}\psi$.

All $\Gamma_{G,\psi}$ ($\psi\in\Psi$) are conjugate by elements of T_{ad}^F: indeed, for any $\psi,\psi'\in\Psi$, there is a unique $x\in T_{\mathrm{ad}}^F$ such that $\psi'={}^x\psi$ (§1.1); then $e_{\psi'}={}^x(e_\psi)$, and thus, upon identifying $\Gamma_{G,\psi}=\overline{\mathbb{Z}}[\frac{1}{p}]G^Fe_\psi$ via (1.2.2),

$$\Gamma_{G,\psi'}=\overline{\mathbb{Z}}[\tfrac{1}{p}]G^Fe_{\psi'}={}^x(\overline{\mathbb{Z}}[\tfrac{1}{p}]G^Fe_\psi)={}^x(\Gamma_{G,\psi}). \tag{1.2.3}$$

If the center Z of G is connected (that is, if $H^1(F,Z)=0$), then $T_{\mathrm{ad}}^F = T^F/Z^F$, so all $\Gamma_{G,\psi}$ are conjugate by elements of T^F and are thus all isomorphic in $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F)$.

Theorem 1.1 ([DiMi, Thm. 14.49])**.** *Let $\psi \in \Psi$. Then the $\overline{\mathbb{Q}}G^F$-module $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ is multiplicity-free; more precisely, we have a $\overline{\mathbb{Q}}G^F$-module decomposition*

$$\overline{\mathbb{Q}}\Gamma_{G,\psi} = \bigoplus_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \rho_{\psi,[x]},$$

where for each $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$, $\rho_{\psi,[x]}$ is an irreducible $\overline{\mathbb{Q}}G^F$-module whose character $\chi_{\psi,[x]}$ lies in the rational Lusztig series $\mathcal{E}(G^F,[x])$.

Lemma 1.2. *Let $\psi, \psi' \in \Psi$, and suppose that $\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$ is a common irreducible character of the $\overline{\mathbb{Q}}G^F$-modules $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ and $\overline{\mathbb{Q}}\Gamma_{G,\psi'}$. Then for the unique $y \in T_{\mathrm{ad}}^F$ such that $\psi' = {}^y\psi$ (§1.1), we have ${}^y\chi = \chi$.*

Proof. By Theorem 1.1, χ lies in some rational Lusztig series $\mathcal{E}(G^F,[x])$; in other words, there is an F-stable maximal torus S and a $\theta \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ such that θ corresponds to $[x]$ under the chosen duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$ (§0.7), and such that $\langle \chi, R_S^G(\theta)\rangle_{G^F} \neq 0$.

Let us show that ${}^y(R_S^G(\theta)) = R_S^G(\theta)$. Indeed, as $y \in T_{\mathrm{ad}}^F$, if we choose a $t \in T$ such that $y = tZ$, then $t^{-1}F(t) \in Z \subset S$, so the Lang-Steinberg theorem enables us to write $t^{-1}F(t) = sF(s)^{-1}$ for some $s \in S$; thus, upon setting $x := ts \in G^F$,

$${}^y(R_S^G(\theta)) = {}^t(R_S^G(\theta)) = R_{{}^tS}^G({}^t\theta) = R_{{}^xS}^G({}^x\theta) = {}^x(R_S^G(\theta)) = R_S^G(\theta).$$

By the above discussion, we have

$$\langle {}^y\chi, R_S^G(\theta)\rangle_{G^F} = \langle {}^y\chi, {}^y(R_S^G(\theta))\rangle_{G^F} = \langle \chi, R_S^G(\theta)\rangle_{G^F} \neq 0.$$

On the other hand, with the help of (1.2.3),

$$\langle {}^y\chi, \overline{\mathbb{Q}}\Gamma_{G,\psi'}\rangle_{G^F} = \langle {}^y\chi, {}^y(\overline{\mathbb{Q}}\Gamma_{G,\psi})\rangle_{G^F} = \langle \chi, \overline{\mathbb{Q}}\Gamma_{G,\psi}\rangle_{G^F} \neq 0.$$

Therefore, $\gamma = \chi$ and $\gamma = {}^y\chi$ both verify $\langle \gamma, R_S^G(\theta)\rangle_{G^F} \neq 0$ and $\langle \gamma, \overline{\mathbb{Q}}\Gamma_{G,\psi'}\rangle_{G^F} \neq 0$; but Theorem 1.1 tells us that there is at most one such $\gamma \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$, so ${}^y\chi$ and χ must coincide. ☐

The endomorphism algebra of $\Gamma_{G,\psi}$ and its $\mathbb{Z}$-model E_G.

1.3. **The endomorphism algebras $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ and $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$.** By §1.2, all endomorphism algebras

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} := \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\Gamma_{G,\psi}) \ \ (\psi \in \Psi)$$

are isomorphic as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras. Later in §1.5, we shall introduce a $\mathbb{Z}$-algebra E_G which is independent of ψ and is such that $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ for all $\psi \in \Psi$.

Let $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} := \overline{\mathbb{Q}} \otimes_{\overline{\mathbb{Z}}[\frac{1}{p}]} \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$. By Theorem 1.1 and Schur's lemma, we may decompose the $\overline{\mathbb{Q}}$-algebra $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ as:

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = \mathrm{End}_{\overline{\mathbb{Q}}G^F}(\overline{\mathbb{Q}}\Gamma_{G,\psi}) \simeq \prod_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \mathrm{End}_{\overline{\mathbb{Q}}G^F}(\rho_{\psi,[x]}) \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}. \tag{1.3.1}$$

Thus $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ and $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ are commutative reduced rings. In terms of algebraic varieties, (1.3.1) identifies $\mathrm{Specm}(\overline{\mathbb{Q}}\mathsf{E}_G)$ (consisting of maximal ideals of $\overline{\mathbb{Q}}\mathsf{E}_G$) with $G_{\mathrm{ss}}^{*F^*}/\sim$.

1.4. **Structures of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ and $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ via idempotents.** Using the identification $\Gamma_{G,\psi} = \overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ in (1.2.2), we shall identify

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F$$

as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras: indeed, there is a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra anti-isomorphism

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi) \xrightarrow{\sim} e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi, \quad \theta \longmapsto \theta(e_\psi), \tag{1.4.1}$$

but $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ is a commutative ring (§1.3), so (1.4.1) is also an isomorphism.

(a) By [CuRe, Prop. 11.30], $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ is a free $\overline{\mathbb{Z}}[\frac{1}{p}]$-module with the following $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis: let $\{x_1, \cdots, x_r\} \subset G^F$ be a set of representatives of $U^F \backslash G^F / U^F$ (so G^F is the disjoint union of $\{U^F x_i U^F : i = 1, \cdots, r\}$), and let

$$J = \{j : 1 \leq j \leq r,\ {}^{x_j}\psi = \psi \text{ on } U^F \cap {}^{x_j}(U^F)\};$$

then $\{e_\psi x_j e_\psi : j \in J\}$ is a $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$.

(b) By [Cu, Sec. 3], $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ may be described via idempotents: for $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$, set

$$e^G_{\psi,[x]} := \frac{\chi_{\psi,[x]}(1)}{|G^F|} \sum_{g \in G^F} \chi_{\psi,[x]}(g^{-1}) g \in \overline{\mathbb{Q}}G^F \quad \text{and} \quad e^{\mathsf{E}}_{\psi,[x]} := e^G_{\psi,[x]} e_\psi \in \overline{\mathbb{Q}}\mathsf{E}_{G,\psi},$$

which are respectively primitive central idempotents of $\overline{\mathbb{Q}}G^F$ and of $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$; then the inclusions $\overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]} \subset \overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ induce a $\overline{\mathbb{Q}}$-algebra isomorphism

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = \prod_{[x] \in G_{\mathrm{ss}}^{*F^*}/\sim} \overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]}. \tag{1.4.2}$$

Combining (1.3.1) and (1.4.2), we obtain $\overline{\mathbb{Q}}$-algebra isomorphisms

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \simeq \prod_{[x] \in G_{\mathrm{ss}}^{*F^*}/\sim} \overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]} \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}, \tag{1.4.3}$$

where each $e^{\mathsf{E}}_{\psi,[x]}$ corresponds to the characteristic function $\mathbf{1}_{\{[x]\}}$ on $G_{\mathrm{ss}}^{*F^*}/\sim$.

1.5. **Definition of the $\mathbb{Z}$-algebra E_G.** The decomposition $U^F/U^F_\bullet = \prod_{i \in \Delta/F} U_i^F$ (§1.1) identifies each U_i^F as a subgroup of $U^F/U^F_\bullet$. Consider the (left) $\mathbb{Z}U^F$-module

$$\mathcal{F}_0 = \{f : U^F/U^F_\bullet \longrightarrow \mathbb{Z} : \sum_{x \in U_i^F} f(x) = 0 \text{ for each } i \in \Delta/F\}$$

on which U^F acts by left translation: $u \cdot f := f(u^{-1}(\cdot))$ ($u \in U^F$, $f \in \mathcal{F}_0$). The groups T^F and T^F_{ad} both act on each U_i^F ($i \in \Delta/F$) by left adjoint action; this induces left T^F-actions and left T^F_{ad}-actions on $\mathcal{F}_0$, $\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0)$ and $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))$. The center of the $\mathbb{Z}$-algebra $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))$, denoted by $Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))\right)$, is fixed by the T^F-action on $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))$ and thus admits an $H^1(F,Z)$-action (recall that $H^1(F,Z) \simeq T^F_{\mathrm{ad}}/T^F$); we then set

$$\mathsf{E}_G := Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))\right)^{H^1(F,Z)}, \tag{1.5.1}$$

which consists of elements of $Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\mathcal{F}_0))\right)$ fixed by the $H^1(F,Z)$-action. This E_G is a $\mathbb{Z}$-algebra and is also a free $\mathbb{Z}$-module of finite rank.

After describing the $\overline{\mathbb{Z}}[\frac{1}{p}]G^F$-module $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ in Lemma 1.3, we shall show in Lemma 1.4 that the $\mathbb{Z}$-algebra E_G in (1.5.1) is consistent with the definition of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ in §1.3.

Lemma 1.3. *The natural inclusion $\mathcal{F}_0 \subset \mathbb{Z}U^F$ induces an embedding of rings $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0) \subset \overline{\mathbb{Z}}[\frac{1}{p}]G^F$, under which the actions of T^F and T_{ad}^F on $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ are exactly the restrictions of their respective adjoint actions on $\overline{\mathbb{Z}}[\frac{1}{p}]G^F$. Moreover, we have a decomposition of $\overline{\mathbb{Z}}[\frac{1}{p}]G^F$-modules:*

$$\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0) = \bigoplus_{\psi\in\Psi} \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi = \bigoplus_{\psi\in\Psi} \Gamma_{G,\psi} \in \mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F).$$

Proof. The description of the actions of T^F and T_{ad}^F on $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ is clear from construction, so let us prove the decomposition formula directly.

(1) (Compare [DLM, (2.4.8)].) Each linear character $\psi : U^F/U_\bullet^F = \prod_{i\in\Delta/F} U_i^F \longrightarrow \overline{\mathbb{Z}}^\times$ is decomposed as $\psi = \prod_{i\in\Delta/F}\psi_i$ with $\psi_i = \psi|_{U_i^F}$ $(i \in \Delta/F)$; such a ψ gives an element in Ψ if and only if each ψ_i is non-trivial.

(2) For each $\psi \in \Psi$, let e_ψ be its primitive central idempotent of $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$ as in (1.2.1). Using (1) and the orthogonality of characters, one can check that $\{e_\psi \mid \psi \in \Psi\}$ is a $\overline{\mathbb{Q}}$-linearly independent subset of $\overline{\mathbb{Q}}\mathcal{F}_0$. By definition of $\mathcal{F}_0$ and (1), we have

$$\dim_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}\mathcal{F}_0) = \prod_{i\in\Delta/F}(|U_i^F| - 1) = |\Psi|,$$

so $\{e_\psi \mid \psi \in \Psi\}$ is in fact a $\overline{\mathbb{Q}}$-linear basis of $\overline{\mathbb{Q}}\mathcal{F}_0$; identifying $\overline{\mathbb{Q}}\mathcal{F}_0 \subset \overline{\mathbb{Q}}U^F$, we have the following decomposition of $\overline{\mathbb{Q}}\mathcal{F}_0$:

$$\overline{\mathbb{Q}}\mathcal{F}_0 = \bigoplus_{\psi\in\Psi} \overline{\mathbb{Q}}e_\psi; \quad \text{for } f \in \overline{\mathbb{Q}}\mathcal{F}_0,\ f = \sum_{\psi\in\Psi} fe_\psi \text{ with each } fe_\psi \in \overline{\mathbb{Q}}e_\psi. \tag{1.5.2}$$

As all e_ψ $(\psi \in \Psi)$ lie in $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$, the decomposition (1.5.2) also holds in $\overline{\mathbb{Z}}[\frac{1}{p}]$-coefficients, from which the desired decomposition of $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ follows.

□

Lemma 1.4. *For each $\psi \in \Psi$, $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras. Explicitly: upon identifying $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ via (1.4.1) and identifying analogously*

$$\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0)) = \bigoplus_{\psi,\psi'\in\Psi} e_{\psi'}\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F,$$

we have a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \xrightarrow{\sim} \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G, \quad e_\psi f e_\psi \longmapsto \sum_{y\in T_{\mathrm{ad}}^F} e_{{}^y\psi}\cdot {}^y f \cdot e_{{}^y\psi} \quad (f \in \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F).$$

Proof. (1) An element in the center of $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$ commutes with all e_ψ ($\psi \in \Psi$), so the orthogonality of idempotents implies that

$$Z\left(\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0))\right) \subset \bigoplus_{\psi\in\Psi} e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi. \tag{1.5.3}$$

(2) Let $y \in T_{\mathrm{ad}}^F$. For $\theta \in \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$, its image $y\cdot\theta$ under the action of y on $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$ is given by $(y\cdot\theta)(\varphi) = {}^y(\theta({}^{y^{-1}}\varphi))$ ($\varphi \in \mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$). Then the subring $\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ is stable under this action of y: indeed, let $\theta \in \bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ and write $\theta = \sum_{\psi\in\Psi}\theta_\psi$ where each $\theta_\psi = e_\psi\theta e_\psi \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$; then it can be checked that $y\cdot\theta = \sum_{\psi\in\Psi}(y\cdot\theta)_\psi$ with each $(y\cdot\theta)_\psi = y\cdot(\theta_{{}^{y^{-1}}\psi}) = {}^y(\theta_{{}^{y^{-1}}\psi}) \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$. We thus deduce from (1.5.3) that

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \subset \left(\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi\right)^{T_{\mathrm{ad}}^F} =: \mathcal{A}. \tag{1.5.4}$$

(3) For each $\theta\in\mathcal{A}$, if we write $\theta = \sum_{\psi\in\Psi}\theta_\psi$ with each $\theta_\psi \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ as in (2), then $\theta_\psi = {}^y(\theta_{{}^{y^{-1}}\psi})$ for all $y\in T_{\mathrm{ad}}^F$; as T_{ad}^F acts regularly on Ψ (§1.1), we see that for every $\psi\in\Psi$ we have a ring isomorphism

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \xrightarrow{\sim} \mathcal{A}, \quad e_\psi f e_\psi \longmapsto \sum_{y\in T_{\mathrm{ad}}^F} e_{{}^y\psi}\cdot {}^y f\cdot e_{{}^y\psi} \ \ (f\in\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F). \tag{1.5.5}$$

(4) By virtue of (1.5.4) and (1.5.5), in order to complete the proof of lemma, it remains to establish the inclusion

$$\overline{\mathbb{Q}}\mathcal{A} = \left(\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Q}}G^F e_\psi\right)^{T_{\mathrm{ad}}^F} \subset Z\left(\bigoplus_{\psi,\psi'\in\Psi} e_{\psi'}\overline{\mathbb{Q}}G^F e_\psi\right). \tag{1.5.6}$$

For each $\psi,\psi'\in\Psi$, Theorem 1.1 (with notations in §1.4) tells us that the space $e_{\psi'}\overline{\mathbb{Q}}G^F e_\psi$ is the direct sum of $e_{\psi'}\overline{\mathbb{Q}}G^F e^G_{\psi,[x]}e_\psi$ for $[x]$ ranging over a (possibly empty) subset of $G^{*F^*}_{\mathrm{ss}}/\sim$. To show (1.5.6), it then suffices to show that

$$e_{{}^y\psi} f e^G_{\psi,[x']} e_\psi \cdot e_\psi e^G_{\psi,[x]} e_\psi = e_{{}^y\psi}({}^y e^G_{\psi,[x]}) e_{{}^y\psi}\cdot e_{{}^y\psi} f e^G_{\psi,[x']} e_\psi \tag{1.5.7}$$

for every $\psi\in\Psi$, $y\in T_{\mathrm{ad}}^F$, $[x],[x']\in G^{*F^*}_{\mathrm{ss}}/\sim$ and $f\in\overline{\mathbb{Q}}G^F$.

