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SKEW CELLULARITY OF THE HECKE ALGEBRAS OF TYPE $G(\ell, p, n)$

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Abstract. This paper introduces (graded) skew cellular algebras, which generalise Graham and Lehrer's cellular algebras. We show that all of the main results from the theory of cellular algebras extend to skew cellular algebras and we develop a "cellular algebra Clifford theory" for the skew cellular algebras that arise as fixed point subalgebras of cellular algebras.

As an application of this general theory, the main result of this paper proves that the Hecke algebras of type $G(\ell, p, n)$ are graded skew cellular algebras. In the special case when $p = 2$ this implies that the Hecke algebras of type $G(\ell, 2, n)$ are graded cellular algebras. The proofs of all of these results rely, in a crucial way, on the diagrammatic Cherednik algebras of Webster and Bowman. Our main theorem extends Geck's result that the one parameter Iwahori-Hecke algebras are cellular algebras in two ways. First, our result applies to all cyclotomic Hecke algebras in the infinite series in the Shephard-Todd classification of complex reflection groups. Secondly, we lift cellularity to the graded setting.

As applications of our main theorem, we show that the graded decomposition matrices of the Hecke algebras of type $G(\ell, p, n)$ are unitriangular, we construct and classify their graded simple modules and we prove the existence of "adjustment matrices" in positive characteristic.

1. Introduction

The Hecke algebras of complex reflection groups were introduced by Ariki and Koike [1, 2] and Broué and Malle [6], as generalisations of the Iwahori-Hecke algebras of Coxeter groups. Cyclotomic Hecke algebras have been studied extensively both because of their rich representation theory and because of their connections to reductive groups [5]. Interest in these algebras intensified with the introduction of the quiver Hecke algebras, or KLR algebras, which categorify the integrable highest weight representations of Kac-Moody algebras [22, 31]. In particular, Brundan–Kleshchev [7] and Rouquier [31] proved that the Ariki-Koike algebras, which are the Hecke algebras associated to the complex reflection groups $G(\ell, 1, n)$ in the classification of Shephard and Todd [2, 6], are isomorphic to the quiver Hecke algebras $\mathscr{R}_n^{\mathbf{\Lambda}}$ of type A.

The theory of cellular algebras, which was introduced by Graham and Lehrer [15], gives a framework for constructing all the irreducible modules of an algebra. In

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particular, Graham and Lehrer proved that the Ariki–Koike algebras are cellular. Later, Hu and Mathas [18], the first two named authors of this paper, extended this result to show that these algebras are *graded* cellular algebras.

Rostam [29], the third named author of this paper, introduced the quiver Hecke algebras $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ of type $G(\ell, p, n)$ as fixed point subalgebras of $\mathscr{R}^{\mathbf{\Lambda}}_{n}$. Extending Brundan and Kleshchev's graded isomorphism theorem, Rostam proved that $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is isomorphic to a cyclotomic Hecke algebra of type $G(\ell, p, n)$. Under some strong assumptions on the parameters, Rostam used [18] to prove that $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is a cellular algebra. For example, he showed that $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is a graded cellular algebra if p and n are coprime. In general, he proved that a natural basis of $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$, arising from a particular cellular basis of the Ariki–Koike algebra cannot be an "adapted" cellular basis; see [30, §5.2].

We can now state the main result of this paper, which shows that the Hecke algebras of type $G(\ell, p, n)$ are graded skew cellular algebras, with the important case $p = 2$.

Main Theorem. *Let $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ be the KLR algebra of type $G(\ell, p, n)$. Then $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is a graded skew cellular algebra. Moreover, if $p = 2$ then $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is a graded cellular algebra.*

This result recovers and generalises known results from the (ungraded) representation theory of the Hecke algebra of $G(\ell, p, n)$ to the graded setting. In particular, this proves that the graded decomposition matrices are unitriangular, extending [14], and using [3, §10] and [21] we obtain a new construction and classification of the graded simple modules, extending [14, 16].

As important special cases, our Main Theorem shows that the Iwahori-Hecke algebras of types A_{n-1}, $B_n = C_n$, D_n and $I_2(n)$ are graded skew cellular algebras. These are the complex reflection groups of types $G(1, 1, n), G(2, 1, n), G(2, 2, n)$ and $G(n, n, 2)$, respectively, in the Shephard–Todd classification. This result extends Geck's theorem [11], which shows that the Iwahori-Hecke algebras of finite Coxeter groups are cellular algebras. In particular, our main theorem gives a new graded cellular algebra structure on the Iwahori-Hecke algebras of type D_n, which are the Hecke algebras of type $G(2, 2, n)$. More generally, we show that the Hecke algebras of type $G(2d, 2, n)$ are graded cellular algebras, for $d, n \geq 1$. When $d > 1$, this result is completely new, even in the ungraded setting. If $d = 1$ and $n \geq 2$ then we generalise Geck's result to the graded setting. Geck's proof relies on Kazhdan–Lusztig theory, which does not exist for complex reflection groups. The proof of our main theorem relies in a crucial way on the diagram calculus introduced by Webster [32] and Bowman [3]. In related work, LePage and Webster [24, §4] generalise Webster's diagrammatic algebras to give diagrammatic Cherednik algebras of type $G(\ell, p, n)$ but they do not consider questions relating to cellularity. Finally, note that since the first version of this paper appeared online, Lehrer and Lyu [23] used our theory of skew cellular algebra to prove that the generalised Temperley–Lieb algebra of type $G(r, p, n)$ is graded cellular.

To prove our main theorem, Definition 2.2 introduces *(graded) skew cellular algebras*, which can be viewed as an analogue of Clifford theory for cellular algebras. More precisely, skew cellular algebras generalise the cellular algebra framework to certain fixed-point subalgebras of cellular algebras. We show that the main structural results of cellular algebras hold for skew cellular algebras. In particular, we show that:

- each (graded) skew cellular algebra has a family of (graded) skew cell modules
- the (graded) simple modules of a (graded) skew cellular algebra arise in a unique way as quotients of the (graded) skew cell modules
- the (graded) decomposition matrices of skew cellular algebras are unitriangular.

In contrast to cellular algebras, the simple modules of a skew cellular algebra are not necessarily self-dual; see Proposition 2.19 for a precise statement.

The outline of this paper is as follows. Section 2 introduces and then develops the representation theory of skew cellular algebras, together with the closely related notion of a shift automorphism of a cellular algebra. Section 2.4 develops Clifford theory in this setting. Section 3 recalls and extends the definitions and known results about the cyclotomic KLR algebras of type $G(\ell,p,n)$ and about the Webster-Bowman diagram calculus for the diagrammatic Cherednik algebras. Section 4 is the technical heart of the paper where we use the diagrammatic Cherednik algebras to define an explicit diagrammatic basis of $\mathscr{R}_n^{\mathbf{\Lambda}}$ (Definition 4.36), which has the properties that we need to prove that $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$ is a skew cellular algebra. Section 5 uses the diagram basis of $\mathscr{R}_n^{\mathbf{\Lambda}}$ constructed in Section 4 to show that $\mathscr{R}_n^{\mathbf{\Lambda}}$ has a shift automorphism. Using the results from Section 2, this implies that $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$ is a skew cellular algebra, establishing our Main Theorem. Finally, as two applications of our main results, Section 5.3 gives an "adjustment matrix" result for the Hecke algebras of type $G(\ell,p,n)$ and Section 5.4 gives the classification of the graded simple $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$-modules.

An index of notation can be found at the end of the paper.

2. Skew cellular algebras

This chapter defines and then develops the representation theory of graded skew cellular algebras. The first section sets our notation for graded algebras. The second section, which is the heart of the chapter, defines skew cellular algebras and shows how to extend the general theory of graded cellular algebras [15, 18] to the skew setting. In the third section we study graded cellular algebras with shift automorphism that, like Clifford theory, provides a general tool for showing that fixed-point subalgebras of cellular algebras are skew cellular algebras. In the fourth section we study Clifford theory for the skew cellular algebras arising from a graded cellular algebra A with a shift automorphism σ, especially when σ_A is ε-splittable (in the sense of Definition 2.39).

2.1. **Graded algebras.** Throughout this paper we fix a commutative integral domain R with one. In this paper a **graded R-module** is a $\mathbb{Z}$-graded R-module $M = \bigoplus_{d\in\mathbb{Z}} M_d$. If $m \in M_d$ then m is **homogeneous** of **degree** d, for $d \in \mathbb{Z}$. If M is a graded R-module and $s \in \mathbb{Z}$ let $M\langle s\rangle$ be the graded R-module that is equal to M as an (ungraded) R-module but where the grading is shifted so that the homogeneous component of $M\langle s\rangle$ of degree d is $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

A **graded algebra** will always mean a $\mathbb{Z}$-graded algebra, which is a graded R-module such that $A_cA_d \subseteq A_{c+d}$, for $c, d \in \mathbb{Z}$. A (graded) A-module is a graded

R-module $M = \bigoplus_{d\in\mathbb{Z}} M_d$ together with an A-action $A \times M \to M$ such that $A_c M_d \subseteq M_{c+d}$, for $c, d \in \mathbb{Z}$.

The category of graded A-modules has objects the graded A-modules and morphisms the homogeneous A-module maps of degree 0. The **graded dimension** of a graded module $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is the Laurent polynomial

$$\dim_t M = \sum_{d\in\mathbb{Z}} (\dim M_d)\, t^d \quad \in \mathbb{Z}[t, t^{-1}].$$

If the algebra A comes equipped with an anti-involution then the **dual** of a graded A-module M is the graded A-module

$$M^* = \operatorname{Hom}_A(M, R) = \bigoplus_{d\in\mathbb{Z}} \operatorname{Hom}_A(M_d, R),$$

where R is in degree 0 and A acts on M^* via its anti-involution. A graded A-module M is **self-dual** if $M \cong M^*$ as graded A-modules.

Finally, if M is a graded R-module let $\underline{M}$ be the ungraded R-module obtained by forgetting the grading. In particular, if M is a graded A-module then $\underline{M}$ is an ungraded $\underline{A}$-module.

2.2. **Skew cellular algebras.** Like cellular algebras, skew cellular algebras are defined in terms of a skew cell datum. To describe these we first need some basic notation.

Recall that a **poset**, or **partially ordered set**, is an ordered pair $(\mathcal{P}, \trianglerighteq)$, where $\mathcal{P}$ is a set and $\trianglerighteq$ is a reflexive, antisymmetric and transitive relation on $\mathcal{P}$. If $x, y \in \mathcal{P}$ and $x \trianglerighteq y$ then we write $y \trianglelefteq x$. In addition, if $x \neq y$ then write $x \triangleright y$ and $y \triangleleft x$.

Definition 2.1. A **poset automorphism** of $(\mathcal{P}, \trianglerighteq)$ is a permutation σ of $\mathcal{P}$ such that

$$\lambda \trianglerighteq \mu \text{ if and only if } \sigma(\lambda) \trianglerighteq \sigma(\mu), \qquad \text{for all } \lambda, \mu \in \mathcal{P}.$$

If $\sigma = \iota$ is an involution we say that ι is a **poset involution** of $(\mathcal{P}, \trianglerighteq)$.

Note that if σ a poset automorphism of $(\mathcal{P}, \trianglerighteq)$ then $\lambda \triangleright \mu \iff \sigma(\lambda) \triangleright \sigma(\mu)$. Following [15, 18], we can now define graded skew cellular algebras.

Definition 2.2 ($\mathbb{Z}$-graded skew cellular algebras)**.** Let R be an integral domain and A a $\mathbb{Z}$-graded R-algebra that is free and of finite rank as R-module.

A **graded skew cell datum** for A is an ordered quintuple $(\mathcal{P}, \iota, T, C, \deg)$ where $(\mathcal{P}, \trianglerighteq)$ is a poset, ι is a poset involution of $\mathcal{P}$, for each $\lambda \in \mathcal{P}$ there is a finite set $T(\lambda)$ together with a bijection

$$\iota_\lambda : T(\lambda) \longrightarrow T(\iota(\lambda)); \mathfrak{s} \mapsto \iota(\mathfrak{s}) = \iota_\lambda(\mathfrak{s}), \qquad \text{for all } \mathfrak{s} \in T(\lambda),$$

such that $\iota_{\iota(\lambda)} \circ \iota_\lambda = \mathrm{id}_{T(\lambda)}$, and

$$C : \coprod_{\lambda\in\mathcal{P}} T(\lambda) \times T(\lambda) \longrightarrow A; (\mathfrak{s}, \mathfrak{t}) \mapsto c_{\mathfrak{s}\mathfrak{t}}, \quad \text{and} \quad \deg : \coprod_{\lambda\in\mathcal{P}} T(\lambda) \to \mathbb{Z}$$

are two functions such that C is injective and

(C_1) Each element $c_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg c_{\mathfrak{s}\mathfrak{t}} = \deg \mathfrak{s} + \deg \mathfrak{t}$, for $\lambda \in \mathcal{P}$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$.

(C_2) The set $\{c_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in T(\lambda), \lambda \in \mathcal{P}\}$ is an R-basis of A.

(C_3) If $a \in A$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for $\lambda \in \mathcal{P}$, then there exist scalars $r_{\mathfrak{v}\mathfrak{s}}(a)$, which do not depend on $\mathfrak{t}$, such that

$$ac_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{v} \in T(\lambda)} r_{\mathfrak{v}\mathfrak{s}}(a) c_{\mathfrak{v}\mathfrak{t}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda}$ is the R-submodule of A spanned by $\{c_{\mathfrak{a}\mathfrak{b}} \mid \mathfrak{a}, \mathfrak{b} \in T(\mu) \text{ for } \lambda \lhd \mu \in \mathcal{P}\}$.

(C_4) There is a unique R-algebra anti-isomorphism $* : A \longrightarrow A$ such that $(c_{\mathfrak{s}\mathfrak{t}})^* = c_{\iota(\mathfrak{t})\iota(\mathfrak{s})}$, for all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $\lambda \in \mathcal{P}$.

A **$\mathbb{Z}$-graded skew cellular algebra** is a graded algebra that has a graded skew cell datum. The basis $\{c_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathcal{P} \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$ is a **$\mathbb{Z}$-graded skew cellular basis** of A.

Applying the anti-isomorphism $*$ to relation (C_3) and using (C_4) together with the assumption that ι is a poset involution shows that if $a \in A$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for $\lambda \in \mathcal{P}$, then

$$(C_3') \qquad c_{\iota(\mathfrak{s})\iota(\mathfrak{t})} a^* = \sum_{\mathfrak{u} \in T(\lambda)} r_{\mathfrak{u}\mathfrak{t}}(a) c_{\iota(\mathfrak{s})\iota(\mathfrak{u})} \pmod{A^{\rhd\iota(\lambda)}}.$$

Therefore, after relabelling, if $a \in A$ and $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for $\lambda \in \mathcal{P}$, then

$$(2.3) \qquad c_{\mathfrak{s}\mathfrak{t}} a = \sum_{\mathfrak{u} \in T(\lambda)} r_{\iota(\mathfrak{u}),\iota(\mathfrak{t})}(a^*) c_{\mathfrak{s}\mathfrak{u}} \pmod{A^{\rhd\lambda}},$$

where the scalars $r_{\iota(\mathfrak{u}),\iota(\mathfrak{t})}(a^*)$ are the same scalars appearing in (C_3). In particular, $r_{\iota(\mathfrak{u}),\iota(\mathfrak{t})}(a^*)$ does not depend on $\mathfrak{s}$.

Remark 2.4. If $\iota = \mathrm{id}_{\mathcal{P}}$ is the identity map, and $\iota_\lambda = \mathrm{id}_{T(\lambda)}$ for all $\lambda \in \mathcal{P}$, then Definition 2.2 recovers the definition of *graded cellular algebras* from [18]. If, in addition, we forget the grading on A then Definition 2.2 reduces to Graham and Lehrer's original definition of cellular algebras [15]. Thus, graded cellular algebras are given by a graded cellular datum $(\mathcal{P}, T, C, \deg)$ and cellular algebras are given by a cell datum $(\mathcal{P}, T, C)$. A **skew cellular algebra** is a graded skew cellular algebra that is concentrated in degree 0. In particular, skew cellular algebras are a generalisation of Graham and Lehrer's definition of cellular algebras.

The reader might find it helpful to refer to Example 2.5 when reading this section. More complicated examples of skew cellular algebras are given in Example 2.33.

Example 2.5. We give a "toy example". Let R be any ring and let x and y be indeterminates over R. Fix an integer $m \geq 1$ and set $A = R[x]/(x^m) \oplus R[y]/(y^m)$. Let $(\mathcal{P}, \trianglerighteq)$ be the poset $\mathcal{P} = \mathbb{Z}_2 \times \{0, 1, \dots, m-1\}$ with $(i,k) \trianglerighteq (i',k')$ only if $i = i'$ and $k \geq k'$ (as integers). Define the poset involution $\iota : \mathcal{P} \longrightarrow \mathcal{P}$ by $\iota(i,k) = (i+1,k)$. For $\lambda = (i,k) \in \mathcal{P}$ set $T(\lambda) = \{k\}$ and $\deg(k) = k$. In particular, $\iota_\lambda(k) = k$. Then, for $\lambda = (i,k) \in \mathcal{P}$ we have $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ only if $\mathfrak{s} = \mathfrak{t} = k$, so define

$$C_{\mathfrak{s}\mathfrak{t}} = C_{kk} = \begin{cases} x^k, & \text{if } i = 0, \\ y^k, & \text{if } i = 1. \end{cases}$$

Then $(\mathcal{P}, \iota, T, C, \deg)$ is a $\mathbb{Z}$-graded skew cell datum for A.

For the rest of this section fix a graded skew cellular algebra A with skew cell datum $(\mathcal{P}, \iota, T, C, \deg)$. We now study the graded representation theory of A, generalising the results of [15, 18].

Definition 2.6. Let $\lambda\in\mathcal{P}$. The **(left) graded skew cell module** C_λ is the left graded A-module with basis $\{c_\mathfrak{s}\mid\mathfrak{s}\in T(\lambda)\}$ and with A-action determined by

$$ac_\mathfrak{s}=\sum_{\mathfrak{t}\in T(\lambda)} r_{\mathfrak{ts}}(a)c_\mathfrak{t},\qquad\text{for } a\in A \text{ and } \mathfrak{s}\in T(\lambda),$$

where $r_{\mathfrak{ts}}(a)\in R$ is the scalar defined in (C_3).

Remark 2.7. The name "skew cell module" also appears in [4] but where the term "skew" refers to skew Young diagrams. A priori, these are different objects.

Let $\lambda\in\mathcal{P}$. Then $C_\lambda=\bigoplus_{d\in\mathbb{Z}}(C_\lambda)_d$ is a graded A-module, where $(C_\lambda)_d$ is the free R-module with basis $\{c_\mathfrak{s}\mid\mathfrak{s}\in T(\lambda)$ with $\deg\mathfrak{s}=d\}$.

If $\mathfrak{u},\mathfrak{s},\mathfrak{t},\mathfrak{v}\in T(\lambda)$ then, by (C_3) and (2.3),

$$c_{\mathfrak{us}}c_{\mathfrak{tv}}\equiv\sum_{\mathfrak{a}\in T(\lambda)} r_{\mathfrak{at}}(c_{\mathfrak{us}})c_{\mathfrak{av}}\equiv\sum_{\mathfrak{b}\in T(\lambda)} r_{\iota(\mathfrak{b})\iota(\mathfrak{s})}(c^*_{\mathfrak{tv}})c_{\mathfrak{ub}}\pmod{A^{\rhd\lambda}}.$$

It follows that

$$r_{\mathfrak{at}}(c_{\mathfrak{us}})\neq 0 \text{ only if } \mathfrak{a}=\mathfrak{u},\tag{2.8}$$

and $r_{\iota(\mathfrak{b})\iota(\mathfrak{s})}(c^*_{\mathfrak{tv}})\neq 0$ only if $\mathfrak{b}=\mathfrak{v}$ and, consequently, that

$$r_{\mathfrak{ut}}(c_{\mathfrak{us}})=r_{\iota(\mathfrak{v})\iota(\mathfrak{s})}(c^*_{\mathfrak{tv}}).$$

By (C_3) and (2.3), the scalar $r_{\mathfrak{ut}}(c_{\mathfrak{us}})$ depends only on $\mathfrak{s}$ and $\mathfrak{t}$ and *not* on the choice of $\mathfrak{u}$ and $\mathfrak{v}$.

Definition 2.9 is motivated by [15, Definition 2.3] and the calculations above.

Definition 2.9. Let $\lambda\in\mathcal{P}$. Let $\phi=\phi_\lambda\colon C_\lambda\times C_\lambda\longrightarrow R$ be the R-bilinear map determined by

$$\phi(c_\mathfrak{s},c_\mathfrak{t}):=r_{\mathfrak{ut}}(c_{\mathfrak{us}})=r_{\iota(\mathfrak{v})\iota(\mathfrak{s})}(c^*_{\mathfrak{tv}})\in R,\qquad\text{for all } \mathfrak{s},\mathfrak{t}\in T(\lambda).$$

Then, by the calculations above,

$$c_{\mathfrak{us}}c_{\mathfrak{tv}}\equiv\phi_\lambda(c_\mathfrak{s},c_\mathfrak{t})c_{\mathfrak{uv}}\pmod{A^{\rhd\lambda}}.\tag{2.10}$$

To better understand ϕ_λ we abuse notation and extend the map $\iota_\lambda\colon T(\lambda)\longrightarrow T(\iota(\lambda))$ to an R-linear isomorphism

$$\iota_\lambda\colon C_\lambda\longrightarrow C_{\iota(\lambda)};c_\mathfrak{s}\mapsto c_{\iota(\mathfrak{s})},\qquad\text{for } s\in T(\lambda).$$

In general, the R-linear map $\iota_\lambda\colon C_\lambda\longrightarrow C_{\iota(\lambda)}$ is not an A-module homomorphism. If $\lambda\in\mathcal{P}$ is fixed and $x\in C_\lambda$ then we simplify our notation and write $\iota(x)=\iota_\lambda(x)\in C_{\iota(\lambda)}$. In particular, $\iota(c_\mathfrak{s})=\iota_\lambda(c_\mathfrak{s})$, for $\mathfrak{s}\in T(\lambda)$.

Lemma 2.11, which gives the main properties of ϕ_λ, is modelled on [26, Proposition 2.9]. However, note that for skew cellular algebras the bilinear form ϕ_λ *not* necessarily symmetric.

Lemma 2.11. *Let $\lambda\in\mathcal{P}$ and $x,y\in C_\lambda$.*

(a) *We have $\phi_\lambda(x,y)=\phi_{\iota(\lambda)}\big(\iota_\lambda(y),\iota_\lambda(x)\big)$.*

(b) *If $a\in A$ then $\phi_\lambda(x,ay)=\phi_{\iota(\lambda)}\big(\iota_\lambda(y),a^*\iota_\lambda(x)\big)=\phi_\lambda\big(\iota_{\iota(\lambda)}(a^*\iota_\lambda(x)),y\big)$.*

(c) *If $\mathfrak{s},\mathfrak{u}\in T(\lambda)$ then $c_{\mathfrak{us}}x=\phi_\lambda(c_\mathfrak{s},x)c_\mathfrak{u}$.*

(d) *The form ϕ_λ is homogeneous of degree* 0.

Proof. Since ϕ_λ is bilinear, and ι_λ is linear, it suffices to consider the cases when $x = c_{\mathfrak{s}}$ and $y = c_{\mathfrak{t}}$ for $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$.

By the discussion above Definition 2.9, for any $\mathfrak{u}, \mathfrak{v} \in T(\lambda)$ we have

$$\phi_\lambda(c_{\mathfrak{s}}, c_{\mathfrak{t}}) = r_{\mathfrak{u}\mathfrak{t}}(c_{\mathfrak{u}\mathfrak{s}}) = r_{\iota(\mathfrak{v}),\iota(\mathfrak{s})}(c^*_{\mathfrak{t}\mathfrak{v}}) = r_{\iota(\mathfrak{v}),\iota(\mathfrak{s})}(c_{\iota(\mathfrak{v}),\iota(\mathfrak{t})}) = \phi_{\iota(\lambda)}(c_{\iota(\mathfrak{t})}, c_{\iota(\mathfrak{s})}),$$

proving (a). Now consider part (b). Working modulo $A^{\rhd\lambda}$, for any $\mathfrak{u}, \mathfrak{v} \in T(\lambda)$

$$\begin{aligned}
\phi_\lambda(c_{\mathfrak{s}}, ac_{\mathfrak{t}})c_{\mathfrak{u}\mathfrak{v}} &= \sum_{\mathfrak{t}' \in T(\lambda)} r_{\mathfrak{t}'\mathfrak{t}}(a)\phi_\lambda(c_{\mathfrak{s}}, c_{\mathfrak{t}'})c_{\mathfrak{u}\mathfrak{v}} && \text{by Definition 2.6} \\
&\equiv \sum_{\mathfrak{t}' \in T(\lambda)} r_{\mathfrak{t}'\mathfrak{t}}(a)c_{\mathfrak{u}\mathfrak{s}}c_{\mathfrak{t}'\mathfrak{v}} && \text{by (2.10)} \\
&\equiv c_{\mathfrak{u}\mathfrak{s}}(ac_{\mathfrak{t}\mathfrak{v}}) && \text{by (C}_3\text{)} \\
&= (c_{\mathfrak{u}\mathfrak{s}}a)c_{\mathfrak{t}\mathfrak{v}} \\
&\equiv \sum_{\mathfrak{s}' \in T(\lambda)} r_{\iota(\mathfrak{s}'),\iota(\mathfrak{s})}(a^*)c_{\mathfrak{u}\mathfrak{s}'}c_{\mathfrak{t}\mathfrak{v}} && \text{by Section 2.2} \\
&\equiv \sum_{\mathfrak{s}' \in T(\lambda)} r_{\iota(\mathfrak{s}'),\iota(\mathfrak{s})}(a^*)\phi_\lambda(c_{\mathfrak{s}'}, c_{\mathfrak{t}})c_{\mathfrak{u}\mathfrak{v}} && \text{by (2.10)} \\
&\equiv \sum_{\mathfrak{s}' \in T(\lambda)} r_{\iota(\mathfrak{s}'),\iota(\mathfrak{s})}(a^*)\phi_{\iota(\lambda)}(c_{\iota(\mathfrak{t})}, c_{\iota(\mathfrak{s}')})c_{\mathfrak{u}\mathfrak{v}} && \text{by part (a)} \\
&= \phi_{\iota(\lambda)}(c_{\iota(\mathfrak{t})}, a^*c_{\iota(\mathfrak{s})})c_{\mathfrak{u}\mathfrak{v}} \pmod{A^{\rhd\lambda}} && \text{by Definition 2.6.}
\end{aligned}$$

Therefore, $\phi_\lambda(c_{\mathfrak{s}}, ac_{\mathfrak{t}}) = \phi_{\iota(\lambda)}(c_{\iota(\mathfrak{t})}, a^*c_{\iota(\mathfrak{s})})$. Hence, $\phi_\lambda(x, ay) = \phi_{\iota(\lambda)}(\iota(y), a^*\iota(x))$, proving the first identity in (b). Applying part (a), we deduce that

$$\phi_\lambda(x, ay) = \phi_{\iota(\lambda)}(\iota_\lambda(y), a^*\iota_\lambda(x)) = \phi_\lambda\big(\iota_{\iota(\lambda)}(a^*\iota_\lambda(x)), y\big),$$

completing the proof of (b).

Finally, part (c) follows immediately from (2.10) and part (d) follows by comparing the degrees on the left and right hand sides of (2.10). $\square$

Remark 2.12. If $\mathfrak{s} \in T(\lambda)$ and $a \in A$ then $\iota_{\iota(\lambda)}\big(a\iota_\lambda(x)\big) = ax$ if and only if $\iota_\lambda : C_\lambda \longrightarrow C_{\iota(\lambda)}$ is an A-module homomorphism. Hence, by Lemma 2.11(b), the bilinear form ϕ_λ is associative if and only if ι_λ is an A-module homomorphism. In particular, ϕ_λ is symmetric and associative when $\iota_\lambda = \mathrm{id}_{C_\lambda}$, as is the case for (non-skew) cellular algebras.

For any $\lambda \in \mathcal{P}$, and for any ring R, the radical of C_λ is:

$$\text{(2.13)} \qquad \operatorname{rad}(C_\lambda) \coloneqq \{y \in C_\lambda \mid \phi_\lambda(x, y) = 0 \text{ for all } x \in C_\lambda\}.$$

Proposition 2.14 is the skew cellular algebra analogue of [18, Lemma 2.7].

Proposition 2.14. *Let $\lambda \in \mathcal{P}$. Then the radical* $\operatorname{rad}(C_\lambda)$ *is a graded A-submodule of C_λ.*

Proof. If $y \in \operatorname{rad}(C_\lambda)$ and $a \in A$ then, by Lemma 2.11(b),

$$\phi(x, ay) = \phi\big(\iota_{\iota(\lambda)}(a^*\iota_\lambda(x)), y\big) = 0, \qquad \text{for any } x \in C_\lambda.$$

Therefore, $ay \in \operatorname{rad}(C_\lambda)$, showing that $\operatorname{rad}(C_\lambda)$ is an A-submodule of C_λ. By Lemma 2.11(d), the form ϕ_λ is homogeneous of degree 0, so $\operatorname{rad}(C_\lambda)$ is a graded submodule of C_λ. $\square$

Remark 2.15. Note that because ϕ_λ is not symmetric this is the *right radical* of ϕ_λ and, *a priori*, this is different from the *left radical* of ϕ_λ. It is not clear if the left radical of ϕ_λ is an A-submodule of C_λ because there is no obvious left-handed analogue of Lemma 2.11(b).

Definition 2.16. For $\lambda \in \mathcal{P}$ set $D_\lambda = C_\lambda / \operatorname{rad}(C_\lambda)$. Let $\mathcal{P}_0 = \{\lambda \in \mathcal{P} \mid D_\lambda \neq 0\}$.

By Proposition 2.14, D_λ is a graded A-module. The next result extends the arguments of [18, §2.2], to characterise the graded simple A-modules. Recall that the **Jacobson radical** of an A-module M is the intersection of its maximal A-submodules.

Theorem 2.17. *Suppose that R is a field.*

(a) *If $\lambda \in \mathcal{P}_0$, W is an A-submodule of $\underline{C}_\lambda$ and $\theta \in \operatorname{Hom}_A(\underline{C}_\lambda, \underline{C}_\lambda/W)$, then there exists a unit $r \in R^\times$ such that $\theta(x) = rx + W$ for all $x \in \underline{C}_\lambda$.*

(b) *If $\lambda \in \mathcal{P}_0$ then D_λ is an absolutely irreducible graded A-module. Moreover, $\operatorname{rad} C_\lambda$ is the Jacobson radical of C_λ and, consequently, D_λ is the unique simple head of C_λ.*

(c) *If $\lambda, \mu \in \mathcal{P}_0$ and $D_\lambda \simeq D_\mu\langle k\rangle$, for some $k \in \mathbb{Z}$, then $\lambda = \mu$ and $k = 0$.*

(d) *The set $\{D_\lambda\langle k\rangle \mid \lambda \in \mathcal{P}_0 \text{ and } k \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple A-modules.*

Proof. Part (a) follows the same argument from the proof of [15, Proposition (2.6)]. Similarly, the proof of [15, Proposition (3.2)(ii)] and [18, §2.2] show that if $\lambda \in \mathcal{P}_0$ then D_λ is an absolutely irreducible graded A-module. Now using (a) we deduce that D_λ is the unique simple head of C_λ and, hence, that $\operatorname{rad}(C_\lambda)$ is the unique maximal submodule of C_λ, so $\operatorname{rad} C_\lambda$ is the Jacobson radical of C_λ. The remaining parts of the theorem follow using the arguments from [18, §2.2]. $\square$

Corollary 2.18. *Suppose that R is a field. Then $\{\underline{D}_\lambda \mid \lambda \in \mathcal{P}_0\}$ is a complete set of pairwise non-isomorphic ungraded simple A-modules.*

The next result describes the duals of the simple modules of skew cellular algebras.

Proposition 2.19. *Suppose that R is a field. Then $D_\lambda^* \simeq D_{\iota(\lambda)}$ as graded A-modules. In particular, $\lambda \in \mathcal{P}_0$ if and only if $\iota(\lambda) \in \mathcal{P}_0$.*

Proof. Let $\lambda \in \mathcal{P}_0$. Then, by definition, there exist $x, y \in C_\lambda$ such that $\phi_\lambda(x, y) \neq 0$. By Lemma 2.11(a), we have $\phi_{\iota(\lambda)}(\iota_\lambda(y), \iota_\lambda(x)) = \phi_\lambda(x, y) \neq 0$ and thus $\iota(\lambda) \in \mathcal{P}_0$.

Now for each $y \in C_{\iota(\lambda)}$ there is a well-defined R-linear map from $\theta_y : C_\lambda \longrightarrow R$ given by

$$\theta_y(x) = \phi_\lambda(\iota_{\iota(\lambda)}(y), x), \qquad \text{for } x \in C_\lambda.$$

By definition, $\theta_y(x) = 0$ if $x \in \operatorname{rad} C_\lambda$, so we can consider θ_y as a map from D_λ to R. Again by Lemma 2.11(a), $\phi_\lambda(\iota_{\iota(\lambda)}(y), x) = \phi_{\iota(\lambda)}(\iota_\lambda(x), y)$, so $\theta_y = 0$ if $y \in \operatorname{rad} C_{\iota(\lambda)}$. Hence, there is a well-defined map $\Theta_\lambda : D_{\iota(\lambda)} \longrightarrow D_\lambda^*$ given by $\Theta_\lambda(y + \operatorname{rad} C_{\iota(\lambda)}) = \theta_y$, for $y \in C_{\iota(\lambda)}$. The map Θ_λ is homogeneous of degree 0 since ϕ_λ is homogeneous of degree 0 by Lemma 2.11(d). Moreover, by Lemma 2.11(b), if $a \in A$ then

$$\theta_{ay}(x) = \phi_\lambda\big(\iota_{\iota(\lambda)}(ay), x\big) = \phi_\lambda(\iota_{\iota(\lambda)}(y), a^* x) = \theta_y(a^* x) = a\theta_y(x).$$

So Θ_λ is a morphism of A-modules. Moreover, since $\iota_{\iota(\lambda)}$ is a bijection, it follows from the equality $\phi_\lambda(\iota_{\iota(\lambda)}(y), x) = \phi_{\iota(\lambda)}(\iota_\lambda(x), y)$ that $\Theta_\lambda : D_{\iota(\lambda)} \longrightarrow D_\lambda^*$ is injective.

We now assume that R is a field. Since Θ_λ is injective we deduce that $\dim_R D_{\iota(\lambda)} \le \dim_R D_\lambda$. By the same argument, the map $\Theta_{\iota(\lambda)} : D_\lambda \to D^*_{\iota(\lambda)}$ is also injective, which gives the reverse inequality. We deduce that Θ_λ is an isomorphism and this concludes the proof. □

By Proposition 2.19, if R is a field and $\lambda \in \mathcal{P}_0$ then the graded A-module D_λ is self-dual if and only if $\lambda = \iota(\lambda)$. In the special case when A is a cellular algebra, this recovers the well-known result that the simple modules of cellular algebras are self-dual since the involution ι is the identity map in this case by Remark 2.4.

Finally, as in [18, §2.3], define the **graded decomposition matrix** of A to be the matrix

(2.20)
$$D_A(t) = \big(d_{\lambda\mu}(t)\big), \quad \text{where} \quad d_{\lambda\mu}(t) := \sum_{k\in\mathbb{Z}} [C_\lambda : D_\mu\langle k\rangle] t^k, \text{ for } \lambda \in \mathcal{P} \text{ and } \mu \in \mathcal{P}_0.$$

We order the rows and columns of $D_A(t)$ by any total order $\ge$ that extends $\trianglerighteq$; that is, if $\lambda \trianglerighteq \mu$ then $\lambda \ge \mu$, for $\lambda, \mu \in \mathcal{P}$. The arguments for cellular algebras now generalise to prove the following.

Proposition 2.21. *Let $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}_0$. Then:*
(a) $d_{\lambda\mu}(t) \in \mathbb{Z}_{\ge 0}[t, t^{-1}]$;
(b) $d_{\lambda\mu}(1) = [\underline{C}_\lambda : \underline{D}_\mu]$;
(c) $d_{\mu\mu}(t) = 1$ *and* $d_{\lambda\mu}(t) \ne 0$ *only if* $\lambda \trianglerighteq \mu$.
In particular, the graded decomposition matrix $D_A(t)$ is upper unitriangular.

