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REALIZATIONS OF $A_1^{(1)}$-MODULES IN CATEGORY $\widetilde{\mathcal{O}}$

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Abstract. In this paper, we give an explicit realization of all irreducible modules in Chari's category $\widetilde{\mathcal{O}}$ for the affine Kac-Moody algebra $A_1^{(1)}$ by using the idea of free fields. We work on a much more general setting which also gives us explicit realizations of all simple weight modules for certain current algebra of $\mathfrak{sl}_2(\mathbb{C})$ with finite weight multiplicities, including the polynomial current algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]$, the loop algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]$ and the three-point Lie algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1},(t-1)^{-1}]$ arisen in the work by Kazhdan-Lusztig.

1. Introduction

Let S be any multiplicatively closed subset of the polynomial ring $\mathbb{C}[t]$ in one variable, and let $\mathbb{C}[t]_S$ be the localization of $\mathbb{C}[t]$ at S. We will construct representations for the current Lie algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$ and the affine Kac-Moody algebra $A_1^{(1)}$. By specializing S, this gives us representations for the polynomial current algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]$, the loop algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]$ and the three-point Lie algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1},(t-1)^{-1}]$ appearing in the study of the tensor structure of affine Kac-Moody algebras by Kazhdan-Lusztig [34].

The purpose of this paper is two-fold. The first one is to present an explicit realization of all irreducible $A_1^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$. The category $\widetilde{\mathcal{O}}$, introduced by Chari [8], is an analog of the BGG category $\mathcal{O}$ [3] corresponding to the natural Borel subalgebra of $A_1^{(1)}$. The category $\widetilde{\mathcal{O}}$ appeared naturally in the study of level 0 modules for affine Kac-Moody Lie algebras [8, 10, 14]. The irreducible integrable objects in the category $\widetilde{\mathcal{O}}$ for $A_1^{(1)}$ were classified in [8] and then realized in [13], which exhaust all irreducible level 0 integrable $A_1^{(1)}$-modules with finite weight multiplicities. Moreover, it was proved in [30, 31] that any level 0 unitarizable highest weight module for $A_1^{(1)}$ (without derivations) must be an irreducible module induced from the natural Borel subalgebra. In contrast to the irreducible highest weight modules in the category $\mathcal{O}$ of $A_1^{(1)}$ [32], the irreducible modules in the category $\widetilde{\mathcal{O}}$ have both finite and infinite weight multiplicities and much more complicated structure [8, 26]. A character formula for the irreducible $A_1^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$ with finite weight multiplicities was obtained in

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[40]. In Section 6 of this paper, we give a free field(-like) realization of all (non-integrable) irreducible $A_1^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$.

Free field realization of modules for affine Kac-Moody algebras plays an important role in representation theory and conformal field theory. In [30], Jakobsen-Kac gave a free field construction for certain level 0 Verma type $A_1^{(1)}$-modules, which is referred as imaginary Verma modules [25]. The free field realization of the imaginary Verma $A_1^{(1)}$-modules at an arbitrary level was given by Bernard-Felder in [2] and then extended in [15] to the case of $A_n^{(1)}$. To prove the Kac-Kazhdan conjecture on the characters of irreducible highest weight $A_1^{(1)}$-modules at the critical level, Wakimoto gave in [39] a remarkable free field construction of $A_1^{(1)}$-modules at an arbitrary level. Since then, the Wakimoto modules for general affine Lie algebras have been extensively studied [17, 19, 20, 22, 23, 27–29, 33, 36, 38]. In this paper, for the purpose of realizing the irreducible $A_1^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$ with finite weight multiplicities, we introduce a free field(-like) construction of level 0 $A_1^{(1)}$-modules.

The free field constructions of $A_1^{(1)}$-modules given in [2, 30, 39] are all realized on the polynomial rings in infinitely many variables, in terms of infinite sums of partial differential operators. In contrast to the constructions given in [2, 30, 39], in this paper we realize a class of $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}]$-modules on the polynomial rings in finitely many variables, in terms of finite sums of partial differential operators that are glued together by certain Lagrange interpolation polynomials. Explicitly, for any $f(t) \in \mathbb{C}[t], \lambda \in \mathbb{C}^\times$ and $r \in \mathbb{N}$ with $r \geq \deg f(t)$, we prove in Section 4.2 that the map

$$\begin{aligned}
&\mathrm{f} \otimes t^n \mapsto \sum_{i=0}^{r} \lambda^n \, \ell_{i,r}(n) \, y_i, \\
&\mathrm{h} \otimes t^n \mapsto f(n)\, \lambda^n - 2 \sum_{i,j=0}^{r} \lambda^n \, \ell_{j,r}(n+i) \, y_j \, \frac{\partial}{\partial y_i}, \\
&\mathrm{e} \otimes t^n \mapsto \sum_{i=0}^{r} \left(\lambda^n f(n+i) - \sum_{j,k=0}^{r} \lambda^n \, \ell_{k,r}(n+i+j) \, y_k \, \frac{\partial}{\partial y_j} \right) \frac{\partial}{\partial y_i}
\end{aligned} \tag{1.1}$$

defines an $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}]$-module structure on the polynomial ring $\mathcal{P}_r = \mathbb{C}[y_0, y_1, \cdots, y_r]$, where $\ell_{j,r}(t)$ stand for the fundamental Lagrange interpolation polynomials of degree r at the points $j = 0, 1, \cdots, r$, and

$$\mathrm{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

Especially, by applying Chari-Pressley's loop module construction [13, 21], we construct in this way an inverse system

$$\cdots \twoheadrightarrow \mathcal{P}_{r+1} \otimes \mathbb{C}[t,t^{-1}] \twoheadrightarrow \mathcal{P}_r \otimes \mathbb{C}[t,t^{-1}] \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{P}_{\deg f(t)} \otimes \mathbb{C}[t,t^{-1}] \tag{1.3}$$

of level 0 $A_1^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$ with finite weight multiplicities. We emphasize that if $\deg f(t) \geq 1$, then the last term in (1.3) is irreducible.

Besides the algebra $A_1^{(1)}$, the free field constructions for $A_n^{(1)}$ with $n \geq 1$ have been given in [15] (see also [17]). By combining the above construction with that given in [15], one can also realize a class of irreducible $A_n^{(1)}$-modules in the category $\widetilde{\mathcal{O}}$ with finite weight multiplicities. In particular, the character formula of such $A_n^{(1)}$-modules will be obtained, which are not known in general. Details for general $A_n^{(1)}$ will be given in another paper.

The second goal of this paper is to provide an explicit realization of all Harish-Chandra modules for the current algebra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$. An $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-module is said to be Harish-Chandra if it is irreducible and decomposes into finite dimensional common eigenspaces with respect to $\mathbb{C}h\otimes 1$. Recently, Lau gave in [37] the classification of Harish-Chandra modules for the general current algebra $\mathfrak{g}\otimes\mathcal{R}$, where $\mathfrak{g}$ is a reductive Lie algebra and $\mathcal{R}$ is a finitely generated commutative algebra. In Section 7, we give a free field realization of all Harish-Chandra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-modules. By taking $S=\{t^n \mid n\in\mathbb{N}\}$, this leads to a free field realization of all Harish-Chandra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]$-modules. Besides the loop algebras which appeared naturally in the affine Kac-Moody algebra theory, there are some other current algebras of particular importance, including the polynomial current algebras and the $(N+1)$-point Lie algebras. Motivated by its relationship with the representation theory of affine and quantum affine algebras, the representation theory of the polynomial current algebra $\mathfrak{g}\otimes\mathbb{C}[t]$ is now extensively studied [1, 9, 11, 12, 24]. By choosing $S=\{1\}$, we obtain a realization of all Harish-Chandra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]$-modules. On the other hand, by taking $S=\{(t-a_1)^{m_1}\cdots(t-a_N)^{m_N} \mid m_1,\cdots,m_N\in\mathbb{N}\}$, we obtain a realization of all Harish-Chandra modules for the $(N+1)$-point Lie algebras $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,(t-a_1)^{-1},\cdots,(t-a_N)^{-1}]$ [5,6]. The three and four point Lie algebras appeared naturally in the work of Kazhdan-Lusztig [34] on the tensor structure of modules over affine Kac-Moody algebras. And, by generalizing Wakimoto's construction, the free field realizations of modules for the three and four point Lie algebras were given respectively in [18] and [16].

We now outline the structure of the paper. In Sections 2 and 3, we collect some elementary facts on the highest weight $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-modules. In Section 4, we give a free field realization of certain quasi-finite highest weight $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-modules. Using this and a result of Jakobsen-Kac, we give an explicit realization of all irreducible highest weight $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-modules in Section 5. As applications, in Section 6 we present an explicit realization of all irreducible $A_1^{(1)}$-modules in Chari's category $\widetilde{\mathcal{O}}$, and in Section 7 we give an explicit realization of all Harish-Chandra $\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t]_S$-modules.

In this paper, let $\mathbb{C}$, $\mathbb{C}^\times$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of complex numbers, nonzero complex numbers, integers, nonnegative integers and positive integers, respectively.

2. Highest weight theory for current algebras of $\mathfrak{sl}_2(\mathbb{C})$

Throughout this section, let $\mathcal{R}$ be a unital commutative associative algebra over $\mathbb{C}$ and let $\chi:\mathcal{R}\to\mathbb{C}$ be a $\mathbb{C}$-valued linear function on $\mathcal{R}$.

2.1. **Highest weight $\mathfrak{g}\otimes\mathcal{R}$-modules.** For any Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, let $\mathfrak{a}\otimes\mathcal{R}$ be the current algebra of $\mathfrak{a}$, with the commutator relations given by

$$[x\otimes r, y\otimes r'] = [x,y]\otimes rr',$$

where $x,y\in\mathfrak{a}$ and $r,r'\in\mathcal{R}$. Let

$$\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$$

be the Lie algebra of traceless 2×2-matrices over $\mathbb{C}$. We fix a basis $\{\mathrm{e}, \mathrm{h}, \mathrm{f}\}$ of $\mathfrak{g}$ as in (1.2). Let $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ be the standard triangular decomposition of $\mathfrak{g}$, where $\mathfrak{n}_+=\mathbb{C}\mathrm{e}, \mathfrak{h}=\mathbb{C}\mathrm{h}$ and $\mathfrak{n}_-=\mathbb{C}\mathrm{f}$. Then we have the triangular decomposition:

$$\mathfrak{g}\otimes\mathcal{R}=(\mathfrak{n}_+\otimes\mathcal{R})\oplus(\mathfrak{h}\otimes\mathcal{R})\oplus(\mathfrak{n}_-\otimes\mathcal{R}).$$

A $\mathfrak{g}\otimes\mathcal{R}$-module V is said to be a weight module if $V=\oplus_{\beta\in\mathfrak{h}^*}V_\beta$, where $V_\beta=\{v\in V\mid \mathrm{h}\cdot v=\beta(\mathrm{h})v\}$. For a weight $\mathfrak{g}\otimes\mathcal{R}$-module V, we denote by

$$\mathcal{P}(V)=\{\beta\in\mathfrak{h}^*\mid V_\beta\neq 0\}$$

the set of weights on V.

Definition 2.1.

(i) A weight $\mathfrak{g}\otimes\mathcal{R}$-module V is called quasi-finite if $\dim V_\beta<\infty$ for every $\beta\in\mathcal{P}(V)$, and is called Harish-Chandra if V is in addition irreducible.

(ii) A $\mathfrak{g}\otimes\mathcal{R}$-module V is called a highest weight module with highest weight χ if there exists a nonzero vector $v\in V$, called the highest weight vector, such that

$$\mathcal{U}(\mathfrak{g}\otimes\mathcal{R})v=V,\quad (\mathfrak{n}_+\otimes\mathcal{R})v=0\quad\text{and}\quad (\mathrm{h}\otimes r)v=\chi(r)v,\ r\in\mathcal{R}.$$

Note that any highest weight $\mathfrak{g}\otimes\mathcal{R}$-module with highest weight χ is a weight module and the weights have the form $\chi(1)\alpha/2-n\alpha$, where $n\in\mathbb{N}$ and $\alpha\in\mathfrak{h}^*$ is the root of $\mathfrak{g}$ defined by $\alpha(\mathrm{h})=2$.

Remark 2.2. We define a notion of lowest weight $\mathfrak{g}\otimes\mathcal{R}$-module with lowest weight χ by replacing $\mathfrak{n}_+$ with $\mathfrak{n}_-$ in Definition 2.1(ii). Extend the Chevalley involution

$$\omega:\mathfrak{g}\to\mathfrak{g},\quad \mathrm{e}\mapsto\mathrm{f},\ \mathrm{f}\mapsto\mathrm{e},\ \mathrm{h}\mapsto-\mathrm{h}$$

to an involution of $\mathfrak{g}\otimes\mathcal{R}$, still denoted as ω, such that $\omega(x\otimes r)=\omega(x)\otimes r$ for $x\in\mathfrak{g}$, $r\in\mathcal{R}$. Let V be a highest weight $\mathfrak{g}\otimes\mathcal{R}$-module with highest weight χ. By twisting the involution ω, one obtains a new $\mathfrak{g}\otimes\mathcal{R}$-module structure on V, denoted as V^ω. One can check that V^ω is a lowest weight $\mathfrak{g}\otimes\mathcal{R}$-module with lowest weight $-\chi$, and is irreducible provided that V is irreducible.

2.2. **Verma type highest weight $\mathfrak{g}\otimes\mathcal{R}$-modules.** Let $\mathbb{C}v_\chi$ be the one dimensional $\mathfrak{h}\otimes\mathcal{R}$-module defined by $(\mathrm{h}\otimes r)v_\chi=\chi(r)v_\chi$ for $r\in\mathcal{R}$. We extend $\mathbb{C}v_\chi$ to an $((\mathfrak{n}_+\oplus\mathfrak{h})\otimes\mathcal{R})$-module with $\mathfrak{n}_+\otimes\mathcal{R}$ acting trivially. Form the induced $\mathfrak{g}\otimes\mathcal{R}$-module

$$M(\mathcal{R},\chi)=\mathcal{U}(\mathfrak{g}\otimes\mathcal{R})\otimes_{\mathcal{U}((\mathfrak{n}_+\oplus\mathfrak{h})\otimes\mathcal{R})}\mathbb{C}v_\chi.$$

Note that any highest weight $\mathfrak{g}\otimes\mathcal{R}$-module with highest weight χ is a quotient of $M(\mathcal{R},\chi)$, and the $\mathfrak{g}\otimes\mathcal{R}$-module $M(\mathcal{R},\chi)$ is quasi-finite if and only if $\dim\mathcal{R}<\infty$. Thus, if $\dim\mathcal{R}<\infty$, then every highest weight $\mathfrak{g}\otimes\mathcal{R}$-module is quasi-finite.

We view the dual space $\mathcal{R}^*$ of $\mathcal{R}$ as an $\mathcal{R}$-module under the natural action

$$r\cdot\varphi(r')=\varphi(r'r)\quad\text{for } r,r'\in\mathcal{R},\ \varphi\in\mathcal{R}^*.$$

For any $\varphi\in\mathcal{R}^*$, denote by

$$\mathrm{Ann}_\varphi(\mathcal{R})=\{r\in\mathcal{R}\mid r\cdot\varphi=0\}$$

the annihilator ideal of $\mathcal{R}$ associated to φ. Set

$$\psi_{\mathcal{R},\chi}:\mathcal{R}\to M(\mathcal{R},\chi)_{\chi(1)\alpha/2-\alpha},\quad r\mapsto(\mathrm{f}\otimes r).v_\chi\ (r\in\mathcal{R}),$$

a linear isomorphism of vector spaces.