To prove (1.5.7), suppose that the element $e_{{}^y\psi} f e^G_{\psi,[x']} e_\psi \in \mathrm{Hom}_{\overline{\mathbb{Q}}G^F}(\overline{\mathbb{Q}}\Gamma_{{}^y\psi}, \rho_{\psi,[x']})$ is not zero (as the opposite case is trivial). Then $\chi_{\psi,[x']}$ is a common irreducible character of $\overline{\mathbb{Q}}\Gamma_{{}^y\psi}$ and $\overline{\mathbb{Q}}\Gamma_\psi$, so Lemma 1.2 gives us ${}^y(\chi_{\psi,[x']}) = \chi_{\psi,[x']}$ and then ${}^y e^G_{\psi,[x']} = e^G_{\psi,[x']}$. We may also suppose that $[x]=[x']$, for otherwise both sides of (1.5.7) are zero because the idempotents $e^G_{\psi,[x]}$ are central and orthogonal in $\overline{\mathbb{Q}}G^F$. Under the above assumptions, the two sides of (1.5.7) are both $e_{{}^y\psi} f e^G_{\psi,[x']} e_\psi$; this justifies (1.5.7).

The equality (1.5.7) being proved, we obtain the inclusion (1.5.6) as well as the equality $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \mathcal{A}$; thus (1.5.5) gives the desired $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism. □

1.6. **The Galois action on $\overline{\mathbb{Q}}\mathsf{E}_G$.** The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}\mathsf{E}_G$; this induces an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}\mathsf{E}_G$, and this action coincides with that induced by the identification

$$\overline{\mathbb{Q}}\mathsf{E}_G = Z\left(\mathrm{End}_{\overline{\mathbb{Q}}G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Q}}\mathcal{F}_0))\right)^{H^1(F,Z)}$$

and by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}\mathcal{F}_0$.

In terms of idempotents, the above Galois action on $\overline{\mathbb{Q}}\mathsf{E}_G$ is, upon fixing a $\psi \in \Psi$ and identifying $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Q}}G^F e_\psi$ (Lemma 1.4):

$$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\ e_\psi f e_\psi \in \overline{\mathbb{Q}}\mathsf{E}_{G,\psi}\ (f \in \overline{\mathbb{Q}}G^F) \Longrightarrow \sigma \cdot (e_\psi f e_\psi) = e_\psi \cdot \sigma({}^{y_\sigma^{-1}} f) \cdot e_\psi,$$

where y_σ is the unique element in T_{ad}^F such that $\sigma \cdot \psi = {}^{y_\sigma}\psi \in \Psi$ (σ acts on $\overline{\mathbb{Z}}^\times$ and hence on Ψ; then recall from §1.1 that T_{ad}^F acts regularly on Ψ), and where, for each $h \in \overline{\mathbb{Q}}G^F$, $\sigma(h)$ denotes its image under the action of σ on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}G^F$.

Curtis homomorphisms.

1.7. **The Curtis homomorphism** Cur_S^G **[Cu, Sec. 4].** Hereafter, fix a $\psi \in \Psi$. For S an F-stable maximal torus of G, Curtis has constructed a $\overline{\mathbb{Q}}$-algebra homomorphism

$$\mathrm{Cur}_S^G : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Q}}S^F$$

(which he called f_S in his paper), which is the unique $\overline{\mathbb{Q}}$-algebra homomorphism from $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ to $\overline{\mathbb{Q}}S^F$ such that, for every irreducible character $\theta \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ corresponding to $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$ under the chosen duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$ (§0.7), the following diagram of $\overline{\mathbb{Q}}$-algebras commutes:

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \arrow[r, "\chi_{\psi,[x]}|_{\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}}"] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}} \\ \overline{\mathbb{Q}}S^F \arrow[ur, "\theta"'] & \end{tikzcd} \tag{1.7.1}$$

The homomorphism Cur_S^G is determined by §1.4(a) and by the following formula:

Curtis' formula. For all $n \in G^F$, $\mathrm{Cur}_S^G(e_\psi n e_\psi) = \sum_{s \in S^F} \mathrm{Cur}_S^G(e_\psi n e_\psi)(s)s \in \overline{\mathbb{Q}}S^F$ with

$$\mathrm{Cur}_S^G(e_\psi n e_\psi)(s) = \frac{1}{\langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F}} \frac{1}{|U^F|} \frac{1}{|(C_G(s)^\circ)^F|} \sum_{\substack{g \in G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}).$$

In the above formula, $Q_S^G : G^F \longrightarrow \mathbb{Z}$ is the Green function, which is the function supported on the set of unipotent elements of G^F defined by $Q_S^G(u) = R_S^G(\mathrm{Id})(u)$ for unipotent elements u of G^F (see [DeLu, Def. 4.1]); $C_G(s)^\circ$ means the connected identity component of the centralizer of s in G; for $g \in G^F$, g_{ss} (resp. g_{u}) denotes its semisimple part (resp. unipotent part).

Lemma 1.5. *Let S be an F-stable maximal torus of G.*

(a) (Compare [BoKe, Thm. 2.7(b)].) Cur_S^G *is defined over* $\overline{\mathbb{Z}}[\frac{1}{p}]$ *in the sense that* $\mathrm{Cur}_S^G(\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}) \subset \overline{\mathbb{Z}}[\frac{1}{p}]S^F$.

(b) *Identify* $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ *(Lemma 1.4). Then* $\mathrm{Cur}_S^G : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}S^F$ *is equivariant under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *(§1.6). Thus (with the help of (a)) Cur_S^G is defined over* $\mathbb{Z}[\frac{1}{p}]$*:* $\mathrm{Cur}_S^G(\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G) \subset \mathbb{Z}[\frac{1}{p}]S^F$*.*

Proof. (a) Using the $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ in §1.4(a), it suffices to show that $\mathrm{Cur}_S^G(e_\psi n e_\psi) \in \overline{\mathbb{Z}}[\frac{1}{p}]S^F$ for every $n \in G^F$. In Curtis' formula for $\mathrm{Cur}_S^G(e_\psi n e_\psi)$ (§1.7), observe that $\langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F} = \langle R_S^G(\mathrm{Id}), \Gamma_{G,\psi}\rangle_{G^F} = \pm 1$ (Theorem 1.1) and that $|U^F|$ is a power of p, so to prove this lemma it remains to show that the number

$$C(s) := \frac{1}{|(C_G(s)^\circ)^F|} \sum_{\substack{g\in G^F,\, u\in U^F \\ (gung^{-1})_{\mathrm{ss}}=s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}})$$

lies in $\overline{\mathbb{Z}}[\frac{1}{p}]$ for all $s \in S^F$. If $(gung^{-1})_{\mathrm{ss}} = s$ ($g, n \in G^F$, $u \in U^F$), for every $c \in C_G(s)^\circ$ we have $((cg)un(cg)^{-1})_{\mathrm{ss}} = s$ and $((cg)un(cg)^{-1})_{\mathrm{u}} = c.(gung^{-1})_{\mathrm{u}}.c^{-1}$, so

$$C(s) = \sum_{\substack{g\in (C_G(s)^\circ)^F\backslash G^F,\, u\in U^F \\ (gung^{-1})_{\mathrm{ss}}=s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \in \overline{\mathbb{Z}}[\tfrac{1}{p}].$$

(b) We have to show that

$$\sigma(\mathrm{Cur}_S^G(e_\psi n e_\psi)(s)) = \mathrm{Cur}_S^G(\sigma\cdot(e_\psi n e_\psi))(s) \tag{1.7.2}$$

for every $n \in G^F$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $s \in S^F$. Again, we apply Curtis' formula: using §1.6 and the notation therein, we have $\sigma\cdot\psi = {}^{y_\sigma}\psi$ with $y_\sigma \in T_{\mathrm{ad}}^F$; choose a $t \in T$ such that $y_\sigma = tZ$, and choose a $c \in S$ such that $x := t^{-1}c \in G^F$ (see the proof of Lemma 1.2); then

$$\begin{aligned}
&\langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F} |U^F||(C_G(s)^\circ)^F| \cdot \sigma(\mathrm{Cur}_S^G(e_\psi n e_\psi)(s)) \\
&= \sum_{\substack{g\in G^F,\, u\in U^F \\ (gung^{-1})_{\mathrm{ss}}=s}} \sigma(\psi(u^{-1})) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \\
&= \sum_{\substack{g\in G^F,\, u\in U^F \\ (gung^{-1})_{\mathrm{ss}}=s}} ({}^{y_\sigma}\psi)(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \\
&= \sum_{\substack{g\in G^F,\, u\in U^F \\ (gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}}={}^{y_\sigma^{-1}}s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}({}^{y_\sigma}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}}))
\end{aligned}$$

(where we have performed $u \mapsto {}^{y_\sigma}u$ and $g \mapsto {}^{y_\sigma}g$)

$$= \sum_{\substack{g\in G^F,\, u\in U^F \\ {}^{x}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}})={}^{t^{-1}}s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}({}^{y_\sigma x}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}}))$$

(where we have performed $g \mapsto xg$)

$$= \sum_{\substack{g \in G^F,\, u \in U^F \\ (gu.^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gu.^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}})$$

(where we have used ${}^{x^{-1}y_\sigma^{-1}}(Q_S^{C_G(s)^\circ}) = Q_S^{C_G(s)^\circ}$; see the proof of Lemma 1.2)

$$= \langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F} |U^F||(C_G(s)^\circ)^F| \cdot \mathrm{Cur}_S^G(\sigma \cdot (e_\psi n e_\psi))(s).$$

$\square$

Lemma 1.6 ([BoKe, Prop. 2.1]). *Let S be an F-stable maximal torus of G. For $z \in S^{*F^*}$, let $\widehat{z} : S^F \longrightarrow \overline{\mathbb{Q}}$ be its corresponding irreducible character of S^F obtained from the duality $S^{*F^*} \simeq \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ (§0.7), and let $e_{\widehat{z}}^S := \frac{1}{|S^F|}\sum_{t \in S^F} \widehat{z}(t^{-1})t \in \overline{\mathbb{Q}}S^F$ be the primitive central idempotent associated to $\widehat{z}$. Then*

$$\mathrm{Cur}_S^G(e_{\psi,[x]}^{\mathsf{E}}) = \sum_{z \in [x] \cap S^{*F^*}} e_{\widehat{z}}^S \in \overline{\mathbb{Q}}S^F \quad \textit{for every } [x] \in G_{\mathrm{ss}}^{*F^*}/\sim . \tag{1.7.3}$$

As a consequence, Cur_S^G may be interpreted as a restriction map: using the identification $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}$ (§1.4) as well as a similar identification $\overline{\mathbb{Q}}S^F = \overline{\mathbb{Q}}^{S^{*F^*}}$ (such that $e_{\widehat{z}}^S$ corresponds to the characteristic function $\mathbf{1}_{\{z\}}$ on S^{*F^*}), we have the following commutative diagram of $\overline{\mathbb{Q}}$-algebras ("Res" means the restriction map):

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, "\mathrm{Res}"] \\ \overline{\mathbb{Q}}S^F \arrow[r, leftrightarrow, "\sim"] & \overline{\mathbb{Q}}^{S^{*F^*}} \end{tikzcd} \tag{1.7.4}$$

The formula (1.7.3)*, or equivalently the commutativity of* (1.7.4)*, gives an alternative definition of Cur_S^G. Note that the two horizontal isomorphisms in* (1.7.4) *are not compatible with obvious $\mathbb{Q}$-structures (the $\mathbb{Q}$-structure of $\overline{\mathbb{Q}}\mathsf{E}_G$ is given by* §1.6*); the two vertical maps in* (1.7.4)*, however, are defined over $\mathbb{Q}$ (see Lemma* 1.5*(b)).*

Corollary 1.7 (of Lemma 1.6; compare [BoKe, Prop. 3.2]). *If for each $w \in W$ we choose an F-stable maximal torus T_w of G whose G^F-conjugacy class corresponds to w (recall that the G^F-conjugacy classes of F-stable maximal tori of G are parametrized by F-conjugacy classes of W; see* §0.6*), then the product map*

$$\mathrm{Cur}^G := (\mathrm{Cur}_{T_w}^G)_{w \in W} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \prod_{w \in W} \overline{\mathbb{Q}}T_w^F$$

is an injective $\overline{\mathbb{Q}}$-algebra homomorphism.

1.8. **Symmetric algebras and symmetrizing forms.** [Br, Sec. 2] Let Λ be a commutative ring with unity. By symmetric Λ-algebra, we mean a pair (A, τ) where:

(i) A is a Λ-algebra which is finitely generated and projective as a Λ-module;

(ii) $\tau : A \longrightarrow \Lambda$ is a central Λ-module homomorphism ("central" means that $\tau(ab) = \tau(ba)$ for all $a, b \in A$) such that the map

$$\widehat{\tau} : A \longrightarrow \mathrm{Hom}_\Lambda(A, \Lambda), \quad a \longmapsto [\widehat{\tau}(a) : A \longrightarrow \Lambda, \ \ b \longmapsto \tau(ab)]$$

is a Λ-module isomorphism.

In this case, τ is called a symmetrizing form on A over Λ.

Now let (A, τ) be a symmetric algebra, let P be a finitely generated projective A-module and let $E = \mathrm{End}_A(P)$ be its endomorphism ring. Then τ induces a symmetrizing form $\tau_E : E \longrightarrow \Lambda$ via $\tau_E = [E \simeq \mathrm{Hom}_A(P, A) \otimes_A P \xrightarrow{\text{natural pairing}} A \xrightarrow{\tau} \Lambda]$, so that (E, τ_E) is a symmetric Λ-algebra.

Application to $\Lambda\mathsf{E}_{G,\psi}$. Let Λ be an integral domain containing $\overline{\mathbb{Z}}[\frac{1}{p}]$. The evaluation at identity $\mathrm{ev}_1 : \Lambda G^F \longrightarrow \Lambda$ is a symmetrizing form on ΛG^F; the Gelfand-Graev representation $\Gamma_{G,\psi} = \Lambda G^F e_\psi$ being a projective ΛG^F-module, the previous discussion implies that ev_1 induces a symmetrizing form τ on $\Lambda\mathsf{E}_{G,\psi} = \mathrm{End}_{\Lambda G^F}(\Gamma_{G,\psi})$; under the identification $\Lambda\mathsf{E}_{G,\psi} = e_\psi \Lambda G^F e_\psi \subset \Lambda G^F$, the form $\tau : \Lambda\mathsf{E}_{G,\psi} \longrightarrow \Lambda$ is just the restriction of ev_1 on $\Lambda\mathsf{E}_{G,\psi}$.

Lemma 1.8 ([BoKe, Lem. 3.8])**.** *Let $\Lambda_1 \subset \Lambda_2$ be an inclusion of commutative rings with unity. Let (A, τ) be a symmetric Λ_1-algebra which is free as a Λ_1-module, and denote the Λ_2-linear extension of τ to $\Lambda_2 A$ again by τ. Let A' be a Λ_1-algebra which is free as Λ_1-module, such that $A \subset A' \subset \Lambda_2 A$ and $\tau(A') \subset \Lambda_1$. Then $A = A'$.*

Theorem 1.9 ([BoKe, Thm. 3.7])**.** *Recall the injective Curtis homomorphism*

$$\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w \in W} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \hookrightarrow \prod_{w \in W} \overline{\mathbb{Q}} T_w^F$$

in Corollary 1.7. *The map* Cur^G *is saturated over* $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ *(with respect to its field of fractions* $\overline{\mathbb{Q}}$*) in the sense that*

$$\mathrm{Cur}^G(\overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{E}_{G,\psi}) = \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}) \cap \left(\prod_{w \in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}] T_w^F \right).$$

Remark. There are examples of G making Cur^G not saturated over $\overline{\mathbb{Z}}[\frac{1}{p}]$, such as $G = \mathrm{SL}_2(\overline{\mathbb{F}_q})$ with q odd (see [BoKe, Rmk. 3.9]).