2.3. **Shift automorphisms of graded cellular algebras.** This section defines *shift automorphisms* of graded cellular algebras, which provides a general framework for constructing skew cellular algebras from cellular algebras. This framework is used to prove all of the main results in this paper.

As in the last section, let R be an integral domain with one. Recall from Remark 2.4 that a graded cellular algebra A is determined by a graded cell datum $(\mathcal{P}, T, C, \deg)$.

Definition 2.22. Let A be a $\mathbb{Z}$-graded cellular R-algebra with graded cell datum $(\mathcal{P}, T, C, \deg)$. A **shift automorphism** of A is a triple of automorphisms $\sigma = (\sigma_A, \sigma_{\mathcal{P}}, \sigma_T)$ where σ_A is an R-algebra automorphism of A, $\sigma_{\mathcal{P}}$ is a poset automorphism of $\mathcal{P}$ and σ_T is an automorphism of the set $T = \coprod_\lambda T(\lambda)$ such that:
(a) If $\mathfrak{s} \in T(\lambda)$ then $\sigma_T(\mathfrak{s}) \in T(\sigma_P(\lambda))$ and $\deg(\sigma_T(\mathfrak{s})) = \deg(\mathfrak{s})$.
(b) If $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ then $\sigma_A(c_{\mathfrak{s}\mathfrak{t}}) = c_{\sigma_T(s)\sigma_T(\mathfrak{t})}$.
(c) If $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$, for $\lambda \in \mathcal{P}$, then $\sigma_T^k(\mathfrak{t}) = \mathfrak{t}$ if and only if $\sigma_T^k(\mathfrak{s}) = \mathfrak{s}$, for $k \in \mathbb{Z}$.

Throughout this section we fix a graded cellular algebra A with a shift automorphism σ. The algebra of **σ_A-fixed points** in A is
$$A^\sigma = \{a \in A \mid \sigma_A(a) = a\}.$$
The aim of this section is to show that A^σ is a skew cellular algebra.

In practice, a shift automorphism $\sigma = (\sigma_A, \sigma_P, \sigma_T)$ is completely determined by the map σ_T. Part (c) of Definition 2.22, which is used in (2.27), is a non-trivial assumption that ensures that whenever σ_T^k restricts to give an automorphism of $T(\lambda)$ then all of the σ_T^k-orbits in $T(\lambda)$ have the same size. As the meaning will always be clear from context, we often abuse notation and simply write σ instead of σ_A, $\sigma_{\mathcal{P}}$ or σ_T.

A trivial example of a shift automorphism is given by taking the identity maps $(\mathrm{id}_A, \mathrm{id}_{\mathcal{P}}, \mathrm{id}_T)$. Here is a less trivial example.

Example 2.23. For $n \in \mathbb{Z}_{\geq 1}$, the full matrix algebra $A = \mathrm{Mat}_n(R)$ is a cellular algebra, with $\mathcal{P} = \{n\}$ and $T(n) = \{1, \dots, n\}$, where the cellular basis elements are given by the set of elementary matrices E_{ij}, where $1 \leq i, j \leq n$. Following [27, Example 2.1.3], fix integers $d_1, \dots, d_n \in \mathbb{Z}$ such that $d_i + d_{n+1-i} = 0$ and define a $\mathbb{Z}$-grading on A by setting $\deg E_{ij} = d_i + d_{n+1-j}$ for $1 \leq i, j \leq n$. Then A is a graded cellular algebra with graded cellular basis $\{F_{ij}\}$, where $F_{ij} = E_{i(n+1-j)}$ and $\deg(i) = d_i$, for $1 \leq i, j \leq n$. The condition $d_j + d_{n+1-j} = 0$ ensures that the degrees add in the relation $F_{ij}F_{(n+1-j)k} = F_{ik}$.

To define a shift automorphism of A, suppose that $w \in \mathfrak{S}_n$ is any permutation that is a product of p disjoint $\frac{n}{p}$-cycles such that $w(n+1-i) = n+1-w(i)$, for $1 \leq i \leq n$. For example, w could be the permutation given by $w(i) = n+1-i$, for $1 \leq i \leq n$, when n is even and $p = \frac{n}{2}$. Then A has a unique shift automorphism $\sigma = (\sigma_A, \sigma_P, \sigma_T)$ such that $\sigma_T(i) = w(i)$, for $1 \leq i \leq n$. Explicitly, $\sigma_T = w$, $\sigma_{\mathcal{P}} = \mathrm{id}_{\mathcal{P}}$ and $\sigma_A(F_{ij}) = F_{w(i)w(j)}$, for $1 \leq i, j \leq n$. The assumption that w is the product of p disjoint $\frac{n}{p}$-cycles ensures that all of the σ_T-orbits have size p, in accordance with Definition 2.22(c). The second condition on w is forced by the requirement that σ_A respect the relations $F_{ij}F_{(n+1-j)k} = F_{ik}$. If n is odd then by looking at the sum $\frac{n(n+1)}{2} = \sum_{i=1}^{n} w(i)$, it follows that $w(\frac{n+1}{2}) = \frac{n+1}{2}$, which forces $w = 1_{\mathfrak{S}_n}$. So, $w \neq 1$ only if n is even.

If we drop the requirement that A be a *graded* algebra then we do not need to assume that $w(i) = n+1-i$, for $1 \leq i \leq n$.

The following properties of shift automorphisms are immediate from Definition 2.22.

Lemma 2.24. *Suppose that $(\sigma_A, \sigma_{\mathcal{P}}, \sigma_T)$ is a shift automorphism of the graded cellular algebra A. Then σ_A is homogeneous automorphism of A of degree zero such that $\sigma_A(A^{\rhd\lambda}) = A^{\rhd\sigma_{\mathcal{P}}(\lambda)}$. Moreover, $\sigma_A(a^*) = (\sigma_A(a))^*$, for all $a \in A$.*

For the rest of this section fix a graded cellular algebra A with graded cell datum $(\mathcal{P}, T, C, \deg)$ and a skew cellular algebra automorphism $\sigma = (\sigma_A, \sigma_{\mathcal{P}}, \sigma_T)$. Let

$$\mathsf{p} = |\sigma_A| \qquad \text{and} \qquad \mathsf{p}_{\mathcal{P}} = |\sigma_{\mathcal{P}}|$$

be the orders of the automorphisms σ_A and $\sigma_{\mathcal{P}}$, respectively. Note that p is also the order of σ_T, since by Definition 2.22(b) if $k \in \mathbb{Z}$ then

$$\begin{aligned}
\sigma_A^k = \mathrm{id}_A &\iff \sigma_A^k(c_{\mathfrak{s}\mathfrak{t}}) = c_{\mathfrak{s}\mathfrak{t}} && \text{for all } \mathfrak{s}, \mathfrak{t} \in T(\lambda) \text{ and } \lambda \in \mathcal{P}\\
&\iff c_{\sigma_T^k(\mathfrak{s}), \sigma_T^k(\mathfrak{t})} = c_{\mathfrak{s}\mathfrak{t}} && \text{for all } \mathfrak{s}, \mathfrak{t} \in T(\lambda) \text{ and } \lambda \in \mathcal{P}\\
&\iff \sigma_T^k(\mathfrak{s}) = \mathfrak{s} && \text{for all } \mathfrak{s} \in T(\lambda) \text{ and } \lambda \in \mathcal{P}\\
&\iff \sigma_T^k = \mathrm{id}_T\,.
\end{aligned}$$

In particular, both $\mathsf{p}_{\mathcal{P}}$ and p are finite since $\mathcal{P}$ and T are finite sets. Finally, by Definition 2.22(a) we have that $\mathsf{p}_{\mathcal{P}}$ divides p. For the rest of this section we assume that R contains a primitive pth root of unity ε and $\mathsf{p} \cdot 1_R$ is invertible in R.

The cyclic group $\mathbb{Z}_{\mathcal{P}} = \langle \sigma_{\mathcal{P}} \rangle \cong \mathbb{Z}/\mathsf{p}_{\mathcal{P}}\mathbb{Z}$ acts on $\mathcal{P}$, let $\mathcal{P}_\sigma$ be a set of representatives for this action. For example, if $\leq$ is any total order refining $\trianglelefteq$ then one could take

$$\mathcal{P}_\sigma = \{\lambda \in \mathcal{P} \mid \sigma_{\mathcal{P}}^k(\lambda) \leq \lambda \text{ for } 0 \leq k < \mathsf{p}_{\mathcal{P}}\}.$$

For each $\lambda \in \mathcal{P}$ let

$$\mathsf{o}_\lambda = \min\{k \geq 1 \mid \sigma_{\mathcal{P}}^k(\lambda) = \lambda\} \qquad \text{and} \qquad \mathsf{p}_\lambda = \mathsf{p}/\mathsf{o}_\lambda.$$

Then o_λ is the size of the $\mathbb{Z}_{\mathcal{P}}$-orbit of λ, so o_λ divides $\mathsf{p}_{\mathcal{P}}$ and $\mathsf{p}_\lambda \in \mathbb{N}$.

Lemma 2.25. *The elements in the same $\mathbb{Z}_{\mathcal{P}}$-orbit are not comparable under $\trianglelefteq$. That is, if $\lambda \in \mathcal{P}$ and $k \in \{1, \dots, \mathsf{o}_\lambda - 1\}$, the elements λ and $\sigma^k\lambda$ are not comparable under $\trianglelefteq$.*

Proof. Let $\lambda \in \mathcal{P}$ and let $[\lambda] = \{\sigma_{\mathcal{P}}^k(\lambda) \mid 0 \leq k < \mathsf{o}_\lambda\}$ be the $\mathbb{Z}_{\mathcal{P}}$-orbit of λ. By way of contradiction, suppose that there exist $k, l \in \mathbb{Z}$ such that $\sigma_{\mathcal{P}}^k\lambda \triangleleft \sigma_{\mathcal{P}}^l\lambda$. Since $\sigma_{\mathcal{P}}$ is a poset automorphism, for any $m \in \mathbb{Z}$

$$\sigma_{\mathcal{P}}^m\lambda = \sigma_{\mathcal{P}}^{m-k}\left(\sigma_{\mathcal{P}}^k\lambda\right) \triangleleft \sigma_{\mathcal{P}}^{m-k}\left(\sigma_{\mathcal{P}}^l\lambda\right) = \sigma_{\mathcal{P}}^{m-k+l}\lambda.$$

We have shown that for any $\mu \in [\lambda]$, there exists $\nu \in [\lambda]$ such that $\mu \triangleleft \nu$. But this is absurd because this implies that the finite poset $([\lambda], \trianglelefteq)$ has no maximal element. □

Define the binary relations $\triangleleft_\sigma$ and $\trianglelefteq_\sigma$ on $\mathcal{P}_\sigma$ by

$$\lambda \triangleleft_\sigma \mu \iff \text{there exists } k \in \mathbb{Z}, \sigma_{\mathcal{P}}^k\lambda \triangleleft \mu,$$

and

$$\lambda \trianglelefteq_\sigma \mu \iff \lambda = \mu \text{ or } \lambda \triangleleft_\sigma \mu,$$

for any $\lambda, \mu \in \mathcal{P}_\sigma$. Since $\sigma_{\mathcal{P}}$ is a poset automorphism,

$$\begin{aligned}
\lambda \triangleleft_\sigma \mu &\iff \text{there exists } l \text{ such that } \lambda \triangleleft \sigma_{\mathcal{P}}^l\mu \\
&\iff \text{there exist } k, l \text{ such that } \sigma_{\mathcal{P}}^k\lambda \triangleleft \sigma_{\mathcal{P}}^l\mu \\
&\iff \text{for all } l, \text{ there exists } k \text{ such that } \sigma_{\mathcal{P}}^k\lambda \triangleleft \sigma_{\mathcal{P}}^l\mu \\
&\iff \text{for all } k, \text{ there exists } l \text{ such that } \sigma_{\mathcal{P}}^k\lambda \triangleleft \sigma_{\mathcal{P}}^l\mu.
\end{aligned}$$

Proposition 2.26. *The binary relation $\trianglelefteq_\sigma$ is a partial order on $\mathcal{P}_\sigma$.*

Proof. By definition, $\lambda \trianglelefteq_\sigma \lambda$ for all $\lambda \in \mathcal{P}_\sigma$. To show that $\trianglelefteq_\sigma$ is transitive suppose that $\lambda \trianglelefteq_\sigma \mu \trianglelefteq_\sigma \nu$, for $\lambda, \mu, \nu \in \mathcal{P}_\sigma$. If either $\lambda = \mu$ or $\mu = \nu$ then $\lambda \trianglelefteq_\sigma \nu$, so we can assume that $\lambda \neq \mu \neq \nu$. Then there exist k, l such that $\sigma_{\mathcal{P}}^k\lambda \triangleleft \mu \triangleleft \sigma_{\mathcal{P}}^l\nu$, and hence $\sigma_{\mathcal{P}}^k\lambda \triangleleft \sigma_{\mathcal{P}}^l\nu$ (since $\triangleleft$ is transitive). So, $\lambda \triangleleft_\sigma \nu$. Finally, if $\lambda \trianglelefteq_\sigma \mu \trianglelefteq_\sigma \lambda$ then if $\lambda \neq \mu$ there exist k, l such that $\lambda \triangleleft \sigma_{\mathcal{P}}^k\mu \triangleleft \sigma_{\mathcal{P}}^l\lambda$. Using again the transitivity of $\triangleleft$ we obtain $\lambda \triangleleft \sigma_{\mathcal{P}}^l\lambda$ which contradicts Lemma 2.25. □

Let $\sigma_\lambda = \sigma_T^{\mathsf{o}_\lambda}$. Then the cyclic group $\mathbb{Z}_\lambda = \langle\sigma_\lambda\rangle \cong \mathbb{Z}/\mathsf{p}_\lambda\mathbb{Z}$ acts on $T(\lambda)$. Let $T_\sigma(\lambda)$ be any set of representatives for the $\mathbb{Z}_\lambda$-orbits of $T(\lambda)$. By Definition 2.22(c), all of the $\mathbb{Z}_\lambda$-orbits in $T(\lambda)$ have the same size, so $|T_\sigma(\lambda)|$ divides $|T(\lambda)|$. Let $\mathsf{o}_T(\lambda) = |T(\lambda)|/|T_\sigma(\lambda)|$ be the size of any $\mathbb{Z}_\lambda$-orbit in $T(\lambda)$. Note that $\mathsf{o}_T(\lambda)$ divides $|\mathbb{Z}_\lambda| = \mathsf{p}_\lambda$, in particular $\mathsf{o}_T(\lambda)$ divides p since p_λ divides p. If λ and μ are in the same $\mathbb{Z}_{\mathcal{P}}$-orbit then it is easy to see that $\mathsf{o}_\lambda = \mathsf{o}_\mu$ and $\mathsf{o}_T(\lambda) = \mathsf{o}_T(\mu)$.

Let $\mathcal{P}_{\sigma,\mathsf{p}} = \{(\lambda, k) \mid \lambda \in \mathcal{P}_\sigma \text{ and } k \in \mathbb{Z}/\mathsf{o}_T(\lambda)\mathbb{Z}\}$, considered as a poset with ordering $\trianglelefteq_\sigma$ given by

$$(\lambda, k) \trianglelefteq_\sigma (\mu, l) \iff (\lambda, k) = (\mu, l) \text{ or } \lambda \triangleleft_\sigma \mu,$$

for all $(\lambda,k),(\mu,l)\in\mathcal{P}_{\sigma,\mathsf{p}}$. We write $(\lambda,k)\lhd_\sigma(\mu,l)$ if $(\lambda,k)\neq(\mu,l)$ and $(\lambda,k)\unlhd_\sigma(\mu,l)$, that is, if $\lambda\lhd_\sigma\mu$. For $(\lambda,k)\in\mathcal{P}_{\sigma,\mathsf{p}}$, define $T_\sigma(\lambda,k)=T_\sigma(\lambda)$. Finally, set

$$c^{(k)}_{\mathfrak{s}\mathfrak{t}}=\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}\overline{\sigma}_A(c_{\mathfrak{s},\sigma^j_\lambda(\mathfrak{t})}),\qquad \text{for } \mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda,k), \tag{2.27}$$

where $\varepsilon_\lambda:=\varepsilon^{\mathsf{p}/\mathsf{o}_T(\lambda)}$ and $\overline{\sigma}_A:=\sum_{l=0}^{\mathsf{p}-1}\sigma_A^l$. To complete the definition of a skew cell datum for A^σ, let ι_σ be the poset involution of $\mathcal{P}_{\sigma,\mathsf{p}}$ given by $\iota_\sigma(\lambda,k)=(\lambda,-k)$ and let $(\iota_\sigma)_{(\lambda,k)}:T_\sigma(\lambda,k)\to T_\sigma(\lambda,-k)$ be given by the identity map of $T_\sigma(\lambda)$, for $(\lambda,k)\in\mathcal{P}_{\sigma,\mathsf{p}}$. Finally, if $\mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda,k)$ set $C_\sigma(\mathfrak{s},\mathfrak{t})=c^{(k)}_{\mathfrak{s}\mathfrak{t}}$ and $\deg_\sigma(\mathfrak{s})=\deg(\mathfrak{s})$.

We can now show that a (graded) cellular algebra with a shift automorphism gives rise to a (graded) skew cellular algebra in the sense of Definition 2.2. This result can be viewed as a cellular algebra analogue of Clifford theory.

Theorem 2.28. *Suppose that A is a graded cellular algebra with graded cell datum $(\mathcal{P},T,C,\deg)$ and shift automorphism $\sigma=(\sigma_A,\sigma_{\mathcal{P}},\sigma_T)$ over the integral domain R containing a primitive pth root of unity ε, where p is the order of σ_A. Assume that $\mathsf{p}\cdot 1_R\in R^\times$. Then A^σ is a graded skew cellular algebra with skew cellular datum $(\mathcal{P}_{\sigma,\mathsf{p}},\iota_\sigma,T_\sigma,C_\sigma,\deg_\sigma)$.*

Proof. By construction, the fixed point subalgebra A^σ is an R-subalgebra of A, so it remains to check that the quintuple $(\mathcal{P}_{\sigma,\mathsf{p}},\iota_\sigma,T_\sigma,C_\sigma,\deg_\sigma)$ satisfies the assumptions of Definition 2.2.

First note that ι_σ is a poset automorphism of $\mathcal{P}_{\sigma,\mathsf{p}}$ since $(\lambda,k)\rhd_\sigma(\mu,l)$ if and only if $\lambda\rhd_\sigma\mu$, which is if and only if $(\lambda,-k)\rhd_\sigma(\mu,-l)$. We now check (C_1), (C_2), (C_3) and (C_4) from Definition 2.2. The first of these properties is easy but the others require more work.

First, if $\mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda,k)$ then $\overline{\sigma}_A$ is homogeneous of degree 0 by Lemma 2.24 and $\deg(\sigma_T(\mathfrak{s}))=\deg(\mathfrak{s})$, for all $s\in T(\lambda)$ by Definition 2.22(a). Therefore, $\deg(c^{(k)}_{\mathfrak{s}\mathfrak{t}})=\deg(c_{\mathfrak{s}\mathfrak{t}})=\deg(\mathfrak{s})+\deg(\mathfrak{t})$, for all $\mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda,k)$. Hence, (C_1) holds.

Next consider (C_2). If $a\in A$ then $\overline{\sigma}_A(a)\in A^\sigma$, so by (2.27) we have $c^{(k)}_{\mathfrak{s}\mathfrak{t}}\in A^\sigma$, for all $\mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda,k)$ and $(\lambda,k)\in\mathcal{P}_{\sigma,\mathsf{p}}$. To show that $\{c^{(k)}_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of A^σ first observe that because $\mathsf{p}\cdot 1_R\in R^\times$, for any $b\in A^\sigma$, $b=\mathsf{p}^{-1}(\mathsf{p}b)=\mathsf{p}^{-1}\sum_{k=0}^{\mathsf{p}-1}\sigma_A^k(b)=\mathsf{p}^{-1}\overline{\sigma}_A(b)\in\operatorname{span}_R\{\overline{\sigma}_A(a)\mid a\in A\}$. It follows that the fixed point subalgebra A^σ is spanned by $\{\overline{\sigma}_A(c_{\mathfrak{s}\mathfrak{t}})\}$. By definition,

$$\overline{\sigma}_A(c_{\mathfrak{s}\mathfrak{t}})=\sum_{k=0}^{\mathsf{p}-1}\sigma_A^k(c_{\mathfrak{s}\mathfrak{t}})=\sum_{k=0}^{\mathsf{p}-1}c_{\sigma_T^k(\mathfrak{s})\sigma_T^k(\mathfrak{t})}. \tag{2.29}$$

In particular, $\overline{\sigma}_A(c_{\mathfrak{s}\mathfrak{t}})=\overline{\sigma}_A(c_{\mathfrak{s}'\mathfrak{t}'})$ whenever $\mathfrak{s}'=\sigma_T^k(\mathfrak{s})$ and $\mathfrak{t}'=\sigma_T^k(\mathfrak{t})$, for some $k\ge 0$. It follows that the algebra A^σ is spanned by the set

$$\begin{aligned}\mathcal{C}&=\big\{\overline{\sigma}_A(c_{\mathfrak{s}\mathfrak{t}})\mid \mathfrak{s}\in T_\sigma(\lambda),\mathfrak{t}\in T(\lambda)\text{ and }\lambda\in\mathcal{P}_\sigma\big\}\\&=\big\{\overline{\sigma}_A(c_{\mathfrak{s}\sigma^j_\lambda(\mathfrak{t})})\mid \mathfrak{s},\mathfrak{t}\in T_\sigma(\lambda),0\le j<\mathsf{o}_T(\lambda)\text{ and }\lambda\in\mathcal{P}_\sigma\big\},\end{aligned}$$

where the second equality follows because $\sigma_\lambda=\sigma_T^{\mathsf{o}_\lambda}:T(\lambda)\xrightarrow{\sim}T(\lambda)$ is a bijection. We claim that $\mathcal{C}$ is linearly independent and hence of basis of A^σ. By (2.29), when we expand $\overline{\sigma}_A(c_{\mathfrak{s}\mathfrak{t}})$ in the c-basis of A the supports of the different elements of $\mathcal{C}$ are disjoint, so $\mathcal{C}$ is linearly independent because $\{c_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of A. Finally, observe that, by (2.27), the transition matrix between the basis in $\mathcal{C}$ and $\{c^{(k)}_{\mathfrak{s}\mathfrak{t}}\}$ is given by Vandermonde matrices in $\{\varepsilon_\lambda^{kj} \mid 0 \le k, j < \mathsf{o}_T(\lambda)\}$, which are invertible over R since $\mathsf{o}_T(\lambda)\cdot 1_R = \prod_{i=1}^{\mathsf{o}_T(\lambda)-1}(1-\varepsilon_\lambda^i)$ and $\mathsf{o}_T(\lambda)$ divides p in $\mathbb{Z}$. More precisely, for any $\lambda \in \mathcal{P}_\sigma$ and $\mathfrak{s},\mathfrak{t} \in T_\sigma(\lambda)$

(2.30)
$$\operatorname{span}_R\big\{c_{\mathfrak{st}}^{(k)} \mid k \in \{1,\dots,\mathsf{o}_T(\lambda)\}\big\} = \operatorname{span}_R\big\{\overline{\sigma}_A(c_{\mathfrak{s},\sigma_\lambda^j(\mathfrak{t})}) : j \in \{1,\dots,\mathsf{o}_T(\lambda)\}\big\}.$$

Hence, $\{c_{\mathfrak{st}}^{(k)}\}$ is a basis of A^σ, so (C_2) holds.

We now verify (C_3). Fix $(\lambda,k) \in \mathcal{P}_{\sigma,\mathsf{p}}$ and $\mathfrak{s},\mathfrak{t} \in T_\sigma(\lambda,k)$. Let $a \in A^\sigma$. Using (C_3) for the c-basis of A and the fact that $\overline{\sigma}_A$ is A^σ-linear

$$\begin{aligned}
ac_{\mathfrak{st}}^{(k)} &= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\overline{\sigma}_A(ac_{\mathfrak{s},\sigma_\lambda^j(\mathfrak{t})})\\
&= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{\mathfrak{v}\in T(\lambda)} \varepsilon_\lambda^{kj} r_{\mathfrak{vs}}(a)\overline{\sigma}_A(c_{\mathfrak{v},\sigma_\lambda^j(\mathfrak{t})}) + \overline{\sigma}_A(b),
\end{aligned}$$

for some $b \in A^{\rhd\lambda}$.

By direct calculation,

$$\begin{aligned}
ac_{\mathfrak{st}}^{(k)} - \overline{\sigma}_A(b) &= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{\mathfrak{v}\in T(\lambda)} \varepsilon_\lambda^{kj} r_{\mathfrak{vs}}(a)\overline{\sigma}_A(c_{\mathfrak{v},\sigma_\lambda^j(\mathfrak{t})})\\
&= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{\mathfrak{v}\in T^\sigma(\lambda)}\sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a)\overline{\sigma}_A(c_{\sigma_\lambda^l(\mathfrak{v}),\sigma_\lambda^j(\mathfrak{t})})\\
&= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{\mathfrak{v}\in T^\sigma(\lambda)}\sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a)\overline{\sigma}_A(c_{\mathfrak{v},\sigma_\lambda^{(j-l)}\mathfrak{t}})\\
&= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{\mathfrak{v}\in T^\sigma(\lambda)}\sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{k(j+l)} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a)\overline{\sigma}_A(c_{\mathfrak{v},\sigma_\lambda^j(\mathfrak{t})})\\
&= \sum_{\mathfrak{v}\in T^\sigma(\lambda)}\sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kl} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a)\sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\overline{\sigma}_A(c_{\mathfrak{v},\sigma_\lambda^j(\mathfrak{t})})\\
&= \sum_{\mathfrak{v}\in T^\sigma(\lambda)} r'_{\mathfrak{vs}}(a)c_{\mathfrak{vt}}^{(k)},
\end{aligned}$$

where

$$r'_{\mathfrak{vs}}(a) = \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kl} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a) \tag{2.31}$$

does not depend on $\mathfrak{t}$. To complete the proof of (C_3) it suffices to prove that $\overline{\sigma}_A(b) \in (A^\sigma)^{\rhd_\sigma(\lambda,k)}$, where $(A^\sigma)^{\rhd_\sigma(\lambda,k)}$ is the R-submodule of A^σ spanned by

$$\big\{c_{\mathfrak{uv}}^{(l)} \mid \mathfrak{u},\mathfrak{v} \in T_\sigma(\mu,l) \text{ for } (\mu,l) \in \mathcal{P}_{\sigma,\mathsf{p}} \text{ with } (\mu,l) \rhd_\sigma (\lambda,k)\big\}.$$

Since $b \in A^{\rhd\lambda}$, it suffices to prove that $\overline{\sigma}_A(c_{\mathfrak{uv}}) \in (A^\sigma)^{\rhd_\sigma(\lambda,k)}$, whenever $\mathfrak{u},\mathfrak{v} \in T(\mu)$ and $\mu \rhd \lambda$. Let $\mu_0 \in \mathcal{P}_\sigma$ be the representative of μ under the action of $\mathbb{Z}_{\mathcal{P}}$ on $\mathcal{P}$. Then $\mu = \sigma^k\mu_0$ for some $k \in \mathbb{Z}$ and since $\mu \rhd \lambda$ we deduce that $\mu_0 \rhd_\sigma \lambda$

(recalling that $\lambda \in \mathcal{P}_\sigma$). Finally, if $l \in \mathbb{Z}$ is such that $\mathfrak{u}_0 := \sigma^l\mathfrak{u} \in T_\sigma(\mu_0)$ then $\overline{\sigma}_A(c_{\mathfrak{u}\mathfrak{v}}) = \overline{\sigma}_A(c_{\mathfrak{u}_0\mathfrak{v}_0})$, where $\mathfrak{v}_0 = \sigma^l\mathfrak{v}$. Now if $\mathfrak{v}_0 = \sigma_{\mu_0}^m\mathfrak{v}_1$ with $\mathfrak{v}_1 \in T_\sigma(\mu_0)$, by (2.30) the element $\overline{\sigma}_A(c_{\mathfrak{u}\mathfrak{v}}) = \overline{\sigma}_A(c_{\mathfrak{u}_0\mathfrak{v}_0})$ is in

$$\operatorname{span}_R\{c^{(j)}_{\mathfrak{u}_0\mathfrak{v}_1} \mid j \in \{1,\dots,\mathsf{o}_T(\mu_0)\}\}.$$

Thus, $\overline{\sigma}_A(c_{\mathfrak{u}\mathfrak{v}}) \in (A^\sigma)^{\rhd_\sigma(\lambda,k)}$ since $\mu_0 \rhd_\sigma \lambda$. This proves that A^σ satisfies (C_3).

Finally, we prove (C_4). By Lemma 2.24, the anti-isomorphism $*$ of A restricts to an anti-isomorphism of A^σ. Moreover, if $\mathfrak{s},\mathfrak{t} \in T_\sigma(\lambda,k)$ and $(\lambda,k) \in \mathcal{P}_{\sigma,\mathsf{p}}$ then

$$\begin{aligned}
\big(c^{(k)}_{\mathfrak{s}\mathfrak{t}}\big)^* &= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\overline{\sigma}_A(c^*_{\mathfrak{s},\sigma^j_\lambda(\mathfrak{t})}) = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\overline{\sigma}_A(c_{\sigma^j_\lambda(\mathfrak{t}),\mathfrak{s}}) \\
&= \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\overline{\sigma}_A(c_{\mathfrak{t},\sigma_\lambda^{-j\mathsf{o}_\lambda}\mathfrak{s}}) = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kj}\overline{\sigma}_A(c_{\mathfrak{t},\sigma^j_\lambda(\mathfrak{s})}) \\
&= c^{(-k)}_{\mathfrak{t}\mathfrak{s}}.
\end{aligned}$$

This completes the proof that A^σ is a graded skew cellular algebra. □

It is an interesting question whether every skew cellular algebra arises in this way, that is, as the fixed point subalgebra of a cellular algebra.

Corollary 2.32. *In the setting of Theorem* 2.28*, if* $\mathsf{o}_T(\lambda) \le 2$ *for all* $\lambda \in \mathcal{P}$ *then* A^σ *is a graded* ***cellular*** *algebra with cellular datum* $(\mathcal{P}_{\sigma,\mathsf{p}}, T_\sigma, C_\sigma, \deg_\sigma)$*. In particular, if* 2 *is invertible in* R *and* A *is a graded cellular algebra with graded cell datum* $(\mathcal{P}, T, C, \deg)$ *and shift automorphism* $\sigma = (\sigma_A, \sigma_{\mathcal{P}}, \sigma_T)$ *such that* σ_A *has order* 2 *then* A^σ *is a graded cellular algebra with cell datum* $(\mathcal{P}_{\sigma,\mathsf{p}}, T_\sigma, C_\sigma, \deg_\sigma)$*.*

Proof. By Theorem 2.28, the algebra A^σ is graded skew cellular with skew cellular datum $(\mathcal{P}_{\sigma,\mathsf{p}}, \iota_\sigma, T_\sigma, C_\sigma, \deg_\sigma)$. By construction, the involution ι_σ of $\mathcal{P}_{\sigma,\mathsf{p}}$ is given by $\iota_\sigma(\lambda,k) = (\lambda,-k)$. Since $\mathsf{o}_T(\lambda) \le 2$ we have $k = -k$ in $\mathbb{Z}/\mathsf{o}_T(\lambda)\mathbb{Z}$, so $\iota_\sigma = \mathrm{id}_{\mathcal{P}_{\sigma,\mathsf{p}}}$. Moreover, still by construction, the map $(\iota_\sigma)_{(\lambda,k)} : T_\sigma(\lambda,k) \longrightarrow T_\sigma(\lambda,-k)$ is the identity map of $T_\sigma(\lambda)$. Hence, $(\iota_\sigma)_{(\lambda,k)}$ is the identity map of $T_\sigma(\lambda,k)$. Recalling Remark 2.4, this proves the first statement. We deduce the second statement by noting that if $2 \in R^\times$ then $-1_R \ne 1_R$ in R, so -1_R is a primitive square root of unity in R. □

Example 2.33. Maintain the notation from Example 2.23. In particular, $A = \mathrm{Mat}_n(R)$ has graded cellular basis $\{F_{ij} \mid 1 \le i,j \le n\}$ and σ is the shift automorphism of A given by $\sigma(F_{ij}) = F_{w(i)w(j)}$, where $w \in \mathfrak{S}_n$ is a permutation such that w is the product of p disjoint $\frac{n}{p}$-cycles and $w(n+1-i) = n+1-w(i)$, for $1 \le i \le n$. Hence, $\mathsf{p} = p$ and we need to assume that R contains a primitive pth root of unity and that $p \cdot 1_R \in R^\times$. The reader can check that the skew cellular subalgebra of σ-fixed points is

$$A^\sigma = \big\{M = (m_{ij}) \in A \mid m_{ij} = m_{w(i)w(j)} \text{ for } 1 \le i,j \le n\big\}.$$

Possible choices for w include the permutations $(1,2)(3,4)$, $(1,3)(2,4)$ or $(1,4)(2,3)$ when $n = 4$ and $(1,2,3)(6,5,4)$ when $n = 6$. In these examples we have $p = 2$ thus A^σ is in fact cellular by Corollary 2.32. An example where A^σ is skew cellular but where Corollary 2.32 does not apply is with $w = (1,3)(2,5)(4,6)$, where $p = 3$. Of course, if w is trivial (which happens when $p = n$) then $A^\sigma = A$ is a cellular algebra. For example, w must be trivial if n is odd by Example 2.23.

We note that Xi and Zhang [34, Theorem 4.5] have shown that the algebra A^σ is cellular; their proof is based on Jordan reduction.

2.4. **Clifford theory.** In this section we explicitly describe how Clifford theory works for skew cellular algebras that are obtained using a shift automorphism, as in Theorem 2.28. The results in this section should be compared with [19, §3.7].

If M is any A-module, let ${}^\sigma M$ be the A-module M where the action of A is twisted by σ. In other words, for any $m \in M$ and $a \in A$ we have

$$a \cdot_{{}^\sigma M} m = \sigma(a) \cdot_M m.$$

Let $M{\downarrow}$ be the restriction of M to an A^σ-module. If N is an A^σ-module, we write $N{\uparrow}$ for the induced A-module.

Recall that A is a graded cellular algebra with a shift automorphism σ.

Lemma 2.34. *Using the notation of* (C_3), $r_{\sigma\mathfrak{v},\sigma\mathfrak{s}}\big(\sigma(a)\big) = r_{\mathfrak{v}\mathfrak{s}}(a)$, *for all* $a \in A$, $\lambda \in \mathcal{P}$ *and* $\mathfrak{v}, \mathfrak{s} \in T(\lambda)$. *Consequently,* $\phi_\lambda(c_\mathfrak{v}, c_\mathfrak{s}) = \phi_{\sigma\lambda}(c_{\sigma(\mathfrak{v})}, c_{\sigma(\mathfrak{s})})$.

Proof. By (C_3) we have

$$ac_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{v}\mathfrak{s}}(a) c_{\mathfrak{v}\mathfrak{t}} \pmod{A^{\rhd\lambda}}.$$

Applying σ using Definition 2.22(b) and Lemma 2.24,

$$\sigma(a) c_{\sigma\mathfrak{s},\sigma\mathfrak{t}} = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{v}\mathfrak{s}}(a) c_{\sigma\mathfrak{v},\sigma\mathfrak{t}} \pmod{A^{\rhd\sigma\lambda}}.$$

Hence, the first equality follows because $\{c_{\mathfrak{u}\mathfrak{v}}\}$ is an R-basis of A. In turn, this implies the second equality by Definition 2.9. □

Proposition 2.35. *Let* $\lambda \in \mathcal{P}$. *The* R*-linear map* $\gamma_\lambda : C_\lambda \longrightarrow C_{\sigma\lambda}$ *defined by* $c_\mathfrak{s} \mapsto c_{\sigma\mathfrak{s}}$, *for* $\mathfrak{s} \in T(\lambda)$, *induces isomorphisms of graded* A*-modules* $C_\lambda \simeq {}^\sigma C_{\sigma\lambda}$ *and* $D_\lambda \simeq {}^\sigma D_{\sigma\lambda}$.

