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FAMILY OF $\mathscr{D}$-MODULES AND REPRESENTATIONS WITH A BOUNDEDNESS PROPERTY

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Abstract. In the representation theory of real reductive Lie groups, many objects have finiteness properties. For example, the lengths of Verma modules and principal series representations are finite, and more precisely, they are bounded. In this paper, we introduce a notion of uniformly bounded families of holonomic $\mathscr{D}$-modules to explain and find such boundedness properties.

A uniform bounded family has good properties. For instance, the lengths of modules in the family are bounded and the uniform boundedness is preserved by direct images and inverse images. By the Beilinson–Bernstein correspondence, we deduce several boundedness results about the representation theory of complex reductive Lie algebras from corresponding results of uniformly bounded families of $\mathscr{D}$-modules. In this paper, we concentrate on proving fundamental properties of uniformly bounded families, and preparing abstract results for applications to the branching problem and harmonic analysis.

1. Introduction

In this paper, we introduce a notion of uniformly bounded families of $\mathscr{D}$-modules, which are good families of holonomic $\mathscr{D}$-modules with bounded lengths. We show that the uniform boundedness is preserved by fundamental operations of $\mathscr{D}$-modules such as direct images, inverse images and taking subquotients. By the Beilinson–Bernstein correspondence [5], we deduce several boundedness results about the representation theory of complex reductive Lie algebras from corresponding results of uniformly bounded families of $\mathscr{D}$-modules.

In the representation theory of real reductive Lie groups, finiteness results about lengths of modules and multiplicities in branching laws are fundamental and enable us to study Harish-Chandra modules and unitary representations. We list typical examples of the results: finiteness of the lengths of Verma modules and principal series representations, Harish-Chandra's admissibility theorem [17], irreducibility of $\mathcal{U}(\mathfrak{g})^K$-actions on K-isotypic components, and finiteness of multiplicities in the Plancherel formula of symmetric spaces [35, 47].

Our main concern is that the finiteness is uniform. The length of a Verma module is bounded by some constant independent of its highest weight, and a similar result holds for principal series representations. The former is an easy

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consequence of Soergel's theorem [43] (see also Remark 7.14), and the latter is proved by Kobayashi–Oshima in [35].

In [35], T. Kobayashi and T. Oshima give criteria for the finiteness and the uniform boundedness of multiplicities in the branching problem and harmonic analysis of real reductive Lie groups. The criteria are given by conditions on the existence of open orbits in flag varieties, and proved by using hyperfunction boundary value maps. A. Aizenbud, D. Gourevitch and A. Minchenko give similar results using families of holonomic $\mathscr{D}$-modules and Schwartz distributions in [2]. T. Tauchi proves similar results based on the finiteness of hyperfunction solutions in [44]. Their results are one of our motivations.

In this paper, we do not deal with concrete applications to the branching problem and harmonic analysis. We concentrate on providing fundamental properties of uniformly bounded families, and preparing abstract results for such applications. See Proposition 7.8 and Remark 7.9 for an easy application to the estimate of multiplicities.

Let us state the definition of uniformly bounded families and their properties. Our definition is based on Bernstein's work [6]. In the paper, he has introduced the multiplicity $m(M)$ of a module M of the Weyl algebra $\mathcal{D}_{\mathbb{C}^n}$, and proved that the multiplicity is well-behaved for direct images, inverse images and taking subquotients. We denote by $\mathrm{Mod}_h(\mathscr{D}_X)$ the category of holonomic $\mathscr{D}$-modules on a smooth variety X. Let $f\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism of algebraic varieties of degree d' and set $d = \max(1, d')$. Let Df_+ (resp. Lf^*) denote the direct (resp. inverse) image functor. Then we have

$$\sum_i m(D^i f_+(\mathcal{M})) \le d^{n+m} m(\mathcal{M}) \text{ and } \sum_i m(L_i f^*(\mathcal{N})) \le d^{n+m} m(\mathcal{N}),$$

for any $\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n})$ and $\mathcal{N} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^m})$ (see Fact 4.4). Here we put $m(\mathcal{M}) := m(\Gamma(\mathcal{M}))$. The key point is that the coefficient d^{n+m} is independent of $\mathcal{M}$ (or $\mathcal{N}$). In other words, the estimates of the multiplicities are uniform with respect to $\mathcal{M}$ (or $\mathcal{N}$). This is the starting point of our definition.

Let $\mathscr{A}_{X,\Lambda} := (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ be a family of algebras of twisted differential operators on a smooth variety X over $\mathbb{C}$. We say that (U, φ, Φ) is a trivialization of $\mathscr{A}_{X,\Lambda}$ if $\varphi\colon U \to X$ is a surjective étale morphism and $\Phi_\lambda\colon \varphi^{\#}\mathscr{A}_{X,\lambda} \to \mathscr{D}_U$ is an isomorphism. Here $\varphi^{\#}$ is the pull-back of algebras of twisted differential operators by φ. Take a trivialization (U, φ, Φ) with affine U and a closed embedding $\iota\colon U \to \mathbb{C}^n$. Then for a family $\mathcal{M} \in \prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$, we can consider a function

$$\Lambda \ni \lambda \mapsto m(\iota_+(\varphi^*(\mathcal{M}_\lambda))) \in \mathbb{N}. \tag{1.0.1}$$

The boundedness of the function does not depend on ι (see Proposition 4.7), and does depend on the isomorphisms Φ.

We introduce a relation $\sim$ of trivializations. For two trivializations (U, φ, Φ) and (V, ψ, Ψ), we write $(U, \varphi, \Phi) \sim (V, \psi, \Psi)$ if the set

$$\left\{ \widetilde{\varphi}^{\#}\Psi_\lambda \circ (\widetilde{\psi}^{\#}\Phi_\lambda)^{-1} : \lambda \in \Lambda \right\} \subset \mathrm{Aut}(\mathscr{D}_{U\times_X V}) \simeq \mathcal{Z}(U \times_X V)$$

spans a finite-dimensional subspace of the space $\mathcal{Z}(U \times_X V)$ of closed 1-forms. Here $\widetilde{\varphi}\colon U \times_X V \to V$ and $\widetilde{\psi}\colon U \times_X V \to U$ are the projections of the fiber product. See Definition 4.11.

We say that a trivialization T is bounded if $T \sim T$. Although the relation is not an equivalence relation of trivializations, it is an equivalence relation of

bounded trivializations. Moreover, if two bounded trivializations $T = (U, \varphi, \Phi)$ and $S = (V, \psi, \Psi)$ with affine V and U are equivalent, the boundedness of the function (1.0.1) defined for T is equivalent to that for S. An equivalence class of bounded trivializations is called a bornology of $\mathscr{A}_{X,\Lambda}$ (see Definition 4.16).

For a bornology $\mathcal{B}$ of $\mathscr{A}_{X,\lambda}$, we say that $\mathcal{M} \in \prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$ is a uniformly bounded family with respect to $\mathcal{B}$ if the function (1.0.1) defined for any/some $T \in \mathcal{B}$ is bounded. We denote by $\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$ the full subcategory of $\prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$ whose objects are uniformly bounded. Similarly, we define a derived category version $D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$. See Definition 4.20 for the details.

Corresponding to operations of $\mathscr{A}_{X,\Lambda}$, we define operations of bornologies by natural ways: pull-back $f^{\#}\mathcal{B}$, external tensor product $\mathcal{B}\boxtimes\mathcal{B}'$, twisting by an invertible sheaf $\mathcal{B}^{\mathcal{L}}$ and opposite $\mathcal{B}^{\mathrm{op}}$. Theorem 1.1 is a fundamental result about uniformly bounded families. See Propositions 4.22 and 4.31, and Theorems 4.26 and 4.28.

Theorem 1.1. *The uniform boundedness is preserved by direct images, inverse images, external tensor products, twisting by an invertible sheaf and taking subquotients.*

For example, for a morphism $f\colon Y \to X$ of smooth varieties, we can define a direct image functor

$$Df_+\colon D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}) \to D^b_{ub}(f^{\#}\mathscr{A}_{X,\Lambda}, f^{\#}\mathcal{B})$$

via $Df_+(\mathcal{M}) = (Df_+(\mathcal{M}_\lambda))_{\lambda\in\Lambda}$, which is the restriction of the direct product of the direct image functors on $D^b_h(\mathscr{A}_{X,\lambda})$ $(\lambda \in \Lambda)$. Here $f^{\#}\mathscr{A}_{X,\Lambda}$ is the family $(f^{\#}\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$.

The proofs for the last three operations in Theorem 1.1 are easy by the definition of uniformly bounded families. The proofs for the others are essentially the same as a proof of preserving holonomicity (see e.g. [12, VII. §12] and [21, 3.2]).

When each $\mathscr{A}_{X,\lambda}$ is G-equivariant, we can define a notion of G-equivariant bornologies by a natural way (Definition 5.6). The G-equivariance is preserved by the pull-back by a G-equivariant morphism. It is important for the representation theory that if X is a homogeneous variety G/H, there exists a unique G-equivariant bornology of a family of G-equivariant algebras of twisted differential operators (see Proposition 5.10).

In Beilinson–Bernstein's paper [5], they give a way to classify equivariant $\mathscr{D}$-modules. Combining the classification and the notion of G-equivariant bornologies, we obtain

Theorem 1.2 (Theorem 5.15)**.** *Let $\mathcal{B}$ be a bornology of $\mathscr{A}_{X,\Lambda}$. Suppose that X is a G-variety of an affine algebraic group G, and $\mathcal{B}$ and $\mathscr{A}_{X,\Lambda}$ are G-equivariant. If G has finitely many orbits in X, then any family of $(\mathscr{A}_{X,\lambda}, G)$-modules with bounded lengths is uniformly bounded with respect to $\mathcal{B}$.*

In Section 5, we give several methods to construct bornologies and uniformly bounded families from algebraic group actions. In particular, we will see that there are many uniformly bounded families.

Let us state applications of uniformly bounded families to the representation theory. Let G be a connected reductive algebraic group over $\mathbb{C}$ and B a Borel subgroup of G. By the Beilinson–Bernstein correspondence, any $\mathfrak{g}$-module with an infinitesimal character is isomorphic to $\Gamma(\mathcal{M})$ for some twisted $\mathscr{D}$-module on G/B. We always choose the twist of $\mathscr{D}$ such that Γ is exact on the category of

quasi-coherent twisted $\mathscr{D}$-modules. We say that a family $(M_i)_{i\in I}$ of $\mathfrak{g}$-modules is uniformly bounded if the lengths of M_i are bounded and the localization of the family of all composition factors of all M_i is a uniformly bounded family on G/B.

The uniform boundedness is preserved by several operations of $\mathfrak{g}$-modules such as taking subquotients, tensoring with finite-dimensional modules and cohomological parabolic inductions. This follows from corresponding results for $\mathscr{D}$-modules in Theorem 1.1. By Theorem 1.2, any family of Harish-Chandra modules (or objects in the BGG category $\mathcal{O}$) with bounded lengths is uniformly bounded. This implies that many families in the representation theory of real reductive Lie groups are uniformly bounded.

We shall state the preservation result for cohomological parabolic inductions. Let P be a parabolic subgroup of G and L a Levi subgroup of P. Let $(\mathfrak{g}, K)$ be a pair (see Definition 2.5). Take a reductive subgroup $K_L \subset K$ such that $\mathfrak{k}_L \subset \mathfrak{k} \cap \mathfrak{l}$ and K_L normalizes $\mathfrak{l}$ and $\mathfrak{p}$.

Theorem 1.3 (Theorem 7.15)**.** *Let $(M_i)_{i\in I}$ be a uniformly bounded family of $(\mathfrak{l}, K_L)$-modules. Then the family $(\mathbb{D}^j\Gamma^K_{K_L}(\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})} M_i))_{i\in I, j\in\mathbb{Z}}$ is uniformly bounded, where $\mathbb{D}^j\Gamma^K_{K_L}$ is the j-th Zuckerman derived functor.*

As mentioned before, the lengths of Verma modules (or principal series representations) are bounded, which is a special case of Theorem 1.3. It is well-known that the length of a cohomologically induced module is finite (see e.g. [32, Theorem 0.46]).

For the proof of Theorem 1.3, we need the localization of the Zuckerman derived functor. In this paper, we construct the localization following F. Bien [10]. A conceptual treatment of the localization using the equivariant derived category is given by S. N. Kitchen [31]. See also [39]. We do not treat the equivariant derived category in this paper.

The algebra of invariant differential operators plays an important role in the representation theory of real reductive Lie groups such as the Schur–Weyl duality and the compact Howe duality [22], a characterization of compact Gelfand pair [45], and Harish-Chandra's study of $(\mathfrak{g}, K)$-modules [17,18]. If $(\mathfrak{g}, K)$ is a pair with connected reductive group K, then the $\mathcal{U}(\mathfrak{g})^K$-action on a non-zero K-isotypic component of an irreducible $(\mathfrak{g}, K)$-module is irreducible. This is a classical result that follows from the Jacobson density theorem and complete reducibility of the K-action (see e.g. [16, Section 4.2]). Theorem 1.4 can be considered as a generalization of the result.

Let G' be a reductive subgroup of G and $(\mathfrak{g}', K')$ a subpair of $(\mathfrak{g}, K)$. Suppose that K' is a reductive subgroup of K and $\mathrm{Ad}_{\mathfrak{g}}(K')$ is contained in $\mathrm{Ad}_{\mathfrak{g}}(G')$.

Theorem 1.4 (Theorem 7.22)**.** *Let $(V_i)_{i\in I}$ and $(V'_i)_{i\in I}$ be uniformly bounded families of $(\mathfrak{g}, K)$-modules and $(\mathfrak{g}', K')$-modules, respectively (e.g. families of irreducible Harish-Chandra modules). Then there exists a constant C such that for any $i \in I$ and $j \in \mathbb{Z}$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(H_j(\mathfrak{g}', K'; V_i \otimes V'_i)) \le C,$$

where $\mathrm{Len}(\cdot)$ *means the length of a module.*

One of our motivations of Theorem 1.4 is to study multiplicities in the branching problem and harmonic analysis of real reductive Lie groups. Theorem 1.4 asserts that the multiplicities are roughly controlled by $\mathcal{U}(\mathfrak{g})^{G'}$. We can give criteria for the

uniform boundedness of the multiplicities by a ring-theoretic invariant of a quotient of $\mathcal{U}(\mathfrak{g})^{G'}$. See [29] for this kind of applications.

Another motivation of Theorem 1.4 is the Howe duality [23]. If V is the Segal–Weil–Shale representation and (G', H') is a reductive dual pair of G (i.e. $Z_G(G') = H', Z_G(H') = G'$), then Theorem 1.4 asserts that (higher) theta lifts $\Theta_i(V') = H_i(\mathfrak{g}', K'; V \otimes V'^{\vee})$ are of finite length as $\mathfrak{h}'$-modules and the lengths are bounded. By our method, we cannot prove one of important parts of Howe's theorem that the theta lift $\Theta_0(V')$ has a unique irreducible quotient. However our theorem enables us to define the Euler–Poincaré characteristic of the higher theta lifts. See Theorem 7.26. We remark that the Euler-Poincaré characteristic of the theta lifting for p-adic groups is studied in [1].

Let G be a connected reductive algebraic group over $\mathbb{C}$ and K its connected reductive subgroup. I. Penkov and G. Zuckerman call a $\mathfrak{g}$-module M a generalized Harish-Chandra module if M is locally finite, completely reducible and admissible as a $\mathfrak{k}$-module (see e.g. [40]). A relation between generalized Harish-Chandra modules and supports of $\mathscr{D}$-modules on G/B is studied by A. V. Petukhov [41].

A motivation of our study of the category of generalized Harish-Chandra modules is to study the category of $\mathcal{U}(\mathfrak{g})^K$-modules. By Lepowsky–McCollum's result [37], the category of $\mathcal{U}(\mathfrak{g})^K$-modules can be embedded in the category of $(\mathfrak{g} \oplus \mathfrak{k}, \Delta(K))$-modules. Hence this relates the branching problem and harmonic analysis to the study of $(\mathfrak{g}\oplus\mathfrak{k}, \Delta(K))$-modules. As an application of uniformly bounded families, we prove fundamental results of the category of generalized Harish-Chandra modules: finiteness of equivalence classes of irreducible objects (Corollary 7.36), boundedness of the Loewy lengths of modules (Theorem 7.41), and existence of projective objects (Proposition 7.43).

This paper is organized as follows. In Section 2, we review the notions of generalized pairs, pairs $(\mathfrak{g}, K)$, relative Lie algebra cohomologies and truncation functors. In Section 3, we recall the definition of algebras of twisted differential operators and their operations. At the end of the section, we study the direct image functors with respect to the projections of principal G-bundles. The definition of uniformly bounded families of $\mathscr{D}$-modules is in Section 4. Theorem 1.1 is proved here. Section 5 is devoted to constructions of bornologies and uniformly bounded families. The proof of Theorem 1.2 is given here. In Section 6, we review the localization of the Zuckerman derived functor following [10]. Applications to the representation theory are given in Section 7.

Notation and convention. In this paper, any algebraic variety is assumed to be quasi-projective and defined over $\mathbb{C}$. Let $\mathcal{O}_X$ and $\mathcal{O}(X)$ denote the structure sheaf of a variety X and the algebra of its global sections, respectively. Let $\mathbb{C}_X$ denote the constant sheaf on X. Suppose that X is smooth. We write $\mathscr{D}_X$ for the algebra of non-twisted (local) differential operators. We express algebras of twisted differential operators and the spaces of their global sections by script letters and calligraphic letters, respectively. For example, the spaces of global sections of algebras $\mathscr{A}_X, \mathscr{B}_{X,\lambda}$ and $\mathscr{D}_X$ are denoted as $\mathcal{A}_X, \mathcal{B}_{X,\lambda}$ and $\mathcal{D}_X$, respectively.

Any representation and module in this paper are assumed to be defined over $\mathbb{C}$. We express affine algebraic groups and their Lie algebras by Roman alphabets and corresponding German letters. For example, the Lie algebras of affine algebraic groups G, K and H are denoted as $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$. For a complex Lie algebra $\mathfrak{g}$, we write $\mathcal{U}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{g})$ for the universal enveloping algebra and its center, respectively. For

an affine algebraic group G, let G_0 denote the identity component of G. For a G-module (resp. a $\mathfrak{g}$-module) V, we write V^G (resp. $V^{\mathfrak{g}}$) for the space of all invariant vectors in V.

We denote by $\mathrm{Mod}(\mathscr{A}_X)$, $\mathrm{Mod}(\mathcal{A}, G)$, $\mathrm{Mod}(\mathfrak{g}, K)$ and $\mathrm{Mod}(\mathfrak{g})$ the categories of (left) modules of a sheaf $\mathscr{A}_X$ of algebras, a generalized pair $(\mathcal{A}, G)$, a pair $(\mathfrak{g}, K)$ and a Lie algebra $\mathfrak{g}$, respectively. We write $\mathrm{Len}_{\mathcal{R}}(V)$ for the length of an $\mathcal{R}$-module V in each category, e.g. $\mathrm{Len}_{\mathscr{A}_X}(V)$, $\mathrm{Len}_{\mathcal{A},G}(V)$, $\mathrm{Len}_{\mathfrak{g},K}(V)$ and $\mathrm{Len}_{\mathfrak{g}}(V)$. We denote by $\mathrm{Mod}_{qc}(\mathscr{A}_X)$ (resp. $\mathrm{Mod}_h(\mathscr{A}_X)$) the category of quasi-coherent modules (resp. holonomic modules) of an algebra $\mathscr{A}_X$ of twisted differential operators. We use the same notation for categories of equivariant modules such as $\mathrm{Mod}_{qc}(\mathscr{A}_X, G)$ and $\mathrm{Mod}_h(\mathscr{A}_X, G)$.

For an algebra $\mathscr{A}_X$ of twisted differential operators on a smooth variety X, let $D^b_{qc}(\mathscr{A}_X)$ (resp. $D^b_h(\mathscr{A}_X)$) denote the full subcategory of the derived category $D(\mathrm{Mod}(\mathscr{A}_X))$ consisting of objects $\mathcal{M}^\bullet$ whose cohomologies $H^i(\mathcal{M}^\bullet)$ are quasi-coherent (resp. holonomic) and vanish for any $|i| \gg 0$. We list operations of sheaves:

- $\mathcal{L}^\vee$: the dual of an invertible sheaf $\mathcal{L}$
- Γ, $R\Gamma$: the global section functor and its right derived functor of sheaves
- f^{-1}: the inverse image functor of sheaves
- f_*, Rf_*: the direct image functor and its right derived functor of sheaves
- f^*, Lf^*: the inverse image functor and its left derived functor of $\mathcal{O}_Y$-modules (or twisted $\mathscr{D}$-modules)
- Df_+: the direct image functor of twisted $\mathscr{D}$-modules.

Here $f\colon X \to Y$ is a morphism of smooth varieties. We denote by $R^i f_*$, $L_{-i} f^*$ and $D^i f_+$ the compositions $H^i \circ Rf_*$, $H^i \circ Lf^*$ and $H^i \circ Df_+$, respectively.

Let $(\cdot) \otimes (\cdot)$ (without subscript) denote the tensor product over $\mathbb{C}$. For an $\mathcal{R}$-module M and an $\mathcal{S}$-module N, we write $M \boxtimes N$ for the external tensor product of M and N.

2. Preliminary

In this section, we prepare several known results and definitions. We deal with generalized pairs, Lie algebra cohomology groups and truncation functors.

2.1. **Generalized pair.** In this subsection, we recall the definitions of generalized pairs and $(\mathcal{A}, G)$-modules, and show easy propositions related to generalized pairs. We refer the reader to [32, p.96].

In this paper, any algebraic group is affine and defined over $\mathbb{C}$, and any $\mathbb{C}$-algebra is associative and unital without Lie algebras. For a representation V of an affine algebraic group G as an abstract group, we say that V is a G-module or G acts rationally on V if the G-action is locally finite and any finite-dimensional G-subrepresentation of V is a representation of an algebraic group.

Definition 2.1. Let $\mathcal{A}$ be a $\mathbb{C}$-algebra and G an affine algebraic group over $\mathbb{C}$ acting rationally on $\mathcal{A}$ by algebra automorphisms. We say that the pair $(\mathcal{A}, G)$ equipped with a G-equivariant algebra homomorphism $\iota\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ is a generalized pair if the adjoint action of $\mathfrak{g}$ on $\mathcal{A}$ determined by ι coincides with the differential of the action of G on $\mathcal{A}$.

For a generalized pair $(\mathcal{A}, G)$, we denote by $\mathrm{Ad}_{\mathcal{A}}$ (or Ad) the action of G on $\mathcal{A}$. For example, if G' is a closed subgroup of G, then $(\mathcal{U}(\mathfrak{g}), G')$ is a generalized

pair. For a G-equivariant algebra $\mathscr{A}_X$ of twisted differential operators on a smooth variety X, $(\Gamma(\mathscr{A}_X), G)$ forms a generalized pair.

Definition 2.2. Let $(\mathcal{A}, G)$ be a generalized pair. We say that an $\mathcal{A}$-module V equipped with a rational G-action is an $(\mathcal{A}, G)$-module if the following two conditions hold.

(i) The differential of the G-action on V coincides with the $\mathfrak{g}$-action via the composition $\mathfrak{g} \xrightarrow{\iota} \mathcal{A} \to \mathrm{End}_{\mathbb{C}}(V)$ and
(ii) $gXv = \mathrm{Ad}(g)(X)gv$ holds for any $g \in G, X \in \mathcal{A}$ and $v \in V$.

We denote by $\mathrm{Mod}(\mathcal{A}, G)$ the category of $(\mathcal{A}, G)$-modules.

If G is reductive, any $(\mathcal{A}, G)$-module is completely reducible as a G-module. Hence the functor $\mathrm{Mod}(\mathcal{A}, G) \ni V \mapsto V^G \in \mathrm{Mod}(\mathcal{A}^G)$ is exact, where V^G is the space of all G-invariant vectors in V. Moreover, it is easy to see that the functor sends an irreducible object to zero or irreducible one. See e.g. [16, Theorem 4.2.1]. Hence we have Proposition 2.3.

Proposition 2.3. *Let $(\mathcal{A}, G)$ be a generalized pair with reductive G. Then for any $(\mathcal{A}, G)$-module V, we have*

$$\mathrm{Len}_{\mathcal{A}^G}(V^G) \leq \mathrm{Len}_{\mathcal{A},G}(V),$$

where Len *means the length of a module.*

We will reduce some propositions about $(\mathcal{A}, G)$-modules to those for $(\mathcal{A}, G_0)$-module. To do this, we need the following easy lemma.

Lemma 2.4. *Let $(\mathcal{A}, G)$ be a generalized pair and V an $(\mathcal{A}, G)$-module. Then we have*

$$\mathrm{Len}_{\mathcal{A},G}(V) \leq \mathrm{Len}_{\mathcal{A},G_0}(V) \leq |G/G_0| \mathrm{Len}_{\mathcal{A},G}(V).$$

Proof. The first inequality is trivial. It is enough to show the second inequality when V is an irreducible $(\mathcal{A}, G)$-module.

By Zorn's lemma, we can take a proper $(\mathcal{A}, G_0)$-submodule W such that any non-zero $(\mathcal{A}, G_0)$-submodule of V/W contains a unique irreducible $(\mathcal{A}, G_0)$-submodule. Take a maximal subset $S \subset G/G_0$ such that $W_S := \bigcap_{g \in S} gW$ is non-zero. Since V is irreducible as an $(\mathcal{A}, G)$-module, S is a proper subset of G/G_0. Fix $g \in G/G_0 - S$. Since $W_S \cap gW = 0$, the composition $W_S \hookrightarrow V \twoheadrightarrow V/gW$ is injective. Hence W_S contains an irreducible $(\mathcal{A}, G_0)$-submodule V_0.

Since V is an irreducible $(\mathcal{A}, G)$-module, we have $V = \sum_{g \in G/G_0} gV_0$. This implies that V is completely reducible as an $(\mathcal{A}, G_0)$-module, and hence the length as an $(\mathcal{A}, G_0)$-module is less than or equal to $|G/G_0|$. □

2.2. $(\mathfrak{g}, K)$**-module.** We review the notion of pairs $(\mathfrak{g}, K)$ and the relative Lie algebra cohomology groups. We refer the reader to [32, Chapters I, IV] and [13, Chapter I].

Definition 2.5. Let $\mathfrak{g}$ be a complex Lie algebra and K an affine algebraic group with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. We say that $(\mathfrak{g}, K)$ is a *pair* if the following two conditions hold.

(i) A rational K-action on $\mathfrak{g}$ by Lie algebra automorphisms is given, whose restriction to $\mathfrak{k}$ is equal to the adjoint action of K on $\mathfrak{k}$.

(ii) The differential of the K-action on $\mathfrak{g}$ coincides with the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$.

We denote by $\mathrm{Ad}_{\mathfrak{g}}$ (or simply Ad) the action of K on $\mathfrak{g}$.

If $(\mathfrak{g}, K)$ is a pair, then $(\mathcal{U}(\mathfrak{g}), K)$ forms a generalized pair. A $(\mathcal{U}(\mathfrak{g}), K)$-module is called a $(\mathfrak{g}, K)$-*module*.

For a complex Lie algebra $\mathfrak{g}$, the functor $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\cdot,\cdot)$ can be computed by an explicit complex called the Chevalley–Eilenberg chain complex. We recall the complex in the relative setting. Let $(\mathfrak{g}, K)$ be a pair. Remark that the following results also hold if K is replaced with its Lie algebra $\mathfrak{k}$. The relative Chevalley–Eilenberg chain complex is a sequence

$$\cdots \xrightarrow{\partial_{k+1}} \mathrm{CE}_k(\mathfrak{g}, K) \xrightarrow{\partial_k} \mathrm{CE}_{k-1}(\mathfrak{g}, K) \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} \mathrm{CE}_0(\mathfrak{g}, K) \to 0,$$

where $\mathrm{CE}_k(\mathfrak{g}, K) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \wedge^k(\mathfrak{g}/\mathfrak{k})$. The differential ∂_k is given by

$$\begin{aligned}
&\partial_k(v \otimes X_1 \wedge \cdots \wedge X_k)\\
&= \sum_i (-1)^{i+1} v\widetilde{X}_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k\\
&+ \sum_{i<j} (-1)^{i+j} v \otimes ([\widetilde{X}_i, \widetilde{X}_j] + \mathfrak{k}) \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k,
\end{aligned}$$

where each X_i is in $\mathfrak{g}/\mathfrak{k}$ and each $\widetilde{X}_i$ is a representative of X_i in $\mathfrak{g}$. For a $(\mathfrak{g}, K)$-module V, we set

$$\begin{aligned}
H_i(\mathfrak{g}, K; V) &:= H_i((V \otimes_{\mathcal{U}(\mathfrak{g})} \mathrm{CE}_\bullet(\mathfrak{g}, K))^K) \simeq H_i((V \otimes_{\mathcal{U}(\mathfrak{k})} \wedge^\bullet(\mathfrak{g}/\mathfrak{k}))^K),\\
H^i(\mathfrak{g}, K; V) &:= H^i(\mathrm{Hom}_{\mathfrak{g},K}(\mathrm{CE}_\bullet(\mathfrak{g}, K), V)) \simeq H^i(\mathrm{Hom}_K(\wedge^\bullet(\mathfrak{g}/\mathfrak{k}), V)).
\end{aligned}$$

We call them the relative Lie algebra homology and cohomology of V, respectively.

If K is reductive, the complex $(\mathrm{CE}_\bullet(\mathfrak{g}, K), \partial_\bullet)$ is a projective resolution of $\mathbb{C}$ in $\mathrm{Mod}(\mathfrak{g}, K)$. Hence we can compute Tor and Ext by the complex. See [32, Proposition 2.117] and [36, Lemma 3.1.9].

Fact 2.6. Let V and W be $(\mathfrak{g}, K)$-modules. If K is reductive, then we have natural isomorphisms

$$\begin{aligned}
H_i(\mathfrak{g}, K; V \otimes W) &\simeq \mathrm{Tor}_i^{\mathfrak{g},K}(V, W),\\
H^i(\mathfrak{g}, K; \mathrm{Hom}_{\mathbb{C}}(V, W)) &\simeq \mathrm{Ext}^i_{\mathfrak{g},K}(V, W).
\end{aligned}$$

The following result is called the Poincaré duality. See [32, Corollary 3.6].

Fact 2.7. Let V be a $(\mathfrak{g}, K)$-module. Put $n = \dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{k})$. If K is reductive, then we have a natural isomorphism

$$H^i(\mathfrak{g}, K; V \otimes \wedge^n(\mathfrak{g}/\mathfrak{k})) \simeq H_{n-i}(\mathfrak{g}, K; V),$$

where $\wedge^n(\mathfrak{g}/\mathfrak{k})$ is a $(\mathfrak{g}, K)$-module with the natural K-action and the $\mathfrak{g}$-action given by the character $X \mapsto \mathrm{tr}(\mathrm{Ad}_{\mathfrak{g}}(X))$.

2.3. **Truncation functor.** In many places, we reduce assertions about a complex to those about a single object. To do so, we need the truncation functors. We refer the reader to [27, Definitions 11.3.11 and 12.3.1].

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ the category of complexes in $\mathcal{A}$. For a complex $(C^\bullet, d^\bullet) \in \mathcal{C}(\mathcal{A})$, we set

$$\begin{aligned}
\tau^{\leq k}C^\bullet &:= \cdots \to C^{k-2} \to C^{k-1} \to \mathrm{Ker}(d^k) \to 0 \to 0 \to \cdots, \\
\tau^{>k}C^\bullet &:= \cdots \to 0 \to 0 \to \mathrm{Im}(d^k) \to C^{k+1} \to C^{k+2} \to \cdots, \\
\tau_s^{\leq k}C^\bullet &:= \cdots \to C^{k-2} \to C^{k-1} \to C^k \to 0 \to 0 \to \cdots, \\
\tau_s^{>k}C^\bullet &:= \cdots \to 0 \to 0 \to 0 \to C^{k+1} \to C^{k+2} \to \cdots.
\end{aligned}$$

$\tau^{\leq k}$ and $\tau^{>k}$ are called *truncation functors*, and $\tau_s^{\leq k}$ and $\tau_s^{>k}$ are called *stupid truncation functors*. Then we have distinguished triangles

$$\begin{aligned}
\tau^{\leq k}C^\bullet \to C^\bullet \to \tau^{>k}C^\bullet \xrightarrow{+1}, \\
\tau_s^{\leq k}C^\bullet \to C^\bullet \to \tau_s^{>k}C^\bullet \xrightarrow{+1}.
\end{aligned}$$

Lemma 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and m a $\mathbb{C}$-valued additive function on the Grothendieck group of $\mathcal{A}$. Assume $m(M) \geq 0$ for any $M \in \mathcal{A}$.*

(i) *For any distinguished triangle $(N^\bullet \to M^\bullet \to L^\bullet \xrightarrow{+1})$ in $D^b(\mathcal{A})$ and $i \in \mathbb{Z}$, we have*

$$m(H^i(M^\bullet)) \leq m(H^i(N^\bullet)) + m(H^i(L^\bullet)).$$

(ii) *For any functor $F\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$ of triangulated categories, complex $M^\bullet \in D^b(\mathcal{B})$ and $i \in \mathbb{Z}$, we have*

$$m(H^i(F(M^\bullet))) \leq \sum_j m(H^{i-j}(F(H^j(M^\bullet)))).$$

(iii) *For any functor $F\colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$ of triangulated categories, bounded complex $M^\bullet \in C(\mathcal{B})$ and $i \in \mathbb{Z}$, we have*

$$m(H^i(F(M^\bullet))) \leq \sum_j m(H^{i-j}(F(M^j))).$$

Proof. (i) is clear by the long exact sequence associated to the distinguished triangle $(N^\bullet \to M^\bullet \to L^\bullet \xrightarrow{+1})$. Set $l(M^\bullet) := |\{n \in \mathbb{Z} : H^n(M^\bullet) \neq 0\}|$. By induction on $l(M^\bullet)$ and the truncation functors, we can reduce the assertion (ii) to the case $M^\bullet \simeq N[n]$ for some $N \in \mathcal{B}$ and $n \in \mathbb{Z}$. In fact, we can take $k \in \mathbb{Z}$ such that $l(\tau^{\leq k}C^\bullet), l(\tau^{>k}C^\bullet) < l(C^\bullet)$. Applying (i) to the following distinguished triangle iteratively, we obtain (ii):

$$F(\tau^{\leq k}M^\bullet) \to F(M^\bullet) \to F(\tau^{>k}M^\bullet) \xrightarrow{+1}.$$

Similarly, using the stupid truncation functors, we obtain (iii). □

3. $\mathscr{D}$-module and its operations

The purpose of this section is to summarize fundamental results about $\mathscr{D}$-modules.

3.1. **Twisted $\mathscr{D}$-module.** We review algebras of twisted differential operators and their operations. We refer to [4], [28, Section 1], [26, Sections 3 and 4] and [25] for (G-equivariant) algebras of twisted differential operators. In this paper, we denote by $\mathscr{D}_X$ the algebra of non-twisted (local) differential operators on a smooth variety X. Any variety in this paper is assumed to be quasi-projective.