Remark 2.3. Since $M(\mathcal{R},\chi)_{\chi(1)\alpha/2} = \mathbb{C}v_\chi$ and $M(\mathcal{R},\chi)$ is generated by v_χ, it follows that any proper $\mathfrak{g}\otimes\mathcal{R}$-submodule of $M(\mathcal{R},\chi)$ must intersect $\mathbb{C}v_\chi$ trivially. Thus, there is a unique maximal proper $\mathfrak{g}\otimes\mathcal{R}$-submodule, denoted as $\overline{M(\mathcal{R},\chi)}$, of $M(\mathcal{R},\chi)$.

We have:

Lemma 2.4. *One has that*

$$\mathrm{Ann}_\chi(\mathcal{R}) = \psi^{-1}_{\mathcal{R},\chi}(\overline{M(\mathcal{R},\chi)}_{\chi(1)\alpha/2-\alpha}).$$

Proof. Let $v \in M(\mathcal{R},\chi)_{\chi(1)\alpha/2-\alpha}$. It is obvious that $v \in \overline{M(\mathcal{R},\chi)}$ if and only if $(\mathfrak{n}_+\otimes\mathcal{R})v = 0$. Moreover, for any $r, r' \in \mathcal{R}$, one has

$$(\mathrm{e}\otimes r)\cdot\psi_{\mathcal{R},\chi}(r') = (\mathrm{e}\otimes r)(\mathrm{f}\otimes r')\cdot v_\chi = (\mathrm{h}\otimes r\,r')\cdot v_\chi = \chi(r\,r') = (r'\cdot\chi)(r).$$

Thus we have $\psi_{\mathcal{R},\chi}(r') \in \overline{M(\mathcal{R},\chi)}_{\chi(1)\alpha/2-\alpha}$ if and only if $r' \in \mathrm{Ann}_\chi(\mathcal{R})$. □

2.3. **Irreducible quasi-finite highest weight $\mathfrak{g}\otimes\mathcal{R}$-modules.** We denote by

$$V(\mathcal{R},\chi) = M(\mathcal{R},\chi)/\overline{M(\mathcal{R},\chi)}$$

the irreducible quotient of the Verma type $\mathfrak{g}\otimes\mathcal{R}$-module $M(\mathcal{R},\chi)$. In this subsection, we give a sufficient and necessary condition for $V(\mathcal{R},\chi)$ to be quasi-finite.

Let $\mathcal{R}'$ be another commutative associative algebra, and let $\rho : \mathcal{R} \to \mathcal{R}'$ be an algebra homomorphism. For any $\mathfrak{g}\otimes\mathcal{R}'$-module W, the pull back of ρ yields a natural $\mathfrak{g}\otimes\mathcal{R}$-module structure on W with

$$(x\otimes r)\cdot w = (x\otimes\rho(r))\cdot w, \quad x\in\mathfrak{g},\ r\in\mathcal{R},\ w\in W. \tag{2.1}$$

We denote by $\rho^{-1}(W)$ the resulting $\mathfrak{g}\otimes\mathcal{R}$-module. The following result is obvious.

Lemma 2.5. *Let W be a highest weight $\mathfrak{g}\otimes\mathcal{R}'$-module with highest weight χ'. Assume that the homomorphism $\rho : \mathcal{R} \to \mathcal{R}'$ is surjective. Then $\rho^{-1}(W)$ is a highest weight $\mathfrak{g}\otimes\mathcal{R}$-module with highest weight*

$$\rho^{-1}(\chi') : \mathcal{R} \to \mathbb{C}, \quad r \mapsto \chi'(\rho(r)).$$

Set $\mathcal{R}_\chi = \mathcal{R}/\mathrm{Ann}_\chi(\mathcal{R})$. Since χ vanishes on $\mathrm{Ann}_\chi(\mathcal{R})$, it induces a linear map

$$\bar\chi : \mathcal{R}_\chi \to \mathbb{C}, \quad r + \mathrm{Ann}_\chi(\mathcal{R}) \mapsto \chi(r). \tag{2.2}$$

In view of Lemma 2.5, one has the following $\mathfrak{g}\otimes\mathcal{R}$-module isomorphism

$$V(\mathcal{R},\chi) \cong \rho^{-1}_{\mathcal{R},\chi}(V(\mathcal{R}_\chi,\bar\chi)), \tag{2.3}$$

where $\rho_{\mathcal{R},\chi} : \mathcal{R} \to \mathcal{R}_\chi$ is the quotient map. We denote by

$$\mathcal{E}(\mathcal{R}) = \{\chi\in\mathcal{R}^* \mid \dim\mathcal{R}_\chi < \infty\}. \tag{2.4}$$

Proposition 2.6. *The irreducible highest weight $\mathfrak{g}\otimes\mathcal{R}$-module $V(\mathcal{R},\chi)$ is quasi-finite if and only if $\chi\in\mathcal{E}(\mathcal{R})$.*

Proof. If $\chi\in\mathcal{E}(\mathcal{R})$, then the $\mathfrak{g}\otimes\mathcal{R}_\chi$-module $V(\mathcal{R}_\chi,\bar\chi)$ is quasi-finite and so is the $\mathfrak{g}\otimes\mathcal{R}$-module $V(\mathcal{R},\chi)$ (see (2.3)). Conversely, if the $\mathfrak{g}\otimes\mathcal{R}$-module $V(\mathcal{R},\chi)$ is quasi-finite, then one can conclude from Lemma 2.4 that

$$\begin{aligned}\dim\mathcal{R}_\chi &= \dim\mathcal{R}/\mathrm{Ann}_\chi(\mathcal{R}) = \dim M(\mathcal{R},\chi)_{\chi(1)\alpha/2-\alpha}/\overline{M(\mathcal{R},\chi)}_{\chi(1)\alpha/2-\alpha}\\ &= \dim V(\mathcal{R},\chi)_{\chi(1)\alpha/2-\alpha} < \infty.\end{aligned}$$

This finishes the proof of the proposition. □

3. The space $\mathcal{E}(\mathcal{C}_S)$

In the rest of this paper, let S be a multiplicatively closed subset of the polynomial ring $\mathcal{C} = \mathbb{C}[t]$, and let $\mathcal{C}_S$ be the localization of $\mathcal{C}$ at S. Recall that S is a subset of $\mathcal{C}$ such that $1 \in S$, $0 \notin S$ and for any $f(t), g(t) \in \mathcal{C}$, the product $f(t)g(t)$ is also in S. In this section we give a description of the space $\mathcal{E}(\mathcal{C}_S)$ (see (2.4)).

3.1. **Basics on the algebra $\mathcal{C}_S$.** Set

$$\mathbb{C}_S = \{\mu \in \mathbb{C} \mid t-\mu \text{ divides } f(t) \text{ for some } f(t) \in S\} \quad \text{and} \quad \overline{\mathbb{C}}_S = \mathbb{C} \setminus \mathbb{C}_S.$$

Notice that if the set $\mathbb{C}_S$ is empty, then $S \subset \mathcal{C}^\times$ (the set of units in $\mathcal{C}$) and $\mathcal{C}_S = \mathcal{C}$. And if the set $\mathbb{C}_S$ is nonempty, then for any $\mu \in \mathbb{C}_S$, the element $t-\mu$ has an inverse in $\mathcal{C}_S$. The following result is elementary (see [5] for example).

Lemma 3.1. *The elements*

$$t^n,\ (t-\mu)^{-m}, \quad n \in \mathbb{N},\ m \in \mathbb{Z}_+,\ \mu \in \mathbb{C}_S \tag{3.1}$$

form a basis of $\mathcal{C}_S$. Moreover, for $k \in \mathbb{N}$, $l \in \mathbb{Z}_+$ and $\mu \in \mathbb{C}_S$, we have

$$t^k\,(t-\mu)^{-l} = \sum_{m=1}^{l} \binom{k}{l-m} \mu^{k-l+m}(t-\mu)^{-m} + \sum_{n=0}^{k-l} \binom{k-n-1}{l-1} \mu^{k-l-n} t^n,$$

and, for $k, l \in \mathbb{Z}_+$, $\mu \neq \mu' \in \mathbb{C}_S$, we have

$$\begin{aligned}(t-\mu)^{-k}(t-\mu')^{-l} = &\sum_{m=1}^{k} (-1)^l \binom{k+l-m-1}{l-1} (\mu'-\mu)^{m-k-l}(t-\mu)^{-m}\\ &+ \sum_{m=1}^{l} (-1)^{l+m} \binom{k+l-m-1}{k-1} (\mu'-\mu)^{m-k-l}(t-\mu')^{-m}.\end{aligned}$$

We note that if $\mathbb{C}_S = \{a_1, a_2, \cdots, a_N\}$ is a finite set, then

$$\mathcal{C}_S = \mathbb{C}[t, (t-a_1)^{-1}, (t-a_n)^{-1}, \cdots, (t-a_N)^{-1}].$$

In this case, $\mathfrak{g} \otimes \mathcal{C}_S$ is called the $(N+1)$-point Lie algebra [5, 6]. In particular, if $N = 1$ and $a_1 = 0$, then $\mathcal{C}_S = \mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials and $\mathfrak{g} \otimes \mathcal{C}_S$ is the usual loop algebra of $\mathfrak{g}$. For convenience, we will also write

$$\mathcal{L} = \mathbb{C}[t, t^{-1}].$$

On the other hand, if $\overline{\mathbb{C}}_S = \{a\}$ consists of a single point, then $S = \mathbb{C}[t] \setminus (t-a)\mathbb{C}[t]$ and $\mathcal{C}_S$ is the local ring associated to the maximal ideal $(t-a)\mathbb{C}[t]$.

Given a linear function χ on $\mathcal{C}_S$. Since $\mathcal{C}_S$ is a principal ideal domain, there is a unique monic polynomial $p_\chi(t)$ of minimal degree such that $p_\chi(t)$ generates the annihilator ideal $\mathrm{Ann}_\chi(\mathcal{C}_S)$. We call $p_\chi(t)$ the characteristic polynomial associated to the function χ, which lies in $\langle t-\lambda \mid \lambda \in \overline{\mathbb{C}}_S\rangle$, the subalgebra of $\mathcal{C}$ generated by the polynomials $t-\lambda$, $\lambda \in \overline{\mathbb{C}}_S$.

Note that the natural inclusion

$$\mathcal{C} = \mathbb{C}[t] \to \mathcal{C}_S, \quad f(t) \mapsto \frac{f(t)}{1}$$

induces an injective homomorphism

$$\mathbb{C}[t]/p_\chi(t)\mathbb{C}[t] \to \mathcal{C}_{S,\chi} = \mathcal{C}_S/\mathrm{Ann}_\chi(\mathcal{C}_S) = \mathcal{C}_S/p_\chi(t)\mathcal{C}_S. \tag{3.2}$$

Given an element $\frac{f(t)}{g(t)} \in \mathcal{C}_S$. If $p_\chi(t) \neq 0$, then there exist suitable polynomials $a(t), b(t)$ such that $g(t)a(t)+p_\chi(t)b(t) = 1$. Let $c(t), d(t) \in \mathbb{C}[t]$ such that $\deg d(t) < \deg p_\chi(t)$ and $f(t)a(t) = p_\chi(t)c(t) + d(t)$. Then we have

$$\frac{f(t)}{g(t)} = f(t)a(t) + p_\chi(t)\frac{f(t)b(t)}{g(t)} = d(t) + p_\chi(t)\left(c(t) + \frac{f(t)b(t)}{g(t)}\right).$$

Thus, if $p_\chi(t) \neq 0$, then the map (3.2) is an isomorphism (of finite dimensional algebras). This together with Proposition 2.6 gives the following result.

Lemma 3.2. *Let $\chi \in \mathcal{C}_S^*$. Then the following conditions are equivalent:*

1. *the irreducible highest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module $V(\mathcal{C}_S, \chi)$ is quasi-finite;*
2. *the annihilator ideal $\mathrm{Ann}_\chi(\mathcal{C}_S)$ of $\mathcal{C}_S$ is nonzero;*
3. *the characteristic polynomial $p_\chi(t)$ of χ is nonzero.*

3.2. **Solving linear recurrence relations.** Before describing the space $\mathcal{E}(\mathcal{C}_S)$, here we recall the general solution of homogeneous linear recurrence relations with constant coefficients for later use.

Let $\mathrm{Z} = \mathbb{Z}$ or $\mathbb{N}$, and let

$$y : \mathrm{Z} \to \mathbb{C}, \quad n \mapsto y(n)$$

be a $\mathbb{C}$-valued function on Z. An order r ($r \geq 1$) homogeneous linear recurrence relation with constant coefficients is an equation of the form

$$c_0\, y(n) + c_1\, y(n+1) + \cdots + c_{r-1}\, y(n+r-1) + y(n+r) = 0, \quad n \in \mathrm{Z}, \tag{3.3}$$

where the coefficients c_i are all constants and $c_0 \neq 0$ if $\mathrm{Z} = \mathbb{Z}$. We denote by

$$p(t) = c_0 + c_1\, t + \cdots + c_{r-1}\, t^{r-1} + t^r$$

the characteristic polynomial of equation (3.3), and let $\lambda_1, \cdots, \lambda_\nu$ be the distinct roots of $p(t)$ with multiplicities $r_1+1, \cdots, r_\nu+1$, respectively. Note that $\lambda_1, \cdots, \lambda_\nu$ are all nonzero if $\mathrm{Z} = \mathbb{Z}$. For $i \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ ($\lambda \neq 0$ when $\mathrm{Z} = \mathbb{Z}$), set

$$\varepsilon_{i,\lambda} : \mathrm{Z} \to \mathbb{C}, \quad n \mapsto \begin{cases} \delta_{i,n}, & \text{if } \lambda = 0; \\ \lambda^n\, n^i, & \text{if } \lambda \neq 0. \end{cases}$$

The following result is elementary.

Lemma 3.3. *The general solution of the recurrence relation* (3.3) *is*

$$y = \sum_{j=1}^{\nu} \sum_{i=0}^{r_j} c_{ij}\, \varepsilon_{i,\lambda_j}, \quad c_{ij} \in \mathbb{C}. \tag{3.4}$$

Following [4], we say that a $\mathbb{C}$-valued function on Z is exp-polynomial if it is a linear combination of the basic exp-polynomials $\varepsilon_{i,\lambda}$ for some $i \in \mathbb{N}$ and $\lambda \in \mathbb{C}^\times$. When $\mathrm{Z} = \mathbb{Z}$, it follows from Lemma 3.3 that y is a solution of the recurrence relation (3.3) if and only if it is an exp-polynomial function. However, when $\mathrm{Z} = \mathbb{N}$, there exist some additional discrete solutions.