Proof. By definition, γ_λ is an isomorphism of R-modules. To show that it is an A-module isomorphism, suppose that $a \in A$. Then, using Lemma 2.34,

$$\gamma_\lambda(ac_\mathfrak{s}) = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{v}\mathfrak{s}}(a)\gamma_\lambda(c_\mathfrak{v}) = \sum_{\mathfrak{v}\in T(\lambda)} r_{\sigma\mathfrak{v},\sigma\mathfrak{s}}\big(\sigma(a)\big) c_{\sigma\mathfrak{v}} = \sigma(a) c_{\sigma\mathfrak{s}},$$

which proves the first isomorphism. The fact that this is an isomorphism of graded modules comes from Definition 2.22(a). To prove that $D_\lambda \simeq {}^\sigma D_{\sigma\lambda}$ it suffices to show that $\gamma_\lambda\big(\mathrm{rad}(C_\lambda)\big) = \mathrm{rad}(C_{\sigma\lambda})$, which follows because $\phi_\lambda(c_\mathfrak{s}, c_\mathfrak{t}) = \phi_{\sigma\lambda}(c_{\sigma\mathfrak{s}}, c_{\sigma\mathfrak{t}})$ by Lemma 2.34. □

We now assume that $\varepsilon \in R$ is a primitive p-th root of unity and that $\mathsf{p}\cdot 1_R \in R^\times$, so that Theorem 2.28 applies. For any $(\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}$, let $C_\lambda^{(k)}$ be the associated skew cell module of A^σ, with R-basis $\{c_\mathfrak{s}^{(k)} \mid \mathfrak{s} \in T_\sigma(\lambda, k)\}$ and R-bilinear form $\phi_\lambda^{(k)}$, which is not symmetric in general.

Lemma 2.36. *Let* $(\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}$ *and* $\mathfrak{s}, \mathfrak{t} \in T_\sigma(\lambda, k)$. *Then*

$$\phi_\lambda^{(k)}\big(c_\mathfrak{s}^{(k)}, c_\mathfrak{t}^{(k)}\big) = \mathsf{p}_\lambda \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kl} \phi_\lambda\big(c_{\sigma_\lambda^l(\mathfrak{s})}, c_\mathfrak{t}\big).$$

Proof. Note that the maps $a \mapsto r_{\mathfrak{st}}(a)$ and $b \mapsto r'_{\mathfrak{st}}(b)$ for $a \in A$ and $b \in A^\sigma$ are R-linear. Unravelling the definitions, if $\mathfrak{u} \in T_\sigma(\lambda)$ then

$$\phi_\lambda^{(k)}\big(c_{\mathfrak{s}}^{(k)}, c_{\mathfrak{t}}^{(k)}\big) = r'_{\mathfrak{ut}}\big(c_{\mathfrak{us}}^{(k)}\big) = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} r'_{\mathfrak{ut}}\big(\overline{\sigma}_A(c_{\mathfrak{u},\sigma_\lambda^j(\mathfrak{s})})\big).$$

Therefore, by (2.31),

$$\begin{aligned}\phi_\lambda^{(k)}\big(c_{\mathfrak{s}}^{(k)}, c_{\mathfrak{t}}^{(k)}\big) &= \sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj}\varepsilon_\lambda^{kl} r_{\sigma_\lambda^l(\mathfrak{u}),\mathfrak{t}}\big(\overline{\sigma}_A(c_{\mathfrak{u},\sigma_\lambda^j(\mathfrak{s})})\big)\\ &= \sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1}\sum_{m=0}^{\mathsf{p}-1} \varepsilon_\lambda^{k(j+l)} r_{\sigma_\lambda^l(\mathfrak{u}),\mathfrak{t}}\big(c_{\sigma^m(\mathfrak{u}),\sigma^m\sigma_\lambda^j(\mathfrak{s})}\big).\end{aligned}$$

Now by (2.8) we have $r_{\sigma_\lambda^l(\mathfrak{u}),\mathfrak{t}}\big(c_{\sigma^m\mathfrak{u},\sigma^m\sigma_\lambda^j(\mathfrak{s})}\big) \neq 0$ only if $\sigma_\lambda^l(\mathfrak{u}) = \sigma^m(\mathfrak{u})$. In particular, this implies that $\sigma^m(\mathfrak{u}) \in T(\lambda)$, so $\mathsf{o}_\lambda \mid m$. Writing $m = \mathsf{o}_\lambda m'$ we obtain $\sigma_\lambda^l(\mathfrak{u}) = \sigma_\lambda^{m'}(\mathfrak{u})$. Now write $m' = a\mathsf{o}_T(\lambda) + m''$, with $0 \le m'' < \mathsf{o}_T(\lambda)$. Then $\sigma_\lambda^{m'}(\mathfrak{u}) = \sigma_\lambda^{m''}(\mathfrak{u})$ and so $m'' = l$, since the $\mathbb{Z}_\lambda$-orbit of $\mathfrak{u}$ has exactly size $\mathsf{o}_T(\lambda)$. Recalling that $\mathsf{o}_T(\lambda)$ divides p_λ, we have $r_{\sigma_\lambda^l(\mathfrak{u}),\mathfrak{t}}\big(c_{\sigma^m\mathfrak{u},\sigma^m\sigma_\lambda^j(\mathfrak{s})}\big) \neq 0$, for $0 \le m < \mathsf{p}$ only if $m = \mathsf{o}_\lambda(a\mathsf{o}_T(\lambda) + l)$ with $0 \le a < \frac{\mathsf{p}_\lambda}{\mathsf{o}_T(\lambda)}$, in which case $\sigma^m(\mathfrak{v}) = \sigma_\lambda^l(\mathfrak{v})$ for all $\mathfrak{v} \in T(\lambda)$. We thus obtain

$$\begin{aligned}\phi_\lambda^{(k)}\big(c_{\mathfrak{s}}^{(k)}, c_{\mathfrak{t}}^{(k)}\big) &= \frac{\mathsf{p}_\lambda}{\mathsf{o}_T(\lambda)} \sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{k(j+l)} r_{\sigma_\lambda^l(\mathfrak{u}),\mathfrak{t}}\big(c_{\sigma_\lambda^l(\mathfrak{u}),\sigma_\lambda^{l+j}(\mathfrak{s})}\big)\\ &= \frac{\mathsf{p}_\lambda}{\mathsf{o}_T(\lambda)} \sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{k(j+l)} \phi_\lambda\big(c_{\sigma_\lambda^{l+j}(\mathfrak{s})}, c_{\mathfrak{t}}\big)\\ &= \frac{\mathsf{p}_\lambda}{\mathsf{o}_T(\lambda)} \sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kl} \phi_\lambda\big(c_{\sigma_\lambda^{l}(\mathfrak{s})}, c_{\mathfrak{t}}\big)\\ &= \mathsf{p}_\lambda \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kl} \phi_\lambda\big(c_{\sigma_\lambda^{l}(\mathfrak{s})}, c_{\mathfrak{t}}\big),\end{aligned}$$

as desired. ∎

Proposition 2.37. *Let $\lambda \in \mathcal{P}_\sigma$. The R-linear map*

$$\gamma'_\lambda : \bigoplus_{k=0}^{\mathsf{o}_T(\lambda)-1} C_\lambda^{(k)} \longrightarrow C_\lambda; \quad c_{\mathfrak{s}}^{(k)} \longmapsto \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kj} c_{\sigma_\lambda^j(\mathfrak{s})}$$

induces isomorphisms of graded A^σ-modules

$$\bigoplus_{k=0}^{\mathsf{o}_T(\lambda)-1} C_\lambda^{(k)} \simeq C_\lambda{\downarrow}, \qquad \textit{and} \qquad \bigoplus_{k=0}^{\mathsf{o}_T(\lambda)-1} D_\lambda^{(k)} \simeq D_\lambda{\downarrow}\,.$$

Proof. First note that γ'_λ is homogeneous since $\deg^\sigma(\mathfrak{s}) = \deg \mathfrak{s} = \deg \sigma^j(\mathfrak{s})$ for all $\mathfrak{s} \in T_\sigma(\lambda, k) = T_\sigma(\lambda)$ and all $j \in \mathbb{Z}$ by Definition 2.22(a). By the Vandermonde determinant argument that we used in the proof of Theorem 2.28, the map γ'_λ sends a basis to a basis, so is an R-module isomorphism.

We prove that $\bigoplus_k C_\lambda^{(k)} \cong C_\lambda\!\downarrow$ as A^σ-modules. Recall that if $\mathfrak{s} \in T(\lambda)$ and $a \in A$ then

$$ac_{\mathfrak{s}} = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{vs}}(a)c_{\mathfrak{v}} = \sum_{\mathfrak{v}\in T_\sigma(\lambda)} \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} r_{\sigma_\lambda^l(\mathfrak{v}),\mathfrak{s}}(a)c_{\sigma_\lambda^l(\mathfrak{v})},$$

in C_λ. Similarly, if $\mathfrak{s} \in T_\sigma(\lambda,k)$ and $a \in A^\sigma$ then

$$ac_{\mathfrak{s}}^{(k)} = \sum_{\mathfrak{v}\in T_\sigma(\lambda,k)} r'_{\mathfrak{vs}}(a)c_{\mathfrak{v}}^{(k)}, \qquad \text{where} \qquad r'_{\mathfrak{vs}}(a) = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} r_{\sigma_\lambda^j(\mathfrak{v}),\mathfrak{s}}(a),$$

in $C_\lambda^{(k)}$. For any $k \in \mathbb{Z}/\mathsf{o}_T(\lambda)\mathbb{Z}$, $\mathfrak{s} \in T_\sigma(\lambda,k)$ and $a \in A^\sigma$ we have

$$a\gamma'_\lambda\big(c_{\mathfrak{s}}^{(k)}\big) = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kj} ac_{\sigma_\lambda^j(\mathfrak{s})} = \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kj} \sum_{\mathfrak{v}\in T_\sigma(\lambda)} \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} r_{\sigma_\lambda^l(\mathfrak{v}),\sigma_\lambda^j(\mathfrak{s})}(a)c_{\sigma_\lambda^l(\mathfrak{v})}.$$

Now by Lemma 2.34 we have $r_{\sigma_\lambda^l(\mathfrak{v}),\sigma_\lambda^j(\mathfrak{s})}(a) = r_{\sigma_\lambda^{l-j}(\mathfrak{v}),\mathfrak{s}}(a)$ since $a \in A^\sigma$. Thus, we obtain, recalling that ε_λ is an $\mathsf{o}_T(\lambda)$-th root of unity,

$$\begin{aligned}
a\gamma'_\lambda\big(c_{\mathfrak{s}}^{(k)}\big) &= \sum_{\mathfrak{v}\in T_\sigma(\lambda)} \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kl} \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{k(l-j)} r_{\sigma_\lambda^{l-j}(\mathfrak{v}),\mathfrak{s}}(a)c_{\sigma_\lambda^l(\mathfrak{v})} \\
&= \sum_{\mathfrak{v}\in T_\sigma(\lambda)} \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kl} \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} r_{\sigma_\lambda^{j}(\mathfrak{v}),\mathfrak{s}}(a)c_{\sigma_\lambda^l(\mathfrak{v})} \\
&= \sum_{\mathfrak{v}\in T_\sigma(\lambda)} \sum_{l=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{-kl} r'_{\mathfrak{vs}}(a)c_{\sigma_\lambda^l(\mathfrak{v})} \\
&= \sum_{v\in T_\sigma(\lambda)} r'_{\mathfrak{vs}}(a)\gamma'_\lambda\big(c_{\mathfrak{v}}^{(k)}\big) \\
&= \gamma'_\lambda\big(ac_{\mathfrak{s}}^{(k)}\big).
\end{aligned}$$

This proves that γ'_λ is A^σ-linear, thus establishing that $\bigoplus_k C_\lambda^{(k)} \cong C_\lambda\!\downarrow$ as A^σ-modules.

To prove the second isomorphism, it suffices to prove that

$$\bigoplus_{k=0}^{\mathsf{o}_T(\lambda)-1} \gamma'_\lambda\left(\operatorname{rad} C_\lambda^{(k)}\right) = \operatorname{rad} C_\lambda. \tag{2.38}$$

Let $x = \sum_{\mathfrak{s}\in T_\sigma(\lambda)} x_{\mathfrak{s}} c_{\mathfrak{s}}^{(k)} \in \operatorname{rad} C_\lambda^{(k)}$ with $x_{\mathfrak{s}} \in R$.

By Lemma 2.36, if $\mathfrak{t} \in T_\sigma(\lambda)$ then

$$0 = \phi_\lambda^{(k)}\big(c_{\mathfrak{t}}^{(k)}, x\big) = \sum_{\mathfrak{s}\in T_\sigma(\lambda)} x_{\mathfrak{s}}\phi_\lambda^{(k)}\big(c_{\mathfrak{t}}^{(k)}, c_{\mathfrak{s}}^{(k)}\big) = \mathsf{p}_\lambda \sum_{\mathfrak{s}\in T_\sigma(\lambda)} \sum_{j=0}^{\mathsf{o}_T(\lambda)-1} \varepsilon_\lambda^{kj} x_{\mathfrak{s}}\phi_\lambda\big(c_{\sigma_\lambda^j(\mathfrak{t})}, c_{\mathfrak{s}}\big).$$

Thus, if $\mathfrak{t}\in T_\sigma(\lambda)$ and $0\le l<\mathsf{o}_T(\lambda)$ then, using Lemma 2.34,

$$\begin{aligned}
\phi_\lambda\big(c_{\sigma_\lambda^l\mathfrak{t}},\gamma'_\lambda(x)\big) &= \sum_{\mathfrak{s}\in T_\sigma(\lambda)} x_\mathfrak{s}\phi_\lambda\left(c_{\sigma_\lambda^l(\mathfrak{t})},\gamma'_\lambda(c_\mathfrak{s}^{(k)})\right)\\
&= \sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{-kj}x_\mathfrak{s}\phi_\lambda\left(c_{\sigma_\lambda^l(\mathfrak{t})},c_{\sigma_\lambda^j(\mathfrak{s})}\right)\\
&= \varepsilon_\lambda^{-kl}\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{k(l-j)}x_\mathfrak{s}\phi_\lambda\left(c_{\sigma_\lambda^{l-j}(\mathfrak{t})},c_\mathfrak{s}\right)\\
&= \varepsilon_\lambda^{-kl}\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}x_\mathfrak{s}\phi_\lambda\left(c_{\sigma_\lambda^{j}(\mathfrak{t})},c_\mathfrak{s}\right)\\
&= 0,
\end{aligned}$$

proving that $\gamma'_\lambda(x)\in\operatorname{rad}C_\lambda$. To prove that $\operatorname{rad}C_\lambda\subseteq\oplus_k\gamma'_\lambda\big(\operatorname{rad}C_\lambda^{(k)}\big)$, first note that if $\mathfrak{s}\in T_\sigma(\lambda)$ and $j\in\mathbb{Z}/\mathsf{o}_T(\lambda)\mathbb{Z}$ then

$$\gamma_\lambda'^{-1}\big(c_{\sigma_\lambda^j\mathfrak{s}}\big)=\frac{1}{\mathsf{o}_T(\lambda)}\sum_{k=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}c_\mathfrak{s}^{(k)}.$$

Let $x=\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}x_{\mathfrak{s},j}c_{\sigma_\lambda^j(\mathfrak{s})}\in\operatorname{rad}C_\lambda$. We have

$$\gamma_\lambda'^{-1}(x)=\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}x_{\mathfrak{s},j}\gamma_\lambda'^{-1}\big(c_{\sigma_\lambda^j(\mathfrak{s})}\big)=\frac{1}{\mathsf{o}_T(\lambda)}\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j,k=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}x_{\mathfrak{s},j}c_\mathfrak{s}^{(k)}.$$

So, to complete the proof it is enough to show that if $0\le k<\mathsf{o}_T(\lambda)$ then

$$\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}x_{\mathfrak{s},j}c_\mathfrak{s}^{(k)}\in\operatorname{rad}C_\lambda^{(k)}.$$

Using Lemma 2.36 and Lemma 2.34, if $\mathfrak{t}\in T_\sigma(\lambda)$ then

$$\begin{aligned}
\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kj}x_{\mathfrak{s},j}\phi_\lambda^{(k)}\big(c_\mathfrak{t}^{(k)},c_\mathfrak{s}^{(k)}\big) &= \mathsf{p}_\lambda\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{k(j+l)}x_{\mathfrak{s},j}\phi_\lambda\big(c_{\sigma_\lambda^l(\mathfrak{t})},c_\mathfrak{s}\big)\\
&= \mathsf{p}_\lambda\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{k(j+l)}x_{\mathfrak{s},j}\phi_\lambda\big(c_{\sigma_\lambda^{l+j}(\mathfrak{t})},c_{\sigma_\lambda^j(\mathfrak{s})}\big)\\
&= \mathsf{p}_\lambda\sum_{\mathfrak{s}\in T_\sigma(\lambda)}\sum_{j,l=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kl}x_{\mathfrak{s},j}\phi_\lambda\big(c_{\sigma_\lambda^l(\mathfrak{t})},c_{\sigma_\lambda^j(\mathfrak{s})}\big)\\
&= \mathsf{p}_\lambda\sum_{l=0}^{\mathsf{o}_T(\lambda)-1}\varepsilon_\lambda^{kl}\phi_\lambda\big(c_{\sigma_\lambda^l(\mathfrak{t})},x\big)\\
&= 0,
\end{aligned}$$

where the last equality comes from the fact that $x\in\operatorname{rad}C_\lambda$. $\square$

Definition 2.39. The automorphism σ_A is **ε-splittable** if there exists an invertible element $z \in A^\times$ that is homogeneous of degree 0 such that $\sigma_A(z) = \varepsilon z$.

Fix $z \in A^\times$ as in Definition 2.39. Then $z^i A^\sigma = \ker(\sigma_A - \varepsilon^i)$, for $i \geq 0$. The terminology of Definition 2.39 is justified because if R is a field then we can decompose A into a direct sum of σ_A-eigenspaces

$$A = \bigoplus_{i=0}^{\mathsf{p}-1} z^i A^\sigma, \tag{2.40}$$

since σ_A has order p. In particular, if σ_A is ε-splittable then A is free, and hence projective, as an A^σ-module.

Recall from Proposition 2.35 that $\gamma_\lambda : C_\lambda \longrightarrow C_{\sigma\lambda}$ is an R-linear isomorphism such that $\gamma_\lambda(ax) = \sigma(a)\gamma_\lambda(x)$, for all $a \in A$ and $x \in C_\lambda$. For $j \geq 0$ define the R-isomorphism $\gamma_{\lambda,j} : C_\lambda \longrightarrow C_{\sigma^j\lambda}$ by

$$\gamma_{\lambda,j} = \gamma_{\sigma^{j-1}\lambda} \circ \cdots \circ \gamma_\lambda, \tag{2.41}$$

and set $\sigma_{C,\lambda} := \gamma_{\lambda,\mathsf{o}_\lambda}$, an R-automorphism of C_λ. Then $\sigma_{C,\lambda}(c_\mathfrak{s}) = c_{\sigma_\lambda \mathfrak{s}}$, for all $\mathfrak{s} \in T(\lambda)$. In particular $\sigma_{C,\lambda}$ has order $\mathsf{o}_T(\lambda)$. The R-linear isomorphism $\gamma_{\lambda,j}$ satisfies

$$\gamma_{\lambda,j}(ax) = \sigma_A^j(a)\gamma_{\lambda,j}(x), \qquad \text{for all } a \in A \text{ and } x \in C_\lambda. \tag{2.42}$$

In particular $\sigma_{C,\lambda}$ satisfies

$$\sigma_{C,\lambda}(ax) = \sigma_A^{\mathsf{o}_\lambda}(a)\sigma_{C,\lambda}(x), \qquad \text{for all } a \in A \text{ and } x \in C_\lambda. \tag{2.43}$$

Proposition 2.44. *Suppose that σ_A is ε-splittable. Then $\mathsf{o}_T(\lambda) = \mathsf{p}_\lambda$ for all $\lambda \in \mathcal{P}$.*

Proof. Under the isomorphism γ'_λ of Proposition 2.37, the cell module $C_\lambda^{(k)}$ is sent into the eigenspace $\ker(\sigma_{C,\lambda} - \varepsilon_\lambda^k)$. Since $\varepsilon_\lambda \in R^\times$, these eigenspaces are in direct sum and we conclude that

$$C_\lambda = \bigoplus_{k=0}^{\mathsf{o}_T(\lambda)-1} \ker(\sigma_{C,\lambda} - \varepsilon_\lambda^k). \tag{2.45}$$

Now let $x \in C_\lambda$ be an eigenvector for $\sigma_{C,\lambda}$ with eigenvalue ε_λ^k. By (2.43), we have

$$\sigma_{C,\lambda}(zx) = \sigma_A^{\mathsf{o}_\lambda}(z)\sigma_{C,\lambda}(x) = \varepsilon^{\mathsf{o}_\lambda} z \varepsilon_\lambda^k x = \varepsilon^{\mathsf{o}_\lambda}\varepsilon_\lambda^k zx.$$

Since $z \in A^\times$, we have $zx \neq 0$ and thus zx is an eigenvector for $\sigma_{C,\lambda}$ with eigenvalue $\varepsilon^{\mathsf{o}_\lambda}\varepsilon_\lambda^k$. By (2.45), this implies that $\varepsilon^{\mathsf{o}_\lambda} \in \langle \varepsilon_\lambda \rangle$, thus $\varepsilon^{\mathsf{o}_\lambda}$ is an $\mathsf{o}_T(\lambda)$-th root of unity. But $\varepsilon^{\mathsf{o}_\lambda}$ has order $\frac{\mathsf{p}}{\mathsf{o}_\lambda} = \mathsf{p}_\lambda$, and hence $\mathsf{p}_\lambda \mid \mathsf{o}_T(\lambda)$, and hence $\mathsf{p}_\lambda = \mathsf{o}_T(\lambda)$ since $\mathsf{o}_T(\lambda) \mid \mathsf{p}_\lambda$. □

In particular, Proposition 2.44 implies that $\varepsilon_\lambda = \varepsilon^{\mathsf{o}_\lambda}$. Now let τ be the homogeneous automorphism of A given by $\tau(a) = zaz^{-1}$, for $a \in A$. Note that A^σ is stable under τ. The next result is a complement to Proposition 2.37.

Proposition 2.46. *Let $(\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}$. The map*

$$\gamma_\lambda^{(k)} : C_\lambda^{(k)} \to C_\lambda^{(k+1)}; \quad x \mapsto {\gamma'_\lambda}^{-1}\big(z\gamma'_\lambda(x)\big)$$

induces graded A^σ-module isomorphisms

$$C_\lambda^{(k)} \simeq {}^\tau C_\lambda^{(k+1)} \qquad \textit{and} \qquad D_\lambda^{(k)} \simeq {}^\tau D_\lambda^{(k+1)}.$$

Proof. During the proof of Proposition 2.44 we obtained $\gamma'_\lambda\big(C_\lambda^{(k)}\big) = \ker\big(\sigma_{C,\lambda} - \varepsilon_\lambda^k\big)$ and

$$z\gamma'_\lambda\big(C_\lambda^{(k)}\big) = \gamma'_\lambda\big(C_\lambda^{(k+1)}\big). \tag{2.47}$$

Thus, the map $\gamma_\lambda^{(k)}$ is well-defined. Moreover, it is clearly bijective since $z \in A^\times$. Equation (2.47) implies that there is an A^σ-module isomorphism $C_\lambda^{(k)} \simeq {}^\tau C_\lambda^{(k+1)}$ because if $x \in C_\lambda^{(k)}$ and $a \in A^\sigma$ then

$$\begin{aligned}\gamma_\lambda^{(k)}(ax) &= {\gamma'_\lambda}^{-1}\big(z\gamma'_\lambda(ax)\big) = {\gamma'_\lambda}^{-1}\big(za\gamma'_\lambda(x)\big) = {\gamma'_\lambda}^{-1}\big(\tau(a)z\gamma'_\lambda(x)\big)\\ &= \tau(a){\gamma'_\lambda}^{-1}\big(z\gamma'_\lambda(x)\big) = \tau(a)\gamma_\lambda^{(k)}(x).\end{aligned}$$

Moreover the map $\gamma_\lambda^{(k)}$ is homogeneous of degree 0 since z has degree zero (and γ'_λ is homogeneous).

Finally, by Theorem 2.17(b), $\operatorname{rad} C_\lambda^{(l)}$ is the Jacobson radical of $C_\lambda^{(l)}$ for all l so, since $\gamma_\lambda^{(k)}$ is an A^σ-module isomorphism, $\gamma_\lambda^{(k)}\big(\operatorname{rad} C_\lambda^{(k)}\big) = \operatorname{rad} C_\lambda^{(k+1)}$. Hence, $\gamma_\lambda^{(k)}$ induces an isomorphism $D_\lambda^{(k)} \simeq {}^\tau D_\lambda^{(k+1)}$ of A^σ-modules. □

Since A^σ is a skew cellular algebra, Theorem 2.17 gives a classification of the graded simple A^σ-modules. Combining the results above we obtain the following classification of the simple A^σ-modules in terms of the simple A-modules.

Theorem 2.48. *Let R be a field containing a primitive pth root of unity ε. Suppose that A has graded cell datum $(\mathcal{P}, T, C, \deg)$ and a shift automorphism $\sigma = (\sigma_A, \sigma_{\mathcal{P}}, \sigma_T)$ such that σ_A is ε-splittable and has order p. Then*

$$\{D_\lambda^{(k)}\langle s\rangle \mid D^\lambda \neq 0 \textit{ for } (\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}, \textit{ and } s \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic graded simple A^σ-modules.

Proof. First note that because R contains a primitive pth root of unity the characteristic of R cannot divide p, so $\mathsf{p} \cdot 1_R$ is invertible in R. Therefore, A^σ is a skew cellular algebra with skew cellular datum $(\mathcal{P}_{\sigma,\mathsf{p}}, \iota_\sigma, T_\sigma, C_\sigma, \deg_\sigma)$ by Theorem 2.28. Therefore, by Theorem 2.17, a complete set of pairwise non-isomorphic graded simple A^σ-modules is given by the non-zero modules in the set $\{D_\lambda^{(k)}\langle s\rangle \mid (\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}, s \in \mathbb{Z}\}$. By Proposition 2.37, Proposition 2.44 and Proposition 2.46, if $\lambda \in \mathcal{P}_\sigma$ then

$$D^\lambda \neq 0 \iff D_\lambda^{(k)} \neq 0 \text{ for some } 0 \le k < \mathsf{p}_\lambda \iff D_\lambda^{(k)} \neq 0 \text{ for all } 0 \le k < \mathsf{p}_\lambda.$$

Hence, the result follows. □

By (2.40), if M is an A^σ-module then the induced A-module $M{\uparrow}$ is given by

$$M{\uparrow} \simeq \bigoplus_{i=0}^{\mathsf{p}-1} z^i M,$$

where the action of $a \in A$ on $z^i M$ is given by

$$ax = \sum_{j=0}^{\mathsf{p}-1} z^j a_j x, \qquad \text{for } x \in M,$$

where $az^i = \sum_j z^j a_j$ with $a_j \in A^\sigma$.

Recall the R-linear maps $\gamma_{\lambda,j} : C_\lambda \longrightarrow C_{\sigma^j\lambda}$ from (2.41).

Proposition 2.49. *Assume that R is a field and that σ_A is ε-splittable. For $(\lambda, k) \in \mathcal{P}_{\sigma,\mathsf{p}}$, let*

$$\hat{\gamma}''_\lambda : C_\lambda^{(k)} \to \bigoplus_{j=0}^{\mathsf{o}_\lambda - 1} C_{\sigma^j\lambda}$$

be the A^σ-linear map whose j-th component is given by $\gamma_{\lambda,j} \circ \gamma'_\lambda$ for $0 \le j < \mathsf{o}_\lambda$. The unique corresponding A-linear map $\gamma''_\lambda : C_\lambda^{(k)}\big\uparrow \longrightarrow \bigoplus_{j=0}^{\mathsf{o}_\lambda-1} C_{\sigma^j\lambda}$ induces isomorphisms of graded A-modules

$$C_\lambda^{(k)}\Big\uparrow \simeq \bigoplus_{j=0}^{\mathsf{o}_\lambda-1} C_{\sigma^j\lambda} \qquad \text{and} \qquad D_\lambda^{(k)}\Big\uparrow \simeq \bigoplus_{j=0}^{\mathsf{o}_\lambda-1} D_{\sigma^j\lambda}.$$

Proof. Since $C_\lambda^{(k)}\Big\uparrow \simeq \oplus_{i=0}^{\mathsf{p}} z^i C_\lambda^{(k)}$ as an A-module, if $x_i \in C_\lambda^{(k)}$, where $0 \le i < \mathsf{p}$, by Frobenius reciprocity we have

$$\gamma''_\lambda\left(\sum_{i=0}^{\mathsf{p}-1} z^i x_i\right) = \sum_{i=0}^{\mathsf{p}-1} z^i \hat{\gamma}''_\lambda(x_i).$$

The map γ''_λ is A-linear by construction. Recalling from Proposition 2.44 that $\mathsf{o}_T(\lambda) = \mathsf{p}_\lambda$, thus

$$\mathsf{p}|T_\sigma(\lambda)| = \mathsf{o}_\lambda \mathsf{p}_\lambda |T_\sigma(\lambda)| = \mathsf{o}_\lambda |T(\lambda)|.$$

It follows that the starting and ending R-vector spaces have the same dimension. To prove that γ''_λ is bijective it suffices to prove that it is injective.

Let $(x_i)_{0 \le i < \mathsf{p}}$ be as above and assume that $\gamma''_\lambda\left(\sum_{i=0}^{\mathsf{p}-1} z^i x_i\right) = 0$. Since $\hat{\gamma}''_\lambda(x) = \sum_{j=0}^{\mathsf{o}_\lambda-1} \gamma_{\lambda,j}\big(\gamma'_\lambda(x)\big)$ for all $x \in C_\lambda^{(k)}$, we deduce that for all $0 \le j < \mathsf{o}_\lambda$ we have

$$\sum_{i=0}^{\mathsf{p}-1} z^i \gamma_{\lambda,j}\big(\gamma'_\lambda(x_i)\big) = 0.$$

Now using Definition 2.39 and (2.42), we deduce that

$$\sum_{i=0}^{\mathsf{p}-1} \varepsilon^{-ij} \gamma_{\lambda,j}\big(z^i \gamma'_\lambda(x_i)\big) = 0.$$

Since $\gamma_{\lambda,j}$ is an R-isomorphism, we deduce that for all $0 \le j < \mathsf{o}_\lambda$ we have

$$\sum_{i=0}^{\mathsf{p}-1} \varepsilon^{-ij} z^i \gamma'_\lambda(x_i) = 0.$$

Writing $i = a\mathsf{p}_\lambda + b$ for $0 \le a < \mathsf{o}_\lambda$ and $0 \le b < \mathsf{p}_\lambda$,

$$\sum_{a=0}^{\mathsf{o}_\lambda-1} \sum_{b=0}^{\mathsf{p}_\lambda-1} \varepsilon^{-(a\mathsf{p}_\lambda+b)j} z^{a\mathsf{p}_\lambda+b} \gamma'_\lambda(x_{a\mathsf{p}_\lambda+b}) = 0, \qquad \text{for all } 0 \le j < \mathsf{o}_\lambda.$$

By (2.47), $z^{a\mathsf{p}_\lambda+b}\gamma'_\lambda(C_\lambda^{(k)}) = \gamma'_\lambda(C_\lambda^{(k+b)})$. Thus, using Proposition 2.37,

$$\sum_{a=0}^{\mathsf{o}_\lambda-1} \varepsilon^{-aj\mathsf{p}_\lambda} z^{a\mathsf{p}_\lambda} \gamma'_\lambda(x_{a\mathsf{p}_\lambda+b}) = 0, \qquad \text{for } 0 \le j < \mathsf{o}_\lambda \text{ and } 0 \le b < \mathsf{p}_\lambda.$$

Since $\varepsilon^{\mathsf{p}_\lambda}$ is a primitive o_λ-th root of unity, for a fixed $0 \le b < \mathsf{p}_\lambda$ we obtain an invertible linear system, so $z^{a\mathsf{p}_\lambda}\gamma'_\lambda(x_{a\mathsf{p}_\lambda+b}) = 0$ in C_λ, for $0 \le j, a < \mathsf{o}_\lambda$ and

$0 \le b < \mathsf{p}_\lambda$. Since $z \in A^\times$ we deduce that $\gamma'_\lambda(x_i) = 0$, for $0 \le i < \mathsf{p}$. Hence, $x_i = 0$ in $C^{(k)}_\lambda$ since γ'_λ is injective. We conclude that γ''_λ is injective, proving the first isomorphism of A-modules. Note that this isomorphism is homogeneous of degree 0 since $\deg(z) = 0$ and $\hat{\gamma}''_\lambda$ is homogeneous.

To prove the second isomorphism, by Proposition 2.35 and Proposition 2.37 we have

$$\gamma''_\lambda\left(\bigoplus_{i=0}^{\mathsf{p}-1} z^i \operatorname{rad} C^{(k)}_\lambda\right) \subseteq \bigoplus_{j=0}^{\mathsf{o}_\lambda - 1} \operatorname{rad} C_{\sigma^j\lambda}.$$

Moreover we also obtain that

$$\dim_R \operatorname{rad} C_{\sigma^j\lambda} = \dim_R \operatorname{rad} C_\lambda = \mathsf{p}_\lambda \dim_R \operatorname{rad} C^{(k)}_\lambda,$$

for all $0 \le j < \mathsf{o}_\lambda$. Thus, the above inclusion is an equality and the proof is complete. □

Remark 2.50. With a little more care it is possible to prove Proposition 2.49 over an integral domain that contains ε. As in [19, §3.7], the existence of the isomorphism of Proposition 2.49 can be deduced from more general results such as [13, Proposition 2.2] (and [16, Appendix]). The point of Proposition 2.49 is to give an explicit isomorphism.

3. Hecke algebras and diagrammatic Cherednik algebras

Having set up the machinery of skew cellular algebras we are now ready to tackle the main results of this paper, which show that the Hecke algebras of type $G(\ell,p,n)$ are graded skew cellular algebras. To do this we use the cyclotomic KLR algebras of type A, together with the diagrammatic Cherednik algebras, to construct a shift automorphism of these algebras.

3.1. **Hecke algebras.** This section recalls the definitions and results from the literature that we need about the Hecke algebras of type $G(\ell,p,n)$. Throughout this paper we fix positive integers n, p and d, with $p \ge 2$. Recall that R is a commutative integral ring with 1. Let $K = \operatorname{Frac}(R)$ be the field of fractions of R. We assume that K contains a primitive pth root of unity ε. Set $\ell = pd$ and fix **cyclotomic parameters** $Q_1, \dots, Q_d \in K$. Set $\mathbf{Q} = (Q_1, \dots, Q_d)$ and

$$\mathbf{Q}^{\vee\varepsilon} = (\varepsilon Q_1, \varepsilon^2 Q_1, \dots, \varepsilon^p Q_1, \varepsilon Q_2, \dots, \varepsilon^p Q_2, \dots, \varepsilon Q_d, \dots, \varepsilon^p Q_d).$$

Finally, fix an invertible **Hecke parameter** $q \in K$.

Definition 3.1 (Ariki and Koike [2], Broué and Malle [6])**.** The **Hecke algebra** of type $G(\ell,1,n)$ with Hecke parameter q and cyclotomic parameters $\mathbf{Q}^{\vee\varepsilon}$ is the unital associative K-algebra $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ with generators $T_0, T_1, \dots, T_{n-1}$ and relations:

$$\begin{aligned} \prod_{k=1}^{d} (T_0^p - Q_k^p) &= 0, & T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\ (T_r - q)(T_r + 1) &= 0, & T_kT_{k+1}T_k &= T_{k+1}T_kT_{k+1}, \\ & & T_iT_j = T_jT_i \quad \text{if} \quad |i-j| > 1, \end{aligned}$$

where $1 \le r < n, 1 \le k < n-1$ and $1 \le i, j < n$.

Remark 3.2. The algebra $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ is in fact a special case of a Hecke algebra of type $G(\ell,1,n)$, which can have ℓ arbitrary cyclotomic parameters $Q_1, \dots, Q_\ell \in K$.

Inspecting the relations, $\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})$ has a unique automorphism σ of order p such that

$$\sigma(T_0)=\varepsilon T_0 \quad\text{and}\quad \sigma(T_i)=T_i \text{ for } 1\le i<n. \tag{3.3}$$

We can now define the main (ungraded) algebras of interest in this paper.

