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UNITARY REPRESENTATIONS OF TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS SATISFYING OLʹŠANSKIĬ'S FACTORISATION

LANCELOT SEMAL

Abstract. Inspired by Olʹšanskiĭ's work, we provide an axiomatic framework to describe certain irreducible unitary representations of non-discrete unimodular totally disconnected locally compact groups. We look at the applications to certain groups of automorphisms of locally finite trees and semi-regular right-angled buildings.

Contents

1. Introduction

In this document topological groups are second-countable, locally compact groups are Hausdorff and the word "representation" stands for strongly continuous unitary representation on a separable complex Hilbert space.

1.1. **Motivation.** Despite the fact that the theory of representations of locally compact groups has been an active domain of research since the 50's, this field of mathematics is still mostly composed by vast uncharted territories. Just as for finite groups the notion of irreducible representations is of central importance. However, for general locally compact groups, the theory is well behaved with respect to irreducible representations only when the group is of type I (Bernstein and Kirillov have used the term "tame" to qualify type I groups, and "wild" for those

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which are not type I). Loosely speaking, type I groups are characterised among locally compact groups by the property that the direct integral decompositions of their representations into irreducible representations are all essentially unique. Moreover, the determination of the equivalence classes of irreducible representations of a locally compact group is known to be an intractable problem unless the group is of type I. Discrete groups are known to be of type I precisely when they are virtually abelian [Tho68]. For non-discrete groups, prominent examples of type I groups are given by compact groups, abelian groups, nilpotent connected Lie groups [Dix59] and semi-simple connected Lie groups [Kna86]. Concerning groups of automorphisms of trees, recent achievements have highlighted that the closed non-amenable subgroups $G \leq \mathrm{Aut}(T)$ acting minimally on the tree T which are not 2-transitive on the boundary ∂T are never of type I see [HR19] and [CKM22]. By contrast, rank one semi-simple algebraic groups over local-fields [Ber74] and groups satisfying the Tits independence property and acting transitively on the boundary ∂T such as the full group $\mathrm{Aut}(T)$ are known to be of type I [Ama03]. This last result was originally informed by G. I. Ol′šanskiĭ who provided a classification of their irreducible representations [Ol′77] and deduced that these groups are of type I in [Ol'80]. This classification is the starting point of our papers.

Let T be a regular tree all of whose vertices have degree at least 3, $V(T)$ be the set of vertices of T, $E(T)$ be its set of edges and $G \leq \mathrm{Aut}(T)$ be a closed subgroup of $\mathrm{Aut}(T)$. An irreducible representation π of G is:

- **spherical** if there exists a vertex $v \in V(T)$ such that π admits a non-zero $\mathrm{Fix}_G(v)$-invariant vector where $\mathrm{Fix}_G(v) = \{g \in G | gv = v\}$.
- **special** if it is not spherical and there exists an edge $e \in E(T)$ such that π admits a non-zero $\mathrm{Fix}_G(e)$-invariant vector where $\mathrm{Fix}_G(e) = \{g \in G | gv = v \ \forall v \in e\}$ is the pointwise stabiliser of the edge e.
- **cuspidal** if it is neither spherical nor special.

It is clear from the definition that each irreducible representation of G is of exactly one of the above three types. The classification of spherical and special representations of any closed subgroup $G \leq \mathrm{Aut}(T)$ acting 2-transitively on the boundary ∂T is a classical result that was achieved in the 70's due to R. Godement, P. Cartier and H. Matsumoto, to cite just a few see [FTN91] and [Mat77]. A few years later, G. I. Ol′šanskiĭ provided a classification of the cuspidal representations of the full group of automorphisms of any regular tree by exploiting the fact that these groups satisfy the Tits independence property. Among other things, the resulting description ensures that the cuspidal representations of those groups are induced from compact open subgroups. In particular, these are associated to compactly supported functions of positive type, they belong to the discrete series of G and their equivalence classes are isolated in the unitary dual for the Fell topology. The key step behind this result is that the Tits independence property can be translated as a factorisation property on a basis of neighbourhoods of the identity consisting of compact open subgroup, see Lemma 3.7. In the first part of this paper, consisting of Sections 1 and 2, we develop an abstraction of Ol′šanskiĭ's framework allowing one to describe the irreducible representations of totally disconnected locally compact groups all of whose isotropy groups are "small" provided the existence of a basis of neighbourhoods of the identity consisting of compact open subgroups and satisfying the same kind of factorisation. In Sections 3, 4, 5 and 6 we provide applications of this machinery to groups of tree automorphisms and to groups of automorphisms of right-angled building. These include applications to groups whose representation theory is new to the literature such as groups of automorphisms of trees satisfying the property IP_k [BEW15] and universal groups of certain semi-regular right-angled buildings [DMSS18]. In addition, this machinery is used in [Sem22] to obtain a description of the irreducible representations of the groups of automorphisms of tree classified by Radu in [Rad17] in order to contribute to Nebbia's conjecture on trees [Neb99].

1.2. **The axiomatic framework.** The purpose of this section is to establish the formalism of our axiomatic framework. Let G be a locally compact group, let π be a representation of G and let H be a closed subgroup of G. We denote by $\mathcal{H}_\pi$ the representation space of π and by $\mathcal{H}_\pi^H$ the subspace of H-invariant vectors of $\mathcal{H}_\pi$. We recall that π is **H-spherical** if the space $\mathcal{H}_\pi^H$ is not reduced to $\{0\}$. Lemma 1.1 shows that this notion carries a particular flavour among totally disconnected locally compact groups.

Lemma 1.1. *Let π be a representation of G and let $\mathcal{S}$ be a basis of neighbourhoods of the identity consisting of compact open subgroups. Then,*

$$\bigcup_{U\in\mathcal{S}} \mathcal{H}_\pi^U = \{\xi \in \mathcal{H}_\pi \mid \exists U \in \mathcal{S} \text{ such that } \pi(h)\xi = \xi \ \forall h \in U\}$$

is dense in $\mathcal{H}_\pi$.

A proof of this result can be found in [FTN91, pg.85] for the full group of automorphisms of a thick regular tree and the basis of neighbourhoods consisting of pointwise stabilisers of complete finite subtrees. Since their reasoning applies to the above setup and is probably well known by the experts, we do not give a proof in these notes.

We recall that totally disconnected locally compact groups are characterised among locally compact groups by **Van Dantzig**'s Theorem which ensures that a locally compact group is totally disconnected if and only if it admits a basis of neighbourhoods of the identity consisting of compact open subgroups. In particular, Lemma 1.1 leads to the observation that every representation π of a totally disconnected locally compact group G admits non-zero invariant vectors for some compact open subgroup of G. Notice, however, that the statement loses its relevance for discrete groups. It is natural to ask whether further information can be obtained if the basis of neighbourhoods of the identity has more properties. The purpose of this paper is to provide a positive answer to that question. We now make a few minor observations and establish our formalism.

Let G be a totally disconnected locally compact group, $U \leq G$ be a closed subgroup and π be a representation of G admitting a non-zero U-invariant vector $\xi \in \mathcal{H}_\pi$. Notice that for each $g \in G$, that the vector $\pi(g)\xi$ defines a non-zero gUg^{-1}-invariant vector of $\mathcal{H}_\pi$. In particular, the existence of a non-zero U-invariant vector is an invariant property of the conjugacy class of U. In light of this observation, we let $\mathcal{B}$ be the set of compact open subgroups of G, $P(\mathcal{B})$ be the power set of $\mathcal{B}$ and

$$\mathcal{C} : \mathcal{B} \to P(\mathcal{B})$$

be the map sending a compact open subgroup to its conjugacy class in G. Let $\mathcal{S}$ be a basis of neighbourhoods of the identity consisting of compact open subgroups of G and let $\mathcal{F}_\mathcal{S} = \{\mathcal{C}(U) | U \in \mathcal{S}\}$. In order to properly state our factorisation property, we require a notion of relative size for the elements of $\mathcal{S}$ that is well behaved with respect to these conjugacy classes. To this end, we equip $\mathcal{F}_{\mathcal{S}}$ with the partial order given by the reverse inclusion of representatives. In other words, we say that $\mathcal{C}(U) \leq \mathcal{C}(V)$ if there exists $\tilde{U} \in \mathcal{C}(U)$ and $\tilde{V} \in \mathcal{C}(V)$ such that $\tilde{V} \subseteq \tilde{U}$. For a poset $(P, \leq)$ and an element $x \in P$, we let L_x be the maximal length of a strictly increasing chain in $P_{\leq x} = \{y \in P | y \leq x\}$. If L_x is finite, we say that x has **height** $L_x - 1$ in $(P, \leq)$. Otherwise, we say that x has infinite height in $(P, \leq)$.

Definition 1.2. A basis of neighbourhoods of the identity $\mathcal{S}$ consisting of compact open subgroups of G is called a **generic filtration** of G if the height of every element in $\mathcal{F}_{\mathcal{S}}$ is finite.

Lemma 1.3 ensures that such a generic filtration exists for unimodular groups.

Lemma 1.3. *Let G be a unimodular totally disconnected locally compact group, μ be a bi-invariant Haar measure on G and $\mathcal{S}$ be a basis of neighbourhoods of the identity consisting of compact open subgroups such that $\mu(U) \leq 1$ for all $U \in \mathcal{S}$. Then, $\mathcal{S}$ is a generic filtration of G.*

Proof. Let $\mathcal{C}(U) \in \mathcal{F}_{\mathcal{S}}$. Since G is unimodular, the measure $\mu(U)$ does not depend on the choice of representative $U \in \mathcal{C}(U)$. Now, let $\mathcal{C}(U_0) \leq \mathcal{C}(U_1) \leq \cdots \leq \mathcal{C}(U_{n-1}) \leq \mathcal{C}(U)$ be a strictly increasing chain of elements of $\mathcal{F}_{\mathcal{S}}$. Changing representatives if needed, we can suppose that $U \subseteq U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0$. In particular, notice that

$$[U_0 : U] = [U_0 : U_1] \cdots [U_{n-1} : U] \geq 2^n.$$

On the other hand, since U and U_0 are both compact open subgroups of G observe that

$$[U_0 : U] = \frac{\mu(U_0)}{\mu(U)} \leq \frac{1}{\mu(U)}.$$

This proves that $n \leq -\log_2(\mu(U))$ and therefore that the height of $\mathcal{C}(U)$ in $\mathcal{F}_{\mathcal{S}}$ is finite. □

Every generic filtration $\mathcal{S}$ of G splits as a disjoint union $\mathcal{S} = \bigsqcup_{l \in \mathbb{N}} \mathcal{S}[l]$ where $\mathcal{S}[l]$ denotes the set of elements $U \in \mathcal{S}$ such that $\mathcal{C}(U)$ has height l in $\mathcal{F}_{\mathcal{S}}$. The elements of $\mathcal{S}[l]$ are called the elements at **depth** l. For every representation π of G there exists a smallest non-negative integer $l_\pi \in \mathbb{N}$ such that π admits non-zero U-invariant vectors for some $U \in \mathcal{S}[l_\pi]$. This l_π is called the **depth** of π with respect to $\mathcal{S}$ in analogy with the similar notion of depth for representations of reductive groups over non-Archimedean fields introduced by Moy-Prasad in [MP96].

Example 1.4. Let T be a thick regular tree and consider the full group of automorphisms $\mathrm{Aut}(T)$ of T. This group is a non-discrete unimodular totally disconnected locally compact group for the permutation topology. We recall that a subtree $\mathcal{T} \subseteq T$ is **complete** if each of its vertices $v \in V(\mathcal{T})$ has degree $0, 1$ or has the same degree in $\mathcal{T}$ as in T. Let $\mathfrak{T}$ be the set of all complete finite subtrees of T and let $\mathcal{S} = \{\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}\}$ be the basis of neighbourhoods of the identity consisting of the groups

$$\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}) = \{g \in \mathrm{Aut}(T) |\ gv = v\ \forall v \in V(\mathcal{T})\},$$

where $V(\mathcal{T})$ denotes the set of vertices of $\mathcal{T}$. In Section 3.2 we show that $\mathcal{S}$ is a generic filtration. Furthermore, according to the terminology introduced on page 357 and as a consequence of Lemma 3.4, the depth l_π with respect to $\mathcal{S}$ of a representation π of $\mathrm{Aut}(T)$ can be interpreted as follows:

(1) π is a spherical representation of $\mathrm{Aut}(T)$ if and only if $l_\pi = 0$.
(2) π is a special representation of $\mathrm{Aut}(T)$ if and only if $l_\pi = 1$.
(3) π is a cuspidal representation of $\mathrm{Aut}(T)$ if and only if $l_\pi \geq 2$. Furthermore, l_π is the smallest positive integer l for which there exists a complete finite subtree $\mathcal{T}$ of T with $l-1$ of interior vertices such that π has a non-zero $\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T})$-invariant vector.

We now introduce the notion of **Ol′šhanskiĭ's factorisation**.

Definition 1.5. Let G be a non-discrete unimodular totally disconnected locally compact group, $\mathcal{S}$ be a generic filtration of G and l be a strictly positive integer. We say that $\mathcal{S}$ **factorises at depth** l if for all $U \in \mathcal{S}[l]$ the following conditions are satisfied:

(1) For every V in the conjugacy class of an element of $\mathcal{S}$ with $V \not\subseteq U$, there exists a subgroup W in the conjugacy class of an element of $\mathcal{S}[l-1]$ satisfying that
$$U \subseteq W \subseteq VU = \{vu | v \in V, u \in U\}.$$
(2) For every V in the conjugacy class of an element of $\mathcal{S}$, the set
$$N_G(U,V) = \{g \in G | g^{-1}Vg \subseteq U\}$$
is compact.

Furthermore, the generic filtration $\mathcal{S}$ of G is said to **factorise$^+$ at depth** l if in addition for all $U \in \mathcal{S}[l]$ and every W in the conjugacy class of an element of $\mathcal{S}[l-1]$ such that $U \subseteq W$ we have
$$W \subseteq N_G(U,U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

Remark 1.6. The factorisation at depth l defined here depends on the entire generic filtration $\mathcal{S}$ and not only on the elements of $\mathcal{S}[l]$ and $\mathcal{S}[l-1]$.

Remark 1.7. Since G is unimodular, notice that the set $N_G(U,U)$ coincides with the normaliser $N_G(U)$ of U in G. Furthermore, notice that the conditions (1) and (2) are satisfied for some $U \in \mathcal{S}[l]$ if and only if they are satisfied for each of its conjugates. In particular, a generic filtration $\mathcal{S}$ factorises at depth l if and only if the conditions (1) and (2) are satisfied for each subgroup U that is conjugate to an element of $\mathcal{S}[l]$. The same remark holds for the notion of factorisation$^+$.

1.3. **Main results and structure of the paper.** This paper is divided in two parts. The purpose of the first part of the paper (Sections 1 and 2) is to establish our axiomatic framework and to prove Theorem A.

Theorem A. *Let G be a non-discrete unimodular totally disconnected locally compact group and $\mathcal{S}$ be a generic filtration of G factorising at depth l. Then, every irreducible representation π of G at depth l satisfies the following:*

(1) *There exists a unique $C_\pi \in \mathcal{F}_{\mathcal{S}} = \{\mathcal{C}(U) | U \in \mathcal{S}\}$ with height l such that for all $U \in C_\pi$, π admits a non-zero U-invariant vector.*
(2) *For every $U \in C_\pi$, π admits a non-zero diagonal matrix coefficient supported in the compact open subgroup $N_G(U)$ of G. In particular, π is induced from an irreducible representation of $N_G(U)$, belongs to the discrete series of G and its equivalence class is isolated in the unitary dual $\widehat{G}$ for the Fell topology.*

Furthermore, if $\mathcal{S}$ factorises$^+$ at depth l, there is a bijective correspondence between the equivalence classes of irreducible representations π of G at depth l with $C_\pi = \mathcal{C}(U)$ and the equivalence classes of $\mathcal{S}$-standard representations of the finite group $N_G(U)/U$. This bijective correspondence is explicitly given by Theorem 2.22.

We refer to Section 2.4 for a proper definition of $\mathcal{S}$-standard representations and for the details of the bijective correspondence given by Theorem 2.22.

Remark 1.8. Theorem A does not ensure the existence of irreducible representations at depth l.

Remark 1.9. Different generic filtrations might factorise simultaneously and lead to a different description of the same representations. A concrete example of this phenomenon is given on the full group of automorphisms of a (d_0, d_1)-regular tree in Sections 3, 4 and 5 if $d_0 \neq d_1$. Furthermore, different generic filtrations might also describe different sets of irreducible representations. A concrete example of this phenomenon occurs for instance if we replace a generic filtration $\mathcal{S}$ of G factorising at all positive depths with the generic filtration $\mathcal{S}' = \mathcal{S} - \mathcal{S}[0]$. In that case, $\mathcal{S}'[l] = \mathcal{S}[l+1]$ $\forall l \in \mathbb{N}$. In particular, when applied to $\mathcal{S}'$, Theorem A does not describe the irreducible representations admitting a non-zero U-invariant vector if $U \in \mathcal{S}[1]$ while it does when applied to $\mathcal{S}$.

In the second part of the paper, we look at various applications of the axiomatic framework developed in the first part of these notes. In Section 3 we recover the classification of the cuspidal representations of the full group of automorphisms of a semi-regular tree made by Ol′šanskiĭ [Ol′77] and later by Amann [Ama03]. In particular, the content of this section is redundant from the point of view of new results. It serves instead as a section allowing the reader to understand and interpret the abstract framework developed in the first part of these notes in a concrete and well-understood case.

The purpose of Section 4 is to apply our axiomatic framework to groups of automorphisms of a thick semi-regular tree and satisfying the property IP_k (Definition 4.1) as defined in [BEW15]. Loosely speaking, this property ensures that the pointwise stabiliser of any ball of radius $k-1$ around an edge decomposes as a direct product of the subgroup fixing all the vertices on one side of the edge and the subgroup fixing all the vertices on the other side. The main contributions of this section are Theorems B and C which provide two ways to build generic filtrations factorising$^+$. To be more precise, let T be a thick semi-regular tree with set of vertices $V(T)$. For every finite subtree $\mathcal{T}$ of T and each integer $r \geq 0$, we denote by $\mathcal{T}^{(r)}$ the ball of radius r around $\mathcal{T}$ for the natural metric d_T on $V(T)$ that is

$$\mathcal{T}^{(r)} = \{v \in V(T) | \exists w \in V(\mathcal{T}) \text{ s.t. } d_T(v,w) \leq r\}.$$

Theorem B. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$, $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the property IP_k for some $k \geq 1$, $\mathcal{P}$ be a complete finite subtree of T containing an interior vertex, $\Sigma_{\mathcal{P}}$ be the set of maximal complete proper subtrees of $\mathcal{P}$ and*

$$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{R} \in \Sigma_{\mathcal{P}} | \mathrm{Fix}_G((\mathcal{R}')^{(k-1)}) \not\subseteq \mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \ \forall \mathcal{R}' \in \Sigma_{\mathcal{P}} - \{\mathcal{R}\}\}.$$

Suppose in addition that:

(1) *$\forall \mathcal{R}, \mathcal{R}' \in \mathfrak{T}_{\mathcal{P}}, \forall g \in G$, we do not have* $\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subsetneq \mathrm{Fix}_G(g(\mathcal{R}')^{(k-1)})$.

(2) *For all* $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$, $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$. *Furthermore, if* $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \subsetneq \mathrm{Fix}_G(g\mathcal{R}^{(k-1)})$ *we have* $\mathcal{P} \subseteq g\mathcal{R}^{(k-1)}$.
(3) $\forall n \in \mathbb{N}, \forall v \in V(T)$, $\mathrm{Fix}_G(v^{(n)}) \subseteq \mathrm{Fix}_G(\mathcal{P}^{(k-1)})$ *implies* $\mathcal{P}^{(k-1)} \subseteq v^{(n)}$.
(4) *For every* $g \in G$ *such that* $g\mathcal{P} \neq \mathcal{P}$, $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mathrm{Fix}_G(g\mathcal{P}^{(k-1)})$.

Then, there exists a generic filtration $\mathcal{S}_{\mathcal{P}}$ *of* G *factorising*$^+$ *at depth* 1 *with*

$$\mathcal{S}_{\mathcal{P}}[0] = \{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\},$$
$$\mathcal{S}_{\mathcal{P}}[1] = \{\mathrm{Fix}_G(\mathcal{P}^{(k-1)})\}.$$

The author would like to underline how realistic the assumptions of Theorem B are. Indeed, any closed non-discrete unimodular subgroup $G \leq \mathrm{Aut}(T)$ satisfying the property IP_k and such that $\mathrm{Fix}_G(\mathcal{T})$ does not admit any fixed point other than the vertices of $\mathcal{T}$ for every complete finite subtree $\mathcal{T}$ of T satisfies the hypotheses of Theorem B. In light of Theorem A this allows one to describe the irreducible representations of G which admit a non-zero $\mathrm{Fix}_G(\mathcal{P}^{(k-1)})$ invariant vector but do not admit a non-zero $\mathrm{Fix}_G(\mathcal{R}^{(k-1)})$-invariant vector for any $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$. Notice furthermore that Theorem B can be applied inductively with different $\mathcal{P}$. This is done for instance in Example 4.14.

Under a stronger hypothesis on G (the hypothesis H_q (Definition 4.2)), we are even able to explicit a generic filtration factorising$^+$ at all sufficiently large depths. To be more precise, we have the following result.

Theorem C. *Let* T *be a* (d_0, d_1)*-semi-regular tree with* $d_0, d_1 \geq 3$ *and* $G \leq \mathrm{Aut}(T)$ *be a closed non-discrete unimodular subgroup satisfying the hypothesis* H_q *and the property* IP_k *for some integers* $q \geq 0$ *and* $k \geq 1$. *Then, there exists a generic filtration* $\mathcal{S}_q$ *of* G *factorising*$^+$ *at all depths* $l \geq L_{q,k}$ *where*

$$L_{q,k} = \begin{cases} \max\{1, 2k - q - 1\} & \text{if } q \text{ is even.} \\ \max\{1, 2k - q\} & \text{if } q \text{ is odd.} \end{cases}$$

The generic filtration $\mathcal{S}_q$ is explicitly described on page 384. For concrete applications of Theorem C, we refer to Examples 4.12, 4.13 and 4.14. Furthermore, we treat some existence criteria for the irreducible representations at depth l with respect to $\mathcal{S}_q$ in Section 4.3. Among the new applications we have the following (Corollary 4.3) where $\mathcal{S}_0$ is $\mathcal{S}_q$ with $q = 0$.

Corollary. *Let* T *be a* (d_0, d_1)*-semi-regular tree with* $d_0, d_1 \geq 6$ *and* $G \leq \mathrm{Aut}(T)$ *be a closed subgroup acting* 2*-transitively on the boundary* ∂T *and whose local action at each vertex contains the alternating group of corresponding degree. Then, there exists a generic filtration* $\mathcal{S}_0$ *of* G *and a constant* $k \in \mathbb{N}$ *such that the* $\mathcal{S}_0$ *factorises*$^+$ *at all depths* $l \geq 2k - 1$.

The groups appearing in the above corollary were extensively studied by Radu in [Rad17] and we refer to them as **Radu groups**. Among other things, Radu completely classified them and proved that each Radu group G is k-closed for some constant $k \in \mathbb{N}$ depending on G. Since Radu groups also satisfy the hypothesis H_0 and since k-closed groups satisfy the property IP_k, the corollary follows. Since this family of groups plays a central role in Nebbia's CCR conjecture on trees [Neb99] it is natural to ask whether more can be said. The following Theorem provides a positive answer without relying on the property IP_k.

Theorem ([Sem22]). *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$ and $G \leq \mathrm{Aut}(T)$ be a simple Radu group. Then, the generic filtration $\mathcal{S}_0$ factorises$^+$ at all positive depths.*

In particular, this factorisation leads to a description of the cuspidal representations of any simple Radu group. However, the proof of this theorem is quite technical, relies heavily on Radu's classification and is unrelated to the property IP_k. Furthermore, since various important consequences such as the description of the cuspidal representations of non-simple Radu groups and the progress related to Nebbia's CCR conjecture on trees need to be tackled in light of the result, the author decided to present a proof in another paper (see [Sem22]).

The purpose of Section 5 is to apply our axiomatic framework to groups of type-preserving automorphisms of a locally finite tree and satisfying the property IP_{V_1}. This serves as a preamble for Section 6 where we show that the universal groups of certain semi-regular right-angled buildings can be realised as such groups. The main result of Section 5 is the following.

Theorem D. *Let T be a locally finite tree and G be a closed non-discrete unimodular group of type-preserving automorphisms of T satisfying the property* IP_{V_1} *(Definition* 5.1*) and the hypothesis H_{V_1} (Definition* 5.3*). Then, there exists a generic filtration $\mathcal{S}_{V_1}$ of G factorising$^+$ at all depths $l \geq 1$.*

The generic filtration $\mathcal{S}_{V_1}$ is defined on page 396 and an existence criterion for the irreducibles at depth l with respect to $\mathcal{S}_{V_1}$ is given in Section 5.3.

The purpose of Section 6 is to prove that the universal groups of certain semi-regular right-angled buildings as introduced in [DMSS18] can be realised as groups of type-preserving automorphisms of a locally finite tree T in such a way that they satisfy the hypotheses of Theorem D. We refer to Section 6.1 for a proper reminder on the notion of universal groups of semi-regular right-angled buildings and state the main result of this section assuming that the reader is familiar with this notion. We let (W, I) be a finitely generated right-angled Coxeter system and suppose that I can be partitioned as $I = \bigsqcup_{k=1}^r I_k$ in such a way that $I_k = \{i\} \cup \{i\}^\perp$ for every $i \in I_k$ and for all $k = 1, \ldots, r$ where $\{i\}^\perp = \{j \in I | ij = ji\}$. In particular, W is virtually free and isomorphic to a free product $W_1 * W_2 * \cdots * W_r$ where each of the W_k is a direct product of finitely many copies of the group of order 2. Let $(q_i)_{i\in I}$ be a set of integers greater than 2 and Δ be a semi-regular building of type (W, I) and prescribed thickness $(q_i)_{i\in I}$. Let $(h_i)_{i\in I}$ be a set of legal colorings of Δ, Y_i be a set of cardinal q_i and $G_i \leq \mathrm{Sym}(Y_i)$ for every $i \in I$.

Theorem E. *There exists a locally finite tree T such that the universal group $\mathcal{U}((h_i, G_i)_{i\in I})$ embeds as a closed subgroup of the group $\mathrm{Aut}(T)^+$ of type-preserving automorphisms of T satisfying the property* IP_{V_1}. *Furthermore, if G_i is 2-transitive on Y_i for each $i \in I$, this group of automorphisms of tree corresponding to $\mathcal{U}((h_i, G_i)_{i\in I})$ satisfies the hypothesis H_{V_1} and is unimodular.*

Together with Theorem D, this proves that every non-discrete universal group $\mathcal{U}((h_i, G_i)_{i\in I})$ with *2*-transitive G_i admits a generic filtration factorising$^+$ at all strictly positive depths.

Part 1. Ol′šhanskiĭ's factorisation

2. The proof of Theorem A

2.1. **Direct consequences of Ol′šhanskiĭ's factorisation.** Let G be a non-discrete unimodular totally disconnected locally compact group, let μ be a Haar measure on G and let $\mathcal{S}$ be a generic filtration of G. This section explores the first consequences of a factorisation of $\mathcal{S}$ at depth l. We begin with the following key lemma.

Lemma 2.1. *Suppose that $\mathcal{S}$ factorises at depth l. Let U be conjugate to an element of $\mathcal{S}[l]$ and let $V \leq G$ be conjugate to an element of $\mathcal{S}$. Suppose that $\varphi : G \to \mathbb{C}$ is a U-right-invariant, V-left-invariant function satisfying*

$$\int_W \varphi(gh) \ \mathrm{d}\,\mu(h) = 0 \quad \forall g \in G$$

for every W that is conjugate to an element of $\mathcal{S}[l-1]$ and such that $U \subseteq W$. Then, φ is compactly supported and

$$\operatorname{supp}(\varphi) \subseteq N_G(U, V) = \{g \in G \mid g^{-1}Vg \subseteq U\}.$$

Remark 2.2. Since U is a compact open subgroup and since φ is U-right-invariant, notice that φ is automatically continuous. In particular, the integrals $\int_W \varphi(gh) \ \mathrm{d}\,\mu(h)$ are all well-defined.

Proof. Since $\mathcal{S}$ factorises at depth l notice that $N_G(U, V)$ is a compact set. Let $g \notin N_G(U, V)$ and notice that $g^{-1}Vg \not\subseteq U$. In particular, there exists W in the conjugacy class of an element of $\mathcal{S}[l-1]$ such that $U \subseteq W \subseteq g^{-1}VgU$. Hence, $gW \subseteq VgU$ and we have by U-right-invariance and V-left-invariance that $\varphi(gh) = \varphi(g)$ for all $h \in W$. It follows that

$$\varphi(g) = \frac{1}{\mu(W)} \int_W \varphi(gh) \ \mathrm{d}\,\mu(h) = 0,$$

which proves as desired that

$$\operatorname{supp}(\varphi) \subseteq N_G(U, V) = \{g \in G \mid g^{-1}Vg \subseteq U\}.$$

□

Lemma 2.3 follows directly and proves Theorem A(1).

Lemma 2.3. *Let π be an irreducible representation of G at depth l and suppose that $\mathcal{S}$ factorises at depth l. Then, there exists a unique $C_\pi \in \mathcal{F}_{\mathcal{S}}$ with height l such that for all $U \in C_\pi$, π has a non-zero U-invariant vector.*

Proof. Since π is at depth l, there exists a compact open subgroup $U \in \mathcal{S}[l]$ at depth l and a non-zero vector $\xi \in \mathcal{H}_\pi^U$. Now, let $V \leq G$ be conjugate to an element of $\mathcal{S}[l]$ and admitting a non-zero invariant vector $\xi' \in \mathcal{H}_\pi$ and let us show that $\mathcal{C}(U) = \mathcal{C}(V)$. The function

$$\varphi_{\xi,\xi'} : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi, \xi' \rangle$$

is clearly U-right-invariant and V-left-invariant. On the other hand, since π does not admit a non-zero W-invariant vector for any subgroup $W \leq G$ that is conjugate to an element of $\mathcal{S}[l-1]$ we have that

$$\int_W \varphi_{\xi,\xi'}(gh) \ \mathrm{d}\,\mu(h) = \langle \int_W \pi(g)\pi(h)\xi \, \mathrm{d}\,\mu(h), \xi' \rangle = 0 \quad \forall g \in G$$

for each such W. It follows from Lemma 2.1 that $\varphi_{\xi,\xi'}$ is supported inside $N_G(U,V) = \{g \in G \mid g^{-1}Vg \subseteq U\}$. On the other hand, since π is irreducible, ξ is cyclic and the function $\varphi_{\xi,\xi'}$ is not identically zero. This implies the existence of an element $g \in G$ such that $g^{-1}Vg \subseteq U$. Considering now the function $\varphi_{\xi',\xi} : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi', \xi\rangle$ we obtain by symmetry the existence of an element $h \in G$ such that $h^{-1}Uh \subseteq V$. In particular, $h^{-1}Uh \subseteq V \subseteq gUg^{-1}$. Hence, $g^{-1}h^{-1}Uhg \subseteq U$. Since U is a compact open subgroup of G and since G is unimodular, this implies that $g^{-1}h^{-1}Uhg = U$. In particular, we have that $h^{-1}Uh = V$ which implies that $\mathcal{C}(U) = \mathcal{C}(V)$. □

In light of this result we make Definition 2.4.

Definition 2.4. The unique element $C_\pi \in \mathcal{F}_{\mathcal{S}}$ with height l such that for all $U \in C_\pi$, π admits a non-zero U-invariant vector is called the **seed** of π.

Proposition 2.5 proves Theorem A(2).

Proposition 2.5. *Let π be an irreducible representation of G at depth l and suppose that $\mathcal{S}$ factorises at depth l. Then, for every $U \in C_\pi$, π admits a non-zero diagonal matrix coefficient supported in the compact open subgroup $N_G(U)$ of G. In particular, π is induced from an irreducible representation of $N_G(U)$, belongs to the discrete series of G and its equivalence class is isolated in the unitary dual $\widehat{G}$ for the Fell topology.*

Proof. Since π is at depth l, there exists a compact open subgroup $U \in \mathcal{S}[l]$ and a non-zero vector $\xi \in \mathcal{H}_\pi^U$. Now, Lemma 2.3 ensures that the diagonal matrix coefficient

$$\varphi_{\xi,\xi} : G \to \mathbb{C} : g \mapsto \langle \pi(g)\xi, \xi\rangle$$

is supported inside the compact set $N_G(U,U)$. Furthermore, notice that $N_G(U,U) = N_G(U)$ is a compact open subgroup of G. In particular, the GNS construction implies that $\pi \simeq \mathrm{Ind}_{N_G(U)}^G(\sigma)$ where σ is the irreducible representation of $N_G(U)$ corresponding to $\varphi_{\xi,\xi}\big|_{N_G(U)}$. Furthermore, since $\varphi_{\xi,\xi}$ is compactly supported, the representation π is both square-integrable and integrable. Hence, π belongs to the discrete series of G and [DM76, Corollary 1 pg.223] ensures that its equivalence class is open in the unitary dual $\widehat{G}$ for the Fell topology. □

Remark 2.6. Notice that the irreducible representation σ of $N_G(U)$ such that $\pi \simeq \mathrm{Ind}_{N_G(U)}^G(\sigma)$ is the inflation of an irreducible representation of a finite quotient of $N_G(U)$. Indeed, as σ is an irreducible of a compact group, it is finite dimensional. In particular, $\sigma(N_G(U))$ is a closed subgroup of the Lie group $\mathcal{U}(d)$ of unitary operators of the d dimensional complex Hilbert space for some positive integer $d \in \mathbb{N}$. On the other hand, $\sigma(N_G(U))$ is a quotient of a totally disconnected compact group. In particular, $\sigma(N_G(U))$ is a totally disconnected compact Lie group and is therefore finite. This implies that $\mathrm{Ker}(\sigma)$ is an open subgroup of finite index of $N_G(U)$ and therefore that σ is lifted to $N_G(U)$ from an irreducible representation of the finite group $N_G(U)/\mathrm{Ker}(\sigma)$. The purpose of the rest of this section is to describe more explicitly the irreducible representations of $N_G(U)$ that arise in this manner if $\mathcal{S}$ factorises$^+$ at depth l and show that under this hypothesis every equivalence class of irreducible representations of G at depth l can be obtained from this procedure.