3.3. **Some exp-polynomials on $\mathbb{Z}$.** For $n, k \in \mathbb{N}$, we denote by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}\, i^n \tag{3.5}$$

the Stirling number of the second kind, which is the number of ways to partition a set of n labelled objects into k unlabelled subsets. The Stirling number can be characterized as the numbers that arise when one expresses powers of an indeterminate t in terms of the falling factorials $t^{(k)} = t\,(t-1)\,\cdots\,(t-k+1)$, i.e.,

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^{(k)}. \tag{3.6}$$

Similar to the number of binomial coefficients, the following holds true

$$\left\{ {n \atop 0} \right\} = \delta_{n,0}, \quad \left\{ {n \atop n} \right\} = 1 \quad \text{and} \quad \left\{ {n \atop k} \right\} = 0 \quad \text{if} \quad n < k. \tag{3.7}$$

For each $i \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \mu$, we define an exp-polynomial function on $\mathbb{Z}$ as follows:

$$\varepsilon_{i,\lambda,\mu} : \mathbb{Z} \to \mathbb{C}, \quad n \mapsto \begin{cases} (-\mu)^n \cdot \left(\frac{n^{(i)}}{i!} (-\mu)^{-i} \right), & \text{if } \lambda = 0, \\ (\lambda-\mu)^n \cdot \left(\sum_{j=0}^{i} \left(\frac{\lambda}{\lambda-\mu} \right)^j \left\{ {i \atop j} \right\} n^{(j)} \right), & \text{if } \lambda \neq 0, \end{cases} \tag{3.8}$$

where $n^{(i)} = t^{(i)}|_{t=n}$. By definition, $\varepsilon_{i,\lambda,\mu}$ is a linear combination of the basic exp-polynomials $\varepsilon_{j,\lambda-\mu}$, $j = 0, \cdots, i$ with a nonzero coefficient of $\varepsilon_{i,\lambda-\mu}$.

Remark 3.4. If $\mu = 0$ (and so $\lambda \neq 0$), then it follows from (3.6) that the function $\varepsilon_{i,\lambda,0}$ is just the basic exp-polynomial function $\varepsilon_{i,\lambda}$.

3.4. **The space $\mathcal{E}(\mathcal{C}_S)$.** Recall that the basis elements of $\mathcal{C}_S$ are given in (3.1). For each $i \in \mathbb{N}$ and $\lambda \in \overline{\mathbb{C}}_S$, we define a linear function $\theta_{i,\lambda}$ on $\mathcal{C}_S$ by

$$\theta_{i,\lambda}(t^n) = \begin{cases} \delta_{i,n}, & \text{if } \lambda = 0, \\ n^i\, \lambda^n, & \text{if } \lambda \neq 0, \end{cases} \qquad \theta_{i,\lambda}((t-\mu)^{-m}) = \varepsilon_{i,\lambda,\mu}(-m),$$

where $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$ and $\mu \in \mathbb{C}_S$.

Lemma 3.5. *For $i \in \mathbb{N}$, $\lambda \in \overline{\mathbb{C}}_S$ and $\mu \in \mathbb{C}_S$, we have*

$$\theta_{i,\lambda}((t-\mu)^n) = \varepsilon_{i,\lambda,\mu}(n), \quad n \in \mathbb{N}. \tag{3.9}$$

Proof. Let i, λ, μ be as in lemma. The assertion is obvious when $\lambda = 0$. Assume now that $\lambda \neq 0$. Then it follows from (3.5) that for any $j \in \mathbb{N}$,

$$\theta_{i,\lambda}((t-\lambda)^j) = \lambda^j \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} k^i = \lambda^j\, j!\, \left\{ {i \atop j} \right\}. \tag{3.10}$$

Using this and the last equality in (3.7), one gets that for any $n \in \mathbb{N}$,

$$\begin{aligned}
&\theta_{i,\lambda}((t-\mu)^n)\\
&= \theta_{i,\lambda}(((t-\lambda)+(\lambda-\mu))^n) = \theta_{i,\lambda}\left(\sum_{j=0}^{n}\binom{n}{j}(t-\lambda)^j\,(\lambda-\mu)^{n-j}\right)\\
&= \sum_{j=0}^{n}\binom{n}{j}\theta_{i,\lambda}((t-\lambda)^j)\,(\lambda-\mu)^{n-j} = \sum_{j=0}^{n}\binom{n}{j}\lambda^j\,j!\,\begin{Bmatrix} i\\ j\end{Bmatrix}(\lambda-\mu)^{n-j}\\
&= (\lambda-\mu)^n\left(\sum_{j=0}^{i} n^{(j)}\begin{Bmatrix} i\\ j\end{Bmatrix}\left(\frac{\lambda}{\lambda-\mu}\right)^j\right) = \varepsilon_{i,\lambda,\mu}(n).
\end{aligned}$$

□

Proposition 3.6. *The elements $\theta_{i,\lambda}$, $i \in \mathbb{N}$, $\lambda \in \overline{\mathbb{C}}_S$ form a basis of $\mathcal{E}(\mathcal{C}_S)$.*

Proof. Let $\chi \in \mathcal{C}_S^*$ be fixed, which induces a family

$$\{\chi_+ : \mathbb{N} \to \mathbb{C},\ \chi_\mu : \mathbb{Z} \to \mathbb{C} \mid \mu \in \mathbb{C}_S\} \tag{3.11}$$

of $\mathbb{C}$-valued functions on Z ($= \mathbb{N}$ or $\mathbb{Z}$), where

$$\chi_+(n) = \chi(t^n),\ n \in \mathbb{N}, \quad \text{and} \quad \chi_\mu(m) = \chi((t-\mu)^m),\ m \in \mathbb{Z},\ \mu \in \mathbb{C}_S.$$

Note that these functions satisfy the following compatible conditions

$$\chi_\mu(n) = \sum_{j=0}^{n}\binom{n}{j}(-\mu)^{n-j}\chi_+(j), \tag{3.12}$$

for $n \in \mathbb{N}$, $\mu \in \mathbb{C}_S$. Moreover, due to Lemma 3.5, one has

$$(\theta_{i,\lambda})_+ = \varepsilon_{i,\lambda} \quad \text{and} \quad (\theta_{i,\lambda})_\mu = \varepsilon_{i,\lambda,\mu}, \tag{3.13}$$

where $i \in \mathbb{N}$, $\lambda \in \overline{\mathbb{C}}_S$ and $\mu \in \mathbb{C}_S$.

Let $r \in \mathbb{Z}_+$, and let $f(t) = c_0 + c_1\,t + \cdots + c_{r-1}\,t^{r-1} + t^r$ be a monic polynomial in $\langle t-\lambda \mid \lambda \in \overline{\mathbb{C}}_S\rangle$ with degree r. For each $\mu \in \mathbb{C}_S$, set

$$f_\mu(t) = f(t+\mu) = c_{0,\mu} + c_{1,\mu}\,t + \cdots + c_{r-1,\mu}\,t^{r-1} + t^r.$$

Note that the constant coefficient $c_{0,\mu} \neq 0$. Furthermore, we have

$$\begin{aligned}
(f(t)\cdot\chi)(t^n) &= c_0\,\chi(t^n) + c_1\,\chi(t^{n+1}) + \cdots + \chi(t^{n+r}),\\
(f(t)\cdot\chi)((t-\mu)^m) &= c_{0,\mu}\,\chi((t-\mu)^m) + c_{1,\mu}\,\chi((t-\mu)^{m+1}) + \cdots + \chi((t-\mu)^{m+r}),
\end{aligned}$$

for $n \in \mathbb{N}$, $\mu \in \mathbb{C}_S$ and $m \in \mathbb{Z}$. This implies that the polynomial $f(t) \in \mathrm{Ann}_\chi(\mathcal{C}_S)$ if and only if the family (3.11) satisfies the following linear recurrence relations

$$c_0\,\chi_+(n) + c_1\,\chi_+(n+1) + \cdots + \chi_+(n+r) = 0,\ n \in \mathbb{N}, \tag{3.14}$$

$$c_{0,\mu}\,\chi_\mu(m) + c_{1,\mu}\,\chi_\mu(m+1) + \cdots + \chi_\mu(m+r) = 0,\ \mu \in \mathbb{C}_S,\ m \in \mathbb{Z}. \tag{3.15}$$

Now, for each $i \in \mathbb{N}$ and $\lambda \in \overline{\mathbb{C}}_S$, one can conclude from Lemma 3.3 and (3.13) that the family $\{(\theta_{i,\lambda})_+,\ (\theta_{i,\lambda})_\mu \mid \mu \in \mathbb{C}_S\}$ satisfies the relations (3.14) and (3.15) with $f(t) = (t-\lambda)^{i+1}$. So we have

$$(t-\lambda)^{i+1} \in \mathrm{Ann}_{\theta_{i,\lambda}}(\mathcal{C}_S). \tag{3.16}$$

This together with Lemma 3.2 gives that $\theta_{i,\lambda} \in \mathcal{E}(\mathcal{C}_S)$. Therefore, it remains to prove that any $\chi \in \mathcal{E}(\mathcal{C}_S)$ is a linear combination of the $\theta_{i,\lambda}$'s. Indeed, recall from

Lemma 3.2 that in this case the characteristic polynomial $p_\chi(t)$ of χ is nonzero. Observe that if $p_\chi(t)=1$, then $\chi=0$. Thus, in what follows, we assume that $p_\chi(t)$ has degree $r\geq 1$.

In this case, one notices that the family (3.11) satisfies the relations (3.14) and (3.15) (with $f(t)=p_\chi(t)$). By applying Lemma 3.3 to the relation (3.14), we get that $\chi_+=\sum_{i=1}^{\nu}\sum_{j=0}^{r_i} c_{ij}\,\varepsilon_{i,\lambda_j}$ for some $\lambda_j\in\overline{\mathbb{C}}_S$, $r_i\in\mathbb{N}$ and $c_{ij}\in\mathbb{C}$. Moreover, for any $\mu\in\mathbb{C}_S$, χ_μ is the unique solution of the linear recurrence relation (3.15) with the initial condition (3.12). Therefore, one easily checks that $\chi_\mu=\sum_{i=1}^{\nu}\sum_{j=0}^{r_i} c_{ij}\,\varepsilon_{i,\lambda_j,\mu}$. In summary, χ has the following desired expression

$$\chi=\sum_{i=1}^{\nu}\sum_{j=0}^{r_i} c_{ij}\,\theta_{i,\lambda_j}. \tag{3.17}$$

□

Remark 3.7. When $\mathcal{C}_S=\mathcal{L}$, Proposition 3.6 was proved in [40] (see also [4]). In this case, it follows from Remark 3.4 that the elements

$$\theta_{i,\lambda}:\mathcal{L}\to\mathbb{C},\quad t^n\mapsto n^i\,\lambda^n\quad (i\in\mathbb{N},\,\lambda\in\mathbb{C}^\times)$$

form a basis of $\mathcal{E}(\mathcal{L})$.

Remark 3.8. For $i\in\mathbb{N}$ and $\lambda\in\overline{\mathbb{C}}_S$, by using (3.10) and the second equality in (3.7), one obtains that $\theta_{i,\lambda}((t-\lambda)^i)=\lambda^i\,i!\neq 0$. This together with (3.16) implies that the characteristic polynomial $p_{\theta_{i,\lambda}}(t)$ of $\theta_{i,\lambda}$ is $(t-\lambda)^{i+1}$. In general, if χ is as in (3.17), then $p_\chi(t)=(t-\lambda_1)^{r_1+1}(t-\lambda_2)^{r_2+1}\cdots(t-\lambda_\nu)^{r_\nu+1}$.

4. New realization of $\mathfrak{g}\otimes\mathcal{C}_S$-modules

For $r\in\mathbb{N}$, let $\mathcal{P}_r=\mathbb{C}[y_0,y_1,\cdots,y_r]$ be the polynomial ring in the variables $y_0,y_1,\cdots,y_r$. In this section, by using the idea of free fields, we realize a class of quasi-finite highest weight $\mathfrak{g}\otimes\mathcal{C}_S$-modules on the Fock space $\mathcal{P}_r$.

4.1. **The main construction.**

Throughout this section, we fix a triple

$$(f(t),\lambda,r)\in\mathcal{C}\times\overline{\mathbb{C}}_S\times\mathbb{N}\quad\text{with}\quad r\geq\deg f(t). \tag{4.1}$$

We will first construct a Fock $\mathfrak{g}\otimes\mathcal{C}_S$-module associated to the triple (4.1) (see Theorem 4.1) and then give a characterization of this $\mathfrak{g}\otimes\mathcal{C}_S$-module (see Theorem 4.2). The proof of Theorems 4.1 and 4.2 will be given in next three subsections.

We start with some notations. Firstly, as in Section 1, let

$$\ell_{j,r}(t)=\prod_{0\leq i\neq j\leq r}\frac{t-i}{j-i},\quad j=0,1,\cdots,r, \tag{4.2}$$

stand for the fundamental Lagrange interpolation polynomials of degree r at the points $j=0,1,\cdots,r$. Next, for $n,j\in\mathbb{N}$ with $j\leq r$, we define

$$\ell_{j,r}^{\lambda,n}:\mathbb{N}\to\mathbb{C},\quad m\mapsto\begin{cases}\delta_{n+m,j}, & \text{if }\lambda=0,\\ \lambda^n\cdot\ell_{j,r}(n+m), & \text{if }\lambda\neq 0.\end{cases} \tag{4.3}$$

Finally, for $j=0,1,\cdots,r$ and $\mu\in\mathbb{C}_S$, we set

$$y_{j,\mu}=y_{j,\mu}^{\lambda}=\begin{cases}\sum_{k=0}^{j}\binom{j}{k}(-\mu)^{-k}y_k, & \text{if }\lambda=0,\\ \sum_{k=0}^{j}\binom{j}{k}\frac{\lambda^k(-\mu)^{j-k}}{(\lambda-\mu)^j}y_k, & \text{if }\lambda\neq 0.\end{cases} \tag{4.4}$$

By definition, for any $\mu \in \mathbb{C}_S$, the elements $y_{j,\mu}$ also form a basis of $\mathcal{P}_r$. That is,

$$\mathbb{C}[y_{0,\mu}, y_{1,\mu}, \cdots, y_{r,\mu}] = \mathbb{C}[y_0, y_1, \cdots, y_r].$$

We define a linear map $\theta_{f(t),\lambda} : \mathcal{C}_S \to \mathbb{C}$ by

$$\theta_{f(t),\lambda} = c_0\,\theta_{0,\lambda} + c_1\,\theta_{1,\lambda} + \cdots + c_s\,\theta_{s,\lambda} \quad \text{if} \quad f(t) = c_0 + c_1\,t + \cdots + c_s\,t^s. \tag{4.5}$$

Using Remark 3.8, one knows that the characteristic polynomial of $\theta_{f(t),\lambda}$ is

$$p_{\theta_{f(t),\lambda}}(t) = (t-\lambda)^{\deg(f(t))+1}. \tag{4.6}$$

Let $(t-\lambda)^{r+1}\mathcal{C}_S$ be the ideal of $\mathcal{C}_S$ generated by $(t-\lambda)^{r+1}$, $\mathcal{C}_{S,\lambda,r}$ the quotient algebra of $\mathcal{C}_S$ modulo the ideal $(t-\lambda)^{r+1}\mathcal{C}_S$, and $\rho_{\lambda,r} : \mathcal{C}_S \to \mathcal{C}_{S,\lambda,r}$ the quotient map. From (4.6), it follows that $\theta_{f(t),\lambda}$ vanishes on $(t-\lambda)^{r+1}\mathcal{C}_S$. So, it induces a linear function

$$\bar{\theta}_{f(t),\lambda,r} : \mathcal{C}_{S,\lambda,r} \to \mathbb{C}, \quad x + (t-\lambda)^{r+1}\mathcal{C}_S \mapsto \theta_{f(t),\lambda}(x), \quad x \in \mathcal{C}_S. \tag{4.7}$$