Definition 3.4 (Ariki [1], Broué and Malle [6]). The **Hecke algebra** of type $G(\ell,p,n)$ with parameters $q\in K^\times$ and $\mathbf{Q}^{\vee\varepsilon}\in K^\ell$ is the fixed-point subalgebra

$$\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})=\big(\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})\big)^\sigma=\{h\in\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})\mid \sigma(h)=h\}.$$

Equivalently, $\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$ is the subalgebra of $\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})$ that is generated by T_0^p, $T_0^{-1}T_1T_0$ and $T_1,\dots,T_{n-1}$. Notice that $\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$ is an Iwahori-Hecke algebra of type D when $\ell=p=2$.

From the relations it is clear that if $c\in K^\times$ is any non-zero scalar then $\mathscr{H}_n(q,c\mathbf{Q}^{\vee\varepsilon})\cong\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})$ and hence that $\mathscr{H}_{p,n}(q,c\mathbf{Q}^{\vee\varepsilon})\cong\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$, where $c\mathbf{Q}^{\vee\varepsilon}=(c\varepsilon Q_1,\dots,c\varepsilon^pQ_d)$. Moreover, by [17], we can assume that the cyclotomic parameters $Q_1,\cdots,Q_d$ are in a single (ε,q)-orbit. That is, $Q_i/Q_j\in\varepsilon^{\mathbb{Z}}q^{\mathbb{Z}}$ for any $1\le i,j\le d$.

Let $e=\min\{e>0\mid 1+q+\cdots+q^{e-1}=0\}$ and set $e=\infty$ if no such integer exists. Using Clifford theory, as discussed on [16, Page 3383], we can further assume that $Q_i=q^{\rho_i}$,

$$\mathbf{Q}^{\vee\varepsilon}=(\varepsilon q^{\rho_1},\varepsilon^2q^{\rho_1},\dots,\varepsilon^pq^{\rho_1},\varepsilon q^{\rho_2},\dots,\varepsilon^pq^{\rho_2},\dots,\varepsilon q^{\rho_d},\dots,\varepsilon^pq^{\rho_d}). \tag{3.5}$$

and that we are in one of Cases 1 and 2:

Case 1 ($q^{\mathbb{Z}}\cap\varepsilon^{\mathbb{Z}}\ne\{1\}$). Equivalently, $e<\infty$ and $\gcd(e,p)>1$. Let $m=\gcd(e,p)$ and write $p=mp'$. We may assume that $e=me'$ and that $\varepsilon^{p'}=q^{e'}$ is a primitive mth root of unity in K. Note that $p'=\min\{1\le a\le p\mid \varepsilon^a\in q^{\mathbb{Z}}\}$.

Case 2 ($q^{\mathbb{Z}}\cap\varepsilon^{\mathbb{Z}}=\{1\}$). Equivalently, either $e<\infty$ and $\gcd(e,p)=1$, or $e=\infty$. For consistency of notation with Case 1, we assume that $0=\rho_1\le\rho_2\le\cdots\le\rho_d$ and set $p'=p$, $e'=e$ and $m=1$. In fact, as noted in [20, Corollary 2.10], if $e=\infty$ then we can replace q with an $\hat{e}$ root of unity for some sufficiently large $\hat{e}$ without changing the (graded) isomorphism type of $\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})$. Henceforth, we assume that e is finite.

Permuting the integers $\rho_1,\dots,\rho_d$ does not affect $\mathscr{H}_n(q,\mathbf{Q}^{\vee\varepsilon})$ up to isomorphism and, similarly, we can replace ρ_a with ρ_a+e since $q^{\rho_a+e}=q^{\rho_a}$. In order to be able to construct the basis that we need to prove our main results we adopt the following convention.

Definition 3.6. A d**-charge** is a d-tuple of integers $\boldsymbol{\rho}=(\rho_1,\dots,\rho_d)\in\mathbb{Z}^d$ such that

$$\rho_{a+1}-\rho_a\ge(2n+3)e,\qquad \text{for } 1\le a<d.$$

We assume that we have a fixed $\boldsymbol{\rho}$ for the rest of this paper. For convenience, we assume that $\rho_1=0$. In particular, this implies that $0=\rho_1<\rho_2<\cdots<\rho_d$.

3.2. **Quiver Hecke algebras.** The quiver Hecke algebras, or KLR algebras, are a remarkable family of $\mathbb{Z}$-graded algebras that were introduced by Khovanov and Lauda [22] and Rouquier [31]. Following [29], and to a lesser extent [7], this section defines the quiver Hecke algebras of type $G(\ell,1,n)$ that we need to study $\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$.

Definition 3.7. Set $\mathcal{I} = \{\varepsilon^j q^i \mid 0 \le j < p \text{ and } 0 \le i < e\}$. An **$\mathcal{I}$-composition** of n is a finitely supported tuple $\alpha = (\alpha_i)_{i \in \mathcal{I}}$ of non-negative integers that sum to n. Let $\mathscr{C}_n^\ell$ be the set of $\mathcal{I}$-compositions of n. If $\alpha = (\alpha_i)_{i\in\mathcal{I}} \in \mathscr{C}_n^\ell$ let

$$\mathcal{I}^\alpha = \big\{\mathbf{i} = (i_1, \dots, i_n) \in \mathcal{I}^n \mid \alpha_i = \#\{1 \le k \le n \mid i_k = i\} \text{ for all } i \in \mathcal{I}\big\}.$$

A **residue sequence** is an element of $\mathcal{I}^n$.

Let $\mathfrak{S}_n$ be the **symmetric group** of degree n. As a Coxeter group, $\mathfrak{S}_n$ is generated by $s_1, \dots, s_{n-1}$, where $s_r = (r, r+1)$. If $w \in \mathfrak{S}_n$ then a **reduced expression** for w is any word $w = s_{r_1} \dots s_{r_k}$ with k minimal.

The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{I}^n$ by place permutations and the sets $\mathcal{I}^\alpha$, for $\alpha \in \mathscr{C}_n^\ell$, are the $\mathfrak{S}_n$-orbits of $\mathcal{I}^n$. In particular, the sets $\mathcal{I}^\alpha$ are finite. Set

$$J' = \{1, 2, \cdots, p'\} \quad \text{and} \quad I = \mathbb{Z}/e\mathbb{Z}.$$

As noted in [29], there is a natural bijection $I \times J' \xrightarrow{\sim} \mathcal{I}$ given by sending (i, j) to $\varepsilon^j q^i$, for $(i, j) \in I \times J'$. Henceforth, we identify $I \times J'$ and $\mathcal{I}$ using this bijection.

Definition 3.8 (Rouquier [31, §3.2.5])**.** Let Γ be the quiver with vertex set $\mathcal{I}$ and edges $i \longrightarrow qi$, for $i \in \mathcal{I}$. Let Γ_e be the full subquiver of Γ with vertex set $\{q^i \mid i \in I\}$.

Notice that there is an isomorphism of quivers $\Gamma_e \cong \Gamma_e^{(j)}$, where $\Gamma_e^{(j)}$ has vertex set $\{\varepsilon^j q^i \mid i \in I\}$, for $1 \le j \le p'$. Moreover, there are no edges between the vertices of $\Gamma_e^{(j)}$ and the vertices of $\Gamma_e^{(k)}$ if $j \ne k \in J'$. Hence, $\Gamma = \Gamma_e^{(1)} \sqcup \cdots \sqcup \Gamma_e^{(p')}$ is the disjoint union of p' copies of the quiver Γ_e, which is the affine quiver of type $A_e^{(1)}$.

Following Rouquier [31, §3.2.4], for $i, j \in \mathcal{I}$ define homogeneous polynomials $Q_{i,j}(u, v) \in R[u, v]$, where u and v are indeterminates, by

$$Q_{i,j}(u,v) = \begin{cases} (u-v)(v-u), & \text{if } i \leftrightarrows j, \\ (v-u), & \text{if } i \to j, \\ (u-v), & \text{if } i \leftarrow j, \\ 1, & \text{if } i \not\!\!- j, \\ 0, & \text{if } i = j, \end{cases}$$

where all edges are in the quiver Γ. The **degree** of $Q_{i,j}(u, v)$ is its homogeneous degree.

For $\iota = (i, j) \in I \times J' = \mathcal{I}$, define Λ_ι to be the multiplicity of $\varepsilon^j q^i$ in $\mathbf{Q}^{\vee\varepsilon}$. Set $\mathbf{\Lambda} = (\Lambda_\iota)_{\iota\in\mathcal{I}}$.

Definition 3.9 (Khovanov and Lauda [22], Rouquier [31])**.** Let α be an $\mathcal{I}$-composition of n. The **quiver Hecke algebra of type** $G(\ell, 1, n)$ **and weight $\mathbf{\Lambda}$** is the unital associative R-algebra $\mathscr{R}_\alpha^{\mathbf{\Lambda}}$ with generators

$$\{e(\mathbf{i}) \mid \mathbf{i} \in \mathcal{I}^\alpha\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

and relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) = 1, \qquad y_1^{\Lambda_{i_1}} e(\mathbf{i}) = 0,$$
$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$

$$\begin{aligned} \psi_r \psi_s &= \psi_s \psi_r, && \text{if } |r - s| > 1, \\ \psi_r y_s &= y_s \psi_r, && \text{if } s \ne r, r+1, \end{aligned}$$

$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r\psi_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \quad y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf{i}),$$

$$\psi_r^2 e(\mathbf{i}) = Q_{i_r,i_{r+1}}(y_r, y_{r+1})e(\mathbf{i}),$$

$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)e(\mathbf{i}) = \delta_{i_r i_{r+2}} \frac{Q_{i_r,i_{r+1}}(y_r, y_{r+1}) - Q_{i_r,i_{r+1}}(y_{r+2}, y_{r+1})}{y_r - y_{r+2}} e(\mathbf{i})$$

for all admissible r, s and $\mathbf{i}, \mathbf{j} \in \mathcal{I}^\alpha$. Set $\mathscr{R}_n^{\mathbf{\Lambda}} = \bigoplus_{\alpha\in\mathscr{C}_n^\ell} \mathscr{R}_\alpha^{\mathbf{\Lambda}}$.

Remark 3.10. In the literature, $\mathscr{R}_n^{\mathbf{\Lambda}}$ is often called a cyclotomic Hecke algebra of type A and weight $\mathbf{\Lambda}$. Our naming convention reflects the close connections between the algebras $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ and $\mathscr{R}_n^{\mathbf{\Lambda}}$.

An important consequence of these relations is that $\mathscr{R}_n^{\mathbf{\Lambda}}$ is a $\mathbb{Z}$-graded algebra with

$$\deg e(\mathbf{i}) = 0, \quad \deg y_r = 2 \quad \text{and} \quad \deg \psi_r e(\mathbf{i}) = \begin{cases} \deg Q_{i_r,i_{r+1}}(u, v), & \text{if } i_r \neq i_{r+1}, \\ -2, & \text{if } i_r = i_{r+1}. \end{cases}$$

Following the reformulation in [29], we can now state the main result of [7] that we need in order to apply this result to the algebra $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$.

Theorem 3.11 (Brundan and Kleshchev's isomorphism theorem [7, 29])**.** *Assume that $R = K$ is a field. Then there is an isomorphism of K-algebras $f\colon \mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon}) \xrightarrow{\sim} \mathscr{R}_n^{\mathbf{\Lambda}}$.*

Motivated in part by Definition 3.4, the third named author [29] generalised this result to show that $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$ is isomorphic to the fixed-point subalgebra of $\mathscr{R}_n^{\mathbf{\Lambda}}$ under a certain homogeneous automorphism of order p. Recall the automorphism σ of $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ from (3.3). By definition, σ has order p, so $\langle\sigma\rangle \cong \langle\varepsilon\rangle$, which is a cyclic group of order p.

If $\alpha \in \mathscr{C}_n^\ell$ let $\sigma \cdot \alpha$ be the $\mathcal{I}$-composition of n given by

$$(\sigma \cdot \alpha)_i = \alpha_{\varepsilon^{-1}i}, \qquad \text{for all } i \in \mathcal{I}. \tag{3.12}$$

Observe that left multiplication by ε gives a map $\mathcal{I}^\alpha \longrightarrow \mathcal{I}^{\sigma\cdot\alpha}; \mathbf{i} \mapsto \varepsilon\mathbf{i} = (\varepsilon i_1, \dots, \varepsilon i_n)$. Moreover, by (3.5), $\Lambda_{\varepsilon\iota} = \Lambda_\iota$ for all $\iota \in \mathcal{I}$.

Theorem 3.13 (Rostam [29], [28, §1.4])**.** *Let $\alpha \in \mathscr{C}_n^\ell$. There is a unique homogeneous R-algebra isomorphism $\sigma_\alpha^{\mathbf{\Lambda}} : \mathscr{R}_\alpha^{\mathbf{\Lambda}} \longrightarrow \mathscr{R}_{\sigma\cdot\alpha}^{\mathbf{\Lambda}}$ such that*

$$\sigma_\alpha^{\mathbf{\Lambda}}\big(e(\mathbf{i})\big) = e(\varepsilon\mathbf{i}), \quad \sigma_\alpha^{\mathbf{\Lambda}}(y_r) = y_r \quad \textit{and} \quad \sigma_\alpha^{\mathbf{\Lambda}}(\psi_s) = \psi_s,$$

for all $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in \mathcal{I}^\alpha$.

Set $\sigma_n^{\mathbf{\Lambda}} = \bigoplus_\alpha \sigma_\alpha^{\mathbf{\Lambda}}$, so that $\sigma_n^{\mathbf{\Lambda}}$ is an automorphism of $\mathscr{R}_n^{\mathbf{\Lambda}} = \bigoplus_\alpha \mathscr{R}_\alpha^{\mathbf{\Lambda}}$. To ease the notation, we normally write $\sigma = \sigma_n^{\mathbf{\Lambda}}$. We are abusing notation here because the automorphism σ of $\mathscr{R}_n^{\mathbf{\Lambda}}$ is not equal to the automorphism σ of $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ that was defined in (3.3). This abuse is justified by Theorem 3.14.

If α is an $\mathcal{I}$-composition let $[\alpha] = \{\sigma^k \cdot \alpha \mid 1 \le k \le p\}$ be the orbit of α under the action of $\langle\sigma\rangle \simeq \mathbb{Z}/p\mathbb{Z}$. Let $\mathcal{I}_\sigma^n = \{[\alpha] \mid \alpha \in \mathscr{C}_n^\ell\}$ be the set of σ-orbits of $\mathscr{C}_n^\ell$ and if $[\alpha] \in \mathcal{I}_\sigma^n$ set $\mathscr{R}_{[\alpha]}^{\mathbf{\Lambda}} = \bigoplus_{\beta\in[\alpha]} \mathscr{R}_\beta^{\mathbf{\Lambda}}$. By definition,

$$\mathscr{R}_n^{\mathbf{\Lambda}} = \bigoplus_{\alpha\in\mathscr{C}_n^\ell} \mathscr{R}_\alpha^{\mathbf{\Lambda}} = \bigoplus_{[\alpha]\in\mathcal{I}_\sigma^n} \mathscr{R}_{[\alpha]}^{\mathbf{\Lambda}}$$

and the isomorphism σ of Theorem 3.13 restricts to an automorphism σ of $\mathscr{R}^{\mathbf{\Lambda}}_{[\alpha]}$. Hence, we can consider σ as both an automorphism of $\mathscr{R}^{\mathbf{\Lambda}}_{[\alpha]}$ and as an automorphism of $\mathscr{R}^{\mathbf{\Lambda}}_{n}$.

Theorem 3.14 (Rostam [29, Theorem 4.14, Corollary 4.16]). *Assume that $R = K$ is a field. We can choose the isomorphism $f \colon \mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon}) \xrightarrow{\sim} \mathscr{R}^{\mathbf{\Lambda}}_n$ of Theorem* 3.11 *so that the following diagram commutes*

$$\begin{tikzcd}
\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon}) \arrow[r, "f"] \arrow[d, "\sigma"'] & \mathscr{R}^{\mathbf{\Lambda}}_n \arrow[d, "\sigma"] \\
\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon}) \arrow[r, "f"'] & \mathscr{R}^{\mathbf{\Lambda}}_n
\end{tikzcd}$$

Consequently, f induces an isomorphism $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon}) \cong \bigoplus_{[\alpha]\in\mathcal{I}^n_\sigma} \left(\mathscr{R}^{\mathbf{\Lambda}}_{[\alpha]}\right)^\sigma.$

Definition 3.15 (Rostam [29], [28, §1.4]). The **quiver Hecke algebra of type** $G(\ell, p, n)$ of weight $\mathbf{\Lambda}$ is the R-algebra

$$\mathscr{R}^{\mathbf{\Lambda}}_{p,n} = \left(\mathscr{R}^{\mathbf{\Lambda}}_n\right)^\sigma = \bigoplus_{[\alpha]\in\mathcal{I}^n_\sigma} \left(\mathscr{R}^{\mathbf{\Lambda}}_{[\alpha]}\right)^\sigma .$$

For $[\alpha] \in \mathcal{I}^n_\sigma$ let $\mathscr{R}^{\mathbf{\Lambda}}_{p,\alpha} = \left(\mathscr{R}^{\mathbf{\Lambda}}_{[\alpha]}\right)^\sigma$.

The algebra $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ inherits a $\mathbb{Z}$-grading from $\mathscr{R}^{\mathbf{\Lambda}}_n$ since σ is a homogeneous automorphism of $\mathscr{R}^{\mathbf{\Lambda}}_n$. The aim of this paper is to better understand the algebra $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$. Our main tool is the diagrammatic Cherednik algebra introduced by Webster [32] and Bowman [3].

3.3. **Loadings, multicharges and ℓ-partitions.** This section introduces the combinatorics that underpins Webster's diagrammatic Cherednik algebras.

After Definition 3.4 we fixed integers $(d, e, e', p, p', m, n, \rho_1, \dots, \rho_d)$ subject to Definition 3.6 that determine q and $\mathbf{Q}^{\vee\varepsilon}$. Using this data we now fix a choice of multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell)$ that we use in Section 3.4 to single out a diagrammatic Cherednik algebra that is particularly well adapted to studying $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$.

Definition 3.16. The **multicharge** of $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$ is the sequence $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell)$ with

$$\kappa_l = \rho_{a+1} + \frac{be'}{p'},$$

where $l = ap + b + 1$ with $0 \le a < d$ and $0 \le b < p$.

Remark 3.17. If $d = 1$, the multicharge $\boldsymbol{\kappa}$ is an example of a "FLOTW charge" (see [3, Example 1.6]).

For the rest of this paper we fix the multicharge $\boldsymbol{\kappa}$ of Definition 3.16 and we identify ε and $q^{e'/p'}$. Recalling Definition 3.6, Definition 3.16 implies that $\mathbf{Q}^{\vee\varepsilon} = (\varepsilon^{\kappa_1}, \dots, \varepsilon^{\kappa_\ell})$.

Note that $b\frac{e'}{p'} < \frac{pe'}{p'} = me' = e$, for $0 \le b < p$. Therefore, $0 = \kappa_1 < \kappa_2 < \cdots < \kappa_\ell$ since $\rho_{a+1} - \rho_a \ge e$ by Definition 3.6.

Examples 3.18.

- Consider $G(p,p,n)$ and suppose that $\gcd(e,p)=1$. Then $\ell=p=p'$ and $m=d=1$ so that $e'=e$ and $\boldsymbol{\kappa}=(0,\frac{e}{p},\frac{2e}{p}\dots,\frac{(p-1)e}{p})$.
- Suppose that $p'=3$ and $m=2$ with $e=2e'<\infty$. Then
$$\boldsymbol{\kappa}=\big(0,\tfrac{e'}{3},\dots,\tfrac{5e'}{3},\rho_2,\dots,\rho_2+\tfrac{5e'}{3},\dots,\rho_d,\dots,\rho_d+\tfrac{5e'}{3}\big).$$

The set of **ℓ-nodes** is the set of all ordered triples

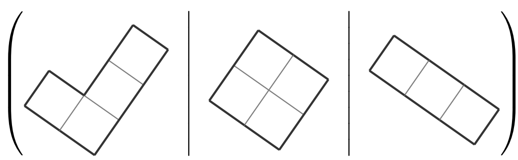
$$\mathscr{N}_n^\ell=\{(r,c,l)\in\mathbb{N}^3\mid 0\le r,c\le n \text{ and } 1\le l\le\ell\}. \tag{3.19}$$

A **partition** of n is a sequence $\mu=(\mu_1,\dots,\mu_h)$ of non-negative integers satisfying $\mu_1\ge\dots\ge\mu_h$ and $|\mu|:=\mu_1+\dots+\mu_h=n$. Let (0) be the empty partition (where $h=0$) and use exponentiation for repeated parts. An **ℓ-partition** of n is an ℓ-tuple $\boldsymbol{\mu}=(\mu^{(1)}|\dots|\mu^{(\ell)})$ of partitions such that $|\mu^{(1)}|+\dots+|\mu^{(\ell)}|=n$. Let $\mathscr{P}_n^\ell$ be the set of ℓ-partitions of n. An ℓ-partition $\boldsymbol{\mu}\in\mathscr{P}_n^\ell$ is identified with its **diagram**, which is the set of nodes

$$\boldsymbol{\mu}=\{(r,c,l)\in\mathscr{N}_n^\ell\mid 1\le r\le\mu_c^{(l)}\}.$$

We draw ℓ-partitions as an array of boxes in plane using Bowman's variation of the *Russian convention*, as in Example 3.20.

Example 3.20. Let $\boldsymbol{\mu}=(3,1|2^2|1^3)$. The diagram of $\boldsymbol{\mu}$ is:



Fix an integer N with

$$N>2nep'(\ell+1). \tag{3.21}$$

Using the multicharge $\boldsymbol{\kappa}=(\kappa_1,\dots,\kappa_\ell)$ define a **loading** function $\mathsf{x}_{\boldsymbol{\rho}}:\mathscr{N}_n^\ell\longrightarrow\mathbb{Q}$ by

$$\mathsf{x}_{\boldsymbol{\rho}}(r,c,l)=c-r+\tfrac{1}{e}\big(\kappa_l-\tfrac{l-1}{\ell+1}\big)-\tfrac{r+c}{N}. \tag{3.22}$$

Remark 3.23. Bowman [3, §1.3] defines his loading function as $\mathsf{x}_{\boldsymbol{\rho}}(r,c,l)=\kappa_l-\frac{l}{\ell}+c-r-\frac{1}{N}(r+c)$. The term $\frac{l-1}{\ell+1}$ is there to separate the nodes (r,c,l) and (r,c,l'), when $l\ne l'$. We divide by $\ell+1$, rather than ℓ as Bowman does, precisely because if $m=1=d$ then $p=p'=\ell$ so $\kappa_l=\rho_{a+1}+\frac{be'}{\ell}$. The $l-1$ in the numerator is a convenient renormalisation so that $\mathsf{x}_{\boldsymbol{\rho}}(0,0,1)=\kappa_1=0$.

Lemma 3.24. *The function $\{1,\dots,\ell\}\to\mathbb{R};l\mapsto\kappa_l-\frac{l-1}{\ell+1}$ is strictly increasing. Moreover, if $l=ap+b+1$, with $0\le a<d$ and $0\le b<p$, then*

$$\rho_{a+1}-\frac{ap}{\ell+1}\le\kappa_l-\frac{l-1}{\ell+1}<\rho_{a+1}+e-\frac{(a+1)p}{\ell+1}.$$

Proof. Let $l,l'\in\{1,\dots,\ell\}$ and write $l=ap+b+1$ and $l'=a'p+b'+1$ with $0\le a,a'<d$ and $0\le b,b'<p$. Without loss of generality, assume that $l<l'$. If $a<a'$ then by the inequality of Definition 3.16, together with the observation that $p\frac{e'}{p'}=me'=e$, we obtain

$$\kappa_{l'}-\kappa_l-\frac{l'-l}{\ell+1}=\rho_{a'+1}-\rho_{a+1}+(b'-b)\frac{e'}{p'}-\frac{l'-l}{\ell+1}\ge(2n+3)e-e-1>0.$$

Now if $a = a'$ and $b < b'$ we have

$$\kappa_{l'} - \kappa_l - \frac{l' - l}{\ell + 1} = (b' - b)\left(\frac{e'}{p'} - \frac{1}{\ell + 1}\right) = (b' - b)\left(\frac{e}{p} - \frac{1}{\ell + 1}\right) > 0,$$

since $p \le \ell$, proving the first claim. If $l = ap + b + 1$, we have $ap + 1 \le l \le (a+1)p$ and we deduce that

$$\rho_{a+1} - \frac{ap}{\ell + 1} \le \kappa_l - \frac{l - 1}{\ell + 1} \le \rho_{a+1} + e - \frac{e}{p} - \frac{(a+1)p - 1}{\ell + 1}.$$

Thus, we deduce the result since $\frac{1}{\ell+1} < \frac{e}{p}$. $\square$

The key properties of the $\mathsf{x}_{\boldsymbol{\rho}}$-coordinate function are given by Lemma 3.25.

Lemma 3.25. *Let $\gamma = (r, c, l), \gamma' = (r', c', l') \in \mathscr{N}_n^\ell$. Let $a, a' \in \{0, \dots, d-1\}$ such that $l - ap, l' - a'p \in \{1, \dots, p\}$.*

(a) *If $\gamma \ne \gamma'$ then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) \notin \{\mathsf{x}_{\boldsymbol{\rho}}(\gamma'), \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \pm 1\}$.*

(b) *If $a > a'$ then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma') + 1$.*

(c) *If $a = a'$ and $c - r > c' - r'$ then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$.*

Proof. First consider part (a). Write $l = ap + b + 1$, $l' = a'p + b' + 1$, where $0 \le b, b' < p$. Suppose that $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) = \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \pm 1$. Then we have

$$\frac{1}{e}(\rho_{a+1} - \rho_{a'+1}) + (c - r - c' + r') + (b - b')\frac{1}{p} - \frac{l - l'}{(\ell + 1)e} - \frac{r + c - r' - c'}{N} = \pm 1.$$

Applying Definition 3.6, (3.19) and (3.21), we can deduce from the above equality that $a = a'$ and $c - r = c' - r' \pm 1$. Since $|\frac{r+c-r'-c'}{N}| < \frac{1}{ep'(\ell+1)}$, it follows that $b = b'$ and hence $c + r = c' + r'$, which is impossible because $c - r = c' - r' \pm 1$. This proves that $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) \ne \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \pm 1$. In the case where ± 1 is replaced by 0, then a similar argument shows that $a = a'$, $b = b'$, $c - r = c' - r'$ and $c + r = c' + r'$. Thus $c = c'$ and $r = r'$, which contradicts the fact that $\gamma \ne \gamma'$. This proves that $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) \ne \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$. This completes the proof of (a).

For part (b), as above write $l' = a'p + b' + 1$, where $0 \le b' < p$. Recall that $\rho_{a+1} - \rho_{a'+1} \ge (2n+3)e$ by Definition 3.6 since $a > a'$. Therefore,

$$\begin{aligned}
&\mathsf{x}_{\boldsymbol{\rho}}(\gamma) - \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \\
&\quad = (c - r - c' + r') + \tfrac{1}{e}\big(\rho_{a+1} - \rho_{a'+1}\big) + \tfrac{(b-b')e'}{ep'} - \tfrac{l-l'}{e(\ell+1)} + \tfrac{r'+c'-r-c}{N} \\
&\quad \ge -2n + (2n+3) + \tfrac{1-p}{e}\tfrac{e'}{p'} - \tfrac{\ell-1}{e(\ell+1)} - \tfrac{2n}{N}, && \text{since } |b - b'| < p \\
&\quad \ge 3 + \tfrac{e'}{ep'} - \tfrac{pe'}{ep'} - \tfrac{\ell}{e(\ell+1)}, && \text{since } N \ge 2ne(\ell+1) \\
&\quad > 2 - \tfrac{1}{p} - \tfrac{1}{e}, && \text{since } \tfrac{e}{e'} = m = \tfrac{p}{p'} \text{ and } \ell = pd.
\end{aligned}$$

Hence, $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma') + 1$ since $p \ge 2$ and $e \ge 2$.

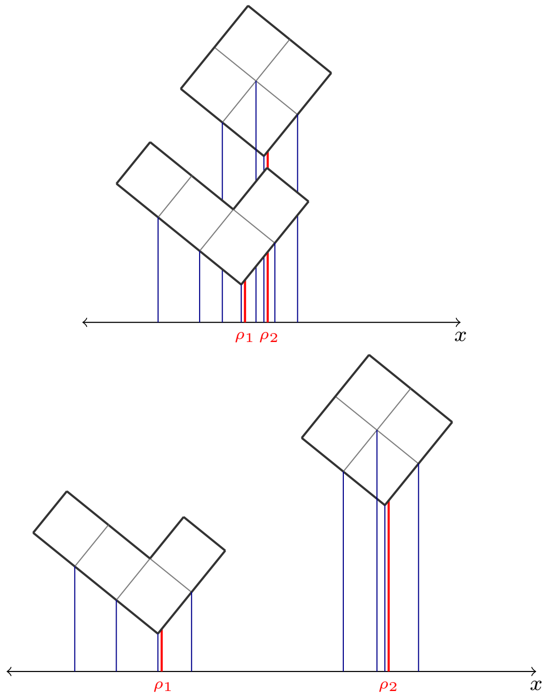
Finally, part (c) is immediate from the definition of $\mathsf{x}_{\boldsymbol{\rho}}$ because $0 \le \kappa_l - \rho_{a+1} < p\frac{e'}{p'} = me' = e$. $\square$

In particular, Lemma 3.25(a) shows that $\mathsf{x}_{\boldsymbol{\rho}}$ defines a total order on the set of nodes.

Let $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$ be an ℓ-partition. Abusing notation slightly, the **loading** of $\boldsymbol{\lambda}$ is the set

$$\mathsf{x}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}) = \{\mathsf{x}_{\boldsymbol{\rho}}(r, c, l) \mid (r, c, l) \in \boldsymbol{\lambda}\}. \tag{3.26}$$

Example 3.27. Let $\boldsymbol{\lambda} = (2,1,1|2^2)$ and suppose that $p = 1$ so that $\boldsymbol{\kappa} = \boldsymbol{\rho}$. The following diagram shows the loadings $\mathsf{x}_{\boldsymbol{\rho}}(\boldsymbol{\lambda})$ for two different choices of $\boldsymbol{\rho}$.



In both diagrams, the line from a node (r,c,l) to the x-axis gives the loading $\mathsf{x}_{\boldsymbol{\rho}}(r,c,l)$. The different components of $\boldsymbol{\lambda}$ are drawn with different heights to make it easier to distinguish between them. The next section explains the significance of this diagram and the red strings.

Extending this notation slightly, define a **generalised partition** to be a finite subset $\widehat{\boldsymbol{\nu}} \subseteq \mathbb{R}\times\mathbb{R}\times\{1,\dots,\ell\}$ such that $\mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}})$ has the same cardinality as $\widehat{\boldsymbol{\nu}}$ and for any $1 \le l \le \ell$,

(3.28)
$$\mathsf{x}_{\boldsymbol{\rho}}(0,0,l) \notin \mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}), \quad \mathsf{x}_{\boldsymbol{\rho}}(0,0,l) - 1 \notin \mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}) \quad \text{and} \quad x+1 \notin \mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}) \text{ for all } x \in \mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}).$$

By Lemma 3.25, if $\boldsymbol{\lambda} \in \mathscr{P}^\ell_n$ then $\boldsymbol{\lambda}$ is a generalised partition. Conversely, if $\widehat{\boldsymbol{\nu}} \notin \mathscr{P}^\ell_n$ is a generalised partition then $\widehat{\boldsymbol{\nu}}$ need not satisfy the conclusions of Lemma 3.25. When we consider generalised partitions below we will only be interested in the set $\mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}})$.

In order to define a partial order on $\mathscr{P}^\ell_n$, define the **residue** of $(r,c,l) \in \mathscr{N}^\ell_n$ to be

$$\operatorname{res}(r,c,l) := q^{\kappa_l + c - r} \in \mathcal{I}. \tag{3.29}$$

Recall that $\varepsilon = q^{e'/p'}$, so $\operatorname{res}(r,c,l) = \varepsilon^b q^{\rho_{a+1}+c-r}$, where $l = ap+b+1$ for $0 \le a < d$ and $0 \le b < p$.

If $\boldsymbol{\lambda} \subseteq \mathscr{N}^\ell_n$ write $\boldsymbol{\lambda} = \{\gamma^{\boldsymbol{\lambda}}_1, \dots, \gamma^{\boldsymbol{\lambda}}_n\}$ so that the nodes $\gamma^{\boldsymbol{\lambda}}_1, \dots, \gamma^{\boldsymbol{\lambda}}_n$ are sorted by decreasing loading function, that is, $\mathsf{x}_{\boldsymbol{\rho}}(\gamma^{\boldsymbol{\lambda}}_n) < \cdots < \mathsf{x}_{\boldsymbol{\rho}}(\gamma^{\boldsymbol{\lambda}}_1)$. The **residue**

sequence of $\boldsymbol{\lambda}$ is

$$\operatorname{res}(\boldsymbol{\lambda}) = \big(\operatorname{res}(\gamma_1^{\boldsymbol{\lambda}}), \operatorname{res}(\gamma_2^{\boldsymbol{\lambda}}), \dots, \operatorname{res}(\gamma_n^{\boldsymbol{\lambda}})\big) \in \mathcal{I}^n. \tag{3.30}$$

If $\alpha \in \mathscr{C}_n^\ell$ is an $\mathcal{I}$-composition of n set

$$\mathscr{P}_\alpha^\ell = \{\boldsymbol{\lambda} \in \mathscr{P}_n^\ell \mid \operatorname{res}(\boldsymbol{\lambda}) \in \mathcal{I}^\alpha\}.$$

We have the decomposition $\mathscr{P}_n^\ell = \bigsqcup_\alpha \mathscr{P}_\alpha^\ell$ (disjoint union).

Definition 3.31 plays a key role in this paper. In particular, it defines the partial order that appears in our main result, which gives a skew cellular basis for $\mathscr{H}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$.

Definition 3.31 (Webster's ordering [3, Definition 1.3, Proposition 1.4])**.** Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_n^\ell$ be two ℓ-partitions. Then $\boldsymbol{\lambda} \trianglerighteq_{\boldsymbol{\rho}} \boldsymbol{\mu}$ if there exists a bijection $\theta : \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}$ such that

$$\operatorname{res}(\theta(\gamma)) = \operatorname{res}(\gamma) \qquad \text{and} \qquad \mathsf{x}_{\boldsymbol{\rho}}(\theta(\gamma)) \le \mathsf{x}_{\boldsymbol{\rho}}(\gamma), \qquad\qquad \text{for all } \gamma \in \boldsymbol{\lambda}.$$

If $\boldsymbol{\lambda} \trianglerighteq_{\boldsymbol{\rho}} \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ write $\boldsymbol{\lambda} \triangleright_{\boldsymbol{\rho}} \boldsymbol{\mu}$.

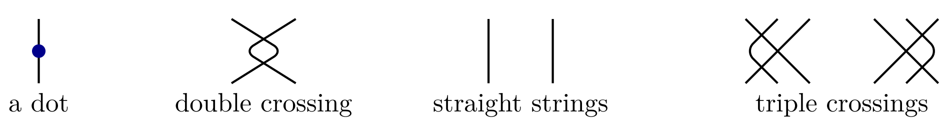
Example 3.32. By Definition 3.6, $0 = \rho_1 < \dots < \rho_d$. Therefore, $\boldsymbol{\lambda} \trianglerighteq_{\boldsymbol{\rho}} (1^n|0|\dots|0)$ whenever $\operatorname{res}(\boldsymbol{\lambda}) = \operatorname{res}(1^n|0|\dots|0)$ and $(0|\dots|0|n) \trianglerighteq_{\boldsymbol{\rho}} \boldsymbol{\mu}$ whenever $\operatorname{res}(\boldsymbol{\mu}) = \operatorname{res}(0|\dots|0|n)$.

3.4. **Diagrammatic Cherednik algebras.** Webster realises the quiver Hecke algebras of type $G(\ell, 1, n)$ as idempotent subalgebras of his diagrammatic Cherednik algebras [32]. Following Bowman [3], we now recall these results, extending them to the slightly more general quiver Γ as we go. We start by defining Webster diagrams.

A **string** in $\mathbb{R}^2$ is a diffeomorphism of the form $[0,1] \longrightarrow \mathbb{R}^2; t \mapsto (\mathsf{s}(t), t)$. By definition, a string is a smooth curve in $\mathbb{R}^2$ with no loops. We sometimes identify a string with the corresponding map $t \mapsto \mathsf{s}(t)$. We regard a string as a directed path from bottom ($t = 0$) to top ($t = 1$).