2.2. **Subrepresentations of the regular representations.** For the rest of this section, let G be a non-discrete unimodular totally disconnected locally compact group, $\mathcal{S}$ be a generic filtration of G factorising at depth l, $U \leq G$ be conjugate to an element of $\mathcal{S}[l]$, μ be a bi-invariant Haar measure of G and let λ_G and ρ_G be respectively the left-regular and right-regular representations of G. The purpose of this section is to study the space of functions of positive type associated with irreducible representations of G with seed $\mathcal{C}(U)$. Lemma 2.1 motivates Definition 2.7.

Definition 2.7. We define $\mathcal{L}_{\mathcal{S}}(U)$ to be the closure in $L^2(G)$ of the set of all functions $\varphi : G \to \mathbb{C}$ satisfying the following properties:

(1) φ is U-right-invariant.
(2) φ is V-left-invariant for some $V \leq G$ conjugate to an element of $\mathcal{S}$.
(3) For every $W \leq G$ containing U and conjugate to an element of $\mathcal{S}[l-1]$ we have that
$$\int_W \varphi(gh) \ \mathrm{d}\,\mu(h) \ = \ 0 \quad \forall g \in G.$$

Equivalently these three properties can be formulated in terms of fixed point subspace and orthogonal complement as follows:

(1) $\varphi \in L^2(G)^{\rho_G(U)}$.
(2) $\varphi \in \bigcup_{W\in\mathcal{S}, V\in\mathcal{C}(W)} L^2(G)^{\lambda_G(V)}$.
(3) $\varphi \in \bigcap_{V\in\mathcal{S}[l-1], U\subseteq W\in\mathcal{C}(V), g\in G} \left(\mathbb{1}_{gW}\right)^{\perp}$.

By definition $\mathcal{L}_{\mathcal{S}}(U)$ is a set of equivalence classes of functions up to negligible sets and not a set of functions. However, Lemma 2.8 ensures the existence of a canonical choice of representative.

Lemma 2.8. *For every $\tilde{\varphi} \in \mathcal{L}_{\mathcal{S}}(U)$, there exists a unique U-right-invariant representative φ of $\tilde{\varphi}$. Furthermore, for every $W \leq G$ containing U and conjugate to an element of $\mathcal{S}[l]$ we have that*
$$\int_W \varphi(gh) \ \mathrm{d}\,\mu(h) \ = \ 0 \quad \forall g \in G.$$

Proof. By the definition of $\mathcal{L}_{\mathcal{S}}(U)$, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of complex valued functions such that:

(1) $\tilde{\varphi}_n \xrightarrow[n\to\infty]{} \tilde{\varphi}$ in $L^2(G)$ where $\tilde{\varphi}_n$ denotes the equivalence class of φ_n.
(2) φ_n is U-right-invariant.
(3) For every $W \leq G$ containing U and conjugate to an element of $\mathcal{S}[l-1]$ we have that
$$\int_W \varphi_n(gh) \ \mathrm{d}\,\mu(h) \ = \ 0 \quad \forall g \in G.$$

Now, let $\varphi' : G \to \mathbb{C}$ be any representative of $\tilde{\varphi}$. The above implies that
$$\int_G |\varphi_n(h) - \varphi'(h)|^2 \ \mathrm{d}\,\mu(h) \xrightarrow[i\to\infty]{} 0.$$
On the other hand, since G is a disjoint union of its U-left-cosets, we have
$$\int_G |\varphi_n(h) - \varphi'(h)|^2 \ \mathrm{d}\,\mu(h) = \sum_{gU\in G/U} \int_{gU} |\varphi_n(h) - \varphi'(h)|^2 \ \mathrm{d}\,\mu(h).$$

Therefore, we obtain for all $g \in G$ that

$$\int_{gU} |\varphi_n(h) - \varphi'(h)|^2 \,\mathrm{d}\,\mu(h) = \int_{U} |\varphi_n(gh) - \varphi'(gh)|^2 \,\mathrm{d}\,\mu(h) \xrightarrow[n\to\infty]{} 0.$$

In particular, for every $g \in G$ this implies that $\varphi_n(gh)$ converges to $\varphi'(gh)$ for almost all $h \in U$. Since the φ_n are constant on U-left-cosets, there exists of a unique representative $\varphi : G \to \mathbb{C}$ of $\tilde{\varphi}$ such that

$$\varphi(gh) = \varphi(g) \ \forall g \in G, \forall h \in U.$$

On the other hand, since U is a compact set, the convergence $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ is uniform on U-left-cosets. Now, let $g \in G$ and let W be conjugate to an element of $\mathcal{S}[l-1]$ and such that $U \subseteq W$. Since gW is compact, it can be covered by finitely many U-left-cosets and the convergence $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ is also uniform on W-left-cosets. This implies as desired that

$$\left| \int_W \varphi(gh) \,\mathrm{d}\,\mu(h) \right| = \left| \int_W \varphi(gh) - \varphi_n(gh) \,\mathrm{d}\,\mu(h) \right| \leq \int_W |\varphi(gh) - \varphi_n(gh)| \mathrm{d}\,\mu(h)$$
$$\leq \mu(W) \sup_{k \in gW} |\varphi(k) - \varphi_n(k)| \xrightarrow[n\to\infty]{} 0 \quad \forall g \in G.$$

□

In light of this result, we identify each equivalence class $\tilde{\varphi} \in \mathcal{L}_{\mathcal{S}}(U)$ with its canonical continuous representative if it leads to no confusion. Lemma 2.9 shows that $\mathcal{L}_{\mathcal{S}}(U)$ is G-left-invariant and therefore defines a subrepresentation of the left-regular representation $(\lambda_G, L^2(G))$.

Lemma 2.9. *$\mathcal{L}_{\mathcal{S}}(U)$ is a closed G-left-invariant subspace of $L^2(G)$.*

Proof. By the definition of $\mathcal{L}_{\mathcal{S}}(U)$, it is enough to prove that the subspace of functions satisfying the three properties of Definition 2.7 is G-left-invariant. Let φ be such a function, $k \in G$ and notice that:

(1) $\lambda_G(k)\varphi \in L^2(G)^{\rho_G(U)}$.
(2) $\lambda_G(k)\varphi \in \bigcup_{\substack{W \in \mathcal{S},\\ V \in \mathcal{C}(W)}} L^2(G)^{\lambda_G(kVk^{-1})} = \bigcup_{\substack{W \in \mathcal{S},\\ V \in \mathcal{C}(W)}} L^2(G)^{\lambda_G(V)}$.
(3) $\lambda_G(k)\varphi \in \bigcap_{\substack{V \in \mathcal{S}[l-1],\\ U \subseteq W \in \mathcal{C}(V),\\ g \in G}} \left(\mathbb{1}_{kgW}\right)^{\perp} = \bigcap_{\substack{V \in \mathcal{S}[l-1],\\ U \subseteq W \in \mathcal{C}(V),\\ g \in G}} \left(\mathbb{1}_{gW}\right)^{\perp}$. □

We denote by $T_{\mathcal{S},U}$ the subrepresentation of λ_G corresponding to $\mathcal{L}_{\mathcal{S}}(U)$. The following result shows that this representation depends, up to equivalence, only on the conjugacy class $\mathcal{C}(U)$ and not on its representative U.

Lemma 2.10. *Let $C \in \mathcal{F}_{\mathcal{S}}$ be a conjugacy class with height l and let $U, U' \in C$. Then, the representations $(T_{\mathcal{S},U}, \mathcal{L}_{\mathcal{S}}(U))$ and $(T_{\mathcal{S},U'}, \mathcal{L}_{\mathcal{S}}(U'))$ are unitarily equivalent.*

Proof. Since U and U' belong to the same conjugacy class C, there exists an element $k \in G$ such that $U' = kUk^{-1}$. Now, notice that $\rho_G(k) : L^2(G) \to L^2(G)$ is a unitary operator and that $\rho_G(k)\mathcal{L}_{\mathcal{S}}(U) = \mathcal{L}_{\mathcal{S}}(U')$. Indeed, for every function $\varphi : G \to \mathbb{C}$ such that $\varphi \in L^2(G)^{\rho_G(U)}$, $\varphi \in \bigcup_{W \in \mathcal{S}, V \in \mathcal{C}(W)} L^2(G)^{\lambda_G(V)}$ and $\varphi \in \bigcap_{V \in \mathcal{S}[l-1], U \subseteq W \in \mathcal{C}(V), g \in G} \left(\mathbb{1}_{gW}\right)^{\perp}$ we have:

(1) $\rho_G(k)\varphi \in L^2(G)^{\rho_G(kUk^{-1})} = L^2(G)^{\rho_G(U')}$.

(2) $\rho_G(k)\varphi \in \bigcup_{W\in\mathcal{S}, V\in\mathcal{C}(W)} L^2(G)^{\lambda_G(V)}$.

(3) $\rho_G(k)\varphi \in \bigcap_{\substack{V\in\mathcal{S}[l-1],\\ U\subseteq W\in\mathcal{C}(V),\\ g\in G}} \left(\mathbb{1}_{gWk^{-1}}\right)^{\perp} = \bigcap_{\substack{V\in\mathcal{S}[l-1],\\ U'\subseteq kWk^{-1}\in\mathcal{C}(V),\\ gk^{-1}\in G}} \left(\mathbb{1}_{gk^{-1}(kWk^{-1})}\right)^{\perp}$.

It follows by density and by continuity of $\rho_G(k)$ that $\rho_G(k)\mathcal{L}_S(U) \subseteq \mathcal{L}_S(U')$. Inverting the role of U and U', we obtain that $\rho_G(k^{-1})\mathcal{L}_S(U') \subseteq \mathcal{L}_S(U)$ and thus that $\rho_G(k)\mathcal{L}_S(U) = \mathcal{L}_S(U')$. Since $\lambda_G(g)\rho_G(k) = \rho_G(k)\lambda_G(g)$ for every $g \in G$, the restriction of $\rho_G(k)$ to $\mathcal{L}_{\mathcal{S}}(U)$ provides the desired intertwining operator between $T_{\mathcal{S},U}$ and $T_{\mathcal{S},U'}$. □

Now, recall from Lemma 2.3 that for every irreducible representation π of G at depth l with a non-zero U-invariant vector ξ, the matrix coefficient $\varphi_{\xi,\xi}$ is a non-zero U-bi-invariant element of $\mathcal{L}_{\mathcal{S}}(U)$. The following result shows that every subrepresentation of $\mathcal{L}_{\mathcal{S}}(U)$ contains such a function.

Lemma 2.11. *Every G-left-invariant subspace of $\mathcal{L}_{\mathcal{S}}(U)$ contains a non-zero U-bi-invariant function.*

Proof. Suppose that M is a non-zero closed G-left-invariant subspace of $\mathcal{L}_{\mathcal{S}}(U)$, consider any non-zero class of functions $\varphi \in M$ and let $t \in G$ be such that $\varphi(t) \neq 0$. Notice that the function

$$\psi = \int_U T_{\mathcal{S},U}(kt^{-1})\varphi \; \mathrm{d}\,\mu(k)$$

is U-left-invariant, U-right-invariant and satisfies

$$\int_W \psi(gh) \; \mathrm{d}\,\mu(h) \;=\; 0 \quad \forall g \in G$$

for every W that is conjugate to an element of $\mathcal{S}[l-1]$ and such that $U \subseteq W$. Since M is a closed G-left-invariant subspace of $\mathcal{L}_{\mathcal{S}}(U)$, we have that $\psi \in M$. Furthermore, this function is not identically zero since the U-right invariance of φ implies that

$$\psi(1_G) = \int_U \varphi(tk^{-1})\,\mathrm{d}\,\mu(k) = \int_U \varphi(t) \;\mathrm{d}\,\mu(k) = \mu(U)\varphi(t) \;\neq 0.$$

□

Proposition 2.12 shows that the space of U-bi-invariant functions of $\mathcal{L}_{\mathcal{S}}(U)$ is finite-dimensional and therefore that $T_{\mathcal{S},U}$ decomposes as a finite sum of irreducible representations of G.

Proposition 2.12. *The subspace of U-bi-invariant functions of $\mathcal{L}_{\mathcal{S}}(U)$ is finite dimensional.*

Proof. Lemma 2.1 ensures that the U-bi-invariant functions of $\mathcal{L}_{\mathcal{S}}(U)$ are supported inside the compact set $N_G(U)$. On the other hand every U-bi-invariant continuous function $\varphi : G \to \mathbb{C}$ is constant on the U-left-cosets and $N_G(U)$ can be covered by finitely such cosets. This proves that the subspace of U-bi-invariant functions of $\mathcal{L}_{\mathcal{S}}(U)$ is in the span of finitely many characteristic functions and is therefore finite dimensional. □

Corollary 2.13. *$T_{\mathcal{S},U}$ decomposes as a finite sum of irreducible square-integrable representations of G with seed $\mathcal{C}(U)$.*

Proof. Lemma 2.11 and Proposition 2.12 ensure that $T_{\mathcal{S},U}$ decomposes as a finite sum of irreducible square-integrable representations of G; each of these containing a non-zero U-invariant vector. Now, let $W \leq G$ be conjugate to an element of $\mathcal{S}[r]$ for some $r \lneq l$. Lemma 2.1 ensures that the W-left-invariant functions of $\mathcal{L}_{\mathcal{S}}(U)$ are supported inside

$$N_G(U, W) = \{g \in G \mid gWg^{-1} \subseteq U\}.$$

On the other hand, since $\mathcal{C}(W)$ has height r, since $\mathcal{C}(U)$ has height l and since $r \lneq l$ the set $N_G(U, W)$ is empty. This proves that $\mathcal{L}_{\mathcal{S}}(U)$ does not contain any non-zero W-left invariant function and thus that every irreducible representation appearing in the decomposition of $T_{\mathcal{S},U}$ has seed $\mathcal{C}(U)$. □

It is a natural to ask whether the reversed implication holds. The following provides a positive answer to this question.

Lemma 2.14. *Every irreducible representation π of G with seed $\mathcal{C}(U)$ is equivalent to a subrepresentation of $T_{\mathcal{S},U}$.*

Proof. Let π be an irreducible representation of G with seed $\mathcal{C}(U)$ and let $\xi \in \mathcal{H}_\pi^U$ be a non-zero vector. Since π has seed $\mathcal{C}(U)$, notice with a similar reasoning as in the proof of Lemma 2.3 that $\widehat{\varphi}_{\xi,\xi}$ is a non-zero U-bi-invariant function of $\mathcal{L}_{\mathcal{S}}(U)$ where $\widehat{\varphi}_{\xi,\xi} : G \to \mathbb{C} : g \mapsto \langle \pi(g^{-1})\xi, \xi\rangle$. Now consider the diagonal matrix coefficient $\psi(\cdot) = \langle T_{\mathcal{S},U}(\cdot)\widehat{\varphi}_{\xi,\xi}, \widehat{\varphi}_{\xi,\xi}\rangle$ of $T_{\mathcal{S},U}$ and notice from [Dix77, Theorem 14.3.3] that

$$\begin{aligned}
\psi(g) &= \int_G T_{\mathcal{S},U}(g)\widehat{\varphi}_{\xi,\xi}(h)\overline{\widehat{\varphi}_{\xi,\xi}(h)}\,\mathrm{d}\,\mu(h) = \int_G \varphi_{\xi,\xi}(h^{-1}g)\overline{\varphi_{\xi,\xi}(h^{-1})}\,\mathrm{d}\,\mu(h)\\
&= \int_G \varphi_{\pi(g)\xi,\xi}(h^{-1})\overline{\varphi_{\xi,\xi}(h^{-1})}\,\mathrm{d}\,\mu(h) = \int_G \varphi_{\pi(g)\xi,\xi}(h)\overline{\varphi_{\xi,\xi}(h)}\,\mathrm{d}\,\mu(h)\\
&= d_\pi^{-1}\|\xi\|^2\langle \pi(g)\xi, \xi\rangle = d_\pi^{-1}\|\xi\|^2\varphi_{\xi,\xi}(g),
\end{aligned}$$

where d_π is the formal dimension of π. The result follows from the uniqueness of the GNS construction [BdlHV08, Theorem C.4.10]. □

Corollary 2.15. *Let $C \in \mathcal{F}_{\mathcal{S}}$ be a conjugacy class with height l. Then, there exist at most finitely many inequivalent classes of irreducible representations of G with seed C.*

To improve the clarity of the exposition the author decided to gather the results of the previous two sections in a theorem. Up to this point, Theorem A(1) and A(2) together with the following result.

Theorem 2.16. *Let G be a non-discrete unimodular totally disconnected locally compact group, $\mathcal{S}$ be a generic filtration of G factorising at depth l and $U \leq G$ be conjugate to an element of $\mathcal{S}[l]$. Then, the following hold:*

1. *The square-integrable representation $(T_{\mathcal{S},U}, \mathcal{L}_{\mathcal{S}}(U))$ depends, up to equivalence, only on the conjugacy class $\mathcal{C}(U)$.*
2. *Every G-left-invariant subspace of $\mathcal{L}_{\mathcal{S}}(U)$ contains a non-zero U-bi-invariant function and every such function is compactly supported inside $N_G(U)$.*
3. *$T_{\mathcal{S},U}$ decomposes as a finite sum of square-integrable irreducible representations of G with seed $\mathcal{C}(U)$.*
4. *Every irreducible representation of G with seed $\mathcal{C}(U)$ is a subrepresentation of $T_{\mathcal{S},U}$.*

In particular, if $\mathcal{S}$ factorises at depth l we have a bijective correspondence between the equivalence classes of irreducible subrepresentations of $(T_{\mathcal{S},U}, \mathcal{L}_{\mathcal{S}}(U))$ and the equivalence classes of irreducible representations of G with seed $\mathcal{C}(U)$. In the following sections, we introduce a family of irreducible representations of the finite group $N_G(U)/U$ which can be lifted to representations of $N_G(U)$ and show that the irreducible representations of G with seed $\mathcal{C}(U)$ are induced from these when $\mathcal{S}$ factorises$^+$ at depth l.

2.3. **Induced representations.** The purpose of this section is to recall the definition of induction and some related results that will be required to end the proof of Theorem A. Even if this notion makes sense in the context of locally compact groups and closed subgroups, this level of generality comes with technicalities that are unnecessary for the rest of our expository. Since most of the complexity vanishes if H is an open subgroup of G (because the quotient space G/H is discrete) and since it is the only setup encountered in these notes, we will work under this hypothesis and refer to [KT13, Chapters 2.1 and 2.2] for more details.

Let G be a locally compact group, let $H \leq G$ be an open subgroup and let σ be a representation of H. The induced representation $\operatorname{Ind}_H^G(\sigma)$ is a representation of G with representation space given by

$$\operatorname{Ind}_H^G(\mathcal{H}_\sigma) = \left\{ \phi : G \to \mathcal{H}_\sigma \middle| \phi(gh) = \sigma(h^{-1})\phi(g), \sum_{gH \in G/H} \langle \phi(g), \phi(g) \rangle < +\infty \right\}.$$

For $\psi, \phi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma)$, we let

$$\langle \psi, \phi \rangle_{\operatorname{Ind}_H^G(\mathcal{H}_\sigma)} = \sum_{gH \in G/H} \langle \psi(g), \phi(g) \rangle.$$

Equipped with this inner product, $\operatorname{Ind}_H^G(\mathcal{H}_\sigma)$ is a separable complex Hilbert space. The induced representation $\operatorname{Ind}_H^G(\sigma)$ is the representation of G on $\operatorname{Ind}_H^G(\mathcal{H}_\sigma)$ defined by

$$\left[\operatorname{Ind}_H^G(\sigma)(h)\right]\phi(g) = \phi(h^{-1}g) \quad \forall \phi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma) \text{ and } \forall g, h \in G.$$

The following three results are classical but the author did not find any convenient sources in the literature and chose to give a proof for instructive purposes and for completeness of the argument.

Lemma 2.17. *Let G be a locally compact group, let $H \leq G$ be an open subgroup and let σ be a representation of H. Then, there exists an isomorphism between the Hilbert spaces $\mathfrak{D}_H^\sigma = \{\phi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma) | \phi(g) = 0 \ \forall g \in G - H\}$ and $\mathcal{H}_\sigma$ that intertwines the representations $\operatorname{Ind}_H^G(\sigma)\Big|_H$ and σ. Furthermore, we have*

$$\operatorname{Ind}_H^G(\mathcal{H}_\sigma) = \overline{\bigoplus_{gH \in G/H} \left[\operatorname{Ind}_H^G(\sigma)(g)\right]\mathfrak{D}_H^\sigma}.$$

Proof. First notice that $\mathfrak{D}_H^\sigma$ is a closed subspace of $\operatorname{Ind}_H^G(\mathcal{H}_\sigma)$ and therefore defines a Hilbert space. Now, for every $\xi \in \mathcal{H}_\sigma$, let

$$\phi_\xi(g) = \begin{cases} \sigma(g^{-1})\xi & \text{if } g \in H, \\ 0 & \text{if } g \notin H \end{cases}$$

and notice that $\phi_\xi = 0$ only if $\xi = 0$ and that $\phi_\xi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma)$. On the other hand, every function $\phi \in \mathfrak{D}_H^\sigma$ is uniquely determined by the value it takes on 1_G since

$\phi(g) = 0\ \forall g \in G - H$ and $\phi(g) = \sigma(g^{-1})\phi(1_G)\ \forall g \in H$. Hence, we have that $\mathfrak{D}_H^\sigma = \{\phi_\xi | \xi \in \mathcal{H}_\sigma\}$ and the map

$$\Phi \ : \ \mathcal{H}_\sigma \ \to \ \mathfrak{D}_H^\sigma \ : \ \xi \ \mapsto \ \phi_\xi$$

is a linear isomorphism between $\mathcal{H}_\sigma$ and $\mathfrak{D}_H^\sigma$. Moreover, Φ is unitary since $\forall \xi, \eta \in \mathcal{H}_\sigma$ we have

$$\langle \Phi(\xi), \Phi(\eta)\rangle_{\mathrm{Ind}_H^G(\mathcal{H}_\sigma)} = \sum_{gH \in G/H} \langle \phi_\xi(g), \phi_\eta(g)\rangle = \langle \phi_\xi(1_G), \phi_\eta(1_G)\rangle = \langle \xi, \eta\rangle.$$

Finally, Φ intertwines σ and $\mathrm{Ind}_H^G(\sigma)\Big|_H$ since $\forall h \in H$, $\forall g \in G$ we have

$$\begin{aligned}\Phi(\sigma(h)\xi)(g) = \phi_{\sigma(h)\xi}(g) &= \begin{cases} \sigma(g^{-1}h)\xi & \text{if } g \in H, \\ 0 & \text{if } g \notin H \end{cases} \\ &= \phi_\xi(h^{-1}g) = \big[\mathrm{Ind}_H^G(\sigma)(h)\big]\phi_\xi(g).\end{aligned}$$

This proves the first part of the claim. To prove the second part of the claim, let $\phi \in \mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ and let us show that

$$\phi = \sum_{gH \in G/H} [\mathrm{Ind}_H^G(\sigma)(g)]\Phi(\phi(g)).$$

First, notice that $\mathrm{Ind}_H^G(\sigma)(g)\mathfrak{D}_H^\sigma$ is the set of functions of $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ that are supported inside gH. On the other hand, $\forall g, t \in G$, $\forall h \in H$ we have that

$$\begin{aligned}\big[\big[\mathrm{Ind}_H^G(\sigma)(gh)\big]\Phi(\phi(gh))\big](t) &= \Phi(\phi(gh))(h^{-1}g^{-1}t) \\ &= \begin{cases} \sigma(t^{-1}gh)\phi(gh) & \text{if } h^{-1}g^{-1}t \in H, \\ 0 & \text{if } h^{-1}g^{-1}t \notin H \end{cases} \\ &= \begin{cases} \sigma(t^{-1}g)\phi(g) & \text{if } g^{-1}t \in H, \\ 0 & \text{if } g^{-1}t \notin H \end{cases} \\ &= \Phi(\phi(g))(g^{-1}t) = \big[[\mathrm{Ind}_H^G(\sigma)(g)]\Phi(\phi(g))\big](t).\end{aligned}$$

This proves that the map $G/H \to \mathrm{Ind}_H^G(\mathcal{H}_\sigma) : gH \mapsto \big[\mathrm{Ind}_H^G(\sigma)(g)\big]\Phi(\phi(g))$ is well-defined. On the other hand, $\forall g \in G$, $\forall h \in H$, we have

$$\big\|\big[\big[\mathrm{Ind}_H^G(\sigma)(g)\big]\Phi(\phi(g))\big](gh)\big\| = \|\Phi(\phi(g))(h)\| = \|\sigma(h^{-1})\phi(g)\| = \|\phi(g)\|.$$

Hence, we obtain that

$$\begin{aligned}\Big\|\sum_{gH \in G/H} \big[\mathrm{Ind}_H^G(\sigma)(g)\big]\Phi(\phi(g))\Big\|^2_{\mathrm{Ind}_H^G(\mathcal{H}_\sigma)} &= \sum_{tH \in G/H} \Big\|\sum_{gH \in G/H} \big[\big[\mathrm{Ind}_H^G(\sigma)(g)\big]\Phi(\phi(g))\big](t)\Big\|^2 \\ = \sum_{tH \in G/H} \big\|\big[\big[\mathrm{Ind}_H^G(\sigma)(t)\big]\Phi(\phi(t))\big](t)\big\|^2 &= \sum_{tH \in G/H} \|\phi(t)\|^2 = \|\phi\|^2_{\mathrm{Ind}_H^G(\mathcal{H}_\sigma)}.\end{aligned}$$

This proves that $\sum_{gH\in G/H}[\mathrm{Ind}_H^G(\sigma)(g)]\Phi(\phi(g))$ belongs to $\overline{\bigoplus_{gH\in G/H} \mathrm{Ind}_H^G(\sigma)(g)\mathfrak{D}_H^\sigma}$. Finally, a direct computation shows that

$$\left\|\phi-\sum_{gH\in G/H}\mathrm{Ind}_H^G(\sigma)(g)\Phi(\phi(g))\right\|^2_{\mathrm{Ind}_H^G(\sigma)}=\sum_{tH\in G/H}\left\|\phi(t)-\sum_{gH\in G/H}\mathrm{Ind}_H^G(\sigma)(g)\Phi(\phi(g))(t)\right\|^2$$
$$=\sum_{tH\in G/H}\left\|\phi(t)-\mathrm{Ind}_H^G(\sigma)(t)\Phi(\phi(t))(t)\right\|^2=\sum_{tH\in G/H}\|\phi(t)-\phi(t)\|^2=0.$$

□

The following provides a useful criterion to check the irreducibility of the induced representation.

Proposition 2.18. *Let G be a locally compact group, let $H\leq G$ be an open subgroup and let σ be an irreducible representation of H. Then, $\mathrm{Ind}_H^G(\sigma)$ is irreducible if and only if every non-zero closed invariant subspace of $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ contains a non-zero function supported in H.*

Proof. The forward implication is trivial. For the other implication, let M be a non-zero closed invariant subspace of $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$. By the hypothesis, $M\cap\mathfrak{D}_H^\sigma\neq\varnothing$. In particular, since σ is irreducible, the correspondence of Lemma 2.17 implies that $\mathfrak{D}_H^\sigma\subseteq M$. We conclude from that same lemma that $M=\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ since M is a closed invariant subspace of $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ and since

$$\mathrm{Ind}_H^G(\mathcal{H}_\sigma)=\overline{\bigoplus_{gH\in G/H}\left[\mathrm{Ind}_H^G(\sigma)(g)\right]\mathfrak{D}_H^\sigma}.$$

□

The following provides a sufficient condition for the induced representations to be inequivalent.

Lemma 2.19. *Let G be a locally compact group, let $H\leq G$ be an open subgroup and let σ_1,σ_2 be two inequivalent irreducible representations of H. Suppose that there exists a subgroup $K\leq H$ such that $\mathfrak{D}_H^{\sigma_i}=\left(\mathrm{Ind}_H^G(\mathcal{H}_{\sigma_i})\right)^K$. Then, $\mathrm{Ind}_H^G(\sigma_1)$ and $\mathrm{Ind}_H^G(\sigma_2)$ are inequivalent.*

Proof. The proof is by contradiction. Suppose that there exists a unitary operator U intertwining $\mathrm{Ind}_H^G(\sigma_1)$ and $\mathrm{Ind}_H^G(\sigma_2)$. In particular, we have $\mathrm{U}\left(\mathrm{Ind}_H^G(\mathcal{H}_{\sigma_1})\right)^K=\left(\mathrm{Ind}_H^G(\mathcal{H}_{\sigma_2})\right)^K$. Let $\Phi_i:\mathcal{H}_{\sigma_i}\to\mathfrak{D}_H^{\sigma_i}$ be the correspondences given by Lemma 2.17 and notice that $\Phi_i(\mathcal{H}_{\sigma_i})=\mathfrak{D}_H^{\sigma_i}=\left(\mathrm{Ind}_H^G(\mathcal{H}_{\sigma_i})\right)^K$. In particular, since Φ_i is a unitary operator intertwining σ_i and $\mathrm{Ind}_H^G(\sigma_i)\Big|_H$, the unitary operator $\Phi_2^{-1}\mathrm{U}\Phi_1$ intertwines σ_1 and σ_2 which leads to a contradiction. □

2.4. **The bijective correspondence of Theorem A.** Let G be a non-discrete unimodular totally disconnected locally compact group and let $\mathcal{S}$ be a generic filtration of G. The purpose of this section is to describe explicitly the bijective correspondence of Theorem A. This requires some formalism that we now introduce.

Let U be conjugate to an element of $\mathcal{S}[l]$ and notice that if $\mathcal{S}$ factorises at depth l, $N_G(U)/U$ is a finite group. Furthermore, notice that

$$N_G(g^{-1}Ug) = g^{-1}N_G(U)g \quad \forall g \in G$$

which implies that

$$N_G(U)/U \simeq N_G(V)/V \quad \forall U, V \in C,\ \forall C \in \mathcal{F}_{\mathcal{S}}.$$

This motivates Definition 2.20.

Definition 2.20. For every conjugacy class $C \in \mathcal{F}_{\mathcal{S}}$, the **group of automorphisms** $\mathrm{Aut}_G(C)$ **of the seed** C is the finite group $N_G(U)/U$ corresponding to any group $U \in C$.

For all $C \in \mathcal{F}_{\mathcal{S}}$ with height l and every $U \in C$, we set

$$\tilde{\mathfrak{H}}_{\mathcal{S}}(U) = \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in \mathcal{S}[l-1] \text{ and } U \subseteq W\},$$

and let $p_U : N_G(U) \mapsto N_G(U)/U$ denote the quotient map. If $\mathcal{S}$ factorises$^+$ at depth l, notice that $p_U(W)$ is a non-trivial (possibly not proper) subgroup of $\mathrm{Aut}_G(C)$ for every $W \in \tilde{\mathfrak{H}}_{\mathcal{S}}(U)$. Moreover, notice that

$$\tilde{\mathfrak{H}}_{\mathcal{S}}(g^{-1}Ug) = g^{-1}\tilde{\mathfrak{H}}_{\mathcal{S}}(U)g \quad \forall g \in G.$$

In particular, the subset of non-trivial subgroups of $\mathrm{Aut}_G(C)$

$$\mathfrak{H}_{\mathcal{S}}(C) = \{p_U(W) | W \in \tilde{\mathfrak{H}}_{\mathcal{S}}(U)\}$$

does not depend on our choice of representative $U \in C$.

Definition 2.21. An irreducible representation ω of $\mathrm{Aut}_G(C)$ is an $\mathcal{S}$**-standard** if it does not have any non-zero H-invariant vector for any $H \in \mathfrak{H}_{\mathcal{S}}(C)$.

Our goal is to describe the irreducible representations of G with seed C from these $\mathcal{S}$**-standard** representations. We recall that every representation ω of $N_G(U)/U$ can be lifted to a representation $\omega \circ p_U$ of $N_G(U)$ acting trivially on U and with representation space $\mathcal{H}_\omega$. Furthermore, notice that $\omega \circ p_U$ is irreducible if and only if ω is irreducible. We can now use the process of induction recalled in Section 2.3. For shortening of the formulation, we denote by

$$T(U, \omega) = \mathrm{Ind}_{N_G(U)}^G(\omega \circ p_U)$$

the resulting representation of G. Our purpose will be to show that $T(U, \omega)$ is an irreducible representation of G with seed $\mathcal{C}(U)$ if ω is $\mathcal{S}$-standard. Conversely, if π is an irreducible representation of G with seed C, notice that $\mathcal{H}_\pi^U$ is a non-zero $N_G(U)$-invariant subspace of $\mathcal{H}_\pi$ for every $U \in C$. In particular, the restriction $(\pi|_{N_G(U)}, \mathcal{H}_\pi^U)$ defines a representation of $N_G(U)$ whose restriction to U is trivial. This representation passes to the quotient group $N_G(U)/U$ and therefore provides a representation ω_π of $\mathrm{Aut}_G(C)$. Theorem 2.22 describes the bijective correspondence of Theorem A using these constructions.