Furthermore, we define a linear map

$$\phi_{f(t),\lambda} : \mathfrak{g} \otimes \mathcal{C}_S \to \operatorname{End}(\mathcal{P}_r) \tag{4.8}$$

by the rules

$$\phi_{f(t),\lambda}(\mathrm{f} \otimes t^n) = \sum_{j=0}^{r} \ell_{j,r}^{\lambda,n}(0)\, y_j,$$

$$\phi_{f(t),\lambda}(\mathrm{h} \otimes t^n) = \theta_{f(t),\lambda}(t^n) - 2\sum_{j=0}^{r}\sum_{j'=0}^{r} \ell_{j',r}^{\lambda,n}(j)\, y_{j'}\, \frac{\partial}{\partial y_j},$$

$$\phi_{f(t),\lambda}(\mathrm{e} \otimes t^n) = \sum_{j=0}^{r} \left(\theta_{f(t),\lambda}(t^{n+j}) - \sum_{j'=0}^{r}\sum_{j''=0}^{r} \ell_{j'',r}^{\lambda,n}(j+j') y_{j''} \frac{\partial}{\partial y_{j'}}\right) \frac{\partial}{\partial y_j},$$

$$\phi_{f(t),\lambda}(\mathrm{f} \otimes (t-\mu)^m) = \sum_{j=0}^{r} (\lambda-\mu)^m\, \ell_{j,r}(m)\, y_{j,\mu},$$

$$\phi_{f(t),\lambda}(\mathrm{h} \otimes (t-\mu)^m) = \theta_{f(t),\lambda}((t-\mu)^m) - 2\sum_{j=0}^{r}\sum_{j'=0}^{r} (\lambda-\mu)^m \ell_{j',r}(m+j) y_{j',\mu} \frac{\partial}{\partial y_{j,\mu}},$$

$$\begin{aligned}\phi_{f(t),\lambda}(\mathrm{e} \otimes (t-\mu)^m) &= \sum_{j=0}^{r} \theta_{f(t),\lambda}((t-\mu)^{m+j})\, \frac{\partial}{\partial y_{j,\mu}} \\ &\quad - \sum_{j=0}^{r}\sum_{j'=0}^{r}\sum_{j''=0}^{r} (\lambda-\mu)^m\, \ell_{j'',r}(m+j+j')\, y_{j'',\mu} \frac{\partial}{\partial y_{j',\mu}} \frac{\partial}{\partial y_{j,\mu}},\end{aligned}$$

where $n \in \mathbb{N}$, $-m \in \mathbb{Z}_+$ and $\mu \in \mathbb{C}_S$. Then we have the following result.

Theorem 4.1. *Let $f(t) \in \mathcal{C}$, $\lambda \in \overline{\mathbb{C}}_S$ and $r \in \mathbb{N}$ with $r \geq \deg(f(t))$. Then the linear map $\phi_{f(t),\lambda} : \mathfrak{g} \otimes \mathcal{C}_S \to \operatorname{End}(\mathcal{P}_r)$ is a Lie algebra homomorphism.*

Denote by

$$W(\mathcal{C}_S, f(t), \lambda, r) = (\phi_{f(t),\lambda}, \mathcal{P}_r)$$

the resulting $\mathfrak{g} \otimes \mathcal{C}_S$-module given in Theorem 4.1. Recall from Lemma 2.5 that the pull back $\rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r}, \bar{\theta}_{f(t),\lambda,r}))$ of the Verma type $\mathfrak{g} \otimes \mathcal{C}_{S,\lambda,r}$-module $M(\mathcal{C}_{S,\lambda,r}, \bar{\theta}_{f(t),\lambda,r})$ is a highest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module with highest weight $\theta_{f(t),\lambda}$. Then we have the following characterization of the $\mathfrak{g} \otimes \mathcal{C}_S$-module $W(\mathcal{C}_S, f(t), \lambda, r)$.

Theorem 4.2. *Let $f(t) \in \mathcal{C}$, $\lambda \in \overline{\mathbb{C}}_S$ and $r \in \mathbb{N}$ with $r \geq \deg(f(t))$. Then the $\mathfrak{g} \otimes \mathcal{C}_S$-module $W(\mathcal{C}_S, f(t), \lambda, r)$ is isomorphic to $\rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r}, \bar{\theta}_{f(t),\lambda,r}))$.*

Remark 4.3. By specifying $\mathcal{C}_S$ to $\mathcal{C}$, we have the quotient algebra

$$\mathcal{C}_{\lambda,r} = \mathbb{C}[t]/(t-\lambda)^{r+1}\mathbb{C}[t]$$

of $\mathcal{C}$. Note that the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}_S$ induces a canonical algebra isomorphism $\mathcal{C}_{\lambda,r} \cong \mathcal{C}_{S,\lambda,r}$. Moreover, under this isomorphism, the function $\bar{\theta}_{f(t),\lambda,r}$ on $\mathcal{C}_{S,\lambda,r}$ coincides with that on $\mathcal{C}_{\lambda,r}$. In view of this, we have the $\mathfrak{g} \otimes \mathcal{C}$-module isomorphism

$$\mathrm{Res}_{\mathfrak{g}\otimes\mathcal{C}}^{\mathfrak{g}\otimes\mathcal{C}_S} \ \rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r}, \bar{\theta}_{f(t),\lambda,r})) \cong \rho_{\lambda,r}^{-1}(M(\mathcal{C}_{\lambda,r}, \bar{\theta}_{f(t),\lambda,r})), \tag{4.9}$$

where the notation $\mathrm{Res}_{\mathfrak{g}\otimes\mathcal{C}}^{\mathfrak{g}\otimes\mathcal{C}_S}$ stands for the restriction functor from the category of $\mathfrak{g} \otimes \mathcal{C}_S$-modules to the category of $\mathfrak{g} \otimes \mathcal{C}$-modules.

4.2. **The case $\mathcal{C}_S = \mathcal{L}$.** Here we give the proof of Theorems 4.1 and 4.2 for the case that $\mathcal{C}_S = \mathcal{L}$. So, throughout this subsection, we assume that $\mathcal{C}_S = \mathcal{L}$. Note that in this case $\lambda \in \overline{\mathbb{C}}_S = \mathbb{C}^\times$ is nonzero and $\mu \in \mathbb{C}_S$ is zero. Furthermore, by definition one has $y_{j,0} = y_j$ for all $j = 0, 1, \cdots, r$. This gives that the map $\phi_{f(t),\lambda} : \mathfrak{g} \otimes \mathcal{L} \to \mathrm{End}(\mathcal{P}_r)$ coincides with that defined in (1.1).

In what follows we are going to prove that the action (1.1) indeed gives a $\mathfrak{g} \otimes \mathcal{L}$-module structure on $\mathcal{P}_r$. We would use the following simple result frequently.

Lemma 4.4. *Let $g(t)$ be any polynomial of degree $\leq r$. Then one has that*

$$g(m+n) = \sum_{j=0}^{r} g(m+j)\, \ell_{j,r}(n), \quad m, n \in \mathbb{Z}. \tag{4.10}$$

Proof. It is well-known that for any polynomial $h(t)$ of degree $\leq r$, one can decompose $h(t)$ as a linear combination of the Lagrange polynomials $\ell_{j,r}(t)$ as follows

$$h(t) = \sum_{j=0}^{r} h(j)\, \ell_{j,r}(t). \tag{4.11}$$

By taking $h(t) = g(m+t)$, $m \in \mathbb{Z}$ in (4.11), we obtain the formula (4.10). $\square$

For $n \in \mathbb{Z}$, we set

$$A(n) = \sum_{i=0}^{r} \ell_{i,r}(n)\, y_i, \quad B(n) = \sum_{i,j=0}^{r} \ell_{j,r}(n+i)\, y_j \frac{\partial}{\partial y_i},$$

$$C(n) = \sum_{i,j,k=0}^{r} \ell_{k,r}(n+j+i)\, y_k \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}, \quad D(n) = \sum_{i=0}^{r} f(n+i) \frac{\partial}{\partial y_i}.$$

The following result is about the commutator relations of these operators.

Lemma 4.5. *For $m, n \in \mathbb{Z}$, one has that*

$$[A(m), A(n)] = 0, \quad [D(m), D(n)] = 0, \quad [B(m), A(n)] = A(m+n), \tag{4.12}$$

$$[C(m), A(n)] = 2B(m+n), \qquad [D(m), A(n)] = f(m+n), \tag{4.13}$$

$$[B(m), B(n)] = 0, \qquad [C(m), B(n)] = C(m+n), \tag{4.14}$$

$$[C(m), C(n)] = 0, \qquad [D(m), B(n)] = D(m+n), \tag{4.15}$$

$$[D(m), C(n)] = \sum_{i,j=0}^{r} f(m+n+i+j) \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}. \tag{4.16}$$

Proof. The first two equalities in (4.12) are trivial and for the third one, we have

$$
\begin{aligned}
[B(m), A(n)] &= [\sum_{i,j=0}^{r} \ell_{j,r}(m+i)\, y_j \,\frac{\partial}{\partial y_i}, \sum_{k=0}^{r} \ell_{k,r}(n)\, y_k] \\
&= \sum_{i,j,k=0}^{r} \ell_{j,r}(m+i)\, \ell_{k,r}(n)\, y_j\, [\frac{\partial}{\partial y_i}, y_k] = \sum_{i,j=0}^{r} \ell_{j,r}(m+i)\, \ell_{i,r}(n)\, y_j \\
&= \sum_{j=0}^{r} \left(\sum_{i=0}^{r} \ell_{j,r}(m+i)\, \ell_{i,r}(n) \right) y_j = \sum_{j=0}^{r} \ell_{j,r}(m+n)\, y_j = A(m+n),
\end{aligned}
$$

where in the second last equality we used the formula (4.10) with $g(t) = \ell_{j,r}(t)$.

The second equality in (4.13) is implied by (4.10) and the assumption that $\deg(f(t)) \le r$. For the first one, we have

$$
\begin{aligned}
[C(m), A(n)] &= [\sum_{j,k=0}^{r} \ell_{k,r}(m+i+j)\, y_k \,\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}, \sum_{l=0}^{r} \ell_{l,r}(n)\, y_l] \\
&= \sum_{i,j,k=0}^{r} \ell_{k,r}(m+i+j)\, y_k \left(\ell_{i,r}(n)\, \frac{\partial}{\partial y_j} + \ell_{j,r}(n)\, \frac{\partial}{\partial y_i} \right) \\
&= \sum_{j,k=0}^{r} \ell_{k,r}(m+n+j)\, y_k \,\frac{\partial}{\partial y_j} + \sum_{i,k=0}^{r} \ell_{k,r}(m+n+i)\, y_k \,\frac{\partial}{\partial y_i} = 2B(m+n),
\end{aligned}
$$

where in the second last equality we used the formula (4.10) with $g(t) = \ell_{k,r}(t+j)$ and $\ell_{k,r}(t+i)$, respectively.

For the second equality in (4.14), one has

$$
\begin{aligned}
[C(m), B(n)] &= [\sum_{i,j,k=0}^{r} \ell_{k,r}(m+i+j)\, y_k \,\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i}, \sum_{i',j'=0}^{r} \ell_{j',r}(n+i')\, y_{j'} \,\frac{\partial}{\partial y_{i'}}] \\
&= \sum_{i',j',i,j,k=0}^{r} \ell_{k,r}(m+i+j)\, \ell_{j',r}(n+i') \Big(y_k\, [\frac{\partial}{\partial y_j}, y_{j'}] \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_{i'}} \\
&\qquad + y_k \,\frac{\partial}{\partial y_j} [\frac{\partial}{\partial y_i}, y_{j'}] \frac{\partial}{\partial y_{i'}} + y_{j'}\, [y_k, \frac{\partial}{\partial y_{i'}}]\, \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} \Big) \\
&= \sum_{i',i,j,k=0}^{r} \ell_{k,r}(m+i+j) \left(\ell_{j,r}(n+i')\, y_k \,\frac{\partial}{\partial y_{i'}}\frac{\partial}{\partial y_i} + \ell_{i,r}(n+i')\, y_k \,\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_{i'}} \right) \\
&\quad - \sum_{j',i,j,k=0}^{r} \ell_{j',r}(n+k)\, \ell_{k,r}(m+i+j) y_{j'} \,\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} \\
&= \sum_{i',i,k=0}^{r} \ell_{k,r}(n+m+i+i') y_k \,\frac{\partial}{\partial y_{i'}}\frac{\partial}{\partial y_i} = C(m+n),
\end{aligned}
$$

where in the second last equality we used the formula (4.10) with $g(t) = \ell_{k,r}(t+i), \ell_{k,r}(t+j)$ and $\ell_{j',r}(t)$, respectively. The remaining equalities in lemma can be checked in a similar way, and we omit the details. ☐

Now, by using the commutator relations given in Lemma 4.5, one can verify that $(\phi_{f(t),\lambda}, \mathcal{P}_r)$ is a representation of $\mathfrak{g} \otimes \mathcal{L}$. This finishes the proof of Theorem 4.1

with $\mathcal{C}_S = \mathcal{L}$. By specifying $\mathcal{C}_S$ to $\mathcal{L}$, we have the quotient algebra $\mathcal{L}_{\lambda,r}$ of $\mathcal{L}$ and the $\mathfrak{g}\otimes\mathcal{L}$-module $\rho_{\lambda,r}^{-1}(M(\mathcal{L}_{\lambda,r},\bar{\theta}_{f(t),\lambda,r}))$.

Lemma 4.6. *$W(\mathcal{L}, f(t), \lambda, r)$ is a highest weight $\mathfrak{g}\otimes\mathcal{L}$-module with highest weight $\theta_{f(t),\lambda}$ and highest weight vector 1.*

Proof. The lemma is implied by the facts that

$$\mathrm{e}\otimes t^n.1 = 0, \quad \mathrm{h}\otimes t^n.1 = \theta_{f(t),\lambda}(t^n)1, \quad n\in\mathbb{Z},$$

and that

$$\mathrm{f}\otimes t^j.1 = \lambda^j\, y_j, \quad j = 0,1,\cdots,r. \tag{4.17}$$

□

Lemma 4.7. *One has that* $\ker\phi_{f(t),\lambda} = \mathfrak{g}\otimes((t-\lambda)^{r+1})$.

Proof. Note that $\ker\phi_{f(t),\lambda}$ is an ideal of $\mathfrak{g}\otimes\mathcal{L}$ and so it has the form $\mathfrak{g}\otimes I$ for some ideal I of $\mathcal{L}$. From (4.17), one knows that the elements $1, t,\cdots, t^r$ are not contained in I. Thus, we now only need to prove that $(t-\lambda)^{r+1}\in I$.