Every string that we consider will be labelled by a **residue** $i \in \mathcal{I}$. An i**-string** is a string of residue i.

A **crossing** of two strings is a point where they intersect. A **dot** on a string is a distinguished point in the image of the string that is not on any crossing or on the start or end points of the string. We will frequently refer to the following configuration of strings when they occur in sufficiently small **local** neighbourhoods of diagrams:



Pulling **apart** the strings in a double crossing gives straight strings whereas pulling the string **through** the crossing in one of the triple crossings gives the other triple crossing. We apply this terminology below to red, solid and ghost strings, which we now define.

Recall that Definition 3.16 fixes the multicharge $\boldsymbol{\kappa} \in \mathbb{Q}^\ell$.

Definition 3.33 (Webster [32, Definition 4.1], Bowman [3, Definition 4.1])**.** Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \subseteq \mathscr{N}_n^\ell$. A **Webster diagram** with multicharge $\boldsymbol{\kappa}$ and type $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and top residue sequence $\mathbf{i} \in \mathcal{I}^n$ consists of the following:

(a) **Red** strings $\mathsf{r}_1, \dots, \mathsf{r}_\ell$ such that r_l has residue q^{κ_l} and $\mathsf{r}_l(t) = \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l)$, for $t \in [0, 1]$.
(b) **Solid** strings $\mathsf{s}_1, \dots, \mathsf{s}_n$, ordered so that $\mathsf{s}_1(1) > \mathsf{s}_2(1) > \dots > \mathsf{s}_n(1)$, such that

$$\mathsf{x}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}) = \{\mathsf{s}_k(1) \mid 1 \le k \le n\} \quad \text{and} \quad \mathsf{x}_{\boldsymbol{\rho}}(\boldsymbol{\mu}) = \{\mathsf{s}_k(0) \mid 1 \le k \le n\}$$

and s_k is an i_k-string, for $1 \le k \le n$.
(c) Each solid i-string has a **ghost** i-string that is obtained by translating the corresponding solid string one unit to the right.

The solid strings in a Webster diagram are decorated with finitely many **dots** on the solid strings, with each dot having a **ghost dot** one unit to the right on the corresponding ghost string. Exactly two strings in a Webster diagram intersect at each crossing and no (red, solid or ghost) string can be tangential to any other string.

Given a Webster diagram D, set $\operatorname{top}(D) = (\boldsymbol{\lambda}, \mathbf{i})$ and $\operatorname{bot}(D) = (\boldsymbol{\mu}, \mathbf{j})$, where $\mathbf{j} = (j_1, \dots, j_n) \in \mathcal{I}^n$ is the residue sequence of the solid strings when read in order from right to left along the bottom of D. Then $\mathbf{i}$ and $\mathbf{j}$ are the **top residue sequence** and **bottom residue sequence** of D, respectively. If $\mathbf{i} = \mathbf{j}$ then $\operatorname{res}(D) = \mathbf{i} = \mathbf{j}$ is the **residue sequence** of D.

To help distinguish between the different types of strings in Webster diagram we draw red strings as thick red strings and ghost strings as dashed gray strings.

Remark 3.34. Ghost dots do not appear in Webster's paper [32] but can be found in Bowman's [3, Remark 4.8]. Including the ghost dots does not change the algebras up to isomorphism and makes the relations easier to write because they are more symmetrical with respect to the dots and ghost dots.

Two Webster diagrams of type $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are equivalent if they have the same residues, same number of dots on each string, when ordered from right to left at the top of the diagram, and they differ by an isotopy, which is a continuous deformation in which all of the intermediate diagrams are Webster diagrams. In particular, the red strings are fixed by isotopy.

Let $\mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the set of (isotopy classes) of Webster diagrams of type $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. If α is an $\mathcal{I}$-composition then set

$$\mathscr{W}_{\boldsymbol{\rho}}(\alpha) = \bigcup_{\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\ell_\alpha} \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu}). \quad \text{and} \quad \mathscr{W}_{\boldsymbol{\rho}}(n) = \bigcup_{\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\ell_n} \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu}). \tag{3.35}$$

Of course, $\mathscr{W}_{\boldsymbol{\rho}}(n) = \bigcup_{\alpha \in \mathscr{C}^\ell_n} \mathscr{W}_{\boldsymbol{\rho}}(\alpha) = \bigcup_{\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^\ell_n} \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Let $D, E \in \mathscr{W}_{\boldsymbol{\rho}}(n)$ be Webster diagrams such that $\operatorname{bot}(D) = \operatorname{top}(E)$. Define $D \circ E$ to be the Webster diagram obtained by identifying the southern points of D with the northern points of E and then rescaling.

There is a distinguished Webster diagram $\mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\lambda})$, for each ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}^\ell_n$ and each residue sequence $\mathbf{i} \in \mathcal{I}^n$, in which all of the strings are vertical. By definition, $\mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}} \circ \mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}} = \mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}}$. If a Webster diagram D of type $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ has top residue sequence $\mathbf{i}$ and bottom residue sequence $\mathbf{i}'$ then $D = \mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}} D \mathbf{1}^{\mathbf{i}'}_{\boldsymbol{\mu}}$.

Example 3.36. Let $\ell = 4$, $e = 3$ and $p = 2$ so that $e' = 3$, $p' = 2$ and let $\rho_2 = 9$ so that $\boldsymbol{\kappa} = (0, 1.5, 9, 10.5)$. Let $\boldsymbol{\lambda} = (2^2|1|0|2)$ and fix $\mathbf{i} \in \mathcal{I}^n$. Then $\mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}}$ is the Webster diagram

$\mathbf{1}^{\mathbf{i}}_{\boldsymbol{\lambda}} \quad =$ 

where the solid strings have residues $i_1, \dots, i_7$ when read from right to left. The $\mathsf{x}_{\boldsymbol{\rho}}$–coordinates of the solid strings are given by the loadings $\mathsf{x}_{\boldsymbol{\rho}}(\gamma)$, for $\gamma \in \boldsymbol{\lambda}$. By (3.22), the $\mathsf{x}_{\boldsymbol{\rho}}$–coordinate of the lth red string is $\frac{1}{e}\big(\kappa_l - \frac{l-1}{\ell+1}\big)$, for $1 \le l \le \ell = 4$. In particular, the leftmost red string has x-coordinate $\mathsf{x}_{\boldsymbol{\rho}}(0,0,1) = \frac{\kappa_1}{3} = 0$.

Lemma 3.37. *Let $\widehat{\boldsymbol{\nu}}$ be a generalised partition and $\mathbf{i} \in \mathcal{I}^n$. Let D be a Webster diagram of type $(\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}})$ that does not contain any crossings. Then D is isotopic to $\mathbf{1}^{\mathbf{i}}_{\widehat{\boldsymbol{\nu}}}$, where $\mathbf{i}$ is the (top) residue sequence of D.*

Proof. In order to construct an isotopy from D to $\mathbf{1}^{\mathbf{i}}_{\widehat{\boldsymbol{\nu}}}$, write $\mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}) = \{x_1 < \cdots < x_n\}$ and let $\mathsf{s}_1, \dots, \mathsf{s}_n$ be the solid strings in D. For $u \in [0,1]$, let $D^{(u)}$ be the Webster diagram of type $(\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}})$ and residue $\mathbf{i}$, which has solid strings $\mathsf{s}^{(u)}_1, \dots, \mathsf{s}^{(u)}_n$ given by

$$\mathsf{s}^{(u)}_k : [0,1] \longrightarrow \mathbb{R} \times [0,1]; t \mapsto \big((1-u)\mathsf{s}_k(t) + u x_k, t\big), \qquad \text{for } 1 \le k \le n.$$

By construction, $D = D^{(0)}$ and $\mathbf{1}^{\mathbf{i}}_{\widehat{\boldsymbol{\nu}}} = D^{(1)}$, so to complete the proof it suffices to prove that the strings in $D^{(u)}$ never intersect, for $u \in [0,1]$. By assumption, the solid strings in D do not intersect, so $\mathsf{s}_k(t) < \mathsf{s}_l(t)$ for $1 \le k < l \le \ell$. Therefore, if $u \in [0,1]$ then $\mathsf{s}^{(u)}_k(t) < \mathsf{s}^{(u)}_l(t)$ for $1 \le k < l \le \ell$, so the solid strings in $D^{(u)}$ do not intersect. Essentially the same argument shows that there are no intersections between any of the solid, ghost and red strings in $D^{(u)}$, completing the proof. $\square$

We can now define Webster's diagrammatic Cherednik algebras.

Definition 3.38 (Webster [32, Definition 4.2], Bowman [3, Definition 4.5])**.** The **diagrammatic Cherednik algebra** $\mathbb{A}^{\boldsymbol{\rho}}_n$ is the R-algebra

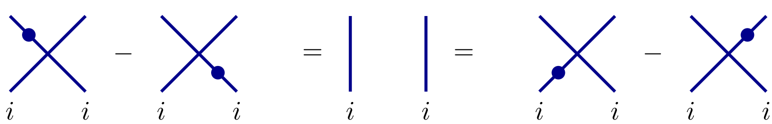
$$\mathbb{A}^{\boldsymbol{\rho}}_n = \bigoplus_{\alpha \in \mathscr{C}^\ell_n} \mathbb{A}^{\boldsymbol{\rho}}_\alpha,$$

where, for each $\mathcal{I}$-composition α, the R-algebra $\mathbb{A}^{\boldsymbol{\rho}}_\alpha$ is the unital associative algebra generated by the Webster diagrams in $\mathscr{W}_{\boldsymbol{\rho}}(\alpha)$ such that

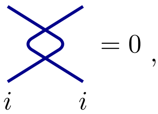
$$DE = \begin{cases} D \circ E, & \text{if } \mathrm{bot}(D) = \mathrm{top}(E), \\ 0, & \text{otherwise,} \end{cases}$$

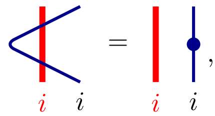
and the following bilocal relations hold:

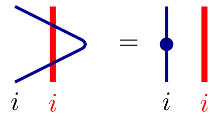
(A) **(Dots and crossings)** Solid and ghost dots can pass through any crossing except:

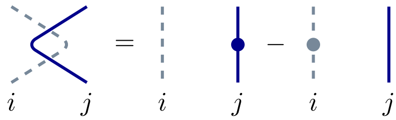
(W$_1$) 

(B) **(Double crossings)** A double crossing between any two strings can be pulled apart except in the following cases:

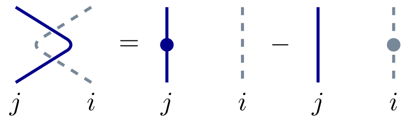
(W$_2$) 

(W$_3$) 

(W$_4$) 

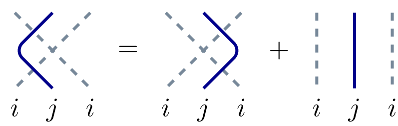
(W$_5$) 

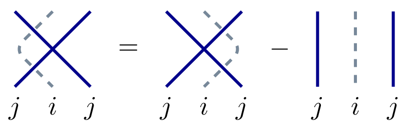
and

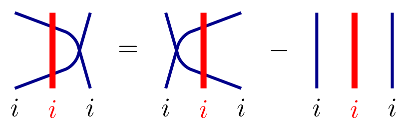
(W$_6$) 

where $j = iq \in \mathcal{I}$

(C) **(Triple crossings)** A string can be pulled through a crossing except in the cases:

(W$_7$) 

(W$_8$) 

(W$_9$) 

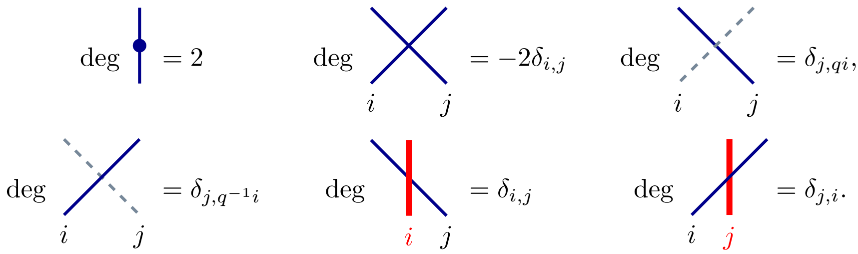
where $j = iq \in \mathcal{I}$

(D) **(Unsteady diagrams)** A Webster diagram is **unsteady** if it contains a solid string that at any point is n units or more to the right of the rightmost red string. Any unsteady diagram is zero.

Solid and ghost strings always occur in pairs, so any solid or ghost strings that are not drawn in the relations above are still part of the relations even though they do not appear. All of the relations in Definition 3.38 are *bilocal* in the sense that the relations need to be applied locally in the regions around the solid strings and their ghost strings. In particular, strings may appear between the solid strings and their ghosts in the double and triple crossing relations.

The relations drawn in (A), (B) and (C) of Definition 3.38 are the **exceptional relations**. The remaining relations are the **non-exceptional** relations of $\mathbb{A}_n^{\boldsymbol{\rho}}$. When they are applied, none of the non-exceptional relations introduce additional diagrams. Explicitly, the non-exceptional relations in (A) allow a dot to be pulled through a crossing, those in (B) allow a double crossing to be pulled apart, and those in (C) allow a string to be pulled through a triple crossing.

As with $\mathscr{R}_n^{\boldsymbol{\Lambda}}$, the algebra $\mathbb{A}_n^{\boldsymbol{\rho}}$ is $\mathbb{Z}$-graded with the grading defined on the Webster diagrams by summing over the contributions from each dot and crossing in the diagram according to the following rules:



All other crossings, and the ghost dots, have degree 0. The algebra $\mathbb{A}_\alpha^{\boldsymbol{\rho}}$ is $\mathbb{Z}$-graded because all of the relations in Definition 3.38 are homogeneous with respect to this degree function.

Remark 3.39. To make some proofs easier to read, we sometime require the diagrams to have their solid strings starting or ending in a set $\mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}})$, where $\widehat{\boldsymbol{\nu}}$ is a generalised partition (cf. (3.28)). In particular, if $D \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a Webster diagram with no dots or intersecting strings on the line $y = H$, where $H \in (0,1)$, then we can factor D as $D = D^+ \circ D^-$ where the diagrams D^+ and D^- are the restrictions of D to $\mathbb{R} \times [H, 1]$ and $\mathbb{R} \times [0, H]$, respectively. We then have $D^+ \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \widehat{\boldsymbol{\nu}})$ and $D^- \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}, \boldsymbol{\mu})$ where $\widehat{\boldsymbol{\nu}}$ is a generalised partition.

Following [3] we now describe a basis of $\mathbb{A}_n^{\boldsymbol{\rho}}$. Recall that we identify an ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$ with its diagram. Let $\boldsymbol{\omega} = (1^n|0|\dots|0) \in \mathscr{P}_n^\ell$; compare with Example 3.32. The ℓ-partition $\boldsymbol{\omega}$ is the unique ℓ-partition such that $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) < 0$, for all $\gamma \in \boldsymbol{\omega}$.

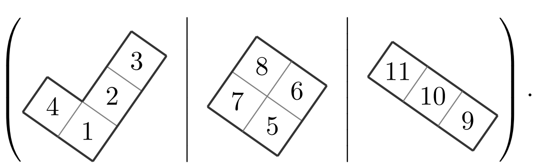
Definition 3.40. Let $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$. A **$\boldsymbol{\lambda}$-tableau** is a bijection $\mathfrak{t} : \boldsymbol{\lambda} \longrightarrow \{1, \dots, n\}$. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau then $\mathfrak{t}$ has **shape $\boldsymbol{\lambda}$** and we write $\operatorname{Shape}(\mathfrak{t}) = \boldsymbol{\lambda}$. A tableau $\mathfrak{t}$ is **standard** if its entries increase along the rows and columns. In other words,

(a) If $(r, c, l), (r-1, c, l) \in \boldsymbol{\lambda}$ then $\mathfrak{t}(r, c, l) \geq \mathfrak{t}(r-1, c, l) + 1$;
(b) If $(r, c, l), (r, c-1, l) \in \boldsymbol{\lambda}$ then $\mathfrak{t}(r, c, l) \geq \mathfrak{t}(r, c-1, l) + 1$.

Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux.

As in Example 3.20, think of standard tableaux as labelled Russian diagrams.

Example 3.41. Let $\boldsymbol{\mu} = (3, 1|2^2|1^3)$. Then one tableau in $\operatorname{Std}(\boldsymbol{\mu})$ is



Let $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $1 \leq k \leq n$. The **residue** of k in $\mathfrak{t}$ is $\operatorname{res}_k(\mathfrak{t}) = \operatorname{res}(\mathfrak{t}^{-1}(k))$ and

$$\operatorname{res}(\mathfrak{t}) = \big(\operatorname{res}_1(\mathfrak{t}), \operatorname{res}_2(\mathfrak{t}), \dots, \operatorname{res}_n(\mathfrak{t})\big) \in \mathcal{I}^n$$

is the **residue sequence** of $\mathfrak{t}$.

Let $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$. A node $\alpha \notin \boldsymbol{\lambda}$ is an **addable** node of $\boldsymbol{\lambda}$ if $\boldsymbol{\lambda} \cup \{\alpha\}$ is (the diagram of) an ℓ-partition. Similarly, $\alpha \in \boldsymbol{\lambda}$ is a **removable** node of $\boldsymbol{\lambda}$ if $\boldsymbol{\lambda} \setminus \{\alpha\}$ is an ℓ-partition. Let $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{R}(\boldsymbol{\lambda})$ be the sets of addable and removable nodes of $\boldsymbol{\lambda}$.

Let $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. If $1 \le l \le n$ let $\mathfrak{t}_{\downarrow k}$ be the restriction of $\mathfrak{t}$ to $\{1, \dots, k\}$ and let $\boldsymbol{\lambda}_k = \operatorname{Shape}(\mathfrak{t}_{\downarrow k})$. Since $\mathfrak{t}$ is standard, $\mathfrak{t}_{\downarrow k}$ is a standard $\boldsymbol{\lambda}_k$-tableau. Define

$$\mathcal{A}_k(\mathfrak{t}) = \{\gamma \in \mathcal{A}(\boldsymbol{\lambda}_k) \mid \operatorname{res}(\gamma) = \operatorname{res}_{\mathfrak{t}}(k) \text{ and } \mathsf{x}_{\boldsymbol{\rho}}(\mathfrak{t}^{-1}(k)) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma)\},$$
$$\mathcal{R}_k(\mathfrak{t}) = \{\gamma \in \mathcal{R}(\boldsymbol{\lambda}_k) \mid \operatorname{res}(\gamma) = \operatorname{res}_{\mathfrak{t}}(k) \text{ and } \mathsf{x}_{\boldsymbol{\rho}}(\mathfrak{t}^{-1}(k)) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma)\}.$$

Following [3, Definition 1.11], and [9, (3.5)], the **degree** of $\mathfrak{t}$ is the integer

$$\deg \mathfrak{t} = \sum_{k=1}^{n} \big(\#\mathcal{A}_k(\mathfrak{t}) - \#\mathcal{R}_k(\mathfrak{t})\big). \tag{3.42}$$

In order to attach a Webster diagram to a standard tableau $\mathfrak{s}$ let $\operatorname{cross}(D)$ be the number of crossings in any diagram $D \in \mathscr{W}_{\boldsymbol{\rho}}(n)$. The number $\operatorname{cross}(D)$ is preserved by isotopy but when we apply the relations in $\mathbb{A}_n^{\boldsymbol{\rho}}$ diagrams with a different number of crossings can appear.

Definition 3.43 ([32, §4.3], [3, Definition 6.1])**.** Let $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$. Let $C_\mathfrak{t} \in \mathbb{A}_n^{\boldsymbol{\rho}}$ be any Webster diagram in $\mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\omega})$ such that:

(a) For each $\gamma \in \boldsymbol{\lambda}$, there is a solid string of residue $\operatorname{res}(\gamma)$ from $(\mathsf{x}_\mathfrak{t}(\gamma), 0)$ to $(\mathsf{x}_{\boldsymbol{\rho}}(\gamma), 1)$, where $\mathsf{x}_\mathfrak{t}(\gamma) = \mathsf{x}_{\boldsymbol{\rho}}(\mathfrak{t}(\gamma), 1, 1)$ and $(\mathfrak{t}(\gamma), 1, 1) \in \boldsymbol{\omega}$.
(b) The diagram $C_\mathfrak{t}$ has no dots on any strings.
(c) If $C'_\mathfrak{t}$ is another diagram satisfying (a) and then $\operatorname{cross}(C_\mathfrak{t}) \le \operatorname{cross}(C'_\mathfrak{t})$.

A **generalised double crossing** in a Webster diagram D is a pair of strings in D that cross twice. In particular, if a diagram $C_\mathfrak{t}$ satisfies part (a) of Definition 3.43 and has no generalised double crossing then $C_\mathfrak{t}$ satisfies (c).

In general, the diagram $C_\mathfrak{t}$ is not uniquely determined by Definition 3.43. In Section 4.5 we give an explicit construction of such diagrams but, for now, we let $C_\mathfrak{t}$ be any Webster diagram satisfying Definition 3.43. Unless stated otherwise, the results that follow do not depend on the choice of diagram for $C_\mathfrak{t}$.

By construction the bottom residue sequence of $C_\mathfrak{t}$ is $\operatorname{res}(\mathfrak{t})$. Moreover, by [3, Theorem 7.1], $\deg C_\mathfrak{t} = \deg \mathfrak{t}$.

Let $* : \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu}) \longrightarrow \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\mu}, \boldsymbol{\lambda})$ be the map that reflects a Webster diagram in the line $y = \frac{1}{2}$. Using Definition 3.38 it is easy to see that $*$ extends to an involution $* : \mathbb{A}_\alpha^{\boldsymbol{\rho}} \longrightarrow \mathbb{A}_\alpha^{\boldsymbol{\rho}}$, for any $\mathcal{I}$-composition α. Hence, we can consider $*$ as a homogeneous automorphism of $\mathbb{A}_n^{\boldsymbol{\rho}}$ of order 2.

Definition 3.44. Let $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Define $C_{\mathfrak{s}\mathfrak{t}} = C_\mathfrak{s}^* C_\mathfrak{t} \in \mathbb{A}_n^{\boldsymbol{\rho}}$.

By the remarks above, $C_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$. The next result shows that a certain idempotent truncation of $\mathbb{A}_n^{\boldsymbol{\rho}}$, which will turn out to be isomorphic to $\mathscr{R}_\alpha^{\boldsymbol{\Lambda}}$, is a graded cellular algebra in the sense of Graham and Lehrer [15,18]. Recall the idempotents $\mathbf{1}_{\boldsymbol{\lambda}}^{\mathbf{i}}$ from before Example 3.36, where $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$ and $\mathbf{i} \in \mathcal{I}^n$. Let α be an $\mathcal{I}$-composition of n. Define the idempotents

$$\mathsf{E}_{\boldsymbol{\omega},\alpha} = \sum_{\mathbf{i} \in \mathcal{I}^\alpha} \mathbf{1}_{\boldsymbol{\omega}}^{\mathbf{i}}, \qquad \mathsf{E}_{\boldsymbol{\omega}} = \sum_{\alpha \in \mathscr{C}_n^\ell} \mathsf{E}_{\boldsymbol{\omega},\alpha}, \tag{3.45}$$

and define the algebras

$$\mathbb{A}_\alpha^{\boldsymbol{\rho}}(\boldsymbol{\omega}) = \mathsf{E}_{\boldsymbol{\omega},\alpha} \mathbb{A}_\alpha^{\boldsymbol{\rho}} \mathsf{E}_{\boldsymbol{\omega},\alpha}, \qquad \mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega}) = \mathsf{E}_{\boldsymbol{\omega}} \mathbb{A}_n^{\boldsymbol{\rho}} \mathsf{E}_{\boldsymbol{\omega}} = \bigoplus_{\alpha \in \mathscr{C}_n^\ell} \mathbb{A}_\alpha^{\boldsymbol{\rho}}(\boldsymbol{\omega}). \tag{3.46}$$

We can now state one of the main results of [3, 32].

Theorem 3.47 ([32, Theorem 4.11], [3, Theorem 7.1])**.** *Let α be an $\mathcal{I}$-composition of n. Then the algebra $\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}(\boldsymbol{\omega})$ is a graded cellular algebra with graded cellular basis*

$$\{C_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathscr{P}^{\ell}_{\alpha}\},$$

with respect to the poset $(\mathscr{P}^{\ell}_{\alpha}, \trianglerighteq_{\boldsymbol{\rho}})$ and homogeneous cellular algebra anti-isomorphism $*$.

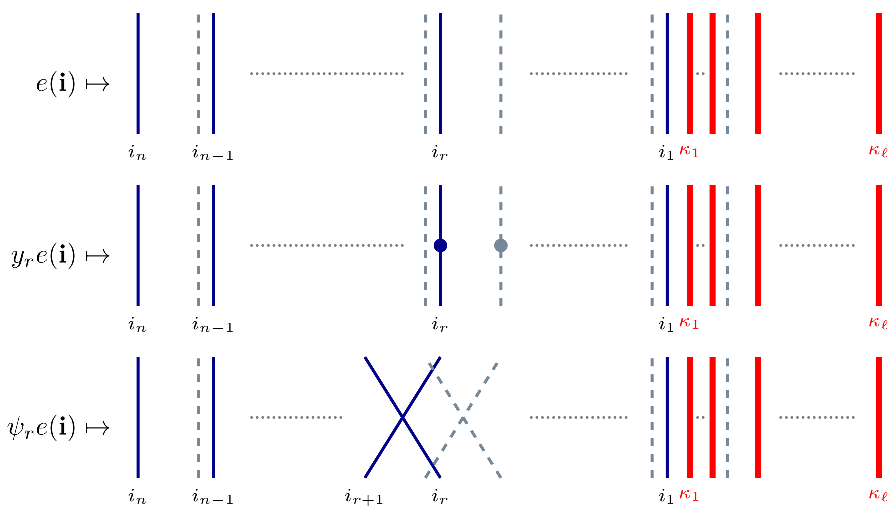
In fact, Bowman and Webster give a cellular basis for the algebra $\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}$. We state only this special case of their result because this is all that we need and because it saves us from having to introduce additional notation.

Remark 3.48. In type A, Bowman [3] and Webster [32, 33] only consider algebras that are attached to the cyclic quiver Γ_e whereas we are considering the more general quiver Γ from Definition 3.8. As the relations in $\mathbb{A}^{\boldsymbol{\rho}}_n$ are local and depend only on the quiver and the choice of residues, it is easy to see that the arguments of these papers apply without change for the quiver Γ. The Webster diagrams, and hence the algebras $\mathbb{A}^{\boldsymbol{\rho}}_n$, also depend on the choice of loading function. In his papers Webster considers arbitrary loadings whereas Bowman fixes a loading that is different from ours; see Remark 3.23. If x and y are two loading functions then it is easy to see that the corresponding Webster algebras $\mathbb{A}^{\mathbf{x}}_n$ and $\mathbb{A}^{\mathbf{y}}_n$ are isomorphic if

$$\mathsf{x}(\gamma) < \mathsf{x}(\gamma') \quad \text{if and only if} \quad \mathsf{y}(\gamma) < \mathsf{y}(\gamma') \qquad \text{for all } \gamma, \gamma' \in \mathscr{N}^{\ell}_n,$$

where an isomorphism is given by conjugating by the "straight line" diagrams that have strings connecting $\mathsf{x}(\gamma)$ to $\mathsf{y}(\gamma)$, for all $\gamma \in \mathscr{N}^{\ell}_n$. Comparing the definition of our loading function $\mathsf{x}_{\boldsymbol{\rho}}$ from (3.22) with Bowman's (see Remark 3.23), it is straightforward to see that our loading function is equivalent to one of Bowman's in the sense that they give isomorphic Webster algebras.

Theorem 3.49 ([32], [3, Theorem 6.17])**.** *Let α be an $\mathcal{I}$-composition of n. There is a unique isomorphism of $\mathbb{Z}$-graded R-algebras $g\colon \mathscr{R}^{\boldsymbol{\Lambda}}_{\alpha} \xrightarrow{\sim} \mathbb{A}^{\boldsymbol{\rho}}_{\alpha}(\boldsymbol{\omega})$ such that*



for $\mathbf{i} \in \mathcal{I}^{\alpha}$.

The idea behind the proof of Theorem 3.49 is to use the relations to pull all of the solid strings in a diagram $D \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\omega}, \boldsymbol{\omega})$ to the left of all of the red strings and then check that the relations are preserved by the map $g : \mathscr{R}^{\boldsymbol{\Lambda}}_{\alpha} \longrightarrow \mathbb{A}^{\boldsymbol{\rho}}_{\alpha}(\boldsymbol{\omega})$ given in the statement of Theorem 3.49. For us the important point is that instead of working in $\mathscr{R}^{\boldsymbol{\Lambda}}_{\alpha}$ we can apply the isomorphism g and work in $\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}(\boldsymbol{\omega})$. We use similar ideas in Section 5.2 to prove our main results.

4. Regular diagrams and shifted tableaux combinatorics

This chapter is the technical heart of this paper. It prepares all the tools we will need in the next section to prove that the graded Hecke algebras of type $G(\ell, 1, n)$ have a shift automorphism, which will imply that the Hecke algebras of type $G(\ell, p, n)$ are skew cellular by Theorem 2.28. All of the calculations take place inside the diagrammatic Cherednik algebra $\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}$. The key point is that the special choice of loading made in Definition 3.16 ensures that the dominance order $\trianglerighteq_{\boldsymbol{\rho}}$ for the cellular basis of $\mathscr{R}^{\boldsymbol{\Lambda}}_{\alpha} \cong \mathsf{E}_{\boldsymbol{\omega},\alpha} \mathbb{A}^{\boldsymbol{\rho}}_{\alpha} \mathsf{E}_{\boldsymbol{\omega},\alpha}$ in Theorem 3.49 is compatible with the shifted tableaux combinatorics that we introduce later in this chapter.

4.1. Regular diagrams.

This section defines a class of diagrams that are easy to work with and which play a key role in the arguments of Section 4.5.

Definition 4.1. Let D be a Webster diagram. A **singular** crossing in D is a crossing between a solid i-string and either:

- another solid i-string
- a red i-string, or
- a ghost iq^{-1}-string.

A crossing is **regular** if it is not singular. A diagram D is a **regular diagram** if D has no dots and all crossings in D are regular. A **singular** diagram is any diagram that is not regular.

In particular, any crossing that does not involve a solid string is regular. Note that regular crossings are preserved by the relations in Definition 3.38 and by isotopy. An element of $\mathbb{A}^{\boldsymbol{\rho}}_{n}$ is **regular** if it is the image of a regular Webster diagram. By assumption, regular diagrams do not contain any of the exceptional crossings in Definition 3.38, so the span of the regular diagrams in $\mathbb{A}^{\boldsymbol{\rho}}_{n}$ is a subalgebra of $\mathbb{A}^{\boldsymbol{\rho}}_{n}$.

The next result is the analogue for regular diagrams of the algorithm for reducing words in the symmetric group.

Proposition 4.2. *Let $\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}'$ be two generalised partitions and let $C \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}')$ be a regular diagram. Then C is equal to a regular diagram with no generalised double crossings.*

Proof. Number the strings in C from left to right along the top of the diagram as $s_1, s_2, \dots, s_{2n+\ell}$.

Each string cuts the diagram into two pieces, say $\mathrm{Left}(s_k)$ and $\mathrm{Right}(s_k)$. In $\mathbb{A}^{\boldsymbol{\rho}}_{n}$, we claim that the diagram C is equal to a diagram that does not have any generalised double crossings in $\mathrm{Left}(s_k)$, for $1 \le k \le 2n + \ell$, and if s_k is a not a ghost string then the only crossings of non-ghost strings in $\mathrm{Left}(s_k)$ are between non-ghost strings s_a and s_b with $\min\{a, b\} < k$. Of course, a non-ghost string is either a red string or a solid string.

We prove the claim by arguing by induction on $k = 1, 2, \dots, 2n+\ell$. If $k = 1$ then we can use the non-exceptional triple crossing relation (C) from Definition 3.38 to pull the string s_1 to the left through any crossings in $\mathrm{Left}(s_1)$. By induction we assume that the claim is true for the strings $s_1,\dots,s_{k-1}$. If s_k is a ghost string there is nothing to prove so we may assume that s_k is a non-ghost string. To show that the claim holds for s_k, use the non-exceptional triple crossing relation (C) from Definition 3.38 to pull the string s_k to the left through any crossing in $\mathrm{Left}(s_k)$ that involve two larger strings. Pulling s_k through a crossing does not destroy any generalised double crossings in the diagram. Moreover, for any generalised double crossing D in $\mathrm{Left}(s_k)$ that involve s_k, any string which goes through the region surrounded by D can be moved away from the region by using the non-exceptional triple crossing relation (C) from Definition 3.38 as C is regular. As a result, we can apply the non-exceptional relation (B) from Definition 3.38 to pull apart any generalised double crossings in $\mathrm{Left}(s_k)$ that involve s_k. Note that the two strings do not have the same residue since all crossings are regular. Observe that the crossings involving string s_j for $j<k$ are unchanged in this process. After a finite number of steps we will show that all of the crossings between non-ghost crossings in $\mathrm{Left}(s_k)$ will involve a string s_j, with $j<k$, and s_k does not meet any other string twice. This completes the proof of the inductive step and hence proves the proposition. □

Corollary 4.3. *Let $\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}'$ be two generalised partitions and let $C,D\in\mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$ be two regular diagrams such that the string starting from $\mathsf{x}_{\boldsymbol{\rho}}(\gamma)$ in C and D has the same residue and the same end points, for all $\gamma\in\widehat{\boldsymbol{\nu}}$. Then $C=D$ in $\mathbb{A}_n^{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$.*

Proof. By Proposition 4.2, in $\mathbb{A}_n^{\boldsymbol{\rho}}$ the diagram C^*D is equal to a diagram E that does not contain any generalised double crossings. By assumption, each string in E starts and ends at the same point, so E does not contain any crossings. Hence, E is an idempotent diagram by Lemma 3.37. By the same argument, CC^* is also equal to an idempotent diagram. Thus, in $\mathbb{A}_n^{\boldsymbol{\rho}}$,

$$D=(CC^*)D=C(C^*D)=C,$$

which completes the proof. □

Hence, once we fix the start and end positions of the n solid strings, together with their residues, then the set of regular diagrams can be identified with a subgroup of $\mathfrak{S}_n$.

4.2. **Shifted tableaux combinatorics.** Recall from Definition 3.15 that $\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}$ is defined as an algebra of σ-fixed points:

$$\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}=\left(\mathscr{R}^{\boldsymbol{\Lambda}}_{n}\right)^{\sigma}.$$

Motivated by Theorem 3.49, we want to consider the σ-fixed point subalgebra of the diagrammatic Cherednik algebra but it is not clear how to extend σ to an automorphism of $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$. This section introduces a combinatorial *shift operator* on the set of nodes that will allow us to extend σ to an automorphism of $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$.

Define a shift operation $\sigma_{\mathscr{N}}$ on the set of nodes $\mathscr{N}_n^{\ell}$ from (3.19) by

$$\sigma_{\mathscr{N}}(r,c,l):=\begin{cases}(r,c,l+1), & \text{if } l\not\equiv 0 \pmod p,\\ (r,c,l+1-p), & \text{if } l\equiv 0 \pmod p.\end{cases}\tag{4.4}$$

We usually abuse notation and write $\sigma=\sigma_{\mathscr{N}}$, as the meaning will be clear from context. Equivalently, if we write $l=ap+b+1$, where $0\le a<d$ and $0\le b<p$,

then

$$\sigma(r,c,ap+b+1) = \begin{cases} (r,c,ap+b+2), & \text{if } b \neq p-1, \\ (r,c,ap+1), & \text{if } b = p-1. \end{cases}$$

Let $\operatorname{Std}(\mathscr{P}_n^\ell) = \bigcup_{\boldsymbol\lambda\in\mathscr{P}_n^\ell} \operatorname{Std}(\boldsymbol\lambda)$.