Theorem 2.22. *Let G be a non-discrete unimodular totally disconnected locally compact group, $\mathcal{S}$ be a generic filtration of G factorising$^+$ at depth l and $C \in \mathcal{F}_{\mathcal{S}}$ be a conjugacy class at height l. There exists a bijective correspondence between the equivalence classes of irreducible representations of G with seed C and the equivalence classes of $\mathcal{S}$-standard representations of* $\mathrm{Aut}_G(C)$*. More precisely, for every $U \in \mathcal{S}$ the following hold:*

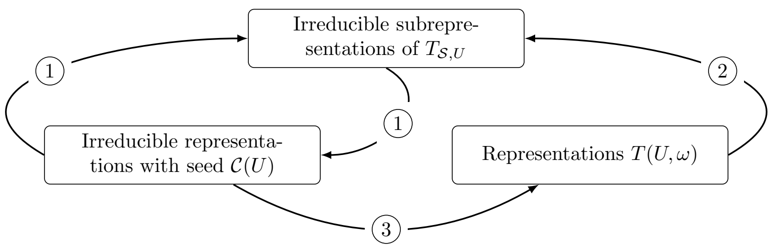
(1) *If π is an irreducible representation of G with seed $\mathcal{C}(U)$, the representation $(\omega_\pi, \mathcal{H}_\pi^U)$ of $\mathrm{Aut}_G(\mathcal{C}(U))$ is an $\mathcal{S}$-standard representation of $\mathrm{Aut}_G(\mathcal{C}(U))$ such that*

$$\pi \simeq T(U, \omega_\pi) = \mathrm{Ind}_{N_G(U)}^G(\omega_\pi \circ p_U).$$

(2) *If ω is an $\mathcal{S}$-standard representation of $\mathrm{Aut}_G(\mathcal{C}(U))$, the representation $T(U,\omega)$ is an irreducible representation of G with seed $\mathcal{C}(U)$.*

Furthermore, if ω_1 and ω_2 are $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(\mathcal{C}(U))$, we have that $T(U,\omega_1) \simeq T(U,\omega_2)$ if and only if $\omega_1 \simeq \omega_2$. In particular, the above two constructions are inverse of one another.

The structure of the proof is given as follows:



(1) Theorem 2.16 (2) Proposition 2.24 (3) Lemma 2.25

The last part of the statement is then handled by Lemma 2.26.

We start the proof with an intermediate result. From now on and for the rest of this section, let $U \leq G$ be conjugate to an element of $\mathcal{S}[l]$ and suppose that $\mathcal{S}$ factorises $^+$ at depth l.

Lemma 2.23. *Let φ be a U-bi-invariant function of $\mathcal{L}_\mathcal{S}(U)$ and let $\widehat{\varphi}$ be the function defined by $\widehat{\varphi}(g) = \varphi(g^{-1})$ $\forall g \in G$. Then, $\widehat{\varphi}$ is a U-bi-invariant function of $\mathcal{L}_\mathcal{S}(U)$.*

Proof. The function $\widehat{\varphi}$ is clearly U-bi-invariant. Furthermore, since G is unimodular and since $\varphi \in L^2(G)$, notice that $\widehat{\varphi} \in L^2(G)$. On the other hand, Lemma 2.1 ensures that φ is supported inside $N_G(U)$. The same holds for $\widehat{\varphi}$ as $N_G(U)$ is symmetric. Now let W be conjugate to an element of $\mathcal{S}[l-1]$ and such that $U \subseteq W$. Since $\mathcal{S}$ factorises$^+$ at depth l, we have that $W \subseteq N_G(U)$. In particular, for every $g \notin N_G(U)$, $gW \cap N_G(U) = \varnothing$ and $\int_W \widehat{\varphi}(gh)\,\mathrm{d}\,\mu(h) = 0$. On the other hand, if $g \in N_G(U)$, notice that $U = gUg^{-1} \subseteq gWg^{-1}$ which implies that

$$\int_W \widehat{\varphi}(gh)\,\mathrm{d}\,\mu(h) = \int_W \varphi(hg^{-1})\,\mathrm{d}\,\mu(h) = \int_{gWg^{-1}} \varphi(g^{-1}h)\,\mathrm{d}\,\mu(h) = 0.$$

□

Proposition 2.24. *Let ω be an $\mathcal{S}$-standard representation of $\mathrm{Aut}_G(\mathcal{C}(U))$. Then, $\mathfrak{D}_{N_G(U)}^{\omega \circ p_U} = \mathcal{H}_{T(U,\omega)}^U$ and $T(U,\omega)$ is equivalent to an irreducible subrepresentation of $T_{\mathcal{S},U}$.*

Proof. We start by showing that $T(U,\omega)$ is equivalent to a subrepresentation of $T_{\mathcal{S},U}$. Let $\xi \in \mathcal{H}_\omega$ be non-zero and consider the function

$$\phi_\xi : G \to \mathbb{C} : g \mapsto \begin{cases} \omega \circ p_U(g^{-1})\xi & \text{if } g \in N_G(U) \\ 0 & \text{if } g \notin N_G(U) \end{cases}.$$

We have that $\phi_\xi \in \mathcal{H}_{T(U,\omega)}$ and since ω is irreducible, Lemma 2.17 ensures that ϕ_ξ is cyclic for $T(U,\omega)$. We consider the diagonal matrix coefficient $\varphi_{\phi_\xi,\phi_\xi}(\cdot) = \langle T(U,\omega)(\cdot)\phi_\xi, \phi_\xi\rangle$ of $T(U,\omega)$ and let

$$\varphi_{\xi,\xi} : G \to \mathbb{C} : g \mapsto \begin{cases} \langle \omega \circ p_U(g)\xi, \xi\rangle_{\mathcal{H}_\omega} & \text{if } g \in N_G(U) \\ 0 & \text{if } g \notin N_G(U) \end{cases}.$$

A straightforward computation shows for every $g \in G$ that

$$\begin{aligned} \varphi_{\phi_\xi,\phi_\xi}(g) &= \langle T(U,\omega)(g)\phi_\xi, \phi_\xi\rangle_{\mathrm{Ind}_{N_G(U)}^G(\mathcal{H}_{\omega\circ p_U})} \\ &= \sum_{tN_G(U)\in G/N_G(U)} \langle T(U,\omega)(g)\phi_\xi(t), \phi_\xi(t)\rangle \\ &= \sum_{tN_G(U)\in G/N_G(U)} \langle \phi_\xi(g^{-1}t), \phi_\xi(t)\rangle = \langle \phi_\xi(g^{-1}), \phi_\xi(1_G)\rangle = \varphi_{\xi,\xi}(g). \end{aligned} \tag{2.1}$$

We are going to show that $\varphi_{\xi,\xi}$ is a U-left-invariant function of $\mathcal{L}_\mathcal{S}(U)$. It is clear from the definition that $\varphi_{\xi,\xi}$ is U-bi-invariant, compactly supported and hence square integrable. Now, let W be conjugate to an element of $\mathcal{S}[l-1]$ and such that $U \subseteq W$. Since $\mathcal{S}$ factorises$^+$ at depth l, notice that $W \subseteq N_G(U)$. In particular, for every $g \notin N_G(U)$ we have that $gN_G(U) \cap N_G(U) = \varnothing$ and thus $\int_W \varphi_{\xi,\xi}(gh)\,\mathrm{d}\,\mu(h) = 0$. On the other hand, for every $g \in N_G(U)$, since ω is an $\mathcal{S}$-standard representation of $\mathrm{Aut}_G(\mathcal{C}(U))$ we have

$$\int_W \varphi_{\xi,\xi}(gh)\,\mathrm{d}\,\mu(h) = \langle \int_W \omega \circ p_U(h)\xi\,\mathrm{d}\,\mu(h), \omega \circ p_U(g^{-1})\xi\rangle_{\mathcal{H}_\omega} = 0.$$

This proves, as desired, that $\varphi_{\xi,\xi}$ is a U-left-invariant function of $\mathcal{L}_\mathcal{S}(U)$. In particular, Lemma 2.23 ensures that $\widehat{\varphi}_{\xi,\xi} \in \mathcal{L}_\mathcal{S}(U)$. Now, consider the diagonal matrix coefficient $\psi(\cdot) = \langle T_{\mathcal{S},U}(\cdot)\widehat{\varphi}_{\xi,\xi}, \widehat{\varphi}_{\xi,\xi}\rangle$ of $T_{\mathcal{S},U}$, let μ be the Haar measure of G renormalised in such a way that $\mu(N_G(U)) = 1$ and notice from [Dix77, Theorem 14.3.3] that

$$\begin{aligned} \psi(g) &= \int_G T_{\mathcal{S},U}(g)\widehat{\varphi}_{\xi,\xi}(h)\overline{\widehat{\varphi}_{\xi,\xi}(h)}\,\mathrm{d}\,\mu(h) = \int_{N_G(U)} \varphi_{\xi,\xi}(h^{-1}g)\overline{\varphi_{\xi,\xi}(h^{-1})}\,\mathrm{d}\,\mu(h) \\ &= \int_{N_G(U)} \varphi_{\xi,\xi}(hg)\overline{\varphi_{\xi,\xi}(h)}\,\mathrm{d}\,\mu(h) = d_\omega^{-1}\|\xi\|^2\varphi_{\xi,\xi}(g), \end{aligned} \tag{2.2}$$

where d_ω is the dimension of ω. In particular, renormalising ξ if needed, the equalities (2.1), (2.2) and the uniqueness of the GNS construction [BdlHV08, Theorem C.4.10] imply that $T(U,\omega)$ is a subrepresentation of $T_{\mathcal{S},U}$.

Now, let us show that $\mathfrak{D}^{\omega\circ p_U}_{N_G(U)} = \big(\mathrm{Ind}_{N_G(U)}^G(\mathcal{H}_{\omega\circ p_U})\big)^U = \mathcal{H}^U_{T(U,\omega)}$. Let

$$\mathcal{U} : \mathcal{H}_{T(U,\omega)} \to \mathcal{L}_\mathcal{S}(U)$$

be a unitary operator intertwining $T(U,\omega)$ and $T_{\mathcal{S},U}$ and notice that

$$\mathfrak{D}^{\omega\circ p_U}_{N_G(U)} \subseteq \mathcal{H}^U_{T(U,\omega)}.$$

Let $g \in G$ and $\phi \in [T(U,\omega)](g)\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}$. Since

$$[T(U,\omega)](g)\mathfrak{D}^{\omega\circ p_U}_{N_G(U)} \subseteq \mathcal{H}^{gUg^{-1}}_{T(U,\omega)},$$

the function $\mathcal{U}(\phi)$ is a gUg^{-1}-left invariant function of $\mathcal{L}_{\mathcal{S}}(U)$ and Lemma 2.1 ensures that it is supported inside

$$N_G(gUg^{-1},U)=\{t\in G|t^{-1}gUg^{-1}t\subseteq U\}=gN_G(U).$$

Now, let $\varphi\in\left(\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}\right)^{\perp}\cap\mathcal{H}^U_{T(U,\omega)}$. By Lemma 2.17, we have that

$$\left(\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}\right)^{\perp}=\overline{\bigoplus_{gN_G(U)\in G/N_G(U)-\{N_G(U)\}}\big[T(U,\omega)\big](g)\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}}.$$

In particular, the above discussion implies that $\operatorname{supp}(\mathcal{U}(\varphi))\subseteq G-N_G(U)$. On the other hand, since $\varphi\in\mathcal{H}^U_{T(U,\omega)}$, the function $\mathcal{U}(\varphi)$ is a U-left invariant function of $\mathcal{L}_{\mathcal{S}}(U)$ and is therefore supported inside $N_G(U)$. This implies that $\mathcal{U}(\varphi)=0$. Hence, $\varphi=0$ and $\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}=\mathcal{H}^U_{T(U,\omega)}$.

Finally, we prove the irreducibility of $T(U,\omega)$ with Proposition 2.18. Let M be a non-zero closed invariant subspace of $\mathcal{H}_{T(U,\omega)}$. Then $\mathcal{U}(M)$ is a non-zero closed invariant subspace of $\mathcal{L}_{\mathcal{S}}(U)$ and Lemma 2.11 ensures the existence of a non-zero U-bi-invariant function $\varphi\in\mathcal{U}(M)$. In particular, $\mathcal{U}^{-1}(\varphi)$ is a non-zero U-invariant function of $\mathcal{H}_{T(U,\omega)}$ contained in M. The result follows from the fact that $\mathcal{H}^U_{T(U,\omega)}=\mathfrak{D}^{\omega\circ p_U}_{N_G(U)}$. □

This proves ②. We now prove ③.

Lemma 2.25. *Let π be an irreducible representation of G with seed $\mathcal{C}(U)$. Then ω_π is an $\mathcal{S}$-standard representation of* $\operatorname{Aut}_G(\mathcal{C}(U))$ *(in particular it is irreducible) and*

$$\pi\ \simeq\ T(U,\omega_\pi)\ =\ \operatorname{Ind}^G_{N_G(U)}(\omega_\pi\circ p_U).$$

Proof. We start by showing that $(\omega_\pi,\mathcal{H}^U_\pi)$ is an $\mathcal{S}$-standard representation of $\operatorname{Aut}_G(\mathcal{C}(U))$. Let M be a non-zero closed $\operatorname{Aut}_G(\mathcal{C}(U))$-invariant subspace of $\mathcal{H}^U_\pi$ for ω_π and let $\xi\in\mathcal{H}^U_\pi$ be a non-zero vector. Since ξ is cyclic for π the function $\varphi_{\xi,\eta}:G\to\mathbb{C}:g\mapsto\langle\pi(g)\xi,\eta\rangle$ is not identically zero for every non-zero $\eta\in M$. On the other hand, Lemma 2.1 ensures that this function is supported inside $N_G(U)$. In particular, there exists an element $g\in N_G(U)$ such that $0\neq\varphi_{\xi,\eta}(g)=\langle\xi,\omega_\pi\circ p_U(g^{-1})\eta\rangle$. Since M is $\operatorname{Aut}_G(\mathcal{C}(U))$-invariant this proves the existence of a vector $\eta'\in M$ such that $\langle\xi,\eta'\rangle\neq0$. It follows that the orthogonal complement of M in $\mathcal{H}^U_\pi$ is trivial. This proves that $M=\mathcal{H}^U_\pi$ and ω_π is irreducible. Now, let $W\in\tilde{\mathfrak{H}}_{\mathcal{S}}(U)$ where we recall that

$$\tilde{\mathfrak{H}}_{\mathcal{S}}(U)=\{W\mid\exists g\in G\text{ s.t. }gWg^{-1}\in\mathcal{S}[l-1]\text{ and }U\subseteq W\}.$$

Since $\pi(g)=\omega_\pi\circ p_U(g)$ for every $g\in N_G(U)$ and since π has seed C, there does not exist any non-zero W-invariant vector in $\mathcal{H}^U_\pi$ for $\omega_\pi\circ p_U$. This proves that ω_π is $\mathcal{S}$-standard.

We now prove that $\pi\simeq T(U,\omega_\pi)$. Let $\xi\in\mathcal{H}^U_\pi$ and consider the function $\varphi_{\xi,\xi}:G\to\mathbb{C}:g\mapsto\langle\pi(g)\xi,\xi\rangle$. The proof of Lemma 2.3 ensures that $\varphi_{\xi,\xi}$ is a U-bi-invariant function of $\mathcal{L}_{\mathcal{S}}(U)$ and is therefore compactly supported inside $N_G(U)$. In particular, we have that

$$\varphi_{\xi,\xi}(g)=\begin{cases}\langle\omega_\pi\circ p_U(g)\xi,\xi\rangle & \text{if } g\in N_G(U)\\ 0 & \text{if } g\notin N_G(U)\end{cases}.$$

On the other hand, the Equality (2.1) ensures that $\varphi_{\xi,\xi}$ is a diagonal matrix coefficient of $\mathrm{Ind}_{N_G(U)}^G(\omega_\pi \circ p_U)$. Since $\mathrm{Ind}_{N_G(U)}^G(\omega_\pi \circ p_U)$ is irreducible by Proposition 2.24, the result follows from the uniqueness of the GNS construction [BdlHV08, Theorem C.4.10]. □

Lemma 2.26. *Let ω_1, ω_2 be $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(\mathcal{C}(U))$. Then, $T(U,\omega_1)$ and $T(U,\omega_2)$ are equivalent if and only if ω_1 and ω_2 are equivalent.*

Proof. Proposition 2.24 ensures that $\mathfrak{D}_{N_G(U)}^{\omega_i \circ p_U} = \mathcal{H}_{T(U,\omega_i)}^U$. The result thus follows from Lemma 2.19 applied with $H = N_G(U)$ and $K = U$. □

2.5. **Some existence criteria.** Let G be a non-discrete unimodular totally disconnected locally compact group and $\mathcal{S}$ be a generic filtration of G. If $\mathcal{S}$ factorises$^+$ at depth l, Theorem A provides a bijective correspondence between the equivalence classes of irreducible representations of G at depth l with seed $C \in \mathcal{F}_{\mathcal{S}}$ and the $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(C)$. However, it does not guarantee the existence of an irreducible representation of G at depth l. The purpose of this section is to provide some existence criteria that will be used in the second part of the paper. The following results were used in [FTN91] to prove the existence of cuspidal representations of the full group of automorphisms $\mathrm{Aut}(T)$ of a regular tree and their proofs are essentially covered by [FTN91, Lemma 3.6, Lemma 3.7, Lemma 3.8 and Theorem 3.9] but we recall them for completeness of the argument.

Lemma 2.27. *Let Q be a finite group with $|Q| \gneq 2$ acting 2-transitively on a finite set $X = \{1, \ldots, d\}$. There exists an irreducible representation of Q without non-zero $\mathrm{Fix}_Q(i)$-invariant vector for all $i \in X$.*

Proof. Since Q acts transitively on X, notice that $\mathrm{Fix}_Q(i)$ and $\mathrm{Fix}_Q(j)$ are conjugate to one another for all $i, j \in X$. In particular, a representation π of Q admits a non-zero $\mathrm{Fix}_Q(i)$-invariant vector for all $i \in X$ if and only if it admits a non-zero $\mathrm{Fix}_Q(1)$-invariant vector. In light of these considerations we are going to prove the existence of an irreducible representation of Q without non-zero $\mathrm{Fix}_Q(1)$-invariant vectors. We recall that the quasi-regular representation σ of $Q/\mathrm{Fix}_Q(1)$ is the representation $\mathrm{Ind}_{\mathrm{Fix}_Q(1)}^Q(1_{\widehat{\mathrm{Fix}_Q(1)}})$ of Q induced by the trivial representation $1_{\widehat{\mathrm{Fix}_Q(1)}}$ of $\mathrm{Fix}_Q(1)$. On the other hand, for every representation π of Q, the Frobenius reciprocity implies that

$$\langle \mathrm{Res}_{\mathrm{Fix}_Q(1)}^Q(\pi), 1_{\widehat{\mathrm{Fix}_Q(1)}} \rangle_{\mathrm{Fix}_Q(1)} = \langle \pi, \mathrm{Ind}_{\mathrm{Fix}_Q(1)}^Q(1_{\widehat{\mathrm{Fix}_Q(1)}}) \rangle_Q = \langle \pi, \sigma \rangle_Q.$$

In particular, every irreducible representation π of Q with a non-zero $\mathrm{Fix}_Q(1)$-invariant vector is a subrepresentation of σ. Moreover, since Q acts 2-transitively on X, [Isa76, Corollary 5.17] ensures the existence of an irreducible representation ρ of Q such that $\sigma = 1_{\widehat{Q}} \oplus \rho$. Suppose for a contradiction that every irreducible representation of Q has a non-zero $\mathrm{Fix}_Q(1)$-invariant vector and is therefore contained in σ. This implies that Q has two conjugacy classes and is therefore isomorphic to the cyclic group of order two which contradicts our hypothesis that $|Q| \gneq 2$. □

The following result provides another useful criterion that we adapt below to the context of trees.

Lemma 2.28 ([FTN91, Lemma 3.7])**.** *Let Q be a finite group, let $H \leq Q$ be a direct product $H_1 \times H_2 \times \cdots \times H_s$ of non-trivial subgroups H_i of Q and suppose that the group of inner automorphisms of Q acts by permutation on the set*

$\{H_1, \dots, H_s\}$. Then, there exists an irreducible representation π of Q without non-zero H_i-invariant vectors for every $i = 1, \dots, s$.

For every locally finite tree T and each subtree $\mathcal{T} \subseteq T$ we set

$$\mathrm{Stab}_G(\mathcal{T}) = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} \text{ and } \mathrm{Fix}_G(\mathcal{T}) = \{g \in G \mid gv = v \ \forall v \in V(\mathcal{T})\}.$$

We obtain Proposition 2.29.

Proposition 2.29. *Let T be a locally finite tree, $G \leq \mathrm{Aut}(T)$ be a closed subgroup, $\mathcal{T}$ be a finite subtree of T and $\{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s\}$ be a set of distinct finite subtrees of T contained in $\mathcal{T}$ such that $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$ for every $i \neq j$. Suppose that $\mathrm{Stab}_G(\mathcal{T})$ acts by permutation on the set $\{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_s\}$ and that $\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i) \subsetneq \mathrm{Stab}_G(\mathcal{T})$. Then, there exists an irreducible representation of $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ without non-zero $\mathrm{Fix}_G(\mathcal{T}_i)/\mathrm{Fix}_G(\mathcal{T})$-invariant vector for every $i = 1, \dots, s$.*

Proof. Since $\mathcal{T}$ is a finite subtree of T, $\mathrm{Stab}_G(\mathcal{T})$ and $\mathrm{Fix}_G(\mathcal{T})$ are compact open subgroups of G. As $\mathrm{Stab}_G(\mathcal{T})$ is the normaliser of $\mathrm{Fix}_G(\mathcal{T})$, $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ is a finite group and our hypothesis ensures that every $H_i = \mathrm{Fix}_G(\mathcal{T}_i)/\mathrm{Fix}_G(\mathcal{T})$ is a non-trivial subgroup of $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$. On the other hand, for every $i \neq j$, $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$ which ensures that $H_i \cap H_j = \{1_{\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})}\}$ and that the supports of elements of $\mathrm{Fix}_G(\mathcal{T}_i)$ and $\mathrm{Fix}_G(\mathcal{T}_j)$ are disjoint from one another. This implies that the elements of H_i and H_j commute with one another and that the subgroup of $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ generated by $\bigcup_{i=1}^s H_i$ is isomorphic to $H_1 \times H_2 \times \cdots \times H_s$. Since the elements of $\mathrm{Stab}_G(\mathcal{T})$ act by permutation on $\{\mathcal{T}_1, \dots, \mathcal{T}_s\}$, notice that the group of inner automorphisms of $\mathrm{Stab}_G(\mathcal{T})$ acts by permutation on $\{\mathrm{Fix}_G(\mathcal{T}_1), \dots, \mathrm{Fix}_G(\mathcal{T}_s)\}$. The result follows from Lemma 2.28. □

Part 2. Applications

3. The full group of automorphisms of a semi-regular tree

The purpose of this section is to apply our axiomatic framework to the full group of automorphisms $\mathrm{Aut}(T)$ of a thick semi-regular tree T and more generally to groups of automorphisms of trees satisfying the Tits independence property (Definition 3.6). This section is therefore redundant from the point of view of new results (see [Ol′77],[Ama03]). It serves instead as a section allowing the reader to interpret the machinery developed in the first part of these notes in a concrete and well-understood case.

Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$, let $\mathrm{Aut}(T)$ be the group of automorphisms of T equipped with the permutation topology and let $\mathfrak{T}$ be the set of non-empty complete finite subtrees of T.

Definition 3.1. A closed subgroup $G \leq \mathrm{Aut}(T)$ is said to satisfy the hypothesis H if for all $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}$ we have that

$$(H) \qquad \mathrm{Fix}_G(\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T}) \text{ if and only if } \mathcal{T} \subseteq \mathcal{T}'.$$

For such groups, we are going to show that $\mathcal{S}$ is a generic filtration and that:

- $\mathcal{S}[0] = \{\mathrm{Fix}_G(v) | v \in V(T)\}$.
- $\mathcal{S}[1] = \{\mathrm{Fix}_G(e) | e \in E(T)\}$.
- $\mathcal{S}[l] = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T} \text{ and } \mathcal{T} \text{ has } l-1 \text{ interior vertices}\} \quad \forall l \geq 2$.

In particular, the spherical, special and cuspidal representations as defined on page 357 correspond respectively to the irreducible representations at depths 0, 1 and at least 2. The purpose of this section is to prove Theorem 3.2.

Theorem 3.2. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H and the Tits independence property, then $\mathcal{S}$ is a generic filtration of G factorising$^+$ at all depths $l \geq 2$.*

Together with Theorem A, this provides a description of the equivalence classes of cuspidal representations in terms of $\mathcal{S}$-standard representations of the group of automorphisms of the corresponding seed. In Section 4, we will give examples of different generic filtrations that also factorise$^+$ for $\mathrm{Aut}(T)$. These will lead to different descriptions of cuspidal representations of $\mathrm{Aut}(T)$.

Remark 3.3. If G is locally 2-transitive, the above generic filtration $\mathcal{S}$ does not factorise at depth 1. Indeed, let e and f be two different edges containing a common vertex $v \in V(T)$, let $U = \mathrm{Fix}_G(e)$, $V = \mathrm{Fix}_G(f)$ and let W be in the conjugacy class of an element of $\mathcal{S}[0]$ such that $U \subseteq W$. We are going to show that $W \not\subseteq VU$. From the definition of W, Lemma 3.4 ensures the existence of a vertex $w \in V(T)$ such that $W = \mathrm{Fix}_G(w)$. Since $U \subseteq W$ and since G satisfies the hypothesis H, we must have $w \in e$. However, if $w \neq v$, there exists an element $g \in W$ that does not fix v (because $\mathrm{Fix}_G(e) \neq \mathrm{Fix}_G(v)$) but every element of VU fixes v. Moreover, if $w = v$, there exists $g \in W$ such that $ge = f$ (because G is locally 2-transitive) but no element of VU maps e to f. In both cases, this proves that $W \not\subseteq VU$ and therefore that $\mathcal{S}$ does not factorise at depth 1.

3.1. **Preliminaries.** The purpose of this section is to settle our formalism on trees and their groups of automorphisms. If Γ is a graph, we denote by $V(\Gamma)$ its set of vertices, by $E(\Gamma)$ its set of edges and we equip $V(\Gamma)$ with the metric d_Γ given by the length of geodesics. An isometry $g : V(\Gamma) \to V(\Gamma)$ for this metric is called an **automorphism** of Γ and we denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ. This group embeds naturally in $\mathrm{Sym}(V(\Gamma))$ and is therefore naturally equipped with the **permutation topology**. A basis of neighbourhoods of the identity of $\mathrm{Aut}(\Gamma)$ for this topology is given by the sets

$$\mathrm{Fix}_{\mathrm{Aut}(\Gamma)}(F) = \{g \in \mathrm{Aut}(\Gamma) \mid gv = v \ \forall v \in F\},$$

where $F \subseteq V(\Gamma)$ is a finite set of vertices. We recall that for a locally finite graph Γ, each $\mathrm{Fix}_{\mathrm{Aut}(\Gamma)}(F)$ with finite $F \subseteq V(\Gamma)$ is a compact open subgroup of $\mathrm{Aut}(\Gamma)$ and that $\mathrm{Aut}(\Gamma)$ is a second-countable totally disconnected locally compact group. A **tree** is connected graph with no loops nor cycles and it is **thick** if the degree of each vertex is at least 3. A **subtree** $\mathcal{T}$ of T is a connected graph that is a connected subgraph of T in the sense that $V(\mathcal{T}) \subseteq V(T)$ and $E(\mathcal{T}) \subseteq E(T)$. Notice that a subtree $\mathcal{T}$ of T is completely determined by its set of vertices. Therefore, when it leads to no confusion, we identify $\mathcal{T}$ with its set of vertices. A tree T is called (d_0, d_1)-**semi-regular** if there exists a bipartition $V(T) = V_0 \sqcup V_1$ of T such that each vertex of V_i has degree d_i and every edge of T contains exactly one vertex in each V_i. Finally, we recall that for a thick semi-regular tree T, the group $\mathrm{Aut}(T)$ is non-discrete and unimodular.

3.2. **Factorisation of the generic filtration $\mathcal{S}$.** The purpose of this section is to prove Theorem 3.2. Let T be a thick semi-regular tree. Let G be a closed subgroup of $\mathrm{Aut}(T)$, let $\mathfrak{T}$ be the set of non-empty complete finite subtrees of T and let $\mathcal{S} = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}\}$. Lemma 3.4 ensures that $\mathcal{S}$ is a generic filtration of G and identifies the elements at height l with respect to $\mathcal{S}$ if G satisfies the hypothesis H.

Lemma 3.4. *Let G be a closed non-discrete unimodular subgroup of* $\mathrm{Aut}(T)$ *satisfying the hypothesis H. Then, $\mathcal{S}$ is a generic filtration of G and:*

- $\mathcal{S}[0] = \{\mathrm{Fix}_G(v) | v \in V(T)\}$.
- $\mathcal{S}[1] = \{\mathrm{Fix}_G(e) | e \in E(T)\}$.
- $\mathcal{S}[l] = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T} \text{ has } l-1 \text{ interior vertices}\} \quad \forall l \geq 2$

Proof. For every $\mathcal{T} \in \mathfrak{T}$ and each $g \in \mathrm{Aut}(T)$, notice that $g\,\mathrm{Fix}_G(\mathcal{T})g^{-1}$ coincides with the pointwise stabiliser $\mathrm{Fix}_G(g\mathcal{T})$. In particular, the elements of $\mathcal{F}_{\mathcal{S}} = \{\mathcal{C}(U) | U \in \mathcal{S}\}$ are of the form

$$\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})) = \{\mathrm{Fix}_G(g\mathcal{T}) | g \in G\}$$

with $\mathcal{T} \in \mathfrak{T}$. Thus, for all $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}$ we have that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exists $g \in G$ such that $\mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(g\mathcal{T}')$. Since G satisfies the hypothesis H, this implies that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exists $g \in G$ such that $g\mathcal{T}' \subseteq \mathcal{T}$. In particular, since $\mathfrak{T}$ is stable under the action of $\mathrm{Aut}(T)$, for every chain $C_0 \lneq C_1 \lneq \cdots \lneq C_{n-1} \lneq C_n$ of elements of $\mathcal{F}_{\mathcal{S}}$, there exists a chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{n-1} \subsetneq \mathcal{T}_n$ of elements of $\mathfrak{T}$ such that $C_r = \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_r))$. Reciprocally, for every chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{n-1} \subsetneq \mathcal{T}$ of elements of $\mathfrak{T}$ contained in a subtree $\mathcal{T} \in \mathfrak{T}$ we obtain a chain $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_0)) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_1)) \lneq \cdots \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_{n-1})) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ of elements of $\mathcal{F}_{\mathcal{S}}$ with maximal element $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$. This proves that the height of $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ is the maximal length of a strictly increasing chain of elements of $\mathfrak{T}$ contained in $\mathcal{T}$. The result then follows from the following observations. If $\mathcal{T}$ is a vertex, it does not contain any non-empty proper subtree. If $\mathcal{T}$ is an edge, every maximal strictly increasing chain of non-empty complete subtree of $\mathcal{T}$ is of the form $\mathcal{T}_0 \subsetneq \mathcal{T}$ where $\mathcal{T}_0$ is a vertex. If $\mathcal{T}$ is a complete finite subtree of T with $n \geq 1$ interior vertices, every maximal strictly increasing chain of non-empty complete subtrees of $\mathcal{T}$ is of the form $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_n \subsetneq \mathcal{T}$ where $\mathcal{T}_0$ is a vertex, $\mathcal{T}_1$ is an edge and $\mathcal{T}_i$ contains $i-1$ interior vertices for every $i \geq 2$. □

Lemma 3.5. $\mathrm{Aut}(T)$ *satisfies the hypothesis H.*

Proof. It is clear that $\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}') \subseteq \mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T})$ if $\mathcal{T} \subseteq \mathcal{T}'$. Now, suppose that $\mathcal{T} \not\subseteq \mathcal{T}'$ and let us show that $\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}') \not\subseteq \mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T})$. Since $\mathcal{T} \not\subseteq \mathcal{T}'$ there exists a vertex $v_{\mathcal{T}} \in V(\mathcal{T}) - V(\mathcal{T}')$. We claim the existence of an element of $\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}')$ that does not fix $v_{\mathcal{T}}$. If $\mathcal{T}'$ is a vertex, this is obvious since $\mathrm{Fix}_G(\mathcal{T}')$ acts transitively on the vertices at distance n from $\mathcal{T}'$. On the other hand, if $\mathcal{T}'$ contains at least two vertices, there exists a unique vertex $v'_{\mathcal{T}} \in V(\mathcal{T}')$ that is closer to $v_{\mathcal{T}}$ than every other vertex of $\mathcal{T}'$ and there exists a unique vertex $w'_{\mathcal{T}}$ of $\mathcal{T}'$ that is at distance one from $v'_{\mathcal{T}}$. Furthermore, notice that $\mathcal{T}'$ is a subtree of the half-tree

$$T(w_{\mathcal{T}}, v'_{\mathcal{T}}) \cup \{v'_{\mathcal{T}}\} = \{v \in V(T) : d_T(w_{\mathcal{T}}, v) < d_T(v'_{\mathcal{T}}, v)\} \cup \{v'_{\mathcal{T}}\}.$$

It follows that $\mathrm{Fix}_{\mathrm{Aut}(T)}(T(w_{\mathcal{T}}, v'_{\mathcal{T}})) \subseteq \mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}')$.