For later use, let us sketch a proof of this theorem. Consider the two $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebras $A := \mathrm{Cur}^G(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_{G,\psi})$ and $A' := \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}) \cap \left(\prod_{w \in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}] T_w^F\right)$; to prove that $A = A'$, the idea is to construct a symmetrizing form on A and then apply Lemma 1.8.

We have $A \subset A'$ as each $\mathrm{Cur}^G_{T_w}$ is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$ (Lemma 1.5). As $\overline{\mathbb{Q}}A = \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi})$, we get the inclusions $A \subset A' \subset \overline{\mathbb{Q}}A$. Consider

$$\tau^{\mathsf{E}} := |U^F| \mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}] \quad (\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}] G^F e_\psi),$$

so that $(\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}, \tau^{\mathsf{E}})$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra (§1.8). Denote the $\overline{\mathbb{Q}}$-linear extension of τ^{E} again by $\tau^{\mathsf{E}} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Q}}$, and set

$$\tau^W := \frac{1}{|W|} \sum_{w \in W} \mathrm{ev}_{1_{T_w^F}} \circ \mathrm{pr}_{\overline{\mathbb{Q}} T_w^F} : \prod_{w \in W} \overline{\mathbb{Q}} T_w^F \longrightarrow \overline{\mathbb{Q}}$$

which is a symmetrizing form on $\prod_{w\in W}\overline{\mathbb{Q}}T_w^F$ and verifies the relation $\tau^{\mathsf{E}}=\tau^W\circ \mathrm{Cur}^G$ on $\overline{\mathbb{Q}}\mathsf{E}_G$ (see [BoKe, Sec. 3.B]).

The last relation and the injectivity of Cur^G together imply that $(A,\tau^W|_A)$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra; on the other hand, since $|W|$ is invertible in $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$, the map τ^W is defined over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ and then we have $\tau^W(A')\subset\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. So Lemma 1.8 implies that $A=A'$, and this completes the proof of the theorem.

Corollary 1.10 (of Theorem 1.9 and Lemma 1.5(b))**.** *Identify* $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G=\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ *(Lemma 1.4). Then the map* Cur^G *is saturated over* $\mathbb{Z}[\frac{1}{p|W|}]$ *(with respect to its field of fractions* $\mathbb{Q}$*):*

$$\mathrm{Cur}^G(\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{E}_G)=\mathrm{Cur}^G(\mathbb{Q}\mathsf{E}_G)\cap\left(\prod_{w\in W}\mathbb{Z}[\tfrac{1}{p|W|}]T_w^F\right).$$

2. Grothendieck groups of representations over defining characteristic

The Grothendieck group K_{G^*}.

2.1. **The p-regular elements.** Let $G^{*F^*}_{p'}$ be the set of p-regular elements of G^{*F^*}; recall that an element of G^{*F^*} is called p-regular if its order in G^{*F^*} is not divisible by p. We have $G^{*F^*}_{p'}=G^{*F^*}_{\mathrm{ss}}$ by the Jordan decomposition. Denote by $G^{*F^*}_{p'}/\sim$ the set of G^{*F^*}-conjugacy classes in $G^{*F^*}_{p'}$; thus $(G^{*F^*}_{p'}/\sim)=(G^{*F^*}_{\mathrm{ss}}/\sim)$.

2.2. **Definition of the algebra K_{G^*}.** Consider $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$, the category of finite-dimensional representations of the finite group G^{*F^*} over the field $\overline{\mathbb{F}_q}$, or equivalently the category of finitely generated $\overline{\mathbb{F}_q}G^{*F^*}$-modules. We define K_{G^*} to be the Grothendieck group of the category $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. The tensor product on $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ induces a multiplication $\otimes$ on K_{G^*}, and then we shall consider $\mathsf{K}_{G^*}=(\mathsf{K}_{G^*},+,\otimes)$ as a $\mathbb{Z}$-algebra.

Denote by $[\cdot]:\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})\longrightarrow\mathsf{K}_{G^*}$ the natural map, which descends to a map $[\cdot]$ on the set of isomorphism classes in $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. Let $\mathrm{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ be the set of isomorphism classes of simple objects in $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. Then K_{G^*} is a free $\mathbb{Z}$-module having the set $\{[M]\in\mathsf{K}_{G^*}:M\in\mathrm{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$ as a basis (see [Se1, Sec. 14.1]).

2.3. **The Brauer character.** [Se1, Sec. 18] Recall that we have fixed an inclusion of multiplicative groups $\kappa:\overline{\mathbb{F}_q}^{\times}\hookrightarrow\overline{\mathbb{Q}}^{\times}$ (§0.7). Let $M\in\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ and $g\in G^{*F^*}_{p'}$ (a p-regular element, see §2.1). Then the action $g:M\longrightarrow M$ (considered as an $\overline{\mathbb{F}_q}$-linear map) is diagonalizable with eigenvalues $\lambda_1,\cdots,\lambda_N\in\overline{\mathbb{F}_q}^{\times}$ (here $N:=\dim_{\overline{\mathbb{F}_q}}M$), and we set $(\mathrm{br}\,M)(g):=\kappa(\lambda_1)+\cdots+\kappa(\lambda_N)\in\overline{\mathbb{Q}}$. We thus get a map $\mathrm{br}\,M:G^{*F^*}_{p'}\longrightarrow\overline{\mathbb{Q}}$ (the Brauer character of M), which descends to an element $\mathrm{br}\,M\in\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$.

The map $M\longmapsto\mathrm{br}\,M$ then induces a ring homomorphism

$$\mathrm{br}:\mathsf{K}_{G^*}\longrightarrow\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

The unique $\overline{\mathbb{Q}}$-linear extension of the map $\mathrm{br}(\cdot)$ is a $\overline{\mathbb{Q}}$-algebra isomorphism:

$$\mathrm{br}:\overline{\mathbb{Q}}\mathsf{K}_{G^*}\xrightarrow{\sim}\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

Thus the rank of the free $\mathbb{Z}$-module K_{G^*} is

$$\operatorname{rank}_{\mathbb{Z}} \mathsf{K}_{G^*} = |\operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})| = |G^{*F^*}_{p'}/\sim| = |G^{*F^*}_{\mathrm{ss}}/\sim|.$$

The above Brauer isomorphism and the canonical inclusion $\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ show that K_{G^*} is reduced and is embedded into $\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$ by the Brauer map $\mathrm{br}(\cdot)$.

2.4. **Projective objects in** $\operatorname{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$[Se1, III^e partie]**.** Let $\operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ be the category of projective $\overline{\mathbb{F}_q}G^{*F^*}$-modules of finite dimension over $\overline{\mathbb{F}_q}$, and let P_{G^*} be the Grothendieck group of the category $\operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. For $M \in \operatorname{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$, denote by $P_M \in \operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ the projective cover of M (which is unique up to isomorphism). Then P_{G^*} is a free $\mathbb{Z}$-module having $\{[P_M] : M \in \operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$ as a basis (the image of $P \in \operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ in P_{G^*} is denoted by $[P]$). Two objects P and P' of $\operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ are isomorphic if and only if $[P] = [P']$ in P_{G^*}. Furthermore:

(a) The pairing

$$\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*}) \times \operatorname{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*}) \longrightarrow \mathbb{Z}, \quad (P,\pi) \longmapsto \dim_{\overline{\mathbb{F}_q}} \operatorname{Hom}_{\overline{\mathbb{F}_q}G^{*F^*}}(P,\pi)$$

descends to a $\mathbb{Z}$-bilinear perfect pairing $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \mathsf{P}_{G^*} \times \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$ with $\{[P_M] : M \in \operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$ and $\{[M] : M \in \operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$ being dual bases with each other.

(b) Let $c : \mathsf{P}_{G^*} \longrightarrow \mathsf{K}_{G^*}$ be the Cartan homomorphism, which is the natural map induced by the inclusion $\operatorname{Proj}_{\overline{\mathbb{F}_q}}(G^{*F^*}) \subset \operatorname{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. Then $c(\mathsf{P}_{G^*}) \supset |G^{*F^*}|_p \mathsf{K}_{G^*}$, the cokernel $\operatorname{coker}(c) = \mathsf{K}_{G^*}/c(\mathsf{P}_{G^*})$ is a finite p-group, and c is an injective map. We shall use this map c to identify P_{G^*} as an ideal of the ring K_{G^*}.

Comparison between $\mathbb{Z}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ and $\mathbb{Z}[\frac{1}{p|W|}]\mathsf{E}_G$.

2.5. **The identification $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{K}_{G^*}$.** From now on, let us fix a $\psi \in \Psi$ and then identify $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ (Lemma 1.4). We shall then write the idempotent $e^{\mathsf{E}}_{\psi,[x]}$ (see §1.4(b)) simply as $e^{\mathsf{E}}_{[x]}$.

Using the Brauer isomorphism in §2.3, the equality $G^{*F^*}_{p'} = G^{*F^*}_{\mathrm{ss}}$ and (1.4.3), we obtain the following identifications of $\overline{\mathbb{Q}}$-algebras:

$$\left\{\begin{array}{ccccccc} \overline{\mathbb{Q}}\mathsf{E}_G & \simeq & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} & = & \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} & \simeq & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \\ e^{\mathsf{E}}_{[x]} & \leftrightarrow & \mathbf{1}_{\{[x]\}} & = & \mathbf{1}_{\{[x]\}} & \leftrightarrow & \mathrm{br}^{-1}(\mathbf{1}_{\{[x]\}}) \end{array}\right\}.$$

When $G = S$ is a torus, the identification $\overline{\mathbb{Q}}\mathsf{E}_S \simeq \overline{\mathbb{Q}}\mathsf{K}_{S^*}$ is already true over $\mathbb{Z}$: indeed, we have $\mathsf{E}_S \simeq \mathbb{Z}S^F$, and the chosen duality $S^F \simeq \operatorname{Irr}_{\overline{\mathbb{F}_q}}(S^{*F^*})$ (§0.7) induces an identification $\mathbb{Z}S^F \simeq \mathsf{K}_{S^*}$.

In the general case, we first consider the coefficients $\overline{\mathbb{Z}}[\frac{1}{p}]$ for the sake of the structure theory of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, and it is expected that we still have a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$. The first idea is to reconstruct the expected isomorphism from the above toric case via the following commutative diagram of rings (where S is an F-stable maximal torus of G):

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Cur}^G_S"'] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, "\mathrm{Res}^{G^*F^*}_{S^*F^*}"] \\ \overline{\mathbb{Q}}S^F \arrow[r, leftrightarrow, "\sim"] & \overline{\mathbb{Q}}\mathsf{K}_{S^*} \end{tikzcd}$$

Proposition 2.1. *Let $\tau^{\mathsf{K}} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-linear form induced by the symmetrizing form $\tau^{\mathsf{E}} = |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}$ (Theorem 1.9) via the canonical identification $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{K}_{G^*}$. Then, with the perfect pairing $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ from §2.4(a),*

$$\tau^{\mathsf{K}}(\pi) = \frac{1}{|W|}\sum_{w\in W}\langle \mathbf{1}, \mathrm{Res}^{G^*F^*}_{T^*_w{}^{F^*}}\pi\rangle_{\mathsf{P},\mathsf{K}} \quad (\pi\in\mathsf{K}_{G^*}),$$

where $\mathbf{1}$ is the trivial representation over $\overline{\mathbb{F}_q}$.

Note that, as $T^{*F^*}_w$ has order coprime to p, each pairing $\langle \mathbf{1}, \mathrm{Res}^{G^*F^*}_{T^*_w{}^{F^*}}\pi\rangle_{\mathsf{P},\mathsf{K}}$ above is simply the usual inner product of the characters for $\mathbf{1}$ and $\mathrm{Res}^{G^*F^*}_{T^*_w{}^{F^*}}\pi$. (Nevertheless, the above expression of $\tau^{\mathsf{K}}(\pi)$ in terms of $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ will be essential in Proposition 2.2.)

Proof of proposition. Under the canonical identification $\overline{\mathbb{Q}}\mathsf{K}_{G^*} = \overline{\mathbb{Q}}\mathsf{E}_G$, $\pi\in\mathsf{K}_{G^*}$ corresponds to $\sum_{x\in G^{*F^*}_{\mathrm{ss}}/\sim}(\mathrm{br}\,\pi)(x)e^{\mathsf{E}}_{[x]}\in\overline{\mathbb{Q}}\mathsf{E}_G$. With the help of the formula $\tau^{\mathsf{E}} = \tau^W\circ\mathrm{Cur}^G$ (Theorem 1.9) as well as Lemma 1.6, we have

$$\tau^{\mathsf{E}}(e^{\mathsf{E}}_{[x]}) = \frac{1}{|W|}\sum_{w\in W}\mathrm{Cur}^G_{T_w}(e^{\mathsf{E}}_{[x]})(1) = \frac{1}{|W|}\sum_{w\in W}\sum_{z\in[x]\cap T^{*F^*}_w}\frac{1}{|T^F_w|},$$

so that

$$\begin{aligned}\tau^{\mathsf{K}}(\pi) &= \sum_{x\in G^{*F^*}_{\mathrm{ss}}/\sim}(\mathrm{br}\,\pi)(x)\cdot\tau^{\mathsf{E}}(e^{\mathsf{E}}_{[x]}) = \frac{1}{|W|}\sum_{x\in G^{*F^*}_{\mathrm{ss}}/\sim}\sum_{w\in W}\sum_{z\in[x]\cap T^{*F^*}_w}\frac{(\mathrm{br}\,\pi)(x)}{|T^F_w|}\\ &= \frac{1}{|W|}\sum_{w\in W}\frac{1}{|T^F_w|}\sum_{x\in G^{*F^*}_{\mathrm{ss}}/\sim}\sum_{z\in[x]\cap T^{*F^*}_w}(\mathrm{br}\,\pi)(z)\\ &= \frac{1}{|W|}\sum_{w\in W}\frac{1}{|T^F_w|}\sum_{z\in T^{*F^*}_w}(\mathrm{br}\,\pi)(z) = \frac{1}{|W|}\sum_{w\in W}\langle\mathbf{1},\mathrm{Res}^{G^*F^*}_{T^*_w{}^{F^*}}\pi\rangle_{\mathsf{P},\mathsf{K}}.\end{aligned}$$

□

2.6. **The Steinberg character.** Let $\mathrm{St}_{G^*}\in\mathrm{Rep}_{\overline{\mathbb{Q}_p}}(G^{*F^*})$ be the Steinberg character of G^{*F^*}, and let $\overline{\mathrm{St}}_{G^*}$ be its reduction modulo p. Recall that $\overline{\mathrm{St}}_{G^*}$ is an irreducible projective $\overline{\mathbb{F}_q}G^{*F^*}$-module and is isomorphic to its dual representation $\overline{\mathrm{St}}^{\vee}_{G^*}$. In addition:

(a) [DeLu, Cor. 7.15] In $\mathbb{Z}[\frac{1}{|W|}]\mathsf{K}_{G^*}$, we have:

$$[\overline{\mathrm{St}}_{G^*}] = \frac{1}{|W|}\sum_{w\in W}(-1)^{\ell(w)}\left[\mathrm{Ind}^{G^*F^*}_{T^*_w{}^{F^*}}\mathbf{1}\right];$$

$$[\overline{\mathrm{St}}_{G^*}]\otimes[\overline{\mathrm{St}}_{G^*}] = \frac{1}{|W|}\sum_{w\in W}\left[\mathrm{Ind}^{G^*F^*}_{T^*_w{}^{F^*}}\mathbf{1}\right].$$

(Here $\ell(w)\in\mathbb{N}$ is the length of $w\in W$.)