Let X be a (quasi-projective) smooth variety over $\mathbb{C}$. Let i_X be the standard homomorphism $\mathcal{O}_X \to \mathscr{D}_X$. There are several definitions of algebras of twisted differential operators on X. We adopt a definition in which the algebras are locally trivial in the étale topology. We refer to [28, 1.1] for the general case, and to [20, A.1] for the locally trivial case in the Zariski topology.

Let $\mathcal{T}_X$ be the tangent sheaf of X and $p\colon T^*X \to X$ the natural projection from the cotangent bundle T^*X to X.

Definition 3.1. We say that a sheaf $\mathscr{A}$ of $\mathbb{C}$-algebras on X is an *algebra of twisted differential operators* if $\mathscr{A}$ is equipped with a $\mathbb{C}$-algebra homomorphism $i\colon \mathcal{O}_X \to \mathscr{A}$ and an increasing filtration $\{F_i(\mathscr{A})\}_{i\in\mathbb{Z}}$ satisfying

(i) $F_i(\mathscr{A}) = 0$ for any $i < 0$,
(ii) $\mathscr{A} = \bigcup_{i\in\mathbb{Z}} F_i(\mathscr{A})$,
(iii) $F_i(\mathscr{A}) \cdot F_j(\mathscr{A}) \subset F_{i+j}(\mathscr{A})$ and $[F_i(\mathscr{A}), F_j(\mathscr{A})] \subset F_{i+j-1}(\mathscr{A})$ for any $i, j \in \mathbb{Z}$,
(iv) $\mathrm{Im}(i) = F_0(\mathscr{A})$, and $\mathcal{O}_X$ is isomorphic to $F_0(\mathscr{A})$ by i,
(v) the morphism $\sigma\colon F_1(\mathscr{A})/F_0(\mathscr{A}) \to \mathcal{T}_X$ given by $\sigma(T)(f) = [T, f]$ ($T \in F_1(\mathscr{A}), f \in \mathcal{O}_X$) is an isomorphism,
(vi) the $\mathcal{O}_X$-algebra homomorphism $p_*\mathcal{O}_{T^*X} \to \mathrm{gr}^F(\mathscr{A})$ induced from $i \oplus \sigma^{-1}\colon \mathcal{O}_X \oplus \mathcal{T}_X \to \mathrm{gr}^F(\mathscr{A})$ is an isomorphism.

In addition to these conditions, we assume that $\mathscr{A}$ is locally trivial in the étale topology, i.e. there exists an étale surjective morphism $\pi\colon \widetilde{X} \to X$ such that the pull-back $\pi^{\#}\mathscr{A}$ (defined below) is isomorphic to $(\mathscr{D}_{\widetilde{X}}, i_{\widetilde{X}})$.

We identify $\mathcal{O}_X$ with $i(\mathcal{O}_X) = F_0(\mathscr{A})$. The filtration F is called the order filtration of $\mathscr{A}$.

Let $f\colon X \to Y$ be a morphism of smooth varieties and $\mathscr{A}_Y$ an algebra of twisted differential operators on Y. We set

$$\begin{aligned}\Omega_f &:= f^{-1}\Omega_Y^{\vee} \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X,\\ \mathscr{A}_{Y\leftarrow X} &:= f^{-1}\mathscr{A}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_f,\\ \mathscr{A}_{X\to Y} &:= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{A}_Y,\end{aligned}$$

where Ω_X (resp. Ω_Y) denotes the canonical sheaf of X (resp. Y). $f^{\#}\mathscr{A}_Y$ denotes the sheaf of all differential endomorphisms of the $\mathcal{O}_X$-module $\mathscr{A}_{X\to Y}$ that commute with the right $f^{-1}\mathscr{A}_Y$-action. Then $f^{\#}\mathscr{A}_Y$ is an algebra of twisted differential operators on X and $\mathscr{A}_{Y\leftarrow X}$ is an $(f^{-1}\mathscr{A}_Y, f^{\#}\mathscr{A}_Y)$-bimodule.

The direct image of $\mathcal{M}^\bullet \in D^b_{qc}(f^{\#}\mathscr{A}_Y)$ is defined by

$$Df_+(\mathcal{M}^\bullet) = Rf_*(\mathscr{A}_{Y\leftarrow X} \otimes^L_{f^{\#}\mathscr{A}_Y} \mathcal{M}^\bullet) \in D^b_{qc}(\mathscr{A}_Y),$$

and the inverse image of $\mathcal{N}^\bullet \in D^b_{qc}(\mathscr{A}_Y)$ is defined by

$$Lf^*(\mathcal{N}^\bullet) = \mathscr{A}_{X\to Y} \otimes^L_{f^{-1}\mathscr{A}_Y} f^{-1}(\mathcal{N}^\bullet) \in D^b_{qc}(f^{\#}\mathscr{A}_Y).$$

It is well-known that the functors Df_+ and Lf^* are local for Y, and preserve holonomicity.

Lemma 3.2. *Let $\mathscr{A}_X$ be an algebra of twisted differential operators on X. Let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{A}_X^{\mathrm{op}})$ and $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathscr{A}_X)$. Suppose that $f : X \to Y$ is an affine morphism and $\mathcal{N}$ is flat. Then $\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}$ is f_*-acyclic and there exists a canonical isomorphism of $\mathbb{C}_Y$-modules*

$$f_*(\mathcal{M}) \otimes_{f_*(\mathscr{A}_X)} f_*(\mathcal{N}) \xrightarrow{\simeq} f_*(\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}).$$

Moreover, $f_*(\mathcal{N})$ is a flat $f_*(\mathscr{A}_X)$-module.

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be as in the assertion. For any open subset $U \subset Y$, there is a canonical morphism

$$\Gamma(f^{-1}(U), \mathcal{M}) \otimes_{\Gamma(f^{-1}(U), \mathscr{A}_X)} \Gamma(f^{-1}(U), \mathcal{N}) \to \Gamma(f^{-1}(U), \mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N})$$

by the definition of the tensor product $\otimes_{\mathscr{A}_X}$. By the universality of the sheafification, this induces a morphism $f_*(\mathcal{M}) \otimes_{f_*(\mathscr{A}_X)} f_*(\mathcal{N}) \to f_*(\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N})$. We shall show the morphism is an isomorphism. Since the assertions are local for Y and f is affine, we can assume that X and Y are affine.

Since X is affine, $\mathcal{M}$ has a free resolution $\cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{M} \to 0$. Then we obtain an exact sequence

$$\cdots \to \mathcal{F}_1 \otimes_{\mathscr{A}_X} \mathcal{N} \to \mathcal{F}_0 \otimes_{\mathscr{A}_X} \mathcal{N} \to \mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N} \to 0$$

of $\mathbb{C}_X$-modules because $\mathcal{N}$ is flat. Since $\mathcal{F}_i \otimes_{\mathscr{A}_X} \mathcal{N}$ admits a quasi-coherent $\mathcal{O}_X$-module structure, $\mathcal{F}_i \otimes_{\mathscr{A}_X} \mathcal{N}$ is f_*-acyclic for any i. Hence $\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}$ is f_*-acyclic and we obtain an exact sequence

$$\cdots \to f_*(\mathcal{F}_1 \otimes_{\mathscr{A}_X} \mathcal{N}) \to f_*(\mathcal{F}_0 \otimes_{\mathscr{A}_X} \mathcal{N}) \to f_*(\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}) \to 0$$

of $\mathbb{C}_Y$-modules. Here we used that $f_* : \mathrm{Mod}(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_Y)$ has finite cohomological dimension, i.e. $R^i f_* = 0$ for any $i \gg 0$ [19, III, Theorem 2.7, Lemma 2.8].

For $\mathcal{L} \in \mathrm{Mod}_{qc}(\mathscr{A}_X)$, set $T(\mathcal{L}) := f_*(\mathcal{L}) \otimes_{f_*(\mathscr{A}_X)} f_*(\mathcal{N})$. Since $\mathcal{F}_i$ is free, $f_*(\mathcal{F}_i \otimes_{\mathscr{A}_X} \mathcal{N})$ is canonically isomorphic to $T(\mathcal{F}_i)$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
T(\mathcal{F}_1) & \longrightarrow & T(\mathcal{F}_0) & \longrightarrow & T(\mathcal{M}) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
f_*(\mathcal{F}_1 \otimes_{\mathscr{A}_X} \mathcal{N}) & \longrightarrow & f_*(\mathcal{F}_0 \otimes_{\mathscr{A}_X} \mathcal{N}) & \longrightarrow & f_*(\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}) & \longrightarrow & 0.
\end{array}$$

We have seen that the lower sequence is exact. The upper sequence is exact since $f_*(\cdot) \otimes_{f_*(\mathscr{A}_X)} f_*(\mathcal{N})$ is right exact. Hence we obtain the desired isomorphism

$$f_*(\mathcal{M}) \otimes_{f_*(\mathscr{A}_X)} f_*(\mathcal{N}) \simeq f_*(\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N}).$$

We shall show that $f_*(\mathcal{N})$ is flat. Let $y \in Y$. For any $L \in \mathrm{Mod}(f_*(\mathscr{A}_X^{\mathrm{op}})_y)$, the canonical homomorphism

$$L \to f_*(\mathscr{A}_X^{\mathrm{op}} \otimes_{\Gamma(\mathscr{A}_X^{\mathrm{op}})} L)_y (\simeq f_*(\mathcal{O}_X \otimes_{\Gamma(\mathcal{O}_X)} L)_y)$$

is an isomorphism since X is affine. Hence the functor $f_*(\cdot)_y \colon \mathrm{Mod}_{qc}(\mathscr{A}_X^{\mathrm{op}}) \to \mathrm{Mod}(f_*(\mathscr{A}_X^{\mathrm{op}})_y)$ is essentially surjective, and similarly, full. Therefore $f_*(\mathcal{N})_y$ is flat and hence $f_*(\mathcal{N})$ is flat. □

Proposition 3.3. *Suppose that $f\colon X \to Y$ is an affine morphism. Then Df_+ is isomorphic to the left derived functor of $f_*(\mathscr{A}_{Y\leftarrow X}) \otimes_{f_*f^\#\mathscr{A}_Y} f_*(\cdot)$, which is $f_*(\mathscr{A}_{Y\leftarrow X}) \otimes^L_{f_*f^\#\mathscr{A}_Y} f_*(\cdot)$.*

Proof. Let $\mathcal{M}^\bullet \in D^b_{qc}(f^\#\mathscr{A}_Y)$. Then $\mathcal{M}^\bullet$ has a locally free resolution $\mathcal{F}^\bullet$ by [21, Corollary 1.4.20]. By Lemma 3.2, $\mathscr{A}_{Y\leftarrow X} \otimes_{f^\#\mathscr{A}_Y} \mathcal{F}^i$ is f_*-acyclic and we have

$$\begin{aligned} Df_+(\mathcal{M}^\bullet) &= f_*(\mathscr{A}_{Y\leftarrow X} \otimes_{f^\#\mathscr{A}_Y} \mathcal{F}^\bullet) \\ &\simeq f_*(\mathscr{A}_{Y\leftarrow X}) \otimes_{f_*f^\#\mathscr{A}_Y} f_*(\mathcal{F}^\bullet) \\ &\simeq f_*(\mathscr{A}_{Y\leftarrow X}) \otimes^L_{f_*f^\#\mathscr{A}_Y} f_*(\mathcal{M}^\bullet). \end{aligned}$$

Note that $f_*(\mathcal{F}^\bullet)$ is a flat resolution of $f_*(\mathcal{M}^\bullet)$ by Lemma 3.2. This shows the proposition. □

Facts 3.4 and 3.5 are fundamental. See [28, Lemma 1.1.7, Propositions 1.2.3 and 1.2.6], and see [21, Propositions 1.5.11 and 1.5.21, and Theorem 1.7.3] for the non-twisted case.

Fact 3.4. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of smooth varieties and $\mathscr{A}_Z$ an algebra of twisted differential operators on Z. Then we have $(g\circ f)^\#\mathscr{A}_Z = f^\#g^\#\mathscr{A}_Z$ and

(i) $Dg_+ \circ Df_+ = D(g\circ f)_+$,
(ii) $Lf^* \circ Lg^* = L(g\circ f)^*$.

To state the base change theorem, we need the shifted inverse image functor $f^\dagger$. For a morphism $f\colon X \to Y$ of smooth varieties and an algebra $\mathscr{A}_Y$ of twisted differential operators, we set

$$f^\dagger(\mathcal{F}^\bullet) = Lf^*(\mathcal{F}^\bullet)[\dim(X) - \dim(Y)] \quad (\mathcal{F}^\bullet \in D^b_{qc}(\mathscr{A}_Y)).$$

Fact 3.5 (Base change theorem)**.** Suppose that we have the following cartesian square of smooth varieties:

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\widetilde{g}} & Y \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X. \end{array}$$

Let $\mathscr{A}_X$ be an algebra of twisted differential operators. Then there exists an isomorphism $g^\dagger \circ Df_+ \simeq D\widetilde{f}_+ \circ \widetilde{g}^\dagger$ of functors.

In this paper, the shift of $f^\dagger$ is not important. When we use the base change theorem, we will say that $Lg^* \circ Df_+$ is isomorphic to $D\widetilde{f}_+ \circ L\widetilde{g}^*$ up to shift.

3.2. **Picard algebroid.** We review the notion of Picard algebroids and describe the action of $f^\#\mathscr{A}_Y$ on $\mathscr{A}_{Y\leftarrow X}$ using Picard algebroids. We refer the reader to [4, §2].

Let Z be a smooth variety and $\mathcal{T}_Z$ the tangent sheaf of Z.

Definition 3.6 ([4, 1.2 and 2.1.3])**.** Let $\widetilde{\mathcal{T}}$ be a quasi-coherent $\mathcal{O}_Z$-module on Z. $\widetilde{\mathcal{T}}$ is called a *Lie algebroid* on Z if $\widetilde{\mathcal{T}}$ is a sheaf of complex Lie algebras equipped with an $\mathcal{O}_Z$-module homomorphism $\sigma\colon \widetilde{\mathcal{T}} \to \mathcal{T}_Z$ such that

$$[T, fT'] = (\sigma(T)f)T' + f[T, T']$$

for any local sections $T, T' \in \widetilde{\mathcal{T}}$ and $f \in \mathcal{O}_Z$.

A Lie algebroid $\widetilde{\mathcal{T}}$ on Z is called a *Picard algebroid* if $\sigma\colon \widetilde{\mathcal{T}} \to \mathcal{T}_Z$ is epimorphic and there is an isomorphism $i\colon \mathcal{O}_Z \to \ker(\sigma)$ of $\mathcal{O}_Z$-modules such that $[T, i(f)] = \sigma(T)f$ for any local sections $T \in \widetilde{\mathcal{T}}$ and $f \in \mathcal{O}_Z$.

The isomorphism i in Definition 3.6 is unique. We identify $\mathcal{O}_X$ with $i(\mathcal{O}_X)$.

For an algebra $(\mathscr{A}_Z, i)$ of twisted differential operators on a smooth variety Z, we denote by $\mathcal{P}(\mathscr{A}_Z)$ the sheaf of sections in $\mathscr{A}_Z$ with the order less than or equal to 1. Then $\mathcal{P}(\mathscr{A}_Z)$ is a Picard algebroid on Z equipped with the homomorphism $i\colon \mathcal{O}_Z \to \mathcal{P}(\mathscr{A}_Z)$. Since $\mathscr{A}_Z$ is generated by $\mathcal{P}(\mathscr{A}_Z)$, to define an action of $\mathscr{A}_Z$, it is enough to define an action of $\mathcal{P}(\mathscr{A}_Z)$ such that $i(1)$ acts by the identity morphism [4, Lemma 2.1.4].

We describe the action $f^{\#}\mathscr{A}_Y$ on $\mathscr{A}_{Y\leftarrow X}$. Let X, Y, f and $\mathscr{A}_Y$ be as in the previous subsection. We denote by $f^{\#}\mathcal{P}(\mathscr{A}_Y)$ the fiber product $f^*\mathcal{P}(\mathscr{A}_Y) \times_{f^*\mathcal{T}_Y} \mathcal{T}_X$ of

$$f^*\mathcal{P}(\mathscr{A}_Y) \xrightarrow{f^*\sigma} f^*\mathcal{T}_Y \leftarrow \mathcal{T}_X.$$

Then $f^{\#}\mathcal{P}(\mathscr{A}_Y)$ is a Picard algebroid on X equipped with $i\colon \mathcal{O}_X \to f^{\#}\mathcal{P}(\mathscr{A}_Y)$ $(h \mapsto (h \otimes 1, 0))$. We can define an action of $f^{\#}\mathcal{P}(\mathscr{A}_Y)$ on $\mathscr{A}_{X\to Y}$ via

$$(\sum_i f_i \otimes T_i, T') \cdot g \otimes S = T'g \otimes S + \sum_i f_i g \otimes T_i S$$

for $(\sum_i f_i \otimes T_i, T') \in f^{\#}\mathcal{P}(\mathscr{A}_Y)$ and $g \otimes S \in \mathscr{A}_{X\to Y}$. This induces a canonical isomorphism $f^{\#}\mathcal{P}(\mathscr{A}_Y) \simeq \mathcal{P}(f^{\#}\mathscr{A}_Y)$ of Picard algebroids. See [4, Lemma 2.2].

Proposition 3.7. *For local sections $(\sum_i f_i \otimes T_i, T') \in f^{\#}\mathcal{P}(\mathscr{A}_Y)$ and $S \otimes \tau \otimes \omega \in \mathscr{A}_{Y\leftarrow X} = f^{-1}(\mathscr{A}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\vee}) \otimes_{\mathcal{O}_X} \Omega_X$, we define*

$$S \otimes \tau \otimes \omega \cdot (\sum_i f_i \otimes T_i, T') = \sum_i ST_i \otimes \tau \otimes f_i\omega - \sum_i S \otimes \sigma(T_i)\tau \otimes f_i\omega - S \otimes \tau \otimes \sigma(T')\omega,$$

where $\sigma(T')\omega$ and $\sigma(T_i)\tau$ are defined by the Lie derivative. Then this induces a right action of $f^{\#}\mathscr{A}_Y$ on $\mathscr{A}_{Y\leftarrow X}$.

Proof. A straightforward computation shows the proposition. Hence we omit the details. □

Remark 3.8. Since $\mathscr{A}_Y$ is locally trivial, we can reduce the computation to the non-twisted case. In the case, the action coincides with that in [21, Lemma 1.3.4]. In [28, 1.1.15], the action in Proposition 3.7 is constructed by a formal computation of algebras of twisted differential operators.

3.3. **G-equivariant module.** In this subsection, we review the notion of G-equivariant $\mathscr{D}$-modules. We refer the reader to [4, 1.8], [25] and [26, Section 3].

Let G be an affine algebraic group and X a smooth G-variety. We write $\mu\colon G \times X \to X$ for the multiplication map and $p_2\colon G \times X \to X$ for the projection onto the second factor. An $\mathcal{O}_X$-module $\mathcal{M}$ is *G-equivariant* if an $\mathcal{O}_{G\times X}$-module isomorphism $\mu^*\mathcal{M} \xrightarrow{\simeq} p_2^*\mathcal{M}$ is specified and satisfies the associative law [26, (3.1.2)]. The G-equivariant structure is sometimes called an (algebraic) G-action on $\mathcal{M}$. In fact, the G-equivariant structure induces a G-action on the set of sections of $\mathcal{M}$. The

G-action on $\mathcal{M}$ is differentiable, that is, it induces a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$.

We say that an algebra $\mathscr{A}$ of twisted differential operators is *G-equivariant* if an algebra homomorphism $i_{\mathfrak{g}} \colon \mathcal{U}(\mathfrak{g}) \to \mathscr{A}$ and a G-action on $\mathscr{A}$ are specified and satisfy the following conditions:

(i) The G-action is given by algebra isomorphisms.
(ii) $i_{\mathfrak{g}}$ is G-equivariant with respect to the adjoint action on $\mathcal{U}(\mathfrak{g})$.
(iii) The differential of the G-action on $\mathscr{A}$ coincides with the adjoint action of $\mathfrak{g}$ on $\mathscr{A}$ coming from $i_{\mathfrak{g}}$.

The G-equivariant structure induces an isomorphism

$$\mu^{\#}\mathscr{A} \simeq p_2^{\#}\mathscr{A} (\simeq \mathscr{D}_G \boxtimes \mathscr{A}) \tag{3.3.1}$$

of algebras satisfying the associative law. See [4, Lemma 1.8.7].

Let $\mathscr{A}_X$ be a G-equivariant algebra of twisted differential operators on X. An $\mathscr{A}$-module $\mathcal{M}$ is called *G-equivariant* or an *$(\mathscr{A}_X, G)$-module* if $\mathcal{M}$ is G-equivariant as an $\mathcal{O}_X$-module and the morphism $\mu^*\mathcal{M} \xrightarrow{\simeq} p_2^*\mathcal{M}$ is a $\mathscr{D}_G \boxtimes \mathscr{A}_X$-isomorphism.

Let $f \colon Y \to X$ be a G-morphism of smooth G-varieties. Then the natural left action of $\mathcal{U}(\mathfrak{g})$ on $\mathscr{A}_{Y \to X}$ induces an algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to f^{\#}\mathscr{A}_X$. Hence the algebra $f^{\#}\mathscr{A}_X$ is G-equivariant. The G-equivariant structure coincides with the one obtained from the canonical isomorphism $\mathcal{P}(f^{\#}\mathscr{A}_X) \simeq f^*\mathcal{P}(\mathscr{A}_X) \times_{f^*\mathcal{T}_X} \mathcal{T}_Y$ of Picard algebroids.

In this setting, the direct image functor and the inverse image functor preserve G-equivariant modules, i.e. we have

$$H^i \circ Df_+ \colon \mathrm{Mod}_{qc}(f^{\#}\mathscr{A}_X, G) \to \mathrm{Mod}_{qc}(\mathscr{A}_X, G),$$
$$H^i \circ Lf^* \colon \mathrm{Mod}_{qc}(\mathscr{A}_X, G) \to \mathrm{Mod}_{qc}(f^{\#}\mathscr{A}_X, G).$$

Although it is more conceptual to use the equivariant derived category, we do not deal with it in this paper.

Let $\mathcal{A}$ be a G-equivariant algebra of twisted differential operators on X. We consider $G \times X$ as a $G \times G$-variety via the action $(a, b) \cdot (g, x) = (agb^{-1}, bx)$, and the codomain X of μ (resp. p_2) as a $G \times G$-variety by letting the second (resp. first) factor of $G \times G$ act trivially. Then μ and p_2 are $G \times G$-equivariant, and hence $\mu^{\#}\mathcal{A}$ and $p_2^{\#}\mathcal{A}$ are $G \times G$-equivariant algebras.

Proposition 3.9. *The isomorphism* (3.3.1) *is $G \times G$-equivariant.*

Proof. The assertion follows from the associative law of the G-equivariant structure on $\mathcal{A}$ and easy diagram chasing. Hence we omit the details. □

3.4. **Principal bundle and direct image.** Let G be an affine algebraic group and $p \colon \widetilde{X} \to X$ a principal G-bundle over a smooth variety X together with a free right action $\widetilde{X} \times G \to \widetilde{X}$. In this paper, a principal bundle over an algebraic variety is assumed to be locally trivial in the étale topology. Then the projection p is affine. In this subsection, we study the direct image functor with respect to the projection p.

Let $\mathscr{A}_{\widetilde{X}}$ be a G-equivariant algebra of twisted differential operators on $\widetilde{X}$ equipped with a G-equivariant algebra homomorphism $R \colon \mathcal{U}(\mathfrak{g}) \to \mathscr{A}_{\widetilde{X}}$.

Proposition 3.10. *Assume that $p\colon \widetilde{X} \to X$ is a trivial bundle, i.e. $\widetilde{X} \simeq X \times G$. Then there exists some algebra $\mathscr{A}_X$ of twisted differential operators on X such that*

$$\mathscr{A}_{\widetilde{X}} \simeq \mathscr{A}_X \boxtimes \mathscr{D}_G$$

under the identification $\widetilde{X} \simeq X \times G$.

Proof. Let $\mu\colon \widetilde{X} \times G \to \widetilde{X}$ be the multiplication map and $p_1\colon \widetilde{X} \times G \to \widetilde{X}$ the projection. We define $s\colon X \to \widetilde{X} = X \times G$ via $x \mapsto (x, e)$ and $t\colon \widetilde{X} \to \widetilde{X} \times G$ via $X \times G \ni (x, g) \mapsto (x, e, g) \in X \times G \times G$. Then we have $\mu \circ t = \mathrm{id}_{\widetilde{X}}$ and a commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{t} & \widetilde{X} \times G & \xrightarrow{\mu} & \widetilde{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \\ X & \xrightarrow{s} & \widetilde{X}. & & \end{array}$$

We regard X and $\widetilde{X}$ at the bottom as G-varieties via the trivial actions and $\widetilde{X} \times G$ via the right translation on G. Then the morphisms are G-equivariant.

Since $\mathscr{A}_{\widetilde{X}}$ is G-equivariant, we have

$$\mu^{\#}\mathscr{A}_{\widetilde{X}} \simeq p_1^{\#}\mathscr{A}_{\widetilde{X}}.$$

We therefore obtain isomorphisms

$$\mathscr{A}_{\widetilde{X}} = (\mu \circ t)^{\#}\mathscr{A}_{\widetilde{X}} \simeq t^{\#}p_1^{\#}\mathscr{A}_{\widetilde{X}} \simeq p^{\#}s^{\#}\mathscr{A}_{\widetilde{X}} \simeq s^{\#}\mathscr{A}_{\widetilde{X}} \boxtimes \mathscr{D}_G$$

of G-equivariant algebras of twisted differential operators. □

For $\lambda \in (\mathfrak{g}^*)^G$, we set $I_\lambda := \mathrm{Ker}(\lambda) \subset \mathcal{U}(\mathfrak{g})$ and

$$\begin{aligned} \mathscr{A}_{X,\lambda} &:= (\mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} p_*\mathscr{A}_{\widetilde{X}})^G \\ &\simeq (p_*\mathscr{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})p_*\mathscr{A}_{\widetilde{X}})^G \\ &\simeq p_*\mathscr{A}_{\widetilde{X}}^G/(R(I_{-\lambda+\delta})p_*\mathscr{A}_{\widetilde{X}})^G \\ &\simeq (p_*\mathscr{A}_{\widetilde{X}}/p_*\mathscr{A}_{\widetilde{X}}R(I_{-\lambda}))^G, \end{aligned} \tag{3.4.1}$$

where $\delta \in (\mathfrak{g}^*)^G$ is the character $Z \mapsto \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(Z))$. Remark that

$$(R(I_{-\lambda+\delta})\mathcal{O}(G))^G = (\mathcal{O}(G)R(I_{-\lambda}))^G \subset \mathcal{D}_G.$$

Then $\mathscr{A}_{X,\lambda}$ is a G-equivariant algebra of twisted differential operators on X equipped with the homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathscr{A}_{X,\lambda}$ $(Z \mapsto \lambda({}^tZ))$. Here we consider X as a G-variety via the trivial action. Note that if λ is a character of G and $\mu \in (\mathfrak{g}^*)^G$, $\mathscr{A}_{X,\lambda+\mu}$ is isomorphic to $\mathcal{L}_\lambda \otimes_{\mathcal{O}_X} \mathscr{A}_{X,\mu} \otimes_{\mathcal{O}_X} \mathcal{L}_{-\lambda}$, where $\mathcal{L}_\lambda$ is the invertible $\mathcal{O}_X$-module corresponding to the line bundle $\widetilde{X} \times_G \mathbb{C}_\lambda \to X$.

For a while, we fix $\lambda \in (\mathfrak{g}^*)^G$. We write $Dp_{+,\lambda}\colon D^b_{qc}(p^{\#}\mathscr{A}_{X,\lambda}) \to D^b_{qc}(\mathscr{A}_{X,\lambda})$ for the direct image functor in Section 3.1. We define a right exact functor

$$p_{+,\lambda}(\mathcal{M}) := \mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{M}) \in \mathrm{Mod}_{qc}(\mathscr{A}_{X,\lambda})$$

for $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$. For a character λ of G, $p_{+,\lambda}(\mathcal{O}_{\widetilde{X}})$ is isomorphic to $\mathcal{L}_\lambda$ if G is reductive.

To see $Dp_{+,\lambda} \simeq Lp_{+,\lambda}$, we need Lemma 3.11.

Lemma 3.11. *$p^{\#}\mathscr{A}_{X,\lambda}$ is canonically isomorphic to $\mathscr{A}_{\widetilde{X}}$ as a G-equivariant algebra of twisted differential operators.*

Proof. Since $p\colon \widetilde{X} \to X$ is locally trivial, the multiplication map induces an isomorphism

$$\mathscr{A}_{\widetilde{X}\to X} = \mathcal{O}_{\widetilde{X}} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathscr{A}_{X,\lambda} \simeq \mathscr{A}_{\widetilde{X}}/\mathscr{A}_{\widetilde{X}} R(I_{-\lambda})$$

of sheaves. This is G-equivariant and an $(\mathcal{O}_{\widetilde{X}} \otimes \mathcal{U}(\mathfrak{g}), p^{-1}\mathscr{A}_{X,\lambda})$-bimodule isomorphism. By the definition of $p^{\#}\mathscr{A}_{X,\lambda}$, we obtain an isomorphism $\mathscr{A}_{\widetilde{X}} \simeq p^{\#}\mathscr{A}_{X,\lambda}$ of G-equivariant algebras of twisted differential operators. □

The isomorphism $\mathscr{A}_{\widetilde{X}} \simeq p^{\#}\mathscr{A}_{X,\lambda}$ can be obtained from the following cartesian square:

$$\begin{array}{ccc} p^*\mathcal{P}(\mathscr{A}_{X,\lambda}) & \xrightarrow{p^*\sigma} & p^*\mathcal{T}_X \\ \uparrow & & \uparrow \\ \mathcal{P}(\mathscr{A}_{\widetilde{X}}) \simeq p^*(p_*(\mathcal{P}(\mathscr{A}_{\widetilde{X}}))^G) & \longrightarrow & \mathcal{T}_{\widetilde{X}} \simeq p^*(p_*(\mathcal{T}_{\widetilde{X}})^G). \end{array}$$

This implies $\mathcal{P}(\mathscr{A}_{\widetilde{X}}) \simeq p^{\#}\mathcal{P}(\mathscr{A}_{X,\lambda})$ (see the discussions above Proposition 3.7). We identify $\mathscr{A}_{\widetilde{X}}$ with $p^{\#}\mathscr{A}_{X,\lambda}$ by the isomorphism.

We shall describe the right $\mathscr{A}_{\widetilde{X}}$-action on $\mathscr{A}_{X\leftarrow\widetilde{X}}$. Let π be the natural projection $\mathcal{T}^G_{\widetilde{X}} \twoheadrightarrow p^{-1}(\mathcal{T}_X)$. Here $\mathcal{T}^G_{\widetilde{X}}$ is the sheaf of G-invariant local sections of $\mathcal{T}_{\widetilde{X}}$, that is, $p^{-1}(p_*(\mathcal{T}_{\widetilde{X}})^G)$. Fix a basis $X_1, X_2, \dots, X_{\dim G} \in \mathfrak{g}$. Since $p\colon \widetilde{X} \to X$ is a principal G-bundle, $\Omega_p = p^{-1}(\Omega_X^{\vee}) \otimes_{p^{-1}\mathcal{O}_X} \Omega_{\widetilde{X}}$ is isomorphic to $\mathcal{O}_{\widetilde{X}}$ as an $\mathcal{O}_{\widetilde{X}}$-module. The isomorphism is given by

$$\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_{\dim X} \otimes \omega \mapsto \omega(\widetilde{\theta_1}, \widetilde{\theta_2}, \dots, \widetilde{\theta_{\dim X}}, R(X_1), R(X_2), \dots, R(X_{\dim G}))$$

for local sections $\theta_1, \theta_2, \dots, \theta_{\dim X} \in p^{-1}(\mathcal{T}_X)$ and $\omega \in \Omega_{\widetilde{X}}$, where each $\widetilde{\theta_i}$ is a local section of $\mathcal{T}^G_{\widetilde{X}}$ such that $\pi(\widetilde{\theta_i}) = \theta_i$. Since $[\mathcal{T}^G_{\widetilde{X}}, R(\mathfrak{g})] = 0$, the isomorphism commutes with the actions of $\mathcal{T}^G_{\widetilde{X}}$ on Ω_p and $\mathcal{O}_{\widetilde{X}}$ defined by the Lie derivative.

Lemma 3.12. *$\mathscr{A}_{X\leftarrow\widetilde{X}}$ is isomorphic to $\mathscr{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathscr{A}_{\widetilde{X}}$ as a $(p^{-1}\mathscr{A}_{X,\lambda}, \mathscr{A}_{\widetilde{X}})$-bimodule.*

Proof. Let $i\colon \Omega_p \to \mathcal{O}_{\widetilde{X}}$ be the above isomorphism. Composing i with the multiplication map, we obtain an isomorphism

$$\mathscr{A}_{X\leftarrow\widetilde{X}} = p^{-1}\mathscr{A}_{X,\lambda} \otimes_{p^{-1}\mathcal{O}_X} \Omega_p \to p^{-1}\mathscr{A}_{X,\lambda} \otimes_{p^{-1}\mathcal{O}_X} \mathcal{O}_{\widetilde{X}} \to \mathscr{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathscr{A}_{\widetilde{X}}$$

of sheaves, and denote it by ι. It is trivial that ι is a $(p^{-1}\mathscr{A}_{X,\lambda}, \mathcal{O}_{\widetilde{X}})$-module homomorphism. By Proposition 3.7, the action of $\theta \in \mathcal{P}(\mathscr{A}_{\widetilde{X}})^G$ is given by

$$(S \otimes \omega) \cdot \theta = S\theta \otimes \omega - S \otimes \sigma(\theta)\omega$$

for $S \otimes \omega \in \mathscr{A}_{X\leftarrow\widetilde{X}}$. Here $\sigma\colon \mathcal{P}(\mathscr{A}_{\widetilde{X}})^G \to \mathcal{T}^G_{\widetilde{X}}$ is the restriction of the morphism $\sigma\colon \mathcal{P}(\mathscr{A}_{\widetilde{X}}) \to \mathcal{T}_{\widetilde{X}}$ attached to the Picard algebroid. Hence ι commutes with the $\mathcal{P}(\mathscr{A}_{\widetilde{X}})^G$-action by the definition of i. Since $\mathcal{P}(\mathscr{A}_{\widetilde{X}})$ is generated by $\mathcal{P}(\mathscr{A}_{\widetilde{X}})^G$ as an $\mathcal{O}_{\widetilde{X}}$-module, ι is a $(p^{-1}\mathscr{A}_{X,\lambda}, \mathscr{A}_{\widetilde{X}})$-bimodule isomorphism. □

Remark 3.13. Suppose that G is not unimodular. Although ι is a right $\mathfrak{g}$-homomorphism, the isomorphism $p^{-1}\mathscr{A}_{X,\lambda} \otimes_{p^{-1}\mathcal{O}_X} \Omega_p \to p^{-1}\mathscr{A}_{X,\lambda} \otimes_{p^{-1}\mathcal{O}_X} \mathcal{O}_{\widetilde{X}}$ is not. This is because $i\colon \Omega_p \to \mathcal{O}_{\widetilde{X}}$ is not G-equivariant.