On the other hand, $\mathrm{Fix}_{\mathrm{Aut}(T)}(T(w_{\mathcal{T}}, v'_{\mathcal{T}}))$ acts transitively on the set

$$\{v \in V(T) - T(w_{\mathcal{T}}, v'_{\mathcal{T}}) : d_T(v'_{\mathcal{T}}, v) = d_T(v'_{\mathcal{T}}, v_{\mathcal{T}})\}.$$

Since T is thick, this set of vertices contains $v_{\mathcal{T}}$ and at least one other vertex. This proves the existence of an element of $\mathrm{Fix}_{\mathrm{Aut}(T)}(T(w_{\mathcal{T}}, v'_{\mathcal{T}}))$ which does not fix $v_{\mathcal{T}}$. It follows that $\mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T}') \nsubseteq \mathrm{Fix}_{\mathrm{Aut}(T)}(\mathcal{T})$. □

Our next task is to prove that $\mathcal{S}$ factorises$^+$ at all depths $l \geq 2$ for groups satisfying the Tits independence property.

Definition 3.6. A group $G \leq \mathrm{Aut}(T)$ satisfies the **Tits independence property** if for any two adjacent vertices $v, v' \in V(T)$, the pointwise stabiliser of the edge $\{v, v'\}$ satisfies

$$\mathrm{Fix}_G(\{v, v'\}) = \mathrm{Fix}_G(T(v, v'))\, \mathrm{Fix}_G(T(v', v)),$$

where $T(w, v) = \{u \in V(T) | d_T(w, u) \lneq d_T(v, u)\}$.

In fact, if G satisfies the Tits independence property and if $\mathcal{T}$ is a complete proper subtree of T containing an edge we have that

$$\mathrm{Fix}_G(\mathcal{T}) = \prod_{f \in \partial E_o(\mathcal{T})} \mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{T}), \tag{3.1}$$

where $\partial E_o(\mathcal{T})$ denotes the set of terminal edges of $\mathcal{T}$ oriented in such a way that the terminal vertex of any $f \in \partial E_o(\mathcal{T})$ is a leaf of $\mathcal{T}$ and where Tf denotes the half-tree of T of vertices which are closer to the origin of f than to its terminal vertex. More general versions of this result are given by Lemma 5.8 and Proposition 4.7. The following result is well known by the expert but we prove it for completeness of the argument. We refer to [FTN91, Lemma 3.1] for the full group of automorphisms of a regular tree and to [Ama03, Lemma 19] for groups satisfying the Tits independence property.

Lemma 3.7. *Let $G \leq \mathrm{Aut}(T)$ be a subgroup satisfying the Tits independence property, let $\mathcal{T}$, $\mathcal{T}'$ be complete finite subtrees of T such that $\mathcal{T}$ has at least one interior vertex and such that $\mathcal{T}'$ does not contain $\mathcal{T}$. Then, there exists a complete proper subtree $\mathcal{R}$ of $\mathcal{T}$ such that $\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\, \mathrm{Fix}_G(\mathcal{T})$. Furthermore, if G satisfies in addition the hypothesis H, $\mathcal{R}$ can be chosen in such a way that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{R}))$ has the height of $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ less 1 in $\mathcal{F}_{\mathcal{S}}$.*

Proof. Since $\mathcal{T}$ and $\mathcal{T}'$ are complete and since $\mathcal{T}'$ does not contain $\mathcal{T}$ there exists a vertex $w_{\mathcal{T}}$ of $\mathcal{T}$ such that all vertices adjacent to $w_{\mathcal{T}}$ but possibly one are leaves of $\mathcal{T}$ and none of those leaves is a vertex of $\mathcal{T}'$. Since $\mathcal{T}$ is complete it contains every oriented edge of T with terminal vertex $w_{\mathcal{T}}$. Furthermore, for one of those oriented edges that we denote by e, the half-tree $Te \cup e$ contains $\mathcal{T}'$. Let $\mathcal{R} = [Te \cup e] \cap \mathcal{T}$. Since $\mathcal{R}$ is complete, contains an edge and since G satisfies the Tits independence property, notice that

$$\mathrm{Fix}_G(\mathcal{R}) = \big[\mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{R})\big] \prod_{f \in \partial E_o(\mathcal{R}) - \{e\}} \big[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R})\big].$$

On the other hand, $\forall f \in \partial E_o(\mathcal{R}) - \{e\}$ we have that $\mathcal{T} \subseteq Tf$ and therefore that $\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T})$. Furthermore, since $\mathcal{T}' \subseteq Te \cup e$, we obtain that $\mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')$. The desired inclusion follows. Now, suppose that G satisfies the hypothesis H. The first part of the proof ensures the existence of a complete proper subtree $\mathcal{P}$ of $\mathcal{T}$ such that

$$\mathrm{Fix}_G(\mathcal{P}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\, \mathrm{Fix}_G(\mathcal{T}).$$

If $\mathcal{T}$ has exactly one interior vertex, there exists an edge $\mathcal{R}$ in $\mathcal{T}$ such that $\mathcal{P} \subseteq \mathcal{R} \subsetneq \mathcal{T}$. On the other hand, if $\mathcal{T}$ has more than one interior vertex, there exists a complete subtree $\mathcal{R}$ of $\mathcal{T}$ with one less interior vertex than $\mathcal{T}$ such that $\mathcal{P} \subseteq \mathcal{R} \subsetneq \mathcal{T}$. In both cases, this implies that $\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{P}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T})$ and $\mathcal{R}$ is as desired. □

Proof of Theorem 3.2. To prove that $\mathcal{S}$ factorises$^+$ a depth $l \geq 2$, we shall successively verify the three conditions of Definition 1.5.

First, we need to prove that for all U in the conjugacy class of an element of $\mathcal{S}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}$ such that $V \not\subseteq U$, there exists a W in the conjugacy class of an element of $\mathcal{S}[l-1]$ and $U \subseteq W \subseteq VU$. Let U and V be as above. Since $\mathfrak{T}$ is stable under the action of G, since G satisfies the hypothesis H and as a consequence of Lemma 3.4 there exist subtrees $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}$ such that $\mathcal{T}$ has $l-1$ interior vertices, $\mathcal{T}'$ does not contain $\mathcal{T}$, $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. Hence, as a consequence of Lemma 3.7, there exists a proper subtree $\mathcal{R}$ of $\mathcal{T}$ such that $\mathrm{Fix}_G(\mathcal{R})$ is conjugate to an element of $\mathcal{S}[l-1]$ and

$$\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

This proves the first condition. Next, we need to prove that

$$N_G(U,V) = \{g \in G | g^{-1}Vg \subseteq U\}$$

is compact for every V in the conjugacy class of an element of $\mathcal{S}$. Just as before, notice that $V = \mathrm{Fix}_G(\mathcal{T}')$ for some $\mathcal{T}' \in \mathfrak{T}$. Furthermore, since G satisfies the hypothesis H we have

$$\begin{aligned} N_G(U,V) &= \{g \in G | g^{-1}Vg \subseteq U\} = \{g \in G | g^{-1}\,\mathrm{Fix}_G(\mathcal{T}')g \subseteq \mathrm{Fix}_G(\mathcal{T})\} \\ &= \{g \in G | \mathrm{Fix}_G(g^{-1}\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T})\} = \{g \in G | g\mathcal{T} \subseteq \mathcal{T}'\} \end{aligned}$$

and this set is compact since both $\mathcal{T}$ and $\mathcal{T}'$ are finite. This proves the second condition. Finally, we need to prove that for any subgroup W in the conjugacy class of an element of $\mathcal{S}[l-1]$ such that $U \subseteq W$ we have

$$W \subseteq N_G(U,U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

For the same reasons as before, there exists a complete subtree $\mathcal{R} \in \mathfrak{T}$ such that $W = \mathrm{Fix}_G(\mathcal{R})$. Furthermore, since $U \subseteq W$ and since G satisfies the hypothesis H, we have that $\mathcal{R} \subseteq \mathcal{T}$. On the other hand, since $\mathcal{R}$ and $\mathcal{T}$ are both complete finite subtrees and since $\mathcal{R}$ has exactly one less interior vertex than $\mathcal{T}$, every interior vertex of $\mathcal{T}$ belongs to $\mathcal{R}$. Hence, $g\mathcal{T} \subseteq \mathcal{T}$ for every $g \in \mathrm{Fix}_G(\mathcal{R})$. This implies that

$$\begin{aligned} W = \mathrm{Fix}_G(\mathcal{R}) &\subseteq \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} = \{g \in G | \mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(g\mathcal{T})\} \\ &= \{g \in G | g^{-1}\,\mathrm{Fix}_G(\mathcal{T})g \subseteq \mathrm{Fix}_G(\mathcal{T})\} \\ &= N_G(\mathrm{Fix}_G(\mathcal{T}), \mathrm{Fix}_G(\mathcal{T})) = N_G(U,U). \end{aligned}$$

□

In particular, for every complete finite subtree $\mathcal{T}$ containing an interior vertex, Theorem A provides a bijective correspondence between the equivalence classes of cuspidal representations of G with seed $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ and the equivalence classes of $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})))$. On the other hand, notice from the above computations that $\mathrm{Aut}_G(\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})))$ can be identified with the group of automorphisms of $\mathcal{T}$ coming from the action of $\mathrm{Stab}_G(\mathcal{T}) = \{g \in G | g\mathcal{T} \subseteq$

$\mathcal{T}\}$ on $\mathcal{T}$ and that, under this identification, the $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})))$ are the irreducible representations that do not admit a non-zero invariant vector for the pointwise stabiliser of any maximal proper complete subtree of $\mathcal{T}$. The existence of $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(C)$ is shown for any seed C at height $l \geq 2$ in [FTN91, Theorem 3.9].

4. Groups of automorphisms of trees with the property IP_k

In this section, we apply our machinery to groups of automorphisms of semi-regular trees satisfying the property IP_k. In particular, the purpose of this section is to prove Theorem B and Theorem C. We use the same notations and terminology as in Section 3. Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$. For every subtree $\mathcal{T}$ of T we denote by $B_T(\mathcal{T}, r)$ or simply by $\mathcal{T}^{(r)}$ the ball of radius $r \geq 0$ around $\mathcal{T}$ that is

$$\mathcal{T}^{(r)} = \{v \in V(T) : \exists w \in V(\mathcal{T}) \text{ s.t. } d_T(v, w) \leq r\}.$$

In addition, for each subtree $\mathcal{T}$ of T we let

$$E_o(\mathcal{T}) = \{(v, w) : v, w \in V(\mathcal{T}),\ v \neq w\}$$

be the set of ordered pairs of distinct adjacent vertices of $\mathcal{T}$. The elements of $E_o(\mathcal{T})$ are called the **oriented edges** of $\mathcal{T}$. For every oriented edge $f = (w, v) \in E_o(\mathcal{T})$, we let $\bar{f} = (v, w)$ be the oriented edge with opposite orientation and we say that w and v are respectively the **origin** and the **terminal** vertex of f. Finally, for every oriented edge $f = (w, v) \in E_o(T)$, we let

$$Tf = T(w, v) = \{u \in V : d_T(w, u) < d_T(v, u)\}.$$

Definition 4.1. Let $k \geq 1$ be a positive integer. A group $G \leq \mathrm{Aut}(T)$ satisfies the **property** IP_k if for all $e \in E_o(T)$ we have that

$$(\mathrm{IP}_k) \qquad \mathrm{Fix}_G(e^{(k-1)}) = \big[\mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(e^{(k-1)})\big]\big[\mathrm{Fix}_G(T\bar{e}) \cap \mathrm{Fix}_G(e^{(k-1)})\big].$$

In particular, under our convention, IP_1 is the Tits independence property as defined in Section 3. Theorem B states the following.

Theorem. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the property IP_k for some $k \geq 1$, $\mathcal{P}$ be a complete finite subtree of T containing an interior vertex, $\Sigma_{\mathcal{P}}$ be the set of maximal complete proper subtrees of $\mathcal{P}$ and*

$$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{R} \in \Sigma_{\mathcal{P}} | \mathrm{Fix}_G((\mathcal{R}')^{(k-1)}) \not\subseteq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})\ \forall \mathcal{R}' \in \Sigma_{\mathcal{P}} - \{\mathcal{R}\}\}.$$

Suppose in addition that:

1. $\forall \mathcal{R}, \mathcal{R}' \in \mathfrak{T}_{\mathcal{P}}, \forall g \in G$, *we do not have* $\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subsetneq \mathrm{Fix}_G(g(\mathcal{R}')^{(k-1)})$.
2. *For all* $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$, $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$. *Furthermore, if* $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \subsetneq \mathrm{Fix}_G(g\mathcal{R}^{(k-1)})$ *we have* $\mathcal{P} \subseteq g\mathcal{R}^{(k-1)}$.
3. $\forall n \in \mathbb{N}, \forall v \in V(T)$, $\mathrm{Fix}_G(v^{(n)}) \subseteq \mathrm{Fix}_G(\mathcal{P}^{(k-1)})$ *implies* $\mathcal{P}^{(k-1)} \subseteq v^{(n)}$.
4. *For every* $g \in G$ *such that* $g\mathcal{P} \neq \mathcal{P}$, $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mathrm{Fix}_G(g\mathcal{P}^{(k-1)})$.

Then, there exists a generic filtration $\mathcal{S}_{\mathcal{P}}$ of G factorising$^+$ at depth 1 with

$$\mathcal{S}_{\mathcal{P}}[0] = \{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\},$$
$$\mathcal{S}_{\mathcal{P}}[1] = \{\mathrm{Fix}_G(\mathcal{P}^{(k-1)})\}.$$

In particular, Theorem A provides a description of the irreducibles of G admitting a non-zero $\mathrm{Fix}_G(\mathcal{P}^{(k-1)})$-invariant vector but which do not admit a non-zero $\mathrm{Fix}_G(\mathcal{R}^{(k-1)})$-invariant vector for any $\mathcal{R} \in \Sigma_{\mathcal{P}}$.

Under stronger hypothesis on G, Theorem C explicits a generic filtration factorising$^+$ at all depths greater than a certain constant. To be more precise, let $q \in \mathbb{N}$ be a non-negative integer. If q is even, let

$$\mathfrak{T}_q = \left\{ B_T(v,r) \middle| v \in V(T), r \geq \frac{q}{2} + 1 \right\} \sqcup \left\{ B_T(e,r) \middle| e \in E(T), r \geq \frac{q}{2} \right\}.$$

If q is odd, let

$$\mathfrak{T}_q = \left\{ B_T(v,r) \middle| v \in V(T), r \geq \frac{q+1}{2} \right\} \sqcup \left\{ B_T(e,r) \middle| e \in E(T), r \geq \frac{q+1}{2} \right\}.$$

For any closed subgroup $G \leq \mathrm{Aut}(T)$, we consider the set

$$\mathcal{S}_q = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_q\}$$

of pointwise stabilisers of those subtrees.

Definition 4.2. A group $G \leq \mathrm{Aut}(T)$ is said to satisfy the hypothesis H_q if for all $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_q$ we have that

$$(H_q) \qquad \mathrm{Fix}_G(\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T}) \text{ if and only if } \mathcal{T} \subseteq \mathcal{T}'.$$

If $G \leq \mathrm{Aut}(T)$ is a closed non-discrete unimodular subgroup of $\mathrm{Aut}(T)$ satisfying the hypothesis H_q, Lemma 4.10 ensures that $\mathcal{S}_q$ is a generic filtration of G and:

- $\mathcal{S}_q[l] = \{\mathrm{Fix}_G(B_T(e, \frac{q+l}{2})) | e \in E(T)\}$ if $q + l$ is even.
- $\mathcal{S}_q[l] = \{\mathrm{Fix}_G(B_T(v, \frac{q+l+1}{2})) | v \in V(T)\}$ if $q + l$ is odd.

Theorem C states the following.

Theorem. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$ and let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_q and the property* IP_k *for some integers $q \geq 0$ and $k \geq 1$. Then $\mathcal{S}_q$ factorises$^+$ at all depths $l \geq L_{q,k}$ where*

$$L_{q,k} = \begin{cases} \max\{1, 2k - q - 1\} & \text{if } q \text{ is even.} \\ \max\{1, 2k - q\} & \text{if } q \text{ is odd.} \end{cases}$$

Together with Theorem A this provides a description of the irreducibles of G that do not admit any non-zero U-invariant vector for any $U \in \mathcal{S}_q$ with depth strictly less than $L_{q,k}$. For $G = \mathrm{Aut}(T)$, one can take $q = 0$ and $k = 1$ so that the generic filtration $\mathcal{S}_0$ factorises$^+$ at all positive depths. This provides a second description of the cuspidal representations of $\mathrm{Aut}(T)$. On the other hand, Theorem C also applies to groups that were not yet treated in the literature. For instance we have Corollary 4.3.

Corollary 4.3. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$ and let $G \leq$* $\mathrm{Aut}(T)$ *be a closed subgroup acting 2-transitively on the boundary ∂T and whose local action at each vertex contains the alternating group of corresponding degree. Then, there exists a constant $k \in \mathbb{N}$ such that the generic filtration $\mathcal{S}_0$ of G factorises$^+$ at all depths $l \geq 2k - 1$.*

In fact, if G is in addition simple, one can even show that the generic filtration $\mathcal{S}_0$ factorises$^+$ at all positive depths [Sem22] which leads to a classification of the cuspidal representations for these groups. However, the proof given in [Sem22] does not (and cannot) rely on the property IP_k.

4.1. **Preliminaries.** The purpose of this section is to provide some reminders and an equivalent characterisation of the property IP_k that will be useful to prove that the generic filtrations below factorise. Let T be a thick semi-regular tree and $k \geq 1$ be an integer. We recall from Definition 4.1 that a subgroup $G \leq \mathrm{Aut}(T)$ has the property IP_k if for every oriented edge $e \in E_o(T)$ we have

$$\mathrm{Fix}_G(e^{(k-1)}) = \left[\mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(e^{(k-1)})\right]\left[\mathrm{Fix}_G(T\bar{e}) \cap \mathrm{Fix}_G(e^{(k-1)})\right].$$

This notion has been extensively studied in [BEW15] and we now recall some of its properties. We start with the following result.

Lemma 4.4 ([BEW15, Proposition 5.3.]). *A group $G \leq \mathrm{Aut}(T)$ satisfying IP_k satisfies $\mathrm{IP}_{k'}$ for every $k' \geq k$.*

In particular, groups satisfying the Tits independence property satisfy the property IP_k for all $k \geq 1$. Furthermore, natural examples of groups satisfying the property IP_k can be constructed by the operation of k-closure.

Definition 4.5. Let $G \leq \mathrm{Aut}(T)$. The k-**closure** $G^{(k)}$ of G in $\mathrm{Aut}(T)$ is

$$G^{(k)} = \left\{h \in \mathrm{Aut}(T) \mid \forall v \in V(T),\ \exists g \in G \text{ with } h\big|_{B_T(v,k)} = g\big|_{B_T(v,k)}\right\}.$$

A group is k-**closed** if it coincides with its k-closure.

Lemma 4.6 ([BEW15, Lemma 3.2, Lemma 3.4 and Proposition 5.2]). *Let $G \leq \mathrm{Aut}(T)$ and let $k \geq 1$ be an integer. The k-closure $G^{(k)}$ is a closed subgroup of $\mathrm{Aut}(T)$ containing G and satisfying the property IP_k.*

Our current purpose is to prove Proposition 4.8 which plays a similar role than Lemma 3.7 in Section 3. To this end, we provide an equivalent formulation of the property IP_k. For every subtree $\mathcal{T}$ of T we denote by $\partial\mathcal{T}$ the boundary of $\mathcal{T}$ and we define the set of terminal edges of $\mathcal{T}$ as the set $\partial E_o(\mathcal{T})$ of oriented edges $e \in E_o(\mathcal{T})$ of $\mathcal{T}$ with terminal vertex in $\partial\mathcal{T}$. For every $H \leq G$, for every complete finite subtree $\mathcal{T} \subseteq T$ and for all $f, f' \in \partial E_o(\mathcal{T})$ notice that the elements of $\mathrm{Fix}_H(Tf)$ and $\mathrm{Fix}_H(Tf')$ commute with one another since their respective supports are disjoint. This remark lifts the ambiguity in the group equality of Proposition 4.7.

Proposition 4.7. *A group $G \leq \mathrm{Aut}(T)$ satisfies the property IP_k if and only if for every complete finite subtree $\mathcal{T}$ of T containing an edge we have*

$$\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) = \prod_{f \in \partial E_o(\mathcal{T})} \left[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})\right].$$

Proof. The reverse implication is trivial. To prove the forward implication we apply an induction on the number of interior vertices of $\mathcal{T}$ and treat multiple cases. If $\mathcal{T}$ does not have an interior vertex, it is an edge and the group equality corresponds exactly to the property IP_k. If $\mathcal{T}$ has exactly one interior vertex, there exists $v \in V(T)$ such that $\mathcal{T} = B_T(v, 1)$. In particular, we have $\mathcal{T}^{(k-1)} = v^{(k)}$. Let $g \in \mathrm{Fix}_G(v^{(k)})$ and let $\partial E_o(\mathcal{T}) = \{f_1, \ldots, f_n\}$. For every $f \in E_o(\mathcal{T})$ notice that $f^{(k-1)} \subseteq v^{(k)}$ and therefore that $\mathrm{Fix}_G(v^{(k)}) \subseteq \mathrm{Fix}_G(f^{(k-1)})$. In particular, since

G satisfies the property IP_k, $g = g_{f_1} g_{\bar{f}_1}$ where $g_{f_1} \in \mathrm{Fix}_G(Tf_1) \cap \mathrm{Fix}_G(f_1^{(k-1)})$ and $g_{\bar{f}_1} \in \mathrm{Fix}_G(T\bar{f}_1) \cap \mathrm{Fix}_G(f_1^{(k-1)})$. On the other hand, since $f_1 \in \partial E_o(\mathcal{T})$ we have that $v^{(k)} \subseteq Tf_1 \cup f_1^{(k-1)}$ which implies that $\mathrm{Fix}_G(Tf_1) \cap \mathrm{Fix}_G(f_1^{(k-1)}) \subseteq \mathrm{Fix}_G(v^{(k)})$. This proves that $g_{f_1} \in \mathrm{Fix}_G(Tf_1) \cap \mathrm{Fix}_G(v^{(k)})$ and since $g \in \mathrm{Fix}_G(v^{(k)})$ we obtain that $g_{\bar{f}_1} \in \mathrm{Fix}_G(T\bar{f}_1) \cap \mathrm{Fix}_G(v^{(k)})$. In particular, $g_{\bar{f}_1} \in \mathrm{Fix}_G(f_2^{(k-1)})$ and since G satisfies the property IP_k we obtain that $g_{\bar{f}_1} = g_{f_2} g_{\bar{f}_2}$ where $g_{f_2} \in \mathrm{Fix}_G(Tf_2) \cap \mathrm{Fix}_G(f_2^{(k-1)})$ and $g_{\bar{f}_2} \in \mathrm{Fix}_G(T\bar{f}_2) \cap \mathrm{Fix}_G(f_2^{(k-1)})$. Furthermore, just as before, since $f_2 \in \partial E_o(\mathcal{T})$, we have that $v^{(k)} \subseteq Tf_2 \cup f_2^{(k-1)}$ which implies that $\mathrm{Fix}_G(T\bar{f}_2) \cap \mathrm{Fix}_G(f_2^{(k-1)}) \subseteq \mathrm{Fix}_G(v^{(k)})$ and we conclude that $g_{\bar{f}_2} \in \mathrm{Fix}_G(T\bar{f}_2) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$. On the other hand, $T\bar{f}_1 \subseteq Tf_2$ which implies that $g_{f_2} \in \mathrm{Fix}_G(T\bar{f}_1)$. Since $g_{\bar{f}_1} \in \mathrm{Fix}_G(T\bar{f}_1)$, the above decomposition of $g_{\bar{f}_1}$ implies that $g_{\bar{f}_2} \in \mathrm{Fix}_G(T\bar{f}_1)$ and we obtain that $g_{\bar{f}_2} \in \mathrm{Fix}_G(T\bar{f}_1 \cup T\bar{f}_2) \cap \mathrm{Fix}_G(v^{(k)})$. Proceeding iteratively, we obtain that

$$g = g_{f_1} g_{f_2} \dots g_{f_n} g_{\bar{f}_n}$$

for some $g_{f_i} \in \mathrm{Fix}(Tf_i) \cap \mathrm{Fix}_G(v^{(k)})$ and some $g_{\bar{f}_n} \in \mathrm{Fix}_G(\bigcup_{i=1}^n T\bar{f}_i) \cap \mathrm{Fix}_G(v^{(k)})$. In particular since $\{f_1, \dots, f_n\} = \partial E_o(\mathcal{T})$, notice that $g_{\bar{f}_n} \in \mathrm{Fix}_G(T) = \{1_G\}$. This proves as desired that

$$g \in \prod_{f \in \partial E_o(\mathcal{T})} \left[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(v^{(k)}) \right].$$

If $\mathcal{T}$ has two interior vertices or more, the reasoning is similar. We let $g \in \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$, let v be an interior vertex of $\mathcal{T}$ at distance 1 form the boundary $\partial\mathcal{T}$ and let $\mathcal{R}$ be the unique maximal proper complete subtree of $\mathcal{T}$ such that v is not an interior vertex. Notice that $\mathcal{R}$ is a complete subtree of T with one interior vertex less. In particular, our induction hypothesis ensures that

$$\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) = \prod_{f \in \partial E_o(\mathcal{R})} \left[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \right].$$

Notice that exactly one extremal edge $e \in \partial E_o(\mathcal{R})$ of $\mathcal{R}$ is not extremal in $\mathcal{T}$; the oriented edge e such that $o(e) = v$. Furthermore, since $\mathcal{R}^{(k-1)} \subseteq \mathcal{T}^{(k-1)}$, we have $g \in \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$ which implies that

$$g = g_e \prod_{f \in \partial E_o(\mathcal{T}) \cap \partial E_o(\mathcal{R})} g_f$$

for some $g_e \in \mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$ and some $g_f \in \mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$. Furthermore, for every $f \in \partial E_o(\mathcal{T}) \cap \partial E_o(\mathcal{R})$ notice that

$$\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)}) = \mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$$

which implies that $g_f \in \mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$. In particular, since $g \in \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$, the above decomposition implies that $g_e \in \mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$. On the other hand, $v^{(k)} \subseteq Te \cup \mathcal{T}^{(k-1)}$ which implies that $g_e \in \mathrm{Fix}_G(v^{(k)})$. Hence, by the first part of the proof, we have

$$g_e = g_{\bar{e}} \prod_{b \in \partial E_o(\mathcal{T}) \cap \partial E_o(v^{(1)})} g_b$$

for some $g_{\bar{e}} \in \mathrm{Fix}_G(T\bar{e}) \cap \mathrm{Fix}_G(v^{(k-1)})$ and some $g_b \in \mathrm{Fix}_G(Tb) \cap \mathrm{Fix}_G(v^{(k-1)})$. Furthermore, for every $b \in \partial E_o(\mathcal{T}) \cap \partial E_o(v^{(1)})$, notice that $\mathcal{T}^{(k-1)} \subseteq Tb \cup v^{(1)}$ and therefore that $g_b \in \mathrm{Fix}_G(Tb) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$. In particular, since $g_e \in \mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})$ the above decomposition of g implies that $g_{\bar{e}} = 1_G$. This proves as desired that g belongs to

$$\prod_{f\in \partial E_o(\mathcal{T})\cap[\partial E_o(\mathcal{R})\cup \partial E_o(v^{(1)})]} \big[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{T}^{(k-1)})\big].$$

□

Proposition 4.8. *Let $G \leq \mathrm{Aut}(T)$ be a subgroup satisfying the property* IP_k*, let $\mathcal{T}$, $\mathcal{T}'$ be complete finite subtrees of T such that $\mathcal{T}$ contains at least one interior vertex and such that $\mathcal{T}'$ does not contain $\mathcal{T}$. Then, there exists a maximal complete proper subtree $\mathcal{R}$ of $\mathcal{T}$ such that*

$$\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G((\mathcal{T}')^{(k-1)})\,\mathrm{Fix}_G(\mathcal{T}^{(k-1)}).$$

Proof. Since $\mathcal{T}$ and $\mathcal{T}'$ are complete and since $\mathcal{T}'$ does not contain $\mathcal{T}$, there exists an extremal edge b of $\mathcal{T}$ which does not belong to $\mathcal{T}'$ and such that $\mathcal{T}' \subseteq Tb$. Let $\mathcal{R}$ be the maximal complete subtree of $\mathcal{T}$ such that $b \not\subseteq \mathcal{R}$. Notice that there is a unique $e \in \partial E_o(\mathcal{R})$ which is not extremal in $\mathcal{T}$. Furthermore, observe that $\mathcal{T}' \subseteq Te \cup e$, that $\mathcal{R}$ is a complete subtree of T containing an edge and that $\mathcal{R}$ has one less interior vertex than $\mathcal{T}$. In particular, Proposition 4.7 ensures that

$$\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) = \prod_{f\in \partial E_o(\mathcal{R})} \big[\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)})\big].$$

However, by construction, we have that $(\mathcal{T}')^{(k-1)} \subseteq Te \cup \mathcal{R}^{(k-1)}$ which implies that

$$\mathrm{Fix}_G(Te) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G((\mathcal{T}')^{(k-1)}).$$

On the other hand, notice that $\partial E_o(\mathcal{R}) - \{e\} \subseteq \partial E_o(\mathcal{T})$. Therefore, for every $f \in \partial E_o(\mathcal{R}) - \{e\}$ we have $\mathcal{T}^{(k-1)} \subseteq Tf \cup \mathcal{R}^{(k-1)}$ which implies that

$$\mathrm{Fix}_G(Tf) \cap \mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G(\mathcal{T}^{(k-1)}).$$

This proves as desired that

$$\begin{aligned}\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) &= \big[\mathrm{Fix}_G(Te)\cap\mathrm{Fix}_G(\mathcal{R}^{(k-1)})\big] \prod_{f\in \partial E_o(\mathcal{R})-\{e\}} \big[\mathrm{Fix}_G(Tf)\cap\mathrm{Fix}_G(\mathcal{R}^{(k-1)})\big]\\ &\subseteq \mathrm{Fix}_G((\mathcal{T}')^{(k-1)})\,\mathrm{Fix}_G(\mathcal{T}^{(k-1)}).\end{aligned}$$

□

4.2. **Factorisation of the generic filtrations.** Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$. Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup and let μ be a Haar measure of G. The purpose of this section is to prove Theorems B and C. This requires some preliminaries.

Lemma 4.9. *Let $\mathcal{P}$ be a complete finite subtree of T containing an interior vertex, let $\Sigma_{\mathcal{P}}$ be the set of maximal complete proper subtrees of $\mathcal{P}$ and let*

$$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{R} \in \Sigma_{\mathcal{P}} | \mathrm{Fix}_G((\mathcal{R}')^{(k-1)}) \not\subseteq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})\ \forall \mathcal{R}' \in \Sigma_{\mathcal{P}} - \{\mathcal{R}\}\}.$$

Suppose that:

(1) $\forall \mathcal{R}, \mathcal{R}' \in \mathfrak{T}_{\mathcal{P}}, \forall g \in G$, *we do not have* $\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subsetneq \mathrm{Fix}_G(g(\mathcal{R}')^{(k-1)})$.

(2) *For every* $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$, $\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$.

Then, there exists a family $\mathfrak{R} \subseteq \mathfrak{T}_{\mathcal{P}} \cup \{\mathcal{P}\} \cup \{v^{(n)} | n \in \mathbb{N}, v \in V(T)\}$ *of complete finite subtrees of* T *such that* $\mathcal{S}_{\mathcal{P}} = \{\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) | \mathcal{T} \in \mathfrak{R}\}$ *is a generic filtration of* G *such that:*

$$\mathcal{S}_{\mathcal{P}}[0] = \{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\}, \ \mathcal{S}_{\mathcal{P}}[1] = \{\mathrm{Fix}_G(\mathcal{P}^{(k-1)})\}$$

and $\mu(\mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \neq \mu(\mathrm{Fix}_G(\mathcal{T}^{(k-1)}))$ *for every* $\mathcal{T} \in \mathfrak{R} - \{\mathcal{P}\}$.