(b) [Lu] The map $(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{P}_{G^*}$ is a $\mathbb{Z}$-module isomorphism.

Proposition 2.2. *Let $\tau^{\mathsf{K}} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-linear form in Proposition* 2.1.

(a) *For every $\pi \in \mathsf{K}_{G^*}$ we have $\tau^{\mathsf{K}}(\pi) = \langle [\overline{\mathrm{St}}_{G^*}] \otimes [\overline{\mathrm{St}}_{G^*}], \pi\rangle_{\mathsf{P},\mathsf{K}} \in \mathbb{Z}$. Thus τ^{K} is defined over $\mathbb{Z}$, and we shall still denote the restriction of τ^{K} to K_{G^*} by $\tau^{\mathsf{K}} : \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$.*

(b) *The $\mathbb{Z}$-linear form $\tau^{\mathsf{K}} : \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$ obtained in* (a) *induces a $\mathbb{Z}$-bilinear form*

$$b^{\mathsf{K}} : \mathsf{K}_{G^*} \times \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}, \quad b^{\mathsf{K}}(x,y) := \tau^{\mathsf{K}}(x\otimes y) \quad (x, y \in \mathsf{K}_{G^*}).$$

The discriminant $\operatorname{disc} b^{\mathsf{K}}$ *of* b^{K} *is then an integer and satisfies*

$$\left|\operatorname{disc} b^{\mathsf{K}}\right| = \left|\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right]\right| = [\mathsf{K}_{G^*} : c(\mathsf{P}_{G^*})] = p^m$$

for some $m \in \mathbb{N}$. (Here $c : \mathsf{P}_{G^*} \longrightarrow \mathsf{K}_{G^*}$ is the Cartan homomorphism in §2.4(b)*.)*

(c) *$\tau^{\mathsf{K}} : \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]$ (well-defined by* (a)*) is a symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$, so $(\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}, \tau^{\mathsf{K}})$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra.*

Proof of proposition. (a) The formula of $\tau^{\mathsf{K}}(\pi)$ follows from Proposition 2.1 and Frobenius reciprocity (we extend $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ $\overline{\mathbb{Q}}$-bilinearly):

$$\tau^{\mathsf{K}}(\pi) = \frac{1}{|W|}\sum_{w\in W} \langle \mathbf{1}, \operatorname{Res}^{G^{*F^*}}_{T^{*F^*}_w}\pi\rangle_{\mathsf{P},\mathsf{K}} = \frac{1}{|W|}\sum_{w\in W}\langle \operatorname{Ind}^{G^{*F^*}}_{T^{*F^*}_w}\mathbf{1},\pi\rangle_{\mathsf{P},\mathsf{K}}$$
$$= \left\langle \frac{1}{|W|}\sum_{w\in W}\left[\operatorname{Ind}^{G^{*F^*}}_{T^{*F^*}_w}\mathbf{1}\right],\pi\right\rangle_{\mathsf{P},\mathsf{K}} = \langle [\overline{\mathrm{St}}_{G^*}]\otimes[\overline{\mathrm{St}}_{G^*}],\pi\rangle_{\mathsf{P},\mathsf{K}}.$$

(b) By (a) and adjunction for all $x, y \in \mathsf{K}_{G^*}$ we have

$$b^{\mathsf{K}}(x,y) = \tau^{\mathsf{K}}(x\otimes y) = \langle [\overline{\mathrm{St}}_{G^*}]\otimes[\overline{\mathrm{St}}_{G^*}], x\otimes y\rangle_{\mathsf{P},\mathsf{K}} = \langle x^\vee\otimes[\overline{\mathrm{St}}_{G^*}], y\otimes[\overline{\mathrm{St}}_{G^*}]\rangle_{\mathsf{P},\mathsf{K}}.$$

Consider the basis $\beta := \{[x] : x \in \operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$ (resp. $P_\beta := \{[P_x] : x \in \operatorname{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})\}$) for the free $\mathbb{Z}$-module K_{G^*} (resp. P_{G^*}); see §2.4). Using the matrices associated to (bi-)linear forms, we have the decomposition

$$\left[b^{\mathsf{K}}\right]_{\beta\times\beta} = {}^tX\cdot Y\cdot Z,$$

where:

$$X := \left[(\cdot)^\vee\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{P}_{G^*}\right]^{P_\beta}_{\beta};$$
$$Y := \left[\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \mathsf{P}_{G^*}\times\mathsf{K}_{G^*} \longrightarrow \mathbb{Z}\right]_{P_\beta,\beta};$$
$$Z := \left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right]^{\beta}_{\beta}.$$

We have seen that $(\cdot)^\vee\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{P}_{G^*}$ is invertible, so $\det X \in \mathbb{Z}^\times = \{\pm 1\}$; by §2.4(a), $\det Y = 1$. We thus obtain:

$$\operatorname{disc} b^{\mathsf{K}} = \det\left(\left[b^{\mathsf{K}}\right]_{\beta\times\beta}\right) = \pm\det Z = \pm\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right].$$

Recall that for a free $\mathbb{Z}$-module A of finite rank and for a $\mathbb{Z}$-module homomorphism $\varphi : A \longrightarrow A$, we have $|\det\varphi| = [A : \varphi(A)]$ (this can be deduced via the Smith normal form of φ). Using this fact and §2.4(b), we get, for some $m\in\mathbb{N}$,

$$\left|\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right]\right| = [\mathsf{K}_{G^*} : c(\mathsf{P}_{G^*})] = p^m.$$

(c) By (b), $\operatorname{disc} b^{\mathsf{K}} = \pm p^m$ is invertible in $\overline{\mathbb{Z}}[\frac{1}{p}]$, and this is equivalent to saying that $\tau^{\mathsf{K}} : \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]$ is a symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$.

□

Remark. The equality

$$\det\left[(\cdot) \otimes [\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right] = \pm p^m$$

(a weaker version of (b) but will be sufficient for the proof of Theorem 2.3) can be directly verified as follows: the map $(\cdot) \otimes [\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}$ corresponds, under the Brauer isomorphism $\mathrm{br} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \xrightarrow{\sim} \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$ (§2.3), to the diagonalizable endomorphism

$$\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} \longrightarrow \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}, \quad f \longmapsto f \cdot (\mathrm{br}\,\overline{\mathrm{St}}_{G^*}),$$

whose eigenvalues are all of the form $(\mathrm{br}\,\overline{\mathrm{St}}_{G^*})(x) = \mathrm{St}_{G^*}(x) = \pm|(C_G(x)^\circ)^F|_p$ with $x \in G^{*F^*}_{\mathrm{ss}}$; the desired equality then follows.

Theorem 2.3. *Consider the following commutative diagram of $\overline{\mathbb{Q}}$-algebras introduced in* §2.5*:*

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w\in W}"'] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res} := \left(\mathrm{Res}^{G^*F^*}_{T^*_w F^*}\right)_{w\in W}"] \\
\prod_{w\in W} \overline{\mathbb{Q}}T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w}
\end{tikzcd} \tag{2.6.1}$$

This diagram restricts itself to the following commutative diagram of $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebras:

$$\begin{tikzcd}
\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "\mathrm{Cur}^G"'] & \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res}"] \\
\prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T^*_w}
\end{tikzcd} \tag{2.6.2}$$

(By Lemma 1.5*, Cur^G is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$.) Moreover, the map*

$$\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w}$$

is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$:

$$\mathrm{Res}(\overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}) = \mathrm{Res}(\overline{\mathbb{Q}}\mathsf{K}_{G^*}) \cap \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{T^*_w}\right).$$

In order to display the symmetry between the E-side and the K-side, we shall give two proofs of this theorem, in both of which the main idea is to use the symmetrizing form Lemma 1.8.

First proof. We first show that the map $\mathrm{Res}: \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. So consider the map

$$i := \frac{1}{|W|} \sum_{w\in W} (-1)^{\ell(w)} \mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}} : \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{T^*_w} \right) \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*},$$

which is well-defined because $|W|$ is invertible in $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. Then the composition

$$\overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*} \xrightarrow{\mathrm{Res}} \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{T^*_w} \right) \xrightarrow{i} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}$$

coincides with $[\overline{\mathrm{St}}_{G^*}] \otimes (\cdot) : \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$, since for $\pi \in \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ we have

$$\begin{aligned}(i\circ \mathrm{Res})(\pi) &= \frac{1}{|W|} \sum_{w\in W} (-1)^{\ell(w)} \mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}} \mathrm{Res}_{T_w^{*F^*}}^{G^{*F^*}} \pi \\ &= \frac{1}{|W|} \sum_{w\in W} (-1)^{\ell(w)} \left(\mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}} \mathbf{1} \right) \otimes \pi = [\overline{\mathrm{St}}_{G^*}] \otimes \pi.\end{aligned}$$

Proposition 2.2(b) (or the remark of Proposition 2.2) then implies: for some $m \in \mathbb{N}$,

$$\det(i\circ\mathrm{Res}) = \det\left[[\overline{\mathrm{St}}_{G^*}] \otimes (\cdot) : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*} \right] = \pm p^m;$$

thus $\det(i\circ\mathrm{Res}) \in \overline{\mathbb{Z}}[\frac{1}{p|W|}]^{\times}$. Hence $\mathrm{Res}: \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$.

The diagram (2.6.1) gives the following commutative diagram:

$$\begin{tikzcd} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[rr, hook] \arrow[d, hook, "\mathrm{Cur}^G"'] & & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res}"] \\ \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[r, hook, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T^*_w} \arrow[r, hook] & \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w} \end{tikzcd}$$

The injective map $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \hookrightarrow \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in this diagram identifies $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$. Using the commutativity of the same diagram as well as the fact that the injective map $\mathrm{Res}: \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T^*_w}$ is saturated, we have $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$. Therefore:

(i) we have the inclusions of rings $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{E}_G$;

(ii) $(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G, \tau^{\mathsf{E}})$ in the proof of Theorem 1.9 is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra;

(iii) $\tau^{\mathsf{E}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}) \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]$ (Proposition 2.2(a)).

Thus Lemma 1.8 implies that $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (under $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$). $\square$

Second proof. This time we prove the $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ first. The diagram (2.6.1) gives the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[d, hook, "\mathrm{Cur}^G"'] & & \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[ll, hook'] \arrow[d, hook, "\mathrm{Res}"] \\
\prod_{w\in W} \overline{\mathbb{Q}}T_w^F & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[l, hook'] \arrow[r, hook, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*}
\end{tikzcd}$$

We then identify $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{E}_G$. As the map Cur^G is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ (Theorem 1.9), the above diagram gives $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G$. Therefore:

(i) we have the inclusions of rings $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$;

(ii) $(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}, \tau^{\mathsf{K}})$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra (Proposition 2.2(c));

(iii) $\tau^{\mathsf{K}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G) = \tau^{\mathsf{E}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G) \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]$.

Thus Lemma 1.8 implies that $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (under $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$), and we also obtain the commutativity of (2.6.2). Using the saturatedness of Cur^G (Theorem 1.9) and the $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$, we see that the map $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. □

Corollary 2.4 (of Proposition 2.3 and Lemma 1.5(b))**.** *The commutative diagram* (2.6.1) *is equivariant under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and induces by restriction the following commutative diagram of* $\mathbb{Z}[\frac{1}{p|W|}]$*-algebras:*

$$\begin{tikzcd}
\mathbb{Z}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[d, "\mathrm{Cur}^G"'] & \mathbb{Z}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[l, hook', "\sim"'] \arrow[d, "\mathrm{Res}"] \\
\prod_{w\in W} \mathbb{Z}[\frac{1}{p|W|}]T_w^F \arrow[r, hook, "\sim"] & \prod_{w\in W} \mathbb{Z}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*}
\end{tikzcd}$$

Moreover, the map $\mathrm{Res} : \mathbb{Q}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \mathbb{Q}\mathsf{K}_{T_w^*}$ *is saturated over* $\mathbb{Z}[\frac{1}{p|W|}]$*:*

$$\mathrm{Res}(\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}) = \mathrm{Res}(\mathbb{Q}\mathsf{K}_{G^*}) \cap \left(\prod_{w\in W} \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{T_w^*}\right).$$

3. Algebras from the invariant theory

Langlands dual groups.

3.1. **The Langlands dual.** Recall the data (G, T, F) and $(X(T), R, Y(T), R^\vee)$ from §0.7. By definition, a Langlands dual of (G, T) is a pair $(G^\vee, T^\vee)$ defined and split over $\mathbb{Z}$ and obtained by assigning its character group (resp. cocharacter group, resp. set of roots, resp. set of coroots) as $Y(T)$ (resp. $X(T)$, resp. $R^\vee$, resp. R). From now on, we fix such a Langlands dual $(G^\vee, T^\vee)$ (all choices are isomorphic).

Note that the Deligne-Lusztig dual pair (G^*, T^*) (§0.7) may be obtained from $(G^\vee, T^\vee)$ through the base change $\mathrm{Spec}(\overline{\mathbb{F}_q}) \longrightarrow \mathrm{Spec}(\mathbb{Z})$. We have the identifications $X(T^\vee) = X(T^*) = Y(T)$ and $Y(T^\vee) = Y(T^*) = X(T)$; moreover, the Weyl groups of $(G^\vee, T^\vee)$ and of (G^*, T^*) are both identified with the Weyl group W of (G, T).

3.2. **The automorphism $\tau^\vee$ and the endomorphism $F^\vee$.** Let τ^* be the automorphism on $X(T^*) = \mathrm{Hom}_{\mathrm{alg}}(T^*, \mathbb{G}_m)$ induced by the arithmetic Frobenius endomorphisms φ on T^* and $\mathbb{G}_m$; more precisely, $\tau^*(\lambda) := \varphi^{-1} \circ \lambda \circ \varphi$ for $\lambda \in X(T^*)$. Via the identification $X(T^\vee) = X(T^*)$ we obtain an automorphism $\tau^\vee$ on $X(T^\vee)$ and hence on $T^\vee$. Using the same identification $X(T^\vee) = X(T^*)$, the Frobenius endomorphism F^* on $X(T^*)$ induces an endomorphism $F^\vee$ on $X(T^\vee)$ and then on $T^\vee$. We have the following properties (compare [DiMi, Ch. 3 & 8]):

(a) On both $X(T^\vee)$ and $T^\vee$, the endomorphisms $\tau^\vee \circ F^\vee$ and $F^\vee \circ \tau^\vee$ are equal; these endomorphisms are the multiplication by q on $X(T^\vee)$ and are $(\cdot)^q$ on $T^\vee$.

(b) The following statements are equivalent: (i) G is split over $\mathbb{F}_q$; (ii) G^* is split over $\mathbb{F}_q$; (iii) $\tau^\vee = \mathrm{id}$ on $X(T^\vee)$; (iv) $F^\vee = q$ on $X(T^\vee)$; (v) $F^\vee = (\cdot)^q$ on $T^\vee$.

The algebra $\mathsf{B}_{G^\vee}$ of the $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$.

3.3. **Definition of the algebra $\mathsf{B}_{G^\vee}$.** As $X(T^\vee)$ is an abelian group of finite rank, the group ring $\mathbb{Z}[X(T^\vee)]$ is a finitely generated commutative $\mathbb{Z}$-algebra, and we know that $T^\vee = \mathrm{Spec}(\mathbb{Z}[X(T^\vee)])$. The Weyl group $W \simeq N_{G^\vee}(T^\vee)/T^\vee$ acts on $T^\vee$ by conjugation and hence on $\mathbb{Z}[X(T^\vee)]$ (by adjoint action). The $\mathbb{Z}$-algebra $\mathbb{Z}[X(T^\vee)]^W$, consisting of elements of $\mathbb{Z}[X(T^\vee)]$ fixed by the W-action, is also finitely generated (see [Se2, pf. of Prop. III.18]). We then consider the categorical quotient $T^\vee /\!\!/ W := \mathrm{Spec}(\mathbb{Z}[X(T^\vee)]^W)$, which is an affine $\mathbb{Z}$-scheme.