We fix the isomorphism $\mathscr{A}_{X\leftarrow\widetilde{X}} \simeq \mathscr{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathscr{A}_{\widetilde{X}}$.

Proposition 3.14. *$Dp_{+,\lambda}$ is isomorphic to the left derived functor $Lp_{+,\lambda}$ of $p_{+,\lambda}$.*

Proof. By Proposition 3.3, $Dp_{+,\lambda}$ is isomorphic to $p_*\mathscr{A}_{X\leftarrow\widetilde{X}} \otimes^L_{p_*\mathscr{A}_{\widetilde{X}}} p_*(\cdot)$. Hence it is enough to show that there is a natural isomorphism

$$p_*(\mathscr{A}_{X\leftarrow\widetilde{X}}) \otimes_{p_*(\mathscr{A}_{\widetilde{X}})} p_*(\mathcal{F}) \simeq \mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F})$$

for any locally free $\mathscr{A}_{\widetilde{X}}$-module $\mathcal{F}$.

Since p is affine, we have $p_*(\mathscr{A}_{X\leftarrow\widetilde{X}}) \simeq p_*(\mathscr{A}_{\widetilde{X}})/R(I_{-\lambda+\delta})p_*(\mathscr{A}_{\widetilde{X}})$ by Lemma 3.12. This implies

$$\begin{aligned} p_*(\mathscr{A}_{X\leftarrow\widetilde{X}}) \otimes_{p_*(\mathscr{A}_{\widetilde{X}})} p_*(\mathcal{F}) &\simeq p_*(\mathcal{F})/R(I_{-\lambda+\delta})p_*(\mathcal{F}) \\ &\simeq \mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F}). \end{aligned}$$

We have proved the proposition. □

Lemma 3.15. *Let U be an open subset of X. If $p_*(\mathscr{A}_{\widetilde{X}})|_U$ is acyclic (e.g. U is affine), $\Gamma(U, p_*\mathscr{A}_{\widetilde{X}})$ is a projective left/right $\mathcal{U}(\mathfrak{g})$-module.*

Proof. Since the bundle $p\colon \widetilde{X} \to X$ is locally trivial in the étale topology, we can take an affine étale covering $\{U_j \to U\}$ such that $U_j \times_X \widetilde{X} \to U_j$ is a trivial principal G-bundle. Since $p_*\mathscr{A}_{\widetilde{X}}$ is a quasi-coherent $\mathcal{O}_X$-module, the cohomology group $H^i(U, p_*\mathscr{A}_{\widetilde{X}})$ is isomorphic to the étale cohomology group $H^i(U_{\text{ét}}, (p_*\mathscr{A}_{\widetilde{X}})_{\text{ét}})$ for any i. Here $(p_*\mathscr{A}_{\widetilde{X}})_{\text{ét}}$ is the étale sheaf associated to $p_*\mathscr{A}_{\widetilde{X}}$. Hence the Čech complex

$$0 \to \Gamma(U, p_*\mathscr{A}_{\widetilde{X}}) \to C^0 \to C^1 \to \cdots$$

associated to the covering $\{U_j \to U\}$ is exact and each term C^j is a free left/right $\mathcal{U}(\mathfrak{g})$-module. $\Gamma(U, p_*\mathscr{A}_{\widetilde{X}})$ is therefore a projective left/right $\mathcal{U}(\mathfrak{g})$-module. □

By Lemma 3.15 and Proposition 3.14, we obtain Theorem 3.16. Note that for a generalized pair $(\mathcal{A}, G)$ and a left $\mathcal{A}$-module M, $\operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, M)$ admits a natural $(\mathcal{A}/I_{-\lambda+\delta}\mathcal{A})^G$-module structure if $\mathcal{A}$ is a flat left $\mathcal{U}(\mathfrak{g})$-module. In fact, $\operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, M)$ can be computed by using a free resolution of the $\mathcal{A}$-module M.

Theorem 3.16. *For any $\mathcal{M} \in \operatorname{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$, we have a natural isomorphism*

$$D^{-i}p_{+,\lambda}(\mathcal{M}) \simeq \operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, p_*(\mathcal{M}))$$

of $\mathscr{A}_{X,\lambda}$-modules, where $\operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, p_*(\mathcal{M}))$ denotes the sheafification of the presheaf $(U \to \operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, \Gamma(p^{-1}(U), \mathcal{M})))$.

By Theorem 3.16, there is a natural homomorphism

$$\operatorname{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, \Gamma(\mathcal{M})) \to \Gamma(D^{-i}p_{+,\lambda}(\mathcal{M}))$$

of $\mathcal{A}_{\widetilde{X}}^G$-modules, where $\mathcal{A}_{\widetilde{X}} = \Gamma(\mathscr{A}_{\widetilde{X}})$. In general, it is not an isomorphism. Under some assumption, we can show that the homomorphism is an isomorphism as follows. Put $\mathcal{A}_{X,\lambda} := \Gamma(\mathscr{A}_{X,\lambda})$.

Lemma 3.17. *Assume that $\mathscr{A}_{\widetilde{X}}$ is acyclic. Then the natural homomorphism $(\mathcal{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathcal{A}_{\widetilde{X}})^G \to \mathcal{A}_{X,\lambda}$ is bijective. Moreover, for a free $\mathscr{A}_{\widetilde{X}}$-module $\mathcal{F}$, the natural homomorphism $\mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} \Gamma(\mathcal{F}) \to \Gamma(p_{+,\lambda}(\mathcal{F}))$ is an isomorphism of $\mathcal{A}_{X,\lambda}$-modules.*

Proof. Let $\mathcal{F}$ be a free $\mathscr{A}_{\widetilde{X}}$-module. Since p is affine and $\mathscr{A}_{\widetilde{X}}$ is acyclic, $p_*(\mathcal{F})$ is also acyclic. Take a free resolution $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{C}_{\lambda-\delta} \to 0$. By Lemma 3.15, the following sequence is exact:

$$\cdots \to P_1 \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F}) \xrightarrow{d_1} P_0 \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F}) \xrightarrow{d_0} p_{+,\lambda}(\mathcal{F}) \to 0.$$

Since all $P_i \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F})$ are acyclic, $\mathrm{Ker}(d_0)$ is also acyclic, and hence

$$\Gamma(P_1 \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F})) \xrightarrow{d_1} \Gamma(P_0 \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F})) \xrightarrow{d_0} \Gamma(p_{+,\lambda}(\mathcal{F})) \to 0$$

is exact. Since each P_i is free, we have $\Gamma(P_i \otimes_{\mathcal{U}(\mathfrak{g})} p_*(\mathcal{F})) \simeq P_i \otimes_{\mathcal{U}(\mathfrak{g})} \Gamma(\mathcal{F})$. This implies that the natural homomorphism $\mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} \Gamma(\mathcal{F}) \to \Gamma(p_{+,\lambda}(\mathcal{F}))$ is bijective. Hence the second assertion follows from the first one.

Since $(\cdot)^G$ is left exact, we have

$$\mathcal{A}_{X,\lambda} = \Gamma(p_{+,\lambda}(\mathscr{A}_{\widetilde{X}})^G) = \Gamma(p_{+,\lambda}(\mathscr{A}_{\widetilde{X}}))^G \simeq (\mathcal{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathcal{A}_{\widetilde{X}})^G.$$

This implies that the natural algebra homomorphism $(\mathcal{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathcal{A}_{\widetilde{X}})^G \to \mathcal{A}_{X,\lambda}$ is an isomorphism. □

Theorem 3.18. *Let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$ and $i \in \mathbb{N}$. Assume that the global section functors are exact on $\mathrm{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$ and $\mathrm{Mod}_{qc}(\mathscr{A}_{X,\lambda})$. Then the natural homomorphism*

$$\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, \Gamma(\mathcal{M})) \to \Gamma(D^{-i}p_{+,\lambda}(\mathcal{M}))$$

is an isomorphism of $\mathcal{A}_{X,\lambda}$-modules.

Proof. Remark that any object in $\mathrm{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$ is acyclic because Γ is exact on $\mathrm{Mod}_{qc}(\mathscr{A}_{\widetilde{X}})$. Take a free resolution $\mathcal{F}^\bullet$ of $\mathcal{M}$. By Lemma 3.15, $\Gamma(\mathcal{F}^\bullet)$ is a projective resolution of $\Gamma(\mathcal{M})$ as a $\mathfrak{g}$-module. Hence we have

$$\begin{aligned}
\Gamma(D^{-i}p_{+,\lambda}(\mathcal{M})) &\simeq \Gamma(H^{-i}(p_{+,\lambda}(\mathcal{F}^\bullet))) \\
&\simeq H^{-i} \circ \Gamma(p_{+,\lambda}(\mathcal{F}^\bullet)) \\
&\simeq H^{-i}(\mathbb{C}_{\lambda-\delta} \otimes_{\mathcal{U}(\mathfrak{g})} \Gamma(\mathcal{F}^\bullet)) \\
&\simeq \mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_{\lambda-\delta}, \Gamma(\mathcal{M})).
\end{aligned}$$

Here the third isomorphism follows from Lemma 3.17. □

Remark 3.19. One can prove a similar result about the commutativity of $R\Gamma$ and $\mathbb{C}_{\lambda-\delta} \otimes^L_{\mathcal{U}(\mathfrak{g})} (\cdot)$ without the exactness of Γ.

Γ is exact for any affine variety and λ, or for any flag variety and good λ (see Fact 7.1). To apply Theorem 3.18 to direct products of such varieties, we shall show Lemma 3.20.

Lemma 3.20. *Let X and Y be smooth varieties and $\mathscr{A}_X$ (resp. $\mathscr{A}_Y$) an algebra of twisted differential operators on X (resp. Y). If the global section functors on $\mathrm{Mod}_{qc}(\mathscr{A}_X)$ and $\mathrm{Mod}_{qc}(\mathscr{A}_Y)$ are exact, then the global section functor on $\mathrm{Mod}_{qc}(\mathscr{A}_X \boxtimes \mathscr{A}_Y)$ is also exact.*

Proof. Let $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ be a short exact sequence of quasi-coherent $\mathscr{A}_X \boxtimes \mathscr{A}_Y$-modules. Let U be an affine open subset of X. We write $p\colon U \times Y \to Y$

for the projection onto the second factor. Then p is affine. Hence we obtain a short exact sequence

$$0 \to p_*(\mathcal{M}_1|_{U\times Y}) \to p_*(\mathcal{M}_2|_{U\times Y}) \to p_*(\mathcal{M}_3|_{U\times Y}) \to 0$$

of quasi-coherent $\mathscr{A}_Y$-modules. Since the global section functor Γ on $\mathrm{Mod}_{qc}(\mathscr{A}_Y)$ is exact, we obtain a short exact sequence

$$0 \to \Gamma(U\times Y, \mathcal{M}_1) \to \Gamma(U\times Y, \mathcal{M}_2) \to \Gamma(U\times Y, \mathcal{M}_3) \to 0. \tag{3.4.2}$$

Let $q\colon X\times Y \to X$ be the projection onto the first factor. By (3.4.2),

$$q_*\colon \mathrm{Mod}_{qc}(\mathscr{A}_X \boxtimes \mathscr{A}_Y) \to \mathrm{Mod}_{qc}(\mathscr{A}_X)$$

is exact. Since the global section functor Γ on $\mathrm{Mod}_{qc}(\mathscr{A}_X)$ is exact, this implies that the sequence $0 \to \Gamma(\mathcal{M}_1) \to \Gamma(\mathcal{M}_2) \to \Gamma(\mathcal{M}_3) \to 0$ is exact. We have proved the lemma. □

4. Uniformly bounded family

The purpose of this section is to reformulate Bernstein's work [6] about the multiplicity of a $\mathcal{D}_{\mathbb{C}^n}$-module. We will introduce the notion of uniformly bounded families of twisted $\mathscr{D}$-modules. A uniformly bounded family is a family with a good boundedness property, which is preserved by direct images and inverse images. We give several applications of uniformly bounded families in Section 7.

4.1. **Multiplicity and functors.** In this subsection, we review Bernstein's work [6] about the multiplicity (or the Bernstein degree) of a $\mathcal{D}_{\mathbb{C}^n}$-module. We refer the reader to [6], [21, 3.2.2] and [11, 1.§3 and §4] for the proof of facts.

Let $\mathscr{D}_{\mathbb{C}^n}$ be the algebra of non-twisted differential operators on $\mathbb{C}^n$ and $\mathcal{D}_{\mathbb{C}^n}$ the algebra of global sections of $\mathscr{D}_{\mathbb{C}^n}$. Let $(x_1, x_2, \dots, x_n)$ be the standard coordinate of $\mathbb{C}^n$ and put $\partial_i = \partial/\partial x_i$. We denote by F the Bernstein filtration of $\mathcal{D}_{\mathbb{C}^n}$, and then $F_0\mathcal{D}_{\mathbb{C}^n} = \mathbb{C}, F_1\mathcal{D}_{\mathbb{C}^n} = \mathrm{span}_{\mathbb{C}}\{1, x_1, x_2, \dots, x_n, \partial_1, \partial_2, \dots, \partial_n\}, F_i\mathcal{D}_{\mathbb{C}^n} = (F_1\mathcal{D}_{\mathbb{C}^n})^i$. Facts 4.1 and 4.2 are essential for our study of a family of $\mathscr{D}$-modules.

Fact 4.1. Let M be a finitely generated $\mathcal{D}_{\mathbb{C}^n}$-module and M_0 a generating subspace of M of finite dimension. Put $F_iM := F_i\mathcal{D}_{\mathbb{C}^n}\cdot M_0$. Then

(i) there exists some polynomial $f \in \mathbb{Q}[t]$ such that $f(i) = \dim_{\mathbb{C}}(F_iM)$ for any $i \gg 0$
(ii) $d(M) := \deg(f)$ does not depend on M_0
(iii) the coefficient $a_{d(M)}$ of $f(t) = a_{d(M)}t^{d(M)} +$ (lower terms) does not depend on M_0
(iv) $m(M) := a_{d(M)}\cdot d(M)!$ is a natural number
(v) $d(M) \geq n$ if M is non-zero.

The integer $m(M)$ is called the *multiplicity* (or the Bernstein degree) of M. A $\mathcal{D}_{\mathbb{C}^n}$-module M is said to be *holonomic* if M is finitely generated and $d(M) = n$ or $d(M) = 0$ holds. A $\mathcal{D}_{\mathbb{C}^n}$-module M is holonomic if and only if the corresponding $\mathscr{D}_{\mathbb{C}^n}$-module $\mathscr{D}_{\mathbb{C}^n}\otimes_{\mathcal{D}_{\mathbb{C}^n}} M$ is holonomic (see [21, Proposition 3.2.11]). We put $m(\mathcal{M}) := m(\Gamma(\mathcal{M}))$ for $\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_{\mathbb{C}^n})$ and

$$m(\mathcal{M}^\bullet) := \sum_i m(H^i(\mathcal{M}^\bullet))$$

for $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_{\mathbb{C}^n})$.

Fact 4.2. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathcal{D}_{\mathbb{C}^n}$-modules. Then we have $d(M) = \max(d(L), d(N))$. If in addition $d(L) = d(N)$, then $m(M) = m(L) + m(N)$ holds. In particular, the length of a holonomic $\mathcal{D}_{\mathbb{C}^n}$-module is less than or equal to its multiplicity.

Fact 4.3 is an easy consequence of the definition of the multiplicity. See the proof of [6, Theorem 3.2].

Fact 4.3. Let N and M be modules of $\mathcal{D}_{\mathbb{C}^n}$ and $\mathcal{D}_{\mathbb{C}^m}$, respectively. If N and M are holonomic, then $N \boxtimes M$ is holonomic and we have $m(N \boxtimes M) = m(N)m(M)$. Conversely, if $N \boxtimes M$ is holonomic, then N and M are holonomic.

We need a derived functor version of [6, Theorem 3.2]. The proof is the same as the original version.

Fact 4.4. Let $f\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism of affine varieties. Set $d := \max(\deg(f), 1)$. Then for any $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_{\mathbb{C}^n})$ and $\mathcal{N}^\bullet \in D^b_h(\mathscr{D}_{\mathbb{C}^m})$, we have

$$m(Df_+(\mathcal{M}^\bullet)) \le d^{n+m} m(\mathcal{M}^\bullet),$$
$$m(Lf^*(\mathcal{N}^\bullet)) \le d^{n+m} m(\mathcal{N}^\bullet).$$

4.2. **$\mathscr{D}$-modules on affine varieties.** In Fact 4.4, we have seen that the multiplicity is well-behaved for operations of $\mathscr{D}$-modules on affine spaces. In this subsection, we consider similar results about $\mathscr{D}$-modules on affine varieties.

We recall the Kashiwara equivalence [21, Theorem 1.6.1].

Fact 4.5. Let $f\colon X \to Y$ be a closed embedding of smooth varieties. Then

$$f_+ := D^0 f_+ \colon \mathrm{Mod}_{qc}(\mathscr{D}_X) \to \mathrm{Mod}^X_{qc}(\mathscr{D}_Y) \text{ and}$$
$$Df_+ \colon D^b_{qc}(\mathscr{D}_X) \to D^{b,X}_{qc}(\mathscr{D}_Y)$$

give equivalences of categories. Here $\mathrm{Mod}^X_{qc}(\mathscr{D}_Y)$ is the full subcategory of $\mathrm{Mod}_{qc}(\mathscr{D}_Y)$ whose objects are supported on X, and $D^{b,X}_{qc}(\mathscr{D}_Y)$ is the full subcategory of $D^b_{qc}(\mathscr{D}_Y)$ consisting of complexes whose cohomologies are supported on X.

For $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$ on a smooth variety X and a closed embedding $\iota\colon X \to \mathbb{C}^n$, we set

$$m_\iota(\mathcal{M}^\bullet) := m(D\iota_+(\mathcal{M}^\bullet)).$$

Proposition 4.6. *Let $f\colon X \to Y$ be a morphism of affine smooth varieties. Fix closed embeddings $\iota\colon X \to \mathbb{C}^n$ and $\iota'\colon Y \to \mathbb{C}^m$. Then there exists a constant $C > 0$ such that*

$$m_{\iota'}(Df_+(\mathcal{M}^\bullet)) \le C \cdot m_\iota(\mathcal{M}^\bullet), \tag{4.2.1}$$
$$m_\iota(Lf^*(\mathcal{N}^\bullet)) \le C \cdot m_{\iota'}(\mathcal{N}^\bullet) \tag{4.2.2}$$

for any $\mathcal{M}^\bullet \in D^b_h(\mathscr{D}_X)$ and $\mathcal{N}^\bullet \in D^b_h(\mathscr{D}_Y)$.

Proof. Fix an extension $\widetilde{f}$ of f to $\mathbb{C}^n$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ \mathbb{C}^n & \xrightarrow{\widetilde{f}} & \mathbb{C}^m \end{array}$$

is commutative. Set $d := \max(\deg(\widetilde{f}), 1)$. By Fact 4.4, we obtain

$$m_{\iota'}(Df_+(\mathcal{M}^\bullet)) = m(D\widetilde{f}_+(D\iota_+(\mathcal{M}^\bullet))) \leq d^{n+m} m_\iota(\mathcal{M}^\bullet).$$

Here we used $D\iota'_+ \circ Df_+ = D\widetilde{f}_+ \circ D\iota_+$ (Fact 3.4(i)). We have shown the first inequality (4.2.1).

Consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & X \times Y & \xrightarrow{f_2} & Y \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota\times\iota'} & & \\ \mathbb{C}^n & \xrightarrow{\widetilde{f}_1} & \mathbb{C}^n \times \mathbb{C}^m, & & \end{array}$$

where $f_1(x) = (x, f(x))$, $f_2(x, y) = y$ and $\widetilde{f}_1(x) = (x, \widetilde{f}(x))$. Since the left square is cartesian, we have an isomorphism $D\iota_+ \circ Lf_1^* \simeq L\widetilde{f}_1^* \circ D(\iota \times \iota')_+$ of functors up to shift by the base change theorem (Fact 3.5). Hence we obtain

$$\begin{aligned} m_{\iota_+}(Lf^*(\mathcal{N}^\bullet))) &= m(L\widetilde{f}_1^* \circ D(\iota \times \iota')_+ \circ Lf_2^*(\mathcal{N}^\bullet)) \\ &\leq d^{n+m} m(D\iota_+(\mathcal{O}_X) \boxtimes D\iota'_+(\mathcal{N}^\bullet)) \\ &= d^{n+m} m_\iota(\mathcal{O}_X) m_{\iota'}(\mathcal{N}^\bullet) \end{aligned}$$

by Fact 4.3 and Fact 4.4, which proves the second inequality (4.2.2). □

In the next subsection, we will consider families of twisted $\mathscr{D}$-modules on general smooth varieties. Although the multiplicity itself is no longer a meaningful value for general smooth varieties, boundedness of multiplicities of twisted $\mathscr{D}$-modules can be defined.

To reduce properties of twisted $\mathscr{D}$-modules on a non-affine variety to that of affine spaces, we have many choices of affine étale coverings, closed embeddings to affine spaces, and local trivializations of an algebra of twisted differential operators. We shall consider the effect on the multiplicity by the choices.

Proposition 4.7. *Let $f\colon X \to Y$ be a surjective étale morphism of affine smooth varieties. Fix closed embeddings $\iota\colon X \to \mathbb{C}^n$ and $\iota'\colon Y \to \mathbb{C}^m$. Then there exists a constant $C > 0$ such that*

$$C^{-1} \cdot m_{\iota'}(\mathcal{N}^\bullet) \leq m_\iota(Lf^*(\mathcal{N}^\bullet)) \leq C \cdot m_{\iota'}(\mathcal{N}^\bullet)$$

for any $\mathcal{N}^\bullet \in D^b_h(\mathscr{D}_Y)$.

Proof. We have proved the second inequality in Proposition 4.6. We shall show the first inequality.

Since f is smooth, f^* is exact [21, Proposition 1.5.13]. Hence we can assume $\mathcal{N} \in \mathrm{Mod}_h(\mathscr{D}_Y)$.

Since f is étale, $f_*(\mathcal{M})$ admits a natural $\mathscr{D}_Y$-module structure for $\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_X)$ and the direct image functor Df_+ is isomorphic to $Rf_* = f_*$ by [15, Theorem 2.2]. Hence the canonical morphism $\mathcal{N} \to f_*(f^*(\mathcal{N}))$ of $\mathcal{O}_Y$-modules is a morphism of $\mathscr{D}_Y$-modules. The morphism is monomorphic since f is surjective. Applying Proposition 4.6 to $f^*(\mathcal{N})$, we obtain

$$m_{\iota'}(\mathcal{N}) \leq m_{\iota'}(f_*(f^*(\mathcal{N}))) \leq C \cdot m_\iota(f^*(\mathcal{N})),$$

where C is a constant independent of $\mathcal{N}$. □

Hereafter we consider the effect on the multiplicity by twisting by automorphisms.

Let X be a smooth affine variety and $\iota\colon X \to \mathbb{C}^n$ a closed embedding. The automorphism group $\mathrm{Aut}(\mathscr{D}_X)$ is isomorphic to the additive group $\mathcal{Z}(X)$ of closed 1-forms on X [5]. For $\omega \in \mathcal{Z}(X)$, we denote by A_ω the corresponding automorphism given by

$$A_\omega(T) = T - \omega(T) \in \mathcal{T}_X \oplus \mathcal{O}_X$$

for $T \in \mathcal{T}_X$. A $\mathscr{D}_X$-module $\mathcal{M}$ can be twisted by A_ω and the twisted module is denoted by $\mathcal{M}^\omega$. We use the same notation for a complex of $\mathscr{D}_X$-modules, e.g. $(\mathcal{M}^\bullet)^\omega$.

Lemma 4.8. *Let W be a finite-dimensional subspace of $\mathcal{Z}(X)$. Then there exists a constant C such that*

$$m_\iota(\mathcal{O}_X^\omega) \le C$$

for any $\omega \in W$.

Proof. Put $\mathcal{M} := \mathcal{O}_X \otimes \mathcal{O}(W)$ equipped with a $\mathcal{T}_X$-action via

$$T \cdot (f \otimes g) = Tf \otimes g - \sum_i \omega_i(T) f \otimes \lambda_i g \qquad (T \in \mathcal{T}_X),$$

where $\{\omega_i\}_i$ is a basis of W and $\{\lambda_i\}_i$ is its dual basis. Then the action on $\mathcal{M}$ extends to a $\mathscr{D}_X \otimes \mathcal{O}(W)$-action.

We denote by $\mathfrak{m}_\omega$ the maximal ideal of $\mathcal{O}(W)$ corresponding to $\omega \in W$. Then by definition, we have $\mathcal{M}/\mathfrak{m}_\omega\mathcal{M} \simeq \mathcal{O}_X^\omega$ for any $\omega \in W$. Since the functors ι_+ and Γ are exact, we have

$$\Gamma(\iota_+(\mathcal{O}_X^\omega)) \simeq \Gamma(\iota_+(\mathcal{M}))/\mathfrak{m}_\omega\Gamma(\iota_+(\mathcal{M})).$$

Put $M := \Gamma(\iota_+(\mathcal{M}))$.

Since the functors ι_+ and Γ preserve the lattice of submodules, M is noetherian and hence finitely generated as a $\mathcal{D}_{\mathbb{C}^n} \otimes \mathcal{O}(W)$-module. Take a finite generating subspace $S \subset M$ and put $F_iM := (F_i\mathcal{D}_{\mathbb{C}^n} \otimes \mathcal{O}(W))S$ for $i \ge 0$. Then the associated graded module $\mathrm{gr}^F M$ is a finitely generated $\mathcal{O}(\mathbb{C}^n \times W)$-module.

By [38, Theorem 24.1], we can take an affine open subset U of W such that $\mathcal{O}(U) \otimes_{\mathcal{O}(W)} \mathrm{gr}^F M$ is a free $\mathcal{O}(U)$-module. Hence $\mathcal{O}(U) \otimes_{\mathcal{O}(W)} F_iM$ is a projective $\mathcal{O}(U)$-module for any $i \ge 0$. This implies that the function

$$W \ni \omega \mapsto \dim_{\mathbb{C}}(F_iM/\mathfrak{m}_\omega F_iM)$$

is constant on U. Hence $U \ni \omega \mapsto m(M/\mathfrak{m}_\omega M)$ is a constant function by the definition of the multiplicity.

Replacing W by $W\backslash U$ and M by $\mathcal{O}(W\backslash U) \otimes_{\mathcal{O}(W)} M$, and repeating this argument, we can see that $m(M/\mathfrak{m}_\omega M)$ is bounded on W. □

Remark 4.9. Lemma 4.8 can be considered as a special case of [2, Theorem 3.18] and the latter half of our proof is essentially the same as theirs.

Corollary 4.10. *Let $\mathcal{M}^\bullet \in D_h^b(\mathscr{D}_X)$ and W be a finite-dimensional subspace of $\mathcal{Z}(X)$. Then there exists a constant C independent of $\mathcal{M}^\bullet$ such that*

$$m_\iota((\mathcal{M}^\bullet)^\omega) \le C \cdot m_\iota(\mathcal{M}^\bullet)$$

for any $\omega \in W$.

Proof. Fix $\omega \in W$. Then we have $(\mathcal{M}^\bullet)^\omega \simeq \mathcal{M}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{O}^\omega_X$. Note that for any smooth variety Y and $\mathcal{N}^\bullet, \mathcal{L}^\bullet \in D^b_{qc}(\mathscr{D}_Y)$, the tensor product $\mathcal{N}^\bullet \otimes^L_{\mathcal{O}_Y} \mathcal{L}^\bullet$ is isomorphic to $L\Delta^*_Y(\mathcal{N}^\bullet \boxtimes \mathcal{L}^\bullet)$, where $\Delta_Y : Y \to Y \times Y$ is the diagonal embedding (see [21, p. 39]). Applying the base change theorem (Fact 3.5) to the following cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota \times \iota} \\ \mathbb{C}^n & \xrightarrow{\Delta_{\mathbb{C}^n}} & \mathbb{C}^n \times \mathbb{C}^n, \end{array}$$

we have

$$D\iota_+((\mathcal{M}^\bullet)^\omega) \simeq D\iota_+(\mathcal{M}^\bullet) \otimes^L_{\mathcal{O}_{\mathbb{C}^n}} \iota_+(\mathcal{O}^\omega_X)$$

up to shift. By Facts 4.3 and 4.4, we obtain

$$m_\iota((\mathcal{M}^\bullet)^\omega) = m(D\iota_+(\mathcal{M}^\bullet) \otimes^L_{\mathcal{O}_{\mathbb{C}^n}} \iota_+(\mathcal{O}^\omega_X)) \le m_\iota(\mathcal{M}^\bullet) m_\iota(\mathcal{O}^\omega_X).$$

This inequality and Lemma 4.8 imply the assertion. □

4.3. **Uniformly bounded family.** We shall define a good local trivialization of a family of algebras of twisted differential operators. For an étale map $\varphi\colon U \to V$, we denote by $(\cdot)|_U$ the functors $\varphi^\#(\cdot)$ and $\varphi^*(\cdot)$ by abuse of notation.

Let $\mathscr{A}_{X,\Lambda} = (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ be a family of algebras of twisted differential operators on a smooth variety X. Hereafter we deal with $\prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})$ the direct product of categories and its derived category. Set

$$\mathrm{Mod}_h(\mathscr{A}_{X,\Lambda}) := \prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda}),$$
$$D^b_h(\mathscr{A}_{X,\Lambda}) := \prod_{\lambda\in\Lambda} D^b_h(\mathscr{A}_{X,\lambda}).$$

We denote by $H^i, Df_+, Lf^*, (\cdot)|_U$ and $f^\#$ the direct products of the corresponding functors by abuse of notation.

Recall that $\mathcal{Z}(X)$ is the space of closed 1-forms on X, which is isomorphic to $\mathrm{Aut}(\mathscr{D}_X)$ as an abelian group.

Definition 4.11. We say that a tuple (U, φ, Φ) is a *trivialization* of $\mathscr{A}_{X,\Lambda}$ if U is a smooth variety, $\varphi\colon U \to X$ is a surjective étale morphism and Φ is a family of isomorphisms $\Phi_\lambda\colon \mathscr{A}_{X,\lambda}|_U \xrightarrow{\simeq} \mathscr{D}_U$.

Let $T_1 = (U, \varphi, \Phi)$ and $T_2 = (V, \psi, \Psi)$ be trivializations of $\mathscr{A}_{X,\Lambda}$. We denote by $\mathcal{Z}(T_1, T_2) \subset \mathcal{Z}(U \times_X V)$ the image of

$$\left\{ \widetilde{\varphi}^\# \Psi_\lambda \circ (\widetilde{\psi}^\# \Phi_\lambda)^{-1} : \lambda \in \Lambda \right\}$$

by the isomorphism $\mathrm{Aut}(\mathscr{D}_{U\times_X V}) \to \mathcal{Z}(U \times_X V)$. Here $\widetilde{\varphi}\colon U \times_X V \to V$ and $\widetilde{\psi}\colon U \times_X V \to U$ are the projections of the fiber product. We write $T_1 \sim T_2$ when $\mathcal{Z}(T_1, T_2)$ spans a finite-dimensional subspace of $\mathcal{Z}(U \times_X V)$.

We say that a trivialization (U, φ, Φ) is *bounded* if $(U, \varphi, \Phi) \sim (U, \varphi, \Phi)$ holds.

Remark 4.12. Let $T = (U, \varphi, \Phi)$ be a trivialization of $\mathscr{A}_{X,\Lambda}$. Then any element of $\mathcal{Z}(T, T)$ is a 1-cocycle of the Čech complex of the sheaf of closed 1-forms on X with respect to the étale covering $\varphi\colon U \to X$. Hence for each $\lambda \in \Lambda$, we have a 1-cocycle $c(\lambda) \in \mathcal{Z}(T, T)$, and the cocycle defines an algebra $\mathscr{D}_{X,c(\lambda)}$ of twisted differential

operators on X. Then Φ_λ extends to an isomorphism $\Phi'_\lambda\colon \mathscr{A}_{X,\lambda} \to \mathscr{D}_{X,c(\lambda)}$. It is obvious that the correspondence

$$(U,\varphi,\Phi) \mapsto (U,\varphi,(c(\lambda))_{\lambda\in\Lambda},(\Phi'_\lambda)_{\lambda\in\Lambda})$$

is one-to-one. One can use such tuples instead of our trivializations.

Definition 4.13. Let $T=(U,\varphi,\Phi)$ be a trivialization of $\mathscr{A}_{X,\Lambda}$ and $f\colon Y\to X$ a morphism of smooth varieties. We set $f^{\#}T := (U\times_X Y, \widetilde{\varphi}, \widetilde{f}^{\#}\Phi)$, where $\widetilde{\varphi}\colon U\times_X Y\to Y$ and $\widetilde{f}\colon U\times_X Y\to U$ are the projections of the fiber product.

It is clear that $f^{\#}T$ is a trivialization of $f^{\#}\mathscr{A}_{X,\Lambda}$.

The relation $\sim$ is clearly symmetric and not reflexive in general. We shall show fundamental properties of bounded trivializations. Lemma 4.14 is well-known and easy.

Lemma 4.14. *Let $f\colon U\to V$ be a morphism of smooth varieties. Then the following diagram of abelian groups is commutative:*

$$\begin{array}{ccc} \mathrm{Aut}(\mathscr{D}_V) & \xrightarrow{f^{\#}} & \mathrm{Aut}(\mathscr{D}_U) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathcal{Z}(V) & \xrightarrow{f^*} & \mathcal{Z}(U). \end{array}$$

If, in addition, f is dominant, then f^* is injective.

Proposition 4.15. *Let $T_i=(U_i,\varphi_i,\Phi_i)$ $(i=1,2,3)$ be trivializations of $\mathscr{A}_{X,\Lambda}$ and $f\colon Y\to X$ a morphism of smooth varieties.*

(i) *$\sim$ is transitive, i.e. $T_1\sim T_2$ and $T_2\sim T_3 \Rightarrow T_1\sim T_3$.*
(ii) *$T_1\sim T_2 \Rightarrow f^{\#}T_1\sim f^{\#}T_2$.*
(iii) *If T_1 is bounded, then so is $f^{\#}T_1$.*
(iv) *If f is dominant, the converse of* (ii) *and* (iii) *is true.*

Proof. To show (i), let $f_{ij}\colon U_1\times_X U_2\times_X U_3\to U_i\times_X U_j$ be the projections of the fiber product for $(i,j)=(1,2),(2,3),(1,3)$. Assume $T_1\sim T_2$ and $T_2\sim T_3$. Then we have

$$f_{13}^*(\mathcal{Z}(T_1,T_3)) \subset f_{12}^*(\mathcal{Z}(T_1,T_2)) + f_{23}^*(\mathcal{Z}(T_2,T_3))$$

by Lemma 4.14. Since f_{13} is surjective, f_{13}^* is injective. Hence $\mathcal{Z}(T_1,T_3)$ spans a finite-dimensional subspace of $\mathcal{Z}(U_1\times_X U_3)$.