Proof. Since G is non-discrete, there exist a vertex $v \in V(T)$ and an integer $N \geq k$ such that $\mathcal{P} \subsetneq v^{(N)}$ and $\mathrm{Fix}_G(v^{(N+k-1)}) \lneq \mathrm{Fix}_G(\mathcal{P}^{(k-1)})$. We set $\mathfrak{R} = \mathfrak{T}_{\mathcal{P}} \sqcup \{\mathcal{P}\} \sqcup \{v^{(n)} | n \geq N\}$ and let $\mathcal{S}_{\mathcal{P}} = \{\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) | \mathcal{T} \in \mathfrak{R}\}$. Notice by construction that for every $\mathcal{T} \in \mathfrak{R}$ there exists $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ such that $\mathcal{R} \subseteq \mathcal{T}$. In particular, for every $U \in \mathcal{S}_{\mathcal{P}}$ there exists $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ such that $U \subseteq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$. On the other hand, since $\mathcal{P}$ is a finite subtree of T notice that $\mathfrak{T}_{\mathcal{P}}$ is finite. This implies that

$$\mu(U) \leq \max_{\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}} \left[\mu(\mathrm{Fix}_G(\mathcal{R}^{(k-1)}))\right] < +\infty \qquad \forall U \in \mathcal{S}_{\mathcal{P}}.$$

Renormalising μ if needed, Lemma 1.3 ensures that $\mathcal{S}_{\mathcal{P}}$ is a generic filtration of G. Now, notice for every $\mathcal{T} \in \mathfrak{R} - (\mathfrak{T}_{\mathcal{P}} \sqcup \{\mathcal{P}\})$ and every $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ we have that $\mathcal{R} \subseteq \mathcal{P} \subseteq \mathcal{T}$. In particular, this implies that $\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) \subseteq \mathrm{Fix}_G(\mathcal{P}^{(k-1)}) \subseteq \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$. On the other hand, by the hypotheses none of those inclusions is an equality which implies that

$$\mu(\mathrm{Fix}_G(\mathcal{T}^{(k-1)})) \lneq \mu(\mathrm{Fix}_G(\mathcal{P}^{(k-1)})) \lneq \mu(\mathrm{Fix}_G(\mathcal{R}^{(k-1)})).$$

This proves, as desired, that $\mu(\mathrm{Fix}_G(\mathcal{P})^{(k-1)}) \neq \mu(\mathrm{Fix}_G(\mathcal{T}^{(k-1)}))$ for every $\mathcal{T} \in \mathfrak{R} - \{\mathcal{P}\}$. Furthermore, since G is unimodular, the measure $\mu(U)$ is an invariant of the conjugacy class $\mathcal{C}(U)$ and one realises that

$$\mathcal{C}(\mathrm{Fix}_G(\mathcal{R}^{(k-1)})) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{P}^{(k-1)})) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}^{(k-1)})).$$

In particular, the depth of every subgroup $\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) \in \mathcal{S}_{\mathcal{P}}$ for which $\mathcal{T} \in \mathfrak{R} - (\mathfrak{T}_{\mathcal{P}} \sqcup \{\mathcal{P}\})$ is at least 2. Now, let us prove that

$$\mathcal{S}_{\mathcal{P}}[0] = \{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\} \text{ and } \mathcal{S}_{\mathcal{P}}[1] = \{\mathrm{Fix}_G(\mathcal{P}^{(k-1)})\}.$$

Let $\mathcal{T} \in \mathfrak{R}$ and $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ be such that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}^{(k-1)})) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{R}^{(k-1)}))$. By the definition, this implies the existence of an element $g \in G$ such that $\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G(g\mathcal{T}^{(k-1)})$ and by the first part of the proof we have that $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$. Our hypotheses imply that $\mathrm{Fix}_G(g\mathcal{T}^{(k-1)}) = \mathrm{Fix}_G(\mathcal{R}^{(k-1)})$ and therefore that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}^{(k-1)})) = \mathcal{C}(\mathrm{Fix}_G(\mathcal{R}^{(k-1)}))$. In particular, this proves that $\{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\} \subseteq \mathcal{S}_{\mathcal{P}}[0]$. On the other hand, we have proved that the depth of every subgroup $\mathrm{Fix}_G(\mathcal{T}^{(k-1)})$ with $\mathcal{T} \in \mathfrak{R} - (\mathfrak{T}_{\mathcal{P}} \sqcup \{\mathcal{P}\})$ is at least 2. Since there must exist an element at depth 1, this implies that $\mathcal{S}_{\mathcal{P}}[1] = \{\mathrm{Fix}_G(\mathcal{P}^{(k-1)})\}$ and it follows that $\mathcal{S}_{\mathcal{P}}[0] = \{\mathrm{Fix}_G(\mathcal{R}^{(k-1)}) | \mathcal{R} \in \mathfrak{T}_{\mathcal{P}}\}$. ☐

Proof of Theorem B. Let $\mathfrak{R}$ be the family of subtrees of T given by Lemma 4.9 and consider the generic filtration $\mathcal{S}_{\mathcal{P}} = \{\mathrm{Fix}_G(\mathcal{T}^{(k-1)}) | \mathcal{T} \in \mathfrak{R}\}$ of G. In order to prove that $\mathcal{S}_{\mathcal{P}}$ factorises$^+$ at depth 1 we shall successively verify the three conditions of Definition 1.5.

First, we need to prove that for all U in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}$ such that $V \not\subseteq U$, there exists a W in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}[l-1]$ such that $U \subseteq W \subseteq VU$. Let U and V be as above. By the definition of $\mathcal{S}_{\mathcal{P}}$ there exist $t, h \in G$ and some

$\mathcal{T} \in \mathfrak{R}$ such that $U = \mathrm{Fix}_G(t\mathcal{P}^{(k-1)})$ and $V = \mathrm{Fix}_G(h\mathcal{T}^{(k-1)})$. Furthermore, since $V \not\subseteq U$, we have that $\mathrm{Fix}_G(t^{-1}h\mathcal{T}^{(k-1)}) \not\subseteq \mathrm{Fix}_G(\mathcal{P}^{(k-1)})$. In particular, we obtain that $\mathcal{P} \not\subseteq t^{-1}h\mathcal{T}$. This follows from hypothesis (3) if $\mathcal{T} = v^{(n)}$ for some $n \geq N$, from hypothesis (4) if $\mathcal{T} = \mathcal{P}$ and from the fact that $t^{-1}h\mathcal{T}^{(k-1)}$ can never contain $\mathcal{P}^{(k-1)}$ if $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$. In particular, Proposition 4.8 ensures the existence of a complete finite subtree $\mathcal{Q} \in \Sigma_{\mathcal{P}}$ such that

$$\mathrm{Fix}_G(t\mathcal{Q}^{(k-1)}) \subseteq \mathrm{Fix}_G(h\mathcal{T}^{(k-1)})\,\mathrm{Fix}_G(t\mathcal{P}^{(k-1)}) = VU.$$

On the other hand, by the definition of $\mathfrak{T}_{\mathcal{P}}$ there exists $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ such that

$$\mathrm{Fix}_G(t\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G(t\mathcal{Q}^{(k-1)}).$$

Furthermore, by the definition of $\mathcal{S}_{\mathcal{P}}$, the group $\mathrm{Fix}_G(t\mathcal{R}^{(k-1)})$ is conjugate to an element of $\mathcal{S}_{\mathcal{P}}[0]$ which proves the first condition.

Next, we need to prove that $N_G(U,V) = \{g \in G | g^{-1}Vg \subseteq U\}$ is compact for every V in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}$. Notice that $V = \mathrm{Fix}_G(h\mathcal{T}^{(k-1)})$ for some $\mathcal{T} \in \mathfrak{R}$ and some $h \in G$ and that

$$N_G(U,V) = \{g \in G | g^{-1}Vg \subseteq U\} = \{g \in G | \mathrm{Fix}_G(g^{-1}h\mathcal{T}^{(k-1)}) \subseteq \mathrm{Fix}_G(t\mathcal{P}^{(k-1)})\}.$$

This leads to three cases. If $\mathcal{T} = v^{(n)}$ for some $n \in \mathbb{N}$, the hypothesis (3) implies that $N_G(U,V) = \{g \in G | t\mathcal{P} \subseteq g^{-1}hv^{(n)}\}$ which is a compact set. If $\mathcal{T} = \mathcal{T}'$, the hypothesis (4) ensures that $N_G(V,U) = \{g \in G | gt\mathcal{P} \subseteq t\mathcal{P}\}$ which is a compact set. If $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$, notice from Lemma 4.9 that $\mu(V)$ is strictly smaller than $\mu(U)$ which implies that $N_G(U,V) = \varnothing$. In every case, this proves the second condition.

Finally, we need to prove that for any subgroup W in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}[0]$ such that $U \subseteq W$ we have

$$W \subseteq N_G(U,U) = \{g \in G \mid g^{-1}Ug \subseteq U\} = \{g \in G \mid g^{-1}Ug = U\}.$$

The hypothesis (4) of the theorem implies that $N_G(U,U) = \{g \in G | gt\mathcal{P} = t\mathcal{P}\}$. On the other hand, by construction of $\mathcal{S}_{\mathcal{P}}$, for every W in the conjugacy class of an element of $\mathcal{S}_{\mathcal{P}}[0]$, there exist an element $h \in G$ and a subtree $\mathcal{R} \in \mathfrak{T}_{\mathcal{P}}$ such that $W = \mathrm{Fix}_G(h\mathcal{R}^{(k-1)})$. Furthermore, if $U \subseteq W$ the hypothesis (2) implies that $\mathcal{P} \subseteq t^{-1}h\mathcal{R}^{(k-1)}$. In particular, we obtain that $W = \mathrm{Fix}_G(h\mathcal{R}^{(k-1)}) \subseteq \mathrm{Fix}_G(t\mathcal{P}) \subseteq N_G(U,U)$ which proves the third condition. □

Our next task is to prove Theorem C concerning groups satisfying the hypothesis H_q (Definition 4.2). Let $q \in \mathbb{N}$ be a non-negative integer. If q is even, let

$$\mathfrak{T}_q = \left\{ B_T(v,r) \middle| v \in V(T), r \geq \frac{q}{2} + 1 \right\} \sqcup \left\{ B_T(e,r) \middle| e \in E(T), r \geq \frac{q}{2} \right\}.$$

If q is odd, let

$$\mathfrak{T}_q = \left\{ B_T(v,r) \middle| v \in V(T), r \geq \frac{q+1}{2} \right\} \sqcup \left\{ B_T(e,r) \middle| e \in E(T), r \geq \frac{q+1}{2} \right\}.$$

For any closed subgroup $G \leq \mathrm{Aut}(T)$, we consider the set

$$\mathcal{S}_q = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_q\}$$

of pointwise stabilisers of those subtrees.

Lemma 4.10. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_q for some integer $q \geq 0$. Then, $\mathcal{S}_q$ is a generic filtration of G and:*

- $\mathcal{S}_q[l] = \{\mathrm{Fix}_G(B_T(e, \frac{q+l}{2}))|e \in E(T)\}$ *if* $q+l$ *is even.*
- $\mathcal{S}_q[l] = \{\mathrm{Fix}_G(B_T(v, (\frac{q+l+1}{2}))) \mid v \in V(T)\}$ *if* $q+l$ *is odd.*

Proof. For shortening of the formulation and readability, for all $v \in V(T)$ and every $e \in E(T)$ we denote by Gv and Ge their respective orbit under the action of G on $V(T)$ and $E(T)$ and by $v^{(r)}$ and $e^{(r)}$ the balls of radius r around v and e respectively. Since $g\,\mathrm{Fix}_G(v^{(r)})g^{-1} = \mathrm{Fix}_G(gv^{(r)})\ \forall g \in G, \forall v \in V(T)$ and $\forall r \in \mathbb{N}$, notice that $\mathcal{C}(\mathrm{Fix}_G(v^{(r)}) = \{\mathrm{Fix}_G(w^{(r)})|w \in Gv\}$. Similarly, we have that $\mathcal{C}(\mathrm{Fix}_G(e^{(r)})) = \{\mathrm{Fix}_G(f^{(r)})|f \in Ge\}\ \forall e \in E(T)$. Furthermore, for every $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_q$, we have that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exist $g \in G$ such that $\mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(g\mathcal{T}')$. Therefore, since G satisfies the hypothesis H_q, we have $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exists $g \in G$ such that $g\mathcal{T}' \subseteq \mathcal{T}$. Furthermore, notice that $\mathcal{T}_q$ is stable under the action of G. In particular, for every increasing chain $C_0 \lneqq C_1 \lneqq \cdots \lneqq C_{n-1} \lneqq C_n$ of elements of $\mathcal{F}_{\mathcal{S}_q}$ there exists a strictly increasing chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{n-1} \subsetneq \mathcal{T}_n$ of elements of $\mathfrak{T}_q$ such that $C_t = \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_t))$. It follows that the height of an element $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})) \in \mathcal{F}_{\mathcal{S}_q}$ is bounded above by the maximal length of a strictly increasing chain of elements of $\mathfrak{T}_q$ contained in $\mathcal{T}$. On the other hand, for every strictly increasing chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_n \subsetneq \mathcal{T}$ of elements of $\mathfrak{T}_q$ contained in $\mathcal{T}$, we can build a strictly increasing chain $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_0)) \lneqq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_1)) \lneqq \cdots \lneqq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_n)) \lneqq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ of elements of $\mathcal{F}_{\mathcal{S}_q}$. This proves that the height of $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ is the maximal length of a strictly increasing chain of elements of $\mathfrak{T}_q$ contained in $\mathcal{T}$. The result therefore follows from the following observation. If q is even, every such chain is of the form

$$e_1^{(\frac{q}{2})} \subseteq v_1^{(\frac{q}{2}+1)} \subseteq e_2^{(\frac{q}{2}+1)} \subseteq v_2^{(\frac{q}{2}+2)} \subseteq \cdots \subseteq \mathcal{T},$$

where $v_t \in V(T)$ and $e_t \in E(T)$ for all t and if q is odd, every such chain is of the form

$$v_1^{(\frac{q+1}{2})} \subseteq e_1^{(\frac{q+1}{2})} \subseteq v_2^{(\frac{q+1}{2}+1)} \subseteq e_2^{(\frac{q+1}{2}+1)} \subseteq \cdots \subseteq \mathcal{T},$$

where the $v_t \in V(T)$ and $e_t \in E(T)$ for every t. □

Lemma 4.11. *Let* $G \leq \mathrm{Aut}(T)$ *be a closed unimodular subgroup satisfying the hypothesis* H_q *and the property* IP_k *for some integers* $q \geq 0$, $k \geq 1$ *and let*

$$L_{q,k} = \begin{cases} \max\{1, 2k-q-1\} & \textit{if } q \textit{ is even.} \\ \max\{1, 2k-q\} & \textit{if } q \textit{ is odd.} \end{cases}$$

Suppose further that $l, l' \in \mathbb{N}$ *are such that* $l \geq L_{q,k}$ *and* $l \leq l'$. *Then, for every* U *in the conjugacy class of an element of* $\mathcal{S}_q[l]$ *and every* V *in the conjugacy class of an element of* $\mathcal{S}_q[l']$ *such that* $V \not\subseteq U$, *there exists* $W \in \mathcal{S}_q[l-1]$ *such that* $U \subseteq W \subseteq VU$.

Proof. For every $t \in \mathbb{N}$, let $\mathfrak{T}_q[t] = \{\mathcal{T} \in \mathfrak{T}_q | \mathrm{Fix}_G(\mathcal{T}) \in \mathcal{S}[t]\}$. Notice that $\mathcal{T}_q[t]$ is stable under the action of G. In particular, there exist $\mathcal{T} \in \mathfrak{T}_q[l]$ and $\mathcal{T}' \in \mathfrak{T}_q[l']$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. Since $V \not\subseteq U$, we have that $\mathcal{T} \not\subseteq \mathcal{T}'$. There are four cases to treat depending on the parity of q and l. We suppose that q is even (the reasoning with odd q is similar). If l is even, let $k' = \frac{l+q}{2}$. Lemma 4.10 implies the existence of an edge $e \in E(T)$ such that $U = \mathrm{Fix}_G(B_T(e, k'))$. Furthermore, since $l \geq L_{q,k}$ and since l is even, we have $k' \geq k$ and Lemma 4.4 implies that G satisfies the property $\mathrm{IP}_{k'}$. Therefore, Proposition 4.8 ensures the existence of a vertex $v \in e$ such that

$$\mathrm{Fix}_G(B_T(v,k')) \subseteq VU$$

and Lemma 4.10 ensures that $\mathrm{Fix}_G(B_T(v,k')) \in \mathcal{S}_q[l-1]$. Finally, notice that $U = \mathrm{Fix}_G(B_T(e,k')) \subseteq \mathrm{Fix}_G(B_T(v,k'))$ since $v \in e$.

If l is odd, let $k' = \frac{l+q+1}{2}$. Lemma 4.10 implies the existence of some $v \in V(T)$ such that $U = \mathrm{Fix}_G(B_T(v,k'))$. Furthermore, since $l \geq L_{q,k}$, we have $k' \geq k$ and Lemma 4.4 implies that G satisfies the property $\mathrm{IP}_{k'}$. Therefore, Proposition 4.8 ensures the existence of an edge $e \subsetneq B_T(v,1)$ such that

$$\mathrm{Fix}_G(B_T(e,k'-1)) \subseteq VU$$

and Lemma 4.10 ensures that $\mathrm{Fix}_G(B_T(e,k'-1) \in \mathcal{S}_q[l-1]$. Finally, since $e \in E(B_T(v,1))$, notice that $\mathrm{Fix}_G(B_T(v,k')) \subseteq \mathrm{Fix}_G(B_T(e,k'-1))$. □

Proof of Theorem C. To prove that $\mathcal{S}_q$ factorises$^+$ at depth $l \geq L_{q,k}$ we shall successively verify the three conditions of Definition 1.5.

First, we need to prove that for every U in the conjugacy class of an element of $\mathcal{S}_q[l]$ and every V in the conjugacy class of an element of $\mathcal{S}_q$ with $V \not\subseteq U$, there exists a W in the conjugacy class of an element of $\mathcal{S}_q[l-1]$ such that $U \subseteq W \subseteq VU$. Let U and V be as above. If V is conjugate to an element of $\mathcal{S}_q[l']$ for some $l' \geq l$ the result follows directly from Lemma 4.11. Therefore, we suppose that $l' \lneq l$. By the definition of $\mathcal{S}_q$ and since $\mathfrak{T}_q$ is stable under the action of G, there exist $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_q$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. We have two cases. Either $\mathcal{T}' \subseteq \mathcal{T}$ and there exists a subtree $\mathcal{R} \in \mathfrak{T}_q$ such that $\mathcal{T}' \subseteq \mathcal{R} \subseteq \mathcal{T}$ and $\mathrm{Fix}_G(\mathcal{R}) \in \mathcal{S}_q[l-1]$. In that case

$$\mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

Or else, $\mathcal{T}' \not\subseteq \mathcal{T}$ and since $l' \lneq l$, this implies the existence of a subtree $\mathcal{P} \in \mathfrak{T}_q$ such that $\mathcal{T}' \subseteq \mathcal{P} \neq \mathcal{T}$ and $\mathrm{Fix}_G(\mathcal{P}) \in \mathcal{S}_q[l]$. In particular, Lemma 4.11 ensures the existence of a $W \in \mathcal{S}_q[l-1]$ such that $U \subseteq W \subseteq \mathrm{Fix}_G(\mathcal{P})U$. Since $\mathrm{Fix}_G(\mathcal{P}) \subseteq \mathrm{Fix}_G(\mathcal{T}')$, this proves the first condition.

Next, we need to prove that $N_G(U,V) = \{g \in G | g^{-1}Vg \subseteq U\}$ is compact for every V in the conjugacy class of an element of $\mathcal{S}_q$. Just as before, notice that $V = \mathrm{Fix}_G(\mathcal{T}')$ for some $\mathcal{T}' \in \mathfrak{T}_q$. Since G satisfies the hypothesis H_q notice that

$$\begin{aligned} N_G(U,V) &= \{g \in G | g^{-1}Vg \subseteq U\} = \{g \in G | g^{-1}\,\mathrm{Fix}_G(\mathcal{T}')g \subseteq \mathrm{Fix}_G(\mathcal{T})\} \\ &= \{g \in G | \mathrm{Fix}_G(g^{-1}\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T})\} = \{g \in G | g\mathcal{T} \subseteq \mathcal{T}'\}. \end{aligned}$$

In particular, since both $\mathcal{T}$ and $\mathcal{T}'$ are finite subtrees of T, $N_G(U,V)$ is a compact subset of G which proves the second condition.

Finally, we need to prove that for every W in the conjugacy class of an element of $\mathcal{S}_q[l-1]$ with $U \subseteq W$ we have

$$W \subseteq N_G(U,U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

For the same reasons as before, there exists $\mathcal{R} \in \mathfrak{T}_q$ such that $W = \mathrm{Fix}_G(\mathcal{R})$. On the other hand, since $U \subseteq W$ and since G satisfies the hypothesis H_q, notice that $\mathcal{R} \subseteq \mathcal{T}$. Furthermore, notice that $\mathrm{Fix}_G(\mathcal{R})$ has depth $l-1$ and therefore that $\mathcal{R}$ contains every interior vertex of $\mathcal{T}$. Since G is unimodular and satisfies the hypothesis H_q this implies that

$$\begin{aligned}\mathrm{Fix}_G(\mathcal{R}) &\subseteq \{h \in G | h\mathcal{T} \subseteq \mathcal{T}\} = \{h \in G | \mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(h\mathcal{T})\} \\ &= \{h \in G | h^{-1}\,\mathrm{Fix}_G(\mathcal{T})h \subseteq \mathrm{Fix}_G(\mathcal{T})\} = N_G(U,U)\end{aligned}$$

which proves the third condition. □

We now give examples of groups satisfying the hypotheses of these Theorems.

Example 4.12. Let T be a thick semi-regular tree and consider the full group $\mathrm{Aut}(T)$. By Lemma 3.5, $\mathrm{Aut}(T)$ satisfies the hypothesis H and hence the hypothesis H_0. Furthermore, $\mathrm{Aut}(T)$ coincides with its 1-closure and therefore satisfies the property IP_k for every integer $k \geq 1$ by Lemmas 4.4 and 4.6. Since $\mathrm{Aut}(T)$ is non-discrete and unimodular Theorem C applies and the generic filtration $\mathcal{S}_0$ factorises$^+$ at all depths $l \geq 1$. In particular, Theorem A provides a description of all the irreducibles at depth $l \geq 1$ for $\mathcal{S}_0$. Due to Lemma 4.10, these are exactly the cuspidal representations of $\mathrm{Aut}(T)$. Notice however that $\mathcal{S}_0$ is quite different from the generic filtration $\mathcal{S}$ of $\mathrm{Aut}(T)$ considered in Section 3 so that this procedure leads to different description of these representations.

Example 4.13 (Proof of Corollary 4.3)**.** Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$. Then, any closed subgroup $G \leq \mathrm{Aut}(T)$ that acts transitively on the boundary and whose local action at every vertex contains the alternating group of the corresponding degree satisfies the hypotheses of Theorem C. Those groups were extensively studied by Radu in [Rad17]. Among other things, he showed that every such group G is non-discrete [Rad17, Theorem G] and k-closed for some k depending on the group (Definition 4.5) [Rad17, Theorem H]. On the other hand, such a group is unimodular and since the local action at each vertex contains the alternating group a similar proof as the one of Lemma 3.5 shows that G satisfies the hypothesis H (hence H_q $\forall q \in \mathbb{N}$). In particular, Theorem C applies and the generic filtration $\mathcal{S}_0$ factorises$^+$ at all depths $l \geq L_{0,k} = 2k-1$. In a forthcoming paper [Sem22], we show without use of the property IP_k that the generic filtration $\mathcal{S}_0$ factorises$^+$ at all positive depths if the group is in addition simple. The proof of that last statement is quite technical, relies heavily on Radu's classification and is unrelated to the property IP_k. Furthermore, since various important consequences such as Nebbia's CCR conjecture on trees need to be discussed in light of the result, the author decided to not present a proof in these notes.

Example 4.14. Let T be a thick semi-regular tree and consider a k-closed group $G \leq \mathrm{Aut}(T)$ (Definition 4.5). Let $\omega \in \partial T$ be an end T and consider the stabiliser of the ω-horocycles

$$G_\omega^0 = \{g \in G : g\omega = \omega \text{ and } \exists v \in V \text{ s.t. } gv = v\}.$$

Notice that G_ω^0 is still k-closed and hence satisfies the property IP_k. Now, consider an infinite geodesic $\gamma = (v_0, v_1, \dots)$ of T with end ω and notice that

$$G_\omega^0 = \bigcup_{n \in \mathbb{N}} \mathrm{Fix}_{G_\omega^0}(v_n).$$

In particular, G_ω^0 is a union of compact groups and is therefore unimodular. If G_ω^0 is non-discrete and satisfies the hypothesis H_q, it satisfies the hypothesis of Theorem C. In particular, in that case, Theorem A provides a description of all the irreducibles of G_ω^0 at depth $l \geq L_{q,k}$ for $\mathcal{S}_q$. However, G_ω^0 never satisfies the hypothesis H_0 since for any edges $e, f \in E$ along an infinite geodesic with end ω we have either that $\mathrm{Fix}_{G^0_\omega}(e) \subseteq \mathrm{Fix}_{G^0_\omega}(f)$ or that $\mathrm{Fix}_{G^0_\omega}(f) \subseteq \mathrm{Fix}_{G^0_\omega}(e)$. Nevertheless, in certain cases, a description of the remaining cuspidal representations of G^0_ω can be obtained using Theorem B. For instance let $G = \mathrm{Aut}(T)$. In that case, G^0_ω satisfies the hypothesis H_1 and the generic filtration $\mathcal{S}_1$ factorises$^+$ at all depths $l \geq 1$. In particular, by Theorem C we obtain a description of the cuspidal representations admitting non-zero invariant vectors for the pointwise stabiliser of a ball of radius one around an edge or bigger but not for the pointwise stabiliser of a ball of radius one around a vertex. To obtain a description of the cuspidal representations admitting non-zero invariant vectors for the pointwise stabiliser of a ball $B_T(v, 1)$ of radius 1 around a vertex $v \in V$, we let $\mathcal{P} = B_T(v, 1)$, notice that $\Sigma_{\mathcal{P}} = \{e\}$ where e is the only edge of $B_T(v, 1)$ contained in the geodesic $[v, \omega]$ and apply Theorem B. For $G = \mathrm{Aut}(T)$, the reaming irreducibles of G^0_ω are all spherical and are classified in [Neb90].

Other applications of Theorem B and Theorem C could be made for instance on the k-closure of certain groups of automorphisms of trees and on the generalisation of Burger-Mozes groups described in [Tor20].

4.3. **Existence of $\mathcal{S}_q$-standard representations.** Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$ and let $q \in \mathbb{N}$ be a non-negative integer. If q is even, let

$$\mathfrak{T}_q = \left\{ B_T(v, r) \middle| v \in V(T), r \geq \frac{q}{2} + 1 \right\} \sqcup \left\{ B_T(e, r) \middle| e \in E(T), r \geq \frac{q}{2} \right\}.$$

If q is odd, let

$$\mathfrak{T}_q = \left\{ B_T(v, r) \middle| v \in V(T), r \geq \frac{q+1}{2} \right\} \sqcup \left\{ B_T(e, r) \middle| e \in E(T), r \geq \frac{q+1}{2} \right\}.$$

For any closed non-discrete unimodular subgroup $G \leq \mathrm{Aut}(T)$ satisfying the hypothesis H_q, we have shown that

$$\mathcal{S}_q = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_q\}$$

is a generic filtration of G. Furthermore, if G satisfies the property IP_k for some integer $k \geq 1$ we have shown that $\mathcal{S}_q$ factorises$^+$ at all depths $l \geq L_{q,k}$ where

$$L_{q,k} = \begin{cases} \max\{1, 2k - q - 1\} & \text{if } q \text{ is even.} \\ \max\{1, 2k - q\} & \text{if } q \text{ is odd.} \end{cases}$$

In particular, Theorem A provides a bijective correspondence between the equivalence classes of irreducible representations of G at depth $l \geq L_{q,k}$ with seed $C \in \mathcal{F}_{\mathcal{S}_q}$ and the $\mathcal{S}_q$-standard representations of $\mathrm{Aut}_G(C)$. However, no result so far ensures the existence of such representations of G. The purpose of the present section is to study the existence of those $\mathcal{S}_q$-standard representations. The following result ensures the existence of $\mathcal{S}_q$-standard representations of $\mathrm{Aut}_G(C)$ for all $C \in \mathcal{F}_{\mathcal{S}_q}$ with height $l \geq L_{q,k}$ if q and l have the same parity.

Proposition 4.15. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_q and the property IP_k for some integers $q \geq 0$, $k \geq 1$ and let $l \geq L_{q,k}$. Suppose that one of the following happens:*

- *q and l are even.*
- *q and l are odd but $l \neq 1$.*

- *q is odd, $l = 1$ and* $\mathrm{Fix}_G(v^{(\frac{q}{2}+1)}) \neq \{g \in G | ge = e\}$ $\forall e \in E(T)$, $\forall v \in e$.

Then, there exists an $\mathcal{S}_q$-standard representation of $\mathrm{Aut}_G(C)$ *for every* $C \in \mathcal{F}_{\mathcal{S}_q}$ *at height l.*

Proof. Let $C \in \mathcal{F}_{\mathcal{S}_q}$ be at height l. Since q and l have the same parity, Lemma 4.10 ensures the existence of an edge $e \in E(T)$ and an integer $r \geq k$ and such that $B_T(e,r) \in \mathfrak{T}_q$ and $C = \mathcal{C}(\mathrm{Fix}_G(B_T(e,r))$. For shortening of the formulation we let $\mathcal{T}$ denote the subtree $B_T(e,r)$. Since G satisfies the hypothesis H_q and as a consequence of Lemma 4.10, notice that

$$N_G(\mathrm{Fix}_G(\mathcal{T})) = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} = \mathrm{Stab}_G(\mathcal{T}) = \{g \in G | ge = e\},$$

that $\mathrm{Aut}_G(C) \simeq \mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ and that

$$\begin{aligned}\tilde{\mathfrak{H}}_{\mathcal{S}_q}(\mathrm{Fix}_G(\mathcal{T})) &= \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in \mathcal{S}_q[l-1] \text{ and } \mathrm{Fix}_G(\mathcal{T}) \subseteq W\} \\ &= \{\mathrm{Fix}_G(B_T(v,r)) | v \in e\}.\end{aligned}$$

Let v_0, v_1 denote the two vertices of e, let $\mathcal{T}_i = B_T(v_i, r)$ and notice that $\mathcal{T}_0 \cup \mathcal{T}_1 = \mathcal{T}$. In particular, the action of $N_G(\mathrm{Fix}_G(\mathcal{T}))$ on $\mathcal{T}$ permutes the subtrees $\{\mathcal{T}_0, \mathcal{T}_1\}$. On the other hand, since G satisfies the hypothesis H_q our hypotheses imply that

$$\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i) \subsetneq \mathrm{Stab}_G(\mathcal{T}).$$

The result therefore follows from Proposition 2.29. □

The following two results ensure the existence of $\mathcal{S}_q$-standard representations of $\mathrm{Aut}_G(C)$ for all $C \in \mathcal{F}_{\mathcal{S}_q}$ with height $l \geq L_{q,k}$ if q and l have opposite parity. We start with the degenerate case $q = 0, k = 1$ and $l = 1$ where Proposition 2.29 does not apply.

Lemma 4.16. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_0, the Tits independence property* IP_1 *and such that* $\mathrm{Fix}_G(v)$ *is 2-transitive on the set of edges of $B_T(v,1)$ for every $v \in V(T)$. Then, there exists an $\mathcal{S}_0$-standard representation of* $\mathrm{Aut}_G(C)$ *for every* $C \in \mathcal{F}_{\mathcal{S}_0}$ *at height* 1.

Proof. Let $C \in \mathcal{F}_{\mathcal{S}_0}$ be at height 1. Lemma 4.10 ensures the existence of a vertex $v \in V(T)$ such that $C = \mathcal{C}(\mathrm{Fix}_G(B_T(v,1))$. Let $U = \mathrm{Fix}_G(B_T(v,1))$ and notice that

$$N_G(U) = \{g \in G \mid gB_T(v,1) \subseteq B_T(v,1)\} = \{g \in G | gv = v\} = \mathrm{Fix}_G(v).$$

Furthermore, since G satisfies the hypothesis H_0, Lemma 4.10 implies that

$$\begin{aligned}\tilde{\mathfrak{H}}_{\mathcal{S}_0}(U) &= \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in \mathcal{S}_q[l-1] \text{ and } \mathrm{Fix}_G(B_T(v,1)) \subseteq W\} \\ &= \{\mathrm{Fix}_G(e) \mid e \in E(B_T(v,1))\},\end{aligned}$$

where $E(B_T(v,1))$ denotes the set of edges of $B_T(v,1)$. Let d be the degree of v in T, let $X = E(B_T(v,1))$ and notice that our hypotheses imply that $\mathrm{Aut}_G(C) \simeq \mathrm{Fix}_G(v)/\mathrm{Fix}_G(B_T(v,1))$ is 2-transitive X. In particular, Lemma 2.27 implies the existence of an irreducible representation σ of $\mathrm{Aut}_G(C)$ without non-zero $\mathrm{Fix}_{\mathrm{Aut}_G(C)}(e)$-invariant vectors for all $e \in X$. Since

$$\mathfrak{H}_{\mathcal{S}_0}(U) = \{p_U(\mathrm{Fix}_G(e)) | e \in X\} = \{\mathrm{Fix}_{\mathrm{Aut}_G(C)}(e) | e \in X\},$$

this proves the existence of an $\mathcal{S}_0$-standard representation of $\mathrm{Aut}_G(C)$. □

The following result treats the remaining cases.

Proposition 4.17. *Let $G \leq \mathrm{Aut}(T)$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_q and the property IP_k for some integers $q \geq 0$, $k \geq 1$ and let $l \geq L_{q,k}$. Suppose further that*

$$\mathrm{Fix}_G((B_T(v,1)-\{w\})^{(r)}) \neq \mathrm{Fix}_G(B_T(v,r+1))\ \forall v \in V(T), \forall w \in B_T(v,1)-\{v\}$$

for all $r \geq \frac{q}{2}$ if q is even and for all $r \geq \frac{q-1}{2}$ if q is odd and that one of the following happens:

- *q is odd and l is even.*
- *q is even, l is odd and $l \neq 1$.*
- *q is even, $q \neq 0$, $l = 1$ and $\mathrm{Fix}_G((B_T(v,1)-\{w\})^{(\frac{q}{2})}) \neq \mathrm{Fix}_G(v)\quad \forall v \in V(T)$, $\forall w \in B_T(v,1)-\{v\}$.*

Then, there exists an $\mathcal{S}_q$-standard representation of $\mathrm{Aut}_G(C)$ for every $C \in \mathcal{F}_{\mathcal{S}_q}$ at height l.