Observe that one can conclude from (3.5) and (3.7) that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\,k^m = 0, \quad \forall\, m,n\in\mathbb{N} \text{ with } n>m.$$

This implies that for any polynomial $g(t)$ with degree $< n$, one has

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\,g(k) = 0. \tag{4.18}$$

Using this formula, we obtain

$$\phi_{f(t),\lambda}(\mathrm{f}\otimes(t-\lambda)^{r+1}) = \sum_{k=0}^{r+1}\binom{r+1}{k}(\phi_{f(t),\lambda}(\mathrm{f}\otimes t^k))(-\lambda)^{r+1-k}$$
$$= \lambda^{r+1}\sum_{i=0}^{r}\left(\sum_{k=0}^{r+1}\binom{r+1}{k}(-1)^{r+1-k}\,\ell_{i,r}(k)\right)y_i = 0.$$

Similarly, one easily checks that

$$\phi_{f(t),\lambda}(\mathrm{h}\otimes(t-\lambda)^{r+1}) = 0 \quad\text{and}\quad \phi_{f(t),\lambda}(\mathrm{e}\otimes(t-\lambda)^{r+1}) = 0,$$

by applying (4.18) with $g(t) = f(t), \ell_{j,r}(t+i)$ and $f(t+i), \ell_{k,r}(t+i+j)$ $(i,j,k = 0,1,\cdots,r)$, respectively. We omit the details. □

Now we are ready to complete the proof of Theorem 4.2 with $\mathcal{C}_S = \mathcal{L}$. In view of Lemma 4.6 and Lemma 4.7, the $\mathfrak{g}\otimes\mathcal{L}$-module $W(\mathcal{L}, f(t), \lambda, r)$ becomes a highest weight $\mathfrak{g}\otimes\mathcal{L}_{\lambda,r}$-module with highest weight $\bar{\theta}_{f(t),\lambda,r}$ and highest weight vector 1. So we have a surjective $\mathfrak{g}\otimes\mathcal{L}_{\lambda,r}$-module homomorphism

$$M(\mathcal{L}_{\lambda,r},\bar{\theta}_{f(t),\lambda,r}) \to W(\mathcal{L}, f(t),\lambda,r), \quad v_{\bar{\theta}_{f(t),\lambda,r}} \mapsto 1. \tag{4.19}$$

Notice that $\mathcal{U}(\mathfrak{n}^-\otimes\mathcal{L}_{\lambda,r}) \cong \mathcal{P}_r$ as commutative algebras. This leads to the fact that the $\mathfrak{g}\otimes\mathcal{L}_{\lambda,r}$-module homomorphism (4.19) is an isomorphism. Furthermore, one can check that this $\mathfrak{g}\otimes\mathcal{L}_{\lambda,r}$-module isomorphism induces a $\mathfrak{g}\otimes\mathcal{L}$-module isomorphism from $\rho_{\lambda,r}^{-1}(M(\mathcal{L}_{\lambda,r},\bar{\theta}_{f(t),\lambda,r}))$ to $W(\mathcal{L}, f(t),\lambda,r)$. This completes the proof.

4.3. **The case $\mathcal{C}_S = \mathcal{C}$.** In this subsection, we give the proof of Theorems 4.1 and 4.2 for the case that $\mathcal{C}_S = \mathcal{C}$. From now on, in this subsection we assume that $\mathcal{C}_S = \mathcal{C}$ and so $\lambda \in \overline{\mathbb{C}}_S = \mathbb{C}$. Assume first that $\lambda \neq 0$. In this case, we have

$$W(\mathcal{C}, f(t), \lambda, r) = \mathrm{Res}_{\mathfrak{g}\otimes\mathcal{C}}^{\mathfrak{g}\otimes\mathcal{L}} W(\mathcal{L}, f(t), \lambda, r).$$

Moreover, it follows from (4.9) that

$$\rho_{\lambda,r}^{-1}(M(\mathcal{C}_{\lambda,r}, \bar{\theta}_{f(t),\lambda,r})) \cong \mathrm{Res}_{\mathfrak{g}\otimes\mathcal{C}}^{\mathfrak{g}\otimes\mathcal{L}} \rho_{\lambda,r}^{-1}(M(\mathcal{L}_{\lambda,r}, \bar{\theta}_{f(t),\lambda,r})).$$

Thus, when $\lambda \neq 0$, the proof of Theorems 4.1 and 4.2 can be reduced to the case that $\mathcal{C}_S = \mathcal{L}$. So we only need to consider the case that $\lambda = 0$. Note that in this case the linear map $\phi_{f(t),\lambda} : \mathfrak{g} \otimes \mathcal{C} \to \mathrm{End}(\mathcal{P}_r)$ is given as follows

$$\begin{aligned}
&\phi_{f(t),0}(\mathrm{f} \otimes t^n) = \sum_{i=0}^{r} \delta_{n,i}\, y_i,\\
&\phi_{f(t),0}(\mathrm{h} \otimes t^n) = \theta_{f(t),0}(t^n) - 2\sum_{i=0}^{r}\sum_{j=0}^{r} \delta_{n+i,j}\, y_j \frac{\partial}{\partial y_i},\\
&\phi_{f(t),0}(\mathrm{e} \otimes t^n) = \sum_{i=0}^{r} \left(\theta_{f(t),0}(t^n) - \sum_{j=0}^{r}\sum_{k=0}^{r} \delta_{n+i+j,k}\, y_k \frac{\partial}{\partial y_j}\right) \frac{\partial}{\partial y_i},
\end{aligned} \tag{4.20}$$

where $n \in \mathbb{N}$.

In order to verify that the action (4.20) determines a $\mathfrak{g} \otimes \mathcal{C}$-module structure on $\mathcal{P}_r$, we need the following analog of Lemma 4.4.

Lemma 4.8. *For $k, r \in \mathbb{N}$ with $k \leq r$, one has*

$$\delta_{m+n,k} = \sum_{j=0}^{r} \delta_{m+j,k}\, \delta_{n,j}, \quad m, n \in \mathbb{N}. \tag{4.21}$$

Proof. The equality (4.21) is trivial when $r \geq n$. And, for the case that $r < n$, one only needs to notice that both sides of the equality (4.21) are zero. □

Similar to the proof of Lemma 4.5, by using Lemma 4.8, one can compute the commutator relations of those operators that appeared in (4.20). According to these commutator relations, one can verify that $(\phi_{f(t),0}, \mathcal{P}_r)$ is indeed a representation of $\mathfrak{g} \otimes \mathcal{C}$ and we omit the details. Moreover, it is easy to see that the $\mathfrak{g} \otimes \mathcal{C}$-module $W(\mathcal{C}, f(t), 0, r)$ is a highest weight $\mathfrak{g} \otimes \mathcal{C}$-module with highest weight $\theta_{f(t),0}$ and highest weight vector 1.

Now, by definition, we have

$$\phi_{f(t),0}(\mathrm{f} \otimes t^n) = \phi_{f(t),0}(\mathrm{h} \otimes t^n) = \phi_{f(t),0}(\mathrm{e} \otimes t^n) = 0 \quad \text{for } n \geq r+1.$$

This implies that $\ker \phi_{f(t),0} = \mathfrak{g} \otimes (t^{r+1}\mathcal{C})$ and so $W(\mathcal{C}, f(t), 0, r)$ becomes a $\mathfrak{g} \otimes \mathcal{C}_{0,r}$-module, which is isomorphic to the Verma type module $M(\mathcal{C}_{0,r}, \bar{\theta}_{f(t),0,r})$. Finally, we can extend this $\mathfrak{g} \otimes \mathcal{C}_{0,r}$-module isomorphism to a $\mathfrak{g} \otimes \mathcal{C}$-module isomorphism from $W(\mathcal{C}, f(t), 0, r)$ to $\rho_{0,r}^{-1}(M(\mathcal{C}_{0,r}, \bar{\theta}_{f(t),0,r}))$. This finishes the proof of Theorems 4.1 and 4.2 with $\mathcal{C}_S = \mathcal{C}$.

4.4. **The general case.** Based on the special cases proved in last two subsections, in this subsection we will complete the proof of Theorems 4.1 and 4.2.

Recall from (4.9) and Section 4.3 that

$$\mathrm{Res}^{\mathfrak{g}\otimes\mathcal{C}_S}_{\mathfrak{g}\otimes\mathcal{C}}\, \rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r},\bar{\theta}_{f(t),\lambda,r})) \cong \rho_{\lambda,r}^{-1}(M(\mathcal{C}_{\lambda,r},\bar{\theta}_{f(t),\lambda,r})) \cong W(\mathcal{C},f(t),\lambda,r).$$

This implies that there is a $\mathfrak{g}\otimes\mathcal{C}_S$-module structure on $\mathcal{P}_r$, denoted as $\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r)$, transferring from $\rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r},\bar{\theta}_{f(t),\lambda,r}))$ such that

$$\mathrm{Res}^{\mathfrak{g}\otimes\mathcal{C}_S}_{\mathfrak{g}\otimes\mathcal{C}}\, \mathcal{P}(\mathcal{C}_S,f(t),\lambda,r) = W(\mathcal{C},f(t),\lambda,r). \tag{4.22}$$

In what follows, we will prove that for $x\in\mathfrak{g}$, $\mu\in\mathbb{C}_S$ and $-m\in\mathbb{Z}_+$,

(∗) the action of $x\otimes(t-\mu)^m$ on $\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r)$ is given by $\phi_{f(t),\lambda}(x\otimes(t-\mu)^m)$.

Note that Theorems 4.1 and 4.2 are implied by this assertion.

Firstly, we fix a $\mu\in\mathbb{C}_S$ and introduce an algebra embedding

$$\zeta_\mu:\mathcal{L}\hookrightarrow\mathcal{C}_S,\quad t^m\mapsto(t-\mu)^m,\ m\in\mathbb{Z}. \tag{4.23}$$

Associated to $(f(t),\mu)$, we define a polynomial $f_\mu(t)$ as follows: if $f(t)=\sum_{i=0}^s c_i\,t^i$, then $f_\mu(t)=\sum_{i=0}^s c_{i,\mu}\,t^i$ with the coefficients $c_{i,\mu}$ defined by the rule

$$\sum_{i=0}^s c_{i,\mu}\,\varepsilon_{i,\lambda-\mu}=\sum_{i=0}^s c_i\,\varepsilon_{i,\lambda,\mu}\quad\text{(see (3.8))}.$$

Note that the degree of $f_\mu(t)$ equals the degree of $f(t)$, and the diagram

$$\begin{tikzcd} \mathcal{C}_S \arrow[dr, "\theta_{f(t),\lambda}"] & \\ & \mathbb{C} \\ \mathcal{L} \arrow[uu, "\zeta_\mu"] \arrow[ur, "\theta_{f_\mu(t),\lambda-\mu}"'] & \end{tikzcd} \tag{4.24}$$

commutes.

Now the algebra homomorphism $\rho_{\lambda,r}\circ\zeta_\mu:\mathcal{L}\hookrightarrow\mathcal{C}_S\to\mathcal{C}_{S,\lambda,r}$ factors through the ideal $(t-(\lambda-\mu))^r\mathcal{L}$ of $\mathcal{L}$. Therefore, it yields a surjective algebra homomorphism $\zeta_{\mu,\lambda,r}:\mathcal{L}_{\lambda-\mu,r}\to\mathcal{C}_{S,\lambda,r}$. Moreover, using this homomorphism, the diagram (4.24) can be extended to the following commutating diagram:

$$\begin{tikzcd} \mathcal{C}_S \arrow[rr, "\rho_{\lambda,r}"] \arrow[dr, "\theta_{f(t),\lambda}"'] & & \mathcal{C}_{S,\lambda,r} \arrow[dl, "\bar{\theta}_{f(t),\lambda,r}"'] \\ & \mathbb{C} & \\ \mathcal{L} \arrow[uu, "\zeta_\mu"] \arrow[ur, "\theta_{f_\mu(t),\lambda-\mu}"'] \arrow[rr, "\rho_{\lambda-\mu,r}"'] & & \mathcal{L}_{\lambda-\mu,r} \arrow[uu, "\zeta_{\mu,\lambda,r}"'] \arrow[ul, "\bar{\theta}_{f_\mu(t),\lambda-\mu,r}"] \end{tikzcd}.$$

Since $\dim\mathcal{C}_{S,\lambda,r}=r+1=\dim\mathcal{L}_{\lambda-\mu,r}$, the surjective homomorphism $\zeta_{\mu,\lambda,r}$ is in fact an isomorphism. This together with the above commutating diagram gives the following $\mathfrak{g}\otimes\mathcal{L}$-module isomorphisms:

$$\begin{aligned}\zeta_\mu^{-1}(\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r)) &\cong \zeta_\mu^{-1}(\rho_{\lambda,r}^{-1}(M(\mathcal{C}_{S,\lambda,r},\bar\theta_{f(t),\lambda,r})))\\ \cong (\rho_{\lambda,r}\circ\zeta_\mu)^{-1}(M(\mathcal{C}_{S,\lambda,r},\bar\theta_{f(t),\lambda,r})) &\cong \rho_{\lambda-\mu,r}^{-1}(M(\mathcal{L}_{\lambda-\mu,r},\bar\theta_{f_\mu(t),\lambda-\mu,r})).\end{aligned} \tag{4.25}$$

Since $\deg f_\mu(t)\le r$ and $\lambda-\mu\neq 0$, it follows from Section 4.2 that there is a $\mathfrak{g}\otimes\mathcal{L}$-module structure $W(\mathcal{L},f_\mu(t),\lambda-\mu,r)$ on $\mathcal{P}_r$ that is isomorphic to $\rho_{\lambda-\mu,r}^{-1}(M(\mathcal{L}_{\lambda-\mu,r},\bar\theta_{f_\mu(t),\lambda-\mu,r}))$. By (4.25), this gives us a $\mathfrak{g}\otimes\mathcal{L}$-module isomorphism

$$\psi_\mu:\zeta_\mu^{-1}(\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r))\to W(\mathcal{L},f_\mu(t),\lambda-\mu,r)$$

such that $\psi_\mu(1)=1$. We claim that

$$\psi_\mu(y_{j_1,\mu}^{n_1}\cdots y_{j_k,\mu}^{n_k})=y_{j_1}^{n_1}\cdots y_{j_k}^{n_k}\quad\text{for } 0\le j_1,\cdots,j_k\le r,\ n_1,\cdots,n_k\in\mathbb{N}. \tag{4.26}$$

Indeed, for each $j=0,1,\cdots,r$, let us set

$$\mathrm{f}_j=(\lambda-\mu)^{-j}(\mathrm{f}\otimes t^j)\in\mathfrak{g}\otimes\mathcal{L}.$$

Then the element f_j acts on $W(\mathcal{L},f_\mu(t),\lambda-\mu,r)$ as the operator y_j. On the other hand, as $\zeta_\mu(t^j)=(t-\mu)^j\in\mathcal{C}$, it follows from (4.22) that the element f_j acts on $\zeta_\mu^{-1}(\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r))$ as the operator

$$(\lambda-\mu)^{-j}\phi_{f(t),\lambda}(\mathrm{f}\otimes(t-\mu)^j)=\begin{cases}\sum_{k=0}^j\binom{j}{k}(-\mu)^{-k}y_k, & \text{if } \lambda=0,\\ \sum_{k=0}^j\binom{j}{k}\frac{\lambda^k(-\mu)^{j-k}}{(\lambda-\mu)^j}y_k, & \text{if } \lambda\neq 0.\end{cases}$$

So f_j acts on $\zeta_\mu^{-1}(\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r))$ as the operator $y_{j,\mu}$, which implies

$$\begin{aligned}\psi_\mu(y_{j_1,\mu}^{n_1}\cdots y_{j_k,\mu}^{n_k})=\psi_\mu(\mathrm{f}_{j_1}^{n_1}\cdots\mathrm{f}_{j_k}^{n_k}.1)=\mathrm{f}_{j_1}^{n_1}\cdots\mathrm{f}_{j_k}^{n_k}.\psi_\mu(1)&=\mathrm{f}_{j_1}^{n_1}\cdots\mathrm{f}_{j_k}^{n_k}.1\\ &=y_{j_1}^{n_1}\cdots y_{j_k}^{n_k}.\end{aligned}$$

This proves the claim (4.26).