By definition, (iii) follows from (ii). We shall show (ii) and (iv). Let $\widetilde{f}\colon U_1\times_X U_2\times_X Y\to U_1\times_X U_2$ be the projection. By Lemma 4.14, we have

$$\widetilde{f}^*(\mathcal{Z}(T_1,T_2)) = \mathcal{Z}(f^{\#}T_1, f^{\#}T_2).$$

This implies (ii) and (iv). □

By Proposition 4.15, the relation $\sim$ is an equivalence relation of bounded trivializations.

Definition 4.16. An equivalence class of bounded trivializations is called a *bornology* of the family $\mathscr{A}_{X,\Lambda}$.

If $\mathcal{B}$ is a bornology of $\mathscr{A}_{X,\Lambda}$ and $(\lambda(i))_{i\in I}$ is a family of elements of Λ, then $T := (U, \varphi, (\Phi_{\lambda(i)})_{i\in I})$ is a bounded trivialization of $(\mathscr{A}_{X,\lambda(i)})_{i\in I}$ for any $(U, \varphi, \Phi) \in \mathcal{B}$. It is clear that the equivalence class of T does not depend on the choice of $(U, \varphi, \Phi) \in \mathcal{B}$. We denote by the same symbol $\mathcal{B}$ the equivalence class of T by abuse of notation.

Definition 4.17. Let $f\colon Y \to X$ be a morphism of smooth varieties and $\mathcal{B}$ a bornology of $\mathscr{A}_{X,\Lambda}$. By Proposition 4.15(ii), the equivalence class of $f^{\#}T$ $(T \in \mathcal{B})$ does not depend on the choice of T. We denote by $f^{\#}\mathcal{B}$ the equivalence class.

Proposition 4.18 is an easy consequence of Definition 4.17.

Proposition 4.18. *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be morphisms of smooth varieties. For any bornology $\mathcal{B}$ of $\mathscr{A}_{X,\Lambda}$, we have $(f\circ g)^{\#}\mathcal{B} = g^{\#}f^{\#}\mathcal{B}$ as bornologies of $(f\circ g)^{\#}\mathscr{A}_{X,\Lambda} = g^{\#}f^{\#}\mathscr{A}_{X,\Lambda}$.*

It is not obvious that a bornology contains enough trivializations for applications. We can make a good bounded trivialization from a bounded trivialization by Proposition 4.19.

Proposition 4.19. *Let $T_U = (U, \varphi, \Phi)$ be a trivialization of $\mathscr{A}_{X,\Lambda}$ and $f\colon V \to U$ a surjective étale morphism. Put $T_V := (V, \varphi\circ f, f^{\#}\Phi)$. Then the following conditions are equivalent:*

(i) *T_U is bounded,*
(ii) *T_V is bounded,*
(iii) *$T_U \sim T_V$.*

In particular, for any bornology $\mathcal{B}$ of $\mathscr{A}_{X,\Lambda}$, there exists a trivialization (W, ψ, Ψ) in $\mathcal{B}$ such that W is affine.

Proof. Let $f_1\colon V \times_X V \to U \times_X V$ and $f_2\colon U \times_X V \to U \times_X U$ be the morphisms determined by the universal property of the fiber products. Then by Lemma 4.14, we have

$$f_2^*(\mathcal{Z}(T_U, T_U)) = \mathcal{Z}(T_U, T_V),$$
$$f_1^*(\mathcal{Z}(T_U, T_V)) = \mathcal{Z}(T_V, T_V).$$

Since f_1 and f_2 are surjective, f_2^* and f_1^* are injective. Hence (i), (ii) and (iii) are equivalent. The second assertion is clear because for any variety U, there is a surjective étale morphism $W \to U$ from an affine variety W. □

Definition 4.20. Let $T = (U, \varphi, \Phi)$ be a trivialization of $\mathscr{A}_{X,\Lambda}$ with affine U. We say that an object $(\mathcal{M}_\lambda)_{\lambda\in\Lambda} \in \mathrm{Mod}_h(\mathscr{A}_{X,\Lambda})$ is *uniformly bounded* with respect to T if for any closed embedding $\iota\colon U \to \mathbb{C}^n$, $m_\iota(\mathcal{M}_\lambda|_U)$ is bounded as a function on Λ. Here we consider an $\mathscr{A}_{X,\lambda}|_U$-module as a $\mathscr{D}_U$-module by the isomorphism $\Phi_\lambda\colon \mathscr{A}_{X,\lambda}|_U \to \mathscr{D}_U$.

We say that an object $\mathcal{M} \in D^b_h(\mathscr{A}_{X,\Lambda})$ is *uniformly bounded* with respect to T if $H^i(\mathcal{M})$ is uniformly bounded for any i and $H^i(\mathcal{M})$ vanishes for any $|i| \gg 0$. Here $H^i(\mathcal{M})$ is the family $(H^i(\mathcal{M}_\lambda))_{\lambda\in\Lambda}$.

We denote by $\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T)$ (resp. $D^b_{ub}(\mathscr{A}_{X,\Lambda}, T)$) the full subcategory of $\mathrm{Mod}_h(\mathscr{A}_{X,\Lambda})$ (resp. $D^b_h(\mathscr{A}_{X,\Lambda})$) consisting of uniformly bounded objects with respect to T.

Remark 4.21. By Proposition 4.7, the boundedness of $m_\iota(\mathcal{M}_\lambda|_U)$ does not depend on the choice of the embedding ι.

Propositions 4.22 and 4.23 are easy consequences of Definition 4.20.

Proposition 4.22. *Let T be a bounded trivialization of $\mathscr{A}_{X,\Lambda}$. Then the following hold.*

(i) $\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T)$ *is abelian.*
(ii) *For a short exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{Mod}_h(\mathscr{A}_{X,\Lambda})$, both L and N are uniformly bounded if and only if so is M.*
(iii) $D^b_{ub}(\mathscr{A}_{X,\Lambda}, T)$ *is a triangulated subcategory of $D^b_h(\mathscr{A}_{X,\Lambda})$.*

Proposition 4.23. *Let $T = (U, \varphi, \Phi)$ be a bounded trivialization with affine U. Then for any $(\mathcal{M}_\lambda)_{\lambda\in\Lambda} \in \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T)$, the function $\mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathcal{M}_\lambda)$ of $\lambda \in \Lambda$ is bounded.*

Proof. Fix a closed embedding $\iota\colon U \to \mathbb{C}^n$. Then we have

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}|_U}(\mathcal{M}_\lambda|_U) = \mathrm{Len}_{\mathscr{D}_{\mathbb{C}^n}}(\iota_+(\mathcal{M}_\lambda|_U)) \le m_\iota(\mathcal{M}_\lambda|_U).$$

The first equality follows from the Kashiwara equivalence (Fact 4.5) and the second inequality from Fact 4.2. By the definition of uniformly bounded family, there is a constant C independent of $\lambda \in \Lambda$ such that

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}|_U}(\mathcal{M}_\lambda|_U) \le m_\iota(\mathcal{M}_\lambda|_U) \le C.$$

Since φ is surjective étale, the inverse image functor φ^* is exact and sends a non-zero module to a non-zero module (see the proof of Proposition 4.7). Hence we obtain

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathcal{M}_\lambda) \le C$$

for any $\lambda \in \Lambda$. □

We will show that the uniform boundedness is preserved by inverse images and direct images. To do so, we need the following basic proposition.

Proposition 4.24. *Let $T_i = (U_i, \varphi_i, \Phi_i)$ $(i = 1, 2)$ be bounded trivializations of $\mathscr{A}_{X,\Lambda}$ with affine U_i. If $T_1 \sim T_2$, then we have*

$$\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_1) = \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_2),$$
$$D^b_{ub}(\mathscr{A}_{X,\Lambda}, T_1) = D^b_{ub}(\mathscr{A}_{X,\Lambda}, T_2).$$

Proof. By definition, the second equality follows from the first one.

Let $p_i\colon U_1 \times_X U_2 \to U_i$ $(i = 1, 2)$ be the projections and put $T_i' := (U_1 \times_X U_2, \varphi_i \circ p_i, p_i^{\#}\Phi_i)$ for $i = 1, 2$. Applying Proposition 4.7 to $f = p_i$, we have

$$\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_i) = \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_i'). \tag{4.3.1}$$

By $T_1 \sim T_2$, $\mathcal{Z}(T_1, T_2)$ spans a finite-dimensional subspace of $\mathcal{Z}(U_1 \times_X U_2)$. Applying Corollary 4.10 to $W = \mathrm{span}_{\mathbb{C}}\mathcal{Z}(T_1, T_2)$, we have

$$\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_1') = \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda}, T_2').$$

This and (4.3.1) imply the desired equality. □

The following is well-defined by Proposition 4.24.

Definition 4.25. Let $\mathcal{B}$ be a bornology of $\mathscr{A}_{X,\Lambda}$. We set

$$\mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B}) := \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda},T),$$
$$D^b_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B}) := D^b_{ub}(\mathscr{A}_{X,\Lambda},T),$$

where $T=(U,\varphi,\Phi)$ is a bounded trivialization in $\mathcal{B}$ with affine U.

Theorem 4.26. *Let $f\colon Y\to X$ be a morphism of smooth varieties and $\mathcal{B}$ a bornology of $\mathscr{A}_{X,\Lambda}$. The direct image functor and the inverse image functor preserve the uniform boundedness, that is, we have functors*

$$Df_+\colon D^b_{ub}(f^{\#}\mathscr{A}_{X,\Lambda},f^{\#}\mathcal{B})\to D^b_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B}),$$
$$Lf^*\colon D^b_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B})\to D^b_{ub}(f^{\#}\mathscr{A}_{X,\Lambda},f^{\#}\mathcal{B}).$$

Proof. Take $T=(U,\varphi,\Phi)\in\mathcal{B}$ with affine U, and a surjective étale morphism $V\to U\times_X Y$ from an affine variety V. Consider the following diagram:

$$\begin{array}{ccccc} V & \xrightarrow{g} & U\times_X Y & \xrightarrow{\widetilde{f}} & U \\ & & \downarrow{\scriptstyle\widetilde{\varphi}} & & \downarrow{\scriptstyle\varphi} \\ & & Y & \xrightarrow{f} & X, \end{array}$$

where $\widetilde{\varphi}$ and $\widetilde{f}$ are the projections. Then $(V,\widetilde{\varphi}\circ g,(\widetilde{f}\circ g)^{\#}\Phi)$ is in $f^{\#}\mathcal{B}$ by Proposition 4.19. Since $L(\widetilde{\varphi}\circ g)^*\circ Lf^*=L(\widetilde{f}\circ g)^*\circ L\varphi^*$ holds (Fact 3.4(ii)), the assertion for the inverse image functor is reduced to Proposition 4.6 for $f=\widetilde{f}\circ g$.

We shall show the assertion for Df_+. Take a finite affine open covering $\{V_i\}_{i=0,1,2,\dots,r}$ of $U\times_X Y$ and replace V with the $(r+1)$-fold fiber product of $\bigsqcup_i V_i$. Let $\mathcal{M}\in\mathrm{Mod}_{ub}(f^{\#}\mathscr{A}_{X,\Lambda},f^{\#}\mathcal{B})$ and fix $\lambda\in\Lambda$. Then $\mathcal{M}_\lambda|_{U\times_X Y}$ is quasi-isomorphic to the Čech complex

$$0\to C^0\to C^1\to\cdots\to C^r\to 0$$

with respect to the covering $\{V_i\}$. By the construction of Čech complex, $\bigoplus_i C^i$ is a direct summand of $g_*(\mathcal{M}_\lambda|_V)$.

By the base change theorem (Fact 3.5), there is an isomorphism $L\varphi^*\circ Df_+\simeq D\widetilde{f}_+\circ L\widetilde{\varphi}^*$ of functors, and hence we have

$$L\varphi^*\circ Df_+(\mathcal{M}_\lambda)\simeq D\widetilde{f}_+(\mathcal{M}_\lambda|_{U\times_X Y})\simeq D\widetilde{f}_+(C^\bullet)$$

up to shift. For a closed embedding $\iota\colon U\to\mathbb{C}^n$, we have

$$\begin{aligned} m_\iota(D\widetilde{f}_+(C^\bullet)) &\le \sum_i m_\iota(D\widetilde{f}_+(C^i)) \\ &\le m_\iota(D\widetilde{f}_+\circ g_*(\mathcal{M}_\lambda|_V)) \\ &= m_\iota(D(\widetilde{f}\circ g)_+(\mathcal{M}_\lambda|_V)). \end{aligned}$$

Here the first inequality follows from Lemma 2.8(iii) for the complex $C^\bullet$. Note that Dg_+ is isomorphic to g_* (see the proof of Proposition 4.7).

By Proposition 4.6 and $\mathcal{M}\in\mathrm{Mod}_{ub}(f^{\#}\mathscr{A}_{X,\Lambda},f^{\#}\mathcal{B})$, there is a constant C independent of λ such that

$$m_\iota(L\varphi^*\circ Df_+(\mathcal{M}_\lambda))\le m_\iota(D(\widetilde{f}\circ g)_+(\mathcal{M}_\lambda|_V))\le C.$$

This shows $Df_+(\mathcal{M})\in D^b_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B})$. □

We study the external tensor product of uniformly bounded families. Let $\mathscr{A}_{Y,\Lambda}$ be a family of algebras of twisted differential operators on a smooth variety Y with the same index set Λ as $\mathscr{A}_{X,\Lambda}$.

Definition 4.27. Let $\mathcal{B}$ and $\mathcal{B}'$ be bornologies of $\mathscr{A}_{X,\Lambda}$ and $\mathscr{A}_{Y,\Lambda}$, respectively. We denote by $\mathcal{B} \boxtimes \mathcal{B}'$ the equivalence class of $(U \times V, \varphi \times \psi, \Phi \boxtimes \Psi = (\Phi_\lambda \boxtimes \Psi_\lambda)_{\lambda\in\Lambda})$ for some $(U, \varphi, \Phi) \in \mathcal{B}$ and $(V, \psi, \Psi) \in \mathcal{B}'$. If $X = Y$, we denote by $\mathcal{B} \# \mathcal{B}'$ the pull-back of $\mathcal{B} \boxtimes \mathcal{B}'$ by the diagonal embedding $X \hookrightarrow X \times X$.

It is easy to see that Definition 4.27 is well-defined. We set $\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda} := (\mathscr{A}_{X,\lambda} \boxtimes \mathscr{A}_{Y,\lambda})_{\lambda\in\Lambda}$ and $\mathcal{M} \boxtimes \mathcal{N} := (\mathcal{M}_\lambda \boxtimes \mathcal{N}_\lambda)_{\lambda\in\Lambda}$ for $\mathcal{M} \in D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$ and $\mathcal{N} \in D^b_{ub}(\mathscr{A}_{Y,\Lambda}, \mathcal{B}')$. By Fact 4.3, we obtain Theorem 4.28.

Theorem 4.28. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bornologies of $\mathscr{A}_{X,\Lambda}$ and $\mathscr{A}_{Y,\Lambda}$, respectively. Then $\mathcal{B} \boxtimes \mathcal{B}'$ is a bornology of $\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda}$ and we have a bifunctor*

$$(\cdot) \boxtimes (\cdot) \colon D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}) \times D^b_{ub}(\mathscr{A}_{Y,\Lambda}, \mathcal{B}') \to D^b_{ub}(\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda}, \mathcal{B} \boxtimes \mathcal{B}').$$

Moreover, for any $\mathcal{M} \in D^b_h(\mathscr{A}_{X,\Lambda})$ and $\mathcal{N} \in D^b_h(\mathscr{A}_{Y,\Lambda})$ such that all $\mathcal{M}_\lambda$ and $\mathcal{N}_\lambda$ have non-zero cohomologies, both $\mathcal{M}$ and $\mathcal{N}$ are uniformly bounded if and only if so is $\mathcal{M} \boxtimes \mathcal{N}$.

4.4. **Twisting, opposite and tensor product.** We consider operations of algebras of twisted differential operators: twisting by an invertible sheaf, taking opposite algebras and tensor products. Corresponding to the operations, we introduce these operations of a bornology.

Let $\mathscr{A}_{X,\Lambda} = (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ be a family of algebras of twisted differential operators on a smooth variety X. Let $\mathcal{B}$ be a bornology of $\mathscr{A}_{X,\Lambda}$ and $\mathcal{L}$ an invertible sheaf on X. Then we have a new family

$$\mathscr{A}^{\mathcal{L}}_{X,\Lambda} := (\mathcal{L} \otimes_{\mathcal{O}_X} \mathscr{A}_{X,\lambda} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)_{\lambda\in\Lambda}.$$

Remark that for a morphism $f \colon Y \to X$ of smooth varieties, there is a canonical isomorphism

$$f^\#(\mathcal{L} \otimes_{\mathcal{O}_X} \mathscr{A}_{X,\lambda} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee) \simeq f^*(\mathcal{L}) \otimes_{\mathcal{O}_Y} f^\#\mathscr{A}_{X,\lambda} \otimes_{\mathcal{O}_Y} f^*(\mathcal{L})^\vee. \tag{4.4.1}$$

See e.g. [28, Lemma 1.1.5].

We shall construct a bornology of $\mathscr{A}^{\mathcal{L}}_{X,\Lambda}$. Since $\mathcal{L}$ is an invertible sheaf, there is a bounded trivialization $T = (U, \varphi, \Phi) \in \mathcal{B}$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$. Take a trivialization $\alpha \colon \mathcal{L}|_U \xrightarrow{\simeq} \mathcal{O}_U$. Then T and α induce an isomorphism $\Phi^{\mathcal{L},\alpha}$ given by

$$(\mathcal{L} \otimes_{\mathcal{O}_X} \mathscr{A}_{X,\Lambda} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee)|_U \xrightarrow{\mathrm{id}\otimes\Phi\otimes\mathrm{id}} \mathcal{L}|_U \otimes_{\mathcal{O}_U} \mathscr{D}_U \otimes_{\mathcal{O}_U} (\mathcal{L}|_U)^\vee \to \mathscr{D}_U.$$

We obtain a trivialization $T^{\mathcal{L},\alpha} = (U, \varphi, \Phi^{\mathcal{L},\alpha})$ of $\mathscr{A}^{\mathcal{L}}_{X,\Lambda}$.

Lemma 4.29. *Take $S = (V, \psi, \Psi) \in \mathcal{B}$ and an isomorphism $\beta \colon \mathcal{L}|_V \to \mathcal{O}_V$. Then we have $S^{\mathcal{L},\beta} \sim T^{\mathcal{L},\alpha}$. In particular, $T^{\mathcal{L},\alpha}$ is a bounded trivialization.*

Proof. α and β induce an isomorphism

$$\mathcal{O}_{U\times_X V} \xrightarrow{\alpha^{-1}|_{U\times_X V}} \mathcal{L}|_{U\times_X V} \xrightarrow{\beta|_{U\times_X V}} \mathcal{O}_{U\times_X V}.$$

We write $f \in \mathcal{O}(U \times_X V)^\times$ for the image of 1 by the isomorphism. Then we have

$$\mathcal{Z}(T^{\mathcal{L},\alpha}, S^{\mathcal{L},\beta}) = f^{-1}df + \mathcal{Z}(T, S)$$

(see Definition 4.11). This shows the lemma. □

By Lemma 4.29, the following is well-defined.

Definition 4.30. We denote by $\mathcal{B}^{\mathcal{L}}$ the equivalence class of $T^{\mathcal{L},\alpha}$.

It is well-known that the functor

$$\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)\colon \mathrm{Mod}_h(\mathscr{A}_{X,\lambda})\to \mathrm{Mod}_h(\mathcal{L}\otimes_{\mathcal{O}_X}\mathscr{A}_{X,\lambda}\otimes_{\mathcal{O}_X}\mathcal{L}^{\vee})$$

gives an equivalence of categories. We denote by $\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)$ the direct product $\prod_{\lambda\in\Lambda}\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)$ of the functors by abuse of notation.

Proposition 4.31. *The functor $\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)$ preserves the uniform boundedness, that is, we have functors*

$$\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)\colon \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B})\to \mathrm{Mod}_{ub}(\mathscr{A}^{\mathcal{L}}_{X,\Lambda},\mathcal{B}^{\mathcal{L}}),$$
$$\mathcal{L}\otimes_{\mathcal{O}_X}(\cdot)\colon D^b_{ub}(\mathscr{A}_{X,\Lambda},\mathcal{B})\to D^b_{ub}(\mathscr{A}^{\mathcal{L}}_{X,\Lambda},\mathcal{B}^{\mathcal{L}}).$$

Moreover, the two functors give equivalences of categories.

Proof. Since the assertion is local for X, we can assume $\mathcal{L}\simeq\mathcal{O}_X$. In this case, the proposition is clear. □

Next we consider the family of the opposite algebras $\mathscr{A}^{\mathrm{op}}_{X,\lambda}$. Note that $(\cdot)^{\mathrm{op}}$ is a functor on the category of algebras of twisted differential operators on X. We set $\mathscr{A}^{\mathrm{op}}_{X,\Lambda}:=(\mathscr{A}^{\mathrm{op}}_{X,\lambda})_{\lambda\in\Lambda}$. We shall construct a bornology of $\mathscr{A}^{\mathrm{op}}_{X,\Lambda}$ from the bornology $\mathcal{B}$.

Recall that there is a canonical isomorphism

$$\mathscr{D}^{\mathrm{op}}_X\simeq\Omega_X\otimes_{\mathcal{O}_X}\mathscr{D}_X\otimes_{\mathcal{O}_X}\Omega^{\vee}_X, \tag{4.4.2}$$

where Ω_X is the canonical sheaf of X. See [21, Lemma 1.2.7]. The isomorphism induces an automorphism of the space $\mathcal{Z}(X)$ of closed 1-forms as

$$\begin{aligned}\mathcal{Z}(X)\xrightarrow{\simeq}\mathrm{Aut}(\mathscr{D}_X)\xrightarrow{(\cdot)^{\mathrm{op}}}\mathrm{Aut}(\mathscr{D}^{\mathrm{op}}_X)\xrightarrow{\simeq}\mathrm{Aut}(\Omega_X\otimes_{\mathcal{O}_X}\mathscr{D}_X\otimes_{\mathcal{O}_X}\Omega^{\vee}_X)\\ \xrightarrow{\simeq}\mathrm{Aut}(\mathscr{D}_X)\xrightarrow{\simeq}\mathcal{Z}(X).\end{aligned}$$

Here the fourth isomorphism comes from the isomorphism $\mathscr{D}_X\simeq\Omega^{\vee}_X\otimes_{\mathcal{O}_X}(\Omega_X\otimes_{\mathcal{O}_X}\mathscr{D}_X\otimes_{\mathcal{O}_X}\Omega^{\vee}_X)\otimes_{\mathcal{O}_X}\Omega_X$.

Lemma 4.32. *The automorphism of $\mathcal{Z}(X)$ is the multiplication map by -1.*

Proof. The lemma can be shown by an easy explicit computation. □

Remark that for an étale morphism $f\colon U\to X$, there are canonical isomorphisms

$$f^*\Omega_X\simeq\Omega_U,$$
$$\Omega_f\simeq\mathcal{O}_U,$$
$$f^{\#}(\mathscr{A}^{\mathrm{op}}_{X,\lambda})\simeq(f^{\#}\mathscr{A}_{X,\lambda})^{\mathrm{op}}.$$

In particular, the canonical isomorphism (4.4.2) commutes with the pull-back by the étale morphism.

Take a bounded trivialization $T=(U,\varphi,\Phi)\in\mathcal{B}$ such that $\Omega_X|_U\simeq\Omega_U$ is isomorphic to $\mathcal{O}_U$. Take a trivialization $\alpha\colon\Omega_U\xrightarrow{\simeq}\mathcal{O}_U$. Then T and α induce an isomorphism $\Phi^{\mathrm{op},\alpha}$ given by

$$\mathscr{D}^{\mathrm{op}}_{X,\Lambda}|_U\xrightarrow{\Phi^{\mathrm{op}}}\mathscr{D}^{\mathrm{op}}_U\xrightarrow{\simeq}\Omega_U\otimes_{\mathcal{O}_U}\mathscr{D}_U\otimes_{\mathcal{O}_U}\Omega^{\vee}_U\to\mathscr{D}_U.$$

We obtain a trivialization $T^{\mathrm{op},\alpha}=(U,\varphi,\Phi^{\mathrm{op},\alpha})$.

Lemma 4.33. *Take $S = (V, \psi, \Psi) \in \mathcal{B}$ and an isomorphism $\beta\colon \Omega_V \to \mathcal{O}_V$. Then we have $S^{\mathrm{op},\beta} \sim T^{\mathrm{op},\alpha}$. In particular, $T^{\mathrm{op},\alpha}$ is bounded.*

Proof. As we have seen in the proof of Lemma 4.29, there is $f \in \mathcal{O}(U \times_X V)^\times$ such that

$$\mathcal{Z}(T^{\mathrm{op},\alpha}, S^{\mathrm{op},\beta}) = f^{-1}df - \mathcal{Z}(T, S).$$

The sign before $\mathcal{Z}(T, S)$ comes from Lemma 4.32. This shows the lemma. □

Definition 4.34. We denote by $\mathcal{B}^{\mathrm{op}}$ the equivalence class of $T^{\mathrm{op},\alpha}$.

Corresponding to canonical isomorphisms of algebras of twisted differential operators, there are identities of bornologies. Let ι be the diagonal embedding $X \hookrightarrow X \times X$. For two algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ of twisted differential operators on X, we set

$$\mathscr{A}_1 \# \mathscr{A}_2 := \iota^{\#}(\mathscr{A}_1 \boxtimes \mathscr{A}_2).$$

We use the same notation for families of algebras.

Let $\mathscr{A}_{X,\Lambda}, \mathscr{B}_{X,\Lambda}$ and $\mathscr{C}_{X,\Lambda}$ be families of algebras of twisted differential operators on X with the same index set Λ. Let $f\colon Y \to X$ be a morphism of smooth varieties. For the constant family $\mathscr{D}_{X,\Lambda} := (\mathscr{D}_X)_{\lambda\in\Lambda}$, we consider a bounded trivialization $(X, \mathrm{id}_X, \mathrm{id})$. We denote by $\mathcal{B}_{\mathrm{id}}$ the equivalence class of the trivialization. We use the same notation for the constant family $\mathscr{D}_{Y,\Lambda}$ on Y. Fix an invertible sheaf $\mathcal{L}$ on X.

By [28, §1], we have canonical isomorphisms

$$\begin{aligned}
(\mathscr{A}^{\mathcal{L}}_{X,\Lambda})^{\mathrm{op}} &\simeq (\mathscr{A}^{\mathrm{op}}_{X,\Lambda})^{\mathcal{L}^\vee},\\
\mathscr{A}_{X,\Lambda} \# \mathscr{B}_{X,\Lambda} &\simeq \mathscr{B}_{X,\Lambda} \# \mathscr{A}_{X,\Lambda},\\
(\mathscr{A}_{X,\Lambda} \# \mathscr{B}_{X,\Lambda}) \# \mathscr{C}_{X,\Lambda} &\simeq \mathscr{A}_{X,\Lambda} \# (\mathscr{B}_{X,\Lambda} \# \mathscr{C}_{X,\Lambda}),\\
\mathscr{D}_{X,\Lambda} \# \mathscr{A}_{X,\Lambda} &\simeq \mathscr{A}_{X,\Lambda},\\
\mathscr{A}^{\mathrm{op}}_{X,\Lambda} \# \mathscr{A}_{X,\Lambda} &\simeq \mathscr{D}^{\mathrm{op}}_{X,\Lambda},\\
f^{\#}\mathscr{D}_{X,\Lambda} &\simeq \mathscr{D}_{Y,\Lambda},\\
f^{\#}(\mathscr{A}^{\mathcal{L}}_{X,\Lambda}) &\simeq (f^{\#}\mathscr{A}_{X,\Lambda})^{f^*\mathcal{L}},\\
f^{\#}(\mathscr{A}^{\mathrm{op}}_{X,\Lambda})^{\Omega_f} &\simeq (f^{\#}(\mathscr{A}_{X,\Lambda}))^{\mathrm{op}}.
\end{aligned}$$

Here we set $\Omega_f = f^{-1}\Omega_X^\vee \otimes_{f^{-1}\mathcal{O}_X} \Omega_Y$ (see Subsection 3.1). Since the isomorphisms are canonical, they are natural in $\mathscr{A}_{X,\Lambda}, \mathscr{B}_{X,\Lambda}$ and $\mathscr{C}_{X,\Lambda}$. It is easy to see that the isomorphisms and the operations commute with the pull-back by the following cartesian square:

$$\begin{array}{ccc}
Y & \xrightarrow{f} & X\\
{\scriptstyle\widetilde{\varphi}}\uparrow & & \uparrow{\scriptstyle\varphi}\\
U \times_X Y & \xrightarrow{\widetilde{f}} & U,
\end{array}$$

where $\varphi\colon U \to X$ is an étale morphism. For example, the following diagram commutes:

$$\begin{array}{ccc} f^{\#}(\mathscr{A}_{X,\Lambda}^{\mathrm{op}})^{\Omega_f}|_{U\times_X Y} & \xrightarrow{\simeq} & (f^{\#}(\mathscr{A}_{X,\Lambda}))^{\mathrm{op}}|_{U\times_X Y} \\ \simeq\uparrow & & \simeq\uparrow \\ \widetilde{f}^{\#}((\mathscr{A}_{X,\Lambda}|_U)^{\mathrm{op}})^{\Omega_{\tilde{f}}} & \xrightarrow{\simeq} & (\widetilde{f}^{\#}(\mathscr{A}_{X,\Lambda}|_U))^{\mathrm{op}}. \end{array}$$

Proposition 4.35. *Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be bornologies of $\mathscr{A}_{X,\Lambda}$, $\mathscr{B}_{X,\Lambda}$ and $\mathscr{C}_{X,\Lambda}$. Under the above identifications, we have*

(i) $(\mathcal{B}_1^{\mathcal{L}})^{\mathrm{op}} = (\mathcal{B}_1^{\mathrm{op}})^{\mathcal{L}^\vee}$,
(ii) $\mathcal{B}_1 \# \mathcal{B}_2 = \mathcal{B}_2 \# \mathcal{B}_1$,
(iii) $(\mathcal{B}_1 \# \mathcal{B}_2) \# \mathcal{B}_3 = \mathcal{B}_1 \# (\mathcal{B}_2 \# \mathcal{B}_3)$,
(iv) $\mathcal{B}_{\mathrm{id}} \# \mathcal{B}_1 = \mathcal{B}_1$,
(v) $\mathcal{B}_1^{\mathrm{op}} \# \mathcal{B}_1 = \mathcal{B}_{\mathrm{id}}^{\mathrm{op}}$,
(vi) $f^{\#}\mathcal{B}_{\mathrm{id}} = \mathcal{B}_{\mathrm{id}}$,
(vii) $f^{\#}(\mathcal{B}_1^{\mathcal{L}}) = (f^{\#}\mathcal{B}_1)^{\mathcal{L}^\vee}$,
(viii) $f^{\#}(\mathcal{B}_1^{\mathrm{op}})^{\Omega_f} = (f^{\#}(\mathcal{B}))^{\mathrm{op}}$.

Proof. The proposition is clear by the constructions of bornologies and the naturality of the canonical isomorphisms as mentioned above. □

4.5. **Integral transform.** We consider integral transforms of $\mathscr{D}$-modules. Let $\mathscr{A}_X$ be an algebra of twisted differential operators on a smooth variety X.

We consider the following canonical isomorphisms:

$$\begin{aligned} \mathscr{A}_X^{\mathrm{op}} \# \mathscr{A}_X &\simeq \mathscr{D}_X^{\mathrm{op}}, \\ \mathscr{D}_X \# \mathscr{A}_X &\simeq \mathscr{A}_X, \\ \Omega_X^\vee \otimes_{\mathcal{O}_X} \mathscr{D}_X^{\mathrm{op}} \otimes_{\mathcal{O}_X} \Omega_X &\simeq \mathscr{D}_X. \end{aligned}$$

We shall describe the isomorphisms explicitly. We write $\iota\colon X \to X \times X$ for the diagonal embedding. Let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{A}_X^{\mathrm{op}})$ and $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathscr{A}_X)$. Then the canonical right action of $\mathcal{T}_X \subset \mathscr{D}_X$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is given by

$$(A \otimes B) \cdot Z = A\widetilde{Z} \otimes B - A \otimes \widetilde{Z}B \tag{4.5.1}$$

for $Z \in \mathcal{T}_X$ and $A \otimes B \in \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$, where $\widetilde{Z}$ is a section of $\mathcal{P}(\mathscr{A}_X)$ such that $\sigma(\widetilde{Z}) = Z$. See Definition 3.6 for the notation of Picard algebroids. Since $\mathscr{A}_X$ is generated by $\mathcal{P}(\mathscr{A}_X)$, we have a canonical isomorphism

$$\mathcal{M} \otimes_{\mathscr{A}_X} \mathcal{N} \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})/(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})\mathcal{T}_X \tag{4.5.2}$$

of $\mathbb{C}_X$-modules by (4.5.1).

Let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_X)$ and $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathscr{A}_X)$. Then the canonical action of $\mathcal{P}(\mathscr{A}_X) \subset \mathscr{A}_X$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is given by

$$Z \cdot (A \otimes B) = \sigma(Z)A \otimes B + A \otimes ZB \tag{4.5.3}$$

for $Z \in \mathcal{P}(\mathscr{A}_X)$ and $A \otimes B \in \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$. Note that if $Z \in \mathcal{O}_X$, then we have $Z \cdot (A \otimes B) = A \otimes ZB$.

Let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_X^{\mathrm{op}})$. Then $\mathscr{D}_X$ acts on $\Omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$ canonically. $\Omega_X^\vee$ is isomorphic to the exterior product $\bigwedge^{\dim(X)} \mathcal{T}_X$ over $\mathcal{O}_X$. Hence $\mathcal{T}_X$ acts on $\bigwedge^{\dim(X)} \mathcal{T}_X \simeq$

$\Omega_X^\vee$ by the Lie bracket. By taking a local coordinate system, we can see that the action of $\mathcal{T}_X \subset \mathscr{D}_X$ on $\Omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$ is given by

$$Z \cdot (A \otimes B) = ZA \otimes B - A \otimes BZ \tag{4.5.4}$$

for $Z \in \mathcal{T}_X$ and $A \otimes B \in \Omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$. We omit the explicit computation. See [21, Lemma 1.2.7].

Lemma 4.36. *For any $\mathcal{A} \in \mathrm{Mod}_{qc}(\mathscr{D}_X^{\mathrm{op}})$, $\mathcal{B} \in \mathrm{Mod}_{qc}(\mathscr{A}_X^{\mathrm{op}})$ and $\mathcal{C} \in \mathrm{Mod}_{qc}(\mathscr{A}_X)$, we have a natural isomorphism*

$$\mathcal{A} \otimes_{\mathscr{D}_X} (\Omega_X^\vee \otimes_{\mathcal{O}_X} (\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C})) \simeq ((\mathcal{A} \otimes_{\mathcal{O}_X} \Omega_X^\vee) \otimes_{\mathcal{O}_X} \mathcal{B}) \otimes_{\mathscr{A}_X} \mathcal{C}$$

of $\mathbb{C}_X$-modules.