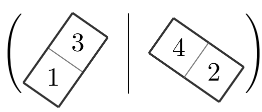
Definition 4.5. Define a map $\sigma_{\mathscr{P}} : \mathscr{P}_n^\ell \longrightarrow \mathscr{P}_n^\ell$ by $\sigma_{\mathscr{P}}(\boldsymbol\lambda) = \{\sigma_{\mathscr{N}}(\gamma) \mid \gamma \in \boldsymbol\lambda\}$, for $\boldsymbol\lambda \in \mathscr{P}_n^\ell$. Similarly, let $\sigma_{\mathrm{Std}} : \operatorname{Std}(\mathscr{P}_n^\ell) \longrightarrow \operatorname{Std}(\mathscr{P}_n^\ell)$ be given by $\sigma_{\mathrm{Std}}(\mathfrak{t}) = \mathfrak{t}\circ\sigma_{\mathscr{P}}$, for $\mathfrak{t} \in \operatorname{Std}(\mathscr{P}_n^\ell)$.

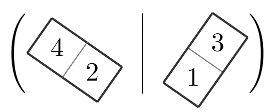
The definitions readily imply that $\sigma_{\mathscr{P}}(\boldsymbol\lambda) \in \mathscr{P}_n^\ell$ if $\boldsymbol\lambda \in \mathscr{P}_n^\ell$, so $\sigma_{\mathscr{P}}$ is well-defined. Namely, the partition $\sigma_{\mathscr{P}}(\boldsymbol\lambda)$ is obtained from $\boldsymbol\lambda$ by a certain permutation of its components. Similarly, if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol\lambda)$ then $\sigma_{\mathrm{Std}}(\mathfrak{t}) \in \operatorname{Std}(\sigma_{\mathscr{P}}(\boldsymbol\lambda))$. As with the automorphism $\sigma = \sigma_n^{\boldsymbol\Lambda}$ of $\mathscr{R}_\alpha^{\boldsymbol\Lambda}$, we usually omit the subscript and write $\sigma = \sigma_{\mathscr{P}}$ and $\sigma = \sigma_{\mathrm{Std}}$. In this way, we think of σ as:

- the automorphism $\sigma_n^{\boldsymbol\Lambda}$ of $\mathscr{R}_\alpha^{\boldsymbol\Lambda}$
- the map $\sigma_{\mathscr{P}}$ of order p on the set $\mathscr{P}_n^\ell$ of ℓ-partitions
- the map $\sigma_{\mathscr{N}}$ on the set $\mathscr{N}_n^\ell$ of nodes
- the map σ_{Std} on the set $\operatorname{Std}(\mathscr{P}_n^\ell)$ of standard tableaux.

This should not cause any ambiguity because the meaning will always be clear from the context. It is not yet clear how the map $\sigma_n^{\boldsymbol\Lambda}$ is related to the other three combinatorially defined maps but we will ultimately see that the triple of maps $(\sigma_n^{\boldsymbol\Lambda}, \sigma_{\mathscr{P}}, \sigma_{\mathrm{Std}})$ is a shift-automorphism in the sense of Definition 2.22 (see Theorem 5.4).

Example 4.6. Suppose that $p = p' = 2$ and $n = 4$. Two standard tableaux $\mathfrak{t} \in \operatorname{Std}(2|1^2)$ and $\mathfrak{s} \in \operatorname{Std}(1^2|2)$ are:

$\mathfrak{t} =$ 

and $\mathfrak{s} =$ 

.

Then $\mathfrak{s} = \sigma\mathfrak{t}$ and $\mathfrak{t} = \sigma\mathfrak{s}$.

Lemma 4.7. *Suppose that* $\gamma \in \mathscr{N}_n^\ell$. *Then* $\operatorname{res}(\sigma(\gamma)) = \varepsilon \operatorname{res}(\gamma)$.

Proof. Let $\gamma = (r,c,l)$ and write $l = ap+b+1$, where $0 \le a < d$ and $0 \le b < p$. By the remarks after (3.29), $\operatorname{res}(\gamma) = \varepsilon^b q^{\rho_{a+1}+c-r}$, which implies the result. ☐

Lemma 4.8. *Suppose that* $\mathsf{x}_{\boldsymbol\rho}(\gamma) > \mathsf{x}_{\boldsymbol\rho}(\gamma') + k$, *where* $\gamma,\gamma' \in \boldsymbol\lambda$ *with* $\boldsymbol\lambda \in \mathscr{P}_n^\ell$ *and* $k \in \{-1,0,1\}$. *Write* $\gamma = (r,c,l)$ *and* $\gamma' = (r',c',l')$, *where* $l = ap+b+1$ *and* $l' = a'p+b'+1$, *with* $0 \le a,a' < d$ *and* $0 \le b,b' < p$. *Then* $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma)) > \mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))+k$ *unless the following four conditions hold:*

$$a = a', \quad 0 \le b' < b = p-1, \quad c-r = c'-r'+k \quad \textit{and} \quad q^k\operatorname{res}(\gamma') = \varepsilon^{b'+1}\operatorname{res}(\gamma).$$

In particular, if $\operatorname{res}(\gamma) = q^k\operatorname{res}(\gamma')$ *then* $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma)) > \mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))+k$.

Proof. By Lemma 3.25(b), $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma)) > \mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))+1$ if $a > a'$, so we may assume that $a = a'$. Recalling the definition of the loading $\mathsf{x}_{\boldsymbol\rho}$ from (3.22), if $b < p-1$ or $b = b'$ then $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma)) - \mathsf{x}_{\boldsymbol\rho}(\gamma) \ge \mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma')) - \mathsf{x}_{\boldsymbol\rho}(\gamma')$ so we can, and do, assume that $b' < b = p-1$.

Notice that $\frac{e'}{ep'}=\frac{1}{mp'}=\frac{1}{p}$ and that $\frac{2n}{N}\le\frac{1}{e(\ell+1)}$ since $N\ge 2ne(\ell+1)$ by (3.21). Using these facts for the third and fourth inequalities,

$$\begin{aligned}k<\mathsf{x}_{\boldsymbol\rho}(r,c,l)-\mathsf{x}_{\boldsymbol\rho}(r',c',l')&=c-r-c'+r'+\tfrac{p-1-b'}{e}\big(\tfrac{e'}{p'}-\tfrac{1}{\ell+1}\big)+\tfrac{r'+c'-c-r}{N}\\ &\le c-r-c'+r'+(p-1)\big(\tfrac{1}{p}-\frac{1}{e(\ell+1)}\big)+\tfrac{2n}{N}\\ &\le c-r-c'+r'+1-\tfrac{1}{p}-\tfrac{p}{e(\ell+1)}+\tfrac{1}{e(\ell+1)}+\tfrac{1}{e(\ell+1))}\\ &\le c-r-c'+r'+1-\tfrac{1}{p}.\end{aligned}$$

Hence, $c-r\ge c'-r'+k$.

A similar calculation, replacing $b=p-1$ with 0 and b' with $b'+1$, shows that $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))-\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))=\frac{b'+1}{e}\big(\frac{1}{\ell+1}-\frac{e'}{p'}\big)+c-r-c'+r'-\frac{r+c-c'-r'}{N}>c-r-c'+r'-\frac{b'+1}{p}$. Consequently, if $c-r>c'-r'+k$ then $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))>\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))+k$.

Therefore, $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))\le\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))+k$ only if $c-r=c'-r'+k$. In this case, by the remarks following (3.29),

$$\operatorname{res}(\gamma')=\varepsilon^{b'}q^{v_{a+1}+c'-r'}=\varepsilon^{b'}q^{v_{a+1}+c-r-k}=q^{-k}\varepsilon^{b'+1}\operatorname{res}(\gamma),$$

completing the proof of the first part of the lemma.

Finally, if $\operatorname{res}(\gamma)=q^k\operatorname{res}(\gamma')$ then for any $0\le b'<p-1$ we have $q^k\operatorname{res}(\gamma')\ne\varepsilon^{b'+1}\operatorname{res}(\gamma)$ since ε has order p, thus $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))>\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))$. □

Remark 4.9. It follows from the proof of Lemma 4.8 that if the four conditions of the lemma are satisfied then $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))-\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))<\frac{p}{e(\ell+1)}$. Hence, it can still happen that $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))>\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma'))$.

Lemma 4.8 implies that σ respects the $\trianglerighteq_{\boldsymbol\rho}$ partial order. More precisely, we have:

Corollary 4.10. *Let $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}^\ell_n$ and suppose that $\boldsymbol\lambda\trianglerighteq_{\boldsymbol\rho}\boldsymbol\mu$. Then $\sigma\boldsymbol\lambda\trianglerighteq_{\boldsymbol\rho}\sigma\boldsymbol\mu$.*

Proof. By Definition 3.31, $\boldsymbol\lambda\trianglerighteq_{\boldsymbol\rho}\boldsymbol\mu$ if and only if there exists a bijection $\theta:\boldsymbol\lambda\longrightarrow\boldsymbol\mu$ such that

$$\operatorname{res}(\theta(\gamma))=\operatorname{res}(\gamma)\qquad\text{and}\qquad \mathsf{x}_{\boldsymbol\rho}(\theta(\gamma))\le\mathsf{x}_{\boldsymbol\rho}(\gamma),\qquad\text{for all }\gamma\in\boldsymbol\lambda.$$

Let $\theta'=\sigma\circ\theta\circ\sigma^{-1}$. Then θ' is a bijection from $\sigma\boldsymbol\lambda$ to $\sigma\boldsymbol\mu$ and if $\gamma\in\sigma\boldsymbol\lambda$ then $\operatorname{res}(\theta'(\gamma))=\operatorname{res}(\gamma)$, by Lemma 4.7, and $\mathsf{x}_{\boldsymbol\rho}(\theta'(\gamma))\le\mathsf{x}_{\boldsymbol\rho}(\gamma)$, by Lemma 4.8. Hence, $\sigma\boldsymbol\lambda\trianglerighteq_{\boldsymbol\rho}\sigma\boldsymbol\mu$. □

Abusing notation we extend σ to a map on the set of standard tableaux.

Definition 4.11. Let $\mathfrak{t}\in\operatorname{Std}(\boldsymbol\lambda)$, for $\boldsymbol\lambda\in\mathscr{P}^\ell_n$. Let $\sigma\mathfrak{t}:\sigma\boldsymbol\lambda\longrightarrow\{1,\dots,n\}$ be the standard $\sigma\boldsymbol\lambda$-tableau given by $\sigma\mathfrak{t}=\mathfrak{t}\circ\sigma^{-1}$.

Note that the $\sigma\boldsymbol\lambda$-tableau $\sigma\mathfrak{t}$ is a standard $\sigma\boldsymbol\lambda$-tableau since σ just permutes the components of $\mathfrak{t}$. We now extend the dominance order $\trianglerighteq_{\boldsymbol\rho}$ on $\mathscr{P}^\ell_n$ to the set of standard tableaux by defining

$$\mathfrak{s}\trianglerighteq_{\boldsymbol\rho}\mathfrak{t}\qquad\text{if}\qquad \operatorname{Shape}(\mathfrak{s}_{\downarrow k})\trianglerighteq_{\boldsymbol\rho}\operatorname{Shape}(\mathfrak{t}_{\downarrow k})\quad\text{for all }1\le k\le n.$$

As before, write $\mathfrak{s}\triangleright_{\boldsymbol\rho}\mathfrak{t}$ if $\mathfrak{s}\trianglerighteq_{\boldsymbol\rho}\mathfrak{t}$ and $\mathfrak{s}\ne\mathfrak{t}$.

Lemma 4.12. *Let $\mathfrak{s}\in\operatorname{Std}(\boldsymbol\mu)$ and $\mathfrak{t}\in\operatorname{Std}(\boldsymbol\lambda)$, for $\boldsymbol\lambda,\boldsymbol\mu\in\mathscr{P}^\ell_n$. Then $\deg\sigma\mathfrak{t}=\deg\mathfrak{t}$ and if $\mathfrak{s}\triangleright_{\boldsymbol\rho}\mathfrak{t}$ then $\sigma\mathfrak{s}\triangleright_{\boldsymbol\rho}\sigma\mathfrak{t}$.*

Proof. It is immediate from Lemma 4.8 that $\sigma\mathfrak{s} \trianglerighteq_{\boldsymbol{\rho}} \sigma\mathfrak{t}$ if $\mathfrak{s} \trianglerighteq_{\boldsymbol{\rho}} \mathfrak{t}$. Recall the definition of $\deg\mathfrak{t}$ from (3.42) and observe that σ induces bijections $\mathcal{A}_k(\mathfrak{t}) \xrightarrow{\sim} \mathcal{A}_k(\sigma\mathfrak{t})$ and $\mathcal{R}_k(\mathfrak{t}) \xrightarrow{\sim} \mathcal{R}_k(\sigma\mathfrak{t})$ by Lemma 4.7 and Lemma 4.8, for $1 \le k \le n$. Hence, $\deg\sigma\mathfrak{t} = \deg\mathfrak{t}$. □

The results in this section suggest that it is not unreasonable to expect that $\sigma(C_{\mathfrak{s}\mathfrak{t}}) = C_{\sigma\mathfrak{s},\sigma\mathfrak{t}}$. Before we can prove this we first need to extend σ to an automorphism of $\mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]}(\boldsymbol{\omega})$ and then construct explicit diagrams $C_{\mathfrak{t}}$ for which this is true. We do this in the next sections.

4.3. **Applying $\sigma^{\boldsymbol{\Lambda}}_n$ in the diagrammatic Cherednik algebra.** This section identifies the image of the automorphism $\sigma^{\boldsymbol{\Lambda}}_n$ of $\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}$ of Theorem 3.13 under the isomorphism of Theorem 3.49. A key ingredient is the diagram automorphism $D \mapsto D^{\mathrm{cyc}}$ introduced in [3], and its generalisations below, that plays a crucial role in the proof of our main results in Section 5.

Let $\alpha \in \mathscr{C}^{\ell}_n$ be an $\mathcal{I}$-composition of n. By Theorem 3.49, there is a $\mathbb{Z}$-graded R-algebra isomorphism $\mathscr{R}^{\boldsymbol{\Lambda}}_{\alpha} \cong \mathsf{E}_{\boldsymbol{\omega},\alpha}\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}\mathsf{E}_{\boldsymbol{\omega},\alpha}$, so we now identify these two algebras. Let $[\alpha]$ be the orbit of α under the action of the finite group $\langle\sigma\rangle$, as described in (3.12). Define

$$\mathscr{R}^{\boldsymbol{\Lambda}}_{[\alpha]} = \bigoplus_{\beta\in[\alpha]} \mathscr{R}^{\boldsymbol{\Lambda}}_{\beta}, \quad \mathsf{E}_{\boldsymbol{\omega},[\alpha]} = \bigoplus_{\beta\in[\alpha]} \mathsf{E}_{\boldsymbol{\omega},\beta} \quad\text{and}\quad \mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]} = \bigoplus_{\beta\in[\alpha]} \mathbb{A}^{\boldsymbol{\rho}}_{\beta}. \tag{4.13}$$

The isomorphism σ in Theorem 3.13 induces an automorphism of $\mathscr{R}^{\boldsymbol{\Lambda}}_{[\alpha]}$. Hence, we can regard σ as a homogeneous R-algebra automorphism of

$$\mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]}(\boldsymbol{\omega}) = \mathsf{E}_{\boldsymbol{\omega},[\alpha]}\mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]}\mathsf{E}_{\boldsymbol{\omega},[\alpha]}.$$

The next result gives a more precise description of σ considered as an automorphism of $\mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]}(\boldsymbol{\omega})$, or equivalently, of $\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$.

Let $\tilde{\sigma}$ be the automorphism of the set of Webster diagrams that sends a diagram D to the diagram that has the same strings but where the residues of the solid and ghost strings are multiplied by ε. Note that $\tilde{\sigma}$ is not well-defined as an automorphism of $\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$, or even as a map $\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega}) \to \mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$, since it is not compatible with the exceptional defining relations (for $\tilde{\sigma}$ does not change the residues of the red strings). Lemma 4.14 is immediate from the definitions.

Lemma 4.14. *If D is any Webster diagram then $(\tilde{\sigma}(D))^* = \tilde{\sigma}(D^*)$.*

Let $\widehat{\boldsymbol{\nu}}$ be a generalised partition and $a \in \{0,\dots,d-1\}$. Then $\widehat{\boldsymbol{\nu}}$ is a-**bounded** if $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(0,0,ap+1)$ for all $\gamma \in \widehat{\boldsymbol{\nu}}$. That is, all the nodes in $\widehat{\boldsymbol{\nu}}$ are to the left of the $(ap+1)$th red string r_{ap+1}. In other words, the nodes are to the right of at most ap red strings. For example, the ℓ-partition $\boldsymbol{\omega}$ is 0-bounded.

Let $a \in \{0,\dots,d-1\}$ and let $\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}'$ be two a-bounded generalised partitions. Let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}')$ be a Webster diagram. Then D is an a-**bounded diagram** if all of its solid string are to the left of r_{ap+1}. Further, D is an $\boldsymbol{\omega}$-**diagram** if $\widehat{\boldsymbol{\nu}} = \widehat{\boldsymbol{\nu}}' = \boldsymbol{\omega}$ and D is 0-bounded. In particular, not all Webster diagrams of type $(\boldsymbol{\omega},\boldsymbol{\omega})$ are $\boldsymbol{\omega}$-diagrams. By Theorem 3.49, $\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$ is spanned by $\boldsymbol{\omega}$-diagrams.

Now let $a \in \{0,\dots,d-1\}$ and let $\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}'$ be two a-bounded generalised partitions. Let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}, \widehat{\boldsymbol{\nu}}')$. Let s be a solid i-string in D and let $l \in \{ap+1,\dots,\ell\}$. Suppose that $q^{\kappa_l} = i$ and that s crosses the lth red string r_l. We pull all solid (and ghost) strings that are to the right of r_{ap+1} to the left of r_{ap+1} while still staying to the right of r_{ap} if $a>0$, and for each crossing involving a solid i-string (from southwest to northeast) and a red string (from southeast to northwest) place a dot at the position of the crossing in D. The argument used in the proof of [3, Proposition 6.19, Figure 18] now shows that by iterating this process we get a single diagram $D^{a,\mathrm{cyc}}$, which is denoted by $\overline{D}$ in [3, Proposition 6.19]. When $a=0$ we simply write $D^{\mathrm{cyc}} := D^{0,\mathrm{cyc}}$. Moreover, by construction,

$$(D^{a+1,\mathrm{cyc}})^{a,\mathrm{cyc}} = D^{a,\mathrm{cyc}}, \qquad \text{whenever } 0 \le a < d-2. \tag{4.15}$$

The following result is implicit in the proof of [3, Proposition 6.19].

Lemma 4.16. *Let $a \in \{0,\dots,d-1\}$, let $\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}'$ be two a-bounded generalised partitions and let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$. In $\mathbb{A}_n^{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$ we have $D = D^{a,\mathrm{cyc}}$. In particular, if D is of type $(\boldsymbol{\omega},\boldsymbol{\omega})$ then $D = D^{\mathrm{cyc}}$ in $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$.*

Proof. This can be proved using the argument of [3, Proposition 6.19, Figure 18]. □

Definition 4.17. Let $a \in \{0,\dots,d-1\}$.

- The a-**comb** in a diagram is the set of red strings $\mathsf{r}_{ap+1},\dots,\mathsf{r}_{(a+1)p}$.
- A diagram D is a-**greedy** if each solid string s either does not cross r_{ap+1} or s crosses all of the strings $\mathsf{r}_{ap+1},\dots,\mathsf{r}_{(a+1)p}$ in the a-comb but does not cross $\mathsf{r}_{(a+1)p+1}$.

Note that an a-bounded diagram is a-greedy. In Example 4.23, $\{\mathsf{r}_1,\mathsf{r}_2\}$ is the 1-comb and $\{\mathsf{r}_3,\mathsf{r}_4\}$ is the 2-comb.

Lemma 4.18. *Let $a \in \{0,\dots,d-1\}$, let $\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}'$ be two a-bounded generalised partitions and let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$. If D is a-greedy then in $\mathbb{A}_n^{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}},\widehat{\boldsymbol{\nu}}')$ we have $\tilde{\sigma}\big(D^{a,\mathrm{cyc}}\big) = \tilde{\sigma}(D)^{a,\mathrm{cyc}}$.*

Proof. Recall that the red strings $\mathsf{r}_{ap+1},\dots,\mathsf{r}_{(a+1)p}$ of the a-comb have residues $q^{\kappa_{ap+1}}, \varepsilon q^{\kappa_{ap+1}},\dots,\varepsilon^{p-1}q^{\kappa_{ap+1}}$. Let s be a solid i-string in D. If s does not cross r_{ap+1}, then $D = D^{a,\mathrm{cyc}}$ and $\tilde{\sigma}\big(D^{a,\mathrm{cyc}}\big) = \tilde{\sigma}(D)^{a,\mathrm{cyc}}$ clearly holds. Suppose that s crosses r_{ap+1}. By assumption, the diagram D is a-greedy, so s crosses $\mathsf{r}_{ap+1},\dots,\mathsf{r}_{(a+1)p}$ and it does not cross $\mathsf{r}_{(a+1)p+1}$. If $i \notin \varepsilon^{\mathbb{Z}}q^{\kappa_{ap+1}}$ then in $D^{a,\mathrm{cyc}}$ the string s of D does not gain a dot, so neither will its image in $\tilde{\sigma}(D)$. On the other hand, if $i = \varepsilon^b q^{\kappa_{ap+1}}$ with $0 \le b < p$, if the string s gains $N \in \mathbb{N}$ dots when it is pulled past r_{ap+b+1} during the operation $D \mapsto D^{a,\mathrm{cyc}}$, then the image of s in $\tilde{\sigma}(D)$ also gains N nodes at the same positions, up to isotopy, during the operation $\tilde{\sigma}(D) \mapsto \tilde{\sigma}(D)^{a,\mathrm{cyc}}$. □

Lemma 4.18 can be thought of as a commutation rule for the operations $D \mapsto D^{a,\mathrm{cyc}}$ and $D \mapsto \tilde{\sigma}(D)$.

Proposition 4.19. *Let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\omega},\boldsymbol{\omega})$ be any diagram such that a solid string crosses a red string in the a-comb only if it crosses every red string in the a-comb, for $0 \le a < d$. Then $\tilde{\sigma}(D^{\mathrm{cyc}}) = \tilde{\sigma}(D)^{\mathrm{cyc}}$ in $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$.*

Proof. The result is clear if there are no solid-red crossings in D since in this case $D = D^{\mathrm{cyc}}$ as Webster diagrams. Let now $\overline{a} \in \{0,\dots,d-1\}$ be maximal such that a solid string crosses a red string of $\mathsf{X}_{\boldsymbol{\rho}}^{\overline{a},0}$. We will prove by reverse induction on $a \in \{0,\dots,\overline{a}\}$ that

$$\tilde{\sigma}(D^{a,\mathrm{cyc}}) = \tilde{\sigma}(D)^{a,\mathrm{cyc}}.$$

The initialisation $a = \overline{a}$ follows directly from Lemma 4.18 since the assumption of D implies that D is $\overline{a}$-greedy. Now let $a \in \{0, \dots, \overline{a}-1\}$ and assume that $\tilde{\sigma}(D^{a+1,\mathrm{cyc}}) = \tilde{\sigma}(D)^{a+1,\mathrm{cyc}}$. The diagram $D^{a+1,\mathrm{cyc}}$ is $(a+1)$-bounded by construction. By assumption of D, the diagram $D^{a+1,\mathrm{cyc}}$ is a-greedy, thus by Lemma 4.18 we have

$$\tilde{\sigma}\big((D^{a+1,\mathrm{cyc}})^{a,\mathrm{cyc}}\big) = \tilde{\sigma}(D^{a+1,\mathrm{cyc}})^{a,\mathrm{cyc}}. \tag{4.20}$$

By (4.15) and the induction hypothesis we deduce that

$$\tilde{\sigma}(D^{a+1,\mathrm{cyc}})^{a,\mathrm{cyc}} = (\tilde{\sigma}(D)^{a+1,\mathrm{cyc}}))^{a,\mathrm{cyc}} = \tilde{\sigma}(D)^{a,\mathrm{cyc}},$$

thus the hereditary property follows from (4.20). The result now follows from the case $a = 0$. $\square$

Proposition 4.21. *Let $D \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\omega}, \boldsymbol{\omega})$. Then $\sigma(D) = \tilde{\sigma}(D^{\mathrm{cyc}})$ in $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$.*

Proof. By Theorem 3.13 and Theorem 3.49, the maps $\tilde{\sigma}$ and σ coincide on the images of the KLR generators in $\mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\omega}, \boldsymbol{\omega})$. Consequently, if D and E are $\boldsymbol{\omega}$-diagrams then

$$\tilde{\sigma}(DE) = \sigma(DE) = \sigma(D)\sigma(E) = \tilde{\sigma}(D)\tilde{\sigma}(E),$$

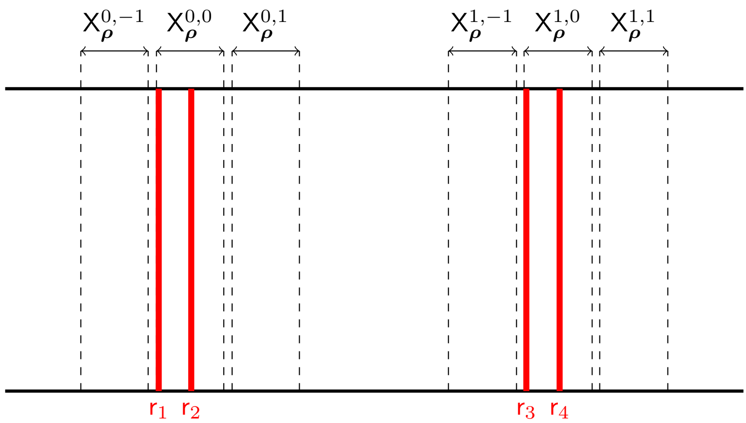
in $\mathbb{A}_n^{\boldsymbol{\rho}}$. Therefore, in view of Lemma 4.16, writing $D^{\mathrm{cyc}} \in \mathbb{A}_n^{\boldsymbol{\rho}}$ as a product of $\boldsymbol{\omega}$-diagrams, the result follows because σ is an algebra automorphism of $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$. $\square$

4.4. **Regions.** Recall that the definition of the set of nodes $\mathscr{N}_n^\ell$ implies that if $(r, c, l) \in \mathscr{N}_n^\ell$ then $r - c \in \{-n+1, \dots, n-1\}$. Definition 4.22 allows us to write the set of nodes, and their $\mathsf{x}_{\boldsymbol{\rho}}$–coordinates, as a disjoint set of **regions**. These regions are the key to defining an explicit diagrammatic basis of $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$ that is compatible with σ, which we will use to prove that $\mathscr{R}_{p,n}^{\boldsymbol{\Lambda}}$ is a graded skew cellular algebra.

Definition 4.22. For integers $0 \le a < d$ and $-n \le \delta \le n$ define

$$\begin{aligned}
\mathscr{N}^{a,\delta} &:= \{\gamma \in \mathscr{N}_n^\ell \mid \gamma = (r, c, l), c - r = \delta, l = ap + b + 1 \text{ with } 0 \le b < p\},\\
\boldsymbol{\lambda}^{a,\delta} &:= \mathscr{N}^{a,\delta} \cap \boldsymbol{\lambda}, \quad \text{for } \boldsymbol{\lambda} \in \mathscr{P}_n^\ell,\\
\mathsf{X}_{\boldsymbol{\rho}}^{a,\delta} &:= \left[\delta + \frac{1}{e}\left(\rho_{a+1} - \frac{ap}{\ell+1}\right) - \frac{2n}{N}, \delta + 1 + \frac{1}{e}\left(\rho_{a+1} - \frac{(a+1)p}{\ell+1}\right)\right] \subseteq \mathbb{R}.
\end{aligned}$$

Example 4.23. Let $\ell = 4, e = 3, p = 2, d = 2, n = 1, N = 61, \rho = (0, 15)$. An illustration of regions in a diagram is given by:



In this diagram the 0-comb is $\{\mathsf{r}_1,\mathsf{r}_2\}$ and the 1-comb is $\{\mathsf{r}_3,\mathsf{r}_4\}$. Hence, the a-comb is contained in the region $\mathsf{X}_{\boldsymbol\rho}^{a,0}$, for $a\in\{0,1\}$.

The node $(0,0,ap+b+1)$, which corresponds to a red string, belongs to $\mathscr{N}^{a,0}$. In particular, note that the red strings in the a-comb are precisely the red strings contained in the region $\mathsf{X}_{\boldsymbol\rho}^{a,0}$. For Lemma 4.24, for any $(a,\delta),(a',\delta')\in\mathbb{Z}^2$, write $(a,\delta)<_{\mathrm{lex}}(a',\delta')$ if (a,δ) is smaller than (a',δ') in the lexicographic order. That is, $(a,\delta)<_{\mathrm{lex}}(a',\delta')$ if $a<a'$, or $a=a'$ and $\delta<\delta'$.

Lemma 4.24. *Suppose that $0\le a<d$, $-n\le\delta\le n$ and $\gamma=(r,c,l)\in\mathscr{N}^{a,\delta}$.*

(a) *Fix $0\le a'<d$ and $-n\le\delta'\le n$ such that $(a,\delta)<_{\mathrm{lex}}(a',\delta')$. Then $\mathsf{X}_{\boldsymbol\rho}^{a,\delta}\cap\mathsf{X}_{\boldsymbol\rho}^{a',\delta'}=\emptyset$. More precisely, if $x\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta}$ and $x'\in\mathsf{X}_{\boldsymbol\rho}^{a',\delta'}$ then $x<x'$.*
(b) *We have $\gamma\in\mathscr{N}^{a,\delta}$ if and only if $\mathsf{x}_{\boldsymbol\rho}(\gamma)\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta}$.*
(c) *If $x\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta}$ and $\delta<n$ then $x+1\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta+1}$.*
(d) *If $\mathsf{x}_{\boldsymbol\rho}(\gamma)\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta}$ then $\mathsf{x}_{\boldsymbol\rho}(\sigma(\gamma))\in\mathsf{X}_{\boldsymbol\rho}^{a,\delta}$. That is, each region is stable under σ.*
(e) *If $\gamma'=(r',c',l')\in\mathscr{N}_n^\ell$, $\gamma,\gamma'\in\mathscr{N}^{a,\delta}$ and $l\ne l'$ then $\operatorname{res}(\gamma)\ne\operatorname{res}(\gamma')$.*
(f) *If $\gamma'=(r',c',l')\in\mathscr{N}_n^\ell$, $\gamma\in\mathscr{N}^{a,\delta}$ and $\gamma'\in\mathscr{N}^{a,\delta-1}$ and $l\ne l'$ then $\operatorname{res}(\gamma)\ne q\operatorname{res}(\gamma')$.*

Proof. Let us prove (a). If $\delta'=\delta+1$ then $\min\mathsf{X}_{\boldsymbol\rho}^{a,\delta+1}-\max\mathsf{X}_{\boldsymbol\rho}^{a,\delta}=-\frac{2n}{N}+\frac{p}{e(\ell+1)}>0$, since $N>2ne(\ell+1)$ by (3.21). Hence, the result follows when $a=a'$. To prove the result when $a<a'$, it suffices to consider the case when $a'=a+1$ and $\delta=n=-\delta'$. We have

$$\begin{aligned}\min\mathsf{X}_{\boldsymbol\rho}^{a+1,-n}&-\max\mathsf{X}_{\boldsymbol\rho}^{a,n}\\&=-n+\frac{1}{e}\left(\rho_{a+2}-\frac{(a+1)p}{\ell+1}\right)-\frac{2n}{N}-\left[n+1+\frac{1}{e}\left(\rho_{a+1}-\frac{(a+1)p}{\ell+1}\right)\right]\\&=-2n-1+\frac{1}{e}(\rho_{a+2}-\rho_{a+1})-\frac{2n}{N}.\end{aligned}$$

Hence,

$$\min\mathsf{X}_{\boldsymbol\rho}^{a+1,-n}-\max\mathsf{X}_{\boldsymbol\rho}^{a,n}>0\iff\rho_{a+2}-\rho_{a+1}>\left(2n+1+\frac{2n}{N}\right)e.$$

The latter inequality holds since $\rho_{a+2}-\rho_{a+1}\ge(2n+3)e$, by Definition 3.6, and $\frac{2n}{N}<\frac{1}{ep'(\ell+1)}<1$ by (3.21). This completes the proof of (a).

We now prove (b). If $\gamma = (r, c, l) \in \mathscr{N}^{a,\delta}$ then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) = \delta + \frac{1}{e}\left(\kappa_l - \frac{l-1}{\ell+1}\right) - \frac{r+c}{N}$, thus by Lemma 3.24 we obtain

$$\delta + \frac{1}{e}\left(\rho_{a+1} - \frac{ap}{\ell+1}\right) - \frac{2n}{N} \le \mathsf{x}_{\boldsymbol{\rho}}(\gamma) \le \delta + 1 + \frac{1}{e}\left(\rho_{a+1} - \frac{(a+1)p}{\ell+1}\right).$$

That is, $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) \in \mathsf{X}_{\boldsymbol{\rho}}^{a,\delta}$ so (b) now follows in view of (a).

Parts (c) and (d) are clear from the definitions. For part (e), writing $l = ap+b+1$, with $0 \le b < p$, we have $\operatorname{res}(\gamma) = q^{\rho_{a+1}+\delta}\varepsilon^b$ by the remark after (3.29). Thus, writing $l' = ap + b' + 1$, with $0 \le b' < p$, we have $b \ne b'$ by assumption and so $\operatorname{res}(\gamma') = q^{\rho_{a+1}+\delta}\varepsilon^{b'} \ne \operatorname{res}(\gamma)$ since ε has order p. The proof of (f) is similar. □

Lemma 4.25 will allow us to consider generalised partitions constructed from $\min \mathsf{X}_{\boldsymbol{\rho}}^{a,\delta}$.

Lemma 4.25. *Let $0 \le a < d$ and $-n \le \delta \le n$.*

(a) *Let $\gamma = (r, c, ap+1) \in \mathbb{R} \times \mathbb{R} \times \{1, \dots, \ell\}$, where $c - r = \delta$ and $c + r = 2n$. Then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) = \min \mathsf{X}_{\boldsymbol{\rho}}^{a,\delta}$.*

(b) *If $l = ap+b+1$ with $0 \le b < p$, then $\min \mathsf{X}_{\boldsymbol{\rho}}^{a,0} < \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l)$ and $\min \mathsf{X}_{\boldsymbol{\rho}}^{a,-1} + 1 < \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l)$.*

(c) *We always have $\min \mathsf{X}_{\boldsymbol{\rho}}^{a,\delta} \notin \{\mathsf{x}_{\boldsymbol{\rho}}(0, 0, l), \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l) - 1\}$.*

Proof. Part (a) is clear. Part (b) follows from Lemma 3.24 and, using Lemma 4.24, we deduce part (c). □

The final result in this section proves a lemma that describes the $\mathsf{x}_{\boldsymbol{\rho}}$-coordinates of nodes in $\boldsymbol{\lambda}^{a,0}$ in terms of the $\mathsf{x}_{\boldsymbol{\rho}}$–coordinates of the adjacent red strings.

Lemma 4.26. *Let $0 \le a < d$ and $\gamma = (r, c, l) \in \boldsymbol{\lambda}^{a,0}$. Then*

$$\mathsf{x}_{\boldsymbol{\rho}}(0, 0, l-1) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l).$$

Proof. The second inequality follows from Lemma 4.25. If $l = ap+1$, then the first equality is clear by Definition 3.6. If $l \not\equiv 1 \pmod p$ then, since $N > 2nep'(\ell+1)$ by (3.21),

$$\begin{aligned}
\mathsf{x}_{\boldsymbol{\rho}}(\gamma) - \mathsf{x}_{\boldsymbol{\rho}}(0, 0, l-1) &= \frac{1}{e}\left(\kappa_l - \frac{l-1}{\ell+1}\right) - \frac{r+c}{N} - \frac{1}{e}\left(\kappa_{l-1} - \frac{l-2}{\ell+1}\right) \\
&= \frac{1}{p} - \frac{1}{e(\ell+1)} - \frac{r+c}{N} \\
&> \frac{1}{p} - \frac{1}{e(\ell+1)} - \frac{1}{ep'(\ell+1)}.
\end{aligned}$$

The last quantity is non-negative since $p \le \ell$, proving the result. □

4.5. **Regular and singular diagrams for tableaux.** We are now ready to start constructing the diagrammatic basis of $\mathbb{A}_n^{\boldsymbol{\rho}}(\boldsymbol{\omega})$ that we will use to prove our main results. In this section we take the first step of fixing a choice of diagrams $\{B_{\mathfrak{t}}\}$, for $\mathfrak{t}$ a standard tableau, in accordance with Definition 3.43. We construct the basis elements $B_{\mathfrak{t}}$ by gluing smaller diagrams together. Lemma 4.27 studies the residues and possible positions of strings at the top of a $B_{\mathfrak{t}}$ diagram.