Proof. Suppose that $C \in \mathcal{F}_{\mathcal{S}_q}$ is at height l. Since q and l have opposite parity, Lemma 4.10 ensures the existence of a vertex $v \in V(T)$ and an integer $r \geq k-1$ such that $B_T(v,r+1) \in \mathfrak{T}_q$ and $C = \mathcal{C}(\mathrm{Fix}_G(B_T(v,r+1)))$. For shortening of the formulation we let $\mathcal{T}$ denote the subtree $B_T(v,r+1)$. Since G satisfies the hypothesis H_q and as a consequence of Lemma 4.10, notice that

$$N_G(\mathrm{Fix}_G(\mathcal{T})) = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} = \mathrm{Stab}_G(\mathcal{T}) = \{g \in G | gv = v\},$$

that $\mathrm{Aut}_G(C) \simeq \mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ and that

$$\begin{aligned}\tilde{\mathfrak{H}}_{\mathcal{S}_q}(\mathrm{Fix}_G(\mathcal{T})) &= \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in \mathcal{S}_q[l-1] \text{ and } \mathrm{Fix}_G(\mathcal{T}) \subseteq W\}\\ &= \{\mathrm{Fix}_G(B_T(e,r)) | e \in E(B_T(v,1))\}.\end{aligned}$$

Now, let $\{w_1, \ldots, w_d\}$ be the leaves of $B_T(v,1)$, let $\mathcal{T}_i = (B_T(v,1)-\{w_i\})^{(r-1)}$ $i = 1, \ldots, d$ and notice that $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}\ \forall i \neq j$. On the other hand, the action of $\mathrm{Stab}_G(\mathcal{T})$ on T permutes the subtrees $\{\mathcal{T}_1, \ldots, \mathcal{T}_d\}$ and since each $\mathcal{T}_i$ contains v we have that $\mathrm{Fix}_G(\mathcal{T}_i) \subseteq \mathrm{Stab}_G(\mathcal{T})\ \forall i = 1, \ldots, d$. Furthermore, the hypotheses on G imply that $\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i) \subsetneq \mathrm{Stab}_G(\mathcal{T})$. In particular, Proposition 2.29 ensures the existence of an irreducible representation σ of $\mathrm{Aut}_G(C)$ without non-zero $p_{\mathrm{Fix}_G(\mathcal{T})}(\mathrm{Fix}_G(\mathcal{T}_i))$-invariant vectors. Moreover, for every edge $e \in E(B_T(v,1))$ there exists some $i \in \{1, \ldots, d\}$ such that $B_T(e,r) \subseteq \mathcal{T}_i$ which implies that $p_{\mathrm{Fix}_G(\mathcal{T})}(\mathrm{Fix}_G(\mathcal{T}_i)) \subseteq p_{\mathrm{Fix}_G(\mathcal{T})}(\mathrm{Fix}_G(B_T(e,r))$. Hence, σ is an $\mathcal{S}_q$-standard representation of $\mathrm{Aut}_G(C)$. □

5. Groups of automorphisms of trees with the property IP_{V_1}

In this section, we apply our machinery to groups of automorphisms of locally finite trees satisfying the property IP_{V_1} (Definition 5.1). We use the same notations and terminology as in Section 3. Let T be a locally finite tree and let $\mathrm{Aut}(T)^+$ be the group of type-preserving automorphisms of T.

Definition 5.1. A group $G \leq \mathrm{Aut}(T)^+$ is said to satisfy the **property** IP_{V_1}, if there exists a bipartition $V(T) = V_0 \sqcup V_1$ such that every edge of T contains exactly one vertex in each V_i and such that $\forall w \in V_1$ we have

$$(\mathrm{IP}_{V_1})\qquad \mathrm{Fix}_G(B_T(w,1)) = \prod_{v \in B_T(w,1)-\{w\}} \mathrm{Fix}_G(T(w,v)),$$

where $T(w,v) = \{u \in V(T) | d_T(w,u) \lneq d_T(v,u)\}$.

Example 5.2. Let T be a locally finite semi-regular tree and let $G \leq \mathrm{Aut}(T)^+$ satisfy the property IP_1. Then, G satisfies the property IP_{V_1}.

Other examples are given in Section 6 where we show that the universal groups of certain semi-regular right-angled buildings can be realised as closed subgroups of $\mathrm{Aut}(T)^+$ satisfying the property IP_{V_1} but where T is in general not semi-regular (Theorem 6.24).

The purpose of the present section is to prove Theorem D which provides an explicit generic filtration factorising$^+$ at all positive depths for subgroups $G \leq \mathrm{Aut}(T)^+$ satisfying the property IP_{V_1} and the hypothesis H_{V_1} (Definition 5.3). This requires some formalism that we now introduce. Let $V(T) = V_0 \sqcup V_1$ be a bipartition of T such that every edge of T contains exactly one vertex in each V_i. For every subtree $\mathcal{T} \subseteq T$, we set

$$Q_{\mathcal{T}} = \{v \in V_0 | B_T(v,2) \subseteq \mathcal{T}\}$$

and we define $\mathfrak{T}_{V_1}$ as follows:

(1) $\mathfrak{T}_{V_1}[0] = \{B_T(v,1) | v \in V_1\}$.
(2) For every $l \in \mathbb{N}$ such that $l \geq 0$, we define iteratively

$$\mathfrak{T}_{V_1}[l+1] = \{\mathcal{T} \subseteq T \mid \exists \mathcal{R} \in \mathfrak{T}_{V_1}[l], \exists w \in (V(\mathcal{R}) - Q_{\mathcal{R}}) \cap V_0 \text{ s.t. } \mathcal{T} = \mathcal{R} \cup B_T(w,2)\}.$$

(3) We set $\mathfrak{T}_{V_1} = \bigsqcup_{l \in \mathbb{N}} \mathfrak{T}_{V_1}[l]$.

For every closed subgroup $G \leq \mathrm{Aut}(T)^+$ we set

$$\mathcal{S}_{V_1} = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_{V_1}\}.$$

Definition 5.3. A group $G \leq \mathrm{Aut}(T)^+$ is said to satisfy the hypothesis H_{V_1} if for all $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_{V_1}$ we have

$$(H_{V_1}) \qquad \mathrm{Fix}_G(\mathcal{T}') \leq \mathrm{Fix}_G(\mathcal{T}) \text{ if and only if } \mathcal{T} \subseteq \mathcal{T}'.$$

If $G \leq \mathrm{Aut}(T)^+$ is a closed non-discrete unimodular subgroup of $\mathrm{Aut}(T)^+$ satisfying the hypothesis H_{V_1}, Lemma 5.7 ensures that $\mathcal{S}_{V_1}$ is a generic filtration of G and that

$$\mathcal{S}_{V_1}[l] = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_{V_1}[l]\}.$$

Theorem D states the following.

Theorem. *Let T be a locally finite tree and let $G \leq \mathrm{Aut}(T)^+$ be a closed non-discrete unimodular subgroup that satisfies the hypothesis H_{V_1} and the property IP_{V_1}. Then, the generic filtration $\mathcal{S}_{V_1}$ factorises$^+$ at all depths $l \geq 1$.*

5.1. **Preliminaries.** Let T be a locally finite tree and let $V(T) = V_0 \sqcup V_1$ be a bipartition of T such that every edge of T contains exactly one vertex in each V_i. The purpose of the present section is to describe further the elements of $\mathfrak{T}_{V_1}$. For every subtree $\mathcal{T} \subseteq T$, we associated a set $Q_{\mathcal{T}} = \{v \in V_0 | B_T(v,2) \subseteq \mathcal{T}\}$. The purpose of Lemmas 5.4 and 5.5 is to give a characterisation of the elements $\mathcal{T} \in \mathfrak{T}_{V_1}$ in terms of their corresponding sets $Q_{\mathcal{T}}$.

Lemma 5.4. *The elements of $\mathfrak{T}_{V_1}$ satisfy the following:*

(i) *Every $\mathcal{T} \in \mathfrak{T}_{V_1}$ is a complete finite subtree of T with leaves in V_0.*
(ii) *For every $\mathcal{T} \in \mathfrak{T}_{V_1} - \mathfrak{T}_{V_1}[0]$ we have that $\mathcal{T} = \bigcup_{v \in Q_{\mathcal{T}}} B_T(v,2)$.*
(iii) *For every $\mathcal{T} \in \mathfrak{T}_{V_1}$, we have $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ if and only if $|Q_{\mathcal{T}}| = l$.*

Proof. Since each element of $\mathfrak{T}_{V_1}$ belong to some $\mathfrak{T}_{V_1}[l]$ for some $l \in \mathbb{N}$, in order to show (iii) it is enough to show that $\forall \mathcal{T} \in \mathfrak{T}_{V_1}[l]$, $|Q_{\mathcal{T}}|= l$. We prove (i), (ii) and that $|Q_{\mathcal{T}}|= l$ for every $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ by induction on l. If $l = 0$, $\mathcal{T} = B_T(v, 1)$ for some $v \in V_1$. Hence, $\mathcal{T}$ is a complete finite subtree with leaves in V_0 and $|Q_{\mathcal{T}}|= 0$. Similarly, if $l = 1$, $\mathcal{T} = B_T(v, 2)$ for some $v \in V_0$. In particular, $\mathcal{T}$ is a complete finite subtree of T with leaves in V_0 and since $Q_{\mathcal{T}} = \{v\}$ we have that $\mathcal{T} = \bigcup_{v\in Q_{\mathcal{T}}} B_T(v, 2)$ and $|Q_{\mathcal{T}}|= 1$. If $l \geq 2$, by construction, there exist $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ and $w \in (V(\mathcal{R}) - Q_{\mathcal{R}}) \cap V_0$ such that $\mathcal{T} = \mathcal{R} \cup B_T(w, 2)$. By the induction hypothesis we have:

(1) $\mathcal{R}$ is a finite complete subtree of T with leaves in V_0.
(2) $\mathcal{R} = \bigcup_{v\in Q_{\mathcal{R}}} B_T(v, 2)$.
(3) $|Q_{\mathcal{R}}|= l - 1$.

Since $\mathcal{T} = \mathcal{R} \cup B_T(w, 2)$, (1) implies that $\mathcal{T}$ is a complete finite subtree with leaves in V_0 which proves (i). On the other hand, (2) implies that $Q_{\mathcal{R}} \cup \{w\} \subseteq Q_{\mathcal{T}}$ and therefore that $\mathcal{T} \subseteq \bigcup_{v\in Q_{\mathcal{T}}} B_T(v, 2)$. The reverse inclusion follows trivially from the definition of $Q_{\mathcal{T}}$ which proves (ii). Now, let $w' \in Q_{\mathcal{T}} - Q_{\mathcal{R}}$. To prove that $|Q_{\mathcal{T}}|= l$ we have to prove that $w' = w$. Since $w' \in Q_{\mathcal{T}} - Q_{\mathcal{R}}$, there exists $u \in B_T(w', 2) \cap V_0$ such that $u \notin V(\mathcal{R})$. Moreover, since the leaves of $\mathcal{T}$ belong to V_0 and since the distance between two vertices of V_0 is even, notice that $d_T(u, \mathcal{R}) = 2$. On the other hand, there exists a unique vertex $x \in V(\mathcal{R}) - Q_{\mathcal{R}}$ such that $u \in B_T(x, 2)$. Since $\mathcal{T} = \mathcal{R} \cup B_T(w, 2)$ this proves that $w = x = w'$, that $Q_{\mathcal{T}} = Q_{\mathcal{R}} \sqcup \{w\}$ and therefore that $|Q_{\mathcal{T}}|= l$. □

Lemma 5.5. *Let $\mathcal{T} = \bigcup_{v\in Q} B_T(v, 2)$ for some finite set $Q \subseteq V_0$ of order $l \geq 1$. Then $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ if and only if $\mathrm{Con}(Q) \cap V_0 = Q$ where $\mathrm{Con}(Q)$ denotes the convex hull of Q in T.*

Proof. Suppose first that $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$. The definition of $Q_{\mathcal{T}}$ implies that $Q \subseteq Q_{\mathcal{T}}$ and Lemma 5.4 ensures that $|Q_{\mathcal{T}}|= l$ which proves that $Q = Q_{\mathcal{T}}$. We prove that $\mathrm{Con}(Q_{\mathcal{T}}) \cap V_0 = Q_{\mathcal{T}}$ by induction on $l \geq 1$. If $l = 1$ the result is trivial. Suppose that $l \geq 2$. By construction, there exist $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ and $w \in (V(\mathcal{R}) \cap V_0) - Q_{\mathcal{R}}$ such that $\mathcal{T} = \mathcal{R} \cup B_T(w, 2)$. Furthermore, since $l-1 \geq 1$, Lemma 5.4 ensures that $\mathcal{R} = \bigcup_{v\in Q_{\mathcal{R}}} B_T(v, 2)$. Since T is a tree and since $w \in \mathcal{R}$, there exists a unique $u \in Q_{\mathcal{R}}$ such that $d_T(u, w) = 2$ and we have that $\mathrm{Con}(Q_{\mathcal{T}}) = \mathrm{Con}(Q_{\mathcal{R}}) \cup [u, w]$. Finally, notice that $[u, w] \cap V_0 = \{u, w\}$ which proves that $\mathrm{Con}(Q_{\mathcal{T}}) \cap V_0 = Q_{\mathcal{R}} \cup \{w\} = Q_{\mathcal{T}}$.

Now, we show by induction on l that $\mathcal{T} = \bigcup_{v\in Q} B_T(v, 2) \in \mathfrak{T}_{V_1}[l]$ if $Q \subseteq V_0$ is a set of order l such that $\mathrm{Con}(Q) \cap V_0 = Q$. If $l = 1$, the result is trivial. Suppose that $l \geq 2$, choose any vertex $v \in Q$ and let $Q^n = \{w \in Q | d_T(w, v) = 2n\}$. Since Q is finite there exists $N \in \mathbb{N}$ such that $Q^N \neq \varnothing$ but $Q^n = \varnothing$ for every $n \gneq N$. In particular, notice that $Q = \bigsqcup_{n\leq N} Q^n$. For every $n \leq N$, we let $S_n = \bigcup_{w\in Q^n, s\leq n} B_T(w, 2)$, $l_n = \left|\bigsqcup_{s\leq n} Q^{s-1}\right|$ and we notice by induction on n that $S_n \in \mathfrak{T}_{V_1}[l_n]$. Notice that the result is trivial for $n = 1$, so let $n \geq 2$ and let $Q^n = \{v_1, \dots, v_{r_n}\}$. Since $\mathrm{Con}(Q) \cap V_0 = Q$, for all $w \in Q^n$ there exists $v_w \in Q^{n-1}$ such that $d_T(w, v_w) = 2$. In particular, starting from our induction hypothesis we obtain iteratively for every $0 \leq t \leq r_n$ that $S_n \cup \left(\bigcup_{i\leq t} B_T(v_i, 2)\right) \in \mathfrak{T}_{V_1}[l_n + t]$. The result follows since $l_N = l$ and $S_N = \mathcal{T}$. □

This description allows one to prove the following result.

Lemma 5.6. *Let $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ and $g \in \mathrm{Aut}(T)^+$, then we have $g\mathcal{T} \in \mathfrak{T}_{V_1}[l]$.*

Proof. If $l = 0$, $\mathcal{T} = B_T(v, 1)$ some $v \in V_1$. Furthermore, $gB_T(v, 1) = B_T(gv, 1)$ and since the elements of $\mathrm{Aut}(T)^+$ are type-preserving, $gv \in V_1$ which proves that $g\mathcal{T} \in \mathfrak{T}_{V_1}[0]$. If $l \geq 1$, Lemma 5.5 ensures that $\mathrm{Con}(Q_{\mathcal{T}}) \cap V_0 = Q_{\mathcal{T}}$. It is clear from the definition that $Q_{g\mathcal{T}} = gQ_{\mathcal{T}}$ and since g is a type-preserving automorphism of a tree we have $\mathrm{Con}(gQ_{\mathcal{T}}) \cap V_0 = gQ_{\mathcal{T}}$. In particular, Lemma 5.5 ensures that $g\mathcal{T} \in \mathfrak{T}_{V_1}[l]$. □

5.2. **Factorisation of the generic filtration $\mathcal{S}_{V_1}$.** The purpose of this section is to prove Theorem D. We adopt the same notations as in the above sections.

Lemma 5.7. *Let $G \leq \mathrm{Aut}(T)^+$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_{V_1}. Then, $\mathcal{S}_{V_1}$ is a generic filtration of G and*

$$\mathcal{S}_{V_1}[l] = \{\mathrm{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_{V_1}[l]\} \quad \forall l \in \mathbb{N}.$$

Proof. For every $\mathcal{T} \in \mathfrak{T}_{V_1}$ notice that $g\,\mathrm{Fix}_G(\mathcal{T})g^{-1} = \mathrm{Fix}_G(g\mathcal{T})$ $\forall g \in G$ and therefore that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})) = \{\mathrm{Fix}_G(g\mathcal{T}) | g \in G\}$. In particular, for every $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_{V_1}$ we have that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exists $g \in G$ such that $\mathrm{Fix}_G(\mathcal{T}) \leq \mathrm{Fix}_G(g\mathcal{T}')$. Since G satisfies the hypothesis H_{V_1}, this implies that $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}')) \leq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ if and only if there exists some $g \in G$ such that $g\mathcal{T}' \subseteq \mathcal{T}$. On the other hand, Lemma 5.6 ensures that $\mathfrak{T}_{V_1}$ is stable under the action of G. In particular, for every strictly increasing chain $C_0 \lneq C_1 \lneq \cdots \lneq C_{n-1} \lneq C_n$ of elements of $\mathcal{F}_{\mathcal{S}_{V_1}}$ there exists a strictly increasing chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{n-1} \subsetneq \mathcal{T}_n$ of elements of $\mathfrak{T}_{V_1}$ such that $C_t = \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_t))$. On the other hand, for every strictly increasing chain $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_n \subseteq \mathcal{T}$ of elements of $\mathfrak{T}_{V_1}$ contained in $\mathcal{T}$ we can build a strictly increasing chain $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_0)) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_1)) \lneq \cdots \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}_n)) \lneq \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ of elements of $\mathcal{F}_{\mathcal{S}_{V_1}}$. This proves that the height of $\mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$ is the maximal length of a strictly increasing chain of elements of $\mathfrak{T}_{V_1}$ contained in $\mathcal{T}$. The result therefore follows from the following observation. Lemma 5.4 ensures that every maximal strictly increasing chain of elements of $\mathfrak{T}_{V_1}$ contained in $\mathcal{T}$ is of the form $\mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_{l-1} \subsetneq \mathcal{T}$ where $\mathcal{T}_t \in \mathfrak{T}_{V_1}[t]$. □

Lemma 5.8 shows that independence properties such as IP_1 or IP_{V_1} can be realised as factorisation properties for the pointwise stabilisers of particular families of complete finite subtrees.

Lemma 5.8. *Let T be a locally finite tree, $G \leq \mathrm{Aut}(T)$, $\mathcal{A}$ be a family of finite subtree of T with at least two vertices such that*

$$\mathrm{Fix}_G(\mathcal{P}) = \prod_{v \in \partial\mathcal{P}} \mathrm{Fix}_G(T(\mathcal{P}, v)) \qquad \forall \mathcal{P} \in \mathcal{A},$$

where $\partial\mathcal{P}$ denotes the set of leaves of $\mathcal{P}$ and $T(\mathcal{P}, v)$ denotes the half-tree $T(\mathcal{P}, v) = \{w \in V(T) | d_T(w, \mathcal{P}) \lneq d_T(w, v)\}$ and let $\mathcal{T}$ be a non-empty complete finite subtree of T such that for every $v \in \partial\mathcal{T}$, there exists a subtree $\mathcal{T}_v \in \mathcal{A}$ with $v \in \mathcal{T}_v \subseteq \mathcal{T}$. Then, we have that

$$\mathrm{Fix}_G(\mathcal{T}) = \prod_{v \in \partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v)).$$

Proof. For every subset $V \subseteq V(T)$, we denote by V^c the complement of V in $V(T)$. First, notice that for every two distinct leaves $v, v' \in \partial\mathcal{T}$, the supports of the elements of $\mathrm{Fix}_G(T(\mathcal{T}, v))$ and $\mathrm{Fix}_G(T(\mathcal{T}, v'))$ are disjoint. In particular, the elements of $\mathrm{Fix}_G(T(\mathcal{T}, v))$ and of $\mathrm{Fix}_G(T(\mathcal{T}, v'))$ commute with one another and $\prod_{v \in \partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v))$ is a well-defined subgroup of G. On the other hand, $\forall v \in \partial\mathcal{T}$ we have $\mathrm{Fix}_G(\mathcal{T}) \supseteq \mathrm{Fix}_G(T(\mathcal{T}, v))$ and therefore that $\mathrm{Fix}_G(\mathcal{T}) \supseteq \prod_{v\in\partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v))$. In order to prove the other inclusion let $g \in \mathrm{Fix}_G(\mathcal{T})$ and let $\partial\mathcal{T} = \{v_1, \dots, v_n\}$. The hypotheses on G imply the existence of a subtree $\mathcal{T}_1 \in \mathcal{A}$ such that $v_1 \in \mathcal{T}_1 \subseteq \mathcal{T}$. Furthermore, since $v_1 \in \partial\mathcal{T}$ and $\mathcal{T}_1 \subseteq \mathcal{T}$, we observe that $v_1 \in \partial\mathcal{T}_1$ and $T(\mathcal{T}_1, v_1) = T(\mathcal{T}, v_1)$. Furthermore, since $\mathcal{T}_1 \subseteq \mathcal{T}$ and $g \in \mathrm{Fix}_G(\mathcal{T})$, notice that $g \in \mathrm{Fix}_G(\mathcal{T}_1)$. Our hypotheses on G ensure the existence of some $h_1 \in \mathrm{Fix}_G(T(\mathcal{T}_1, v_1))$ and

$$g_1 \in \prod_{v\in\partial\mathcal{T}_1-\{v_1\}} \mathrm{Fix}_G(T(\mathcal{T}_1, v)) \subseteq \mathrm{Fix}_G(T(\mathcal{T}_1, v_1)^c)$$

such that $g = h_1 g_1$. Since $g \in \mathrm{Fix}_G(\mathcal{T})$ and $\mathrm{Fix}_G(T(\mathcal{T}_1, v_1)) \subseteq \mathrm{Fix}_G(\mathcal{T})$, this decomposition implies that $g_1 \in \mathrm{Fix}_G(\mathcal{T}) \cap \mathrm{Fix}_G(T(\mathcal{T}, v_1)^c)$. Proceeding iteratively, we prove the existence of some $h_i \in \mathrm{Fix}_G(T(\mathcal{T}, v_i))$ and some $g_i \in \mathrm{Fix}_G(\mathcal{T}) \cap \left(\bigcap_{r\leq i} \mathrm{Fix}_G(T(\mathcal{T}, v_r)^c)\right)$ such that $g_{i-1} = h_i g_i$. To see that $g_i \in \bigcap_{r\leq i} \mathrm{Fix}_G(T(\mathcal{T}, v_r)^c)$, notice by induction that

$$g_{i-1} \in \bigcap_{r\leq i-1} \mathrm{Fix}_G(T(\mathcal{T}, v_r)^c),$$

that $h_i \in \mathrm{Fix}_G(T(\mathcal{T}, v_i))$ and that $\mathrm{Fix}_G(T(\mathcal{T}, v_j)) \subseteq \mathrm{Fix}_G(T(\mathcal{T}, v_i)^c)$ $\forall i \neq j$. This implies that $h_i \in \bigcap_{r\leq i-1} \mathrm{Fix}_G(T(\mathcal{T}, v_r)^c)$ and therefore that $g_i \in \bigcap_{r\leq i} \mathrm{Fix}_G(T(\mathcal{T}, v_r)^c)$. The result follows since we have by construction that $g = h_1 h_2 \dots h_n g_n$, that $h_i \in \mathrm{Fix}_G(T(\mathcal{T}, v_i))$ and that

$$g_n \in \mathrm{Fix}_G(\mathcal{T}) \cap \left(\bigcap_{v\in\partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v)^c) \right) = \mathrm{Fix}_G(T) = \{1_{\mathrm{Aut}(T)}\}.$$

□

This result provides an alternative proof of the group equality (3.1) or equivalently of Proposition 4.7 if $k = 1$ and allows one to prove the following result which is key to the proof of Theorem D.

Proposition 5.9. *Let $G \leq \mathrm{Aut}(T)^+$ be a subgroup satisfying the property IP_{V_1}. Then, for every $\mathcal{T} \in \mathfrak{T}_{V_1}$, we have*

$$\mathrm{Fix}_G(\mathcal{T}) = \prod_{v\in\partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v)).$$

Proof. If $\mathcal{T} \in \mathfrak{T}_{V_1}[0]$, there exists $w \in V_1$ such that $\mathcal{T} = B_T(w, 1)$. Notice that $\partial\mathcal{T} = \{v \in V(T) | d_T(v, w) = 1\}$ and that $T(\mathcal{T}, v) = T(w, v)$ $\forall v \in \partial\mathcal{T}$. Since G satisfies the property IP_{V_1}, we obtain, as desired, that

$$\mathrm{Fix}_G(\mathcal{T}) = \prod_{v\in\partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v)).$$

Now, let $\mathcal{A} = \mathfrak{T}_{V_1}[0]$ and notice from Lemma 5.4(ii) that the hypotheses of Lemma 5.8 are satisfied for every $\mathcal{T} \in \mathfrak{T}_{V_1}$. The result follows. □

Lemma 5.10. *Let $G \leq \mathrm{Aut}(T)^+$ and suppose that*

$$\mathrm{Fix}_G(\mathcal{T}) = \prod_{v\in\partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v)) \quad \forall \mathcal{T} \in \mathfrak{T}_{V_1}.$$

Then, for every integer $l, l' \geq 1$ with $l' \geq l$, $\forall \mathcal{T} \in \mathfrak{T}_{V_1}[l]$ and $\forall \mathcal{T}' \in \mathfrak{T}_{V_1}[l']$ such that $\mathcal{T} \not\subseteq \mathcal{T}'$, there exists a subtree $\mathcal{R} \subseteq \mathcal{T}$ such that $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ and

$$\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

Proof. Let $l, l' \geq 1$ be such that $l' \geq l$, let $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ and let $\mathcal{T}' \in \mathfrak{T}_{V_1}[l']$ be such that $\mathcal{T} \not\subseteq \mathcal{T}'$. If $l = 1$, $Q_{\mathcal{T}} = \{v_{\mathcal{T}}\}$ for some $v_{\mathcal{T}} \in V_0$. Since $\mathcal{T} \not\subseteq \mathcal{T}'$, we have that $v_{\mathcal{T}} \notin Q_{\mathcal{T}'}$. Hence $d_T(v_{\mathcal{T}}, Q_{\mathcal{T}'}) \geq 2$. In particular, there exists a unique vertex $w \in V_1 \cap B_T(v_{\mathcal{T}}, 1)$ such that $\mathcal{T}' \subseteq T(w, v_{\mathcal{T}}) \cup \{v_{\mathcal{T}}\}$. Let $\mathcal{R} = B_T(w, 1)$ and notice that $\mathcal{R} \in \mathfrak{T}_{V_1}[0]$. Our hypotheses on G ensure that

$$\mathrm{Fix}_G(\mathcal{R}) = \prod_{v \in \partial\mathcal{R}} \mathrm{Fix}_G(T(\mathcal{R}, v)).$$

On the other hand, $\partial\mathcal{R} \subseteq \partial\mathcal{T} \cup \{v_{\mathcal{T}}\}$ and $T(\mathcal{R}, v) = T(w, v)$ $\forall v \in \partial\mathcal{R}$. This proves that $\mathrm{Fix}_G(T(\mathcal{R}, v)) \subseteq \mathrm{Fix}_G(\mathcal{T})$ for every leaf $v \in \partial\mathcal{R} - \{v_{\mathcal{T}}\}$ and since $\mathcal{T}' \subseteq T(w, v_{\mathcal{T}}) \cup \{v_{\mathcal{T}}\} = T(\mathcal{R}, v_{\mathcal{T}}) \cup \{v_{\mathcal{T}}\}$ we also have that $\mathrm{Fix}_G(T(\mathcal{R}, v_{\mathcal{T}})) \subseteq \mathrm{Fix}_G(\mathcal{T}')$. This proves, as desired, that

$$\mathrm{Fix}_G(\mathcal{R}) = \prod_{v \in \partial\mathcal{R}} \mathrm{Fix}_G(T(\mathcal{R}, v)) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

If $l \geq 2$, we have that $|Q_{\mathcal{T}}| \geq 2$ and since $\mathcal{T} \not\subseteq \mathcal{T}'$, $Q_{\mathcal{T}} \not\subseteq Q_{\mathcal{T}'}$. In particular, there exists $v_{\mathcal{T}} \in Q_{\mathcal{T}}$ such that $d_T(v_{\mathcal{T}}, Q_{\mathcal{T}'}) = \max\{d_T(v, Q_{\mathcal{T}'}) | v \in Q_{\mathcal{T}}\}$. On the other hand, since $Q_{\mathcal{T}}, Q_{\mathcal{T}'} \subseteq V_0$ the distance $d_T(v_{\mathcal{T}}, Q_{\mathcal{T}'})$ must be even, hence $d_T(v_{\mathcal{T}}, Q_{\mathcal{T}'}) \geq 2$. Now, let $Q_{\mathcal{R}} = Q_{\mathcal{T}} - \{v_{\mathcal{T}}\}$. Notice that $\mathrm{Con}(Q_{\mathcal{R}}) \cap V_0 = Q_{\mathcal{R}}$. Indeed, suppose for a contradiction that there exists $w \in \mathrm{Con}(Q_{\mathcal{R}}) \cap V_0$ such that $w \notin Q_{\mathcal{R}}$. Since $\mathcal{T} \in \mathfrak{T}_{V_1}[|Q_{\mathcal{T}}|]$, Lemma 5.5 guarantees that $\mathrm{Con}(Q_{\mathcal{T}}) \cap V_0 = Q_{\mathcal{T}}$. In particular, we observe that $w \in Q_{\mathcal{T}} - Q_{\mathcal{R}}$ and since $Q_{\mathcal{R}} = Q_{\mathcal{T}} - \{v_{\mathcal{T}}\}$ this implies that $w = v_{\mathcal{T}}$. In particular, $v_{\mathcal{T}} \in \mathrm{Con}(Q_{\mathcal{R}}) \cap V_0$, there exists $w_1, w_2 \in Q_{\mathcal{T}} - \{v_{\mathcal{T}}\}$ such that $([w_1, w_2] \cap V_0) - \{w_1, w_2\} = \{v_{\mathcal{T}}\}$. If $l = 2$, this leads to a contradiction since $Q_{\mathcal{T}}$ contains only two elements. On the other hand, if $l \geq 3$, we obtain a contradiction with our choice of $v_{\mathcal{T}}$. Indeed, for $i = 1, 2$ we have that $d_T(w_i, Q_{\mathcal{T}'}) \leq d_T(v_{\mathcal{T}}, Q_{\mathcal{T}'})$. Since $v_{\mathcal{T}} \notin \mathcal{T}'$, this implies the existence of $\tilde{w}_i \in Q_{\mathcal{T}'} \cap T(w_i, v_{\mathcal{T}})$ and since there exists a unique simple path between $\tilde{w}_1$ and $\tilde{w}_2$, we obtain that $v_{\mathcal{T}} \in [\tilde{w}_1, \tilde{w}_2] \cap V_0 \subseteq \mathrm{Con}(Q_{\mathcal{T}'}) \cap V_0 = Q_{\mathcal{T}'} \cap V_0$. This is a contradiction since $v_{\mathcal{T}} \notin Q_{\mathcal{T}'}$ which proves that $\mathrm{Con}(Q_{\mathcal{R}}) \cap V_0 = Q_{\mathcal{R}}$. In particular, Lemma 5.5 ensures that $\mathcal{R} = \bigcup_{w \in Q_{\mathcal{R}}} B_T(w, 2) \in \mathfrak{T}_{V_1}[l-1]$. On the other hand, by choice of $v_{\mathcal{T}}$, we have $d_T(v_{\mathcal{T}}, w) \geq d_T(v, w)$ $\forall v \in Q_{\mathcal{T}}$, $\forall w \in Q_{\mathcal{T}'}$ which implies that $\mathrm{Fix}_G(T(\mathcal{R}, v_{\mathcal{T}})) \subseteq \mathrm{Fix}_G(\mathcal{T}')$. On the other hand, $\partial\mathcal{R} \subseteq \partial\mathcal{T} \sqcup \{v_{\mathcal{T}}\}$ and $T(\mathcal{R}, v) = T(\mathcal{T}, v)$ $\forall v \in \partial\mathcal{R} \cap \partial\mathcal{T}$. In particular, since $\mathrm{Fix}_G(\mathcal{T}) = \prod_{v \in \partial\mathcal{T}} \mathrm{Fix}_G(T(\mathcal{T}, v))$, we obtain that

$$\mathrm{Fix}_G(\mathcal{R}) = \prod_{v \in \partial\mathcal{R}} \mathrm{Fix}_G(T(\mathcal{R}, v)) \subseteq \mathrm{Fix}_G(T(\mathcal{R}, v_{\mathcal{T}}))\,\mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

□

The following result plays a similar role as Proposition 4.8 in Section 4.