Proof. Both sides of the expression can be regarded as the sheaves of $\mathcal{T}_X$-coinvariants in $\mathcal{A} \otimes_{\mathcal{O}_X} \Omega_X^\vee \otimes_{\mathcal{O}_X} \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C}$ by (4.5.2). It is easy to see that the two actions of $\mathcal{T}_X$ coincide by (4.5.1), (4.5.3) and (4.5.4). □

Let $\mathscr{A}_{X,\Lambda}$ and $\mathscr{A}_{Y,\Lambda}$ be families of algebras of twisted differential operators on smooth varieties X and Y with the same index set Λ, respectively. Fix bornologies $\mathcal{B}_X$ and $\mathcal{B}_Y$ of $\mathscr{A}_{X,\Lambda}$ and $\mathscr{A}_{Y,\Lambda}$, respectively.

Theorem 4.37. *Let $\mathcal{M} \in D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}_X)$ and $\mathcal{N} \in D^b_{ub}(\mathscr{A}_{Y,\Lambda} \boxtimes \mathscr{A}_{X,\Lambda}^{\mathrm{op}}, \mathcal{B}_Y \boxtimes \mathcal{B}_X^{\mathrm{op}})$. Then we have*

$$(Rq_*(\mathcal{N}_\lambda \otimes^L_{p^{-1}\mathscr{A}_{X,\lambda}} p^{-1}\mathcal{M}_\lambda))_{\lambda\in\Lambda} \in D^b_{ub}(\mathscr{A}_{Y,\Lambda}, \mathcal{B}_Y),$$

where p (resp. q) is the projection from $Y \times X$ onto X (resp. Y).

Proof. Fix $\lambda \in \Lambda$. Then we have

$$\begin{aligned}
&Dq_+(p^*\Omega_X^\vee \otimes^L_{\mathcal{O}_{Y\times X}} (\mathcal{N}_\lambda \otimes^L_{\mathcal{O}_{Y\times X}} Lp^*\mathcal{M}_\lambda))\\
\simeq &Rq_*((\mathscr{A}_{Y,\lambda} \boxtimes \Omega_X) \otimes^L_{\mathscr{A}_{Y,\lambda}\boxtimes\mathscr{D}_X} (p^*\Omega_X^\vee \otimes^L_{\mathcal{O}_{Y\times X}} (\mathcal{N}_\lambda \otimes^L_{\mathcal{O}_{Y\times X}} Lp^*\mathcal{M}_\lambda)))\\
\simeq &Rq_*(p^{-1}\Omega_X \otimes^L_{p^{-1}\mathscr{D}_X} (p^{-1}\Omega_X^\vee \otimes^L_{p^{-1}\mathcal{O}_X} (\mathcal{N}_\lambda \otimes^L_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{M}_\lambda)))\\
\simeq &Rq_*((p^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \Omega_X^\vee) \otimes^L_{p^{-1}\mathcal{O}_X} \mathcal{N}_\lambda) \otimes^L_{p^{-1}\mathscr{A}_{X,\lambda}} p^{-1}\mathcal{M}_\lambda)\\
\simeq &Rq_*(\mathcal{N}_\lambda \otimes^L_{p^{-1}\mathscr{A}_{X,\lambda}} p^{-1}\mathcal{M}_\lambda).
\end{aligned}$$

The third isomorphism follows from Lemma 4.36 by taking flat resolutions of Ω_X, $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$. By Proposition 4.35, we have

$$((\mathcal{B}_Y \boxtimes \mathcal{B}_X^{\mathrm{op}}) \# p^\# \mathcal{B}_X)^{p^*\Omega_X^\vee} = (\mathcal{B}_Y \boxtimes \mathcal{B}_{\mathrm{id}}^{\mathrm{op}})^{\mathcal{O}_Y \boxtimes \Omega_X^\vee} = \mathcal{B}_Y \boxtimes \mathcal{B}_{\mathrm{id}} = q^\# \mathcal{B}_Y.$$

Therefore the theorem follows from Theorem 4.26. □

4.6. **Family of easy morphisms.** Retain the notation X, Y, $\mathscr{A}_{X,\Lambda}$, $\mathscr{A}_{Y,\Lambda}$, $\mathcal{B}_X$ and $\mathcal{B}_Y$ in the previous subsection. We consider operations of $\mathscr{D}$-modules by the following family of morphisms:

$$\begin{aligned}
&f_y \colon X \to X \times Y \quad (y \in Y),\\
&f_y(x) = (x, y).
\end{aligned}$$

Proposition 4.38. *For any $\mathcal{M} \in D^b_{ub}(\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda}, \mathcal{B}_X \boxtimes \mathcal{B}_Y)$, the family $(Lf_y^*(\mathcal{M}_\lambda))_{\lambda\in\Lambda, y\in Y}$ is uniformly bounded with respect to the bornology $\mathcal{B}_X$.*

Proof. It is enough to show the assertion for $\mathcal{M} \in \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda}, \mathcal{B}_X \boxtimes \mathcal{B}_Y)$. Fix $\lambda \in \Lambda$ and $y \in Y$.

Take $(U, \varphi, \Phi) \in \mathcal{B}_X$ and $(V, \psi, \Psi) \in \mathcal{B}_Y$ with affine U and V. Fix closed embeddings $\iota_U \colon U \to \mathbb{C}^n$ and $\iota_V \colon V \to \mathbb{C}^m$, and $y' \in \psi^{-1}(y)$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{f''_y} & \mathbb{C}^n \times \mathbb{C}^m \\ {\scriptstyle \iota_U}\uparrow & & \uparrow{\scriptstyle \iota_U \times \iota_V} \\ U & \xrightarrow{f'_y} & U \times V \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi \times \psi} \\ X & \xrightarrow{f_y} & X \times Y, \end{array}$$

where $f'_y(x) = (x, y')$ and $f''_y(x) = (x, \iota_V(y'))$. Remark that the upper square is cartesian.

By Facts 3.4(ii) and 3.5, we have

$$D(\iota_U)_+ \circ L\varphi^* \circ Lf_y^*(\mathcal{M}_\lambda) \simeq L(f''_y)^* \circ D(\iota_U \times \iota_V)_+ \circ L(\varphi \times \psi)^*(\mathcal{M}_\lambda)$$

up to shift. Since the degree of f''_y is 1, we have

$$m_{\iota_U}(L\varphi^* \circ Lf_y^*(\mathcal{M}_\lambda)) \leq m_{\iota_U \times \iota_V}(L(\varphi \times \psi)^*(\mathcal{M}_\lambda))$$

by Fact 4.4. This shows the proposition. □

Proposition 4.39. *Let $\mathcal{N} \in D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}_X)$. The family $(D(f_y)_+(\mathcal{N}_\lambda))_{\lambda \in \Lambda, y \in Y}$ is uniformly bounded with respect to the bornology $\mathcal{B}_X \boxtimes \mathcal{B}_Y$.*

Proof. We retain the notation in the proof of Proposition 4.38. Then we have

$$L(\varphi \times \psi)^* \circ D(f_y)_+(\mathcal{N}_\lambda) \simeq L\varphi^*(\mathcal{N}_\lambda) \boxtimes D\iota_+(\mathcal{O}_{\psi^{-1}(y)}),$$

where $\iota \colon \psi^{-1}(y) \to V$ is the inclusion map. By Fact 4.3, we have

$$m_{\iota_U \times \iota_V}(L\varphi^*(\mathcal{N}_\lambda) \boxtimes D\iota_+(\mathcal{O}_{\psi^{-1}(y)})) = m_{\iota_U}(L\varphi^*(\mathcal{N}_\lambda)) m_{\iota_V}(D\iota_+(\mathcal{O}_{\psi^{-1}(y)})).$$

Since the multiplicity of the unique irreducible holonomic $\mathscr{D}_{\mathbb{C}^m}$-module supported on a point is 1, we have

$$m_{\iota_V}(D\iota_+(\mathcal{O}_{\psi^{-1}(y)})) = |\psi^{-1}(y)|.$$

Since ψ is étale, $|\psi^{-1}(y)|$ is bounded on Y. Therefore we have shown the proposition. □

5. Examples of uniformly bounded family

In general, it is not easy to construct bornologies and uniformly bounded families of twisted $\mathscr{D}$-modules. An easy way to construct them is to use group actions. In this section, we construct uniformly bounded families using principal bundles and group actions with finite orbits.

5.1. **Bornology of a principal bundle.** Let G be an affine algebraic group and $p\colon \widetilde{X} \to X$ a principal G-bundle over a smooth variety X. Let $\mathscr{A}_{\widetilde{X}}$ be a G-equivariant algebra of twisted differential operators. For each $\lambda \in (\mathfrak{g}^*)^G$, we have defined a G-equivariant algebra $\mathscr{A}_{X,\lambda}$ of twisted differential operators on X in (3.4.1). Then we obtain a family $(\mathscr{A}_{X,\lambda})_{\lambda\in(\mathfrak{g}^*)^G}$. Put $\Lambda := (\mathfrak{g}^*)^G$.

In this subsection, we shall show that the family admits a standard bornology determined by the bundle $\widetilde{X} \to X$.

We can take a surjective étale morphism $\varphi\colon U \to X$ such that the pull-back $p\colon U\times_X \widetilde{X} \to U$ of the G-bundle is trivial and $\mathscr{A}_{\widetilde{X}}|_{U\times_X\widetilde{X}}$ is isomorphic to the algebra $\mathscr{D}_{U\times_X\widetilde{X}}$. Fix a section $s\colon U \to U\times_X \widetilde{X}$ and an isomorphism $\alpha\colon s^{\#}(\mathscr{A}_{\widetilde{X}}|_{U\times_X\widetilde{X}}) \to \mathscr{D}_U$.

The section s determines a trivialization $U \times G \simeq U \times_X \widetilde{X}$ and α induces an isomorphism $\mathscr{A}_{\widetilde{X}}|_{U\times_X\widetilde{X}} \simeq \mathscr{D}_U \boxtimes \mathscr{D}_G$ by Proposition 3.10. Then we have an isomorphism

$$\Phi^{U,s,\alpha}_\lambda\colon \mathscr{A}_{X,\lambda}|_U = p_*(\mathscr{A}_{\widetilde{X}}/R(I_{-\lambda+\delta})\mathscr{A}_{\widetilde{X}})^G|_U \xrightarrow{\simeq} \mathscr{D}_U$$

for any $\lambda \in \Lambda$. See (3.4.1) for the notation. Hence we obtain a trivialization $(U, \varphi, \Phi^{U,s,\alpha})$ of $\mathscr{A}_{X,\Lambda}$.

Proposition 5.1. *$(U, \varphi, \Phi^{U,s,\alpha})$ is bounded and its equivalence class does not depend on the choice of $\varphi\colon U \to X$, s and α.*

Proof. Let $(\psi\colon V \to X, t, \beta)$ be another choice of (φ, s, α). By considering the pull-back of $s, \alpha, t, \beta, \Phi^{s,\alpha}$ and $\Phi^{t,\beta}$ to $U\times_X V$, our computation can be done only on $U\times_X V$. Hence we can assume $U = V = X$, $\widetilde{X} = X\times G$ and $\mathscr{A}_{\widetilde{X}} = \mathscr{D}_{\widetilde{X}} = \mathscr{D}_X\boxtimes\mathscr{D}_G$.

We identify $s^{\#}(\mathscr{D}_{\widetilde{X}})$ and $t^{\#}(\mathscr{D}_{\widetilde{X}})$ with $\mathscr{D}_X$ by the canonical isomorphisms. Then α and β are automorphisms of $\mathscr{D}_X$. Since α and β are independent of $\lambda \in \Lambda$, the choice of α and β does not affect the equivalence. Hence we can assume $\alpha = \beta = \mathrm{id}$.

Fix $\lambda \in \Lambda$. By the decomposition $X \times G = s(X)G$, we have a monomorphism $\iota_s\colon \mathscr{D}_X \simeq \mathscr{D}_{s(X)} \to p_*(\mathscr{D}_{X\times G})^G$ and the isomorphism $(\Phi^{U,s,\alpha})^{-1}\colon \mathscr{D}_X \to \mathscr{D}_{X,\lambda}$ factors through the monomorphism. We define ι_t similarly. Then $\Phi^{V,t,\beta}_\lambda \circ (\Phi^{U,s,\alpha}_\lambda)^{-1}$ is given by the following dot arrow:

$$\begin{array}{ccccc}
\mathscr{D}_X \simeq \mathscr{D}_{s(X)} & \xrightarrow{\iota_s} & p_*(\mathscr{D}_{X\times G})^G & \twoheadrightarrow & \mathscr{D}_{X,\lambda} \\
\vdots & & & & \| \\
\mathscr{D}_X \simeq \mathscr{D}_{t(X)} & \xrightarrow{\iota_t} & p_*(\mathscr{D}_{X\times G})^G & \twoheadrightarrow & \mathscr{D}_{X,\lambda}.
\end{array}$$

We write $s(x) = (x, s'(x))$ $(x \in X)$ and define an automorphism a of $X \times G$ by $a(x, g) = (x, s'(x)g)$. For a local section $T \in \mathcal{T}_X$, we denote by T_s the corresponding section of $\mathcal{T}_{s(X)}$.

There exist closed 1-forms $\omega^s_1, \omega^s_2, \cdots, \omega^s_n$ $(n = \dim_{\mathbb{C}}(\mathfrak{g}))$ on X such that for any local sections $T \in \mathcal{T}_X$ and $f \in \mathcal{O}_X \otimes \mathcal{O}(G)$,

$$T_s f = ((a^*)^{-1} \circ T \circ a^*)f = Tf - \sum_i \omega^s_i(T)L(X_i)f,$$

where $\{X_i\}_{i=1,2,\dots,n}$ is a basis of $\mathfrak{g}$ and L is the differential of the left translation on G. Similarly, we define $\{\omega^t_j\}$ for t. Therefore $\left\{\Phi^{V,t,\beta}_\lambda \circ (\Phi^{U,s,\alpha}_\lambda)^{-1}\right\}_{\lambda\in\Lambda}$ is contained

in a finite-dimensional subspace spanned by $\{\omega_i^s\}$ and $\{\omega_j^t\}$ in $\mathcal{Z}(X)$. We have proved the proposition. □

Definition 5.2. We denote by $\mathcal{B}(X, \widetilde{X})$ the equivalence class of the bounded trivialization $(U, \varphi, \Phi^{U,s,\alpha})$.

Let $f\colon Y \to X$ be a morphism of smooth varieties. Then we have a cartesian square

$$\begin{array}{ccc} Y \times_X \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X. \end{array}$$

Put

$$\widetilde{Y} := Y \times_X \widetilde{X},$$
$$\mathscr{A}_{\widetilde{Y}} := \widetilde{f}^{\#}\mathscr{A}_{\widetilde{X}}.$$

It is easy to see that $q\colon \widetilde{Y} \to Y$ is a principal G-bundle and $\mathscr{A}_{\widetilde{Y}}$ is G-equivariant. For each $\lambda \in \Lambda$, we define an algebra $\mathscr{A}_{Y,\lambda}$ of twisted differential operators on Y as in (3.4.1) and we have a canonical isomorphism

$$\mathscr{A}_{Y,\lambda} \simeq f^{\#}\mathscr{A}_{X,\lambda}.$$

We identify the two algebras by the isomorphism. Then we obtain two bornologies $\mathcal{B}(Y, \widetilde{Y})$ and $f^{\#}\mathcal{B}(X, \widetilde{X})$ of $\mathscr{A}_{Y,\Lambda} := (\mathscr{A}_{Y,\lambda})_{\lambda\in\Lambda}$.

Lemma 5.3. *$\mathcal{B}(Y, \widetilde{Y})$ and $f^{\#}\mathcal{B}(X, \widetilde{X})$ are equal.*

Proof. It is clear from the definition of $\mathcal{B}(X, \widetilde{X})$ and its pull-back (Definition 4.17). □

Theorem 5.4 is a consequence of Lemma 5.3 and Theorem 4.26.

Theorem 5.4. *We have functors*

$$Df_+\colon D^b_{ub}(\mathscr{A}_{Y,\Lambda}, \mathcal{B}(Y, \widetilde{Y})) \to D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}(X, \widetilde{X})),$$
$$Lf^*\colon D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B}(X, \widetilde{X})) \to D^b_{ub}(\mathscr{A}_{Y,\Lambda}, \mathcal{B}(Y, \widetilde{Y})),$$

which are the restrictions of the direct image functor and the inverse image functor, respectively.

Corollary 5.5. *Let $\mathcal{M} \in D^b_h(\mathscr{A}_{\widetilde{X}})$. Then the family $(Dp_{+,\lambda}(\mathcal{M}))_{\lambda\in\Lambda}$ is uniformly bounded with respect to $\mathcal{B}(X, \widetilde{X})$.*

Proof. We shall apply Theorem 5.4 to $Y = \widetilde{X}$ and $f = p$. The fiber product $\widetilde{X} \times_X \widetilde{X}$ is canonically isomorphic to the trivial bundle $\widetilde{X} \times G$. The isomorphism is given by $\widetilde{X} \times G \ni (x, g) \mapsto (x, xg) \in \widetilde{X} \times_X \widetilde{X}$. Hence the following diagram is a cartesian square:

$$\begin{array}{ccc} \widetilde{X} \times G & \xrightarrow{m} & \widetilde{X} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle p} \\ \widetilde{X} & \xrightarrow{p} & X, \end{array}$$

where m is the multiplication map and pr is the projection onto the first factor. Then the constant family $(\mathcal{M})_{\lambda\in\Lambda}$ is uniformly bounded with respect to $\mathcal{B}(\widetilde{X}, \widetilde{X} \times G)$. Therefore the assertion follows from Theorem 5.4. □

By Corollary 5.5, we can construct many uniformly bounded families of $\mathscr{D}$-modules parametrized by $(\mathfrak{g}^*)^G$ using principal G-bundles.

5.2. **G-equivariant bornology.** Let X be a smooth G-variety of an affine algebraic group G, and $\mathscr{A}_{X,\Lambda}$ be a family of G-equivariant algebras of twisted differential operators. We write $\pi\colon G \times X \to X$ and $m\colon G \times X \to X$ for the projection and the multiplication map, respectively. Since all $\mathscr{A}_{X,\lambda}$ are G-equivariant, we have a canonical isomorphism

$$\pi^{\#}\mathscr{A}_{X,\Lambda} \simeq m^{\#}\mathscr{A}_{X,\Lambda}.$$

See (3.3.1).

Definition 5.6. We say that a bornology $\mathcal{B}$ of $\mathscr{A}_{X,\Lambda}$ is G-equivariant if $\pi^{\#}\mathcal{B} = m^{\#}\mathcal{B}$ holds under the isomorphism $\pi^{\#}\mathscr{A}_{X,\Lambda} \simeq m^{\#}\mathscr{A}_{X,\Lambda}$.

Proposition 5.7 is clear by Definition 5.6 and Proposition 4.18.

Proposition 5.7. *Let $f\colon Y \to X$ be a morphism of smooth G-varieties and $\mathcal{B}$ a G-equivariant bornology of $\mathscr{A}_{X,\Lambda}$. Then $f^{\#}\mathcal{B}$ is G-equivariant.*

We set $m_g := m(g, \cdot)$ for $g \in G$. Then m_g is an automorphism of X.

Proposition 5.8. *Let $\mathcal{M} \in D^b_{ub}(\mathscr{A}_{X,\Lambda}, \mathcal{B})$. Then $(Lm_g^*(\mathcal{M}_\lambda))_{\lambda\in\Lambda, g\in G}$ is uniformly bounded with respect to $\mathcal{B}$.*

Proof. For $g \in G$, let f_g denote the morphism $f_g\colon X \to G \times X$ defined by $f_g(x) = (g, x)$. Then we have $m_g = m \circ f_g$. Since $\mathcal{B}$ is G-equivariant, $Lm^*(\mathcal{M})$ is uniformly bounded with respect to $\pi^{\#}\mathcal{B} = m^{\#}\mathcal{B}$. By Proposition 4.38, $(Lf_g^* \circ Lm^*(\mathcal{M}_\lambda))_{\lambda\in\Lambda, g\in G}$ is uniformly bounded with respect to $\mathcal{B} = f_g^{\#}\pi^{\#}\mathcal{B}$. This shows the assertion. □

In Subsection 5.1, we have given a way to construct a bornology using a principal bundle. We shall show that the bornology is G-equivariant if the bundle has G-equivariant structure. Let G and T be affine algebraic groups and $p\colon \widetilde{X} \to X$ a principal T-bundle over a smooth variety X. Suppose that $\widetilde{X}$ and X are $G \times T$-varieties and p is $G \times T$-equivariant. Let $\mathscr{A}_{\widetilde{X}}$ be a $G \times T$-equivariant algebra of twisted differential operators on $\widetilde{X}$.

Put $\Lambda := (\mathfrak{t}^*)^T$. Then we have a family $\mathscr{A}_{X,\Lambda} = (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ of $G \times T$-equivariant algebras, and its bornology $\mathcal{B}(X, \widetilde{X})$ as in Subsection 5.1.

Proposition 5.9. *$\mathcal{B}(X, \widetilde{X})$ is G-equivariant.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{X} & \xleftarrow{\pi} & G \times \widetilde{X} & \xrightarrow{m} & \widetilde{X} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \mathrm{id}\times p} & & \downarrow{\scriptstyle p} \\
X & \xleftarrow{\pi} & G \times X & \xrightarrow{m} & X,
\end{array}$$

where π and m are the projection and the multiplication map, respectively. Since $\mathscr{A}_{\widetilde{X}}$ is G-equivariant, $\pi^{\#}\mathscr{A}_{\widetilde{X}}$ and $m^{\#}\mathscr{A}_{\widetilde{X}}$ are canonically isomorphic. We obtain a family $\mathscr{A}_{G\times X,\Lambda}$ constructed from the principal T-bundle $G\times\widetilde{X}\to G\times X$. Then $\pi^{\#}\mathscr{A}_{X,\Lambda}$ and $m^{\#}\mathscr{A}_{X,\Lambda}$ are canonically isomorphic to $\mathscr{A}_{G\times X,\Lambda}=\mathscr{D}_G\boxtimes\mathscr{A}_{X,\Lambda}$. Under this identification, by Lemma 5.3, we have

$$\pi^{\#}\mathcal{B}(X,\widetilde{X})=\mathcal{B}(G\times X,G\times\widetilde{X})=m^{\#}\mathcal{B}(X,\widetilde{X}).$$

This implies that $\mathcal{B}(X,\widetilde{X})$ is G-equivariant. □

We shall show the uniqueness of G-equivariant bornologies on a homogeneous variety. Let G and H be affine algebraic group and its closed subgroup, and $\mathscr{D}_G$ the algebra of non-twisted differential operators. We write $p\colon G\to G/H$ for the natural projection. Then we obtain a G-equivariant algebra $\mathscr{D}_{G/H,\lambda}$ of twisted differential operators on G/H for any $\lambda\in(\mathfrak{h}^*)^H$. See (3.4.1).

It is well-known that any G-equivariant algebra of twisted differential operators is canonically isomorphic to some $\mathscr{D}_{G/H,\lambda}$ (see [25, Theorem 4.9.2]). This is because it is generated by $\mathcal{U}(\mathfrak{g})$ and $\mathcal{O}_{G/H}$. Hence we consider a bornology of a family $\mathscr{D}_{G/H,\Lambda}:=(\mathscr{D}_{G/H,\Lambda(r)})_{r\in R}$ for $\Lambda\colon R\to(\mathfrak{h}^*)^H$.

Proposition 5.10. *There exists a unique G-equivariant bornology of $\mathscr{D}_{G/H,\Lambda}$.*

Proof. The existence is clear because $\mathcal{B}(G/H,G)$ is a G-equivariant bornology of $\mathscr{D}_{G/H,\Lambda}$ by Proposition 5.9.

We shall show the uniqueness. Let $\mathcal{B}$ be a G-equivariant bornology of $\mathscr{D}_{G/H,\Lambda}$. By Proposition 4.15(iv), it is enough to show $p^{\#}\mathcal{B}=p^{\#}\mathcal{B}(G/H,G)$. Let $\pi,m\colon G\times G/H\to G/H$ be the projection and the multiplication map, respectively, and $\iota\colon G\to G\times G/H$ a morphism given by $\iota(g)=(g,eH)$. Using the G-equivariant structure, we identify the following three families:

$$m^{\#}\mathscr{D}_{G/H,\Lambda},\quad \pi^{\#}\mathscr{D}_{G/H,\Lambda},\quad (\mathscr{D}_G\boxtimes\mathscr{D}_{G/H,\lambda(r)})_{r\in R}.$$

Since $m\circ\iota=p$, by Proposition 4.18, we have

$$p^{\#}\mathcal{B}=\iota^{\#}m^{\#}\mathcal{B}=\iota^{\#}\pi^{\#}\mathcal{B}=\iota^{\#}(\mathcal{B}_{\mathrm{id}}\boxtimes\mathcal{B})=\mathcal{B}_{\mathrm{id}},$$

where $\mathcal{B}_{\mathrm{id}}$ is the equivalence class of the trivialization $(G,\mathrm{id}_G,\mathrm{id})$ of the constant family $(\mathscr{D}_G)_{r\in R}$. Therefore we have $p^{\#}\mathcal{B}=\mathcal{B}_{\mathrm{id}}=p^{\#}\mathcal{B}(G/H,G)$. □

5.3. **Uniformly bounded family of irreducible modules.** Let K be an affine algebraic group and X a K-variety. Let $\mathscr{A}_{X,\Lambda}:=(\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ be a family of K-equivariant algebras of twisted differential operators on X. Fix a K-equivariant bornology $\mathcal{B}$ of $\mathscr{A}_{X,\Lambda}$. A classification of K-equivariant $\mathscr{A}_{X,\lambda}$-modules is given by Beilinson–Bernstein [5] (see also [20, Theorem 2.4]).

We review the classification. Fix $\lambda\in\Lambda$ and $x\in X$. We write $i\colon Kx\hookrightarrow X$ and $p\colon K\to Kx$ for the inclusion and the natural surjection, respectively. Let K_x denote the stabilizer of x in K. Since $i^{\#}\mathscr{A}_{X,\lambda}$ is K-equivariant and Kx is homogeneous, there is a unique element $\mu(\lambda)$ of $(\mathfrak{k}_x^*)^{K_x}$ such that $i^{\#}\mathscr{A}_{X,\lambda}$ is canonically isomorphic to

$$\mathscr{D}_{Kx,\mu(\lambda)}:=(p_*(\mathscr{D}_K)\otimes_{\mathcal{U}(\mathfrak{k}_x)}\mathbb{C}_{\mu(\lambda)})^{K_x}.$$

See [25, Theorem 4.9.2]. We identify $i^{\#}\mathscr{A}_{X,\lambda}$ with $\mathscr{D}_{Kx,\mu(\lambda)}$ by the isomorphism.

Fact 5.11. Let $\mathcal{M}$ be an irreducible coherent $(\mathscr{A}_{X,\lambda}, K)$-module whose support is $\overline{Kx}$. Then there exists a unique irreducible K_x-module F such that

(i) $\mathfrak{k}_x$ acts on F by the character $\mu(\lambda)$,

(ii) $\mathcal{M}$ is isomorphic to the unique irreducible submodule of $D^0 i_+(\mathrm{Ind}_{K_x}^K(F))$,

where $\mathrm{Ind}_{K_x}^K(F)$ is the $(\mathscr{D}_{Kx,\mu(\lambda)}, K)$-module of local sections of the associated vector bundle $K \times_{K_x} F$ over $Kx \simeq K/K_x$. In particular, $\mathcal{M}$ is holonomic.

We shall show that a family of $(\mathscr{A}_{X,\lambda}, K)$-modules with bounded lengths is uniformly bounded if K has finitely many orbits in X.

Lemma 5.12. *Let F be an irreducible K_x-module. Put $n = \dim_{\mathbb{C}}(K_x)$. Assume that $\mathfrak{k}_x$ acts on F by the character $\mu(\lambda)$. Then $\mathrm{Ind}_{K_x}^K(F)$ is isomorphic to a direct summand of $D^{-n}p_{+,\mu(\lambda)}(\mathcal{O}_K)$.*

Proof. By Theorem 3.16 and the Poincaré duality (Fact 2.7), we have

$$D^{-n}p_{+,\mu(\lambda)}(\mathcal{O}_K) \simeq \mathrm{Tor}_n^{\mathcal{U}(\mathfrak{k}_x)}(\mathbb{C}_{\mu(\lambda)-\delta}, p_*(\mathcal{O}_K)) \simeq (p_*(\mathcal{O}_K) \otimes \mathbb{C}_{\mu(\lambda)})^{(K_x)_0},$$

where δ is the character $\mathfrak{k}_x \ni X \mapsto \mathrm{tr}(\mathrm{ad}_{\mathfrak{k}_x}(X))$. The assertion follows from the isomorphisms and the Frobenius reciprocity. □

Lemma 5.13. *Let $\mathcal{M}$ be an irreducible coherent $(\mathscr{A}_{X,\lambda}, K)$-module whose support is $\overline{Kx}$. Then $\mathcal{M}$ is isomorphic to a submodule of $H^{-n} \circ Di_+ \circ Dp_{+,\mu(\lambda)}(\mathcal{O}_K)$.*

Proof. Since Kx is locally closed in X, the cohomology $D^k i_+(\mathcal{N})$ vanishes for any $k < 0$ and $\mathcal{N} \in \mathrm{Mod}_{qc}(\mathscr{A}_{X,\lambda})$. Using truncation functors (see Subsection 2.3), we have

$$H^{-n} \circ Di_+ \circ Dp_{+,\mu(\lambda)}(\mathcal{O}_K) \simeq D^0 i_+(D^{-n}p_{+,\mu(\lambda)}(\mathcal{O}_K)).$$

Hence the assertion follows from Fact 5.11 and Lemma 5.12. □

Let π and m be the projection and the multiplication map from $K \times X$ to X, respectively. We write $f_x(g) = (g, x)$ for $g \in K$ and $x \in X$. Then we have $i \circ p = m \circ f_x$. We denote by $D(f_x)_{+,\lambda}$ the direct image functor $D^b_{qc}(\mathscr{D}_G) \to D^b_{qc}(\pi^{\#}\mathscr{A}_{X,\lambda})$. By Proposition 4.39, $(D(f_x)_{+,\lambda}(\mathcal{O}_K))_{x\in X,\lambda\in\Lambda}$ is uniformly bounded with respect to $\pi^{\#}\mathcal{B}$.

Since $\mathcal{B}$ and any algebra in $\mathscr{A}_{X,\Lambda}$ are K-equivariant, we have $m^{\#}\mathscr{A}_{X,\Lambda} \simeq \pi^{\#}\mathscr{A}_{X,\Lambda}$ and $m^{\#}\mathcal{B} = \pi^{\#}\mathcal{B}$. Therefore $(Dm_+ \circ D(f_x)_{+,\lambda}(\mathcal{O}_K))_{x\in X,\lambda\in\Lambda}$ is uniformly bounded with respect to $\mathcal{B}$ by Theorem 4.26. By Lemma 5.13 and $i \circ p = m \circ f_x$, we obtain

Proposition 5.14. *Let $\mathcal{M} \in \prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda}, K)$. Assume that each $\mathcal{M}_\lambda$ is irreducible and its support is the closure of some K-orbit dependent on λ. Then $\mathcal{M}$ is a uniformly bounded family with respect to $\mathcal{B}$.*

Theorem 5.15. *Let $\mathcal{M} \in \prod_{\lambda\in\Lambda} \mathrm{Mod}_h(\mathscr{A}_{X,\lambda}, K)$. Assume that K has finitely many orbits in X and the length of each $\mathcal{M}_\lambda$ is bounded by a constant independent of $\lambda \in \Lambda$. Then $\mathcal{M}$ is a uniformly bounded family with respect to $\mathcal{B}$.*

For the representation theory of real reductive Lie groups, we generalize Theorem 5.15 to the universal covering group of K in a sense. Retain the notation $X, K, \mathscr{A}_{X,\Lambda}, \mathcal{B}$ as above and assume that K is connected.

Fix $\lambda \in \Lambda$ for a while. Let ν be a character of $\mathfrak{k}$ and $\mathcal{M}$ a quasi-coherent $\mathscr{A}_{X,\lambda}$-module. We say that $\mathcal{M}$ is a twisted $(\mathscr{A}_{X,\lambda}, K)$-module with twist ν if the action of $\mathfrak{k}$ on $\mathcal{M} \otimes \mathbb{C}_\nu$ lifts to an action of K. Let $\mathscr{A}_{X,(\lambda,\nu)}$ be the K-equivariant

algebra $\mathscr{A}_{X,\lambda} \otimes \mathrm{End}_{\mathbb{C}}(\mathbb{C}_\nu)$, which is isomorphic to $\mathscr{A}_{X,\lambda}$ without the K-equivariant structures. Then $\mathcal{M}$ is a twisted $(\mathscr{A}_{X,\lambda}, K)$-module with twist ν if and only if $\mathcal{M}$ admits a K-equivariant structure as an $\mathscr{A}_{X,(\lambda,\nu)}$-module.

Take $(U, \varphi, \Phi) \in \mathcal{B}$. Then $(U, \varphi, (\Phi_\lambda)_{\lambda\in\Lambda, \nu\in(\mathfrak{k}^*)^K})$ is a bounded trivialization of $(\mathscr{A}_{X,(\lambda,\nu)})_{\lambda\in\Lambda,\nu\in(\mathfrak{k}^*)^K}$. Since the K-action on $\mathscr{A}_{X,(\lambda,\nu)}$ is the same as that on $\mathscr{A}_{X,\lambda}$, the bornology defined by $(U, \varphi, (\Phi_\lambda)_{\lambda\in\Lambda,\nu\in(\mathfrak{k}^*)^K})$ is K-equivariant.

Corollary 5.16. *Proposition* 5.14 *and Theorem* 5.15 *hold even if all* $\mathcal{M}_\lambda$ *are twisted* $(\mathscr{A}_{X,\lambda}, K)$*-modules.*

5.4. **Finite orbits and uniformly bounded family.** Retain the notation X, K, $\mathscr{A}_{X,\Lambda}, \mathcal{B}$ in the previous subsection. Assume that K has finitely many orbits in X and K is connected. In this subsection, we consider the $\mathscr{A}_{X,\lambda}$-module $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$ for a finite-dimensional $\mathfrak{k}$-module F.

To estimate the length of $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$, we need Lemma 5.17 about a complex of filtered modules.

Lemma 5.17. *Let* $\mathcal{A}$ *be a filtered ring and* $(C^\bullet, d^\bullet)$ *a complex of filtered* $\mathcal{A}$*-modules. Then* $\mathrm{gr}(H^i(C^\bullet))$ *is isomorphic to a subquotient of* $H^i(\mathrm{gr}(C^\bullet))$ *for any* $i \in \mathbb{Z}$.