Lemma 4.27. *Let $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$ and let $\gamma \in \boldsymbol{\lambda}^{a,\delta}$, for $0 \le a < d$ and $-n \le \delta \le n$.*

(SS) *If $\gamma' \in \boldsymbol{\lambda}^{a,\delta}$ and $\operatorname{res}(\gamma) = \operatorname{res}(\gamma')$ and $\mathfrak{t}(\gamma) > \mathfrak{t}(\gamma')$ then $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$.*

(SG) *If* $\gamma' \in \boldsymbol{\lambda}^{a,\delta-1}$ *and* $\operatorname{res}(\gamma) = q\operatorname{res}(\gamma')$ *and* $\mathfrak{t}(\gamma) > \mathfrak{t}(\gamma')$ *then* $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma') + 1$.

(GS) *If* $\gamma' \in \boldsymbol{\lambda}^{a,\delta+1}$ *and* $q\operatorname{res}(\gamma) = \operatorname{res}(\gamma')$ *and* $\mathfrak{t}(\gamma) > \mathfrak{t}(\gamma')$ *then* $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) + 1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$.

(SR) *If* $\delta = 0$ *and* $\gamma' \in \mathscr{N}^{a,0}$ *corresponds to a red string with* $\operatorname{res}(\gamma) = \operatorname{res}(\gamma')$ *then* $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$.

(GR) *If* $\delta = -1$ *and* $\gamma' \in \mathscr{N}^{a,0}$ *corresponds to a red string with* $q\operatorname{res}(\gamma) = \operatorname{res}(\gamma')$ *then* $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) + 1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma')$.

Proof. Write $\gamma = (r, c, l)$ and $\gamma' = (r', c', l')$. For (SS), which stands for "solid-solid", since $\gamma, \gamma' \in \boldsymbol{\lambda}^{a,\delta}$ we have $c - r = c' - r' = \delta$, so $q^{\kappa_l} = q^{\kappa_{l'}}$ since $\operatorname{res}(\gamma) = \operatorname{res}(\gamma')$. If $b, b' \in \{0, \dots, p-1\}$ are such that $l - b - 1 = l' - b' - 1 = ap$ we deduce that $\varepsilon^b = \varepsilon^{b'}$ and thus $b = b'$ since ε has order p and thus $l = l'$. Thus, since $\mathfrak{t}$ is standard, the box corresponding to γ is higher than the box corresponding to γ'. Therefore, $r + c > r' + c'$, so $\mathsf{x}_{\boldsymbol{\rho}}(\gamma) - \mathsf{x}_{\boldsymbol{\rho}}(\gamma') = \frac{r'+c'-r-c}{N} < 0$ as desired.

For (SG), which corresponds to solid-ghost positions, γ, γ' are necessarily in the same component of $\boldsymbol{\lambda}$. Hence, since $\mathfrak{t}(\gamma) > \mathfrak{t}(\gamma')$ we have $r + c > r' + c'$ and thus $\mathsf{x}_{\boldsymbol{\rho}}(\gamma') + 1 - \mathsf{x}_{\boldsymbol{\rho}}(\gamma) = \frac{r+c-r'-c'}{N} > 0$. The (GS) case, or ghost-solid case, is similar.

Finally, the solid-red (SR) case is deduced from the solid-solid (SS), and (GR) is deduced from (GS), using the convention that $\mathfrak{t}(\gamma') \coloneqq 0$. $\square$

Recall the definition of a generalised partition, from (3.28) and the definition of a regular diagram from Section 4.1. In order to make the inductive arguments below clearer, we write $\boldsymbol{\omega}_n = (1^n|0|\dots|0)$, instead of $\boldsymbol{\omega}$, and let $\mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}(n)$ be the interval $\mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}$ defined in Definition 4.22, for any $0 \le a < d$ and $-n \le \delta \le n$.

Definition 4.28. Let $\boldsymbol{\lambda} \in \mathscr{P}^\ell_n$ and fix a standard tableau $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Set $\gamma_k \coloneqq \mathfrak{t}^{-1}(k)$, for $1 \le k \le n$, and let a_k, δ_k be the integers such that $\gamma_k \in \mathsf{X}^{a_k,\delta_k}_{\boldsymbol{\rho}}(k)$. Define $\widehat{\boldsymbol{\nu}}_{\mathfrak{t},k} \coloneqq \min \mathsf{X}^{a_k,\delta_k}_{\boldsymbol{\rho}}(k)$ and let $\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}$ be a generalised partition such that

$$\mathsf{x}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}) = \{\widehat{\boldsymbol{\nu}}_{\mathfrak{t},k} \mid 1 \le k \le n\}.$$

The inequalities $\min \mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}(n) < \min \mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}(n-1)$, for any a and δ, together with Lemma 4.24 and Lemma 4.25, ensure that $\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}$ satisfies the requirements of (3.28) and hence is a generalised partition.

For the rest of this section we fix a partition $\boldsymbol{\lambda} \in \mathscr{P}^\ell_n$ and a standard tableau $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ together with the associated notation from Definition 4.28.

Proposition 4.29. *There exists a regular diagram* $B^{\mathrm{reg}}_{\mathfrak{t}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$ *such that* $B^{\mathrm{reg}}_{\mathfrak{t}}$ *contains a solid string* s_k *of residue* $\operatorname{res}_{\mathfrak{t}}(k)$ *that starts at* $\mathsf{s}_k(0) = \widehat{\boldsymbol{\nu}}_{\mathfrak{t},k}$ *and ends at* $\mathsf{s}_k(1) = \mathsf{x}_{\boldsymbol{\rho}}(\gamma_k)$, *for* $1 \le k \le n$.

Proof. We argue by induction on $n = |\boldsymbol{\lambda}|$ to show that such a diagram $B^{\mathrm{reg}}_{\mathfrak{t}}$ exists for any $\boldsymbol{\lambda} \in \mathscr{P}^\ell_n$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The base case $n = 0$ is immediate because in this case $\mathfrak{t}$ is the empty tableau and we can define $B^{\mathrm{reg}}_{\mathfrak{t}}$ to be the diagram with no solid (and ghost) strings.

Now suppose that $n > 0$ and let $\mathfrak{s}$ be the (standard) tableau obtained from $\mathfrak{t}$ by removing the (removable) box $\gamma_n = \mathfrak{t}^{-1}(n)$. Let $\boldsymbol{\mu} = \operatorname{Shape}(\mathfrak{s}) \in \mathscr{P}^\ell_{n-1}$. By induction, there exists a regular diagram $B^{\mathrm{reg}}_{\mathfrak{s}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\mu}, \widehat{\boldsymbol{\nu}}_{\mathfrak{s}})$ that satisfies the conditions of the proposition.

Define $B^{\mathrm{reg}}_{\mathfrak{t}}$ to be the diagram obtained by adding the solid string s_n of residue $\mathrm{res}_{\mathfrak{t}}(n)$ to the diagram $B^{\mathrm{reg}}_{\mathfrak{s}}$, together with its ghost, where the string s_n is given by

$$\mathsf{s}_n(t) = \begin{cases} \widehat{\boldsymbol{\nu}}_{\mathfrak{t},n}, & \text{if } 0 \le t < 1-\epsilon, \\ \frac{1}{\epsilon}\big(\widehat{\boldsymbol{\nu}}_{\mathfrak{t},n}(1-t) + \mathsf{x}_{\boldsymbol{\rho}}(\gamma_n)(t-1+\epsilon)\big), & \text{if } 1-\epsilon \le t \le 1, \end{cases}$$

where ϵ is sufficiently small. That is, the solid string s_n is vertical for $0 \le t \le 1-\epsilon$ after which it is an almost horizontal line connecting the points $(\widehat{\boldsymbol{\nu}}_{\mathfrak{t},n}, 1-\epsilon)$ and $(\mathsf{x}_{\boldsymbol{\rho}}(\gamma_n), 1)$.

By induction, $B^{\mathrm{reg}}_{\mathfrak{t}}$ has the required endpoints, so it remains to show that $B^{\mathrm{reg}}_{\mathfrak{t}}$ is regular. By construction, the string s_n does not cross any other strings when $0 \le t < 1-\epsilon$ because $\min \mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}(n) < \min \mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}(n-1)$. By definition, $\gamma_n = (r,c,l) \in \mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}$, where for convenience we write $a = a_n$ and $\delta = \delta_n$. By Lemma 4.24, when $1-\epsilon \le t \le 1$ the solid string (resp., the ghost string corresponding to) s_n crosses:

- any solid string $\mathsf{s}_{\gamma'} \in B^{\mathrm{reg}}_{\mathfrak{s}}$ for $\gamma' \in \boldsymbol{\mu}^{a,\delta}$ with $\mathsf{x}_{\boldsymbol{\rho}}(\gamma') < \mathsf{x}_{\boldsymbol{\rho}}(\gamma_n)$;
- any ghost (resp. solid) string corresponding to $\mathsf{s}_{\gamma'} \in B^{\mathrm{reg}}_{\mathfrak{s}}$ for $\gamma' \in \boldsymbol{\mu}^{a,\delta-1}$ (resp. $\gamma' \in \boldsymbol{\mu}^{a,\delta+1}$) with $\mathsf{x}_{\boldsymbol{\rho}}(\gamma') + 1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma_n)$ (resp. $\mathsf{x}_{\boldsymbol{\rho}}(\gamma') < \mathsf{x}_{\boldsymbol{\rho}}(\gamma) + 1$);
- any red string starting at $(\mathsf{x}_{\boldsymbol{\rho}}(0,0,ap+b'+1), 1)$ with $0 \le b' < b$ if $\delta = 0$ (resp. $\delta = -1$), where $b \in \{0, \dots, p-1\}$ is such that $l = ap+b+1$.

Since $\gamma = \gamma_n$ is the box with the highest label in $\mathfrak{t}$, if $\gamma' \in \boldsymbol{\mu}$ then have $\mathfrak{t}(\gamma_n) = n > \mathfrak{t}(\gamma')$. Therefore, by Lemma 4.27, all of the crossings above are regular. Hence, $B^{\mathrm{reg}}_{\mathfrak{t}}$ is regular and the proof is complete. □

An example of a diagram $B^{\mathrm{reg}}_{\mathfrak{t}}$ can be found in Figure 1 and Figure 2. Now Definition 4.28 and Proposition 4.29 immediately imply the following result.

Lemma 4.30. *Each string of $B^{\mathrm{reg}}_{\mathfrak{t}}$ is contained in a single region.*

We now define the second part of the diagram $B_{\mathfrak{t}}$.

Proposition 4.31. *There exists a diagram $B^{\mathrm{sing}}_{\mathfrak{t}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}, \boldsymbol{\omega})$ with solid strings $\mathsf{s}_1, \dots \mathsf{s}_n$, and corresponding ghost strings $\mathsf{g}_1, \dots, \mathsf{g}_n$, such that:*

(a) *The string s_k has residue $\mathrm{res}_{\mathfrak{t}}(k)$, starts at $\mathsf{x}_{\boldsymbol{\rho}}(k,1,1)$ and ends at $\widehat{\boldsymbol{\nu}}_{\mathfrak{t},k}$, for $1 \le k \le n$.*

(b) *In particular, the string s_k crosses a red string in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ for $a \in \{0, \dots, d-1\}$ if and only if it crosses all the red strings in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$.*

(c) *If s_k crosses s_j then $(a_k, \delta_k) \ne (a_j, \delta_j)$. If s_k crosses g_j then $(a_k, \delta_k) \ne (a_j, \delta_j + 1)$. If g_k crosses s_j then $(a_k, \delta_k + 1) \ne (a_j, \delta_j)$.*

(d) *If s_k crosses the red string in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ then $\gamma_k \notin \boldsymbol{\lambda}^{a,0}$. Similarly, if g_k crosses the red string in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ then $\gamma_k \notin \boldsymbol{\lambda}^{a,-1}$.*

Proof. Again we argue by induction on the number of solid strings $n \ge 0$. When $n = 0$ the result is vacuously true so suppose $n > 0$. Let $\mathfrak{s}$ be the (standard) $\boldsymbol{\mu}$-tableau obtained from $\mathfrak{t}$ by removing the (removable) box $\gamma_n = \mathfrak{t}^{-1}(n)$, where $\boldsymbol{\mu} \in \mathscr{P}^{\ell}_{n-1}$ is the ℓ-partition obtained from $\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_n$ by removing $\gamma_n \in \boldsymbol{\lambda}^{a_n,\delta_n}$. By induction, there exists a diagram $B^{\mathrm{sing}}_{\mathfrak{s}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{s}}, \boldsymbol{\omega}_{n-1})$ that satisfies the conditions of Proposition 4.31. As in the proof of Proposition 4.29, define $B^{\mathrm{sing}}_{\mathfrak{t}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}, \boldsymbol{\omega}_n)$ to be the diagram obtained from $B^{\mathrm{sing}}_{\mathfrak{s}}$ by adding the solid string s_n of residue $\mathrm{res}_{\mathfrak{t}}(n)$, and its ghost, going vertically from $(\mathsf{x}_{\boldsymbol{\rho}}(n,1,1), 0)$ until almost the top of the diagram after which s_n goes almost horizontally to the point $(\widehat{\boldsymbol{\nu}}_{\mathfrak{t},n}, 1)$. More explicitly,

$$\mathsf{s}_n(t) = \begin{cases} \mathsf{x}_{\boldsymbol{\rho}}(n,1,1), & \text{if } 0 \le t < 1-\epsilon', \\ \frac{1}{\epsilon'}\big(\mathsf{x}_{\boldsymbol{\rho}}(n,1,1)(1-t) + \widehat{\boldsymbol{\nu}}_{\mathsf{t},n}(t-1+\epsilon')\big), & \text{if } 1-\epsilon' \le t \le 1, \end{cases}$$

for some sufficiently small ϵ'.

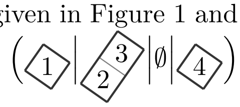
Note that $\mathsf{x}_{\boldsymbol{\rho}}(n,1,1) \in \mathsf{X}_{\boldsymbol{\rho}}^{0,1-n}(n)$ and $\mathsf{x}_{\boldsymbol{\rho}}(n,1,1) < \mathsf{x}_{\boldsymbol{\rho}}(c,1,1)$ for any $1 \le c < n$. By definition, $\delta_n \ge 1-n$. Hence, by Lemma 4.24(a), $\mathsf{x}_{\boldsymbol{\rho}}(n,1,1) \le \widehat{\boldsymbol{\nu}}_{\mathsf{t},n} = \min \mathsf{X}_{\boldsymbol{\rho}}^{a_n,\delta_n}(n)$. In particular, by induction, the diagram $B_{\mathsf{t}}^{\mathrm{sing}}$ satisfies (a). Recalling that $\mathsf{x}_{\boldsymbol{\rho}}(n,1,1) < \mathsf{x}_{\boldsymbol{\rho}}(0,0,l)$ for any $l \in \{1,\dots,\ell\}$, we deduce that if $\delta_n \neq 0$ then s_n satisfies (b), and if $\delta_n = 0$ then s_n also satisfies (b) by Lemma 4.25(b). By the same argument, s_n crosses no solid or ghost strings ending in $\mathsf{X}_{\boldsymbol{\rho}}^{a_n,\delta_n}$ and similarly for the ghost string g_n ending in $\mathsf{X}_{\boldsymbol{\rho}}^{a_n,\delta_n+1}$, so (c) holds. Finally, condition (d) follows from Lemma 4.25 and the explicit construction of s_n. □

An example of a diagram $B_{\mathsf{t}}^{\mathrm{reg}}$ can be found in Figure 1. Composing the diagrams $B_{\mathsf{t}}^{\mathrm{reg}}$ and $B_{\mathsf{t}}^{\mathrm{sing}}$ from Proposition 4.29 and Proposition 4.31 we can now define a diagram B_{t} satisfying the conditions of Definition 3.43.

Definition 4.32. Let $B_{\mathsf{t}} := B_{\mathsf{t}}^{\mathrm{reg}} B_{\mathsf{t}}^{\mathrm{sing}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\omega})$.

Proposition 4.33. *The diagram $B_{\mathsf{t}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\omega})$ satisfies the assumptions of Definition* 3.43*.*

Proof. By construction, for $1 \le k \le n$ the diagram B_{t} has a solid string s_k of residue $\mathrm{res}_{\mathsf{t}}(k)$ from $(\mathsf{x}_{\boldsymbol{\rho}}(\gamma), 1)$ to $(\widehat{\boldsymbol{\nu}}_{\mathsf{t},\mathsf{t}(\gamma)}, \frac{1}{2})$ and from $(\widehat{\boldsymbol{\nu}}_{\mathsf{t},\mathsf{t}(\gamma)}, \frac{1}{2})$ to $\big(\mathsf{x}_{\boldsymbol{\rho}}(\mathsf{t}(\gamma),1,1), 0\big) = (\mathsf{x}_{\mathsf{t}}(\gamma), 0)$, so B_{t} satisfies Definition 3.43(a). The diagram B_{t} has no dots so it satisfies Definition 3.43(b). To prove that B_{t} satisfies Definition 3.43(c), it suffices to show that B_{t} does not have a generalised double crossing. By Lemma 4.30, the crossings in $B_{\mathsf{t}}^{\mathrm{reg}}$ are between strings that belong to the same region, while by Proposition 4.31, the crossings in $B_{\mathsf{t}}^{\mathrm{sing}}$ are between strings that end in different regions. Hence, B_{t} does not contain any generalised double crossings. □

An illustration of Definition 4.32 is given in Figure 1 and Figure 2 for $\ell = 4, e = 3, p = 2, d = 2, N = 241, \rho = (0, 33), \mathsf{t} =$ .

4.6. **Orbit diagrams.** Recall from Section 4.2 that σ is an automorphism of order p that acts on $\mathscr{P}_n^\ell$ and $\mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$. This section uses the diagrams from Definition 4.32 to construct diagrams indexed by orbits of tableaux under σ. Ultimately, this will make easier to compute the image under σ of the diagrams of the cellular basis.

Lemma 4.34. *Let $I \subseteq \mathbb{R}$ be an interval that is neither empty nor a singleton. We can find a map $f_I : \mathscr{N}_n^\ell \to I$ such that if $\gamma \in \mathscr{N}^{a,\delta}$ and $\gamma' \in \mathscr{N}^{a,\delta'}$, for $0 \le a < d$ and $-n \le \delta, \delta' \le n$, then*

$$f_I(\gamma) < f_I(\gamma') \iff \mathsf{x}_{\boldsymbol{\rho}}(\gamma) - \delta + \delta' < \mathsf{x}_{\boldsymbol{\rho}}(\gamma').$$

Proof. This is obvious because we can define $f_I'(\gamma) := \mathsf{x}_{\boldsymbol{\rho}}(\gamma) - \delta$, where $\gamma = (r, c, l)$ with $c - r = \delta$ and then suitably renormalise f_I' so that it has the required properties. □

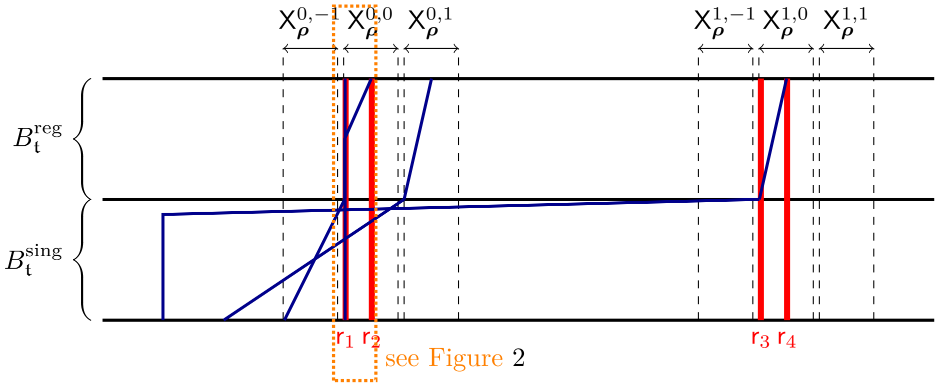


FIGURE 1. An example of a diagram $B_t = B_t^{\mathrm{reg}} B_t^{\mathrm{sing}}$ (the distance between r_2 and r_3 was shortened for expository means)

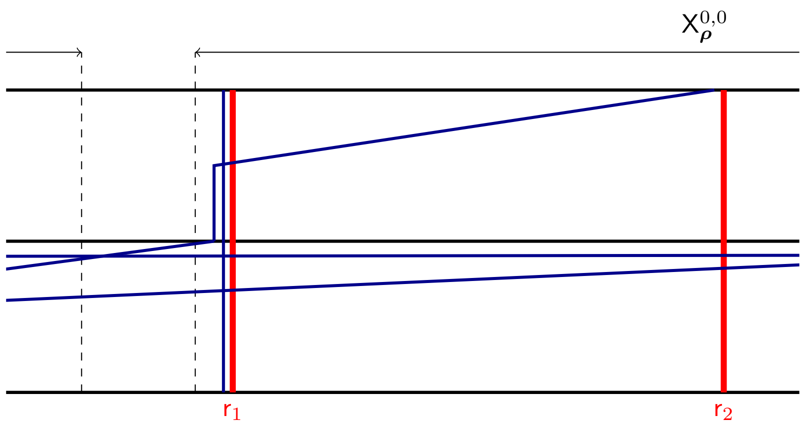


FIGURE 2. Focusing on the orange rectangle of Figure 1

Corollary 4.35. *Let $I \subseteq [0,1]$ be an interval that is neither empty nor a singleton. We can find two maps $f_I, g_I : \mathscr{N}_n^\ell \to I$ so that for any (a,δ) and any $\gamma, \gamma' \in \mathscr{N}_n^\ell$ with $\gamma' \in \mathscr{N}^{a,\delta}$ we have*

$$f_I(\gamma) < f_I(\gamma') \iff g_I(\gamma) > g_I(\gamma') \iff \mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma'), \qquad \text{if } \gamma \in \mathscr{N}^{a,\delta},$$
$$f_I(\gamma) < f_I(\gamma') \iff g_I(\gamma) > g_I(\gamma') \iff \mathsf{x}_{\boldsymbol{\rho}}(\gamma) + 1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma'), \quad \text{if } \gamma \in \mathscr{N}^{a,\delta-1}.$$

Proof. The existence of f_I follows directly from Lemma 4.34. The existence of g_I follows from the existence of the map $1 - f_{1-I}$. □

For each $\boldsymbol{\lambda} \in \mathscr{P}_n^\ell$ let $\mathsf{o}_{\boldsymbol{\lambda}} \in \{1, \dots, p\}$ be the order of $\boldsymbol{\lambda}$ under the action of $\langle \sigma \rangle$ on $\mathscr{P}_n^\ell$ and let $\mathscr{P}_{\sigma,n}^\ell$ be a fixed set of representatives in $\mathscr{P}_n^\ell$ under the action of σ. The integer $\mathsf{o}_{\boldsymbol{\lambda}}$ divides p, so $\mathsf{p}_{\boldsymbol{\lambda}} := \frac{p}{\mathsf{o}_{\boldsymbol{\lambda}}} \in \{1, \dots, p\}$ is an integer. The cyclic group $\mathbb{Z}/p_{\boldsymbol{\lambda}}\mathbb{Z} \cong \langle \sigma^{\mathsf{o}_{\boldsymbol{\lambda}}} \rangle$ generated by $\sigma^{\mathsf{o}_{\boldsymbol{\lambda}}}$ acts on the set $\mathrm{Std}(\boldsymbol{\lambda})$ of standard $\boldsymbol{\lambda}$-tableaux. Let $\mathrm{Std}_\sigma(\boldsymbol{\lambda})$ be a fixed set of $\mathbb{Z}/\mathsf{p}_{\boldsymbol{\lambda}}\mathbb{Z}$-orbit representatives with respect to this action.

Let $\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_n$ and fix $k \in \{1,\dots,p-1\}$. Define a decomposition of $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{[k]}_L \sqcup \boldsymbol{\lambda}^{[k]}_R$ by

$$\begin{aligned}\boldsymbol{\lambda}^{[k]}_L &:= \{\gamma \in \boldsymbol{\lambda} \mid \mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma)\}\\ &= \{\gamma = (r,c,l) \in \boldsymbol{\lambda} \mid l = ap+b+1, 0 \le a < p \text{ and } p-k \le b < p\},\\ \boldsymbol{\lambda}^{[k]}_R &:= \{\gamma \in \boldsymbol{\lambda} \mid \mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma) > \mathsf{x}_{\boldsymbol{\rho}}(\gamma)\}\\ &= \{\gamma = (r,c,l) \in \boldsymbol{\lambda} \mid l = ap+b+1, 0 \le a < p \text{ and } 0 \le b < p-k\}.\end{aligned}$$

In other words, the nodes in $\boldsymbol{\lambda}^{[k]}_L$ and $\boldsymbol{\lambda}^{[k]}_R$ move to the left and right, respectively, after applying σ^k. In the Example of Figure 3, we have $\gamma_1, \gamma_3 \in \boldsymbol{\lambda}^{[1]}_L$ and $\gamma_2 \in \boldsymbol{\lambda}^{[1]}_R$.

Now fix a real number $H \in (0,1)$. Several applications of Corollary 4.35 show that there exist families of real numbers $(x^{[k]}_\gamma)_{\gamma\in\boldsymbol{\lambda}}$, $(y^{[k]}_\gamma)_{\gamma\in\boldsymbol{\lambda}^{[k]}_R}$ and $(z^{[k]}_\gamma)_{\gamma\in\boldsymbol{\lambda}^{[k]}_R}$ such that:

- For any $\gamma, \gamma' \in \boldsymbol{\lambda}^{[k]}_L$ with $\gamma' \in \mathscr{N}^{a,\delta}$ we have
$$\frac{H}{2} < x^{[k]}_\gamma < H,$$
$$\text{if } \gamma \in \mathscr{N}^{a,\delta} \text{ then } \mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \iff x^{[k]}_\gamma < x^{[k]}_{\gamma'},$$
$$\text{if } \gamma \in \mathscr{N}^{a,\delta-1} \text{ then } \mathsf{x}_{\boldsymbol{\rho}}(\gamma)+1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \iff x^{[k]}_\gamma < x^{[k]}_{\gamma'}.$$
- For any $\gamma, \gamma' \in \boldsymbol{\lambda}^{[k]}_R$ with $\gamma' \in \mathscr{N}^{a,\delta}$ we have
$$0 < x^{[k]}_\gamma < \frac{H}{2},$$
$$\mathsf{x}_{\boldsymbol{\rho}}(0,0,l+k-1) - \frac{1}{N} \le y^{[k]}_\gamma < \mathsf{x}_{\boldsymbol{\rho}}(0,0,l+k-1), \text{ where } \gamma = (r,c,l),$$
$$H < z^{[k]}_\gamma < 1,$$
$$\text{if } \gamma \in \mathscr{N}^{a,\delta} \text{ then } \mathsf{x}_{\boldsymbol{\rho}}(\gamma) < \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \implies x^{[k]}_\gamma > x^{[k]}_{\gamma'} \text{ and } y^{[k]}_\gamma < y^{[k]}_{\gamma'} \text{ and } z^{[k]}_\gamma > z^{[k]}_{\gamma'},$$
$$\text{if } \gamma \in \mathscr{N}^{a,\delta-1} \text{ then } \mathsf{x}_{\boldsymbol{\rho}}(\gamma)+1 < \mathsf{x}_{\boldsymbol{\rho}}(\gamma') \implies x^{[k]}_\gamma > x^{[k]}_{\gamma'} \text{ and } y^{[k]}_\gamma < y^{[k]}_{\gamma'} \text{ and } z^{[k]}_\gamma > z^{[k]}_{\gamma'}.$$

Given two points (x_1, y_1) and (x_1, y_2), with $x_2 \ge x_1$ and $y_1 \le y_2$, let $(x_1,y_1) \rightsquigarrow (x_2,y_2)$ be the straight line string that:

$$\begin{cases}\text{goes from } (x_1,y_1) \text{ to } (x_2,y_2), & \text{if } y_1 \neq y_2,\\ \text{goes from } (x_1,y_1) \text{ to } (x_2,y_2+\epsilon), & \text{if } y_1 = y_2,\end{cases}$$

where ϵ is sufficiently small.

Definition 4.36. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_{\sigma,n}$ and $1 \le k < p$. Let $B^{[k]}_{\boldsymbol{\lambda}} \in \mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda}, \boldsymbol{\lambda})$ be the Webster diagram with the following solid strings:

- For any $\gamma \in \boldsymbol{\lambda}^{[k]}_L$, there is a solid string of residue $\operatorname{res}(\gamma)$ given by
$$(\mathsf{x}_{\boldsymbol{\rho}}(\gamma),0) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\gamma), x^{[k]}_\gamma) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma), x^{[k]}_\gamma) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma), 1).$$
- For any $\gamma \in \boldsymbol{\lambda}^{[k]}_R$, there is a solid string of residue $\operatorname{res}(\gamma)$ given by
$$(\mathsf{x}_{\boldsymbol{\rho}}(\gamma),0) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\gamma), x^{[k]}_\gamma) \rightsquigarrow (y^{[k]}_\gamma, x^{[k]}_\gamma) \rightsquigarrow (y^{[k]}_\gamma, z^{[k]}_\gamma) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma), z^{[k]}_\gamma) \rightsquigarrow (\mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma), 1).$$

An illustration of a diagram $B^{[k]}_{\boldsymbol{\lambda}}$ is given in Figure 3. Note the following result, which follows from Lemma 4.24.

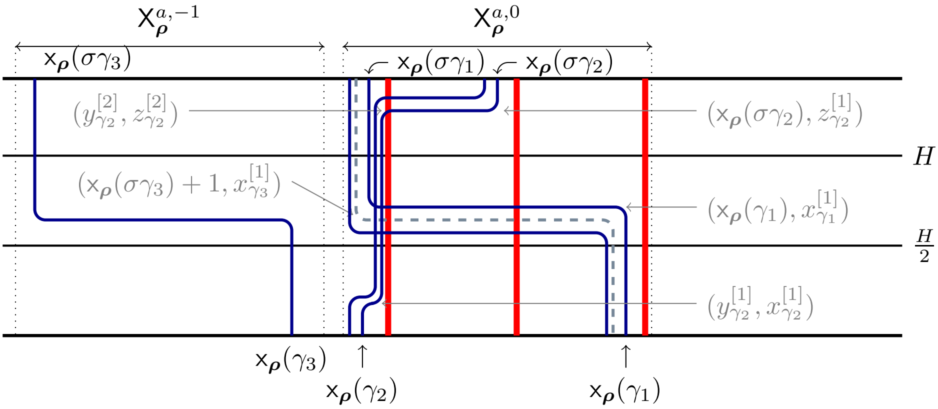


FIGURE 3. Illustration of a diagram $B^{[1]}_{\boldsymbol{\lambda}}$

Lemma 4.37. *Each string of $B^{[k]}_{\boldsymbol{\lambda}}$ is contained in a single region.*

The next result describes all of the crossings the diagrams $B^{[k]}_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_{n}$. Examples of crossings are depicted in Figure 4.

Proposition 4.38. *Let $\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_{\sigma,n}$ and $1 \le k < p$ and fix $0 \le a < p$ and $-n \le \delta \le n$. If $\gamma \in \boldsymbol{\lambda}$ let s_γ be the solid string in $B^{[k]}_{\boldsymbol{\lambda}}$ that starts at $\mathsf{x}_{\boldsymbol{\rho}}(\gamma)$ and let g_γ be its ghost. Let $\mathsf{r}_1, \dots, \mathsf{r}_\ell$ be the red strings in $B^{[k]}_{\boldsymbol{\lambda}}$.*

(a) *The diagram $B^{[k]}_{\boldsymbol{\lambda}} \in \mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda}, \boldsymbol{\lambda})$ does not contain any generalised double crossings.*

(b) *If $\gamma \in \boldsymbol{\lambda}^{[k]}_R \cap \boldsymbol{\lambda}^{a,\delta}$ then s_γ crosses $\mathsf{s}_{\gamma'}$ and $\mathsf{g}_{\gamma''}$, for all $\gamma' \in \boldsymbol{\lambda}^{[k]}_L \cap \boldsymbol{\lambda}^{a,\delta}$ and $\gamma'' \in \boldsymbol{\lambda}^{[k]}_L \cap \boldsymbol{\lambda}^{a,\delta-1}$, and g_γ crosses $\mathsf{s}_{\gamma'''}$, for all $\gamma''' \in \boldsymbol{\lambda}^{[k]}_L \cap \boldsymbol{\lambda}^{a,\delta+1}$. Moreover, all of these crossings are regular and every crossing between the solid and ghost strings in $B^{[k]}_{\boldsymbol{\lambda}}$ is of this form.*

(c) *If $\gamma = (r, c, l) \in \boldsymbol{\lambda}^{[k]}_R \cap \boldsymbol{\lambda}^{a,0}$ then the solid strings s_γ crosses the red string r_j if and only if $l \le j < l + k$. Moreover, the crossing of s_γ with r_l is singular and all of the other solid-red crossings involving s_γ are regular. The remaining solid strings from $\boldsymbol{\lambda}^{[k]}_R$ do not cross any red strings.*

(d) *If $\gamma = (r, c, l) \in \boldsymbol{\lambda}^{[k]}_R \cap \boldsymbol{\lambda}^{a,-1}$ then the ghost string g_γ crosses the red string r_j if and only if $l \le j < l + k$.*

(e) *If $\gamma = (r, c, l) \in \boldsymbol{\lambda}^{[k]}_L \cap \boldsymbol{\lambda}^{a,0}$ then s_γ crosses the red string r_j if and only if $l - p + k \le j < l$. Moreover, all these crossings are regular. The remaining solid strings from $\boldsymbol{\lambda}^{[k]}_L$ do not cross any red strings.*

(f) *If $\gamma = (r, c, l) \in \boldsymbol{\lambda}^{[k]}_L \cap \boldsymbol{\lambda}^{a,-1}$ then the ghost string g_γ crosses the red string r_j if and only if $l - p + k \le j < l$. The remaining ghost strings from $\boldsymbol{\lambda}^{[k]}_L$ do not cross any red strings.*

Proof. All the results follow directly from Definition 4.36 and Lemma 4.24 because:

- if $\gamma \in \boldsymbol{\lambda}^{[k]}_R$ and $\gamma' \in \boldsymbol{\lambda}^{[k]}_L$ then γ and γ' belong to different components of $\boldsymbol{\lambda}$, so if γ and γ' belong to the same $\boldsymbol{\lambda}^{a,\delta}$ then $\operatorname{res}(\gamma) \ne \operatorname{res}(\gamma')$ by Lemma 4.24;

- if $\gamma = (r, c, l) \in \boldsymbol{\lambda}_L^{[k]} \cap \boldsymbol{\lambda}^{a,0}$ then $l = ap + b + 1$ with $p - k \le b < p$ and $\operatorname{res}(\gamma) = q^{\kappa_l} \ne q^{\kappa_{l-i}} = \operatorname{res}(\mathsf{r}_{l-i})$ for all $i \in \{1, \dots, p-k\}$ since $ap + 1 \le l - i = ap + (b - i) + 1 \le ap + (b - 1) + 1$.