The endomorphism $F^\vee$ on $X(T^\vee)$ induces an endomorphism $F^\vee$ on $\mathbb{Z}[X(T^\vee)]^W$ (because $F^\vee(W) = W$) and thus an $F^\vee$-action on $T^\vee /\!\!/ W$, so we have the fixed-point subscheme $(T^\vee /\!\!/ W)^{F^\vee}$, which is also an affine $\mathbb{Z}$-scheme. We define $\mathsf{B}_{G^\vee}$ as the ring of functions of the affine scheme $(T^\vee /\!\!/ W)^{F^\vee}$, so that

$$(T^\vee /\!\!/ W)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee}) \quad \text{and} \quad \mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W / I,$$

where I is the ideal of $\mathbb{Z}[X(T^\vee)]^W$ generated by the subset $\{F^\vee f - f : f \in \mathbb{Z}[X(T^\vee)]^W\}$; then $\mathsf{B}_{G^\vee}$ is a finitely generated commutative $\mathbb{Z}$-algebra.

3.4. **A reducedness problem and the algebra $\mathsf{B}_{G^\vee,\mathrm{red}}$.** In the algebro-geometric viewpoint, we wish to know whether the scheme $(T^\vee /\!\!/ W)^{F^\vee}$ is reduced or not; that is, whether the $\mathbb{Z}$-algebra $\mathsf{B}_{G^\vee}$ is reduced or not. We shall see later (Theorem 3.9) that $\mathsf{B}_{G^\vee}$ is reduced when the derived subgroup $G^\vee_{\mathrm{der}}$ of $G^\vee$ is simply-connected.

Beyond the case of simply-connected $G^\vee_{\mathrm{der}}$, so far we haven't developed a general theory to determine if $\mathsf{B}_{G^\vee}$ is reduced or not, though we have verified directly that $\mathsf{B}_{G^\vee}$ is reduced when $G = \mathrm{SO}_{2n}(\overline{\mathbb{F}_q})$ for q odd (using combinatorics of the root system of SO_{2n}), when $G = \mathrm{SL}_2(\overline{\mathbb{F}_q})$ for all q (simple computation), or when $G = \mathrm{SL}_3(\overline{\mathbb{F}_q})$ for $q \in \{2, 3\}$ (direct calculation via Gröbner basis) — it is thus hoped that a uniform approach can be found even just for all $G = \mathrm{SL}_3(\overline{\mathbb{F}_q})$!

At the moment, let us denote by $\mathsf{B}_{G^\vee,\mathrm{red}}$ the reduced ring derived from $\mathsf{B}_{G^\vee}$; we have $\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[X(T^\vee)]^W/\sqrt{I}$, where $\sqrt{I} = \{f \in \mathbb{Z}[X(T^\vee)]^W : f^m \in I \text{ for some } m \in \mathbb{N}^*\}$ is the radical of I. So $\mathsf{B}_{G^\vee}$ is reduced if and only if $\mathsf{B}_{G^\vee,\mathrm{red}}$ is equal to $\mathsf{B}_{G^\vee}$. Besides, $\mathsf{B}_{G^\vee,\mathrm{red}}$ is also a finitely generated commutative $\mathbb{Z}$-algebra (see §3.3).

3.5. **Decompositions of the algebras $k\mathsf{B}_{G^\vee}$ and $k\mathsf{B}_{G^\vee,\mathrm{red}}$.** Let k be an algebraically closed field. The set of k-points of $(T^\vee /\!\!/ W)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee})$ may be identified as:

$$(T^\vee /\!\!/ W)^{F^\vee}(k) = \mathrm{Specm}(k\mathsf{B}_{G^\vee}) = (T^\vee(k)/W)^{F^\vee} = \left(\bigcup_{w\in W} T^\vee(k)^{wF^\vee}\right)\Big/ W.$$

(Notation: when a set X is equipped with a (left) W-action, we denote by X/W the set of W-orbits in X.) Each $T^\vee(k)^{wF^\vee}$ may be identified as $T^{*wF^*} \simeq T_w^{*F^*}$ (with respect to an embedding $\overline{\mathbb{F}_q}^\times \hookrightarrow k^\times$) and is thus a finite set (compare Lemma 3.1); the set $\mathrm{Specm}(k\mathsf{B}_{G^\vee})$ is thus finite, so $k\mathsf{B}_{G^\vee}$ is a finite-dimensional vector space over k and in particular an Artinian k-algebra. Therefore, if we denote by $(k\mathsf{B}_{G^\vee})_{\mathfrak{m}}$ the localization of $k\mathsf{B}_{G^\vee}$ at the maximal ideal $\mathfrak{m} \in (T^\vee /\!\!/ W)^{F^\vee}(k) = \mathrm{Specm}(k\mathsf{B}_{G^\vee})$, then the map

$$k\mathsf{B}_{G^\vee} \longrightarrow \prod_{\mathfrak{m}\in (T^\vee/\!\!/W)^{F^\vee}(k)} (k\mathsf{B}_{G^\vee})_{\mathfrak{m}}, \quad f \longmapsto (f)_{\mathfrak{m}\in (T^\vee/\!\!/W)^{F^\vee}(k)}$$

is a k-algebra isomorphism, where each $(k\mathsf{B}_{G^\vee})_{\mathfrak{m}} \simeq (k\mathsf{B}_{G^\vee})/\mathfrak{m}^N$ as k-algebras for some $N \in \mathbb{N}^*$ depending on $\mathfrak{m}$. We have the following equivalent conditions:

$$\begin{aligned} k\mathsf{B}_{G^\vee} \text{ is reduced} &\iff \text{each } (k\mathsf{B}_{G^\vee})_{\mathfrak{m}} \simeq (k\mathsf{B}_{G^\vee})/\mathfrak{m} = k \text{ as } k\text{-algebras} \\ &\iff k\mathsf{B}_{G^\vee} \simeq k^{(T^\vee/\!\!/W)^{F^\vee}(k)} \text{ as } k\text{-algebras} \\ &\iff \dim_k(k\mathsf{B}_{G^\vee}) = |(T^\vee /\!\!/ W)^{F^\vee}(k)|. \end{aligned}$$

The above discussion also applies to the reduced version $(T^\vee /\!\!/ W)^{F^\vee}_{\mathrm{red}} = \mathrm{Spec}(\mathsf{B}_{G^\vee,\mathrm{red}})$. When $k = \overline{\mathbb{Q}}$ (or other fields of characteristic zero), the reducedness of $\mathsf{B}_{G^\vee,\mathrm{red}}$ implies that of $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}$, so that:

(a) $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \simeq \overline{\mathbb{Q}}^{(T^\vee/\!\!/W)^{F^\vee}(\overline{\mathbb{Q}})}$ as $\overline{\mathbb{Q}}$-algebras;

(b) $\dim_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}) = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$.

Remark. In general, $k\mathsf{B}_{G^\vee,\mathrm{red}}$ need not be reduced. For example, let $G^\vee = T^\vee = \mathbb{G}_m$ with $F^\vee = (\cdot)^q$, so $\mathsf{B}_{G^\vee} = \mathbb{Z}[X^{\pm1}]/(X^{q-1}-1) = \mathsf{B}_{G^\vee,\mathrm{red}}$ (X an indeterminate); if ℓ is a prime number dividing $q-1$, then $\overline{\mathbb{F}_\ell}\mathsf{B}_{G^\vee,\mathrm{red}} = \overline{\mathbb{F}_\ell}[X^{\pm1}]/((X^{\frac{q-1}{\ell}}-1)^\ell)$ is not reduced.

Lemma 3.1. *Denote by $G^*_{\mathrm{ss}}/\sim$ the set of G^*-conjugacy classes of semisimple elements of G^*, and define $G^\vee(\overline{\mathbb{Q}})_{\mathrm{ss}}/\sim$ in a similar way. Then*

$$|(T^\vee/\!\!/W)^{F^\vee}(\overline{\mathbb{Q}})| = |(T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}_q})| = |(G^\vee(\overline{\mathbb{Q}})_{\mathrm{ss}}/\sim)^{F^\vee}| = |(G^*_{\mathrm{ss}}/\sim)^{F^*}|.$$

Proof. These equalities come from the chosen embedding $\kappa : \overline{\mathbb{F}_q}^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ (§0.7) and the following observations (stated for G^* but also valid for $G^\vee(\overline{\mathbb{Q}})$): (i) G^*_{ss} is the union of all maximal tori of G^*; (ii) for $z \in T^*$, $Wz \in (T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}_q})$ if and only if $z \in T^{*wF^*}$ for some $w \in W$; (iii) elements of each T^{*wF^*} are of finite order prime to p; (iv) The canonical bijection $(G^*_{\mathrm{ss}}/\sim) \simeq T^* /\!\!/ W$ induced by the diagonalization is compatible with respect to F^* (compare [La, Sec. 3.1 & App. B]). $\square$

Lemma 3.2. *The $\mathbb{Z}$-module $\mathsf{B}_{G^\vee,\mathrm{red}}$ has no non-zero $\mathbb{Z}$-torsion elements.*

Proof. Let $0 \neq f \in \mathsf{B}_{G^\vee,\mathrm{red}}$. As $\mathsf{B}_{G^\vee,\mathrm{red}}$ is a finitely generated $\mathbb{Z}$-algebra (§3.4), it is a Jacobson ring, so its Jacobson radical (intersection of maximal ideals) is equal to its nilradical (intersection of prime ideals), which is zero because $\mathsf{B}_{G^\vee,\mathrm{red}}$ is reduced. Thus there is a maximal ideal $\mathfrak{m}$ of $\mathsf{B}_{G^\vee,\mathrm{red}}$ such that $f \notin \mathfrak{m}$. Consider the field $k = \mathsf{B}_{G^\vee,\mathrm{red}}/\mathfrak{m}$, which is a finite field because $\mathsf{B}_{G^\vee,\mathrm{red}}$ is a finitely-generated $\mathbb{Z}$-algebra; for the canonical quotient map $t : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow k$ which represents a k-point of $\mathrm{Spec}(\mathsf{B}_{G^\vee,\mathrm{red}})$, we then have $f(t) = t(f) \neq 0 \in k$. We may and we shall replace k by its algebraic closure, so we shall write $k = \overline{\mathbb{F}_d}$ for some prime number d.

Let us show that the $\overline{\mathbb{F}_d}$-point $t : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{F}_d}$ can be canonically lifted to a $\overline{\mathbb{Z}_d}$-point $t' : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Z}_d}$. As $t \in \mathrm{Specm}(\overline{\mathbb{F}_d}\mathsf{B}_{G^\vee,\mathrm{red}}) = (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{F}_d})$, we may write $t = Ws$ where $s \in T^\vee(\overline{\mathbb{F}_d})^{wF^\vee}$ for some $w \in W$. Using the canonical lifting $i : \overline{\mathbb{F}_d}^\times \hookrightarrow \overline{\mathbb{Z}_d}^\times$ of d'-th roots of unity (Hensel's lemma), our $s \in T^\vee(\overline{\mathbb{F}_d}) = \mathrm{Hom}(X(T^\vee), \overline{\mathbb{F}_d}^\times)$ can be canonically lifted to some $s' \in T^\vee(\overline{\mathbb{Z}_d}) = \mathrm{Hom}(X(T^\vee), \overline{\mathbb{Z}_d}^\times)$, and it can be checked that $s' \in T^\vee(\overline{\mathbb{Z}_d})^{wF^\vee}$, so $t' := Ws' \in (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Z}_d})$; this t' is a $\overline{\mathbb{Z}_d}$-point $t' : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Z}_d}$ which lifts the $\overline{\mathbb{F}_d}$-point t, in the sense that the following diagram is commutative (where r_d is the standard reduction map):

$$\begin{tikzcd} & \overline{\mathbb{Z}_d} \arrow[d, two heads, "r_d"] \\ \mathsf{B}_{G^\vee,\mathrm{red}} \arrow[ur, "t'"] \arrow[r, "t"'] & \overline{\mathbb{F}_d} \end{tikzcd}$$

Now, for our non-zero element $f \in \mathsf{B}_{G^\vee,\mathrm{red}}$ at the beginning of the proof, we have seen that $t(f) \neq 0 \in \overline{\mathbb{F}_d}$, so $t'(f) \neq 0 \in \overline{\mathbb{Z}_d}$ thanks to the above lifting diagram. Suppose that $n \cdot f = 0 \in \mathsf{B}_{G^\vee,\mathrm{red}}$ for some $n \in \mathbb{Z}$. Then $n \cdot t'(f) = t'(n \cdot f) = 0 \in \overline{\mathbb{Z}_d}$ while $t'(f) \neq 0$, whence $n = 0$. This shows that f is not a $\mathbb{Z}$-torsion element of $\mathsf{B}_{G^\vee,\mathrm{red}}$. □

Corollary 3.3 (of Lemma 3.2)**.** *The natural map* $\mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}$ *is injective; combining this with the identification* $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} = \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$*, we obtain a canonical injection of* $\mathsf{B}_{G^\vee,\mathrm{red}}$ *into* $\overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$.

Proposition 3.4. *The* $\mathbb{Z}$*-module* $\mathsf{B}_{G^\vee,\mathrm{red}}$ *is free of rank*

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{B}_{G^\vee,\mathrm{red}} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |(G^*_{\mathrm{ss}}/{\sim})^{F^*}|.$$

Remark. Having just seen that $\mathsf{B}_{G^\vee,\mathrm{red}}$ is without non-zero $\mathbb{Z}$-torsion (Lemma 3.2), the proposition will be proved once we have the finite-generacy of $\mathsf{B}_{G^\vee,\mathrm{red}}$ as a $\mathbb{Z}$-module; however, it seems difficult to prove this finite-generacy directly.

Proof of proposition, admitting Theorem 3.9*.* Let us admit for the moment the results in Theorem 3.9, which say that this proposition is true when the derived group $G^\vee_{\mathrm{der}}$ of $G^\vee$ is simply-connected. By general theories of algebraic groups, G^* fits into an F^*-equivariant exact sequence of reductive groups,

$$1 \longrightarrow S^* \longrightarrow H^* \longrightarrow G^* \longrightarrow 1,$$

such that H^*_{der} is simply-connected and that S^* is central in H^*. This exact sequence induces an F-equivariant exact sequence of reductive groups,

$$1 \longrightarrow G \longrightarrow H \longrightarrow S \longrightarrow 1,$$

which induces finally an exact sequence of reductive groups over $\mathbb{Z}$,

$$1 \longrightarrow S^\vee \longrightarrow H^\vee \longrightarrow G^\vee \longrightarrow 1,$$

with $H^\vee_{\mathrm{der}}$ simply-connected and with $S^\vee$ a torus central in $H^\vee$; then the Weyl group of $H^\vee$ is also W. Let $T^\vee_H$ denote the $F^\vee$-stable maximal torus of $H^\vee_{\mathrm{der}}$ which maps onto $T^\vee$. We then have a surjective morphism of reduced $\mathbb{Z}$-schemes, $(T^\vee_H /\!\!/ W)^{F^\vee}_{\mathrm{red}} \longrightarrow (T^\vee /\!\!/ W)^{F^\vee}_{\mathrm{red}}$, so that in the level of ring we obtain an inclusion of rings $\mathsf{B}_{G^\vee,\mathrm{red}} \subset \mathsf{B}_{H^\vee,\mathrm{red}}$. As $H^\vee_{\mathrm{der}}$ is simply-connected, Theorem 3.9 tells us that $\mathsf{B}_{H^\vee,\mathrm{red}}$ is a free $\mathbb{Z}$-module of finite rank; thus its $\mathbb{Z}$-submodule $\mathsf{B}_{G^\vee,\mathrm{red}}$ is also free of finite rank. This obtained, the rank of $\mathsf{B}_{G^\vee,\mathrm{red}}$ may be calculated from Lemma 3.1. ☐

On the reducedness of $\mathsf{B}_{G^\vee}$.