Proof. Fix $i \in \mathbb{Z}$. It is easy to see that the following canonical homomorphisms are injective:

$$\mathrm{Im}(\mathrm{gr}(d^{i-1})) \to \mathrm{gr}(\mathrm{Im}(d^{i-1})) \to \mathrm{gr}(\mathrm{Ker}(d^i)) \to \mathrm{Ker}(\mathrm{gr}(d^i)), \tag{5.4.1}$$

where $\mathrm{gr}(d^k)\colon \mathrm{gr}(C^k) \to \mathrm{gr}(C^{k+1})$ is the homomorphism induced from $d^k\colon C^k \to C^{k+1}$. The filtrations on $\mathrm{Im}(d^{i-1})$, $\mathrm{Ker}(d^i)$ and $H^i(C^\bullet)$ are induced from that on C^i. Hence we have $\mathrm{gr}(H^i(C^\cdot)) \simeq \mathrm{gr}(\mathrm{Ker}(d^i))/\mathrm{gr}(\mathrm{Im}(d^{i-1}))$. This isomorphism and (5.4.1) show the lemma. □

Let $\pi\colon T^*X \to X$ be the cotangent bundle. We have a homomorphism $\sigma\colon S(\mathfrak{k}) \to \mathcal{O}_{T^*X}$ defined by taking the principal symbol of $\mathscr{A}_{X,\lambda}$. The homomorphism σ does not depend on the choice of the K-equivariant algebra $\mathscr{A}_{X,\lambda}$. In fact, the composition $\mathfrak{k} \to \mathcal{P}(\mathscr{A}_{X,\lambda}) \to \mathcal{T}_X$ coincides with the differential of the K-action on X, and σ is determined by $\sigma|_{\mathfrak{k}}$. Here $\mathcal{P}(\mathscr{A}_{X,\lambda})$ is the Picard algebroid associated to $\mathscr{A}_{X,\lambda}$ (see Subsection 3.2).

Lemma 5.18. *Fix* $\lambda \in \Lambda$. *Let* $\mathcal{M}$ *be an* $\mathscr{A}_{X,\lambda}$*-module with a filtration, and* $\mathcal{N}$ *a coherent* $\pi_*\mathcal{O}_{T^*X}$*-module annihilated by* $\sigma(\mathfrak{k})$. *If* $\mathrm{gr}(\mathcal{M})$ *is isomorphic to a subquotient of* $\mathcal{N}^{\oplus n}$, *then* $\mathcal{M}$ *is holonomic and there exists a constant* $C(\mathcal{N})$ *depending only on* $\mathcal{N}$ *such that*

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathcal{M}) \le C(\mathcal{N}) \cdot n.$$

Proof. Put $\widetilde{\mathcal{N}} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \mathcal{N}$. Since $\sigma(\mathfrak{k})$ annihilates $\widetilde{\mathcal{N}}$ and K has finitely many orbits in X, the support of $\widetilde{\mathcal{N}}$ is contained in the union of the conormal bundles of all K-orbits in X. Since $\mathrm{gr}(\mathcal{M})$ is isomorphic to a subquotient of $\mathcal{N}^{\oplus n}$, the filtration of $\mathcal{M}$ is good, and hence $\mathcal{M}$ is coherent by [21, Theorem 2.1.3]. Moreover, the characteristic variety of $\mathcal{M}$ is a union of the conormal bundles of some K-orbits in X. This shows that $\mathcal{M}$ is holonomic.

Let $C(\mathcal{N})$ be the sum of multiplicities of $\widetilde{\mathcal{N}}$ along the conormal bundles of all K-orbits. Let $m(\mathcal{M})$ be the sum of multiplicities in the characteristic cycle of $\mathcal{M}$. Then we have $m(\mathcal{M}) \le C(\mathcal{N}) \cdot n$. Since the length of $\mathcal{M}$ is bounded by $m(\mathcal{M})$ (see [21, Proposition 5.1.9]), this shows the lemma. □

Let U be the unipotent radical of K. If necessary, replacing K with its finite covering, we may assume that $[K/U, K/U]$ is simply-connected.

Lemma 5.19. *There exists some constant $C > 0$ such that for any finite-dimensional $\mathfrak{k}$-module F, $\lambda \in \Lambda$ and $i \in \mathbb{Z}$, we have*

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)) \leq C \cdot \dim_{\mathbb{C}}(F),$$

where $\mathscr{A}_{X,\lambda}$ is considered as a $\mathcal{U}(\mathfrak{k})$-module by the right action. Moreover, any composition factor of $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$ is a holonomic twisted $(\mathscr{A}_{X,\lambda}, K)$-module.

Remark 5.20. Lemma 5.19 for $i = 0$ is proved in [44].

Proof. Fix F and λ. By induction on the length of F, the assertion can be reduced to the case of irreducible F. Since F is irreducible, $\mathfrak{k}/\mathrm{Ann}_{\mathfrak{k}}(F)$ is reductive, where Ann means the annihilator of a module. Hence we can take a character μ of $\mathfrak{k}$ such that $F \otimes \mathbb{C}_\mu$ lifts to a K-module. This implies that $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$ is a twisted $(\mathscr{A}_{X,\lambda}, K)$-module with twist μ. In fact, the homology can be computed by an h-complex of weak $(\mathscr{A}_{X,(\lambda,\mu)}, K)$-modules in the sense of Bernstein–Lunts [9, 2.5]. See [31, Proposition 3.3] for the complex. Here $\mathscr{A}_{X,(\lambda,\mu)}$ is a K-equivariant algebra defined before Corollary 5.16.

To compute $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$, we shall use the Chevalley–Eilenberg chain complex. See Fact 2.6. Let $(\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^{-\bullet}\mathfrak{k}, d^\bullet)$ be the complex. For any $i \geq 0$, the differential d^{-i} is given by

$$\begin{aligned}
&d^{-i}(P \otimes f \otimes (X_1 \wedge X_2 \wedge \cdots \wedge X_i)) \\
&= \sum_a (-1)^{a+1}(PX_a \otimes f - P \otimes X_a f) \otimes X_1 \wedge X_2 \wedge \cdots \wedge \hat{X}_a \wedge \cdots \wedge X_i \\
&+ \sum_{a<b} (-1)^{a+b} P \otimes f \otimes [X_a, X_b] \wedge X_1 \wedge X_2 \wedge \cdots \wedge \hat{X}_a \wedge \cdots \wedge \hat{X}_b \wedge \cdots \wedge X_i.
\end{aligned} \tag{5.4.2}$$

We denote by G the order filtration of $\mathscr{A}_{X,\lambda}$. It induces a filtration $\widetilde{G}^i$ on $\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^i\mathfrak{k}$ as

$$\widetilde{G}^i_n(\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^i\mathfrak{k}) = G_{n-i}(\mathscr{A}_{X,\lambda}) \otimes F \otimes \wedge^i\mathfrak{k}$$

for any $i \geq 0$. Then the complex $(\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^{-\bullet}\mathfrak{k}, d^\bullet)$ is a complex of filtered $\mathscr{A}_{X,\lambda}$-modules. By (5.4.2), we have an isomorphism

$$(\mathrm{gr}(\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^{-\bullet}\mathfrak{k}), \mathrm{gr}(d^\bullet)) \simeq (\pi_*\mathcal{O}_{T^*X} \otimes F \otimes \wedge^{-\bullet}\mathfrak{k}, d^\bullet)$$

of complexes, where the right hand side is the Koszul complex of the $S(\mathfrak{k})$-module $\pi_*\mathcal{O}_{T^*X} \otimes F \simeq (\pi_*\mathcal{O}_{T^*X})^{\oplus \dim(F)}$. Hence we have

$$H^{-i}(\mathrm{gr}(\mathscr{A}_{X,\lambda} \otimes F \otimes \wedge^{-\bullet}\mathfrak{k})) \simeq \mathrm{Tor}_i^{S(\mathfrak{k})}(\pi_*\mathcal{O}_{T^*X}, \mathbb{C}) \otimes F$$

as $\pi_*\mathcal{O}_{T^*X}$-modules. Note that $\mathrm{Tor}_i^{S(\mathfrak{k})}(\pi_*\mathcal{O}_{T^*X}, \mathbb{C})$ is a coherent $\pi_*\mathcal{O}_{T^*X}$-module because each term of the Koszul complex is coherent.

By Lemma 5.17, $\mathrm{gr}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F))$ is isomorphic to a subquotient of $\mathrm{Tor}_i^{S(\mathfrak{k})}(\pi_*\mathcal{O}_{T^*X}, \mathbb{C}) \otimes F$. We can apply Lemma 5.18 to $\mathcal{M} = \mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)$

and $\mathcal{N} = \mathrm{Tor}_i^{S(\mathfrak{k})}(\pi_*\mathcal{O}_{T^*X}, \mathbb{C})$. Hence there is a constant C_i depending only on $\mathrm{Tor}_i^{S(\mathfrak{k})}(\pi_*\mathcal{O}_{T^*X}, \mathbb{C})$ such that

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F)) \leq C_i \cdot \dim_{\mathbb{C}}(F).$$

$C := \max_i \{C_i\}$ exists because $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\cdot,\cdot)$ vanishes for any $i > \dim_{\mathbb{C}}(\mathfrak{k})$. The assertion in the lemma holds for this C. □

Corollary 5.21 is a direct consequence of Lemma 5.19 and Corollary 5.16.

Corollary 5.21. *Let $\mathcal{F}$ be a set of $\mathfrak{k}$-modules with bounded dimensions. Then the family $(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(\mathscr{A}_{X,\lambda}, F))_{i\in\mathbb{Z},F\in\mathcal{F},\lambda\in\Lambda}$ is uniformly bounded with respect to $\mathcal{B}$.*

Let $\mathscr{A}_{Y,\Lambda}$ be a family of twisted differential operators on a smooth variety Y. Fix a bornology $\mathcal{B}'$ of $\mathscr{A}_{Y,\Lambda}$. We write $q\colon X \times Y \to Y$ for the projection onto the second factor.

Theorem 5.22. *Let $\mathcal{M} \in \mathrm{Mod}_{ub}(\mathscr{A}_{X,\Lambda} \boxtimes \mathscr{A}_{Y,\Lambda}, \mathcal{B} \boxtimes \mathcal{B}')$. If all $\mathcal{M}_\lambda$ are q_*-acyclic, then there exists a constant $C > 0$ such that*

$$\mathrm{Len}_{\mathscr{A}_{Y,\lambda}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, q_*(\mathcal{M}_\lambda))) \leq C \cdot \dim_{\mathbb{C}}(F)$$

for any finite-dimensional $\mathfrak{k}$-module F, $i \in \mathbb{Z}$ and $\lambda \in \Lambda$. Moreover, the family $(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, q_*(\mathcal{M}_\lambda)))_{\lambda\in\Lambda,i\in\mathbb{Z},F\in\mathcal{F}}$ is uniformly bounded with respect to $\mathcal{B}'$. Here $\mathcal{F}$ is a set of finite-dimensional $\mathfrak{k}$-modules whose dimensions are bounded.

Proof. For $\mathcal{N} \in D_h^b(\mathscr{A}_{X,\lambda}^{\mathrm{op}})$, put

$$T_\lambda^i(\mathcal{N}) := R^i q_*(p^{-1}\mathcal{N} \otimes^L_{p^{-1}\mathscr{A}_{X,\lambda}} \mathcal{M}_\lambda),$$

where $p\colon X \times Y \to X$ is the projection onto the first factor.

For $\lambda \in \Lambda$, let I_λ be the set of all (isomorphism classes of) irreducible twisted $(\mathscr{A}_{X,\lambda}^{\mathrm{op}}, K)$-modules. By Corollary 5.16, the family $(\mathcal{N})_{\lambda\in\Lambda,\mathcal{N}\in I_\lambda}$ is a uniformly bounded family with respect to $\mathcal{B}$. Hence by Theorem 4.37, we can define a constant C_1 as

$$C_1 := \max\left\{\mathrm{Len}_{\mathscr{A}_{Y,\lambda}}(T_\lambda^i(\mathcal{N})) : \lambda \in \Lambda, \mathcal{N} \in I_\lambda, i \in \mathbb{Z}\right\}.$$

Fix a finite-dimensional $\mathfrak{k}$-module F. Take a free resolution $J^\bullet$ of the $\mathcal{U}(\mathfrak{k})$-module F. Then we have

$$\begin{aligned} T_\lambda^{-i}(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} \mathscr{A}_{X,\lambda}) &= R^{-i}q_*(p^{-1}(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} \mathscr{A}_{X,\lambda}) \otimes^L_{p^{-1}\mathscr{A}_{X,\lambda}} \mathcal{M}_\lambda) \\ &\simeq R^{-i}q_*(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{M}_\lambda) \\ &\simeq H^{-i}(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} q_*(\mathcal{M}_\lambda)) \\ &\simeq \mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, q_*(\mathcal{M}_\lambda)). \end{aligned}$$

Here the second isomorphism holds because $J^k \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{M}_\lambda$ is isomorphic to a direct sum of some copies of $\mathcal{M}_\lambda$ as a sheaf, and $\mathcal{M}_\lambda$ is q_*-acyclic. This shows the second assertion by Theorem 4.37 and Corollary 5.21.

To show the first assertion, we remark that $H^{-i}(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} \mathscr{A}_{X,\lambda})$ is isomorphic to $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, \mathscr{A}_{X,\lambda})$ as an $\mathscr{A}_{X,\lambda}^{\mathrm{op}}$-module. By Lemma 2.8(ii), we have

$$\begin{aligned}\mathrm{Len}_{\mathscr{A}_{Y,r}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, q_*(\mathcal{M}_\lambda))) &= \mathrm{Len}_{\mathscr{A}_{Y,r}}(T_\lambda^{-i}(J^\bullet \otimes_{\mathcal{U}(\mathfrak{k})} \mathscr{A}_{X,\lambda})) \\ &\le \sum_{j=0}^{\dim_{\mathbb{C}}(\mathfrak{k})} \mathrm{Len}_{\mathscr{A}_{Y,\lambda}}(T_\lambda^{-i+j}(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{k})}(F, \mathscr{A}_{X,\lambda}))).\end{aligned}$$

By Lemma 5.19, there is a constant C_2 independent of F such that

$$\mathrm{Len}_{\mathscr{A}_{X,\lambda}^{\mathrm{op}}}(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{k})}(F, \mathscr{A}_{X,\lambda})) \le C_2 \cdot \dim_{\mathbb{C}}(F)$$

for any $j \in \mathbb{Z}$ and $\lambda \in \Lambda$, and any composition factor of the module is in I_λ. By Lemma 2.8(i), we obtain

$$\mathrm{Len}_{\mathscr{A}_{Y,\lambda}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{k})}(F, q_*(\mathcal{M}_\lambda))) \le C_1 \cdot C_2 \cdot \dim_{\mathbb{C}}(F)(\dim_{\mathbb{C}}(\mathfrak{k}) + 1).$$

We have taken C_1 and C_2 independently of F, i and λ. Therefore we have proved the theorem. □

6. Zuckerman derived functor and its localization

In this section, we review the Zuckerman derived functors and their localization. We use the functors to study the relative Lie algebra cohomology/homology. The localization can be realized by a composition of direct image functors and inverse image functors. Hence we can apply results about uniformly bounded families to study the functors and the cohomologies.

6.1. **Zuckerman functor.** In this subsection we review the Zuckerman derived functor. We refer the reader to [13, I.8] and [48, 6.3] for our construction.

Let $(\mathcal{A}, G)$ be a generalized pair and H a reductive subgroup of G. Then $(\mathcal{A}, H)$ forms a generalized pair and $(\mathfrak{g}, H)$ forms a pair (see Definitions 2.1 and 2.5). Since $\mathcal{A}$ is a G-module, for any $X \in \mathcal{A}$, we can take $f_1, \dots, f_n \in \mathcal{O}(G)$ and $X_1, \dots, X_n \in \mathcal{A}$ such that

$$\mathrm{Ad}(g^{-1})(X) = \sum_i f_i(g) X_i$$

for any $g \in G$.

Let V be an $(\mathcal{A}, H)$-module. We define three actions on $\mathcal{O}(G) \otimes V$ via

$$\begin{aligned}\mu(X)(f \otimes v) &= \sum_i f_i f \otimes X_i v && (X \in \mathcal{A}),\\ r(Y)(f \otimes v) &= R(Y)f \otimes v + f \otimes Yv && (Y \in \mathfrak{g}),\\ r(g)(f \otimes v) &= R(g)f \otimes gv && (g \in H),\\ l(g)(f \otimes v) &= L(g)f \otimes v && (g \in G)\end{aligned}$$

for $f \in \mathcal{O}(G)$ and $v \in V$. Here L (resp. R) denotes the left (resp. right) regular action of G on $\mathcal{O}(G)$ and f_i, X_i are the elements taken above for X. It is easy to see that $\mu(X)$ does not depend on the choice of $\{f_i\}$ and $\{X_i\}$. Note that the actions μ and l commute with r and we have

$$(\mu(X) - l(X))(\mathcal{O}(G) \otimes V)^{r(\mathfrak{g})} = 0$$

for any $X \in \mathfrak{g}$. This implies that $\Gamma_H^G(V) := (\mathcal{O}(G) \otimes V)^{r(\mathfrak{g}), r(H)}$ is an $(\mathcal{A}, G)$-module via μ and l.

For an $(\mathcal{A}, H)$-module V and $i \in \mathbb{N}$, we set

$$\mathbb{D}^i\Gamma_H^G(V) := H^i(\mathfrak{g}, H; \mathcal{O}(G) \otimes V),$$

where $\mathcal{O}(G) \otimes V$ is considered as a $(\mathfrak{g}, H)$-module via the action r to take the relative Lie algebra cohomology. The two actions l and μ satisfy the definition of $(\mathcal{A}, G)$-modules (Definition 2.2), and hence $\mathbb{D}^i\Gamma_H^G(V)$ is an $(\mathcal{A}, G)$-module. See e.g. [13, Proposition I.8.2] and [39, Theorem 1.6]. Remark that we can prove that $\mathbb{D}^i\Gamma_H^G(V)$ is an $(\mathcal{A}, G)$-module under the weaker assumption that G/H is affine without reductivity of H. See Remark 6.6.

Fact 6.1. $\mathbb{D}^i\Gamma_H^G(V)$ admits an $(\mathcal{A}, G)$-module structure defined by μ and l. If, in addition, $\mathcal{A}$ is flat as a right $\mathcal{U}(\mathfrak{g})$-module, then $\mathbb{D}^i\Gamma_H^G$ is isomorphic to the i-th right derived functor of Γ_H^G.

The functors $\mathbb{D}^i\Gamma_H^G$ are called the *Zuckerman derived functors*.

The following property is well-known and easy to see from the above isomorphism and the algebraic Peter–Weyl theorem [16, Theorem 4.2.7]. See e.g. [13, Theorem I.8.8].

Fact 6.2. Let V be an $(\mathcal{A}, H)$-module. Assume that G is reductive. For any $i \in \mathbb{N}$, the irreducible decomposition of $\mathbb{D}^i\Gamma_H^G(V)$ as a G-module is given by

$$\mathbb{D}^i\Gamma_H^G(V) \simeq \bigoplus_F H^i(\mathfrak{g}, H; F \otimes V) \otimes F^*,$$

where the direct sum is over all isomorphism classes of irreducible G-modules. The isomorphism is natural in V.

For a generalized pair $(\mathcal{A}, G)$, we consider $(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}), G)$ as a generalized pair equipped with the diagonal homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{A} \otimes \mathcal{U}(\mathfrak{g})$ and the diagonal action of G on $\mathcal{A} \otimes \mathcal{U}(\mathfrak{g})$.

Lemma 6.3. *Let V be an $(\mathcal{A}, H)$-module. Assume that G is reductive. Then for any $(\mathfrak{g}, H)$-module W and $i \in \mathbb{N}$, there exists a natural isomorphism of $\mathcal{A}^G$-modules*

$$\mathbb{D}^i\Gamma_H^G(V \otimes W)^G \simeq H^i(\mathfrak{g}, H; V \otimes W),$$

where $V \otimes W$ is considered as an $(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}), H)$-module to apply the functor $\mathbb{D}^i\Gamma_H^G$.

Proof. The isomorphism $\mathbb{D}^i\Gamma_H^G(V \otimes W)^G \simeq H^i(\mathfrak{g}, H; V \otimes W)$ of vector spaces in Fact 6.2 is natural in V and W. Hence the isomorphism is also an $\mathcal{A}^G$-homomorphism. □

Corollary 6.4. *Retain the notation in Lemma 6.3. Then for any $(\mathfrak{g}, H)$-module W and $i \in \mathbb{N}$, we have*

$$\mathrm{Len}_{\mathcal{A}^G}(H^i(\mathfrak{g}, H; V \otimes W)) \leq \mathrm{Len}_{\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}), G}(\mathbb{D}^i\Gamma_H^G(V \otimes W)).$$

Proof. We write $\Delta\colon \mathfrak{g} \to \mathcal{A} \otimes \mathcal{U}(\mathfrak{g})$ for the diagonal homomorphism $\Delta(X) = X \otimes 1 + 1 \otimes X$. Then we have $(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))/(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))\Delta(\mathfrak{g}) \simeq \mathcal{A}$.

By Proposition 2.3, we have

$$\mathrm{Len}_{(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))^G}(\mathbb{D}^i\Gamma_H^G(V \otimes W)^G) \leq \mathrm{Len}_{\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}), G}(\mathbb{D}^i\Gamma_H^G(V \otimes W)).$$

The action of $(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))^G$ on $\mathbb{D}^i\Gamma_H^G(V \otimes W)^G$ factors through $(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))^G/(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g})\Delta(\mathfrak{g}))^G$. Since G is reductive, we have

$$(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g}))^G/(\mathcal{A} \otimes \mathcal{U}(\mathfrak{g})\Delta(\mathfrak{g}))^G \simeq (\mathcal{A} \otimes \mathcal{U}(\mathfrak{g})/\mathcal{A} \otimes \mathcal{U}(\mathfrak{g})\Delta(\mathfrak{g}))^G \simeq \mathcal{A}^G.$$

The isomorphism $\mathcal{A}^G \simeq (\mathcal{A}\otimes\mathcal{U}(\mathfrak{g}))^G/(\mathcal{A}\otimes\mathcal{U}(\mathfrak{g})\Delta(\mathfrak{g}))^G$ is given by

$$\mathcal{A}^G \ni a \mapsto a\otimes 1 \in (\mathcal{A}\otimes\mathcal{U}(\mathfrak{g}))^G/(\mathcal{A}\otimes\mathcal{U}(\mathfrak{g})\Delta(\mathfrak{g}))^G.$$

The assertion therefore follows from Lemma 6.3. □

6.2. **Localization of the Zuckerman functor.** We review the localization of the Zuckerman derived functor. We refer to [10, II.4] and to [31, 4.6] for a conceptual treatment using the equivariant derived category (see also [8, 3.7]).

Let G be an affine algebraic group over $\mathbb{C}$ and $\mathscr{A}_X$ a G-equivariant algebra of twisted differential operators on a smooth G-variety X. Let H be a reductive subgroup of G. We construct an $(\mathscr{A}_X, G)$-module from an $(\mathscr{A}_X, H)$-module.

We consider the following diagram:

$$X \xleftarrow{\pi} G\times X \xrightarrow{a} G\times X \xrightarrow{q} G/H\times X \xrightarrow{\pi'} X,$$

where π and π' are the projections, a is the isomorphism given by $a(g,x)=(g,gx)$ and q is the natural projection. We consider the left two X as $G\times H$-varieties letting G act trivially and the others as $G\times H$-varieties letting H act trivially. We consider G as a $G\times H$-variety via the left and right translations. Then π, a, q and π' are $G\times H$-equivariant. Since $\mathscr{A}_X$ is G-equivariant, we have canonical isomorphisms

$$\pi^{\#}\mathscr{A}_X \simeq \mathscr{D}_G \boxtimes \mathscr{A}_X \simeq (\pi'\circ q\circ a)^{\#}\mathscr{A}_X$$

of $G\times G$-equivariant algebras (see Proposition 3.9). In particular, they are $G\times H$-equivariant.

We set $n=\dim_{\mathbb{C}}(\mathfrak{h}), m=\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{h})$ and

$$\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M}) := L_{m-i}\pi'_+(L_nq_+(a_+\pi^*(\mathcal{M}))^{H/H_0}) \in \mathrm{Mod}_{qc}(\mathscr{A}_X, G)$$

for $\mathcal{M}\in\mathrm{Mod}_{qc}(\mathscr{A}_X,H)$ and $i\in\mathbb{N}$. Here $(\cdot)^{H/H_0}$ means taking the H/H_0-invariant part in an H/H_0-equivariant sheaf. Since the functors to define $\mathbb{D}^i\mathrm{L}^G_H$ preserve holonomicity, we can replace Mod_{qc} by Mod_h.

The functors $\mathbb{D}^i\mathrm{L}^G_H$ can be considered as a localization of the Zuckerman functors $\mathbb{D}^i\Gamma^G_H$ (see [10, Theorem 4.4] and [31, Proposition 4.17]).

Proposition 6.5. *Let $\mathcal{M}$ be an object in $\mathrm{Mod}_{qc}(\mathscr{A}_X,H)$. If the global section functor $\Gamma\colon \mathrm{Mod}_{qc}(\mathscr{A}_X)\to\mathrm{Mod}(\mathcal{A}_X)$ is exact, then there exists a natural isomorphism*

$$\Gamma(\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M})) \simeq \mathbb{D}^i\Gamma^G_H(\Gamma(\mathcal{M}))$$

of $(\mathcal{A}_X,G)$-modules for any $i\in\mathbb{N}$.

Proof. Note that G/H is affine by Matsushima's criterion [46, Theorem 3.8]. Fix $i\in\mathbb{N}$.

Since $q\colon G\times X\to G/H\times X$ is a principal H-bundle, $L_nq_+(\cdot)^{H/H_0}$ is isomorphic to $\mathrm{Tor}_n^{\mathcal{U}(\mathfrak{h})}(\mathbb{C},q_*(\cdot))^{H/H_0}$ by Theorem 3.16. Since $\mathfrak{h}$ is reductive, we do not need the shift δ. By $n=\dim_{\mathbb{C}}(\mathfrak{h})$, and Facts 2.6 and 2.7, we have

$$\begin{aligned}\mathrm{Tor}_n^{\mathcal{U}(\mathfrak{h})}(\mathbb{C},q_*(\cdot))^{H/H_0} &\simeq H_n(\mathfrak{h};q_*(\cdot))^{H/H_0}\\ &\simeq H^0(\mathfrak{h};q_*(\cdot))^{H/H_0}\simeq q_*(\cdot)^H.\end{aligned}$$

Hence we have

$$L_nq_+a_+\pi^*(\mathcal{M})^{H/H_0}\simeq q_*(\mathcal{O}_G\boxtimes\mathcal{M})^H.$$

The action of $\mathcal{D}_X \subset \mathcal{A}_{G/H} \otimes \mathcal{A}_X$ on $\Gamma(q_*(\mathcal{O}_G \boxtimes \mathcal{M})^H) \simeq (\mathcal{O}(G) \otimes \Gamma(\mathcal{M}))^H$ is given by

$$A \cdot f \otimes m = \sum_i f_i f \otimes A_i m, \tag{6.2.1}$$

where $\{A_i\} \subset \mathcal{A}_X$ and $\{f_i\} \subset \mathcal{O}(G)$ are finite subsets satisfying $\sum_i f_i(g) A_i = \mathrm{Ad}(g^{-1})A$. The G-action on $(\mathcal{O}(G) \otimes \Gamma(\mathcal{M}))^H$ is given by the left translation on $\mathcal{O}(G)$.

Let $p\colon G/H \times X \to G/H$ be the projection and $q'\colon G \to G/H$ the natural projection. Since π' is projection, we can compute $L_{m-i}\pi'_+$ by the relative de Rham complex (see [21, Lemma 1.5.27]). Since G/H is a homogeneous variety, the tangent sheaf $\mathcal{T}_{G/H}$ is isomorphic to $(q'_*\mathcal{O}_G \otimes \mathfrak{g}/\mathfrak{h})^H$. Hence we can write the relative de Rham complex using Lie algebras as

$$\begin{aligned}
&\Gamma(L_{m-i}\pi'_+(q_*(\mathcal{O}_G \boxtimes \mathcal{M})^H)) \\
\simeq &H^{i-m} \circ \Gamma \circ R\pi'_*(p^{-1}(q'_*\mathcal{O}_G \otimes \wedge^{m+\bullet}(\mathfrak{g}/\mathfrak{h})^*)^H \otimes_{p^{-1}\mathcal{O}_{G/H}} q_*(\mathcal{O}_G \boxtimes \mathcal{M})^H) \\
\simeq &H^i((\mathcal{O}(G) \otimes \wedge^\bullet(\mathfrak{g}/\mathfrak{h})^*)^H \otimes_{\mathcal{O}(G/H)} (\mathcal{O}(G) \otimes \Gamma(\mathcal{M}))^H) \\
\simeq &H^i((\mathcal{O}(G) \otimes \wedge^\bullet(\mathfrak{g}/\mathfrak{h})^* \otimes \Gamma(\mathcal{M}))^H).
\end{aligned}$$

Here the second isomorphism holds because G/H is affine and Γ is exact on $\mathrm{Mod}_{qc}(\mathscr{A}_X)$, and the third isomorphism comes from the tensor product of the two locally free sheaves on the affine variety G/H. The differentials in the above complexes are those induced from the relative de Rham complex. By a straightforward computation, the complex $(\mathcal{O}(G)\otimes\wedge^\bullet(\mathfrak{g}/\mathfrak{h})^*\otimes\Gamma(\mathcal{M}))^H$ is isomorphic to the complex $\mathrm{Hom}_H(\mathrm{CE}_\bullet(\mathfrak{g}, H), \mathcal{O}(G) \otimes \Gamma(\mathcal{M}))$. See Subsection 2.2 for the Chevalley–Eilenberg chain complex $\mathrm{CE}_\bullet(\mathfrak{g}, H)$. Therefore we have

$$\Gamma(L_{m-i}\pi'_+(q_*(\mathcal{O}_G \boxtimes \mathcal{M})^H)) \simeq H^i(\mathfrak{g}, H; \mathcal{O}(G) \otimes \Gamma(\mathcal{M})) \simeq \mathbb{D}^i\Gamma^G_H(\Gamma(\mathcal{M})).$$

As we have seen around (6.2.1), under the isomorphism $\Gamma(\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M})) \simeq \Gamma(\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M}))$ of vector spaces, the $\mathcal{A}_X$-action and the G-action on $\Gamma(\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M}))$ coincide with those on $\mathbb{D}^i\Gamma^G_H(\Gamma(\mathcal{M}))$ given in Fact 6.1. We have therefore proved the proposition. □

Remark 6.6. In the proof, we did not use the reductivity of H. In fact, one can define the Zuckerman functor $\mathbb{D}^i\Gamma^G_H(V)$ by the same way in the previous subsection for any H if G/H is affine.

Let G and H be an affine algebraic group and its reductive subgroup, and X a smooth G-variety. Let $\mathscr{A}_{X,\Lambda} := (\mathscr{A}_{X,\lambda})_{\lambda\in\Lambda}$ be a family of G-equivariant algebras of twisted differential operators on X. Take a G-equivariant bornology $\mathcal{B}$ of $\mathscr{A}_{X,\Lambda}$. See Definition 5.6.

The functor $\mathbb{D}^i\mathrm{L}^G_H$ is defined by a composition of inverse image functors, direct image functors and taking invariants $(\cdot)^{H/H_0}$. Note that taking invariants is a special case of taking a subquotient. Hence $\mathbb{D}^i\mathrm{L}^G_H$ preserves the uniform boundedness.

Theorem 6.7. *Let $(\mathcal{M}_\lambda)_{\lambda\in\Lambda}$ be a family of $(\mathscr{A}_{X,\lambda}, H)$-modules. Suppose that $\mathcal{M}$ is uniformly bounded with respect to $\mathcal{B}$. Then $(\mathbb{D}^i\mathrm{L}^G_H(\mathcal{M}_\lambda))_{i\in\mathbb{Z},\lambda\in\Lambda}$ is uniformly bounded with respect to $\mathcal{B}$.*

Proof. Recall the definition of the morphisms:

$$X \xleftarrow{\pi} G \times X \xrightarrow{a} G \times X \xrightarrow{q} G/H \times X \xrightarrow{\pi'} X.$$

Since $\mathcal{B}$ is G-equivariant, we have $\pi^{\#}\mathcal{B} = (\pi' \circ q \circ a)^{\#}\mathcal{B}$ by Definition 5.6. The uniform boundedness is preserved by direct images, inverse images and taking subquotients by Proposition 4.22 and Theorem 4.26. Hence we have proved the theorem. □

7. Application to representation theory

In this section, we define a notion of uniformly bounded family of $\mathfrak{g}$-modules. A typical example is a family of Harish-Chandra modules with bounded lengths. As an application of results about uniformly bounded families of $\mathscr{D}$-modules, we will show that the uniform boundedness of a family of $\mathfrak{g}$-modules is preserved by several operations such as (cohomologically) parabolic induction and taking coinvariants. We will also prove the boundedness of the lengths of $\mathcal{U}(\mathfrak{g})^{G'}$-modules, which is related to the branching problem and harmonic analysis.

7.1. **Uniformly bounded family of $\mathfrak{g}$-modules.** In this subsection, we introduce the notion of uniformly bounded family of $\mathfrak{g}$-modules.

Let G be a connected reductive algebraic group and B a Borel subgroup of G. Fix a Levi decomposition $B = TU$, where T is a maximal torus and U is the unipotent radical of B. Then the natural projection $p\colon G/U \to G/B$ is a principal T-bundle and G-equivariant.

We will reduce theorems about $\mathfrak{g}$-modules to those about $\mathcal{D}$-modules on G/B. To do so, we review the Beilinson–Bernstein correspondence. Let $\mathscr{D}_{G/U}$ be the algebra of non-twisted differential operators on G/U equipped with the natural $G \times T$-equivariant structure. For a character λ of $\mathfrak{t}$, we set

$$\mathscr{D}_{G/B,\lambda} := (\mathbb{C}_\lambda \otimes_{\mathcal{U}(\mathfrak{t})} p_*\mathscr{D}_{G/U})^T$$

and consider $\mathscr{D}_{G/B,\lambda}$ as a $G \times T$-equivariant algebra of twisted differential operators as in Subsection 3.4. Then $p^{\#}\mathscr{D}_{G/B,\lambda}$ is naturally isomorphic to $\mathscr{D}_{G/U}$. Note that one can explicitly construct a bounded trivialization belonging to $\mathcal{B}(G/B, G)$ using an open covering by the open Bruhat cell and its translations.

$\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ denotes the set of positive roots determined by B and T. We write ρ for half the sum of the positive roots. Let $\mathrm{Mod}^e_{qc}(\mathscr{D}_{G/B,\lambda})$ be a full subcategory of $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$ whose object $\mathcal{M}$ satisfies the following conditions.

(i) The canonical morphism $\mathscr{D}_{G/B,\lambda} \otimes_{\mathcal{U}(\mathfrak{g})} \Gamma(\mathcal{M}) \to \mathcal{M}$ is surjective.
(ii) $\Gamma(\mathcal{N}) \neq 0$ holds for any non-zero submodule $\mathcal{N}$ of $\mathcal{M}$ in $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$.