Proposition 5.11. *Let $G \leq \mathrm{Aut}(T)^+$ be a closed subgroup satisfying the hypothesis H_{V_1} and the property IP_{V_1}. Then, for every integer $l, l' \geq 1$ such that $l' \geq l$, for every U in the conjugacy class of an element of $\mathcal{S}_{V_1}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}_{V_1}[l']$ such that $V \not\subseteq U$, there exists $W \in \mathcal{S}_{V_1}[l-1]$ such that $U \subseteq W \subseteq VU$.*

Proof. Lemma 5.6 ensures that $\mathfrak{T}_{V_1}$ is stable under the action of G. In particular, by Lemma 5.7 there exist $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ and $\mathcal{T}' \in \mathfrak{T}_{V_1}[l']$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. Furthermore, since G satisfies the hypothesis H_{V_1} and since $V \not\subseteq U$ we have $\mathcal{T} \not\subseteq \mathcal{T}'$. In particular, since G satisfies the property IP_{V_1}, Lemma 5.10 ensures the existence of $\mathcal{R} \in \mathcal{T}[l-1]$ such that $\mathcal{R} \subseteq \mathcal{T}$ and $\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T})$. The result follows since, by Lemma 5.7, $W = \mathrm{Fix}_G(\mathcal{R}) \in \mathcal{S}_{V_1}[l-1]$. □

We are finally ready to prove the main result of this Section.

Proof of Theorem D. To prove that $\mathcal{S}_{V_1}$ factorises$^+$ at all depths $l \geq 1$ we shall successively verify the three conditions of Definition 1.5.

First, we need to prove that for every U in the conjugacy class of an element of $\mathcal{S}_{V_1}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}_{V_1}$ such that $V \not\subseteq U$, there exists a W in the conjugacy class of an element of $\mathcal{S}_{V_1}[l-1]$ such that $U \subseteq W \subseteq VU$. Let U, V be as above. If V is conjugate to an element of $\mathcal{S}_{V_1}[l']$ for some $l' \geq l$ the result follows directly from Proposition 5.11. Therefore, we suppose that $l' \lneq l$. Since $\mathfrak{T}_{V_1}$ is stable under the action of G (Lemma 5.6), Lemma 5.7 ensures the existence of $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ and $\mathcal{T}' \in \mathfrak{T}_{V_1}[l']$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. We have two cases.

Either $\mathcal{T}' \subseteq \mathcal{T}$. In that case, we prove the existence of a finite subtree $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ such that $\mathcal{T}' \subseteq \mathcal{R} \subsetneq \mathcal{T}$ by induction on $l - l'$. If $l' - l = 1$ we can take $\mathcal{R} = \mathcal{T}'$ and the result is trivial. On the other hand, if $l' - l \geq 2$, notice that $Q_{\mathcal{T}'} \subseteq Q_{\mathcal{T}'}$ and Lemma 5.4 ensures that $Q_{\mathcal{T}} - Q_{\mathcal{T}'}$ contains $l' - l$ vertices. If $Q_{\mathcal{T}'}$ is empty, there exists $v \in Q_{\mathcal{T}}$ such that $\mathcal{T}' \subseteq B_T(v, 2)$. In particular, we let $\mathcal{P} = B_T(v, 2)$ and notice that $\mathcal{P} \in \mathfrak{T}_{V_1}[l'+1]$ and that $\mathcal{T}' \subseteq \mathcal{P} \subsetneq \mathcal{T}$. If $Q_{\mathcal{T}'}$ is not empty, Lemma 5.5 ensures that $\mathrm{Con}(Q_{\mathcal{T}}) \cap V_0 = Q_{\mathcal{T}}$ and at least one vertex $v \in Q_{\mathcal{T}} - Q_{\mathcal{T}'}$ satisfies that $d_T(v, Q_{\mathcal{T}'}) = 2$. We let $Q = Q_{\mathcal{T}'} \cup \{v\}$ and $\mathcal{P} = \bigcup_{w \in Q} B_T(w, 2)$. Notice that $\mathrm{Con}(Q) \cap V_0 = Q$. In particular, Lemma 5.5 ensures that $\mathcal{P} \in \mathfrak{T}_{V_1}[l'+1]$ and we have, by construction, that $\mathcal{T}' \subseteq \mathcal{P} \subsetneq \mathcal{T}$. In both cases ($Q_{\mathcal{T}}$ is empty or not) our induction hypothesis ensures the existence of a finite subtree $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ such that $\mathcal{T}' \subseteq \mathcal{P} \subseteq \mathcal{R} \subsetneq \mathcal{T}$. In particular, we have $\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}')$ which implies, as desired, that

$$\mathrm{Fix}_G(\mathcal{R}) \subseteq \mathrm{Fix}_G(\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T}')\,\mathrm{Fix}_G(\mathcal{T}).$$

Or else, $\mathcal{T}' \not\subseteq \mathcal{T}$. If $Q_{\mathcal{T}'} = \varnothing$, there exists a vertex $v \in V_0 - Q_{\mathcal{T}}$ such that $\mathcal{T}' \subseteq B_T(v, 2)$. In particular, we choose a set $Q \subseteq V_0$ of order l containing v, such that $\mathrm{Con}(Q) \cap V_0 = Q$ and we set $\mathcal{P} = \bigcup_{w \in Q} B_T(w, 2)$. Similarly, if $Q_{\mathcal{T}'} \neq \varnothing$, since $\mathrm{Con}(Q_{\mathcal{T}'}) \cap V_0 = Q_{\mathcal{T}'}$ by Lemma 5.4, there exists a finite set $Q \subseteq V_0$ of order l containing $Q_{\mathcal{T}'}$ and such that $\mathrm{Con}(Q) \cap V_0 = Q$. We set $\mathcal{P} = \bigcup_{w \in Q} B_T(w, 2)$. In both cases ($Q_{\mathcal{T}}$ is empty or not), Lemma 5.5 ensures that $\mathcal{P} \in \mathfrak{T}_{V_1}[l]$ and we have by construction that $\mathcal{T}' \subseteq \mathcal{P}$. In particular, Proposition 5.11 applied to $\mathrm{Fix}_G(\mathcal{T})$ and $\mathrm{Fix}_G(\mathcal{P})$ ensures the existence of a $W \in \mathcal{S}_{V_1}[l-1]$ such that $\mathrm{Fix}_G(\mathcal{T}) \subseteq W \subseteq \mathrm{Fix}_G(\mathcal{P})\,\mathrm{Fix}_G(\mathcal{T})$. On the other hand, $\mathcal{T}' \subseteq \mathcal{P}$ which implies that $\mathrm{Fix}_G(\mathcal{P}) \leq \mathrm{Fix}_G(\mathcal{T}')$. This proves the first condition.

Next, we need to prove that $N_G(U, V) = \{g \in G | g^{-1}Vg \subseteq U\}$ is compact for every V in the conjugacy class of an element of $\mathcal{S}_{V_1}$. Just as before, notice that $V = \mathrm{Fix}_G(\mathcal{T}')$ for some $\mathcal{T}' \in \mathfrak{T}_{V_1}[l']$. Since G satisfies the hypothesis H_{V_1} notice that

$$\begin{aligned} N_G(U,V) &= \{g \in G | g^{-1}Vg \subseteq U\} = \{g \in G | g^{-1} \operatorname{Fix}_G(\mathcal{T}')g \subseteq \operatorname{Fix}_G(\mathcal{T})\} \\ &= \{g \in G | \operatorname{Fix}_G(g^{-1}\mathcal{T}') \subseteq \operatorname{Fix}_G(\mathcal{T})\} = \{g \in G | g\mathcal{T} \subseteq \mathcal{T}'\}. \end{aligned}$$

Since both $\mathcal{T}$ and $\mathcal{T}'$ are finite subtrees of T, this implies that $N_G(U,V)$ is a compact subset of G which proves the second condition.

Finally, we need to prove that for every W in the conjugacy class of an element of $\mathcal{S}_{V_1}[l-1]$ such that $U \subseteq W$ we have

$$W \subseteq N_G(U,U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

The same reasoning as before ensures the existence of some $\mathcal{R} \in \mathfrak{T}_{V_1}$ such that $W = \operatorname{Fix}_G(\mathcal{R})$. On the other hand, since $U \subseteq W$ and since G satisfies the hypothesis H_{V_1}, notice that $\mathcal{R} \subseteq \mathcal{T}$. We have multiple cases. If $l = 1$, there exist vertices $v \in V_0$ and $w \in V_1$ such that $\mathcal{T} = B_T(v,2)$ and $\mathcal{R} = B_T(w,1)$. In particular, since $\mathcal{R} \subseteq \mathcal{T}$, this implies that $v \in \mathcal{R}$ and therefore that

$$\operatorname{Fix}_G(\mathcal{R}) \subseteq \operatorname{Fix}_G(v) = \{h \in G | hB_T(v,2) \subseteq B_T(v,2)\} = \{h \in G | h\mathcal{T} \subseteq \mathcal{T}\}.$$

Similarly, if $l \geq 2$, notice that $Q_{\mathcal{T}}$ and $Q_{\mathcal{R}}$ are non-empty sets. Furthermore, since $\mathcal{R} \subseteq \mathcal{T}$, we have that $Q_{\mathcal{R}} \subseteq Q_{\mathcal{T}}$ and there exists a unique $v \in Q_{\mathcal{T}} - Q_{\mathcal{R}}$. On the other hand, Lemma 5.5 implies that $d_T(v, Q_{\mathcal{R}}) = 2$ and since $\mathcal{R} = \bigcup_{w \in Q_{\mathcal{R}}} B_T(w,2)$ we observe that $\operatorname{Fix}_G(\mathcal{R}) \subseteq \operatorname{Fix}_G(Q_{\mathcal{T}})$. Since $\mathcal{T} = \bigcup_{w \in Q_{\mathcal{T}}} B_T(w,2)$, this implies that

$$\operatorname{Fix}_G(\mathcal{R}) \subseteq \{h \in G | h\mathcal{T} \subseteq \mathcal{T}\}.$$

In both cases, since G satisfies the hypothesis H_{V_1} we obtain that

$$\begin{aligned} \operatorname{Fix}_G(\mathcal{R}) &\subseteq \{h \in G | h\mathcal{T} \subseteq \mathcal{T}\} = \{h \in G | \operatorname{Fix}_G(\mathcal{T}) \subseteq \operatorname{Fix}_G(h\mathcal{T})\} \\ &= \{h \in G | h^{-1} \operatorname{Fix}_G(\mathcal{T})h \subseteq \operatorname{Fix}_G(\mathcal{T})\} = N_G(U,U) \end{aligned}$$

which proves the third condition. ☐

In particular, if $G \leq \operatorname{Aut}(T)^+$ is a closed non-discrete unimodular subgroup satisfying the hypothesis H_{V_1} and the property IP_{V_1} Theorem A provides a bijective correspondence between the equivalence classes of irreducible representations of G at depth $l \geq 1$ with seed $C \in \mathcal{F}_{\mathcal{S}_{V_1}}$ and the $\mathcal{S}_{V_1}$-standard representations of $\operatorname{Aut}_G(C)$. As a concrete example, the group $\operatorname{Aut}(T)^+$ of type-preserving automorphisms of a (d_0, d_1)-semi-regular tree T with $d_0, d_1 \geq 3$ satisfies the hypotheses of Theorem D. Other examples will be constructed in Section 6.

5.3. **Existence of $\mathcal{S}_{V_1}$-standard representations.** Let T be a locally finite tree and let $V(T) = V_0 \sqcup V_1$ be a bipartition of T such that every edge of T contains exactly one vertex in each V_i. Let $\mathfrak{T}_{V_1}$ be the family of subtrees defined on page 396, let G be a closed non-discrete unimodular subgroup of $\operatorname{Aut}(T)^+$ and let $\mathcal{S}_{V_1} = \{\operatorname{Fix}_G(\mathcal{T}) | \mathcal{T} \in \mathfrak{T}_{V_1}\}$. If $G \leq \operatorname{Aut}(T)^+$ is a closed unimodular subgroup satisfying the hypothesis H_{V_1} and the property IP_{V_1} we have shown that $\mathcal{S}_{V_1}$ is a generic filtration of G factorising$^+$ at all depths $l \geq 1$. In particular, Theorem A ensures the existence of a bijective correspondence between the equivalence classes of irreducible representations of G at depth $l \geq 1$ with seed $C \in \mathcal{F}_{\mathcal{S}_{V_1}}$ and the $\mathcal{S}_{V_1}$-standard representations of $\operatorname{Aut}_G(C)$. The following result treats the existence of $\mathcal{S}_{V_1}$-standard representations of $\operatorname{Aut}_{\mathcal{S}_{V_1}}(C)$ for all $C \in \mathcal{F}_{\mathcal{S}_{V_1}}$ at height $l \geq 1$ provided that G satisfy some geometric property.

Proposition 5.12. *Let $G \leq \mathrm{Aut}(T)^+$ be a closed non-discrete unimodular subgroup satisfying the hypothesis H_{V_1} and the property IP_{V_1}, let $l \geq 1$ and let $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ be such that for every $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ with $\mathcal{R} \subseteq \mathcal{T}$ we have*

$$\mathrm{Fix}_G(\mathcal{R}) \subsetneq \mathrm{Stab}_G(\mathcal{T}) = \{g \in G | g\mathcal{T} \subseteq \mathcal{T}\}.$$

Then, there exists an $\mathcal{S}_{V_1}$-standard representation of $\mathrm{Aut}_{\mathcal{S}_{V_1}}(\mathcal{C}(\mathrm{Fix}_G(\mathcal{T})))$.

Proof. Let $C = \mathcal{C}(\mathrm{Fix}_G(\mathcal{T}))$. Lemma 5.7 ensures that C has height l in $\mathcal{F}_{\mathcal{S}_{V_1}}$. Since G satisfies the hypothesis H_{V_1} and as a consequence of Lemma 5.7, notice that $N_G(\mathrm{Fix}_G(\mathcal{T}), \mathrm{Fix}_G(\mathcal{T})) = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} = \mathrm{Stab}_G(\mathcal{T})$, that $\mathrm{Aut}_G(C) \cong \mathrm{Stab}_G(\mathcal{T}) / \mathrm{Fix}_G(\mathcal{T})$ and that

$$\begin{aligned} \tilde{\mathfrak{H}}_{\mathcal{S}_{V_1}}(\mathrm{Fix}_G(\mathcal{T})) &= \{W \mid \exists g \in G \text{ s.t. } gWg^{-1} \in \mathcal{S}_{V_1}[l-1] \text{ and } \mathrm{Fix}_G(\mathcal{T}) \subseteq W\} \\ &= \{\mathrm{Fix}_G(\mathcal{R}) | \mathcal{R} \in \mathcal{S}_{V_1}[l-1] \text{ s.t. } \mathcal{R} \subseteq \mathcal{T}\}. \end{aligned}$$

Furthermore, the hypotheses on G imply that $\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{R}) \subsetneq \mathrm{Stab}_G(\mathcal{T})$ for every $\mathcal{R} \in \mathfrak{T}_{V_1}[l-1]$ with $\mathcal{R} \subseteq \mathcal{T}$.

We have two cases. If $\mathcal{T} \in \mathfrak{T}_{V_1}[1]$, there exists $v \in V_0$ such that $\mathcal{T} = B_T(v, 2)$ and every subtree $\mathcal{R}$ of $\mathcal{T}$ that belongs to $\mathcal{S}_{V_1}[0]$ is of the form $B_T(w, 1)$ for some $w \in \partial B_T(v, 1)$. Let $\{\mathcal{T}_1, \dots, \mathcal{T}_d\}$ be the set of subtrees of T of the form $\bigcup_{w \in \partial B_T(v,1) - \{u\}} B_T(w, 1)$ for some $u \in \partial B_T(v, 1)$. Notice that each element of $\mathrm{Stab}_G(\mathcal{T}) = \mathrm{Fix}_G(v)$ permutes the vertices of $\partial B_T(v, 1)$ and therefore permutes the elements of $\{\mathcal{T}_1, \dots, \mathcal{T}_d\}$. On the other hand, for every $i, j \in \{1, \dots, d\}$ with $i \neq j$, we have that $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$ and thanks to Lemma 5.8, $\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i) \subsetneq \mathrm{Stab}_G(\mathcal{T})$. In particular, Proposition 2.29 ensures the existence of an irreducible representation σ of $\mathrm{Aut}_G(C) \cong \mathrm{Stab}_G(\mathcal{T}) / \mathrm{Fix}_G(\mathcal{T})$ without non-zero $p_{\mathrm{Fix}_G(\mathcal{T})}(\mathrm{Fix}_G(\mathcal{T}_i))$-invariant vector and therefore without non-zero $p_{\mathrm{Fix}_G(\mathcal{T})}(\mathrm{Fix}_G(\mathcal{R}))$-invariant vector for every subtree $\mathcal{R} \in \mathcal{S}_{V_1}[0]$ such that $\mathcal{R} \subseteq \mathcal{T}$.

If $\mathcal{T} \in \mathfrak{T}_{V_1}[l]$ for some $l \geq 2$, every subtree $\mathcal{R} \in \mathcal{S}_{V_1}[l-1]$ is such that $Q_{\mathcal{R}} \neq \varnothing$. Let $\{\mathcal{T}_1, \dots, \mathcal{T}_d\}$ be the set of subtrees $\mathcal{R}$ of $\mathcal{S}_{V_1}[l-1]$ such that $\mathcal{R} \subseteq \mathcal{T}$ and notice that $Q_{\mathcal{T}_i} \subsetneq Q_{\mathcal{T}}$ $\forall i$. Furthermore, notice the elements of $\mathrm{Stab}_G(\mathcal{T})$ permute the elements of $Q_{\mathcal{T}}$ and therefore the elements of $\{\mathcal{T}_1, \dots, \mathcal{T}_d\}$. On the other hand, for every $i, j \in \{1, \dots, d\}$ with $i \neq j$, we have $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$. In particular, Proposition 2.29 ensures the existence of an irreducible representation σ of $\mathrm{Aut}_G(C) \cong \mathrm{Stab}_G(\mathcal{T}) / \mathrm{Fix}_G(\mathcal{T})$ without non-zero $p_{\mathcal{T}}(\mathrm{Fix}_G(\mathcal{T}_i))$-invariant vectors and the result follows. □

6. Universal groups of certain right-angled buildings

The purpose of this section is to prove that the universal groups of certain semi-regular right-angled buildings embed as closed subgroups of the group $\mathrm{Aut}(T)^+$ of type-preserving automorphisms of a locally finite tree T and that those subgroups satisfy the hypotheses of Theorem D if the prescribed local action is 2-transitive on panels. In particular, for every such group we obtain a generic filtration factorising$^+$ at all depths and the machinery developed in the first part of these notes applies to these groups.

6.1. **Preliminaries.** In this document, we realise buildings as W-metric spaces associated with a Coxeter system (W, I). We now formalise these concepts and refer to [AB08] for more details. In what follows, given a Coxeter system (W, I) and an element $w \in W$, we denote by $l(w)$ the length of w with respect to I.

Definition 6.1. Let (W, I) be a Coxeter system. A **building** Δ of type (W, I) is a couple $(\mathrm{Ch}(\Delta), \delta)$ where $\mathrm{Ch}(\Delta)$ is a set called **the set of chambers** of Δ and where

$$\delta : \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to W$$

is a map satisfying the following conditions for all chambers $c, d \in \mathrm{Ch}(\Delta)$:

(i) $\delta(c, d) = 1_W$ if and only if $c = d$.
(ii) If $\delta(c, d) = w$ and $c' \in \mathrm{Ch}(\Delta)$ satisfies $\delta(c', c) = i \in I$ we have that $\delta(c', d) \in \{w, iw\}$. If, in addition, $l(iw) = l(w) + 1$, then $\delta(c', d) = iw$.
(iii) If $\delta(c, d) = w$, then for any $i \in I$, there is a chamber $c' \in \mathrm{Ch}(\Delta)$ such that $\delta(c', c) = i$ and $\delta(c', d) = iw$.

The map δ is called the **Weyl distance** of Δ.

Definition 6.2. For every subset $J \subseteq I$ we let W_J be the subgroup of W generated by J and we define the J**-residue** of Δ containing a chamber $c \in \mathrm{Ch}(\Delta)$ to be the set

$$\mathcal{R}_J(c) = \{d \in \mathrm{Ch}(\Delta) : \delta(c, d) \in W_J\}.$$

An $\{i\}$-residue is called an i**-panel**. In addition, when $\mathcal{R}$ is a residue of Δ, we denote by $\mathrm{Ch}(\mathcal{R})$ the set of chambers $c \in \mathrm{Ch}(\Delta)$ that belong to $\mathcal{R}$.

In these notes, we will only be interested in right-angled buildings. We recall that a building is **right-angled** if its type (W, I) is given by a right-angled Coxeter system that is if for any two generators $i, j \in I$ we either have that i and j commute or generate a free product $C_2 * C_2$ of two copies of the cyclic group of order two C_2. The following result ensures the existence of a rich family of regular buildings for any right-angled Coxeter system.

Theorem 6.3 ([HP03, Proposition 1.2])**.** *Let (W, I) be a right-angled Coxeter system and $(q_i)_{i\in I}$ be a set of positive integers with $q_i \geq 2$. Then, there exists a right-angled building Δ of type (W, I) such that for every $i \in I$, each i-panel of Δ has size q_i. This building is unique, up to isomorphism.*

Definition 6.4. The essentially unique building Δ given by Theorem 6.3 is the **semi-regular building of prescribed thickness** $(q_i)_{i\in I}$.

Our next task is to recall one of the fundamental features of buildings that is the existence of combinatorial projections between residues. To this end, we make a series of definitions.

Definition 6.5. Two chambers $c, d \in \mathrm{Ch}(\Delta)$ are said to be i**-adjacent** for some $i \in I$ if $\delta(c, d) = i$. A **gallery** in Δ between two chambers c, d is a finite sequence $c_1, \ldots, c_n$ of chambers such that $c_1 = c$, $c_n = d$ and such that c_t and c_{t+1} are i_t-adjacent for all $t \in \{1, \ldots, n-1\}$. In that case, the gallery is said to have **length** $n - 1$ and it is **minimal** if there is no shorter gallery between c and d.

This notion of gallery provides a discrete valued metric

$$d_\Delta : \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to \mathbb{N}$$

on the set of chambers $\mathrm{Ch}(\Delta)$ where $d_\Delta(c, d)$ is the length of a minimal gallery containing both c and d. Now, given a chamber $c \in \mathrm{Ch}(\Delta)$ and a residue $\mathcal{R}$ in a building Δ, the **gate property** ensures the existence of a unique chamber $d \in \mathrm{Ch}(\mathcal{R})$ that is closest to c for the chamber metric d_Δ. This unique chamber is called the **projection** of c on $\mathcal{R}$ and is denoted by $\mathrm{proj}_{\mathcal{R}}(c)$. We refer to [Tit74] for more details about this notion and state some of its properties. To start with, we recall that for any two residues $\mathcal{R}$ and $\mathcal{R}'$ of Δ, the set

$$\{\mathrm{proj}_{\mathcal{R}}(c) : c \in \mathrm{Ch}(\mathcal{R}')\}$$

is the chamber set of a residue of Δ contained in $\mathcal{R}$. Furthermore, as we recall below, if Δ is a right-angled building this notion of projection provides a way to partition the building into convex chamber sets. A subset $C \subseteq \mathrm{Ch}(\Delta)$ is said to be **convex** if for any two chambers $c, d \in C$, C contains each of the chambers appearing in any minimal gallery between c and d.

Definition 6.6. Let Δ be a right-angled building of type (W, I), $J \subseteq I$ and $c \in \mathrm{Ch}(\Delta)$. The **$J$-wing** of Δ containing c is the set

$$X_J(c) = \{d \in \mathrm{Ch}(\Delta) : \mathrm{proj}_{\mathcal{R}_J(c)}(d) = c\}.$$

If $J = \{i\}$ is a singleton, we refer to this set as the **i-wing** $X_i(c)$ of c.

It is shown in [Cap14] that wings are convex chamber sets and that for every J-residue $\mathcal{R}$, $\mathrm{Ch}(\Delta)$ is partitioned by the J-wings $X_J(c)$ with $c \in \mathcal{R}$. We now recall some of their properties.

Lemma 6.7 ([Cap14, Lemma 3.1])**.** *Let Δ be a right-angled building of type (W, I), $J \subseteq I$ be a non-empty set and $c \in \mathrm{Ch}(\Delta)$. The J-wing $X_J(c)$ containing c satisfies the following properties:*

(1) $X_J(c) = \bigcap_{i \in J} X_i(c)$.
(2) $X_J(c) = X_J(c')$ *for all* $c' \in X_J(c) \cap \mathcal{R}_{J \cup J^\perp}(c)$.

Lemma 6.8. *Let Δ be a right-angled building of type (W, I), $J, J' \subseteq I$ be two disjoint subsets and $c \in \mathrm{Ch}(\Delta)$. Then $\mathcal{R}_J(c) \subseteq X_{J'}(c)$.*

Proof. The result follows directly from the fact the two residues $\mathcal{R}_J(c)$ and $\mathcal{R}_{J'}(c)$ contain c and that the intersection of a J-residue and a J'-residue is a $J \cap J'$-residue. In particular, this proves that $\mathrm{proj}_{\mathcal{R}_{J'}(c)}(\mathcal{R}_J(c)) = \{c\}$ and therefore that $\mathcal{R}_J(c) \subseteq X_{J'}(c)$. ☐

Lemma 6.9 ([Cap14, Lemma 3.4])**.** *Let Δ be a right-angled building of type (W, I), $i, i' \in I$ be such that $m_{i,j} = \infty$ and $c, c' \in \mathrm{Ch}(\Delta)$ be such that $c' \in X_i(c)$ but $c \notin X_{i'}(c')$. Then, we have $X_{i'}(c') \subseteq X_i(c)$.*

Another feature of combinatorial projections is given by the relation of parallelism. Two residues $\mathcal{R}$ and $\mathcal{R}'$ in a building Δ are said to be **parallel** if $\mathrm{proj}_{\mathcal{R}}(\mathcal{R}') = \mathcal{R}$ and $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}) = \mathcal{R}'$. Notice that the chamber sets of parallel residues are in bijection under the respective projection maps and that two parallel residues have the same rank. Caprace showed that the relation of parallelism has a particular flavour in right-angled buildings.

Lemma 6.10 ([Cap14, Corollary 2.9])**.** *Let Δ be a right-angled building. Then the relation of parallelism of residues is an equivalence relation.*

Lemma 6.11 ([Cap14, Proposition 2.8])**.** *Let Δ be a right-angled building of type (W, I) and let $J \subseteq I$. Then two J-residues $\mathcal{R}$ and $\mathcal{R}'$ are parallel if and only if they are both contained in a common $J \cup J^\perp$-residue where $J^\perp = \{i \in I : ij = ji \ \forall j \in J\}$.*

We now recall the notion of universal groups of semi-regular right-angled building introduced by Tom De Medts, Ana C. Silva and Koen Struyve in [DMSS18] to generalise the concept of Burger and Mozes universal groups on trees. We start by recalling the definition of type-preserving automorphisms of buildings.

Definition 6.12. Let Δ be a building of type (W, I) and Weyl distance δ. A **type-preserving automorphism** of Δ is a bijection

$$g : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) : c \mapsto gc$$

preserving the Weyl distance in the sense that for all $c, d \in \mathrm{Ch}(\Delta)$ one has

$$\delta(gc, gd) = \delta(c, d).$$

We denote by $\mathrm{Aut}(\Delta)$ the group of type-preserving automorphisms of Δ.

The definition of universal groups of right-angled building requires a notion of coloring of the building. Let Δ be a semi-regular right-angled building of type (W, I) and prescribed thickness $(q_i)_{i\in I}$.

Definition 6.13. For each $i \in I$, let Y_i be a set of size q_i which we will refer to as the set of i-colors of Δ. A **set of legal colorings** of Δ is a set $(h_i)_{i\in I}$ of maps

$$h_i : \mathrm{Ch}(\Delta) \to Y_i$$

such that $h_i\big|_{\mathrm{Ch}(\tau)} : \mathrm{Ch}(\tau) \to Y_i$ is a bijection on each i-panel τ of Δ, and such that $h_i(c) = h_i(c')$ for every $(I - \{i\})$-residue $\mathcal{R}$ and each $c, c' \in \mathrm{Ch}(\mathcal{R})$.

Now, for each $i \in I$, let $G_i \leq \mathrm{Sym}(Y_i)$ be a transitive permutation group and let $(h_i)_{i\in I}$ be a set of legal colorings of Δ.

Definition 6.14. The **universal group** $\mathcal{U}((h_i, G_i)_{i\in I})$ of Δ with respect to the set of legal colorings $(h_i)_{i\in I}$ and prescribed local action $(G_i)_{i\in I}$ is the subgroup of $\mathrm{Aut}(\Delta)$ defined by

$$\{g \in \mathrm{Aut}(\Delta) : (h_i\big|_{g\tau}) \circ g \circ (h_i\big|_{\tau})^{-1} \in G_i \ \forall i \in I \text{ and every } i - \text{panel } \tau \text{ of } \Delta\}.$$

It appears that those groups satisfy a factorisation property similar to the Tits independence property that we now introduce.

Definition 6.15. Let Δ be a right-angled building of type (W, I) and $J \subseteq I$. A subgroup $G \leq \mathrm{Aut}(\Delta)$ is said to satisfy the property IP_J if for all $J \cup J^{\perp}$-residues $\mathcal{R}$ of Δ we have that

$$\mathrm{Fix}_G(\mathcal{R}) = \prod_{c\in\mathcal{R}} \mathrm{Fix}_G(V_J(c)), \qquad (\mathrm{IP}_J)$$

where $V_J(c) = \{d \in \mathrm{Ch}(\Delta) : \mathrm{proj}_{\mathcal{R}}(d) \neq c\}$ is the complement of the J-wing containing c and where $\mathrm{Fix}_G(\mathcal{R}) = \{g \in G : gc = c \ \forall c \in \mathrm{Ch}(\mathcal{R})\}$.

Proposition 6.16 ensures that every universal group of right-angled building satisfies the property $\mathrm{IP}_{\{i\}}$ for every $i \in I$. Furthermore, Proposition 6.17 ensures that they also satisfy the property IP_J for every finite set $J \subseteq I$ such that $\{i\} \cup \{i\}^{\perp} = J$ $\forall i \in J$.

Proposition 6.16 ([DMSS18, Proposition 3.16])**.** *Let G be a universal group of a semi-regular right-angled building Δ. Then G satisfies the property $\mathrm{IP}_{\{i\}}$ for every*

$i \in I$. Furthermore, for all $i \in I$, $g \in \mathrm{Fix}_G(\mathcal{R}_{\{i\}\cup\{i\}^\perp}(c))$ and for each $c \in \mathrm{Ch}(\Delta)$ the type-preserving automorphism

$$g^c : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) : x \mapsto \begin{cases} gx & \text{if } x \in X_i(c), \\ x & \text{if } x \in V_i(c) \end{cases}$$

is an element of G.