3.6. **The derived group G_{der}.** compare [Her, App.] Denote by $G_{\mathrm{der}} = [G,G]$ the derived subgroup of G. We shall freely use the following properties of G_{der}:

(a) we have $G = R(G).G_{\mathrm{der}}$ with $R(G) \cap G_{\mathrm{der}}$ being a finite set (here $R(G)$ is the reductive radical of G; it is the connected identity component of the center of G);

(b) $T_{\mathrm{der}} := T \cap G_{\mathrm{der}}$ is a maximal torus of G_{der}, and $\overline{T} := G/G_{\mathrm{der}}$ is a torus on which the Weyl group $W = N_G(T)/T$ acts trivially; the Weyl group of $(G_{\mathrm{der}}, T_{\mathrm{der}})$ is identified with W;

(c) we have a canonical exact sequence of tori $1 \longrightarrow T_{\mathrm{der}} \longrightarrow T \longrightarrow \overline{T} \longrightarrow 1$, which induces the following exact sequences of groups:

$$1 \longrightarrow (T_{\mathrm{der}})^{wF} \longrightarrow T^{wF} \longrightarrow \overline{T}^{wF} (= \overline{T}^{F}) \longrightarrow 1 \quad (w \in W);$$
$$0 \longrightarrow X(\overline{T}) \longrightarrow X(T) \longrightarrow X(T_{\mathrm{der}}) \longrightarrow 0;$$

the last exact sequence gives the identifications $X(\overline{T}) = X(T)^W = X^0(T)$ where $X^0(T) := \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta\}$.

We shall mainly apply these results on the dual sides $G^*_{\mathrm{der}} := (G^*)_{\mathrm{der}}$ and $G^\vee_{\mathrm{der}} := (G^\vee)_{\mathrm{der}}$. Observe that G^*_{der} is simply-connected if and only if $G^\vee_{\mathrm{der}}$ is.

Theorem 3.5. [St1, pf. of Lem. 3.9], [St2, Cor. 8.5] *If G_{der} is simply-connected, then the centralizer $C_G(x)$ of every semisimple element x of G is connected.*

The citations here address the case where G is simply-connected, while the case of simply-connected G_{der} can be deduced as a corollary via the following observation: for every semisimple element $x = zy \in G$ with $z \in R(G)$ and $y \in G_{\mathrm{der}}$, we have $C_G(x) = R(G).C_{G_{\mathrm{der}}}(y)$.

Lemma 3.6. *Recall that $G^*_{\mathrm{ss}}/\sim$ denotes the set of G^*-conjugacy classes of semisimple elements of G^* (Lemma 3.1). Then:*

(a) [DiMi, Cor. 3.12] *the map $(G^{*F^*}_{\mathrm{ss}}/\sim) \longrightarrow (G^*_{\mathrm{ss}}/\sim)^{F^*}$ induced by the set inclusion $G^{*F^*}_{\mathrm{ss}} \subset G^*_{\mathrm{ss}}$ is surjective.*

(b) [Ca, Prop. 3.7.3 & Thm. 3.7.6] *If G^*_{der} is simply-connected, then the map in (a) is a bijection $(G^{*F^*}_{\mathrm{ss}}/\sim) \xrightarrow{\sim} (G^*_{\mathrm{ss}}/\sim)^{F^*}$, and $|(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}_q})| = q^{\operatorname{rank} G^*_{\mathrm{der}}} \cdot |\overline{T}^{*F^*}|$.*

Similar results hold for the Langlands dual side $(G^\vee(\overline{\mathbb{Q}}), T^\vee(\overline{\mathbb{Q}}))$.

3.7. **Combinatorics of root data.** Elements of $X(T^\vee)$ are also called weights; inside $X(T^\vee)$, we shall need the following two subsets:

$$X^+(T^\vee) := \{\lambda \in X(T^\vee) : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta^\vee\} \quad \text{(dominant weights);}$$
$$X_q^+(T^\vee) := \{\lambda \in X(T^\vee) : 0 \leq \langle \lambda, \alpha^\vee \rangle < q \text{ for all } \alpha \in \Delta^\vee\} \quad \text{(q-restricted weights).}$$

Note that $X^+(T^\vee)$ is identified with the space $X(T^\vee)/W$ of W-orbits in $X(T^\vee)$, in the way that every W-orbit in $X(T^\vee)$ contains exactly one element of $X^+(T^\vee)$.

When $G^\vee$ is semisimple, $\Delta^\vee$ is a $\mathbb{Q}$-linear basis of $X(T^\vee)_\mathbb{Q} := X(T^\vee) \otimes_\mathbb{Z} \mathbb{Q}$, from which we introduce two additional notions:

(a) let $\{\omega_\alpha : \alpha \in \Delta^\vee\} \subset X(T^\vee)_\mathbb{Q}$ be the set of fundamental weights of $(G^\vee, T^\vee, B^\vee)$, characterised by the relations $\langle \omega_\alpha, \beta^\vee \rangle = \left\{ \begin{array}{ll} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{array} \right\}$ for all $\alpha, \beta \in \Delta^\vee$;

(b) let $\mathrm{ht} : X(T^\vee) \longrightarrow \mathbb{Q}$ be the height function with respect to $\Delta^\vee$, defined for every $\lambda \in X(T^\vee)$ by $\mathrm{ht}(\lambda) = \sum_{\alpha\in\Delta^\vee} m_\alpha$ where $\lambda = \sum_{\alpha\in\Delta^\vee} m_\alpha \alpha$ with all $m_\alpha \in \mathbb{Q}$.

Lemma 3.7. *Suppose that $G^\vee$ is semisimple, so that we have the height function $\mathrm{ht} : X(T^\vee) \longrightarrow \mathbb{Q}$ with respect to $\Delta^\vee$. Then:*

(a) *for $\lambda \in X^+(T^\vee)$, we have $\mathrm{ht}(\lambda') < \mathrm{ht}(\lambda)$ for every $\lambda \neq \lambda' \in W\lambda$;*

(b) *for every $0 \neq \lambda \in X^+(T^\vee)$, we have $\mathrm{ht}(\lambda) > 0$.*

Proof. (a) Choose a W-invariant inner product $(\cdot|\cdot)$ on the $\mathbb{R}$-vector space $X(T^\vee)_\mathbb{R} = X(T^\vee) \otimes_\mathbb{Z} \mathbb{R}$. Set $\widetilde{\rho} := \sum_{\alpha\in\Delta^\vee} \dfrac{2\omega_\alpha}{(\alpha|\alpha)} \in X(T^\vee)_\mathbb{R}$. Then one can show that $\mathrm{ht}(\mu) = (\mu|\widetilde{\rho})$ for every $\mu \in X(T^\vee)$. For every $\mu \in X(T^\vee)$, fix a $\sigma_\mu \in W$ such that $\mathrm{ht}(\sigma_\mu\mu) = \max_{\sigma\in W} \mathrm{ht}(\sigma\mu)$. Then $\sigma_\mu\mu \in X^+(T^\vee)$: for every $\alpha \in \Delta^\vee$, we have $\langle \sigma_\mu\mu, \alpha^\vee \rangle \geq 0$ because

$$(\sigma_\mu\mu|\widetilde{\rho}) \geq (s_\alpha\sigma_\mu\mu|\widetilde{\rho}) = (\sigma_\mu\mu|s_\alpha\widetilde{\rho}) = \left(\sigma_\mu\mu \middle| \widetilde{\rho} - \frac{2\alpha}{(\alpha|\alpha)}\right) = (\sigma_\mu\mu|\widetilde{\rho}) - \langle \sigma_\mu\mu, \alpha^\vee \rangle.$$

Now let $\lambda \in X^+(T^\vee)$ and suppose that $w \in W$ is such that $\lambda' := w\lambda \neq \lambda$. The previous discussion tells us that $\mathrm{ht}(\lambda') \leq \mathrm{ht}(\sigma_\lambda\lambda)$ and that $\sigma_\lambda\lambda \in X^+(T^\vee) \cap W\lambda = \{\lambda\}$, so $\sigma_\lambda\lambda = \lambda$ and $\mathrm{ht}(\lambda') \leq \mathrm{ht}(\lambda)$. But $\mathrm{ht}(\lambda') \neq \mathrm{ht}(\lambda)$, for otherwise the last paragraph (with (μ, σ_μ) therein replaced by (λ, w)) would show that $\lambda' \in X^+(T^\vee) \cap W\lambda = \{\lambda\}$ and then $\lambda' = \lambda$, contradicting our hypothesis. Thus $\mathrm{ht}(\lambda') < \mathrm{ht}(\lambda)$ as desired.

(b) Recall that $\{s_\alpha : \alpha \in \Delta^\vee\}$ determines a length function $W \to \mathbb{N}$. Let $w_\circ$ be the longest element of W (which maximize the length function); then it is a fact that the action $-w_\circ : X(T^\vee) \longrightarrow X(T^\vee)$ permutes the elements of $\Delta^\vee$, so that for every $\lambda \in X(T^\vee)$ we have $\mathrm{ht}(\lambda) = \mathrm{ht}(-w_\circ\lambda) = -\mathrm{ht}(w_\circ\lambda)$. Now let $0 \neq \lambda \in X^+(T^\vee)$, so $w_\circ\lambda \neq \lambda$ and hence (a) implies that $\mathrm{ht}(w_\circ\lambda) < \mathrm{ht}(\lambda)$. But by the last paragraph we have $\mathrm{ht}(\lambda) = -\mathrm{ht}(w_\circ\lambda)$, whence $\mathrm{ht}(\lambda) > 0$.

□

3.8. **The canonical $\mathbb{Z}$-basis of $\mathbb{Z}[X(T^\vee)]^W$.** For each $\lambda \in X(T^\vee)$, denote by $e(\lambda)$ its image in the group algebra $\mathbb{Z}[X(T^\vee)]$; thus $e(\lambda)$ is identified with the characteristic function $\mathbf{1}_{\{\lambda\}} : X(T^\vee) \longrightarrow \mathbb{Z}$, and $\{e(\lambda) : \lambda \in X(T^\vee)\}$ is a $\mathbb{Z}$-linear basis for $\mathbb{Z}[X(T^\vee)]$. For $\lambda \in X(T^\vee)$, let $W\lambda \subset X(T^\vee)$ be the W-orbit of λ, let $W_\lambda \subset W$ be the stabilizer of λ under the W-action on $X(T^\vee)$, and set $r(\lambda) := \frac{1}{|W_\lambda|}\sum_{w\in W} e(w\lambda) = \sum_{\mu\in W\lambda} e(\mu)$; then each $r(\lambda)$ lies in $\mathbb{Z}[X(T^\vee)]^W$. The identification $X(T^\vee)/W = X^+(T^\vee)$ implies that $\{r(\lambda) : \lambda \in X^+(T^\vee)\}$ is a $\mathbb{Z}$-linear basis for $\mathbb{Z}[X(T^\vee)]^W$.

Observe also that $F^\vee(r(\lambda)) = r(F^\vee\lambda)$ for every $\lambda \in X(T^\vee)$ (because $F^\vee(W) = W$).

Lemma 3.8. *Let $\pi : X(T^\vee) \twoheadrightarrow X(T^\vee_{\mathrm{der}})$ be the canonical surjection (§3.6), and let $\mathrm{ht} : X(T^\vee_{\mathrm{der}}) \longrightarrow \mathbb{Q}$ be the height function with respect to $\Delta^\vee$ (recall that $G^\vee_{\mathrm{der}}$ is semisimple, and that $\Delta^\vee$ is a set of simple roots for the root system of $(G^\vee_{\mathrm{der}}, T^\vee_{\mathrm{der}})$).*

(a) *For every $\lambda \in X(T^\vee)$, the restriction of π to $W\lambda$ is injective.*

(b) *For every $\lambda_1, \lambda_2 \in X^+(T^\vee)$, in $\mathbb{Z}[X(T^\vee)]^W$ we have*

$$r(\lambda_1)\cdot r(\lambda_2) = r(\lambda_1+\lambda_2) + \sum_{\substack{\mu\in\Omega(\lambda_1,\lambda_2)\\ \mu\neq\lambda_1+\lambda_2}} c_\mu r(\mu),$$

where $\Omega(\lambda_1,\lambda_2) = (W\lambda_1+W\lambda_2)\cap X^+(T^\vee)$ and $c_\mu \in \mathbb{N}^*$ for all μ. Moreover, for every $\mu \in \Omega(\lambda_1,\lambda_2)$ with $\mu \neq \lambda_1+\lambda_2$, we have $\mathrm{ht}(\pi(\mu)) < \mathrm{ht}(\pi(\lambda_1+\lambda_2))$.

Proof. (a) If $w_1, w_2 \in W$ are such that $\pi(w_1\lambda) = \pi(w_2\lambda)$, then $\nu := w_1\lambda - w_2\lambda$ lies in $\ker(\pi)$; as $\ker(\pi) = X(T^\vee)^W$ (§3.6), we have

$$\nu = \frac{1}{|W|}\sum_{w\in W} w\nu = \frac{1}{|W|}\left(\sum_{w\in W} ww_1\lambda - \sum_{w\in W} ww_2\lambda\right) = 0,$$

whence $w_1\lambda = w_2\lambda$.

(b) By definition of $r(\lambda)$, it is clear that

$$r(\lambda_1)\cdot r(\lambda_2) = \sum_{\mu\in\Omega(\lambda_1,\lambda_2)} c_\mu r(\mu)$$

with all coefficients $c_\mu \in \mathbb{N}^*$. For each $\mu \in \Omega(\lambda_1,\lambda_2)$, we have $\mu = \mu_1 + \mu_2$ where $(\mu_1,\mu_2) \in W\lambda_1 \times W\lambda_2$; if $\mu \neq \lambda_1+\lambda_2$, then $(\mu_1,\mu_2) \neq (\lambda_1,\lambda_2)$, so (a) implies that $(\pi(\mu_1),\pi(\mu_2)) \neq (\pi(\lambda_1),\pi(\lambda_2))$ in $W.\pi(\lambda_1)\times W.\pi(\lambda_2)$, and hence Lemma 3.7(a) shows that

$$\mathrm{ht}(\pi(\mu)) = \mathrm{ht}(\pi(\mu_1)) + \mathrm{ht}(\pi(\mu_2)) < \mathrm{ht}(\pi(\lambda_1)) + \mathrm{ht}(\pi(\lambda_2)) = \mathrm{ht}(\pi(\lambda_1+\lambda_2)).$$

It then remains to prove that $c_{\lambda_1+\lambda_2} = 1$. Indeed, $c_{\lambda_1+\lambda_2}$ is the number of pairs (μ_1,μ_2) in $W\lambda_1\times W\lambda_2$ such that $\mu_1+\mu_2 = \lambda_1+\lambda_2$. For such a pair (μ_1,μ_2), we have

$$\mathrm{ht}(\pi(\mu_1)) + \mathrm{ht}(\pi(\mu_2)) = \mathrm{ht}(\pi(\lambda_1)) + \mathrm{ht}(\pi(\lambda_2)),$$

so the argument of the last paragraph tells us that $(\mu_1,\mu_2) = (\lambda_1,\lambda_2)$. This proves the equality $c_{\lambda_1+\lambda_2} = 1$ and completes the proof of the lemma. □

Theorem 3.9. *Suppose that $G^\vee_{\mathrm{der}}$ is simply-connected, so that the fundamental weights ω'_α ($\alpha \in \Delta^\vee$) of $G^\vee_{\mathrm{der}}$ all lie in $X^+(T^\vee_{\mathrm{der}})$. Let $\pi : X^+(T^\vee) \twoheadrightarrow X^+(T^\vee_{\mathrm{der}})$ be the canonical surjection, and identify $X(\overline{T}^\vee) \subset X(T^\vee)$ (§3.6). For each $\alpha \in \Delta^\vee$, choose a lifting of ω'_α to $\omega_\alpha \in X^+(T^\vee)$ via π, so that $\pi(\omega_\alpha) = \omega'_\alpha$. Let also*

$\mathfrak{A} \subset X(\overline{T}^\vee)$ *be a set of representatives of the* $\mathbb{Z}$*-module* $X(\overline{T}^\vee)/(F^\vee - \mathrm{id})X(\overline{T}^\vee)$. *Then* $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ *is a free* $\mathbb{Z}$*-module having the set*

$$\mathfrak{F} := \left\{ r\left(\mu + \sum_{\alpha\in\Delta^\vee} b_\alpha\omega_\alpha\right) + I : \mu \in \mathfrak{A},\, b_\alpha \in \{0,1,\cdots,q-1\}\,(\alpha\in\Delta^\vee)\right\}$$

as its basis, and the rank of $\mathsf{B}_{G^\vee}$ *over* $\mathbb{Z}$ *is*

$$\mathrm{rank}_{\mathbb{Z}}\,\mathsf{B}_{G^\vee} = |\mathfrak{F}| = q^{\mathrm{rank}\,G^\vee_{\mathrm{der}}} \cdot |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}| = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |G^{*F^*}_{\mathrm{ss}}/\sim|.$$

Moreover, $\mathsf{B}_{G^\vee}$ *is a reduced ring, so* $I = \sqrt{I}$ *and* $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\,\mathrm{red}}$.