Fact 7.1 is called the Beilinson–Bernstein correspondence [5].

Fact 7.1. Let λ be a character of $\mathfrak{t}$.

(i) The homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{D}_{G/B,\lambda} (= \Gamma(\mathscr{D}_{G/B,\lambda}))$ is surjective and its kernel is equal to the minimal primitive ideal with infinitesimal character $\lambda - \rho$.
(ii) If $\lambda - \rho$ is anti-dominant, then any quasi-coherent $\mathscr{D}_{G/B,\lambda}$-module $\mathcal{M}$ is acyclic, i.e. $H^i(G/B, \mathcal{M}) = 0$ for any $i > 0$. In particular, the global

section functor Γ is exact on $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$. Moreover, the global section functor

$$\Gamma\colon \mathrm{Mod}^{e}_{qc}(\mathscr{D}_{G/B,\lambda}) \to \mathrm{Mod}(\mathcal{D}_{G/B,\lambda})$$

gives an equivalence of categories.

(iii) If $\lambda - \rho$ is regular and anti-dominant, then the global section functor $\Gamma\colon \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda}) \to \mathrm{Mod}(\mathcal{D}_{G/B,\lambda})$ gives an equivalence of categories.

We need the following consequences of the Beilinson–Bernstein correspondence.

Corollary 7.2. *Let λ be a character of $\mathfrak{t}$ such that $\lambda - \rho$ is anti-dominant.*

(i) *Let M be a $\mathfrak{g}$-module with the infinitesimal character $\lambda - \rho$. Suppose that $\mathcal{L}$ is the maximum submodule of $\mathcal{M} := \mathscr{D}_{G/B,\lambda} \otimes_{\mathcal{U}(\mathfrak{g})} M$ such that $\Gamma(\mathcal{L}) = 0$. Then we have $\mathcal{M}/\mathcal{L} \in \mathrm{Mod}^{e}_{qc}(\mathscr{D}_{G/B,\lambda})$ and $M \simeq \Gamma(\mathcal{M}/\mathcal{L})$. Moreover, if M is irreducible, then so is $\mathcal{M}/\mathcal{L}$ in $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$.*

(ii) *For any $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$, there exists a subquotient $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N} \in \mathrm{Mod}^{e}_{qc}(\mathscr{D}_{G/B,\lambda})$ and $\Gamma(\mathcal{M}) \simeq \Gamma(\mathcal{N})$.*

(iii) *For any irreducible $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$, the $\mathfrak{g}$-module $\Gamma(\mathcal{M})$ is irreducible or zero. In particular, we have $\mathrm{Len}_{\mathfrak{g}}(\Gamma(\mathcal{M})) \leq \mathrm{Len}_{\mathscr{D}_{G/B,\lambda}}(\mathcal{M})$ for any $\mathcal{M} \in \mathrm{Mod}_{h}(\mathscr{D}_{G/B,\lambda})$.*

Proof. The first assertion of (i) is shown in the proof of [21, Corollary 11.2.6]. Assume that M is irreducible and let $\mathcal{N}$ be a non-zero submodule of $\mathcal{M}/\mathcal{L}$ in $\mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$. Then we have $\Gamma(\mathcal{N}) \neq 0$ and hence $\Gamma(\mathcal{N}) = \Gamma(\mathcal{M}/\mathcal{L})$. Since $\mathcal{M}/\mathcal{L}$ is generated by $\Gamma(\mathcal{M}/\mathcal{L}) = \Gamma(\mathcal{N})$, we have $\mathcal{N} = \mathcal{M}/\mathcal{L}$. This shows that $\mathcal{M}/\mathcal{L}$ is irreducible.

To show (ii), let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$. Take the maximum submodule $\mathcal{L}$ of $\mathcal{M}' := \mathscr{D}_{G/B,\lambda}\Gamma(\mathcal{M})$ such that $\Gamma(\mathcal{L}) = 0$. Then $\mathcal{N} := \mathcal{M}'/\mathcal{L}$ satisfies the desired conditions as in (i).

To show (iii), let $\mathcal{M} \in \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$ and assume that $\mathcal{M}$ is irreducible. If $\Gamma(\mathcal{M}) \neq 0$, then $\mathcal{M}$ belongs to $\mathrm{Mod}^{e}_{qc}(\mathscr{D}_{G/B,\lambda})$ by definition, and hence $\Gamma(\mathcal{M})$ is irreducible. □

Definition 7.3. Let λ be a character of $\mathfrak{t}$ such that $\lambda - \rho$ is anti-dominant. For a $\mathfrak{g}$-module M with the infinitesimal character $\lambda - \rho$, we denote by $L^{e}_{\lambda}(M)$ the $\mathscr{D}_{G/B,\lambda}$-module $\mathcal{M}/\mathcal{L}$ in (i) of Corollary 7.2.

If $\lambda - \rho$ is regular, then L^{e}_{λ} is the localization functor $\mathscr{D}_{G/B,\lambda} \otimes_{\mathcal{U}(\mathfrak{g})} (\cdot)$.

Proposition 7.4. *Let λ be a character of $\mathfrak{t}$ such that $\lambda - \rho$ is anti-dominant. Assume $G = G_1 \times G_2$ and decompose $B = B_1 \times B_2$, $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$, $\rho = \rho_1 + \rho_2$ and $\lambda = \lambda_1 + \lambda_2 \in \mathfrak{t}_1^* \oplus \mathfrak{t}_2^*$. Let M_i be a $\mathfrak{g}_i$-module with the infinitesimal character $\lambda_i - \rho_i$ for $i = 1, 2$. Then we have*

$$L^{e}_{\lambda_1}(M_1) \boxtimes L^{e}_{\lambda_2}(M_2) \simeq L^{e}_{\lambda}(M_1 \boxtimes M_2).$$

Proof. If M_1 or M_2 is zero, the assertion is trivial, so we can assume that M_1 and M_2 are non-zero. Since $L^{e}_{\lambda_1}(M_1) \boxtimes L^{e}_{\lambda_2}(M_2)$ is generated by $\Gamma(L^{e}_{\lambda_1}(M_1) \boxtimes L^{e}_{\lambda_2}(M_2)) = M_1 \boxtimes M_2$, there is a natural epimorphism

$$\varphi\colon L^{e}_{\lambda_1}(M_1) \boxtimes L^{e}_{\lambda_2}(M_2) \to L^{e}_{\lambda}(M_1 \boxtimes M_2).$$

We shall show $\mathrm{Ker}(\varphi) = 0$. Let $p\colon G_1/B_1 \times G_2/B_2 \to G_1/B_1$ be the projection onto the first factor. Then we have $p_*(L^{e}_{\lambda_1}(M_1) \boxtimes L^{e}_{\lambda_2}(M_2)) \simeq L^{e}_{\lambda_1}(M_1) \otimes M_2$.

Let $\alpha \in M_2^*$ and put

$$\widetilde{\alpha} := \mathrm{id} \otimes \alpha \colon L^e_{\lambda_1}(M_1) \otimes M_2 \to L^e_{\lambda_1}(M_1).$$

Since $\Gamma(\mathrm{Ker}(\varphi)) = 0$, we have $\Gamma(p_*(\mathrm{Ker}(\varphi))) = 0$. Since Γ is exact, we have

$$\Gamma(\widetilde{\alpha}(p_*(\mathrm{Ker}(\varphi)))) = 0.$$

$\widetilde{\alpha}(p_*(\mathrm{Ker}(\varphi)))$ is a submodule of $L^e_{\lambda_1}(M_1) \in \mathrm{Mod}^e_{qc}(\mathscr{D}_{G_1/B_1,\lambda_1})$. Hence we have $\widetilde{\alpha}(p_*(\mathrm{Ker}(\varphi))) = 0$ for any $\alpha \in M_2^*$. This shows $p_*(\mathrm{Ker}(\varphi)) = 0$.

Let U be an open subset of G_1/B_1. Then we have $\Gamma(U, p_*(\mathrm{Ker}(\varphi))) = \Gamma(U \times G_2/B_2, \mathrm{Ker}(\varphi)) = 0$. Considering the projection $U \times G_2/B_2 \to G_2/B_2$, and by the same discussion, we have $\Gamma(U \times V, \mathrm{Ker}(\varphi)) = 0$ for any open subset $V \subset G_2/B_2$. This shows $\mathrm{Ker}(\varphi) = 0$. □

Motivated by the Beilinson–Bernstein correspondence and the definition of uniformly bounded family of $\mathscr{D}$-modules, we introduce Definition 7.5.

Definition 7.5. Let $(V_i)_{i\in I}$ be a family of $\mathfrak{g}$-modules. We say that $(V_i)_{i\in I}$ is *uniformly bounded* if the following two conditions hold.

(i) The length of V_i is bounded by a constant independent of $i \in I$.
(ii) There exist a family $(\lambda(r))_{r\in R}$ of anti-dominant weights of $\mathfrak{t}$ and a family $\mathcal{N} \in \mathrm{Mod}_{ub}((\mathscr{D}_{G/B,\lambda(r)+\rho})_{r\in R}, \mathcal{B}(G/B, G))$ (see Section 5.1) such that any composition factor of any V_i is isomorphic to some $\Gamma(\mathcal{N}_r)$.

We say that a family of $(\mathfrak{g}, K)$-modules of a pair $(\mathfrak{g}, K)$ is *uniformly bounded* if it is a uniformly bounded family of $\mathfrak{g}$-modules.

The uniform boundedness is preserved by several operations of $\mathfrak{g}$-modules. We shall explain how to show the uniform boundedness.

Strategy. Let G' be a connected reductive algebraic group with a Borel subgroup B', and $F \colon \mathrm{Mod}(\mathfrak{g}) \to \mathrm{Mod}(\mathfrak{g}')$ a functor. Take $\mathfrak{t}'$ and $\rho' \in (\mathfrak{t}')^*$ for G' as $\mathfrak{t}$ and ρ for G. Suppose that we want to show that $(F(V_i))_{i\in I}$ is uniformly bounded for any uniformly bounded family $(V_i)_{i\in I}$ of $\mathfrak{g}$-modules.

Assume that for any exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{Mod}(\mathfrak{g})$, the sequence $F(L) \to F(M) \to F(N)$ is exact. Then it is enough to show that $(F(V_i))_{i\in I}$ is uniformly bounded when V_i is irreducible for any $i \in I$. See Proposition 7.6(i).

Let $(V_i)_{i\in I}$ be a uniformly bounded family of $\mathfrak{g}$-modules such that V_i is irreducible for any $i \in I$. By definition, there exists a uniformly bounded family $(\mathcal{M}_i)_{i\in I}$ of twisted $\mathscr{D}$-modules on G/B such that $\Gamma(\mathcal{M}_i) \simeq V_i$, $\mathcal{M}_i \in \mathrm{Mod}_h(\mathscr{D}_{G/B,\lambda(i)})$ and $\lambda(i) - \rho$ is anti-dominant for any $i \in I$. Remark that

$$\Gamma \text{ is exact on } \mathrm{Mod}_h(\mathscr{D}_{G/B,\lambda(i)})$$

by Fact 7.1(ii).

We need a geometrical version of F. Suppose that there is a functor

$$\mathcal{F} \colon \prod_{i\in I} \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda(i)}) \to \prod_{i\in I} \mathrm{Mod}_{qc}(\mathscr{D}_{G'/B',\lambda'(i)})$$

such that

(i) $\lambda'(i) - \rho'$ is anti-dominant for any $i \in I$,
(ii) $\Gamma \circ \mathcal{F} \simeq F \circ \Gamma$,
(iii) $\mathcal{F}$ preserves the uniform boundedness (for the bornologies $\mathcal{B}(G/B, G)$ and $\mathcal{B}(G'/B', G')$).

Then F also preserves the uniform boundedness. The boundedness of the lengths of $(F(V_i))_{i\in I}$ follows from Corollary 7.2(iii). Note that the condition (i) always holds in this paper and hence Γ is exact on each $\mathrm{Mod}_{qc}(\mathscr{D}_{G'/B',\lambda'(i)})$.

We have seen the conditions (ii) and (iii) for several functors such as the Zuckerman derived functors.

Proposition 7.6. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be complex reductive Lie algebras.*

(i) *For a short exact sequence $0 \to L \to M \to N \to 0$ in $\prod_{i\in I} \mathrm{Mod}(\mathfrak{g})$, both L and N are uniformly bounded if and only if so is M.*
(ii) *For any family $(\lambda_i)_{i\in I}$ of characters of $\mathfrak{g}$ and uniformly bounded family $(V_j)_{j\in J}$ of $\mathfrak{g}$-modules, $(V_j \otimes \mathbb{C}_{\lambda_i})_{i\in I, j\in J}$ is also uniformly bounded.*
(iii) *For any set Φ of inner automorphisms of $\mathfrak{g}$ and uniformly bounded family $(V_j)_{j\in J}$ of $\mathfrak{g}$-modules, $(V_j^{\varphi})_{\varphi\in\Phi, j\in J}$ is also uniformly bounded. Here V_j^{φ} is the $\mathfrak{g}$-module defined by the composition $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V_j)$.*
(iv) *Let $(V_i)_{i\in I}$ (resp. $(W_i)_{i\in I}$) be a family of non-zero $\mathfrak{g}$-modules (resp. $\mathfrak{h}$-modules). Then $(V_i \boxtimes W_i)_{i\in I}$ is a uniformly bounded family of $(\mathfrak{g}\oplus\mathfrak{h})$-modules if and only if both $(V_i)_{i\in I}$ and $(W_i)_{i\in I}$ are uniformly bounded.*

Proof. (i) is trivial by definition.

(ii) is easy from the observation that the twist $(\cdot)\otimes\mathbb{C}_{\lambda_i}$ does not affect the structure of twisted $\mathscr{D}_{G/B}$-modules. See also Proposition 4.31.

Let $g \in G$. The functor $(\cdot)^{\mathrm{Ad}(g)}$ is realized geometrically as the inverse image functor $m_g^*\colon \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda}) \to \mathrm{Mod}_{qc}(\mathscr{D}_{G/B,\lambda})$, where $m_g\colon G/B \to G/B$ is defined by $m_g(x) = gx$. Hence (iii) is a direct consequence of Proposition 5.8.

(iv) follows from Theorem 4.28 and Proposition 7.4. □

Proposition 7.7. *Let $(\mathfrak{g}, K)$ be a pair and M a reductive subgroup of K. Let $(V_i)_{i\in I}$ be a uniformly bounded family of $(\mathfrak{g}, M)$-modules.*

(i) *$(\mathbb{D}^j\Gamma_M^K(V_i))_{i\in I, j\in\mathbb{Z}}$ is uniformly bounded.*
(ii) *If K is reductive, there exists a constant C such that for any $i \in I$ and $j \in \mathbb{Z}$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^K}(H^j(\mathfrak{k}, M; V_i)) \le C.$$

(iii) (ii) *is also true if $(V_i)_{i\in I}$ is a uniformly bounded family of $(\mathfrak{g},\mathfrak{m})$-modules and $H^j(\mathfrak{k}, M; \cdot)$ is replaced by $H^j(\mathfrak{k},\mathfrak{m};\cdot)$.*
(iv) (ii) *is also true if we replace M with its covering $\widetilde{M}$.*

Proof. As we have mentioned in Strategy, we can reduce the assertions to similar results about $\mathscr{D}$-modules on the flag variety. (i) follows from Theorem 6.7. Taking the K-invariant part of (i), (ii) follows from Corollary 6.4.

By the definition of the relative Lie algebra cohomology, we have

$$H^j(\mathfrak{k},\mathfrak{m}; V_i) = H^j(\mathfrak{k}, M_0; (V_i)_{M_0}),$$

where $(V_i)_{M_0}$ is the sum of all $\mathfrak{m}$-submodules in V_i that can lift to M_0-modules. Hence (iii) follows from (ii). (iv) can be reduced to (iii) by

$$H^j(\mathfrak{k}, \widetilde{M}; V_i) = H^j(\mathfrak{k},\mathfrak{m}; V_i)^{\widetilde{M}/\widetilde{M}_0}.$$

We have proved the proposition. □

Let $(\mathfrak{g}, K)$ be a pair. Then K acts on the flag variety of $\mathfrak{g}$, which is isomorphic to G/B. Assume that K has finitely many orbits in G/B.

Proposition 7.8. *Let $\mathfrak{h}$ be a complex reductive Lie algebra and $(M_i)_{i\in I}$ a uniformly bounded family of $(\mathfrak{g}\oplus\mathfrak{h})$-modules. For any set $\mathcal{F}$ of finite-dimensional $\mathfrak{k}$-modules whose dimensions are bounded, the family $(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{k})}(F, M_i))_{i\in I, j\in\mathbb{Z}, F\in\mathcal{F}}$ is a uniformly bounded family of $\mathfrak{h}$-modules. Moreover, there exists a constant C such that for any finite-dimensional $\mathfrak{k}$-module F, $i \in I$ and $j \in \mathbb{Z}$,*

$$\mathrm{Len}_{\mathfrak{h}}(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{k})}(F, M_i)) \leq C \cdot \dim_{\mathbb{C}}(F).$$

Proof. The proposition follows from Theorem 5.22. See Strategy. Remark that the assumption that K has finitely many orbits in G/B is also assumed in Subsection 5.4. □

Remark 7.9. If $\mathfrak{h} = 0$, then $\mathrm{Len}_{\mathfrak{h}}(\mathrm{Tor}_0^{\mathcal{U}(\mathfrak{k})}(F, M_i))$ is the dimension of $F \otimes_{\mathcal{U}(\mathfrak{k})} M_i$. In this case, Proposition 7.8 is a kind of finite multiplicity theorems. Replacing G/B by a partial flag variety G/P and $\mathscr{D}_{G/B,\lambda} \otimes_{\mathcal{U}(\mathfrak{k})} F$ by some holonomic $\mathscr{D}$-module, one can obtain several finite multiplicity theorems. We postpone the results to the sequel.

A typical example of uniformly bounded family is a family of Harish-Chandra modules.

Proposition 7.10. *Any family of $(\mathfrak{g}, \mathfrak{k})$-modules with bounded lengths is uniformly bounded. In particular, so is any family of $(\mathfrak{g}, K)$-modules with bounded lengths.*

Proof. The second assertion follows from the first one because the length of $(\mathfrak{g}, K)$-module V is bounded by $|K/K_0| \cdot \mathrm{Len}_{\mathfrak{g}}(V)$ by Lemma 2.4.

Take a covering K' of K_0 such that $[K'/U_{K'}, K'/U_{K'}]$ is simply-connected, where $U_{K'}$ is the unipotent radical of K'. Then we have a homomorphism $K' \to K \to \mathrm{Aut}(\mathfrak{g})$ and K' has finitely many orbits in G/B.

Let V be an irreducible $(\mathfrak{g}, \mathfrak{k})$-module. We want to realize V as $\Gamma(\mathcal{V})$ for some irreducible $\mathscr{D}$-module $\mathcal{V}$ on G/B. Since the $\mathfrak{k}$-action is locally finite, we can take a finite-dimensional irreducible $\mathfrak{k}$-submodule $F \subset V$. Take a character μ of $\mathfrak{k}$ such that the $\mathfrak{k}$-action on $F \otimes \mathbb{C}_\mu$ lifts to a K'-action. Since V is irreducible, the multiplication map $\mathcal{U}(\mathfrak{g}) \otimes F \to V$ is surjective. Hence the $\mathfrak{k}$-action on $V \otimes \mathbb{C}_\mu$ lifts to a K'-action.

Let λ be a character of $\mathfrak{t}$ such that $\lambda - \rho$ is anti-dominant and $\lambda - \rho$ is the infinitesimal character of V. We can take an irreducible subquotient $\mathcal{V}$ of $\mathscr{D}_{G/B,\lambda} \otimes_{\mathcal{U}(\mathfrak{g})} V$ such that $\Gamma(\mathcal{V}) \simeq V$ (see Corollary 7.2(i)). By construction, $\mathcal{V}$ is an irreducible twisted $(\mathscr{D}_{G/B,\lambda}, K')$-module with twist μ.

Since K' has finitely many orbits in G/B, the proposition follows from Corollary 5.16. □

7.2. **Induction of uniformly bounded family.** In this subsection, we will show uniform boundedness of some family of $\mathfrak{g}$-modules.

Let G be a connected reductive algebraic group and B a Borel subgroup of G with unipotent radical U. Put $T := B/U$. We denote by $\mathcal{I}_\chi$ the minimal primitive ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character χ. Let W_G be the Weyl group of G.

Proposition 7.11. *The family $(\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi)_{\chi\in\mathfrak{t}^*/W_G}$ of $(\mathfrak{g}\oplus\mathfrak{g}, \Delta(G))$-modules is uniformly bounded.*

Proof. $\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi$ is isomorphic to $\Gamma((\mathscr{D}_{G/B,\lambda+\rho} \boxtimes \mathscr{D}_{G/B,\lambda'+\rho}) \otimes_{\mathcal{U}(\Delta(\mathfrak{g}))} \mathbb{C})$, where λ (resp. λ') is an anti-dominant weight in χ (resp. $-\chi$). Since $\Delta(G)$ has finitely many orbits in $G/B \times G/B$, the assertion follows from Corollary 5.21.

□

Remark 7.12. The structure of the $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(G))$-modules can be reduced to that of Verma modules (see [7, Section 6]). Proposition 7.11 can be deduced from this and Soergel's theorem [43, Theorem 11] (see also Remark 7.14).

Proposition 7.13. *Let P be a parabolic subgroup of G containing B with unipotent radical U_P, and $(M_i)_{i\in I}$ a uniformly bounded family of $\mathfrak{p}/\mathfrak{u}_P$-modules. Then $(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M_i)_{i\in I}$ is a uniformly bounded family of $\mathfrak{g}$-modules. In particular, the length of any Verma module is bounded by a constant independent of its highest weight.*

Proof. As we have mentioned in Strategy in the previous subsection, we can assume that each M_i is irreducible and hence has an infinitesimal character. Since P is parabolic, each $\mathfrak{g}$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M_i$ has an infinitesimal character χ_i. Then we have

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M_i \simeq (\mathcal{U}(\mathfrak{g})/\mathcal{I}_{\chi_i} \otimes M_i) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}.$$

$(\mathcal{U}(\mathfrak{g})/\mathcal{I}_{\chi_i} \otimes M_i)_{i\in I}$ is a uniformly bounded family of $(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{p}/\mathfrak{u}_P)$-modules by Propositions 7.11 and 7.6(iv). Since P has finitely many orbits in $G/B \times P/B$, the assertion follows from Proposition 7.8. □

Remark 7.14. The second assertion is an easy consequence of Soergel's theorem [43, Theorem 11]. In fact, the categorical structure of each block of the BGG category $\mathcal{O}$ depends only on a pair of a Coxeter system and a subgroup of W_G determined by the block, and the number of such pairs is finite.

We consider cohomologically induced modules. Let $(\mathfrak{g}, K)$ be a pair and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$. Take a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ and a reductive subgroup K_L of K whose Lie algebra is contained in $\mathfrak{l} \cap \mathfrak{k}$. Assume that K_L normalizes $\mathfrak{p}$ and $\mathfrak{l}$. We consider $\mathfrak{l}$-modules as $\mathfrak{p}$-modules through the natural surjection $\mathfrak{p} \to \mathfrak{l}$.

Theorem 7.15. *Let $(V_i)_{i\in I}$ be a uniformly bounded family of $(\mathfrak{l}, K_L)$-modules, e.g. a family of irreducible Harish-Chandra modules. (See Proposition* 7.10*.) Then $(\mathbb{D}^j \Gamma^K_{K_L}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V_i))_{j\in\mathbb{Z}, i\in I}$ is a uniformly bounded family of $(\mathfrak{g}, K)$-modules. In particular, there exists some constant C such that for any $i \in I$ and $j \in \mathbb{Z}$, we have*

$$\mathrm{Len}_{\mathfrak{g},K}(\mathbb{D}^j \Gamma^K_{K_L}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V_i)) \le C.$$

Proof. The assertion follows from Propositions 7.13 and 7.7(i). □

It is well-known that a $(\mathfrak{g}, K)$-module cohomologically induced from an irreducible module of a parabolic subpair is of finite length (see e.g. [32, Theorem 0.46]). In addition to this fact, we have shown that the lengths of such modules are bounded.

Corollary 7.16 is a special case of Theorem 7.15 because the underlying Harish-Chandra module of any principal series representation can be realized as a cohomologically induced module. See [32, Propositions 11.57 and 11.65].

Corollary 7.16. *Let $G_\mathbb{R}$ be a real reductive Lie group. Then there exists a constant C such that the length of any principal series representation of $G_\mathbb{R}$ is bounded by C.*

Remark 7.17. Corollary 7.16 has been proved in [35, Proposition 4.1] by using the theory of minimal K-types and the translation principle.

7.3. $\mathcal{U}\mathfrak{g}^{G'}$**-modules.** For applications to the branching problem and harmonic analysis, we shall summarize several consequences of the results so far about uniformly bounded families.

Let G be a reductive algebraic group and G' a reductive subgroup of G.

Theorem 7.18. *Let $(V_i)_{i\in I}$ and $(V_i')_{i\in I}$ be uniformly bounded families of $\mathfrak{g}$-modules and $\mathfrak{g}'$-modules, respectively. Then there exists some constant C such that for any $i \in I$ and $j \in \mathbb{N}$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{g}')}(V_i, V_i')) \le C.$$

Proof. By Proposition 7.7(ii) for $K = \Delta(G')$ and $M = \{e\}$, there is a constant C such that for any $i \in I$ and $j \in \mathbb{N}$,

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(H^j(\mathfrak{g}'; V_i \otimes V_i')) \le C.$$

Here we replaced $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}')^{\Delta(G')}$ with $\mathcal{U}(\mathfrak{g})^{G'}$ using the isomorphism

$$\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}')/\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}')\Delta(\mathfrak{g}') \simeq \mathcal{U}(\mathfrak{g})$$

(see also the proof of Corollary 6.4). Put $n = \dim_{\mathbb{C}}(\mathfrak{g}')$. By the Poincaré duality (Fact 2.7), we have

$$H^j(\mathfrak{g}'; V_i \otimes V_i') \simeq H_{n-j}(\mathfrak{g}'; V_i \otimes V_i') \simeq \mathrm{Tor}_{n-j}^{\mathcal{U}(\mathfrak{g}')}(V_i, V_i').$$

Since these isomorphisms are natural in V_i and V_i', the isomorphisms are $\mathcal{U}(\mathfrak{g})^{G'}$-homomorphisms. We have shown the theorem. □

Corollary 7.19. *Let $\mathfrak{b}'$ be a Borel subalgebra of $\mathfrak{g}'$ and $(V_i)_{i\in I}$ a uniformly bounded family of $\mathfrak{g}$-modules. Then there exists some constant C such that for any character λ of $\mathfrak{b}'$, $j \in \mathbb{Z}$ and $i \in I$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{b}')}(V_i, \mathbb{C}_\lambda)) \le C.$$

Moreover, the constant C can be chosen independently of $\mathfrak{b}'$.

Proof. Since $\mathcal{U}(\mathfrak{g}')$ is a free right $\mathcal{U}(\mathfrak{b}')$-module, there is a natural isomorphism

$$\mathrm{Tor}_j^{\mathcal{U}(\mathfrak{b}')}(V_i, \mathbb{C}_\lambda) \simeq \mathrm{Tor}_j^{\mathcal{U}(\mathfrak{g}')}(V_i, \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{b}')} \mathbb{C}_\lambda)$$

of $\mathcal{U}(\mathfrak{g})^{G'}$-modules. The family $(\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{b}')} \mathbb{C}_\lambda)_{\lambda,\mathfrak{b}'}$ is uniformly bounded by Proposition 7.13 and Proposition 7.6(iii). Hence the corollary follows from Theorem 7.18. □

Corollary 7.20. *Let $(V_i)_{i\in I}$ be a uniformly bounded family of $\mathfrak{g}$-modules. There exists some constant C such that for any maximal ideal $\mathcal{I}$ of $\mathcal{Z}(\mathfrak{g}')$, $i \in I$ and $j \in \mathbb{Z}$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'} \otimes \mathcal{U}(\mathfrak{g}')}(\mathrm{Tor}_j^{\mathcal{Z}(\mathfrak{g}')}(\mathcal{Z}(\mathfrak{g}')/\mathcal{I}, V_i)) \le C.$$

Proof. Since $\mathcal{U}(\mathfrak{g}')$ is a free $\mathcal{Z}(\mathfrak{g}')$-module, we have a natural isomorphism

$$\mathrm{Tor}_j^{\mathcal{Z}(\mathfrak{g}')}(\mathcal{Z}(\mathfrak{g}')/\mathcal{I}, V_i) \simeq \mathrm{Tor}_j^{\mathcal{U}(\mathfrak{g}')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I}\mathcal{U}(\mathfrak{g}'), V_i).$$

Hence the corollary follows from Proposition 7.11 and Theorem 7.18. □

Retain the notation G and G' as above. Let $(\mathfrak{g}, K)$ and $(\mathfrak{g}', K')$ be pairs (see Definition 2.5).

Corollary 7.21. *Assume that K and K' have finitely many orbits in the flag varieties of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Then there exists some constant C such that for any $i \in \mathbb{N}$, irreducible $(\mathfrak{g}', K')$-module V' and irreducible $(\mathfrak{g}, K)$-module V, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g}')}(V, V')) \leq C.$$

Proof. By Proposition 7.10, any family of irreducible $(\mathfrak{g}, K)$-modules or irreducible $(\mathfrak{g}', K')$-modules is uniformly bounded. Hence the assertion follows from Theorem 7.18. □

7.4. **Euler–Poincaré characteristic.** We shall define the Euler–Poincaré characteristic in the setting of the branching problem and harmonic analysis. Retain the notation G, G', K and K' in the previous subsection. Assume that K' is reductive and contained in K, and $\mathrm{Ad}_{\mathfrak{g}}(K')$ is contained in $\mathrm{Ad}_{\mathfrak{g}}(G')$.

Theorem 7.22. *Let $(V_i)_{i\in I}$ (resp. $(V_i')_{i\in I}$) be a uniformly bounded family of $(\mathfrak{g}, K)$-modules (resp. $(\mathfrak{g}', K')$-modules). Then there exists some constant C such that for any $i \in I$ and $j \in \mathbb{N}$, we have*

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^{G'}}(H_j(\mathfrak{g}', K'; V_i \otimes V_i')) \leq C.$$

In particular, the Euler–Poincaré characteristic

$$\mathrm{EP}(V_i, V_i') := \sum_i (-1)^i H_i(\mathfrak{g}', K'; V_i \otimes V_i')$$

is well-defined as an element of the Grothendieck group of the category of $\mathcal{U}(\mathfrak{g})^{G'}$-modules of finite length.

Proof. Almost all of the proof is the same as that of Theorem 7.18. We note the difference. In this setting, the Poincaré duality (Fact 2.7) is written as

$$H^{n-j}(\mathfrak{g}', K'; V_i \otimes V_i' \otimes \wedge^n(\mathfrak{g}'/\mathfrak{k}')) \simeq H_j(\mathfrak{g}', K'; V_i \otimes V_i'),$$

where $n = \dim_{\mathbb{C}}(\mathfrak{g}'/\mathfrak{k}')$ and the $\mathfrak{g}'$-action on $\wedge^n(\mathfrak{g}'/\mathfrak{k}')$ is trivial. Hence the twisting by $\wedge^n(\mathfrak{g}'/\mathfrak{k}')$ does not affect the action of $\mathcal{U}(\mathfrak{g})^{G'}$. Therefore we have proved the theorem by Proposition 7.7(ii). □

Remark 7.23. It is clear that for the well-definedness of the Euler–Poincaré characteristic, we do not need the notion of uniformly bounded families. In fact, we need only holonomicity of modules.

Remark 7.24. $H_i(\mathfrak{g}', K'; V \otimes V')^*$ is isomorphic to $\mathrm{Ext}^i_{\mathfrak{g}', K'}(V, (V')^*_{K'})$ as a $\mathcal{U}(\mathfrak{g})^{G'}$-module (see [32, Corollary 3.2]). If $\mathrm{Ext}^i_{\mathfrak{g}', K'}(V, (V')^*_{K'})$ is not finite dimensional, the $\mathcal{U}(\mathfrak{g})^{G'}$-module does not have finite length because it is uncountably infinite dimensional.

If all $H_i(\mathfrak{g}', K'; V \otimes V')$ are finite dimensional, we can define the ($\mathbb{Z}$-valued) Euler–Poincaré characteristic

$$\dim_{\mathbb{C}} \mathrm{EP}(V, V') := \sum_i (-1)^i \dim_{\mathbb{C}}(H_i(\mathfrak{g}', K'; V \otimes V')). \tag{7.4.1}$$

The characteristic for p-adic groups is studied in [42], [3] and [14]. Remark that $\mathrm{EP}(V, V')$ in the papers corresponds to $\dim_{\mathbb{C}} \mathrm{EP}(V, (V')^*_{K'})$ in our notation. We

give sufficient conditions for the well-definedness of the $\mathbb{Z}$-valued characteristic in the sequels [29, Corollary 7.17].

7.5. **Theta lifting.** We apply Theorem 7.22 to the theory of the Howe duality (see [22, 23]).

Let $G_{\mathbb{R}}$ be a double cover of $\mathrm{Sp}(n,\mathbb{R})$. Let $(H_{\mathbb{R}}, H'_{\mathbb{R}})$ be a reductive dual pair of $G_{\mathbb{R}}$, i.e. $H_{\mathbb{R}} = C_{G_{\mathbb{R}}}(H'_{\mathbb{R}})$ and $H'_{\mathbb{R}} = C_{G_{\mathbb{R}}}(H_{\mathbb{R}})$ holds. Here $C_{G_{\mathbb{R}}}(\cdot)$ denotes the centralizer in $G_{\mathbb{R}}$. We write $\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}'_{\mathbb{R}}$ for the Lie algebras of $G_{\mathbb{R}}$, $H_{\mathbb{R}}$ and $H'_{\mathbb{R}}$, respectively.

Fix a Cartan involution θ of $G_{\mathbb{R}}$ which stabilizes $H_{\mathbb{R}}$ and $H'_{\mathbb{R}}$, and put $K_{\mathbb{R}} := G^{\theta}_{\mathbb{R}}, K_{H,\mathbb{R}} := H^{\theta}_{\mathbb{R}}$ and $K_{H',\mathbb{R}} := (H'_{\mathbb{R}})^{\theta}$. Then we have pairs $(\mathfrak{g}, K), (\mathfrak{h}, K_H)$ and $(\mathfrak{h}', K_{H'})$, which are the complexifications of $(\mathfrak{g}_{\mathbb{R}}, K_{\mathbb{R}})$, $(\mathfrak{h}_{\mathbb{R}}, K_{H,\mathbb{R}})$ and $(\mathfrak{h}'_{\mathbb{R}}, K_{H',\mathbb{R}})$, respectively. We write (ω, V) for the underlying Harish-Chandra module of the Segal–Shale–Weil representation of $G_{\mathbb{R}}$. Then, by the classical invariant theory, we have $\omega(\mathcal{U}(\mathfrak{g}))^H = \omega(\mathcal{U}(\mathfrak{h}'))$. Here H is the centralizer in $\mathrm{Sp}(n,\mathbb{C})$ of the image of $H'_{\mathbb{R}}$ by the covering map $G_{\mathbb{R}} \to \mathrm{Sp}(n,\mathbb{R})$.