Finally, by using the claim (4.26) and the $\mathfrak{g}\otimes\mathcal{L}$-module action (1.1) on $W(\mathcal{L},f_\mu(t),\lambda-\mu,r)$, we see that the $\mathfrak{g}\otimes\mathcal{L}$-module action on $\zeta_\mu^{-1}(\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r))$ is given as follows:

$$\begin{aligned}&\mathrm{f}\otimes t^m\mapsto\sum_{i=0}^r(\lambda-\mu)^m\,\ell_{i,r}(m)\,y_{i,\mu},\\ &\mathrm{h}\otimes t^m\mapsto\theta_{f_\mu(t),\lambda-\mu}(t^m)-2\sum_{i=0}^r\sum_{j=0}^r(\lambda-\mu)^m\,\ell_{j,r}(m+i)\,y_{j,\mu}\frac{\partial}{\partial y_{i,\mu}},\\ &\mathrm{e}\otimes t^m\mapsto\sum_{i=0}^r\left(\theta_{f_\mu(t),\lambda-\mu}(t^{m+i})-\sum_{j,k=0}^r(\lambda-\mu)^m\,\ell_{k,r}(m+i+j)\,y_{k,\mu}\frac{\partial}{\partial y_{j,\mu}}\right)\frac{\partial}{\partial y_{i,\mu}},\end{aligned}$$

where $m\in\mathbb{Z}$. Thus, for each $x\in\mathfrak{g}$ and $m\in\mathbb{Z}$, the action of $x\otimes(t-\mu)^m=\zeta_\mu(x\otimes t^m)$ on $\mathcal{P}(\mathcal{C}_S,f(t),\lambda,r)$ is given as above, which by definition is just the operator $\phi_{f(t),\lambda}(x\otimes(t-\mu)^m)$. This proves the assertion $(*)$, as desired.

5. Realization of irreducible highest weight $\mathfrak{g}\otimes\mathcal{C}_S$-modules

Here we give a realization of all irreducible highest weight $\mathfrak{g}\otimes\mathcal{C}_S$-modules.

5.1. **The quasi-finite case.** Given a $\chi \in \mathcal{E}(\mathcal{C}_S)$. From Proposition 3.6, we may assume that χ has the following expression

$$\chi = \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} a_{ij}\, \theta_{j,\lambda_i}, \tag{5.1}$$

where $\lambda_1, \cdots, \lambda_\nu$ are some distinct complex numbers in $\overline{\mathbb{C}}_S$, $r_1, \cdots, r_\nu \in \mathbb{N}$ and a_{ij} are some constants with $a_{ir_i} \neq 0$ for all i. For each $i = 1, \cdots, \nu$, set

$$\chi_i = \sum_{j=0}^{r_i} a_{ij}\, \theta_{j,\lambda_i} \quad \text{and} \quad f_i(t) = \sum_{j=0}^{r_i} a_{ij}\, t^j \tag{5.2}$$

so that $\chi = \sum_{i=1}^{\nu} \chi_i$ and $\chi_i = \theta_{f_i(t),\lambda_i}$ (see (4.5)).

We form the Fock space

$$\mathbb{C}[Y, \chi] = \mathbb{C}[y_{ij};\ i = 1, \cdots, \nu,\ j = 0, \cdots, r_i], \tag{5.3}$$

and define a linear map $\phi_\chi : \mathfrak{g} \otimes \mathcal{C}_S \to \mathrm{End}(\mathbb{C}[Y, \chi])$ in the following way:

$$\begin{aligned}
\phi_\chi(\mathrm{f} \otimes t^n) &= \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \ell^{\lambda,n}_{j,r_i}(0)\, y_{ij},\\
\phi_\chi(\mathrm{h} \otimes t^n) &= \chi(t^n) - 2 \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \sum_{j'=0}^{r_i} \ell^{\lambda,n}_{j',r_i}(j)\, y_{ij'}\, \frac{\partial}{\partial y_{ij}},\\
\phi_\chi(\mathrm{e} \otimes t^n) &= \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \left(\chi_i(t^{n+j}) - \sum_{j'=0}^{r_i} \sum_{j''=0}^{r_i} \ell^{\lambda,n}_{j'',r_i}(j+j')\, y_{ij''} \frac{\partial}{\partial y_{ij'}} \right) \frac{\partial}{\partial y_{ij}},
\end{aligned}$$

$$\begin{aligned}
\phi_\chi(\mathrm{f} \otimes (t-\mu)^m) &= \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} (\lambda_i - \mu)^m\, \ell_{j,r_i}(m)\, y_{ij,\mu},\\
\phi_\chi(\mathrm{h} \otimes (t-\mu)^m) &= \chi((t-\mu)^m) - 2 \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \sum_{j'=0}^{r_i} (\lambda_i - \mu)^m\, \ell_{j',r_i}(m+j)\, y_{ij',\mu}\, \frac{\partial}{\partial y_{ij,\mu}},\\
\phi_\chi(\mathrm{e} \otimes (t-\mu)^m) &= \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \chi_i((t-\mu)^{m+j})\, \frac{\partial}{\partial y_{ij,\mu}}\\
&\quad - \sum_{i=1}^{\nu} \sum_{j=0}^{r_i} \sum_{j'=0}^{r_i} \sum_{j''=0}^{r_i} (\lambda_i - \mu)^m\, \ell_{j'',r_i}(m+j+j')\, y_{ij'',\mu} \frac{\partial}{\partial y_{ij',\mu}} \frac{\partial}{\partial y_{ij,\mu}},
\end{aligned}$$

for $n \in \mathbb{N}$, $-m \in \mathbb{Z}_+$ and $\mu \in \mathbb{C}_S$, where the notations ℓ_{j,r_i}, $\ell^{\lambda_i,n}_{j,r_i}$, are as in (4.2), (4.3), and $y_{ij,\mu} = y^{\lambda_i}_{ij,\mu}$ is a linear combination of $y_{i0}, \cdots, y_{ij}$ as in (4.4).

Set

$$I_\chi = \{i = 1, \cdots, \nu \mid r_i = 0 \text{ and } a_{i0} \in \mathbb{N}\}.$$

For the case that $I_\chi \neq \emptyset$, we write $\mathbb{C}[Y, I_\chi]$ for the ideal of $\mathbb{C}[Y, \chi]$ generated by the elements $y_{i0}^{a_{i0}+1}$, $i \in I_\chi$, and write

$$W(\mathcal{C}_S, \chi) = \begin{cases} \mathbb{C}[Y, \chi], & \text{if } I_\chi = \emptyset,\\ \mathbb{C}[Y, \chi]/\mathbb{C}[Y, I_\chi], & \text{if } I_\chi \neq \emptyset. \end{cases} \tag{5.4}$$

Note that if $I_\chi \neq \emptyset$, then

$$W(\mathcal{C}_S,\chi) \cong \left(\bigotimes_{i\in I_\chi} \mathbb{C}[y_{i0}]/y_{i0}^{a_{i0}+1}\mathbb{C}[y_{i0}]\right) \bigotimes \left(\bigotimes_{i\notin I_\chi} \mathbb{C}[y_{ij} \mid j=0,\cdots,r_i]\right). \tag{5.5}$$

Moreover, in this case, the operators $\phi_\chi(\mathfrak{g}\otimes\mathcal{C}_S)$ stabilize the ideal $\mathbb{C}[Y,I_\chi]$ and hence ϕ_χ induces a linear map, still called ϕ_χ, from $\mathfrak{g}\otimes\mathcal{C}_S$ to $\mathrm{End}(W(\mathcal{C}_S,\chi))$.

Theorem 5.1. *Let $\chi \in \mathcal{E}(\mathcal{C}_S)$ be as in (5.1). Then, under the action of ϕ_χ, $W(\mathcal{C}_S,\chi)$ is an irreducible highest weight $\mathfrak{g}\otimes\mathcal{C}_S$-module with highest weight χ.*

Proof. Using Theorem 4.1, we know that

$$\mathbb{C}[Y,\chi] \cong W(\mathcal{C}_S, f_1(t),\lambda_1,r_1)\otimes\cdots\otimes W(\mathcal{C}_S, f_\nu(t),\lambda_\nu,r_\nu)$$

is a $\mathfrak{g}\otimes\mathcal{C}_S$-module with the module action given by ϕ_χ. Moreover, Theorem 4.2 implies that the $\mathfrak{g}\otimes\mathcal{C}_S$-module $\mathbb{C}[Y,\chi]$ is isomorphic to

(5.6)
$$\begin{aligned} &\rho^{-1}_{f_1(t),\lambda_1}(M(\mathcal{C}_{S,\lambda_1,r_1},\bar\theta_{f_1(t),\lambda_1,r_1}))\otimes\cdots\otimes\rho^{-1}_{f_\nu(t),\lambda_\nu}(M(\mathcal{C}_{S,\lambda_\nu,r_\nu},\bar\theta_{f_\nu(t),\lambda_\nu,r_\nu}))\\ &=\rho^{-1}_{\mathcal{C}_S,\chi_1}(M(\mathcal{C}_{S,\chi_1},\bar\chi_1))\otimes\cdots\otimes\rho^{-1}_{\mathcal{C}_S,\chi_\nu}(M(\mathcal{C}_{S,\chi_\nu},\bar\chi_\nu)). \end{aligned}$$

One may see (2.2), (4.7) and (5.2) for the notations used in (5.6). From Remark 3.8, it follows that

$$p_\chi(t)=\prod_{i=1}^{\nu}(t-\lambda_i)^{r_i+1} \quad\text{and}\quad p_{\chi_i}(t)=(t-\lambda_i)^{r_i+1},\quad i=1,\cdots,\nu.$$

So, by the Chinese Remainder Theorem, we have

$$\mathcal{C}_{S,\chi}\cong\mathcal{C}_{S,\chi_1}\oplus\mathcal{C}_{S,\chi_2}\oplus\cdots\oplus\mathcal{C}_{S,\chi_\nu}.$$

This implies that, for any $\mathfrak{g}\otimes\mathcal{C}_{S,\chi_i}$-module W_i, the tensor product space $W_1\otimes\cdots\otimes W_\nu$ is a $\mathfrak{g}\otimes\mathcal{C}_{S,\chi}$-module. Moreover, one has the $\mathfrak{g}\otimes\mathcal{C}_S$-module isomorphism

$$\rho^{-1}_{\mathcal{C}_S,\chi}(W_1\otimes\cdots\otimes W_\nu)\cong\rho^{-1}_{\mathcal{C}_S,\chi_1}(W_1)\otimes\cdots\otimes\rho^{-1}_{\mathcal{C}_S,\chi_\nu}(W_\nu). \tag{5.7}$$

Now, since the algebras $\mathfrak{g}\otimes\mathcal{C}_{S,\chi_i}$ are finite dimensional, from Schur's lemma, it follows that the tensor product $\mathfrak{g}\otimes\mathcal{C}_{S,\chi}$-module $V(\mathcal{C}_{S,\chi_1},\bar\chi_1)\otimes\cdots\otimes V(\mathcal{C}_{S,\chi_\nu},\bar\chi_\nu)$ is still irreducible. One easily checks that this tensor $\mathfrak{g}\otimes\mathcal{C}_{S,\chi}$-module is a highest weight module with highest weight χ and hence is isomorphic to $V(\mathcal{C}_{S,\chi},\bar\chi)$. This together with (2.3) and (5.7) gives the following $\mathfrak{g}\otimes\mathcal{C}_S$-module isomorphisms

$$\begin{aligned} V(\mathcal{C}_S,\chi)&\cong\rho^{-1}_{\mathcal{C}_S,\chi}(V(\mathcal{C}_{S,\chi},\bar\chi))\\ &\cong\rho^{-1}_{\mathcal{C}_S,\chi}(V(\mathcal{C}_{S,\chi_1},\bar\chi_1)\otimes\cdots\otimes V(\mathcal{C}_{S,\chi_\nu},\bar\chi_\nu))\\ &\cong\rho^{-1}_{\mathcal{C}_S,\chi_1}(V(\mathcal{C}_{S,\chi_1},\bar\chi_1))\otimes\cdots\otimes\rho^{-1}_{\mathcal{C}_S,\chi_\nu}(V(\mathcal{C}_{S,\chi_\nu},\bar\chi_\nu)). \end{aligned}$$

By comparing the above isomorphisms with those given in (5.6) and (5.5), now we only need to show that for each i, $V(\mathcal{C}_{S,\chi_i},\bar\chi_i)=M(\mathcal{C}_{S,\chi_i},\bar\chi_i)$ unless $i\in I_\chi$, in which case $\dim V(\mathcal{C}_{S,\chi_i},\bar\chi_i)=a_{i0}+1$.

Indeed, note that $\mathfrak{g}\otimes\mathcal{C}_{S,\chi_i}$ is isomorphic to the truncated current algebra $\mathfrak{g}\otimes\mathcal{C}/t^{r_i+1}\mathcal{C}$. Moreover, via this isomorphism, $\bar\chi_i$ induces a linear function on $\mathcal{C}/t^{r_i+1}\mathcal{C}$ whose value at $t^{r_i}+t^{r_i+1}\mathcal{C}$ is $a_{ir_i}\neq 0$. If $i\in I_\chi$, then $\mathfrak{g}\otimes\mathcal{C}_{S,\chi_i}\cong\mathfrak{g}$ and the assertion is obvious. If $i\notin I_\chi$, then the assertion is implied by the irreducibility of Verma type highest weight $\mathfrak{g}\otimes\mathcal{C}/t^{r_i+1}\mathcal{C}$-modules given in [41, Proposition A1]. Therefore, we complete the proof of Theorem 5.1. □

5.2. **The general case.** In this subsection, we let χ be an arbitrary $\mathbb{C}$-valued function on $\mathcal{C}_S$. Set

$$\mathbb{I}_S = \{n,\, (m,\mu) \mid n \in \mathbb{N},\, m \in \mathbb{Z}_+,\, \mu \in \mathbb{C}_S\}$$

and for each $i \in \mathbb{I}_S$, set

$$t_i = \begin{cases} t^n, & \text{if } i = n, \\ (t-\mu)^{-m}, & \text{if } i = (m,\mu). \end{cases}$$

Then by Lemma 3.1, $\{t_i \mid i \in \mathbb{I}_S\}$ form a basis of $\mathcal{C}_S$. For each $i, j \in \mathbb{I}_S$, define $c_{i,j}^k, k \in \mathbb{I}_S$ to be the structure constants of $\mathcal{C}_S$ relative to this basis. Namely,

$$t_i\, t_j = \sum_{k\in\mathbb{I}_S} c_{i,j}^k\, t_k, \quad i, j \in \mathbb{I}_S.$$

We remark that the nontrivial structure constants are given in Lemma 3.1.