Remark. In the special case where $G^\vee$ is simply-connected (so that $G^\vee_{\mathrm{der}} = G^\vee$), we have $\pi = \mathrm{id}$, $\overline{T}^\vee = 1$ and $X(\overline{T}^\vee) = 0$; we may choose $\omega_\alpha = \omega'_\alpha$, so $\{\omega_\alpha\}_{\alpha\in\Delta^\vee}$ is a $\mathbb{Z}$-basis of $X(T^\vee)$; thus $\mathsf{B}_{G^\vee}$ is a reduced ring and is also a free $\mathbb{Z}$-module having $\mathfrak{F} = \{r(\lambda) + I : \lambda \in X^+_q(T^\vee)\}$ as its $\mathbb{Z}$-linear basis; the $\mathbb{Z}$-rank of $\mathsf{B}_{G^\vee}$ is $q^{\mathrm{rank}\,G^\vee}$.

Proof of theorem (Compare [Hu, Sec. 5.6–5.7]*).* Let us consider the height function $\mathrm{ht} : X(T^\vee_{\mathrm{der}}) \longrightarrow \mathbb{Q}$ as in Lemma 3.8. For each $f \in \mathbb{Z}[X(T^\vee)]^W$, write $\overline{f} := f + I \in \mathsf{B}_{G^\vee}$, so that $\{\overline{r(\lambda)} : \lambda \in X^+(T^\vee)\}$ generates $\mathsf{B}_{G^\vee}$ as a $\mathbb{Z}$-module. In $\mathsf{B}_{G^\vee}$, we have (using the relation $q = F^\vee \circ \tau^\vee$ from §3.2(a))

$$\overline{r(F^\vee\lambda)} = \overline{r(\lambda)} \quad \text{and} \quad \overline{r(q\lambda)} = \overline{r(\tau^\vee\lambda)} \text{ for every } \lambda \in X(T^\vee).$$

(1) We first prove that $\mathsf{B}_{G^\vee}$ is generated by $\{\overline{r(\lambda)} : \lambda \in X^+_q(T^\vee)\}$ as a $\mathbb{Z}$-module. Suppose that $\lambda \in X^+(T^\vee)$ but $\lambda \notin X^+_q(T^\vee)$. Then there exists an $\alpha \in \Delta^\vee$ such that $\langle\lambda, \alpha^\vee\rangle \geq q$. Consider $\lambda' := \lambda - q\omega_\alpha \in X(T^\vee)$; by hypothesis on λ, we have in fact $\lambda' \in X^+(T^\vee)$. We may then use Lemma 3.8(b) to expand the product $r(\lambda')r(q\omega_\alpha)$ as

$$r(\lambda')r(q\omega_\alpha) = r(\lambda) + \sum_{\lambda\neq\mu\in\Omega(\lambda',q\omega_\alpha)} c_\mu r(\mu) \in \mathbb{Z}[X(T^\vee)]^W,$$

with $c_\mu \in \mathbb{N}^*$ and $\mathrm{ht}(\pi(\mu)) < \mathrm{ht}(\pi(\lambda))$ for all $\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)$. Passing this expansion into $\mathsf{B}_{G^\vee}$ and using the relation $\overline{r(q\omega_\alpha)} = \overline{r(\tau^\vee\omega_\alpha)}$ in $\mathsf{B}_{G^\vee}$, we get

$$\overline{r(\lambda')}\,\overline{r(\tau^\vee\omega_\alpha)} = \overline{r(\lambda)} + \sum_{\lambda\neq\mu\in\Omega(\lambda',q\omega_\alpha)} c_\mu\overline{r(\mu)} \in \mathsf{B}_{G^\vee}.$$

As $\tau^\vee\omega_\alpha \in X^+(T^\vee)$ (T is contained in the F-stable Borel subgroup B, so $X^+(T^\vee)$ is $\tau^\vee$-invariant), we may use Lemma 3.8(b) again to expand the product $r(\lambda')r(\tau^\vee\omega_\alpha)$ as

$$r(\lambda')r(\tau^\vee\omega_\alpha) = r(\lambda' + \tau^\vee\omega_\alpha) + \sum_{\lambda'+\tau^\vee\omega_\alpha\neq\nu\in\Omega(\lambda',\tau^\vee\omega_\alpha)} c'_\nu r(\nu) \in \mathbb{Z}[X(T^\vee)]^W,$$

with $c'_\nu \in \mathbb{N}^*$ and $\mathrm{ht}(\pi(\nu)) < \mathrm{ht}(\pi(\lambda' + \tau^\vee\omega_\alpha))$ for all $\lambda' + \tau^\vee\omega_\alpha \neq \nu \in \Omega(\lambda', \tau^\vee\omega_\alpha)$. Moreover, as $\tau^\vee$ preserves the height function on $X(T^\vee_{\mathrm{der}})$ ($\tau^\vee$ permutes elements of $\Delta^\vee$), we have $\mathrm{ht}(\pi(\tau^\vee\omega_\alpha)) = \mathrm{ht}(\tau^\vee\omega'_\alpha) = \mathrm{ht}(\omega'_\alpha) = \mathrm{ht}(\pi(\omega_\alpha))$, so by Lemma 3.7(b) we have

$$\mathrm{ht}(\pi(\lambda' + \tau^\vee\omega_\alpha)) < \mathrm{ht}(\pi(\lambda' + q\omega_\alpha)) = \mathrm{ht}(\pi(\lambda)).$$

We thus obtain the relation

$$\overline{r(\lambda)} = \overline{r(\lambda' + \tau^\vee\omega_\alpha)} + \sum_{\lambda'+\tau^\vee\omega_\alpha \neq \nu \in \Omega(\lambda', \tau^\vee\omega_\alpha)} c'_\nu \overline{r(\nu)} - \sum_{\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)} c_\mu \overline{r(\mu)} \in \mathsf{B}_{G^\vee},$$

which expresses $\overline{r(\lambda)}$ in terms of a $\mathbb{Z}$-linear combination of some $\overline{r(\gamma)}$ where $\gamma \in X^+(T^\vee)$ with $\mathrm{ht}(\pi(\gamma)) < \mathrm{ht}(\pi(\lambda))$. On the other hand, as $X(T^\vee)$ is a free $\mathbb{Z}$-module of finite rank, we see that $\mathrm{ht}(\pi(X^+(T^\vee))) \subset h^{-1}\mathbb{N}$ for some $h \in \mathbb{N}^*$. We can thus repeat the above reduction process of $\overline{r(\lambda)}$, so that for every $\lambda \in X^+(T^\vee)$ we can eventually express $\overline{r(\lambda)}$ as a $\mathbb{Z}$-linear combination of those $\overline{r(\mu)}$ with $\mu \in X_q^+(T^\vee)$.

(2) Let us use (1) to prove that $\mathsf{B}_{G^\vee}$ is generated by $\mathfrak{F}$ as a $\mathbb{Z}$-module. From the canonical exact sequence $0 \longrightarrow X(\overline{T}^\vee) \longrightarrow X(T^\vee) \longrightarrow X(T^\vee_{\mathrm{der}}) \longrightarrow 0$ (§3.6), each $\lambda \in X_q^+(T^\vee)$ may be expressed as $\lambda = \mu + \sum_{\alpha \in \Delta^\vee} b_\alpha \omega_\alpha$ where $\mu \in X(\overline{T}^\vee)$ and each $b_\alpha \in \{0, 1, \cdots, q-1\}$. Furthermore, for each $\mu \in X(\overline{T}^\vee) \subset X(T^\vee)$, we have $r(\mu) = e(\mu) \in \mathbb{Z}[X(T^\vee)]^W$ and therefore $e((F^\vee - \mathrm{id})\mu) - 1 = e(-\mu)(e(F^\vee\mu) - e(\mu)) \in I$. These observations and (1) together imply that the $\mathbb{Z}$-module is $\mathsf{B}_{G^\vee}$ is generated by $\mathfrak{F}$.

(3) Let us now prove that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module having $\mathfrak{F}$ as its basis and having the desired rank formulae. In (2) we have seen that $\mathfrak{F}$ generates the $\mathbb{Z}$-module $\mathsf{B}_{G^\vee}$, so that $\mathfrak{F}$ also generates the $\overline{\mathbb{Q}}$-linear vector space $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$. By §3.5, we have

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \geq |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|;$$

on the other hand, we have (compare the part of duality of tori in §0.7)

$$|\mathfrak{A}| = [X(\overline{T}^\vee) : (F^\vee - \mathrm{id})X(\overline{T}^\vee)] = |\mathrm{Irr}_{\overline{\mathbb{Q}}}(\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee})| = |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}|,$$

which implies, together with Lemma 3.6 and Lemma 3.1, that

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \leq |\mathfrak{F}| = q^{\mathrm{rank}\, G^\vee_{\mathrm{der}}} \cdot |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}| = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |G_{\mathrm{ss}}^{*F^*}/\sim|.$$

Therefore, $\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$, and $\mathfrak{F}$ is in fact a basis for the $\overline{\mathbb{Q}}$-linear vector space $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$; it follows that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module of basis $\mathfrak{F}$.

(4) Finally, let us prove the reducedness of $\mathsf{B}_{G^\vee}$. We have just proved in (3) that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module, and this implies that the natural map $\mathsf{B}_{G^\vee} \longrightarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ is injective. In (3) we have also shown that $\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$, so the discussion in §3.5 implies that $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ is a reduced ring. Hence $\mathsf{B}_{G^\vee}$ is also reduced.

$\square$

Comparison between $\mathsf{B}_{G^\vee}$ and K_{G^*}.

3.9. **Algebraic representations and formal characters**[Ja, Ch. I.2]**.** Denote by $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ the category of G^*-modules of finite dimension (over the defining field $\overline{\mathbb{F}_q}$ of G^*); an object of $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ is called an algebraic (or rational) representation of G^*. Let $\mathrm{Irr}_{\mathrm{alg}}(G^*)$ be the set of isomorphism classes of simple objects in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$. Denote also by $\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*))$ the Grothendieck group of $\mathrm{Rep}_{\mathrm{alg}}(G^*)$; $\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*))$ is in fact a ring with multiplication given by the tensor product.

For $M \in \mathrm{Rep}_{\mathrm{alg}}(G^*)$, its formal character $\mathrm{ch}\, M$ is defined as follows: M has a weight space decomposition (relative to T^*) $M = \bigoplus_{\lambda \in X(T^*)} M_\lambda$ where the weight spaces $M_\lambda := \{m \in M : zm = \lambda(z)m \text{ for all } z \in T^*\}$ (we call $\lambda \in X(T^*)$ a weight of M if $M_\lambda \neq \{0\}$); then set $\mathrm{ch}\, M := \sum_{\lambda \in X(T^*)} \dim_{\overline{\mathbb{F}_q}} M_\lambda \cdot e(\lambda) \in \mathbb{Z}[X(T^*)]^W$. (We have $M_{w\lambda} = wM_\lambda$ for every $w \in W$, whence the W-invariance of $\mathrm{ch}\, M$.)

3.10. **Highest weights.** [Ja, Ch. II.2] Identify $X(T^*) = X(T^\vee)$, so for $\lambda \in X(T^*)$ we may define $r(\lambda) \in \mathbb{Z}[X(T^*)]^W$ as in §3.8. Let $\leq$ be the standard partial ordering on $X(T^*) = X(T^\vee)$ determined by $\Delta^\vee$. Then:

(a) every $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$ admits a unique highest weight λ_M with respect to the partial ordering $\leq$; we have $\lambda_M \in X^+(T^*)$ and $\dim_{\overline{\mathbb{F}_q}} M_{\lambda_M} = 1$; also, every weight $\lambda \in X(T^*)$ of M satisfies $\lambda \leq \lambda_M$; as a result, for each $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$, we have $\mathrm{ch}\, M \in r(\lambda_M) + \sum_{\lambda \in X^+(T^*),\, \lambda < \lambda_M} \mathbb{Z}.r(\lambda)$;

(b) for every $\lambda \in X^+(T^*)$, there is a unique $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$ which admits λ as its highest weight in the sense of (a); we shall denote this unique M by $L(\lambda)$;

(c) the map $M \longmapsto \mathrm{ch}\, M$ introduced in §3.9 induces a $\mathbb{Z}$-algebra isomorphism
$$\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W.$$

Lemma 3.10. *Let $\lambda \in X^+(T^*)$. Consider the Frobenius twist $L(\lambda)^{[F^*]}$: it is the G^*-module whose underlying set is $L(\lambda)$ and whose G^*-action is given by the composition $G^* \xrightarrow{F^*} G^* \xrightarrow{\rho} \mathrm{GL}(L(\lambda))$ where ρ denotes the G^*-action on $L(\lambda)$. Then $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$.*

Proof. The G^*-module $L(\lambda)^{[F^*]}$ is irreducible by [St1, Thm. 5.1]. As the highest weight of $L(\lambda)^{[F^*]}$ is $F^*\lambda$, we see that $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ as G^*-modules. □

Lemma 3.11 ([St1, Thm. 7.4], [Her, Thm. 3.10])**.** *If G^*_{der} is simply-connected, then every $M \in \mathrm{Irr}_{\overline{\mathbb{F}_q}}(G^{*F^*})$ comes from the restriction of some $L(\lambda) \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$ with $\lambda \in X_q^+(T^*)$, so the restriction map $\mathrm{Res}^{G^*}_{G^{*F^*}} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \longrightarrow \mathsf{K}_{G^*}$ is surjective.*

The first reference establishes the case where G^* is simply-connected, which is generalized in the second reference to the case where G^*_{der} is simply-connected.