For an irreducible $(\mathfrak{h}, K_H)$-module V', we set

$$\Theta_i(V') := H_i(\mathfrak{h}, K_H; V \otimes {V'}^{\vee}),$$

where ${V'}^{\vee}$ is the space of all K_H-finite vectors in $(V')^*$. Then $\Theta_i(V')$ is an $(\mathfrak{h}', K_{H'})$-module. Let $\mathcal{R}(\mathfrak{h}, K_H, \omega)$ be the set of equivalence classes of irreducible $(\mathfrak{h}, K_H)$-modules such that $\Theta_0(V') \neq 0$.

Fact 7.25 (R. Howe [23, Theorem 2.1])**.** For any $V' \in \mathcal{R}(\mathfrak{h}, K_H, \omega)$, $\Theta_0(V')$ is of finite length and has a unique irreducible quotient $\theta(V')$. The correspondence $\mathcal{R}(\mathfrak{h}, K_H, \omega) \ni V' \mapsto \theta(V') \in \mathcal{R}(\mathfrak{h}', K_{H'}, \omega)$ is bijective.

For any $i \in \mathbb{N}$, $\Theta_i(V')$ is of finite length by $\omega(\mathcal{U}(\mathfrak{g}))^H = \omega(\mathcal{U}(\mathfrak{h}'))$ and Theorem 7.22. More precisely, Theorem 7.26 holds.

Theorem 7.26. *Let V' be an irreducible $(\mathfrak{h}, K_H)$-module. Then there exists some constant C independent of V' such that*

$$\mathrm{Len}_{\mathfrak{h}', K_{H'}}(\Theta_i(V')) \leq C$$

for any $i \in \mathbb{N}$. In particular, as an element of the Grothendieck group of the category of $(\mathfrak{h}', K_{H'})$-modules of finite length, the Euler–Poincaré characteristic

$$\mathrm{EP}(V, {V'}^{\vee}) = \sum_i (-1)^i \Theta_i(V')$$

is well-defined.

The well-definedness of the Euler–Poincaré characteristic of the theta lifting for p-adic groups is proved and studied in [1, Proposition 1.1].

7.6. **Uniformly bounded family in branching problem.** Let G be a connected reductive algebraic group and G' a connected reductive subgroup of G. Using the restriction of modules, we shall construct a uniformly bounded family of $\mathfrak{g}'$-modules from one of $\mathfrak{g}$-modules. We consider the embedding $\iota\colon G' \to G' \times G' \times G$ defined by $\iota(g) = (e, g, g)$.

Lemma 7.27. *Let V be a $\mathfrak{g}$-module and V' an irreducible $\mathfrak{g}'$-module, and set $\mathcal{I} := \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g}')}(V')$. If $0 < \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}'}(V, V') < \infty$, then there exists an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module W such that $V' \boxtimes W$ is isomorphic to a subquotient of $\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I} \otimes V)$, where $n := \dim_{\mathbb{C}}(G')$.*

Proof. Take a basis $\{\varphi_i\}$ of $\mathrm{Hom}_{\mathfrak{g}'}(V/\mathcal{I}V, V')(\simeq \mathrm{Hom}_{\mathfrak{g}'}(V, V'))$ and its dual basis $\{\lambda_i\}$ of $\mathrm{Hom}_{\mathfrak{g}'}(V/\mathcal{I}V, V')^*$. Since $\mathrm{Hom}_{\mathfrak{g}'}(V/\mathcal{I}V, V')$ is finite dimensional, we obtain a $\mathcal{U}(\mathfrak{g}') \otimes \mathcal{U}(\mathfrak{g})^{G'}$-module homomorphism

$$V/\mathcal{I}V \to V' \boxtimes \mathrm{Hom}_{\mathfrak{g}'}(V/\mathcal{I}V, V')^*$$

given by $v \mapsto \sum_i \varphi_i(v) \otimes \lambda_i$. Hence the $\mathcal{U}(\mathfrak{g}') \otimes \mathcal{U}(\mathfrak{g})^{G'}$-module $V/\mathcal{I}V$ has an irreducible quotient of the form $V' \boxtimes W_0$ for an irreducible $\mathcal{U}(\mathfrak{g})^{G'}$-module W_0.

By Fact 2.7 and Lemma 6.3, we have

$$\begin{aligned}\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I} \otimes V)^{\iota(G')} &\simeq \mathcal{U}(\mathfrak{g}')/\mathcal{I} \otimes_{\mathcal{U}(\mathfrak{g}')} V \\ &\simeq V/\mathcal{I}V\end{aligned}$$

as $\mathcal{U}(\mathfrak{g}') \otimes \mathcal{U}(\mathfrak{g})^{G'}$-modules. Hence $V' \boxtimes W_0$ is isomorphic to a quotient of

$$\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I} \otimes V)^{\iota(G')}.$$

This implies that we can take an irreducible subquotient X of $\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I} \otimes V)$ such that $X^{\iota(G')} \simeq V' \boxtimes W_0$ (see e.g. [48, Proposition 3.5.4]). Since $X = \mathcal{U}(\mathfrak{g}) X^{\iota(G')}$, the $\mathfrak{g}'$-module $X|_{\mathfrak{g}'}$ is a direct sum of some copies of V'. Hence X is naturally isomorphic to $V' \boxtimes \mathrm{Hom}_{\mathfrak{g}'}(V', X)$ and the natural $(\mathfrak{g}' \oplus \mathfrak{g})$-action on $\mathrm{Hom}_{\mathfrak{g}'}(V', X)$ is irreducible. We have shown the lemma. □

Theorem 7.28. *Let $(V_i)_{i \in I}$ be a uniformly bounded family of $\mathfrak{g}$-modules and $(V_i')_{i \in I}$ a family of irreducible $\mathfrak{g}'$-modules. If $0 < \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{g}'}(V_i, V_i')) < \infty$ for any $i \in I$, then $(V_i')_{i \in I}$ is uniformly bounded.*

Proof. Set $\mathcal{I}_i := \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g}')}(V_i')$. Then $(\mathcal{U}(\mathfrak{g}')/\mathcal{I}_i \otimes V_i)_{i \in I}$ is a uniformly bounded family of $(\mathfrak{g}' \oplus \mathfrak{g}' \oplus \mathfrak{g})$-modules by Proposition 7.11. By Proposition 7.7(i), $(\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I}_i \otimes V_i))_{i \in I}$ is a uniformly bounded family of $(\mathfrak{g}' \oplus \mathfrak{g}' \oplus \mathfrak{g}, \iota(G'))$-modules. Here we set $n := \dim_{\mathbb{C}}(G')$.

By Lemma 7.27, for each $i \in I$, we can take an irreducible $(\mathfrak{g}' \oplus \mathfrak{g}, \Delta(G'))$-module W_i such that $V_i' \boxtimes W_i$ is a subquotient of $\mathbb{D}^n \Gamma_{\{e\}}^{\iota(G')}(\mathcal{U}(\mathfrak{g}')/\mathcal{I}_i \otimes V_i)$. This implies that $(V_i' \boxtimes W_i)_{i \in I}$ is a uniformly bounded family of $(\mathfrak{g}' \oplus \mathfrak{g}' \oplus \mathfrak{g}, \iota(G'))$-modules. By Proposition 7.6(iv), the family $(V_i')_{i \in I}$ is uniformly bounded. □

7.7. **Tensoring with finite-dimensional modules.** Let G be a connected reductive algebraic group. We shall show that the uniform boundedness is preserved by tensoring with finite-dimensional modules. In particular, the uniform boundedness is preserved by the translation functors.

Lemma 7.29. *Let V be an irreducible $\mathfrak{g}$-module. Fix a maximal torus T of G. Consider $\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ as a G-module by the adjoint action. Then for any irreducible G-module F, we have*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(F, \mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V))) \leq \dim_{\mathbb{C}}(\mathrm{Hom}_G(F, \mathcal{O}(G/T))) = \dim_{\mathbb{C}}(F^T).$$

Proof. Let F be an irreducible G-module and χ the infinitesimal character of V. Then $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ contains the minimal primitive ideal $\mathcal{I}_\chi$ with the infinitesimal character χ. Hence it is enough to show

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(F,\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi)) = \dim_{\mathbb{C}}(\mathrm{Hom}_G(F,\mathcal{O}(G/T))).$$

This is shown in [32, Theorem 7.194]. Note that $R(\lambda)$ in the theorem is $\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi$ in our notation. □

Lemma 7.30. *Let $(V_i)_{i\in I}$ be a uniformly bounded family of $\mathfrak{g}$-modules and F a finite-dimensional $\mathfrak{g}$-module. Then there exists a constant $C>0$ independent of F such that*

$$\mathrm{Len}_{\mathfrak{g}}(V_i\otimes F) \le C\cdot \dim_{\mathbb{C}}(F)^2$$

for any $i\in I$.

Proof. Clearly, we can assume that F is completely reducible. Since the lengths of all V_i are bounded by a constant independent of $i\in I$, we can also assume that all V_i are irreducible. Set $n := \dim_{\mathbb{C}}(F)$.

Fix $i\in I$. By Kostant's theorem [32, Theorem 7.133], $V_i\otimes F$ is a direct sum of finitely many submodules $W_1, W_2, \dots, W_m$ with generalized infinitesimal characters $\chi_1,\chi_2,\dots,\chi_m$, respectively. More precisely, we have $m\le n$ and $\mathcal{I}_j^{|W_{\mathfrak{g}}|}W_j = 0$ for any $1\le j\le m$, where $\mathcal{I}_j$ is the maximal ideal of $\mathcal{Z}(\mathfrak{g})$ corresponding to χ_j and $W_{\mathfrak{g}}$ is the Weyl group of $\mathfrak{g}$. There is a $\mathfrak{g}$-module surjection

$$\mathcal{I}_j^k\otimes(W_j/\mathcal{I}_jW_j) \twoheadrightarrow \mathcal{I}_j^kW_j/\mathcal{I}_j^{k+1}W_j$$

for any $k\in\mathbb{N}$, and $\mathcal{I}_j^k$ is generated by r^k elements as a $\mathcal{Z}(\mathfrak{g})$-module. Here r is the rank of $\mathfrak{g}$. Hence we have

$$\mathrm{Len}_{\mathfrak{g}}(V_i\otimes F) = \sum_j \mathrm{Len}_{\mathfrak{g}}(W_j) \tag{7.7.1}$$

$$\le C'\cdot\sum_j \mathrm{Len}_{\mathfrak{g}}(W_j/\mathcal{I}_jW_j) \tag{7.7.2}$$

$$= C'\cdot\sum_j \mathrm{Len}_{\mathfrak{g}}((V_i\otimes F)/\mathcal{I}_j(V_i\otimes F)), \tag{7.7.3}$$

where C' is a constant depending only on $|W_{\mathfrak{g}}|$ and r.

We shall estimate $\mathrm{Len}_{\mathfrak{g}}((V_i\otimes F)/\mathcal{I}_j(V_i\otimes F))$. By Corollary 7.20, there exists a constant C'' depending only on the family $(V_i)_{i\in I}$ such that

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})^{\Delta(G)}}((V_i\otimes F)/\mathcal{I}_j(V_i\otimes F)) \le C'' \tag{7.7.4}$$

for any j. Here the action of $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})^{\Delta(G)}$ on $V_i\otimes F$ factors through

$$(\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_i)\otimes\mathrm{End}_{\mathbb{C}}(F))^{\Delta(G)}.$$

Take a maximal torus T of G. By Lemma 7.29, we have

$$\begin{aligned}\dim_{\mathbb{C}}((\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_i)\otimes\mathrm{End}_{\mathbb{C}}(F))^{\Delta(G)}) &\le \dim_{\mathbb{C}}((\mathcal{O}(G/T)\otimes\mathrm{End}_{\mathbb{C}}(F))^G)\\ &= \dim_{\mathbb{C}}(\mathrm{End}_T(F)) \le \dim_{\mathbb{C}}(F)^2.\end{aligned}$$

In particular, the dimension of any irreducible module of $(\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_i)\otimes\mathrm{End}_{\mathbb{C}}(F))^{\Delta(G)}$ is less than or equal to $\dim_{\mathbb{C}}(F)$. By (7.7.4), we have

$$\mathrm{Len}_{\mathfrak{g}}((V_i\otimes F)/\mathcal{I}_j(V_i\otimes F)) \le C''\cdot\dim_{\mathbb{C}}(F).$$

Combining (7.7.3), we obtain

$$\mathrm{Len}_{\mathfrak{g}}(V_i \otimes F) \leq C'C'' \cdot \dim_{\mathbb{C}}(F)^2. \qquad \square$$

One can prove and refine Lemma 7.30 using twisting of $\mathscr{D}$-modules on the flag variety of $\mathfrak{g}$ or the theory of projective functors [7]. For $(\mathfrak{g}, K)$-modules, a more precise estimate is known [34, Proposition 5.4.1 and its proof].

Theorem 7.31. *Let $(V_i)_{i\in I}$ be a uniformly bounded family of $\mathfrak{g}$-modules and $(F_j)_{j\in J}$ a family of finite-dimensional $\mathfrak{g}$-modules with bounded dimensions. Then $(V_i \otimes F_j)_{i\in I, j\in J}$ is a uniformly bounded family of $\mathfrak{g}$-modules.*

Proof. For $i \in I$ and $j \in J$, let R_{ij} be the set of all composition factors of $V_i \otimes F_j$. By Lemma 7.30, the lengths of all $V_i \otimes F_j$ are bounded by a constant independent of $i \in I$ and $j \in J$. Hence it suffices to show that the family $(W)_{W\in R_{ij}, i\in I, j\in J}$ is uniformly bounded.

As we have seen in the proof of Lemma 7.30, any element of R_{ij} is a subquotient of $(V_i \otimes F_j)/\mathcal{I}(V_i \otimes F_j)$ for a maximal ideal $\mathcal{I}$ of $\mathcal{Z}(\mathfrak{g})$. By Theorem 7.28, the family $(W)_{W\in R_{ij}, i\in I, j\in J}$ is uniformly bounded. Note that although we have proved Theorem 7.28 for a family of irreducible quotients, the proof also works for a family of irreducible subquotients if each V_i in the theorem has finite length as a $\mathfrak{g}'$-module. $\square$

7.8. **Category of $(\mathfrak{g}, \mathfrak{k})$-modules.** Let G be a connected reductive algebraic group and K a finite covering of a connected reductive subgroup of G. Suppose that $[K, K]$ is simply-connected. We denote by $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ the full subcategory of $\mathrm{Mod}(\mathfrak{g})$ whose object is

(i) of finite length,
(ii) locally finite and completely reducible as a $\mathfrak{k}$-module, and
(iii) $\mathfrak{k}$-admissible, i.e. any $\mathfrak{k}$-isotypic component is finite dimensional.

Such a module is called a generalized Harish-Chandra module by I. Penkov and G. Zuckerman (see e.g. [40]). We write $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$ for the full subcategory of $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ whose object has the infinitesimal character χ.

In this subsection, we study the category $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$. It is related to the branching problem and harmonic analysis because the algebra $\mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ roughly controls multiplicities and its modules can be obtained from the $\Delta(K)$-invariant part of $(\mathfrak{g} \oplus \mathfrak{k}, \Delta(K))$-modules. See Theorem 7.18.

Theorem 7.32. *Let $\mathcal{I}$ (resp. $\mathcal{J}$) be a maximal ideal of $\mathcal{Z}(\mathfrak{g})$ (resp. $\mathcal{Z}(\mathfrak{k})$). $\mathcal{A} := \mathcal{U}(\mathfrak{g})^K/(\mathcal{I} + \mathcal{J})\mathcal{U}(\mathfrak{g})^K$ has finite length as an $(\mathcal{A}, \mathcal{A})$-bimodule. In particular, the intersection of all two sided ideals of finite codimension in $\mathcal{A}$ has finite codimension in $\mathcal{A}$, which coincides with the intersection of all left ideals of finite codimension in $\mathcal{A}$.*

Proof. We have a $(\mathcal{U}(\mathfrak{g})^K, \mathcal{U}(\mathfrak{g})^K)$-bimodule isomorphism

$$\begin{aligned}\mathcal{U}(\mathfrak{g})^K/(\mathcal{I} + \mathcal{J})\mathcal{U}(\mathfrak{g})^K &= (\mathcal{U}(\mathfrak{g})/(\mathcal{I}\mathcal{U}(\mathfrak{g}) + \mathcal{J}\mathcal{U}(\mathfrak{g})))^K \\ &\simeq (\mathcal{U}(\mathfrak{k})/\mathcal{J}\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})/\mathcal{I}\mathcal{U}(\mathfrak{g}))^K.\end{aligned}$$

By Proposition 7.11, $(\mathcal{U}(\mathfrak{g})/\mathcal{I}\mathcal{U}(\mathfrak{g}))_{\mathcal{I}}$ and $(\mathcal{U}(\mathfrak{k})/\mathcal{J}\mathcal{U}(\mathfrak{k}))_{\mathcal{J}}$ are uniformly bounded families of $(\mathfrak{g} \oplus \mathfrak{g})$-modules and $(\mathfrak{k} \oplus \mathfrak{k})$-modules, respectively. By applying Theorem

7.18 to $\mathcal{U}(\mathfrak{k})/\mathcal{J}\mathcal{U}(\mathfrak{k}) \otimes \mathcal{U}(\mathfrak{g})/\mathcal{I}\mathcal{U}(\mathfrak{g})$, there exists some constant C independent of $\mathcal{I}$ and $\mathcal{J}$ such that

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})^K \otimes \mathcal{U}(\mathfrak{g})^K}((\mathcal{U}(\mathfrak{k})/\mathcal{J}\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})/\mathcal{I}\mathcal{U}(\mathfrak{g}))^K) \leq C.$$

This shows the first assertion.

Let $\mathcal{K}$ be the intersection of all two sided ideals of finite codimension in $\mathcal{A}$. A two sided ideal of $\mathcal{A}$ is just an $(\mathcal{A}, \mathcal{A})$-submodule of $\mathcal{A}$. By the first assertion, the poset of two sided ideals of $\mathcal{A}$ satisfies the descending chain condition. Hence $\mathcal{K}$ can be written as an intersection of finitely many two sided ideals of finite codimension in $\mathcal{A}$. This implies that $\mathcal{K}$ has finite codimension in $\mathcal{A}$.

Any left ideal $\mathcal{J}$ of $\mathcal{A}$ of finite codimension contains the two sided ideal $\mathrm{Ann}_{\mathcal{A}}(\mathcal{A}/\mathcal{J})$, which is of finite codimension. Hence $\mathcal{K}$ is also the intersection of all left ideals of finite codimension. □

Remark 7.33. If $\mathfrak{k}$ does not contain any non-trivial ideal of $\mathfrak{g}$, the center of $\mathcal{U}(\mathfrak{g})^K$ is equal to $\mathcal{Z}(\mathfrak{g})\mathcal{Z}(\mathfrak{k}) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$ by [33, Theorem 10.1].

Let $\mathcal{I}_\chi$ be the minimal primitive ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character χ.

Theorem 7.34. *Any family of objects in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ with bounded lengths is a uniformly bounded family. In particular, for any irreducible object $V \in \mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$, there exist an anti-dominant $\lambda \in \chi$ and some $\mathcal{M} \in \mathrm{Mod}_h(\mathscr{D}_{G/B,\lambda+\rho})$ such that $V \simeq \Gamma(\mathcal{M})$. (See Subsection* 7.1 *for the notation.)*

Remark 7.35. The second assertion has been proved by A. V. Petukhov [41].

Proof. It is enough to show that the family of all irreducible objects in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ (modulo isomorphism) is uniformly bounded.

Let V be an irreducible object in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ with infinitesimal character χ. Then we can take a character μ of $\mathfrak{k}$ such that $V \otimes \mathbb{C}_\mu$ lifts to a K-module (see the proof of Proposition 7.10). Since $V \otimes \mathbb{C}_\mu$ is an irreducible $(\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{g}, \Delta(K))$-module, replacing V by $V \otimes \mathbb{C}_\mu$ and $(\mathfrak{g}, \mathfrak{k})$ by $(\mathfrak{k}/[\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{g}, \Delta(K))$, we can assume that V is an irreducible $(\mathfrak{g}, K)$-module. See also the proof of Corollary 5.16.

We put $W := \mathbb{D}^n \Gamma^K_{\{e\}}(\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi)$, where $n = \dim_{\mathbb{C}}(\mathfrak{k})$ and we take the functor $\mathbb{D}^n \Gamma^K_{\{e\}}$ with respect to the left $\mathfrak{k}$-action. Then for any irreducible K-module F, we have

$$\mathrm{Hom}_K(F, W) \simeq F^* \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi$$

by Fact 6.2. The action of $0 \oplus \mathfrak{k} \subset \mathfrak{g} \oplus \mathfrak{g}$ on $F^* \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi$ is given by the right multiplication, which is locally finite and lifts canonically to a K-action. W is a $(\mathfrak{g} \oplus \mathfrak{g}, K \times K)$-module since W is isomorphic to the direct sum of the $K \times K$-modules of the form $F \otimes \mathrm{Hom}_K(F, W)$.

$V^\vee$ denotes the subspace of all K-finite vectors in V^*, which is the dual in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$. It is easy to see that $V^\vee$ is irreducible by the K-admissibility. We shall show that $V \boxtimes V^\vee$ is a subquotient of W.

Fix an irreducible K-submodule F of V. Then we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{K\times K}(F \boxtimes F^*, W) &\simeq F^* \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi \otimes_{\mathcal{U}(\mathfrak{k})} F \\
&\simeq (\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi \otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{End}_{\mathbb{C}}(F))^K \\
&\simeq (\mathcal{U}(\mathfrak{g})/(\mathcal{I}_\chi + \mathcal{U}(\mathfrak{g})\mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(F)))^K
\end{aligned}$$

as $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})^{K\times K}$-modules. Here $\mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(F)$ denotes the annihilator of F in $\mathcal{U}(\mathfrak{k})$. Since $\mathrm{Hom}_K(F,V)$ is a finite-dimensional irreducible $\mathcal{U}(\mathfrak{g})^K$-module (see Fact 7.38), we have a surjection

$$(\mathcal{U}(\mathfrak{g})/(\mathcal{I}_\chi+\mathcal{U}(\mathfrak{g})\mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(F)))^K\to\mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_K(F,V))$$

by the Jacobson density theorem. This implies that there is a surjection

$$\begin{aligned}\mathrm{Hom}_{K\times K}(F\boxtimes F^*,W)&\to\mathrm{Hom}_{K\times K}(F\boxtimes F^*,V\boxtimes V^\vee)\\&\simeq\mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_K(F,V))\end{aligned}$$

of $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})^{K\times K}$-modules. By [48, Proposition 3.5.4], $V\boxtimes V^\vee$ is isomorphic to a subquotient of W.

By Propositions 7.11 and 7.7(i), $(\mathbb{D}^n\Gamma^K_{\{e\}}(\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi))_\chi$ is a uniformly bounded family of $(\mathfrak{g}\oplus\mathfrak{g},K\times K)$-modules. By Proposition 7.6(i) the family $(V\boxtimes V^\vee)_{V\in\mathcal{S}}$ is uniformly bounded, where $\mathcal{S}$ is the set of all equivalence classes of irreducible K-admissible $(\mathfrak{g},K)$-modules. From this and Proposition 7.6(iv), the family $(V)_{V\in\mathcal{S}}$ is uniformly bounded. This shows the theorem. □

In the proof, we have proved Corollary 7.36. For a character μ of $\mathfrak{k}$, we denote by $\mathcal{C}_{\chi,\mu}(\mathfrak{g},\mathfrak{k})$ the full subcategory of $\mathcal{C}_\chi(\mathfrak{g},\mathfrak{k})$ whose object V satisfies that $V\otimes\mathbb{C}_\mu$ lifts to a K-module.

Corollary 7.36. *Let μ be a character of $\mathfrak{k}$ and χ an infinitesimal character of $\mathfrak{g}$. The number of equivalence classes of irreducible objects in $\mathcal{C}_{\chi,\mu}(\mathfrak{g},\mathfrak{k})$ is bounded by some constant independent of μ and χ.*

Remark 7.37. The number of equivalence classes of irreducible objects in $\mathcal{C}_\chi(\mathfrak{g},\mathfrak{k})$ may be infinite. $(\mathfrak{g},\mathfrak{k})=(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{so}(2,\mathbb{C}))$ gives an example. See [24, Theorem 1.3.1].

For a $(\mathfrak{g},\mathfrak{k})$-module V and an irreducible finite-dimensional $\mathfrak{k}$-module F, we denote by $V(F)$ the isotypic component with respect to F. Then $V(F)$ is a $\mathcal{U}(\mathfrak{k})\otimes\mathcal{U}(\mathfrak{g})^K$-module. It is well-known that the $\mathfrak{g}$-module structure on V is related to the $\mathcal{U}(\mathfrak{k})\otimes\mathcal{U}(\mathfrak{g})^K$-module structure on $V(F)$. See [48, Lemma 3.5.3] and [37]. Recall that $\mathcal{U}(\mathfrak{k})$ and $\mathcal{U}(\mathfrak{g})^K$ are noetherian.

Fact 7.38. Let V be a $(\mathfrak{g},\mathfrak{k})$-module and F an irreducible finite-dimensional $\mathfrak{k}$-module. For any submodule W of $V(F)$, we have $(\mathcal{U}(\mathfrak{g})W)(F)=W$. In particular, the length of $V(F)$ is less than or equal to that of V, and $V(F)$ is finitely generated if V is finitely generated.

Lemma 7.39. *Let F be an irreducible K-module. Then $\mathcal{U}(\mathfrak{g})(F)$ with respect to the adjoint action is finitely generated as a left/right $\mathcal{U}(\mathfrak{g})^K$-module. In particular, any finitely generated submodule of the $(\mathcal{U}(\mathfrak{g})^K,\mathcal{U}(\mathfrak{g})^K)$-bimodule $\mathcal{U}(\mathfrak{g})$ is finitely generated as a left/right $\mathcal{U}(\mathfrak{g})^K$-module.*

Proof. $F^*\otimes\mathcal{U}(\mathfrak{g})$ is finitely generated left $\mathcal{U}(\mathfrak{g})$-module. By [30, Lemma 2.2], $(F^*\otimes\mathcal{U}(\mathfrak{g}))^K$ is finitely generated left $\mathcal{U}(\mathfrak{g})^K$-module. Since $\mathcal{U}(\mathfrak{g})(F)$ is canonically isomorphic to $F\otimes\mathrm{Hom}_K(F,\mathcal{U}(\mathfrak{g}))$ as a $\mathcal{U}(\mathfrak{g})^K$-module, this shows the first assertion.

Any finitely generated submodule of the $(\mathcal{U}(\mathfrak{g})^K,\mathcal{U}(\mathfrak{g})^K)$-bimodule $\mathcal{U}(\mathfrak{g})$ is contained in a finite sum of some K-isotypic components. Since $\mathcal{U}(\mathfrak{g})^K$ is noetherian, the second assertion follows from the first one. □

Lemma 7.40. *Let V be a $(\mathfrak{g},\mathfrak{k})$-module and F an irreducible finite-dimensional $\mathfrak{k}$-module. If $V(F)$ is finite dimensional and generates V, then V is in $\mathcal{C}(\mathfrak{g},\mathfrak{k})$.*

Proof. Since the multiplication map $\mathcal{U}(\mathfrak{g})V(F) \to V$ is surjective, V is completely reducible as a $\mathfrak{k}$-module. We shall show that V is $\mathfrak{k}$-admissible and of finite length.

Let F' be an irreducible finite-dimensional $\mathfrak{k}$-module. We shall show that $V(F')$ is finite dimensional. Since V is finitely generated, $V(F')$ is finitely generated as a $\mathcal{U}(\mathfrak{g})^K$-module by Fact 7.38. Since V is generated by $V(F)$, we can take a finite-dimensional subspace $X \subset \mathcal{U}(\mathfrak{g})$ such that $V(F') \subset \mathcal{U}(\mathfrak{g})^K XV(F)$. By Lemma 7.39, there exists a finite-dimensional subspace $X' \subset \mathcal{U}(\mathfrak{g})$ such that $\mathcal{U}(\mathfrak{g})^K X\mathcal{U}(\mathfrak{g})^K = X'\mathcal{U}(\mathfrak{g})^K$. Then we have

$$V(F') \subset \mathcal{U}(\mathfrak{g})^K XV(F) = X'\mathcal{U}(\mathfrak{g})^K V(F) = X'V(F),$$

and hence $V(F')$ is finite dimensional. Therefore V is $\mathfrak{k}$-admissible.

We shall show that V is of finite length. Since V is generated by $V(F)$, $\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}(V)$ is of finite codimension in $\mathcal{Z}(\mathfrak{g})$. Hence V is a finite direct sum of $\mathfrak{g}$-submodules with generalized infinitesimal characters. We can assume that V has a generalized infinitesimal character χ.

Since V is generated by $V(F)$, there is a character μ of $\mathfrak{k}$ such that $V \otimes \mathbb{C}_\mu$ lifts to a K-module. Then any irreducible subquotient of V is in $\mathcal{C}_{\chi,\mu}(\mathfrak{g},\mathfrak{k})$. By Corollary 7.36, the number of equivalence classes of irreducible objects in $\mathcal{C}_{\chi,\mu}(\mathfrak{g},\mathfrak{k})$ is finite. Since V is $\mathfrak{k}$-admissible and noetherian, this shows that V is of finite length. □

Suppose that $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = V$ is the socle filtration of $V \in \mathcal{C}(\mathfrak{g},\mathfrak{k})$, that is, each V_i/V_{i-1} is the sum of all irreducible submodules in V/V_{i-1}. The length r is called the Loewy length of V.

Theorem 7.41. *The Loewy length of any object in $\mathcal{C}_\chi(\mathfrak{g},\mathfrak{k})$ is bounded by some constant independent of the object and the infinitesimal character χ.*

Proof. We construct projective objects in $\mathcal{C}_\chi(\mathfrak{g},\mathfrak{k})$ using $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} F$, which is a projective object in the category of all $(\mathfrak{g},\mathfrak{k})$-modules whose $\mathfrak{k}$-actions are completely reducible.

Let F be an irreducible finite-dimensional $\mathfrak{k}$-module and χ an infinitesimal character of $\mathfrak{g}$. Put

$$\widetilde{P}_{F,\chi} := \mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi \otimes_{\mathcal{U}(\mathfrak{k})} F.$$

Then there are canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi}) &\simeq (\mathcal{U}(\mathfrak{g})/(\mathcal{I}_\chi + \mathcal{U}(\mathfrak{g})\mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(F)))^K \\ &\simeq \mathcal{U}(\mathfrak{g})^K/(\mathcal{I}_\chi + \mathcal{U}(\mathfrak{g})\mathrm{Ann}_{\mathcal{U}(\mathfrak{k})}(F))^K\end{aligned}$$

of $\mathcal{U}(\mathfrak{g})^K$-modules. Let $\mathcal{J}$ be the intersection of all $\mathcal{U}(\mathfrak{g})^K$-submodules of finite codimension in $\mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi})$. By Theorem 7.32, $\mathcal{J}$ has finite codimension in $\mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi})$.

There is a canonical isomorphism

$$F \otimes \mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi}) \xrightarrow{\simeq} \widetilde{P}_{F,\chi}(F)$$

of $\mathcal{U}(\mathfrak{k}) \otimes \mathcal{U}(\mathfrak{g})^K$-modules. We consider $F \otimes \mathcal{J}$ as a subspace of $\widetilde{P}_{F,\chi}$ by the isomorphism. Put $\widetilde{J} := \mathcal{U}(\mathfrak{g}) \cdot (F \otimes \mathcal{J})$ and

$$P_{F,\chi} := \widetilde{P}_{F,\chi}/\widetilde{J}.$$

Since $\widetilde{J}(F) = F \otimes J$ by Fact 7.38, we have

$$P_{F,\chi}(F) \simeq F \otimes (\mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi})/\mathcal{J}).$$

Hence $P_{F,\chi}$ is generated by the finite-dimensional subspace $P_{F,\chi}(F)$. By Lemma 7.40, $P_{F,\chi}$ is an object in $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$.

We shall show that $P_{F,\chi}$ is projective in $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$. Let $V \in \mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$ and $f\colon \widetilde{P}_{F,\chi} \to V$ be a $\mathfrak{g}$-homomorphism. Then we have the $\mathcal{U}(\mathfrak{g})^K$-homomorphism

$$\mathrm{Hom}_{\mathfrak{k}}(F, \widetilde{P}_{F,\chi}) \xrightarrow{f^*} \mathrm{Hom}_{\mathfrak{k}}(F, V).$$

The $\mathcal{U}(\mathfrak{g})^K$-module $\mathrm{Hom}_{\mathfrak{k}}(F, V)$ is finite-dimensional. By the definition of $\mathcal{J}$, we have $f^*(\mathcal{J}) = 0$ and hence $f(\widetilde{J}) = 0$. This implies

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}(P_{F,\chi}, V) &\simeq \mathrm{Hom}_{\mathfrak{g}}(\widetilde{P}_{F,\chi}, V)\\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} F, V)\\ &\simeq \mathrm{Hom}_{\mathfrak{k}}(F, V).\end{aligned}$$

Since all objects in $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$ are completely reducible as $\mathfrak{k}$-modules, the functor $\mathrm{Hom}_{\mathfrak{k}}(F, \cdot)$ is exact on $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$. Therefore $P_{F,\chi}$ is projective in $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$.

Any object in $\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$ is isomorphic to a quotient of a finite direct sum of projective objects of the form $P_{F,\chi}$. It is enough to bound the Loewy length of $P_{F,\chi}$. By Proposition 7.11 and Theorem 7.18, there is a constant C independent of χ and F such that

$$\mathrm{Len}_{\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})^K}(P_{F,\chi}) \le \mathrm{Len}_{\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})^K}(\mathcal{U}(\mathfrak{g})/\mathcal{I}_\chi \otimes_{\mathcal{U}(\mathfrak{k})} F) \le C.$$

By Lemma 7.42, the Loewy length of $P_{F,\chi}$ as a $\mathfrak{g}$-module is bounded by C. □

Lemma 7.42. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra and V an irreducible $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{A}$-module. If V is in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ as a $\mathfrak{g}$-module, then V is completely reducible as a $\mathfrak{g}$-module.*

Proof. Since V has finite length as a $\mathfrak{g}$-module, V has an irreducible $\mathfrak{g}$-submodule W. Since V is irreducible as a $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{A}$-module, we have $V = \mathcal{A} \cdot W$. This implies that V is a sum of some copies of W, and hence V is completely reducible as a $\mathfrak{g}$-module. □

In the proof of Theorem 7.41, we have proved Proposition 7.43.

Proposition 7.43. *$\mathcal{C}_\chi(\mathfrak{g}, \mathfrak{k})$ has enough projectives.*

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