Let

$$P(\mathcal{C}_S, \chi) = \mathbb{C}[x_i; i \in \mathbb{I}_S]$$

be the polynomial algebra in the variables $x_i, i \in \mathbb{I}_S$. Set

$$\begin{aligned}
&\pi_\chi(\mathrm{f}\otimes t_i) = x_i,\\
&\pi_\chi(\mathrm{h}\otimes t_i) = \chi(t_i) - 2\sum_{j\in\mathbb{I}_S}\sum_{k\in\mathbb{I}_S} c_{ij}^k\, x_k\, \frac{\partial}{\partial x_j},\\
&\pi_\chi(\mathrm{e}\otimes t_i) = \sum_{j\in\mathbb{I}_S}\left(\chi(t_i\, t_j) - \sum_{k\in\mathbb{I}_S}\sum_{j'\in\mathbb{I}_S}\sum_{k'\in\mathbb{I}_S} c_{jk}^{j'}\, c_{ij'}^{k'}\, x_{k'}\, \frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j},
\end{aligned}$$

for $i \in \mathbb{I}_S$. It was shown in [30, § 5, Remark 2] that the Verma type $\mathfrak{g}\otimes\mathcal{C}_S$-module $M(\mathcal{C}_S, \chi)$ can be realized on the Fock space $P(\mathcal{C}_S, \chi)$ with the action π_χ.

Remark 5.2. When $\mathcal{C}_S = \mathcal{L}$, we will often identify $P(\mathcal{L}, \chi)$ with the Fock space $\mathbb{C}[x_i; i \in \mathbb{Z}]$ so that for each $i \in \mathbb{Z}$ [30],

$$\begin{aligned}
&\pi_\chi(\mathrm{f}\otimes t^i) = x_i, \quad \pi_\chi(\mathrm{h}\otimes t^i) = \chi(t^i) - 2\sum_{j\in\mathbb{Z}} x_{i+j}\, \frac{\partial}{\partial x_j},\\
&\pi_\chi(\mathrm{e}\otimes t^i) = \sum_{j\in\mathbb{Z}}\left(\chi(t^{i+j}) - \sum_{k\in\mathbb{Z}} x_{i+j+k}\, \frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j}.
\end{aligned}$$

The following result is also due to Jakobsen-Kac.

Proposition 5.3 ([30])**.** *Let χ be a linear function on $\mathcal{C}_S$. Then the $\mathfrak{g}\otimes\mathcal{C}_S$-module $P(\mathcal{C}_S, \chi)$-module is irreducible if and only if $\chi \notin \mathcal{E}(\mathcal{C}_S)$.*

Proof. It suffices to show that the Verma type $\mathfrak{g}\otimes\mathcal{C}_S$-module $M(\mathcal{C}_S, \chi)$ is irreducible when $\chi \notin \mathcal{E}(\mathcal{C}_S)$. Note that the algebra $\mathcal{C}_S$ contains no finite dimensional nonzero ideals, and $\chi \in \mathcal{E}(\mathcal{C}_S)$ if and only if χ vanishes on some nonzero ideal of $\mathcal{C}_S$. Thus, if $\chi \notin \mathcal{E}(\mathcal{C}_S)$, then the pair $(\mathcal{C}_S, \chi)$ satisfies the conditions (6.3a) and (6.3b) stated in [30, § 6]. Then the assertion is implied by [30, Proposition 6.2]. ☐

We now summarize the free field realizations of irreducible highest weight $\mathfrak{g}\otimes\mathcal{C}_S$-modules given in this section in Theorem 5.4.

Theorem 5.4. *For any $\chi \in \mathcal{C}_S^*$, one has the $\mathfrak{g} \otimes \mathcal{C}_S$-module isomorphism*

$$V(\mathcal{C}_S, \chi) \cong \begin{cases} P(\mathcal{C}_S, \chi), & \text{if } \chi \notin \mathcal{E}(\mathcal{C}_S), \\ W(\mathcal{C}_S, \chi), & \text{if } \chi \in \mathcal{E}(\mathcal{C}_S). \end{cases}$$

6. Realization of irreducible objects in Chari's category $\widetilde{\mathcal{O}}$

As an application of Theorem 5.4, here we present an explicit realization of all irreducible objects in Chari's category $\widetilde{\mathcal{O}}$ for the affine Kac-Moody algebra $A_1^{(1)}$.

6.1. **Chari's category $\widetilde{\mathcal{O}}$ for $\widetilde{\mathfrak{g}}$.** Let

$$\widetilde{\mathfrak{g}} = (\mathfrak{g} \otimes \mathcal{L}) \oplus \mathbb{C}\mathrm{d}$$

be the centerless affine Kac-Moody algebra associated to $\mathfrak{g}$ with the remaining Lie bracket given by

$$[\mathrm{d}, x \otimes t^m] = m\, x \otimes t^m,$$

for $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Let $\widetilde{\mathfrak{h}} = \mathbb{C}\mathrm{h} \oplus \mathbb{C}\mathrm{d}$ and define $\alpha, \delta \in \widetilde{\mathfrak{h}}^*$ by letting

$$\alpha(\mathrm{h}) = 2,\ \alpha(\mathrm{d}) = 0, \quad \delta(\mathrm{d}) = 1,\ \delta(\mathrm{h}) = 0.$$

A $\widetilde{\mathfrak{g}}$-module V is called a weight module if it admits a weight spaces decomposition $V = \oplus_{\beta \in \widetilde{\mathfrak{h}}^*} V_\beta$, where $V_\beta = \{v \in V \mid h.v = \beta(h)v \text{ for } h \in \widetilde{\mathfrak{h}}\}$. Set

$$\mathcal{P}(V) = \{\beta \in \widetilde{\mathfrak{h}} \mid V_\beta \neq 0\},$$

the set of weights in V. The category $\widetilde{\mathcal{O}}$ introduced by Chari [8] is defined as follows: an object $V \in \widetilde{\mathcal{O}}$ if and only if V is a weight $\widetilde{\mathfrak{g}}$-module and there exist finitely many $\beta_1, \cdots, \beta_r \in \widetilde{\mathfrak{h}}^*$ such that

$$\mathcal{P}(V) \subset \{\beta_i - m\alpha + n\delta \mid 1 \le i \le r, m \in \mathbb{N}, n \in \mathbb{Z}\}.$$

The morphisms of the category $\widetilde{\mathcal{O}}$ are the homomorphisms of $\widetilde{\mathfrak{g}}$-modules.

Set

$$\widehat{\mathfrak{h}} = (\mathfrak{h} \otimes \mathcal{L}) \oplus \mathbb{C}\mathrm{d}.$$

Following [10], we say that a $\widetilde{\mathfrak{g}}$-module V is an l-highest weight $\widetilde{\mathfrak{g}}$-module if there is a nonzero vector $v \in V$ such that

$$\mathcal{U}(\widetilde{\mathfrak{g}})v = V, \quad (\mathfrak{n}_+ \otimes \mathcal{L})v = 0 \quad \text{and} \quad \mathcal{U}(\widehat{\mathfrak{h}})v \text{ is an irreducible weight } \widehat{\mathfrak{h}}\text{-module.}$$

The notation "l-highest weight $\widetilde{\mathfrak{g}}$-module" is used to distinguish the usual highest weight $\widetilde{\mathfrak{g}}$-module in the category $\mathcal{O}$ (cf. [32]). It is obvious that any l-highest weight $\widetilde{\mathfrak{g}}$-module belongs to the category $\widetilde{\mathcal{O}}$. Conversely, it was shown in [8] that any irreducible object in the category $\widetilde{\mathcal{O}}$ is an l-highest weight $\widetilde{\mathfrak{g}}$-module.

6.2. **Irreducible l-highest weight $\widetilde{\mathfrak{g}}$-modules.** For any $\chi \in \mathcal{L}^*$ and $b \in \mathbb{C}$, we define an $\widehat{\mathfrak{h}}$-module structure on $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ with

$$(\mathrm{h} \otimes t^m) \cdot t^n = \chi(t^n)\, t^{m+n}, \quad \mathrm{d} \cdot t^n = (n + b)\, t^n, \tag{6.1}$$

for $m, n \in \mathbb{Z}$. We denote this $\widehat{\mathfrak{h}}$-module by $\mathcal{L}(\chi, b)$. Let $\mathcal{L}_{\chi,b}$ be the $\widehat{\mathfrak{h}}$-submodule of $\mathcal{L}(\chi, b)$ generated by 1. Note that, as vector spaces, $\mathcal{L}_{\chi,b} = \mathcal{L}_{\chi,0}$ for all $b \in \mathbb{C}$.

For each $s \in \mathbb{N}$, we set

$$\mathcal{L}_s = \begin{cases} \mathbb{C}1, & \text{if } s = 0, \\ \mathbb{C}[t^s, t^{-s}], & \text{if } s > 0, \end{cases} \quad \text{and} \quad \mathcal{L}^*(s) = \{\chi \in \mathcal{L}^* \mid \mathcal{L}_{\chi,0} = \mathcal{L}_s\}.$$

We also set

$$\mathcal{L}^*_> = \biguplus_{s>0} \mathcal{L}^*(s) \quad \text{and} \quad \mathcal{L}^*_\geq = \mathcal{L}^*(0) \uplus \mathcal{L}^*_>.$$

The following result is straightforward.

Lemma 6.1. *Let $\chi \in \mathcal{L}^*$ and $b \in \mathbb{C}$. Then*

(i) *The $\widehat{\mathfrak{h}}$-module $\mathcal{L}_{\chi,b}$ is irreducible if and only if $\chi \in \mathcal{L}^*_\geq$;*
(ii) *$\chi \in \mathcal{L}^*(0)$ if and only if $\chi(t^n) = 0$ for all $n \neq 0$;*
(iii) *$\chi \in \mathcal{L}^*_>$ if and only if there exist $n_1 > 0$ and $n_2 < 0$ such that both $\chi(t^{n_1})$ and $\chi(t^{n_2})$ are nonzero.*

It view of Lemma 6.1, for any $\chi \in \mathcal{L}^*_\geq$ and $b \in \mathbb{C}$, $\mathcal{L}_{\chi,b}$ is an irreducible weight $\widehat{\mathfrak{h}}$-module. Conversely, it was proved in [8] that any irreducible weight $\widehat{\mathfrak{h}}$-module has such a form. We remark that if $\chi \in \mathcal{E}(\mathcal{L}) \setminus \{0\}$, then $\chi \in \mathcal{L}^*_>$. For $\chi \in \mathcal{L}^*_\geq$ and $b \in \mathbb{C}$, let us form the induced $\widetilde{\mathfrak{g}}$-module

$$\widetilde{M}(\chi, b) = \mathcal{U}(\widetilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{h}} \oplus (\mathfrak{n}_+ \otimes \mathcal{L}))} \mathcal{L}_{\chi,b},$$

where $\mathcal{L}_{\chi,b}$ is viewed as an $\widehat{\mathfrak{h}} \oplus (\mathfrak{n}_+ \otimes \mathcal{L})$-module by letting $\mathfrak{n}_+ \otimes \mathcal{L}$ acts trivially. Write $\widetilde{V}(\chi, b)$ for the irreducible quotient of $\widetilde{M}(\chi, b)$.

Proposition 6.2 ([8])**.** *Let V be an irreducible object in the category $\widetilde{\mathcal{O}}$. Then V is isomorphic to $\widetilde{V}(\chi, b)$ for some $\chi \in \mathcal{L}^*_\geq$ and $b \in \mathbb{C}$.*

6.3. **Realization of $\widetilde{V}(\chi, b)$.** Given a pair $(\chi, b) \in \mathcal{L}^*_\geq \times \mathbb{C}$. In this subsection we will provide an explicit realization of the $\widetilde{\mathfrak{g}}$-module $\widetilde{V}(\chi, b)$.

We first consider the case that $\chi \in \mathcal{L}^*(0)$. If $\chi = 0$, then it is clear that the $\mathfrak{g} \otimes \mathcal{L}$-module $\widetilde{V}(0, b)$ is isomorphic to $\widetilde{W}(0, b)$, the one dimensional $\widetilde{\mathfrak{g}}$-module on which $\mathfrak{g} \otimes \mathcal{L}$ acts trivially and d acts as the scalar b. If $\chi \neq 0$, then it follows from Lemma 6.1(ii) that there is a $\gamma \in \mathbb{C}^\times$ such that $\chi(t^n) = \gamma\, \delta_{n,0}$ for $n \in \mathbb{Z}$. This implies that the $\mathfrak{g} \otimes \mathcal{L}$-module $P(\mathcal{L}, \chi)$ (see Remark 5.2) can be lifted to a $\widetilde{\mathfrak{g}}$-module with the remaining action given by

$$\mathrm{d} \mapsto b\, \mathrm{Id} + \sum_{n \in \mathbb{Z}} n\, x_n \frac{\partial}{\partial x_n}.$$

This irreducible $\widetilde{\mathfrak{g}}$-module, denoted as $\widetilde{P}(\chi, b)$, is isomorphic to $\widetilde{V}(\chi, b)$.

In what follows, we assume that $\chi \in \mathcal{L}^*(s)$ for a fixed $s \in \mathbb{Z}_+$. For a $\mathfrak{g} \otimes \mathcal{L}$-module W, we define a $\widetilde{\mathfrak{g}}$-module structure on the loop space $W \otimes \mathcal{L}$ by [13]

$$(x \otimes t^n) \cdot w \otimes t^m = ((x \otimes t^n) \cdot w) \otimes t^{m+n}, \quad \mathrm{d} \cdot w \otimes t^m = (m + b) \otimes t^m,$$

where $x \in \mathfrak{g}$, $m, n \in \mathbb{Z}$ and $w \in W$. We denote the resulting $\widetilde{\mathfrak{g}}$-module by $\mathcal{L}(W, b)$.

Recall that v_χ denotes the highest weight vector in the $\mathfrak{g} \otimes \mathcal{L}$-module $V(\mathcal{L}, \chi)$. For $i \in \mathbb{Z}$, let $\mathcal{L}(V(\mathcal{L}, \chi), b)_i$ be the $\widetilde{\mathfrak{g}}$-submodule of $\mathcal{L}(V(\mathcal{L}, \chi), b)$ generated by $v_\chi \otimes t^i$. The following result is standard, one may see [13] or [10] for a proof.

Lemma 6.3. *For each $i \in \mathbb{Z}$, the $\widetilde{\mathfrak{g}}$-module $\mathcal{L}(V(\mathcal{L},\chi),b)_i$ is isomorphic to the irreducible l-highest weight $\widetilde{\mathfrak{g}}$-module $\widetilde{V}(\chi, b+i)$. Moreover, one has*

$$\mathcal{L}(V(\mathcal{L},\chi),b) = \bigoplus_{i=0}^{s-1} \mathcal{L}(V(\mathcal{L},\chi),b)_i \cong \bigoplus_{i=0}^{s-1} \widetilde{V}(\chi, b+i).$$

In view of Lemma 6.3 and Theorem 5.4, we now only need to determine the structure of the $\widetilde{\mathfrak{g}}$-module $\mathcal{L}(P(\mathcal{L},\chi),b)_0$ (if $\chi \notin \mathcal{E}(\mathcal{L})$) or $\mathcal{L}(W(\mathcal{L},\chi),b)_0$ (if $\chi \in \mathcal{E}(\mathcal{L})$). Consider first the case that $\chi \notin \mathcal{E}(\mathcal{L})$. Note that we have

$$\mathcal{U}(\widehat{\mathfrak{h}}) \cdot 1 \otimes 1 = 1 \otimes \mathcal{L}_s \subset \mathcal{L}(P(\mathcal{L},\chi),b).$$

Using this, one can check that

$$\widetilde{P}(\chi,b) = \mathrm{Span}_{\mathbb{C}}\{x_{i_1}\cdots x_{i_n} \otimes t^m \mid i_1,\cdots,i_n \in \mathbb{Z}, i_1+\cdots+i_n - m \in s\mathbb{Z}\}$$

is the $\widetilde{\mathfrak{g}}$-submodule of $\mathcal{L}(P(\mathcal{L},\chi),b)$ generated by $1\otimes 1$, and hence is isomorphic to the l-highest weight $\widetilde{\mathfrak{g}}$-module $\widetilde{V}(\chi,b)$.