Proposition 6.17. *Let $G \leq \mathrm{Aut}(\Delta)$ be a universal group of a semi-regular right-angled building Δ. Then, G satisfies the property IP_J for every finite set $J \subseteq I$ such that $\{i\} \cup \{i\}^\perp = J\ \forall i \in J$.*

Proof. If $|J|= 1$, the results follows directly from Proposition 6.16. Suppose therefore that $|J|\geq 2$ and let us show that

$$\mathrm{Fix}_G(\mathcal{R}) = \prod_{c\in\mathcal{R}} \mathrm{Fix}_G(V_J(c))$$

for every J-residue $\mathcal{R}$. First, notice that $\mathrm{Fix}_G(V_J(c))$ is a subgroup of $\mathrm{Fix}_G(\mathcal{R})$ for every $c \in \mathcal{R}$. Lemma 6.7 ensures that $X_J(c) = \bigcap_{i\in J} X_i(c)$ for every $c \in \mathrm{Ch}(\mathcal{R})$. In particular, taking the complement, we obtain that $V_J(c) = \bigcup_{i\in J} V_i(c)$ and therefore that $\mathrm{Fix}_G(V_J(c)) = \bigcap_{i\in J} \mathrm{Fix}_G(V_i(c))$. Let $i \in J$ and recall from Proposition 6.16 that G satisfies the property $\mathrm{IP}_{\{i\}}$ which implies that $\mathrm{Fix}_G(V_i(c)) \subseteq \mathrm{Fix}_G(\mathcal{R}_{\{i\}\cup\{i\}^\perp}(c))$. Furthermore, since $J = \{i\} \cup \{i\}^\perp$, we obtain that $\mathcal{R} = \mathcal{R}_{\{i\}\cup\{i\}^\perp}(c)$ and therefore that $\mathrm{Fix}_G(V_J(c)) \subseteq \mathrm{Fix}_G(\mathcal{R})$. Notice that for every two distinct $c, d \in \mathrm{Ch}(\mathcal{R})$, the supports of the elements of $\mathrm{Fix}_G(V_i(c))$ and $\mathrm{Fix}_G(V_i(d))$ are disjoint from one another which proves that $\prod_{c\in\mathcal{R}} \mathrm{Fix}_G(V_J(c))$ is a well-defined subgroup of G. The above discussion proves that $\prod_{c\in\mathcal{R}} \mathrm{Fix}_G(V_J(c)) \subseteq \mathrm{Fix}_G(\mathcal{R})$. To prove the other inclusion, let $g \in \mathrm{Fix}_G(\mathcal{R})$ and let $J = \{i_1, \dots, i_n\}$. For any $i \in J$ and $c \in \mathrm{Ch}(\mathcal{R})$, let $\mathcal{R}_i(c)$ be the i-panel containing c in $\mathcal{R}$. Let us fix some chamber $c \in \mathrm{Ch}(\mathcal{R})$ and let

$$g_1^d : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) : x \mapsto \begin{cases} gx & \text{if } x \in X_{i_1}(d), \\ x & \text{if } x \in V_{i_1}(d) \end{cases}$$

for every $d \in \mathcal{R}_{i_1}(c)$. Proposition 6.16 ensures that $g_1^d \in G\ \forall d \in \mathcal{R}_{i_1}(c)$ and that $g = \prod_{d\in\mathcal{R}_{i_1}(c)} g_1^d$. On the other hand, for every $d \in \mathcal{R}_{i_1}(c)$, there exists a unique i_2-panel $\mathcal{R}_{i_2}(d)$ such that $d \in \mathcal{R}_{i_2}(d)$. Since $g_1^d \in \mathrm{Fix}_G(V_{i_1}(d)) \subseteq \mathrm{Fix}_G(\mathcal{R})$, we can repeat the above argument and we obtain that $g_1^d = \prod_{d'\in\mathcal{R}_{i_2}(d)} g_2^{d'}$ where

$$g_2^{d'} : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) : x \mapsto \begin{cases} g_1^d x & \text{if } x \in X_{i_2}(d'), \\ x & \text{if } x \in V_{i_2}(d') \end{cases}$$

for every $d' \in \mathcal{R}_{i_2}(d)$. Just as before, Proposition 6.16 ensures that $g_2^{d'} \in G$ for every $d' \in \mathcal{R}_{i_2}(d)$. On the other hand, $\mathcal{R}_{i_1}(c) \cap \mathcal{R}_{i_2}(d) = \mathcal{R}_{i_1}(d) \cap \mathcal{R}_{i_2}(d) = \{d\}$. In particular, this implies that $\mathrm{proj}_{\mathcal{R}_{i_1}(c)}(\mathcal{R}_{i_2}(d)) = \{d\}$ and $d' \in X_{i_1}(d)$. Since $\mathcal{R}$ is an $\{i_1\} \cup \{i_1\}^\perp$-residue, Lemma 6.7 ensures that $X_{i_1}(d) = X_{i_1}(d')$. This proves that $g_2^{d'}$ has support in $X_{i_1}(d') \cap X_{i_2}(d') = X_{\{i_1,i_2\}}(d')$ and therefore that $g_2^{d'} \in \mathrm{Fix}_G(V_{\{i_1,i_2\}}(d'))$. Proceeding iteratively, for any of the constructed g_k^d with

$k \in \{2, \dots, n-1\}$, we set

$$g_{k+1}^{d'} : \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) : x \mapsto \begin{cases} g_k^d x & \text{if } x \in X_{i_{k+1}}(d'), \\ x & \text{if } x \in V_{i_{k+1}}(d') \end{cases}$$

for every $d' \in \mathcal{R}_{i_{k+1}}(d)$. Once more, Proposition 6.16 ensures that $g_{k+1}^{d'} \in G$ $\forall d' \in \mathcal{R}_{i_{k+1}}(d)$ and that $g_k^d = \prod_{d' \in \mathcal{R}_{i_{k+1}}(d)} g_{k+1}^{d'}$. On the other hand, $\mathcal{R}_{i_l}(d) \cap \mathcal{R}_{i_{k+1}}(d) = \{d\}$ for every $l = 1, \dots, k$ which implies that $d' \in X_{i_l}(d)$. Since $\mathcal{R}$ is an $\{i_l\} \cup \{i_l\}^{\perp}$-residue, Lemma 6.7 ensures that $X_{i_l}(d) = X_{i_l}(d')$. Finally, since g_k^d has support in $X_{i_1}(d) \cap X_{i_2}(d) \cap \dots \cap X_{i_k}(d)$ this proves that $g_{k+1}^{d'}$ has support in $X_{i_1}(d') \cap \dots X_{i_k}(d') \cap X_{i_{k+1}}(d') = X_{\{i_1,\dots,i_{k+1}\}}(d')$ and therefore that $g_{k+1}^{d'} \in \mathrm{Fix}_G(V_{\{i_1,i_2,\dots,i_{k+1}\}}(d'))$. □

6.2. **Groups of automorphisms of certain right-angled buildings as groups of automorphisms of trees.** The purpose of this section is to show that the group of type-preserving automorphisms $\mathrm{Aut}(\Delta)$ of certain semi-regular right-angled buildings Δ can be realised as closed subgroups of the group $\mathrm{Aut}(T)^+$ of type-preserving automorphisms of a locally finite tree T in such a way that the universal groups of those buildings embed as closed subgroups $G \leq \mathrm{Aut}(T)^+$ satisfying the property IP_{V_1}. This applies only to certain Coxeter types and motivates Definition 6.18.

Definition 6.18. A right-angled Coxeter system (W, I) is said to satisfy the hypothesis $\star$ if it is finitely generated and there exists $r \geq 2$ such that

$$(\star) \qquad I = \bigsqcup_{k=1}^{r} I_k$$

for some $I_k = \{i\} \cup \{i\}^{\perp}$ $\forall i \in I_k$ and $\forall k = 1, \dots, r$.

Remark 6.19. A right-angled Coxeter system satisfying the hypothesis $\star$ is isomorphic to a free product $W_1 * W_2 * \dots * W_r$ where each of the W_k is a direct product of finitely many copies of the group of order 2. In particular, W is virtually free.

Let (W, I) be a right-angled Coxeter system satisfying the hypothesis $\star$, let $(q_i)_{i \in I}$ be a set of positive integers $q_i \geq 2$ and let Δ be a semi-regular building of type (W, I) and prescribed thickness $(q_i)_{i \in I}$. We associate a locally finite bipartite graph to Δ as follows. We let $V_0 = \mathrm{Ch}(\Delta)$,

$$V_1 = \{\mathcal{R} | \mathcal{R} \text{ is an } I_k - \text{residue of } \Delta \text{ for some } k \in \{1, \dots, r\}\}$$

and we define T as the bipartite graph with vertex set $V(T) = V_0 \sqcup V_1$ and where a chamber $c \in V_0$ is adjacent to a residue $\mathcal{R} \in V_1$ if $c \in \mathcal{R}$.

Lemma 6.20. *The graph T is a locally finite tree.*

Proof. The graph T is locally finite since each chamber is contained in finitely many residues and since each I_k-residue is finite. The graph T is path connected since every two chambers of Δ are connected by a gallery and since each such gallery corresponds naturally to a path in T. We now show that T does not contain any cycle. Suppose for a contradiction that there is a simple cycle in T, say

$$c_1 - \mathcal{R}_1 - c_2 - \dots - \mathcal{R}_n - c_1.$$

Since each chamber $c \in \mathrm{Ch}(\Delta)$ is contained in a unique residue $\mathcal{R}$ of type I_t and since the cycle is simple, notice that $\mathcal{R}_1$ and $\mathcal{R}_n$ have different types. In particular, Lemma 6.8 ensures that $\mathcal{R}_n \subseteq X_{J_1}(c_1)$ where J_1 is the type of $\mathcal{R}_1$. On the other hand, as we show below, $\mathcal{R}_n \subseteq X_{J_1}(c_2)$. For now, we assume this inclusion that is $\mathcal{R}_n \subseteq X_{J_1}(c_1) \cap X_{J_1}(c_2)$ and we show this leads to a contradiction. Since the cycle is simple, we have that $c_1 \neq c_2$. Furthermore, since $c_1, c_2 \in \mathcal{R}_1$, there exists some $i \in J_1$ such that $c_1 \notin X_i(c_2)$. Hence, we have that $X_i(c_1) \cap X_i(c_2) = \varnothing$ and therefore that $X_{J_1}(c_1) \cap X_{J_1}(c_2) = \varnothing$. The desired contradiction follows from our inclusion.

Now, let us prove that $\mathcal{R}_n \subseteq X_{J_1}(c_2)$. To this end, we show that $X_{J_{t+1}}(c_{t+2}) \subseteq X_{J_t}(c_{t+1})$ for every $t \in \{1, \ldots, n-2\}$ where J_t is the type of $\mathcal{R}_t$. Since the cycle is simple notice that $\mathcal{R}_t$ and $\mathcal{R}_{t+1}$ have different types and that $\mathcal{R}_{t+1} \subseteq X_{J_t}(c_{t+1})$ for every $t = 1, \ldots, n-2$. On the other hand, since $c_{t+1} \neq c_{t+2}$, there exists some $i' \in J_{t+1}$ such that $c_{t+1} \notin X_{i'}(c_{t+2})$. Notice for every $i \in J_t$ that $m_{i,i'} = \infty$ and that $c_{t+2} \in \mathcal{R}_{t+1} \subseteq X_i(c_{t+1})$. In particular, Lemma 6.9 implies that $X_{J_{t+1}}(c_{t+2}) \subseteq X_{i'}(c_{t+2}) \subseteq \bigcap_{i \in J_t} X_i(c_{t+1}) = X_{J_t}(c_{t+1})$ which completes the induction. This proves as desired that $\mathcal{R}_n \subseteq X_{J_{n-1}}(c_n) \subseteq \cdots \subseteq X_{J_1}(c_2)$. □

Our next goal is to explicit an injective map $\alpha : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(T)^+$ defining an homeomorphism on its image. Notice that any type-preserving automorphism $g \in \mathrm{Aut}(\Delta)$ is bijective on the set of chambers $\mathrm{Ch}(\Delta)$ but also on the I_k-residues of Δ for any fixed k. For every $g \in \mathrm{Aut}(\Delta)$ we define the map $\alpha(g) : V(T) \to V(T)$ as follows:

- If $v \in V_0$ then v is a chamber $c \in \mathrm{Ch}(\Delta)$ and we define $\alpha(g)v = gc$.
- If $v \in V_1$ then v is an I_k-residue $\mathcal{R}$ of Δ for some $k \in \{1, \ldots, r\}$ and we define $\alpha(g)v = g\mathcal{R}$.

The map $\alpha(g)$ clearly defines a type-preserving bijection on $V(T)$. In fact, $\alpha(g)$ is a tree automorphism of T and $\alpha : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(T)^+$ is a well-defined group homomorphism since for every $g \in \mathrm{Aut}(\Delta)$, every residue $\mathcal{R}$ of Δ and every $c \in \mathrm{Ch}(\Delta)$, we have that $c \in \mathcal{R}$ if and only if $gc \in g\mathcal{R}$.

Proposition 6.21. *The map* $\alpha : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(T)^+$ *is an injective group homomorphism;* $\alpha(\mathrm{Aut}(\Delta))$ *is a closed subgroup of* $\mathrm{Aut}(T)^+$ *and* α *defines an homeomorphism between* $\mathrm{Aut}(\Delta)$ *and* $\alpha(\mathrm{Aut}(\Delta))$.

Proof. The homomorphism α is injective, since

$$\begin{aligned}\ker(\alpha) &= \{g \in \mathrm{Aut}(\Delta) | \alpha(g) = 1_{\mathrm{Aut}(T)}\} \\ &\subseteq \{g \in \mathrm{Aut}(\Delta) | gc = c \ \forall c \in \mathrm{Ch}(\Delta)\} = \{1_{\mathrm{Aut}(\Delta)}\}.\end{aligned}$$

We recall that the sets

$$U_T(F_T) = \{g \in \mathrm{Aut}(T)^+ | gv = v \ \forall v \in F_T\}$$

with finite subset $F_T \subsetneq V_0$ form a basis of open neighbourhoods of the identity in $\mathrm{Aut}(T)^+$. On the other hand, an element $h \in \mathrm{Aut}(T)^+$ belongs to $\mathrm{Aut}(T)^+ - \alpha(\mathrm{Aut}(\Delta))$ if and only if there exist $i \in I$ and two i-adjacent chambers $c, d \in V_0$ such that hc and hd are not i-adjacent. In particular, for every such automorphism h, the set $hU_T(\{c, d\})$ is an open neighbourhood of h in $\mathrm{Aut}(T)^+ - \alpha(\mathrm{Aut}(\Delta))$. This proves that the complement of $\alpha(\mathrm{Aut}(\Delta))$ is an open set and therefore that $\alpha(\mathrm{Aut}(\Delta))$ is a closed subgroup of $\mathrm{Aut}(T)^+$.

Let $\Phi : \mathrm{Ch}(\Delta) \to V_0$ be the map sending a chamber of Δ to the corresponding vertex of $V_0 \subseteq V(T)$ and recall that the sets

$$U_\Delta(F_\Delta) = \{g \in \mathrm{Aut}(\Delta) | gc = c \ \forall c \in F_\Delta\}$$

where $F_\Delta \subseteq \mathrm{Ch}(\Delta)$ is finite form a basis of open neighbourhoods of the identity in $\mathrm{Aut}(\Delta)$. In particular, notice that $\alpha : \mathrm{Aut}(\Delta) \to \alpha(\mathrm{Aut}(\Delta))$ is continuous since for every finite subset $F_T \subseteq V_0$ we have that $\alpha^{-1}(U_T(F_T) \cap \alpha(\mathrm{Aut}(\Delta)) = U_\Delta(\Phi^{-1}(F_T))$. Finally, notice that $\alpha : \mathrm{Aut}(\Delta) \to \alpha(\mathrm{Aut}(\Delta))$ is an open map since, for every finite set $F_\Delta \subsetneq \mathrm{Ch}(\Delta)$ we have that $\alpha(U_\Delta(F_\Delta)) = U_\Delta(\Phi(F_\Delta)) \cap \alpha(\mathrm{Aut}(\Delta))$. □

Proposition 6.22 shows that under this correspondence, the property IP_{V_1} of groups of type-preserving automorphisms of trees is tightly related to the property IP_J of groups of type-preserving automorphisms of right-angled buildings.

Proposition 6.22. *Let $G \leq \mathrm{Aut}(\Delta)$ be a closed subgroup satisfying the property IP_{I_k} for every $k = 1, \ldots, r$. Then, $\alpha(G)$ is a closed subgroup of $\mathrm{Aut}(T)^+$ satisfying the property IP_{V_1}.*

Proof. Proposition 6.21 ensures that $\alpha(G)$ is a closed subgroup of $\alpha(\mathrm{Aut}(\Delta))$ and therefore of $\mathrm{Aut}(T)^+$. Let $\Phi : \mathrm{Ch}(\Delta) \to V_0$ be the map sending a chamber of Δ to the corresponding vertex of $V_0 \subseteq V(T)$ and notice that:

- For every residue $\mathcal{R} \in V_1$, $\Phi(\mathcal{R}) = \{v \in V_0 | v \in \mathcal{R}\} = B_T(\mathcal{R}, 1) \cap V_0$. In particular, we have that
$$\begin{aligned}\alpha(\mathrm{Fix}_G(\mathcal{R})) &= \mathrm{Fix}_{\alpha(G)}(\Phi(\mathcal{R})) = \mathrm{Fix}_{\alpha(G)}(B_T(\mathcal{R}, 1) \cap V_0)\\ &= \mathrm{Fix}_{\alpha(G)}(B_T(\mathcal{R}, 1)).\end{aligned}$$
- For every $k \in \{1, \ldots, r\}$ and every chamber $c \in \mathrm{Ch}(\Delta)$ we have
$$\begin{aligned}\Phi(\mathrm{Ch}(\Delta) - X_{I_k}(c)) &= \Phi(\{d \in \mathrm{Ch}(\Delta) | \mathrm{proj}_{\mathcal{R}_{I_k}(c)}(d) \neq c\})\\ &= \{v \in V_0 | d_T(v, \mathcal{R}_{I_k}(c)) \lneq d_T(v, c)\}\\ &= T(\mathcal{R}_{I_k}(c), c) \cap V_0.\end{aligned}$$
  In particular, we obtain that
$$\begin{aligned}\alpha(\mathrm{Fix}_G(\mathrm{Ch}(\Delta) - X_{I_k}(c))) &= \mathrm{Fix}_{\alpha(G)}(\Phi(\mathrm{Ch}(\Delta) - X_{I_k}(c)))\\ &= \mathrm{Fix}_{\alpha(G)}(T(\mathcal{R}_{I_k}(c), c)).\end{aligned}$$

The results follows from the definitions of the properties IP_{I_k} and IP_{V_1}. □

The proof of Theorem E requires one last preliminary. Let (W, I) be a right-angled Coxeter system. Let $(q_i)_{i\in I}$ be a set of positive integers $q_i \geq 2$ and let Δ be the semi-regular building of type (W, I) and prescribed thickness $(q_i)_{i\in I}$. Finally, let $(h_i)_{i\in I}$ be a set of legal colorings of Δ with i-colors given by a set Y_i of size q_i and let $G_i \leq \mathrm{Sym}(Y_i)$ be transitive on Y_i.

Proposition 6.23. *For every $J \subseteq I$, every $(g_j)_{j\in J} \in \prod_{j\in J} G_j$ and every $c \in \mathrm{Ch}(\Delta)$, there exists $g \in \mathcal{U}((h_i, G_i)_{i\in I})$ such that $g\mathcal{R}_J(c) = \mathcal{R}_J(c)$, $h_j \circ g = g_j \circ h_j$ for all $j \in J$ and $h_i \circ g = h_i$ for all $i \in I - J$.*

Proof. Let $(h'_i)_{i\in I}$ be the set of legal colorings obtained from $(h_i)_{i\in I}$ by replacing h_j by $g_j \circ h_j$ for all $j \in J$ and leaving the other colorings unchanged. Notice that h'_j is still a legal coloring of G for every $j \in J$ since for all $I - \{j\}$-residues $\mathcal{R}$ and for all $d, d' \in \mathcal{R}$ we have $h'_j(c) = g_j \circ h_j(d) = g_j \circ h_j(d') = h'_j(d')$. Now, let c' be the chamber of $\mathcal{R}_J(c)$ with colors $h_j(c') = g_j \circ h_j(c)$ for every $j \in J$. Since $h'_i(c') = h_i(c)$ for every $i \in I$, [DMSS18, Proposition 2.44] ensures the existence of an automorphism $g \in \mathrm{Aut}(\Delta)$ mapping c to c' and such that $h_i \circ g = h'_i$ for all $i \in I$. Since $c' \in \mathcal{R}_J(c)$ notice that g stabilises $\mathcal{R}_J(c)$. Finally, notice that g acts locally as the identity on i-panels for all $i \in I - J$ and as g_j on j-panels for all $j \in J$. Hence, g is as desired. $\square$

The following result proves Theorem E.

Theorem 6.24. *Suppose that (W, I) satisfies the hypothesis $\star$. The group $\alpha(\mathcal{U}((h_i, G_i)_{i\in I}))$ is a closed subgroup of $\mathrm{Aut}(T)^+$ satisfying the property IP_{V_1}. In addition, if G_i is 2-transitive on Y_i for every $i \in I$, $\alpha(\mathcal{U}((h_i, G_i)_{i\in I}))$ is unimodular and satisfies the hypothesis H_{V_1}.*

Proof. Let $G = \mathcal{U}((h_i, G_i)_{i\in I})$. The first part of the theorem follows directly from Proposition 6.17 and Proposition 6.22. Now, suppose that G_i is 2-transitive on Y_i for every $i \in I$, let $\mathfrak{T}_{V_1}$ be the family of subtrees defined on page 396 and let $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_{V_1}$. If $\mathcal{T} \subseteq \mathcal{T}'$ we clearly have that $\mathrm{Fix}_{\alpha(G)}(\mathcal{T}') \subseteq \mathrm{Fix}_{\alpha(G)}(\mathcal{T})$. On the other hand, if $\mathcal{T} \not\subseteq \mathcal{T}'$, let $v_{\mathcal{T}}$ be a vertex of $V_1 \cap \mathcal{T}$ that is at maximal distance from $\mathcal{T}'$ and let $w_{\mathcal{T}} \in B_T(v_{\mathcal{T}}, 1) - \{v_{\mathcal{T}}\}$ be such that $\mathcal{T}' \subseteq T(w_{\mathcal{T}}, v_{\mathcal{T}})$. In particular, we have that $\mathrm{Fix}_{\alpha(G)}(T(w_{\mathcal{T}}, v_{\mathcal{T}})) \leq \mathrm{Fix}_{\alpha(G)}(\mathcal{T}')$. Let I_k denote the type of $v_{\mathcal{T}}$ seen as a residue of Δ and let $j \in I_k$. Consider an element $g_j \in G_i$ that is not trivial and such that $g_j \circ h_j(w_{\mathcal{T}}) = h_j(w_{\mathcal{T}})$ and let $g_i = \mathrm{id}_{Y_i}$ for every $i \in I - \{j\}$. Proposition 6.23 ensures the existence of an element $g \in \mathcal{U}((h_i, G_i)_{i\in I})$ such that $g\mathcal{R}_J(c) = \mathcal{R}_J(c)$ and $h_i \circ g = g_i \circ h_i$ for every $i \in I$. Now, notice that there exists a unique vertex $v'_{\mathcal{T}}$ that is adjacent to $w_{\mathcal{T}}$ and such that $\mathcal{T}' \subseteq T(v'_{\mathcal{T}}, w_{\mathcal{T}}) \cup \{w_{\mathcal{T}}\}$. Now, notice that $v'_{\mathcal{T}}$ is a residue of type $I_{k'}$ with $I_{k'} \neq I_k$ and we realise from the definition that g fixes every chamber of $v'_{\mathcal{T}}$ or equivalently that $\alpha(g)$ fixes $B_T(v'_{\mathcal{T}}, 1)$ pointwise. Proposition 6.22 implies the existence of an element $h \in \mathrm{Fix}_{\alpha(G)}(B_T(v'_{\mathcal{T}}), 1) \cap \mathrm{Fix}_G(T(v'_{\mathcal{T}}, w_{\mathcal{T}}))$ such that $hv = \alpha(g)v$ for every $v \in T(w_{\mathcal{T}}, v'_{\mathcal{T}})$. In particular, we have that $h \in \mathrm{Fix}_{\alpha(G)}(\mathcal{T}')$ but $h \notin \mathrm{Fix}_{\alpha(G)}(\mathcal{T})$ since by the definition g does not fix every chamber of $v_{\mathcal{T}}$. This proves as desired that $\mathrm{Fix}_{\alpha(G)}(\mathcal{T}') \subsetneq \mathrm{Fix}_{\alpha(G)}(\mathcal{T})$.

To prove that $\alpha(G)$ is unimodular, we apply [BRW07, Corollary 5]. This result ensures that a group G which acts δ-2-transitively on the set of chambers of a locally finite building is unimodular. Choose a chamber $c \in \mathrm{Ch}(\Delta)$. Since G is transitive on the chambers of Δ, we need to show for any two chambers $d_1, d_2 \in \mathrm{Ch}(\Delta)$ that are W-equidistant from c that there exists an element $g \in \mathrm{Fix}_G(c)$ such that $gd_1 = d_2$. First of all, notice from the hypothesis $\star$ and the solution of the word problem in Coxeter groups that every $w \in W$ admits a unique decomposition $w = w_1 \dots w_n$ with $w_t \in W_{I_{k_t}} - \{1_W\}$ such that $I_{k_t} \neq I_{k_{t+1}}$ for every t. Suppose that $d_1, d_2 \in \mathrm{Ch}(\Delta)$ have W-distance $w_1 \dots w_n$ from c with $w_t \in W_{I_{k_t}}$ and let us show the existence of g by induction on n. If $n = 1$, the result follows from the first part of the proof since for every $j \in I_{k_1}$, there exists an element $g_j \in G_j$ such that $g_j \circ h_j(c) = h_j(c)$ and $g_j \circ h_j(d_1) = h_j(d_2)$. Hence, there exists an element $g \in G$ such that $h_j \circ g_j = g \circ h_j$ for every $j \in J$. In particular, $gc = c$, $gd_1 = d_2$ and the result follows. If $n \geq 2$, we let $d'_s = \mathrm{proj}_{\mathcal{R}_{I_{k_n}}(d_s)}(c)$ and notice that $\delta(c, d'_s) = w_1 \dots w_{n-1}$. Our induction hypothesis therefore ensures the existence of a $g' \in G$ such that $g'c = c$ and $g'd'_1 = d'_2$. Now, notice that $\delta(d'_2, d_2) = w_n$ and $\delta(d'_2, g'd_1) = \delta(g'd'_1, g'd_1) = \delta(d'_1, d_1) = w_n$. In particular, the first part of the proof ensures the existence of an element $h \in G$ such that $hd = d$ for every $d \in \mathcal{R}_{I_{k_{(n-1)}}}(d'_2)$ and $hg'd_1 = d_2$. Since G satisfies the property $\mathrm{IP}_{I_{k_{(n-1)}}}$ by Proposition 6.17, this implies the existence of an element $h' \in \mathrm{Fix}_G(V_{I_{k_{(n-1)}}}(d'_2))$ such that

$h'b = hb$ for every $b \in X_{I_{k_{(n-1)}}}(d_2')$. Since $c \in V_{I_{k_{(n-1)}}}(d_2')$, the automorphism $h'g' \in G$ satisfies that $h'g'c = c$ and that $h'g'd_1 = d_2$. The result follows. □

In particular, if $\mathcal{U}((h_i, G_i)_{i\in I})$ is non-discrete, Theorem D applies to $\alpha(G)$ and Theorem A provides a bijective correspondence between the equivalence classes of irreducible representations of $\alpha(G)$ at depth $l \geq 1$ with seed $C \in \mathcal{F}_{\mathcal{S}_{V_1}}$ and the $\mathcal{S}_{V_1}$-standard representations of $\mathrm{Aut}_{\alpha(G)}(C)$. We recall further that an existence criterion for those representations was given in Section 5.3.

Since α is a homeomorphism on its image, the same holds for the representations of G. Notice that under the correspondence given by α^{-1}, the generic filtration $\mathcal{S}_{V_1}$ describes a generic filtration $\mathcal{S}_\Delta$ of G that factorises$^+$ at all depths $l \geq 1$ and which can be interpreted as follows. We explicit this correspondence below. Let δ denote the W-distance of Δ and let us consider the set

$$\mathcal{R}'(c) = \{d \in \mathrm{Ch}(\Delta) | \delta(c,d) = w \text{ s.t. } \exists k \in \{1,\dots,r\} \text{ for which } w \in W_{I_k}\}$$

for every chamber $c \in \mathrm{Ch}(\Delta)$. By use of the correspondence $\Phi : \mathrm{Ch}(\Delta) \to V_0$ between chambers of Δ and vertices of V_0, notice, for every $c \in V_0$, that $\Phi^{-1}(B_T(\Phi(c),2) \cap V_0)) = \mathcal{R}'(c)$. We define a family $\mathcal{T}_\Delta$ of subsets of $\mathrm{Ch}(\Delta)$ as follows:

(1) $\mathcal{T}_\Delta[0] = \{\mathcal{R}_{I_k}(c) | k \in \{1,\dots,r\}, c \in \mathrm{Ch}(\Delta)\}$.

(2) For every l such that $l \geq 0$, we define iteratively:

$$\mathcal{T}_\Delta[l+1] = \{\mathcal{R} \subseteq \mathrm{Ch}(\Delta) | \exists \mathcal{Q} \in \mathcal{T}_\Delta[l],\ \exists c \in \mathcal{Q} \text{ s.t. } \mathcal{R}'(c) \not\subseteq \mathcal{Q} \text{ and } \mathcal{R} = \mathcal{Q} \cup \mathcal{R}'(c)\}.$$

(3) We set $\mathcal{T}_\Delta = \bigsqcup_{l\in\mathbb{N}} \mathcal{T}_\Delta[l]$.

It is quite easy to realise that $\mathcal{S}_\Delta = \{\mathrm{Fix}_G(\mathcal{R}) | \mathcal{R} \in \mathfrak{T}_\Delta\}$ is the generic filtration of G corresponding to $\mathcal{S}_{V_1}$ under the correspondence given by α^{-1} and that

$$\mathcal{S}_\Delta[l] = \alpha^{-1}\big(\mathcal{S}_{V_1}[l]\big) = \{\mathrm{Fix}_G(\mathcal{R}) | \mathcal{R} \in \mathcal{T}_\Delta[l]\}.$$

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References

[AB08] Peter Abramenko and Kenneth S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008. Theory and applications, DOI 10.1007/978-0-387-78835-7. MR2439729

[Ama03] Olivier Eric Amann, *Groups of tree-automorphisms and their unitary representations*, Ph.D. Thesis, ETH Zurich, 2003.

[BEW15] Christopher Banks, Murray Elder, and George A. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, J. Group Theory **18** (2015), no. 2, 235–261, DOI 10.1515/jgth-2014-0041. MR3318536

[BRW07] Udo Baumgartner, Bertrand Rémy, and George A. Willis, *Flat rank of automorphism groups of buildings*, Transform. Groups **12** (2007), no. 3, 413–436, DOI 10.1007/s00031-006-0050-3. MR2356316

[BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008, DOI 10.1017/CBO9780511542749. MR2415834

[Ber74] I. N. Bernšteĭn, *All reductive $\mathfrak{p}$-adic groups are of type I* (Russian), Funkcional. Anal. i Priložen. **8** (1974), no. 2, 3–6. MR0348045

[Cap14] Pierre-Emmanuel Caprace, *Automorphism groups of right-angled buildings: simplicity and local splittings*, Fund. Math. **224** (2014), no. 1, 17–51, DOI 10.4064/fm224-1-2. MR3164745

[CKM22] Pierre-Emmanuel Caprace, Mehrdad Kalantar, and Nicolas Monod, *A type i conjecture and boundary representations of hyperbolic groups*, Preprint, arXiv:2110.00190, 2022.

[DMSS18] Tom De Medts, Ana C. Silva, and Koen Struyve, *Universal groups for right-angled buildings*, Groups Geom. Dyn. **12** (2018), no. 1, 231–287, DOI 10.4171/GGD/443. MR3781422

[Dix59] J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents. I* (French), Amer. J. Math. **81** (1959), 160–170, DOI 10.2307/2372853. MR103943

[Dix77] Jacques Dixmier, *C^*-algebras*, North-Holland Mathematical Library, Vol. 15, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett. MR0458185

[DM76] M. Duflo and Calvin C. Moore, *On the regular representation of a nonunimodular locally compact group*, J. Functional Analysis **21** (1976), no. 2, 209–243, DOI 10.1016/0022-1236(76)90079-3. MR0393335

[FTN91] Alessandro Figà-Talamanca and Claudio Nebbia, *Harmonic analysis and representation theory for groups acting on homogeneous trees*, London Mathematical Society Lecture Note Series, vol. 162, Cambridge University Press, Cambridge, 1991, DOI 10.1017/CBO9780511662324. MR1152801

[HP03] Frédéric Haglund and Frédéric Paulin, *Constructions arborescentes d'immeubles* (French, with English summary), Math. Ann. **325** (2003), no. 1, 137–164, DOI 10.1007/s00208-002-0373-x. MR1957268

[HR19] Cyril Houdayer and Sven Raum, *Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras*, Comment. Math. Helv. **94** (2019), no. 1, 185–219, DOI 10.4171/CMH/458. MR3941470

[Isa76] I. Martin Isaacs, *Character theory of finite groups*, Pure and Applied Mathematics, No. 69, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. MR0460423

[KT13] Eberhard Kaniuth and Keith F. Taylor, *Induced representations of locally compact groups*, Cambridge Tracts in Mathematics, vol. 197, Cambridge University Press, Cambridge, 2013. MR3012851

[Kna86] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986. An overview based on examples, DOI 10.1515/9781400883974. MR855239

[Mat77] Hideya Matsumoto, *Analyse harmonique dans les systèmes de Tits bornologiques de type affine* (French), Lecture Notes in Mathematics, Vol. 590, Springer-Verlag, Berlin-New York, 1977. MR0579177

[MP96] Allen Moy and Gopal Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121, DOI 10.1007/BF02566411. MR1371680

[Neb90] Claudio Nebbia, *Classification of all irreducible unitary representations of the stabilizer of the horicycles [horocycles] of a tree*, Israel J. Math. **70** (1990), no. 3, 343–351, DOI 10.1007/BF02801468. MR1074496

[Neb99] Claudio Nebbia, *Groups of isometries of a tree and the CCR property*, Rocky Mountain J. Math. **29** (1999), no. 1, 311–316, DOI 10.1216/rmjm/1181071692. MR1687668

[Ol′77] G. I. Ol′šanskiĭ, *Classification of the irreducible representations of the automorphism groups of Bruhat-Tits trees* (Russian), Funkcional. Anal. i Priložen. **11** (1977), no. 1, 32–42, 96. MR0578650

[Ol'80] G. I. Ol′šanskiĭ, *New "large" groups of type* I, Current problems in mathematics, Vol. 16 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980, pp. 31–52, 228. MR611159

[Rad17] Nicolas Radu, *A classification theorem for boundary 2-transitive automorphism groups of trees*, Invent. Math. **209** (2017), no. 1, 1–60, DOI 10.1007/s00222-016-0704-2. MR3660305

[Sem22] Lancelot Semal, *Radu groups acting on trees are CCR*, Preprint, `arXiv:2203.04189`, 2022.

[Tho68] Elmar Thoma, *Eine Charakterisierung diskreter Gruppen vom Typ I* (German), Invent. Math. **6** (1968), 190–196, DOI 10.1007/BF01404824. MR248288

[Tit74] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin-New York, 1974. MR0470099

[Tor20] Stephan Tornier, *Groups acting on trees with prescribed local action*, Preprint, `arXiv:2002.09876`, 2020.

UCLouvain, Chemin du Cyclotron 2/L7.01.02, 1348 Louvain-la-Neuve, Belgium
Email address: `lancelot.semal@uclouvain.be`