Proposition 3.12. *Identify $X(T^*) = X(T^\vee)$ and let $\pi : \mathbb{Z}[X(T^*)]^W \twoheadrightarrow \mathsf{B}_{G^\vee,\mathrm{red}}$ be the induced quotient map. Observe that the maps*
$$(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \xrightarrow[\text{Lemma 3.1}]{\sim} (G^*_{\mathrm{ss}}/\sim)^{F^*} \xrightarrow[\text{Lemma 3.6}]{} \!\!\!\!\!\!\to (G^{*F^*}_{\mathrm{ss}}/\sim) \overline{\overline{\;\;_{\S 2.1}\;\;}} (G^{*F^*}_{p'}/\sim)$$
induce an injection of $\overline{\mathbb{Q}}$-algebras
$$j : \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})} \hookrightarrow \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

Then there is an injective $\mathbb{Z}$-algebra homomorphism $\overline{\gamma} : \mathsf{B}_{G^\vee,\mathrm{red}} \hookrightarrow \mathsf{K}_{G^*}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\pi"] \\
\mathsf{K}_{G^*} \arrow[d, hook, "(\S 2.3)\ \mathrm{br}"'] & \mathsf{B}_{G^\vee,\mathrm{red}} \arrow[l, hook', "\overline{\gamma}"'] \arrow[d, hook, "\mathit{Corollary}\ 3.3"] \\
\overline{\mathbb{Q}}^{G^{*F^*}_{p'}}/\sim & \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})} \arrow[l, hook', "j"']
\end{tikzcd} \tag{3.10.1}$$

Proof. (1) Let $\lambda \in X^+(T^*)$. Then $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ (Lemma 3.10). As F^* acts trivially on G^{*F^*}, we have

$$\mathrm{Res}^{G^*}_{G^{*F^*}}(L(\lambda)^{[F^*]}) = \mathrm{Res}^{G^*}_{G^{*F^*}}(L(\lambda))$$

in $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$, so $\mathrm{Res}^{G^*}_{G^{*F^*}} L(F^*\lambda) = \mathrm{Res}^{G^*}_{G^{*F^*}} L(\lambda)$ in $\mathrm{Rep}_{\overline{\mathbb{F}_q}}(G^{*F^*})$. Therefore, if we denote by J the ideal of $\mathbb{Z}[X(T^*)]^W$ generated by $\{\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) : \lambda \in X^+(T^*)\}$, then J lies in the kernel of the composition $\gamma := \mathrm{Res}^{G^*}_{G^{*F^*}} \circ \mathrm{ch}^{-1} : \mathbb{Z}[X(T^*)]^W \longrightarrow \mathsf{K}_{G^*}$. As K_{G^*} is reduced (§2.3), γ descends to $\overline{\gamma} : \mathbb{Z}[X(T^*)]^W/\sqrt{J} \longrightarrow \mathsf{K}_{G^*}$, and we obtain the following commutative diagram of $\mathbb{Z}$-algebras:

$$\begin{tikzcd}
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\pi"] \\
\mathsf{K}_{G^*} & \mathbb{Z}[X(T^*)]^W/\sqrt{J} \arrow[l, "\overline{\gamma}"]
\end{tikzcd} \tag{3.10.2}$$

(2) We now show that under the identification $\mathbb{Z}[X(T^*)]^W = \mathbb{Z}[X(T^\vee)]^W$ (induced by the identification $X(T^*) = X(T^\vee)$), the ideal $J \subset \mathbb{Z}[X(T^*)]^W$ corresponds to the ideal $I \subset \mathbb{Z}[X(T^\vee)]^W$ appearing in the definition of $\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[X(T^\vee)]^W/\sqrt{I}$ (§3.4). By §3.3 and §3.8, the ideal I is generated by $\{r(F^\vee\lambda) - r(\lambda) : \lambda \in X^+(T^\vee)\}$. For $\lambda \in X^+(T^*)$, we may write $\mathrm{ch}\, L(\lambda) = r(\lambda) + \sum_{\mu\in X^+(T^*),\ \mu<\lambda} c_\mu r(\mu)$ where only a finite number of c_μ is not zero. Then $\mathrm{ch}\, L(F^*\lambda) = r(F^*\lambda) + \sum_{\mu\in X^+(T^*),\ \mu<\lambda} c_\mu r(F^*\mu)$: in fact, again by Lemma 3.10 we have $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$, thus all the weights of $L(F^*\lambda)$ are of the form $F^*\mu$ where μ is a weight of $L(\lambda)$, and moreover $L(F^*\lambda)_{F^*\mu} \simeq L(\lambda)_\mu$ for all $\mu \in X(T^*)$, whence the desired expression for $\mathrm{ch}\, L(F^*\lambda)$. We thus have:

$$\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) = \\ (r(F^*\lambda) - r(\lambda)) + \sum_{\mu\in X^+(T^*),\ \mu<\lambda} c_\mu.(r(F^*\mu) - r(\mu)) \quad (\lambda \in X^+(T^*)).$$

This implies that the transition matrix from $\{\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) : \lambda \in X^+(T^*)\}$ to $\{r(F^*\lambda) - r(\lambda) : \lambda \in X^+(T^*)\}$ is triangular with all diagonal elements being 1 (with respect to the partial ordering $\leq$ on $X^+(T^*)$), whence $J = I$.

(3) The equality $J = I$ established in (2) implies that $\mathbb{Z}[X(T^*)]^W/\sqrt{J} = \mathsf{B}_{G^\vee,\mathrm{red}}$ under the identification $X(T^*) = X(T^\vee)$, so (3.10.2) is exactly the upper part of (3.10.1), where we still need to show the injectivity of $\overline{\gamma}$.

(4) To establish (3.10.1) (in particular the injectivity of $\overline{\gamma}$), it remains to prove that the outermost rectangle diagram in (3.10.1) commutes, that is, to prove the following diagram commutes:

$$\begin{tikzcd} \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{br}\circ\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\mathrm{canonical}"] \\ \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} & \overline{\mathbb{Q}}^{(T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}_q})} \arrow[l, hook', "j"'] \end{tikzcd} \tag{3.10.3}$$

(We have identified $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) = (T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}_q})$.) To do this, we need the help of the following properties:

(i) $G^{*F^*}_{p'} = G^{*F^*}_{\mathrm{ss}}$ is the union of all S^{*F^*} where S^* runs through elements of $\mathcal{T}^* := \{F^*\text{-stable maximal tori of } G^*\}$;

(ii) $(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}_q})$ is the set of W-orbits in $\bigcup_{w\in W} T^{*wF^*}$;

(iii) For each F^*-stable maximal torus S^* of G^*, we can find a $g^* \in G^*$ such that $g^*S^*(g^*)^{-1} = T^*$, so that the map $x^* \longmapsto g^*x^*(g^*)^{-1}$ establishes an isomorphism $S^{*F^*} \xrightarrow{\sim} T^{*wF^*}$ where $w \in W$ is the quotient image of $g^*F^*(g^*)^{-1} \in N_{G^*}(T^*)$.

These properties enable us to integrate (3.10.3) into the following cubic diagram:

$$\begin{tikzcd} & \prod\limits_{S^*\in\mathcal{T}^*} \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(S^*)) \arrow[rr, "\sim", "\mathrm{ch}"'] \arrow[dd, "\mathrm{br}\circ\mathrm{Res}"] & & \prod\limits_{S^*\in\mathcal{T}^*} \mathbb{Z}[X(S^*)] \arrow[dd, "\mathrm{canonical}"] \\ \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[ur, "\mathrm{Res}"] \arrow[rr, "\sim", "\mathrm{ch}"'] \arrow[dd, "\mathrm{br}\circ\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & & \mathbb{Z}[X(T^*)]^W \arrow[ur, "\mathrm{(iii)}"'] \arrow[dd, two heads, "\mathrm{canonical}"'] & \\ & \prod\limits_{S^*\in\mathcal{T}^*} \overline{\mathbb{Q}}^{S^{*F^*}} \arrow[rr, equal] & & \prod\limits_{S^*\in\mathcal{T}^*} \overline{\mathbb{Q}}^{S^{*F^*}} \\ \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} \arrow[ur, hook, "\mathrm{Res}"] & & \overline{\mathbb{Q}}^{(T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}_q})} \arrow[ll, hook', "j"'] \arrow[ur, hook, "\mathrm{(iii)}"'] & \end{tikzcd}$$

(Here "Res" means the natural restriction maps, and the maps "(iii)" on the right face are the natural maps induced by the bijection in (iii) above.) In the above cubic diagram, it can be checked that all the five faces other than the front face (3.10.3) are commutative diagrams; thus the front face (3.10.3) is also commutative.

□

Theorem 3.13. *If G^*_{der} is simply-connected, then the formal character isomorphism* $\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W$ *and the bijection of finite sets* $(G^{*F^*}_{p'}/\sim$

) $\simeq (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$ (§2.1, Lemma 3.1 and Lemma 3.6) both induce, via the commutative diagram in Proposition 3.12, the same $\mathbb{Z}$-algebra isomorphism $\mathsf{B}_{G^\vee} \simeq \mathsf{K}_{G^*}$.

Proof. The simple-connectedness of G^*_{der} has two consequences: (i) the restriction map $\mathrm{Res}^{G^*}_{G^{*F^*}} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \longrightarrow \mathsf{K}_{G^*}$ is surjective (Lemma 3.11), so the injective map $\overline{\gamma} : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \mathsf{K}_{G^*}$ in Proposition 3.12 is also surjective and is therefore a $\mathbb{Z}$-algebra isomorphism; (ii) $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}}$ (Theorem 3.9). Thus the map $\overline{\gamma}$, coming from the formal character map, establishes a $\mathbb{Z}$-algebra isomorphism $\mathsf{B}_{G^\vee} \simeq \mathsf{K}_{G^*}$. □

Corollary 3.14 (of Corollary 2.4 and Theorem 3.13)**.** *The maps*

$$(G^{*F^*}_{\mathrm{ss}}/\sim) =\!=\!= (G^{*F^*}_{p'}/\sim) \xrightarrow[\text{Lemma 3.6}]{\text{Lemma 3.1}} (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$$

induce $\overline{\mathbb{Q}}$-algebra homomorphisms

$$\overline{\mathbb{Q}}\mathsf{E}_G \xleftrightarrow{\sim} \overline{\mathbb{Q}}\mathsf{K}_{G^*} \hookleftarrow (\overline{\mathbb{Q}}\mathsf{B}_{G^\vee})_{\mathrm{red}} \overset{\S 3.5}{=\!=\!=} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadleftarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$$

which descend to $\mathbb{Z}[\frac{1}{p|W|}]$-algebra homomorphisms:

(3.10.4)

$$\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{E}_G \xleftrightarrow{\sim} \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*} \hookleftarrow \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadleftarrow \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{B}_{G^\vee}.$$

If G^*_{der} is simply-connected, all maps in (3.10.4) are $\mathbb{Z}[\frac{1}{p|W|}]$-algebra isomorphisms:

$$\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{E}_G \simeq \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*} \simeq \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{B}_{G^\vee}.$$

3.11. **Remark on toric graduations.** For a general connected reductive group $G^\vee$ (over $\mathbb{Z}$), use the proof of Proposition 3.4 to integrate $G^\vee$ into an exact sequence of reductive groups over $\mathbb{Z}$,

$$1 \longrightarrow S^\vee \longrightarrow H^\vee \longrightarrow G^\vee \longrightarrow 1,$$

with $H^\vee_{\mathrm{der}}$ simply-connected and with $S^\vee$ a torus central in $H^\vee$. Denote by $G^\vee /\!\!/ G^\vee$ the categorical quotient induced by the adjoint action of $G^\vee$ on itself; note that $G^\vee /\!\!/ G^\vee \simeq T^\vee /\!\!/ W$ by Chevalley's restriction theorem. The multiplication action of $S^{\vee F^\vee}$ on $(H^\vee /\!\!/ H^\vee)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{H^\vee})$ induces an $S^{\vee F^\vee}$-action on $\mathsf{B}_{H^\vee}$, from which the ring $\mathsf{B}_{H^\vee}$ admits an $X(S^{\vee F^\vee})$-graded decomposition $\mathsf{B}_{H^\vee} = \bigoplus_{\lambda \in X(S^{\vee F^\vee})} (\mathsf{B}_{H^\vee})_\lambda$. Observe that $(\mathsf{B}_{H^\vee})_0 = (\mathsf{B}_{H^\vee})^{S^{\vee F^\vee}}$, so the canonical surjection $(H^\vee /\!\!/ H^\vee)^{F^\vee} /\!\!/ S^{\vee F^\vee} \twoheadrightarrow (G^\vee /\!\!/ G^\vee)^{F^\vee}$ induces an inclusion of rings $\mathsf{B}_{G^\vee,\mathrm{red}} \hookrightarrow (\mathsf{B}_{H^\vee})_0$.

We have analogous discussion on the K-side: as S^{*F^*} lies in the center of H^{*F^*}, the map associating a character of H^{*F^*} to its central character induces an $X(S^{*F^*})$-graded decomposition $\mathsf{K}_{H^*} = \bigoplus_{\lambda \in X(S^{*F^*})} (\mathsf{K}_{H^*})_\lambda$, while this time we have canonical ring isomorphisms $(\mathsf{K}_{H^*})_0 \simeq \mathsf{K}_{G^*}$. With the identification $X(S^{\vee F^\vee}) \simeq X(S^{*F^*})$, the ring isomorphism $\mathsf{B}_{H^\vee} \simeq \mathsf{K}_{H^*}$ established in Theorem 3.13 respects the above graded structures and restricts itself to a ring isomorphism $(\mathsf{B}_{H^\vee})_0 \simeq (\mathsf{K}_{H^*})_0$.

In summary, we have the following commutative diagram of rings:

$$\begin{tikzcd}
\mathsf{B}_{G^\vee\,\mathrm{red}} \arrow[r, hook] \arrow[d, hook, "\text{Proposition 3.12}"'] & (\mathsf{B}_{H^\vee})_0 \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"] & \mathsf{B}_{H^\vee} \arrow[d, leftrightarrow, "\wr"', "\text{graded (Theorem 3.13)}"] \\
\mathsf{K}_{G^*} \arrow[r, leftrightarrow, "\sim"] & (\mathsf{K}_{H^*})_0 \arrow[r, hook] & \mathsf{K}_{H^*}
\end{tikzcd}$$

Acknowledgments

I would like to thank Professor Jean-François Dat for introducing me to this subject and for having many stimulating discussions with me. I would also like to thank the anonymous referee for helpful suggestions on this article.

References

[BoKe] Cédric Bonnafé and Radha Kessar, *On the endomorphism algebras of modular Gelfand-Graev representations*, J. Algebra **320** (2008), no. 7, 2847–2870, DOI 10.1016/j.jalgebra.2008.05.029. MR2441999

[Br] Michel Broué, *Higman's criterion revisited*, Michigan Math. J. **58** (2009), no. 1, 125–179, DOI 10.1307/mmj/1242071686. MR2526081

[Ca] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters; A Wiley-Interscience Publication. MR794307

[Cu] Charles W. Curtis, *On the Gel′fand-Graev representations of a reductive group over a finite field*, J. Algebra **157** (1993), no. 2, 517–533, DOI 10.1006/jabr.1993.1113. MR1220781

[CuRe] Charles W. Curtis and Irving Reiner, *Methods of representation theory. Vol. I*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1981. With applications to finite groups and orders. MR632548

[DHKM] J.-F. Dat, D. Helm, R. Kurinczuk, and G. Moss, *Local Langlands correspondence in families*, In preparation.

[DeLu] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161, DOI 10.2307/1971021. MR393266

[DLM] F. Digne, G. I. Lehrer, and J. Michel, *The characters of the group of rational points of a reductive group with nonconnected centre*, J. Reine Angew. Math. **425** (1992), 155–192, DOI 10.1515/crll.1992.425.155. MR1151318

[DiMi] François Digne and Jean Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, Cambridge, 1991, DOI 10.1017/CBO9781139172417. MR1118841

[FaSc] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, Preprint, `arXiv:2102.13459`, 2021.

[Hel] David Helm, *Curtis homomorphisms and the integral Bernstein center for* GL_n, Algebra Number Theory **14** (2020), no. 10, 2607–2645, DOI 10.2140/ant.2020.14.2607. MR4190413

[HeMo] David Helm and Gilbert Moss, *Converse theorems and the local Langlands correspondence in families*, Invent. Math. **214** (2018), no. 2, 999–1022, DOI 10.1007/s00222-018-0816-y. MR3867634

[Her] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116, DOI 10.1215/00127094-2009-036. MR2541127

[Hu] James E. Humphreys, *Modular representations of finite groups of Lie type*, London Mathematical Society Lecture Note Series, vol. 326, Cambridge University Press, Cambridge, 2006. MR2199819

[Ja] Jens Carsten Jantzen, *Representations of algebraic groups*, Pure and Applied Mathematics, vol. 131, Academic Press, Inc., Boston, MA, 1987. MR899071

[La] Thomas Lanard, *Sur les* ℓ*-blocs de niveau zéro des groupes* p*-adiques* (French, with English summary), Compos. Math. **154** (2018), no. 7, 1473–1507, DOI 10.1112/s0010437x18007133. MR3826462

[Lu] G. Lusztig, *Divisibility of projective modules of finite Chevalley groups by the Steinberg module*, Bull. London Math. Soc. **8** (1976), no. 2, 130–134, DOI 10.1112/blms/8.2.130. MR401900

[Se1] Jean-Pierre Serre, *Représentations linéaires des groupes finis* (French), Third revised edition, Hermann, Paris, 1978. MR543841

[Se2] Jean-Pierre Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics, vol. 117, Springer-Verlag, New York, 1988. Translated from the French, DOI 10.1007/978-1-4612-1035-1. MR918564

[Sp] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics, vol. 9, Birkhäuser Boston, Inc., Boston, MA, 1998, DOI 10.1007/978-0-8176-4840-4. MR1642713

[St1] Robert Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56. MR155937

[St2] Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR0230728

Institut de Mathématiques de Jussieu-Paris Rive Gauche (IMJ-PRG), 4 place Jussieu, 75252 Paris cedex 05, France

Email address: `tzu-jan.li@imj-prg.fr`