Next, for the case that $\chi \in \mathcal{E}(\mathcal{L})$, we may assume that χ is as in (5.1). It was shown in [40] (see also [13]) that $\nu \equiv 0 \,(\bmod s)$. Moreover, there exist a permutation τ of $\{1,2,\cdots,\nu\}$ and complex numbers $\bar\lambda_0,\cdots,\bar\lambda_{p-1}$ such that

$$f_{\tau(si+1)}(t) = f_{\tau(si+2)}(t) = \cdots = f_{\tau((i+1)s)}(t), \quad \text{(see (5.2))},$$
$$\lambda_{\tau(si+1)} = \varepsilon\bar\lambda_i,\ \lambda_{\tau(si+2)} = \varepsilon^2\bar\lambda_i,\cdots,\lambda_{\tau((i+1)s)} = \varepsilon^s\bar\lambda_i,$$

for $i = 0,1,\cdots,p-1$, where $p = \nu/s$, and ε is a primitive s-th root of unity. For convenience, up to a permutation, we may assume that

$$f_1(t) = \cdots = f_s(t),\ f_{s+1}(t) = \cdots = f_{2s}(t),\cdots,f_{\nu-s+1}(t) = \cdots = f_\nu(t).$$

Let σ be the following permutation:

$$\sigma = (1,2,\cdots,s)(s+1,s+2,\cdots,2s)\cdots(\nu-s+1,\nu-s+2,\cdots,\nu).$$

Then the assignment

$$y_{ij} \mapsto y_{\sigma(i)j}, \qquad 1 \le i \le \nu,\ 1 \le j \le r_i, \tag{6.2}$$

determines a (unique) algebra automorphism $\tilde\sigma$ of $\mathbb{C}[Y,\chi]$ (see (5.3)). If $I_\chi \neq \emptyset$, this automorphism stabilizes the ideal $\mathbb{C}[Y,I_\chi]$ and hence induces an automorphism, still called $\tilde\sigma$, of the corresponding quotient algebra $W(\mathcal{L},\chi)$ of $\mathbb{C}[Y,\chi]$ (see (5.4)).

For any $i \in \mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z}$, let

$$W(\mathcal{L},\chi)_{(i)} = \mathrm{Span}_{\mathbb{C}}\{v_{(i)} = \sum_{0\le q\le s-1} \varepsilon^{-iq}\,\tilde\sigma^q(v) \mid v \in W(\mathcal{L},\chi)\}$$

be the $\tilde\sigma$-eigenvector subspace of $W(\mathcal{L},\chi)$ with eigenvalue ε^i. Notice that in this case the actions of $\mathrm{f}\otimes t^n$, $n \in \mathbb{Z}$ on $W(\mathcal{L},\chi)$ are given by

$$\phi_\chi(\mathrm{f}\otimes t^n) = \sum_{i=0}^{p-1}\sum_{j=1}^{r_{si+1}} \bar\lambda_i^n\,\ell_{j,r_{si+1}}(n)\,y_{ij(\overline{n})},$$

where $\overline{n} \in \mathbb{Z}_s$ is the image of n under the homomorphism $\mathbb{Z} \to \mathbb{Z}_s$. This implies

$$\phi_\chi(\mathrm{f}\otimes t^n)(W(\mathcal{L},\chi)_{(\overline{m})}) = W(\mathcal{L},\chi)_{(\overline{m+n})}$$

for $m, n \in \mathbb{Z}$. It then follows that

$$\widetilde{W}(\chi, b) = \bigoplus_{m\in\mathbb{Z}} W(\mathcal{L}, \chi)_{(\overline{m})} \otimes \mathbb{C}\, t^m$$

is the submodule of $\mathcal{L}(W(\mathcal{L}, \chi), b)$ generated by $1 \otimes 1$, and hence is isomorphic to the l-highest weight $\widetilde{\mathfrak{g}}$-module $\widetilde{V}(\chi, b)$.

We summarize the above realization of irreducible l-highest weight $\widetilde{\mathfrak{g}}$-modules in Theorem 6.4. Due to Proposition 6.2, this in turn gives an explicit realization of all irreducible objects in the category $\widetilde{\mathcal{O}}$.

Theorem 6.4. *For any $\chi \in \mathcal{L}^*_{\geq}$ and $b \in \mathbb{C}$, one has the $\widetilde{\mathfrak{g}}$-module isomorphism*

$$\widetilde{V}(\chi, b) \cong \begin{cases} \widetilde{P}(\chi, b), & \text{if } \chi \notin \mathcal{E}, \\ \widetilde{W}(\chi, b), & \text{if } \chi \in \mathcal{E}. \end{cases}$$

7. Realization of Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-modules

In this section, as an application of Theorem 5.4, we give an explicit realization of all Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-modules.

7.1. **Classification of Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-modules.** For $\lambda \in \overline{\mathbb{C}}_S$ and $u \in \mathcal{C}_S$, we denote by M_λ the maximal ideal of $\mathcal{C}_S$ generated by $t - \lambda$, and denote by $u_\lambda \in \mathbb{C}$ the residue of u modulo M_λ. Namely, $u_\lambda = u + M_\lambda$ as an element of $M/M_\lambda \cong \mathbb{C}$. In particular, we have

$$(t^n)_\lambda = \lambda^n,\ ((t-\mu)^m)_\lambda = (\lambda-\mu)^m, \quad n \in \mathbb{N}, -m \in \mathbb{Z}_+, \mu \in \mathbb{C}_S. \tag{7.1}$$

Let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_\nu)$ be a ν-tuple of distinct elements in $\overline{\mathbb{C}}_S$. Then we have the following evaluation map

$$\mathrm{ev}_{\boldsymbol{\lambda}} : \mathfrak{g} \otimes \mathcal{C}_S \to \mathfrak{g}^{\oplus\nu}, \quad x \otimes u \mapsto (u_{\lambda_1} x, \cdots, u_{\lambda_\nu} x).$$

Following [7], we call the pull back $\mathrm{ev}_{\boldsymbol{\lambda}}^{-1}(V_1 \otimes \cdots \otimes V_\nu)$ of the $\mathfrak{g}^{\oplus\nu}$-module $V_1 \otimes \cdots \otimes V_\nu$ an evaluation (weight) $\mathfrak{g} \otimes \mathcal{C}_S$-module, where $V_1, \cdots, V_\nu$ are some Harish-Chandra $\mathfrak{g}$-modules. Recall the notion of "lowest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module" introduced in Remark 2.2. Then we have

Proposition 7.1. *Let V be a Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-module. Then either V is isomorphic to an evaluation $\mathfrak{g} \otimes \mathcal{C}_S$-module with bounded weight multiplicities or V is isomorphic to an irreducible quasi-finite highest/lowest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module.*

Proof. This classification result was proved in [37] for any current algebra $\mathfrak{g} \otimes \mathcal{R}$ with the assumption that $\mathcal{R}$ is finitely generated. Since the commutative algebra $\mathcal{C}_S$ is not necessary finitely generated, here we give a sketch of the proof for this proposition. Let V be a Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-module with the action given by $\phi : \mathfrak{g} \otimes \mathcal{C}_S \to \mathrm{End}(V)$. Then $\ker \phi = \mathfrak{g} \otimes I$ for some ideal I of $\mathcal{C}_S$. By the same proof as that in [7, Proposition 4.3], one finds that I is nonzero (or equivalently, cofinite) in $\mathcal{C}_S$. Thus, there exists a nonzero monic polynomial $p(t) \in \langle t - \lambda \mid \lambda \in \overline{\mathbb{C}}_S \rangle$ that generates I.

Now V descends to a faithful Harish-Chandra module for the truncated current algebra $\mathfrak{g} \otimes \mathcal{C}_S/I \cong \mathfrak{g} \otimes \mathcal{C}/p(t)\mathcal{C}$. By [37, Theorem 4.3], as a $\mathfrak{g} \otimes \mathcal{C}_S/I$-module, V is either a highest/lowest weight module, or an evaluation module with bounded weight multiplicities. In the former case, one immediate gets that V must be a highest/lowest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module. For the latter case, notice that a faithful evaluation module for the algebra $\mathfrak{g}\otimes\mathcal{C}_S/I$ exists only when $p(t)=\prod_{i=1}^{\nu}(t-\lambda_i)$ for some distinct $\lambda_i\in\overline{\mathbb{C}}_S$. Thus we have a natural Lie algebra isomorphism

$$\mathfrak{g}\otimes\mathcal{C}_S/I=\oplus_{i=1}^{\nu}\mathfrak{g}\otimes\mathcal{C}_S/M_{\lambda_i}\to\mathfrak{g}^{\oplus\nu}. \tag{7.2}$$

This implies that V is a simple weight $\mathfrak{g}^{\oplus\nu}$-module (with respect to $\mathfrak{h}^{\oplus\nu}$). By [7, Proposition 3.4], as $\mathfrak{g}^{\oplus\nu}$-modules, V is isomorphic to $V_1\otimes\cdots\otimes V_\nu$ for some Harish-Chandra $\mathfrak{g}$-modules $V_1,\cdots,V_\nu$. This gives that V is an evaluation $\mathfrak{g}\otimes\mathcal{C}_S$-module with bounded weight multiplicities, which completes the proof. □

7.2. **Realization of Harish-Chandra $\mathfrak{g}\otimes\mathcal{C}_S$-modules.** For any pair

$$(\boldsymbol{a},\boldsymbol{\lambda})=((a_1',a_1,a_2,\cdots,a_\nu),(\lambda_1,\cdots,\lambda_\nu))\in\mathbb{C}^{\nu+1}\times(\overline{\mathbb{C}}_S)^\nu, \tag{7.3}$$

we define certain operators on the Fock space $\mathbb{C}[y_1,y_1^{-1},y_2,y_3,\cdots,y_\nu]$ as follows

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{f}\otimes t^n)=\sum_{i=1}^{\nu}\lambda_i^n\left(\delta_{i,1}\frac{a_1'}{y_1}-\frac{\partial}{\partial y_i}\right),$$

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{h}\otimes t^n)=\sum_{i=1}^{\nu}\lambda_i^n\left(a_i-\delta_{i,1}a_1'+2y_i\frac{\partial}{\partial y_i}\right),$$

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{e}\otimes t^n)=\sum_{i=1}^{\nu}\lambda_i^n\left(y_i^2\frac{\partial}{\partial y_i}+a_iy_i\right),$$

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{f}\otimes (t-\mu)^m)=\sum_{i=1}^{\nu}(\lambda_i-\mu)^m\left(\delta_{i,1}\frac{a_1'}{y_1}-\frac{\partial}{\partial y_i}\right),$$

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{h}\otimes (t-\mu)^m)=\sum_{i=1}^{\nu}(\lambda_i-\mu)^m\left(a_i-\delta_{i,1}a_1'+2y_i\frac{\partial}{\partial y_i}\right),$$

$$\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}(\mathrm{e}\otimes (t-\mu)^m)=\sum_{i=1}^{\nu}(\lambda_i-\mu)^m\left(y_i^2\frac{\partial}{\partial y_i}+a_iy_i\right),$$

where $n\in\mathbb{N}$, $-m\in\mathbb{Z}_+$ and $\mu\in\mathbb{C}_S$. Using Lemma 3.1, it is straightforward to see that $\mathbb{C}[y_1,y_1^{-1},y_2,y_3,\cdots,y_\nu]$ is a $\mathfrak{g}\otimes\mathcal{C}_S$-module under the action of $\varphi_{\boldsymbol{a},\boldsymbol{\lambda}}$.

Assume now that the pair $(\boldsymbol{\lambda},\boldsymbol{a})$ satisfies the following condition

$$a_1,a_1'\notin\mathbb{Z},\quad a_2,\dots,a_\nu\in\mathbb{N}\quad\text{and}\quad\lambda_1,\dots,\lambda_\nu\text{ are distinct.} \tag{7.4}$$

Then one can check that

$$W(\mathcal{C}_S,\boldsymbol{a},\boldsymbol{\lambda})=\mathrm{Span}_{\mathbb{C}}\{y_1^{i_1}y_2^{i_2}\cdots y_\nu^{i_\nu}\mid i_1\in\mathbb{Z},\ 0\le i_j\le a_j,\ 2\le j\le\nu\} \tag{7.5}$$

is a $\mathfrak{g}\otimes\mathcal{C}_S$-submodule of $\mathbb{C}[y_1,y_1^{-1},y_2,\dots,y_\nu]$. Moreover, $W(\mathcal{C}_S,\boldsymbol{a},\boldsymbol{\lambda})$ is an evaluation weight $\mathfrak{g}\otimes\mathcal{C}_S$-module with bounded weight multiplicities (see (7.1)). We remark that $W(\mathcal{C}_S,\boldsymbol{a},\boldsymbol{\lambda})$ is not a highest/lowest weight $\mathfrak{g}\otimes\mathcal{C}_S$-module.

In literature, a Harish-Chandra $\mathfrak{g}$-module is called cuspidal if it is not a highest weight $\mathfrak{g}$-module. It was known that (see [35, Section 4] for example) any cuspidal $\mathfrak{g}$-module can be realized on the Fock space $\mathbb{C}[y_1,y_1^{-1}]$ with

$$\mathrm{f}\mapsto\frac{a_1'}{y_1}-\frac{\partial}{\partial y_1},\quad\mathrm{h}\mapsto a_1-a_1'+2y_1\frac{\partial}{\partial y_1},\quad\mathrm{e}\mapsto y_1^2\frac{\partial}{\partial y_1}+a_1y_1, \tag{7.6}$$

for some $a_1,a_1'\in\mathbb{C}\setminus\mathbb{Z}$. From [7, Proposition 3.13], it follows that an evaluation $\mathfrak{g}\otimes\mathcal{C}_S$-module $\mathrm{ev}_{\boldsymbol{\lambda}}^{-1}(V_1\otimes\cdots\otimes V_\nu)$ has bounded weight multiplicities if and only if at most one of the V_i is infinite dimensional. Moreover, if the $\mathfrak{g}\otimes\mathcal{C}_S$-module

$\mathrm{ev}_{\boldsymbol{\lambda}}^{-1}(V_1 \otimes \cdots \otimes V_\nu)$ is in addition not a highest/lowest weight $\mathfrak{g} \otimes \mathcal{C}_S$-module, then exactly one of the $\mathfrak{g}$-modules V_i is cuspidal. This together with (7.6), Proposition 7.1, Theorem 5.1 and Remark 2.2 gives the following realization theorem.

Theorem 7.2. *Let V be a Harish-Chandra $\mathfrak{g} \otimes \mathcal{C}_S$-module. Then V is isomorphic to one of the following $\mathfrak{g} \otimes \mathcal{C}_S$-modules:*

(1) *$W(\mathcal{C}_S, \boldsymbol{a}, \boldsymbol{\lambda})$ for some $(\boldsymbol{a}, \boldsymbol{\lambda})$ satisfies the condition* (7.4)*;*
(2) *$W(\mathcal{C}_S, \chi)$ for some $\chi \in \mathcal{E}(\mathcal{C}_S)$;*
(3) *$W(\mathcal{C}_S, \chi)^\omega$ for some $\chi \in \mathcal{E}(\mathcal{C}_S)$.*

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