$\square$

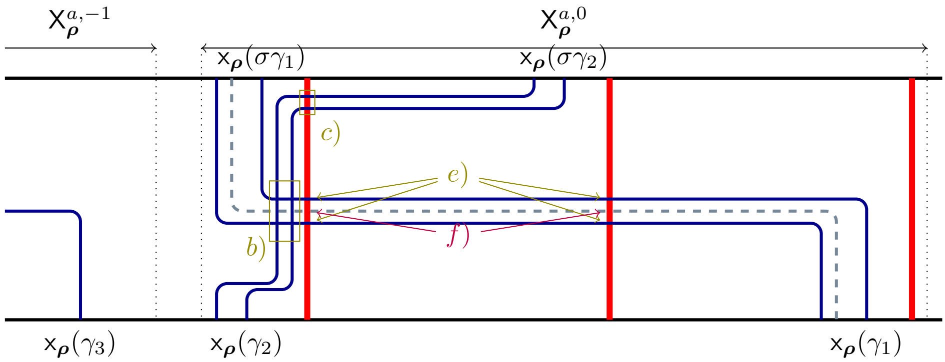


FIGURE 4. Illustration of Proposition 4.38 for a diagram $B_{\boldsymbol{\lambda}}^{[1]}$

Corollary 4.39. *There exist a generalised partition $\widehat{\boldsymbol{\nu}}_{\boldsymbol{\lambda}}^k$ and diagrams $B_{\boldsymbol{\lambda}}^{[k],\mathrm{sing}} \in \mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda}, \widehat{\boldsymbol{\nu}}_{\boldsymbol{\lambda}}^k)$ and $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\boldsymbol{\lambda}}^k, \boldsymbol{\lambda})$ such that $B_{\boldsymbol{\lambda}}^{[k]} = B_{\boldsymbol{\lambda}}^{[k],\mathrm{sing}} B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}}$, and any crossing in $B_{\boldsymbol{\lambda}}^{[k]}$ between a solid or ghost string from $\boldsymbol{\lambda}_R^{[k]}$ and a red string is contained in $B_{\boldsymbol{\lambda}}^{[k],\mathrm{sing}}$. In particular, the diagram $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}}$ is regular.*

Proof. Following Definition 4.36, define the diagram $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}}$ to be the subdiagram of $B_{\boldsymbol{\lambda}}^{[k]}$ that is below the line $y = H$ and, similarly, define $B_{\boldsymbol{\lambda}}^{[k],\mathrm{sing}}$ to be the subdiagram that is above this line. By Corollary 4.39, these two diagrams satisfy the requirements of the corollary. $\square$

Let $\boldsymbol{\lambda} \in \mathscr{P}_{\sigma,n}^{\ell}$, let $\mathfrak{t} \in \operatorname{Std}_\sigma(\boldsymbol{\lambda})$ and fix $k \in \{1, \dots, p-1\}$. Recall that we constructed a regular diagram $B_{\mathfrak{t}}^{\mathrm{reg}}$ in Proposition 4.29. The diagram $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}} B_{\mathfrak{t}}^{\mathrm{reg}}$ is non-zero but, in general, it can contain generalised double crossings. However, $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}}$ and $B_{\mathfrak{t}}^{\mathrm{reg}}$ are both regular diagrams so, by Proposition 4.2, there is a regular diagram $B_{\boldsymbol{\lambda},\mathfrak{t}}^{[k],\mathrm{reg}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}^k, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$ that does not have any generalised double crossings such that

$$B_{\boldsymbol{\lambda},\mathfrak{t}}^{[k],\mathrm{reg}} = B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}} B_{\mathfrak{t}}^{\mathrm{reg}} \quad \text{in } \mathbb{A}_n^{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}^k, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}}). \tag{4.40}$$

Lemma 4.41. *Let $\mathfrak{t} \in \operatorname{Std}_\sigma(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \in \mathscr{P}_{\sigma,n}^{\ell}$, and let $1 \le k < p$. Suppose that a solid i-string in $B_{\boldsymbol{\lambda},\mathfrak{t}}^{[k],\mathrm{reg}}$ crosses a red j-string. Then $j = \varepsilon^c i$, where $0 < c < k$. In particular, there are no solid-red crossings in $B_{\boldsymbol{\lambda},\mathfrak{t}}^{[k],\mathrm{reg}}$ when $k = 1$.*

Proof. By Proposition 4.38(d), the only solid-red crossings in $B_{\boldsymbol{\lambda}}^{[k],\mathrm{reg}}$ are between a solid string s_γ, for $\gamma = (r, c, l) \in \boldsymbol{\lambda}_L^{[k]} \cap \boldsymbol{\lambda}^{a,0}$, and the red strings $\mathsf{r}_{l+k-p}, \dots, \mathsf{r}_{l-1}$, which all belong to $\mathsf{X}_{\boldsymbol{\rho}}^{a,0}$. By Proposition 4.29 and Lemma 4.26, in $B_{\mathfrak{t}}^{\mathrm{reg}}$ the solid string s_γ crosses only the red strings $\mathsf{r}_{ap+1}, \dots, \mathsf{r}_{l-1}$. Since $\gamma \in \boldsymbol{\lambda}_L^{[k]}$ we have $l+k-p \ge$

$ap+1$. Therefore, since $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$ has no generalised double crossings, the only crossings that remain are for the red strings $\mathsf{r}_{ap+1},\dots,\mathsf{r}_{l+k-p-1}$. If $k=1$ this set is always empty since $l-p\le ap$, thus we now assume $k\ge 2$. We also assume $l+k-p>ap+1$, since the latter set of red strings is empty if $l+k-p=ap+1$. We now write $l=ap+b+1$ with $0\le b<p$. Since $l+k-p>ap+1$ we have $b+k-p>0$. Set $\alpha=q^{\kappa_{ap+1}}$, then the red strings have residues $\alpha,\varepsilon\alpha,\dots,\varepsilon^{b+k-p-1}\alpha$. As $i=\operatorname{res}(\gamma)=\varepsilon^b\alpha$ these residues can be written as $\varepsilon^{-b}i,\varepsilon^{1-b}i,\dots,\varepsilon^{k-p-1}i$, or as $\varepsilon^{p-b}i,\varepsilon^{p-b+1},\dots,\varepsilon^{k-1}i$ since ε has order p. As $0\le b<p$ the result follows. □

Remark 4.42. The statement of Lemma 4.41 also holds for ghost-red crossings.

Lemma 4.43. *The diagram $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$ has no generalised double crossings.*

Proof. By construction, neither of the diagrams $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$ and $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$ has a generalised double crossing, so it suffices to prove that any crossing that appears in $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$ does not appear in $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$. By construction, the Webster diagram $B^{[k]}_{\boldsymbol{\lambda}}=B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}$ has no generalised double crossings, thus it suffices to prove that any crossing in $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$ does not appear in $B^{\mathrm{reg}}_{\mathfrak{t}}$, since $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}=B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}B^{\mathrm{reg}}_{\mathfrak{t}}$. As $B^{\mathrm{reg}}_{\mathfrak{t}}$ is regular, we only need to consider the regular crossings in $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$. By Proposition 4.38(c), the only regular crossings in $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$ are crossings of the solid strings s_γ, for $\gamma=(r,c,l)\in\boldsymbol{\lambda}^{[k]}_R\cap\boldsymbol{\lambda}^{a,0}$, with the red strings $\mathsf{r}_{l+1},\dots,\mathsf{r}_{l+k-1}$. On the other hand, by Proposition 4.29 and Lemma 4.26 the corresponding solid strings in $B^{\mathrm{reg}}_{\mathfrak{t}}$ only cross the red strings $\mathsf{r}_{l'}$ for $l'<l$. This completes the proof. □

5. A skew cellular basis for $\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}$

We are finally ready to define the basis of $\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$ that we need to prove our main results. Recall from Section 4.3 that $\tilde\sigma$ is the automorphism on the set of Webster diagrams that multiplies the residues of all solid and ghost strings by ε and recall the definitions of the diagrams $B_{\mathfrak{t}}$, $B^{\mathrm{reg}}_{\mathfrak{t}}$ and $B^{\mathrm{sing}}_{\mathfrak{t}}$ from Section 4.5. For the readers' convenience, we summarise the relationships between the different diagrams we defined in the last chapter:

$$\begin{aligned}
B^{\mathrm{reg}}_{\mathfrak{t}}&\in\mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}) &&\text{(Proposition 4.29)},\\
B^{\mathrm{sing}}_{\mathfrak{t}}&\in\mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{t}},\boldsymbol{\omega}) &&\text{(Proposition 4.31)},\\
B_{\mathfrak{t}}=B^{\mathrm{reg}}_{\mathfrak{t}}B^{\mathrm{sing}}_{\mathfrak{t}}&\in\mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\lambda},\boldsymbol{\omega}) &&\text{(Definition 4.32)},\\
B^{[k]}_{\boldsymbol{\lambda}}=B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}&\in\mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda},\boldsymbol{\lambda}) &&\text{(Definition 4.36 and Corollary 4.39)},\\
B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}=B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}B^{\mathrm{reg}}_{\mathfrak{t}}&\in\mathbb{A}^{\boldsymbol{\rho}}_n(\widehat{\boldsymbol{\nu}}^k_{\mathfrak{t}},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}) &&\text{(4.40)}.
\end{aligned}$$

5.1. **A particular cellular basis of $\mathscr{R}^{\boldsymbol{\Lambda}}_n$.** To show that $\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}$ is a skew cellular algebra we first show that $\mathscr{R}^{\boldsymbol{\Lambda}}_n$ has a shift-automorphism. To do this we first use the diagrams defined in Section 4 to define a particular basis of $\{C_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathscr{R}^{\boldsymbol{\Lambda}}_n\cong\mathbb{A}^{\boldsymbol{\rho}}_n(\boldsymbol{\omega})$ that is compatible with Definition 3.43 and then show that $\sigma(C_{\mathfrak{s}\mathfrak{t}})=C_{\sigma(\mathfrak{s})\sigma(\mathfrak{t})}$, for all pairs $(\mathfrak{s},\mathfrak{t})$ of standard tableaux of the same shape.

Definition 5.1. Suppose that $\boldsymbol{\lambda}\in\mathscr{P}^{\ell}_{\sigma,n}$, $\mathfrak{t}\in\operatorname{Std}_\sigma(\boldsymbol{\lambda})$ and $0\le k<p$. Set

$$C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}\coloneqq\begin{cases}\tilde{\sigma}^k\big(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)\in\mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}), & \text{if } k\neq 0,\\ B^{\mathrm{reg}}_{\mathfrak{t}}, & \text{if } k=0,\end{cases}$$

so that $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}\in\mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$. Define

$$C_{\sigma^k\mathfrak{t}}\coloneqq C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}\circ\tilde{\sigma}^k\big(B^{\mathrm{sing}}_{\mathfrak{t}}\big)\in\mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda},\boldsymbol{\omega}).$$

Notice that the diagrams $\{C_{\sigma^k\mathfrak{t}}\mid 0\le k<p\}$ are in a single $\mathbb{Z}/p\mathbb{Z}$-orbit under $\tilde{\sigma}$. By definition, $C_{\mathfrak{t}}=C_{\sigma^k\mathfrak{t}}|_{k=0}=B_{\mathfrak{t}}$.

Lemma 5.2. *Let $\boldsymbol{\lambda}\in\mathscr{P}^{\ell}_{\sigma,n}$, $\mathfrak{t}\in\operatorname{Std}_\sigma(\boldsymbol{\lambda})$ and $0\le k<p$. Then the diagram $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}$ is regular.*

Proof. If $k=0$ then $C^{\mathrm{reg}}_{\mathfrak{t}}=B^{\mathrm{reg}}_{\mathfrak{t}}$ and the result follows from Proposition 4.33, so we can assume that $k>0$. By definition, since $\tilde{\sigma}$ is compatible with the concatenation of Webster diagrams we have $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}=\tilde{\sigma}^k\big(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}\big)\circ\tilde{\sigma}^k\big(B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)$. It suffices to prove that both $\tilde{\sigma}^k\big(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}\big)$ and $\tilde{\sigma}^k\big(B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)$ are regular. By Proposition 4.38(c), the only crossings in $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}$ are solid-red (and ghost-red) where the solid string s_γ, for $\gamma\in\boldsymbol{\lambda}^{[k]}_R$, has residue i and the red string has residue $\varepsilon^c i$ for $0\le c<k$. So, the only crossings in $\tilde{\sigma}^k\left(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}\right)$ are solid-red (and ghost-red) where the solid string has residue $\varepsilon^k i$ and the red string has residue $\varepsilon^c i$. In particular, $\tilde{\sigma}^k\big(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}\big)$ is a regular diagram since $0\le c<k<p$.

By definition, the diagram $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$ is regular, so to prove that $\tilde{\sigma}^k\big(B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)$ is regular it suffices to consider the solid-red crossings in $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$. By Lemma 4.41, if a solid $\varepsilon^k i$-string crosses a red j-string in $\tilde{\sigma}^k\big(B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)$ then $j=\varepsilon^{c'}i$, where $0<c'<k$. All of these crossing are regular, so the lemma is proved. $\square$

Proposition 5.3. *Let $\boldsymbol{\lambda}\in\mathscr{P}^{\ell}_{\sigma,n}$, $\mathfrak{t}\in\operatorname{Std}_\sigma(\boldsymbol{\lambda})$ and $0\le k<p$. Then the diagram $C_{\sigma^k\mathfrak{t}}$ satisfies the assumptions of Definition* 3.43.

Proof. When $k=0$ the result is just Proposition 4.33, so we can assume that $0<k<p$. By construction the diagram $C_{\sigma^k\mathfrak{t}}$ has no dots and a solid string of residue $\varepsilon^k\operatorname{res}(\gamma)=\operatorname{res}(\sigma^k\gamma)$ from $(\mathsf{x}_{\mathfrak{t}}(\gamma),0)$ to $(\mathsf{x}_{\boldsymbol{\rho}}(\sigma^k\gamma),1)$, for all $\gamma\in\boldsymbol{\lambda}$. Hence, $C_{\sigma^k\mathfrak{t}}$ satisfies parts (a) and (b) of Definition 3.43 and it remains to verify (c). That is, we need to show that $C_{\sigma^k\mathfrak{t}}$ does not contain a generalised double crossing. By Lemma 4.43 and Proposition 4.31, respectively, neither of the diagrams $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}$ and $\tilde{\sigma}^k\big(B^{\mathrm{sing}}_{\mathfrak{t}}\big)$ contains a generalised double crossing. Therefore, it suffices to prove that the diagrams $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}$ and $\tilde{\sigma}^k\big(B^{\mathrm{sing}}_{\mathfrak{t}}\big)$ do not have any crossings in common.

Recall that $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}=\tilde{\sigma}^k\big(B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}\big)$ in $\mathscr{W}_{\boldsymbol{\rho}}(\sigma^k\boldsymbol{\lambda},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$, where

$$B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}=B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}B^{\mathrm{reg}}_{\mathfrak{t}}\in\mathbb{A}^{\boldsymbol{\rho}}_n(\widehat{\boldsymbol{\nu}}^k_{\mathfrak{t}},\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}).$$

By Lemma 4.30 and Proposition 4.38, any crossing in $C^{\mathrm{reg}}_{\sigma^k\mathfrak{t}}$ is between two strings (solid, ghost or red) that begin in a same region $\mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}$. On the contrary, by Proposition 4.31 any two strings in $B^{\mathrm{sing}}_{\mathfrak{t}}$ that end in a same region $\mathsf{X}^{a,\delta}_{\boldsymbol{\rho}}$ do not cross, hence do not cross in $\tilde{\sigma}^k\big(B^{\mathrm{sing}}_{\mathfrak{t}}\big)$ either. This completes the proof. $\square$

Now we can apply Proposition 5.3 to Definition 3.43, Definition 3.44 and Theorem 3.47 to get a particular $\mathbb{Z}$-graded cellular basis $\{C_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathscr{R}_n^{\mathbf{\Lambda}}$. This particular $\mathbb{Z}$-graded cellular basis will play a key role in the proof of our main result in the next section.

5.2. **Main results: Proof of skew cellularity.** We are now ready to prove our main theorem from Section 1, which says that $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$ is a graded skew cellular algebra. As a consequence, we deduce that the (ungraded) Iwahori-Hecke algebras of type D are cellular algebras, under weaker assumption than in the literature.

We can now state the main technical result of this paper, which implies all our main results. Recall from Definition 2.22 that a shift automorphism of a graded cellular algebra A is a triple of maps $\sigma = (\sigma_A, \sigma_{\mathcal{P}}, \sigma_{\mathrm{Std}})$ that satisfies three requirements, the most important of which is that $\sigma_A(c_{\mathfrak{s}\mathfrak{t}}) = c_{\sigma_{\mathrm{Std}}(\mathfrak{s})\sigma_{\mathrm{Std}}(\mathfrak{t})}$, for all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $\lambda \in \mathcal{P}$. Recall the definitions of the map $\sigma_n^{\mathbf{\Lambda}}$ from Theorem 3.13 and the maps $\sigma_{\mathscr{P}}$ and σ_{Std} from Definition 4.5. The graded cellular structure that we consider on $\mathscr{R}_n^{\mathbf{\Lambda}}$ is the one described in Section 3.4, with the graded cellular basis obtained from Proposition 5.3.

Theorem 5.4. *The triple of maps $\sigma = (\sigma_n^{\mathbf{\Lambda}}, \sigma_{\mathscr{P}}, \sigma_{\mathrm{Std}})$ is a shift automorphism of $\mathscr{R}_n^{\mathbf{\Lambda}}$.*

Proof. First, note that by Corollary 4.10 we know that $\sigma_{\mathscr{P}}$ is a poset automorphism of $(\mathscr{P}_n^{\ell}, \trianglerighteq_{\boldsymbol{\rho}})$. By Definition 2.22, we need to show that:

(a) If $\mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $\sigma_{\mathrm{Std}}(\mathfrak{s}) \in \mathrm{Std}(\sigma_{\mathscr{P}}(\boldsymbol{\lambda}))$ and $\deg(\sigma_{\mathrm{Std}}(\mathfrak{s})) = \deg(\mathfrak{s})$.
(b) If $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $\sigma_{\mathscr{R}_n^{\mathbf{\Lambda}}}(c_{\mathfrak{s}\mathfrak{t}}) = c_{\sigma_{\mathrm{Std}}(s)\sigma_{\mathrm{Std}}(\mathfrak{t})}$.
(c) If $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathscr{P}_n^{\ell}$, then $\sigma_{\mathrm{Std}}^k(\mathfrak{t}) = \mathfrak{t}$ if and only if $\sigma_{\mathrm{Std}}^k(\mathfrak{s}) = \mathfrak{s}$, for $k \in \mathbb{Z}$.

The first requirement in part (a) is immediate from Definition 4.5 and the second requirement that the fact that σ_{Std} is homogeneous follows from Lemma 4.12. Part (c) follows because all tableaux have order p under the action of σ_{Std}. It remains to check part (b). That is, we need to show that

$$\sigma\left(C_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}\right) = C_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}, \qquad \text{for any } \boldsymbol{\lambda} \in \mathscr{P}_{\sigma,n}^{\ell},\ \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\sigma}(\boldsymbol{\lambda}) \text{ and } k, l \in \mathbb{Z}/p\mathbb{Z}.$$

Equivalently, we need to show that $\sigma\left(C_{\sigma^k\mathfrak{s}}^* C_{\sigma^l\mathfrak{t}}\right) = C_{\sigma^{k+1}\mathfrak{s}}^* C_{\sigma^{l+1}\mathfrak{t}}$. By Definition 5.1 and Lemma 4.14,

$$C_{\sigma^k\mathfrak{s}}^* = \tilde{\sigma}^k\left(B_{\mathfrak{s}}^{\mathrm{sing}}\right)^*\left(C_{\sigma^k\mathfrak{s}}^{\mathrm{reg}}\right)^*, \qquad C_{\sigma^l\mathfrak{t}} = C_{\sigma^l\mathfrak{t}}^{\mathrm{reg}}\,\tilde{\sigma}^l\left(B_{\mathfrak{t}}^{\mathrm{sing}}\right),$$
$$C_{\sigma^{k+1}\mathfrak{s}}^* = \tilde{\sigma}^{k+1}\left(B_{\mathfrak{s}}^{\mathrm{sing}}\right)^*\left(C_{\sigma^{k+1}\mathfrak{s}}^{\mathrm{reg}}\right)^*, \qquad C_{\sigma^{l+1}\mathfrak{t}} = C_{\sigma^{l+1}\mathfrak{t}}^{\mathrm{reg}}\,\tilde{\sigma}^{l+1}\left(B_{\mathfrak{t}}^{\mathrm{sing}}\right).$$

By Lemma 4.30 and Lemma 4.37, any string in $(C_{\mathfrak{s}}^{\mathrm{reg}})^* C_{\mathfrak{t}}^{\mathrm{reg}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{s}}, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$ stays inside a single region. Moreover, by Lemma 4.25, any solid or ghost string of $(C_{\mathfrak{s}}^{\mathrm{reg}})^* C_{\mathfrak{t}}^{\mathrm{reg}} \in \mathscr{W}_{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\mathfrak{s}}, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$ in the region $\mathsf{X}_{\boldsymbol{\rho}}^{a,0}$ starts (and ends) at the left of all the red strings in $\mathsf{X}_{\boldsymbol{\rho}}^{a,0}$, for $0 \leq a < d$. By Definition 5.1 and (4.40), this is also true in the two diagrams $\left(C_{\sigma^k\mathfrak{s}}^{\mathrm{reg}}\right)^* C_{\sigma^l\mathfrak{t}}^{\mathrm{reg}}$ and $\left(C_{\sigma^{k+1}\mathfrak{s}}^{\mathrm{reg}}\right)^* C_{\sigma^{l+1}\mathfrak{t}}^{\mathrm{reg}}$. As both of these diagrams are regular by Lemma 5.2, we can apply Proposition 4.2 to find two regular Webster diagrams $D_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}^{\mathrm{reg}}$ and $D_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}^{\mathrm{reg}}$ that do not contain any generalised double crossings such that in $\mathbb{A}_n^{\boldsymbol{\rho}}(\widehat{\boldsymbol{\nu}}_{\sigma^k\mathfrak{s}}, \widehat{\boldsymbol{\nu}}_{\sigma^l\mathfrak{t}})$

$$D_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}^{\mathrm{reg}} = \left(C_{\sigma^k\mathfrak{s}}^{\mathrm{reg}}\right)^* C_{\sigma^l\mathfrak{t}}^{\mathrm{reg}} \qquad \text{and} \qquad D_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}^{\mathrm{reg}} = \left(C_{\sigma^{k+1}\mathfrak{s}}^{\mathrm{reg}}\right)^* C_{\sigma^{l+1}\mathfrak{t}}^{\mathrm{reg}}$$

and all of the solid and ghost strings in any region $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ are to the left of all of the red strings in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$. Since $D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}$ is a regular diagram that has no crossings involving red strings, the diagram $\tilde{\sigma}(D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}})$ is also regular. Hence, the diagrams $\tilde{\sigma}(D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}})$ and $D^{\mathrm{reg}}_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}$ satisfy the assumptions of Corollary 4.3, and so $\tilde{\sigma}(D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}) = D^{\mathrm{reg}}_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}$ in $\mathbb{A}^{\boldsymbol{\rho}}_n(\widehat{\boldsymbol{\nu}}_{\sigma^k\mathfrak{s}},\widehat{\boldsymbol{\nu}}_{\sigma^l\mathfrak{t}})$. We have proved so far that

$$C_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}} = \tilde{\sigma}^k\left(B^{\mathrm{sing}}_{\mathfrak{s}}\right)^* D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}\,\tilde{\sigma}^l\left(B^{\mathrm{sing}}_{\mathfrak{t}}\right),$$
$$C_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}} = \tilde{\sigma}^{k+1}\left(B^{\mathrm{sing}}_{\mathfrak{s}}\right)^*\tilde{\sigma}\left(D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}\right)\tilde{\sigma}^{l+1}\left(B^{\mathrm{sing}}_{\mathfrak{t}}\right).$$

Consequently,

$$\tilde{\sigma}\left(C_{\sigma^k\mathfrak{s},\sigma^k\mathfrak{t}}\right) = C_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}}. \tag{5.5}$$

Recall that in $D^{\mathrm{reg}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}$, all of the solid and ghost strings that start or, equivalently, end in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ are to the left of all of the red strings in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$. Similarly, since $\left(B^{\mathrm{sing}}_{\mathfrak{s}}\right)^* B^{\mathrm{sing}}_{\mathfrak{t}} \in \mathscr{W}_{\boldsymbol{\rho}}(\boldsymbol{\omega},\boldsymbol{\omega})$, all the solid strings begin and end to the left of all red strings. Moreover by Proposition 4.31(b), in this diagram any solid string that crosses a red string in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$ must cross all of the red strings in $\mathsf{X}^{a,0}_{\boldsymbol{\rho}}$, for $0 \le a < d$. Thus, we can apply Proposition 4.19, which gives

$$\tilde{\sigma}\left(C^{\mathrm{cyc}}_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}\right) = \tilde{\sigma}\left(C_{\sigma^k\mathfrak{s},\sigma^l\mathfrak{t}}\right)^{\mathrm{cyc}}.$$

Combining (5.5) with Lemma 4.16 and Proposition 4.21, we find that

$$\sigma\left(C_{\sigma^k\mathfrak{s},\sigma^k\mathfrak{t}}\right) = C^{\mathrm{cyc}}_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}} = C_{\sigma^{k+1}\mathfrak{s},\sigma^{l+1}\mathfrak{t}},$$

as desired. □

Remark 5.6. A very particular case of Theorem 5.4 can be found in [3, Example 7.5]. Namely, for $\ell = e = 2$ and the 2-charge obtained from $(0,1)$, the basis elements are of the form

$c_{\mathfrak{s},\mathfrak{s}} = e(\mathbf{i})$	$c_{\sigma\mathfrak{s},\sigma\mathfrak{s}} = e(-\mathbf{i})$,
$c_{\mathfrak{t},\mathfrak{t}} = y_2 e(\mathbf{i})$	$c_{\sigma\mathfrak{t},\sigma\mathfrak{t}} = y_2 e(-\mathbf{i})$,
$c_{\sigma\mathfrak{t},\mathfrak{t}} = \psi_1 e(\mathbf{i})$	$c_{\mathfrak{t},\sigma\mathfrak{t}} = \psi_1 e(-\mathbf{i})$,
$c_{\mathfrak{u},\mathfrak{u}} = y_2^2 e(\mathbf{i})$	$c_{\sigma\mathfrak{u},\sigma\mathfrak{u}} = y_2^2 e(-\mathbf{i})$,

where $\mathfrak{s}$ (resp. $\mathfrak{u}$) is the only element in $\mathrm{Std}(1^2|0)$ (resp. $\mathrm{Std}(2|0)$), $\mathbf{i} \in \{(1,-1),(-1,1)\}$ and the tableau $\mathfrak{t}$ is a particular element of $\mathrm{Std}(1|1)$. Recall that σ fixes y_1, y_2, ψ_1 and that $\sigma\big(e(\mathbf{j})\big) = e(-\mathbf{j})$.

We can now prove our Main Theorem from Section 1.

Corollary 5.7. *Assume that R contains a primitive p-th root of unity and $p\cdot 1_R$ is invertible in R. The algebra $\mathscr{R}^{\boldsymbol{\Lambda}}_{p,n}$ is a graded skew cellular algebra. In particular, if $2\cdot 1_R \in R^\times$ then $\mathscr{R}^{\boldsymbol{\Lambda}}_{2d,2,n}$ is graded cellular.*

Proof. This is an immediate consequence of Theorem 5.4, Corollary 2.32 and Theorem 2.28 applied for the graded cellular algebra $\mathscr{R}^{\boldsymbol{\Lambda}}_n$. Note that $\sigma^{\boldsymbol{\Lambda}}_n$ has order p indeed. □

Combining the last result with Proposition 2.21 yields:

Corollary 5.8. *The graded decomposition matrix of $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ is unitriangular.*

We now explicitly describe the graded skew cellular datum of $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$, following the construction given in Theorem 2.28. Recall that $\mathscr{P}^{\ell}_{\sigma,n}$ is any set of representatives for the action of $\langle\sigma_{\mathscr{P}}\rangle$ on $\mathscr{P}^{\ell}_{n}$. We endow $\mathscr{P}^{\ell}_{\sigma,n}$ with the partial order $\trianglerighteq_{\boldsymbol{\rho}}$ so that $a \vartriangleright_{\boldsymbol{\rho}} b$ for $a,b\in\mathscr{P}^{\ell}_{\sigma,n}$ if and only if there exist $\boldsymbol{\lambda}\in a$ and $\boldsymbol{\mu}\in b$ such that $\boldsymbol{\lambda}\vartriangleright_{\boldsymbol{\rho}}\boldsymbol{\mu}$. Define

$$\mathscr{P}^{\ell}_{\sigma,p,n} := \mathscr{P}^{\ell}_{\sigma,n}\times \mathbb{Z}/\mathsf{o}_{\boldsymbol{\lambda}}\mathbb{Z}, \tag{5.9}$$

where $\mathsf{o}_{\boldsymbol{\lambda}}$ is the size of the orbit of $\boldsymbol{\lambda}$ under the action of $\langle\sigma_{\mathscr{P}}\rangle$. The order $\trianglerighteq_{\boldsymbol{\rho}}$ extends to an order of $\mathscr{P}^{\ell}_{\sigma,p,n}$ with the rule $(\boldsymbol{\lambda},k)\vartriangleright_{\boldsymbol{\rho}}(\boldsymbol{\mu},l) \iff \boldsymbol{\lambda}\vartriangleright_{\boldsymbol{\rho}}\boldsymbol{\mu}$. Let ι be the involution on $\mathscr{P}^{\ell}_{\sigma,p,n}$ given by $(\boldsymbol{\lambda},k)\mapsto(\boldsymbol{\lambda},-k)$. For $(\boldsymbol{\lambda},l)\in\mathscr{P}^{\ell}_{\sigma,p,n}$ let $\mathrm{Std}_{\sigma}(\boldsymbol{\lambda},k) := \mathrm{Std}_{\sigma}(\boldsymbol{\lambda})$. For $(\boldsymbol{\lambda},k)\in\mathscr{P}^{\ell}_{\sigma,p,n}$ and $\mathfrak{s},\mathfrak{t}\in\mathrm{Std}_{\sigma}(\boldsymbol{\lambda},k)$, define

$$D^{(k)}_{\mathfrak{s}\mathfrak{t}} := \sum_{j=0}^{\mathsf{p}_{\boldsymbol{\lambda}}-1} \varepsilon^{kj\mathsf{o}_{\boldsymbol{\lambda}}}\overline{\sigma}_A(C_{\mathfrak{s},\sigma^{j\mathsf{o}_{\boldsymbol{\lambda}}}\mathfrak{t}}),$$

where

$$\overline{\sigma}_A := \sum_{k=0}^{p-1}\sigma^k \quad\text{and}\quad \mathsf{p}_{\boldsymbol{\lambda}} = \frac{p}{\mathsf{o}_{\boldsymbol{\lambda}}}.$$

In particular, note that $\mathsf{o}_T(\boldsymbol{\lambda}) = \mathsf{p}_{\boldsymbol{\lambda}}$, since $\mathsf{p}=\mathsf{p}_{\mathcal{P}}=p$, so $\frac{p}{\mathsf{o}_T(\boldsymbol{\lambda})} = \frac{p}{\mathsf{p}_{\boldsymbol{\lambda}}} = \mathsf{o}_{\boldsymbol{\lambda}}$ and $\varepsilon_{\boldsymbol{\lambda}} = \varepsilon^{\mathsf{o}_{\boldsymbol{\lambda}}}$. Finally, define the map

$$D\colon \mathrm{Std}_{\sigma}(\boldsymbol{\lambda},k)\times\mathrm{Std}_{\sigma}(\boldsymbol{\lambda},k)\longrightarrow\mathscr{R}^{\mathbf{\Lambda}}_{p,n}; (\mathfrak{s},\mathfrak{t})\mapsto D^{(k)}_{\mathfrak{s}\mathfrak{t}},$$

and let $\iota^{(k)}_{\boldsymbol{\lambda}}$ be the identity map on $\mathrm{Std}_{\sigma}(\boldsymbol{\lambda},k)$. Then

$$\big(\mathscr{P}^{\ell}_{\sigma,p,n},\iota,\mathrm{Std}_{\sigma},D,\deg\big) \tag{5.10}$$

is a graded skew cell datum for $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$. In particular, $\mathscr{R}^{\mathbf{\Lambda}}_{n}$ has cell modules $C^{(k)}_{\boldsymbol{\lambda}}$, and simple modules the non-zero quotients $D^{(k)}_{\boldsymbol{\lambda}} = C^{(k)}_{\boldsymbol{\lambda}}/\operatorname{rad} C^{(k)}_{\boldsymbol{\lambda}}$, for $(\boldsymbol{\lambda},k)\in\mathscr{P}^{\ell}_{\sigma,p,n}$.

Note that the graded skew cell datum for $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$ restricts to give a graded skew cell data for the blocks $\mathscr{R}^{\mathbf{\Lambda}}_{p,\alpha}$ and that the graded decomposition matrix of $\mathscr{R}^{\mathbf{\Lambda}}_{p,\alpha}$ is unitriangular, for $\alpha\in\mathcal{I}^n_{\sigma}$.

Recall from Definition 3.7 that $\mathcal{I} = \{\varepsilon^j q^i \mid 0\le j<p \text{ and } 0\le i<e\}$. Define

$$z := \sum_{\mathbf{i}\in\mathcal{I}^n} i_1^{-1}e(\mathbf{i}) \in \mathscr{R}^{\mathbf{\Lambda}}_{n}.$$

Lemma 5.11. *The element $z\in\mathscr{R}^{\mathbf{\Lambda}}_{n}$ is homogeneous of degree 0, invertible and $\sigma(z)=\varepsilon z$.*

Proof. First, recall from Section 3.1 that e is finite thus $\mathcal{I}$ is finite and z is well-defined. We have $\deg z = 0$ since $\deg e(\mathbf{i}) = 0$ for all $\mathbf{i}\in\mathcal{I}^n$. Using the fact that $\{e(\mathbf{i}) : \mathbf{i}\in\mathcal{I}^n\}$ is a complete set of orthogonal idempotents we find that z is invertible in $\mathscr{R}_n^{\mathbf{\Lambda}}$ with inverse $\sum_{(i_1,\dots,i_n)\in\mathcal{I}^n} i_1 e(i_1,\dots,i_n)$. Finally, we have

$$\begin{aligned}\sigma(z) &= \sum_{(i_1,\dots,i_n)\in\mathcal{I}^n} i_1^{-1} e(\varepsilon i_1,\dots,\varepsilon i_n)\\ &= \sum_{(i_1,\dots,i_n)\in\mathcal{I}^n} \left(\varepsilon^{-1} i_1\right)^{-1} e(i_1,\dots,i_n)\\ &= \varepsilon z,\end{aligned}$$

which completes the proof. □

Corollary 5.12. *The automorphism σ is ε-splittable in the sense of Definition* 2.39*.*

Hence, all the results in Section 2.4 now apply to $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$. Note that multiplication by z induces an analogue of Lemma 5.11 for the blocks $\mathscr{R}_\alpha^{\mathbf{\Lambda}}$ of $\mathscr{R}_n^{\mathbf{\Lambda}}$, so the results in Section 2.4 also apply to the blocks $\mathscr{R}_{p,\alpha}^{\mathbf{\Lambda}}$ of $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$, for $[\alpha]\in\mathcal{I}_\sigma^n$.

By Theorem 3.14, all the results concerning the skew cellularity of $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$, over a ring R containing ε, can be deduced from $\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$ over the field K, which contains ε by assumption. For example, the following holds:

Corollary 5.13. *The algebra $\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$ is a graded skew cellular algebra. Moreover, if $p=2$ then $\mathscr{H}_{2,n}(q,\mathbf{Q}^{\vee\varepsilon})$ is a graded cellular algebra.*

Geck [11] has proved that the Iwahori–Hecke algebra of a finite Coxeter group is always an (ungraded) cellular algebra. By Corollary 5.13, the Iwahori–Hecke algebras of the Coxeter groups of types A_{n-1}, $B_n=C_n$, D_n and $I_2(n)$ are graded skew cellular algebras. (These Coxeter groups are the complex reflection groups of types $G(1,1,n), G(2,1,n), G(2,2,n)$ and $G(n,n,2)$, respectively.)

The Iwahori-Hecke algebras of types A_{n-1} and B_n are graded cellular algebras by [18]. For the algebras of types D_n and $I_2(n)$, Geck's proof of the cellularity of the Iwahori-Hecke algebras $\mathscr{H}_q(D_n)$ assumes that $q^{1/2}\in K$. Corollary 5.14 generalises Geck's result to the graded case and removes the assumption that $q^{1/2}\in K$.

Corollary 5.14. *Suppose* $\operatorname{char} K\neq 2$*. The Iwahori-Hecke algebra $\mathscr{H}_q(D_n)$ of type D_n is a $\mathbb{Z}$-graded cellular algebra.*

Proof. The Coxeter group of type D_n is the complex reflection group of type $G(2,2,n)$ in the Shephard–Todd classification. The result thus directly follows from Corollary 5.13 for $d=1$. □

It is an interesting open question whether the algebras $\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$ are graded cellular algebras when $p>2$ (and $n>2$ if $p=\ell$).

5.3. **Adjustment matrices.** As a final application we use the bases in this paper to describe "adjustment matrices" for the Hecke algebras of type $G(\ell,p,n)$, which relate decomposition matrices in different characteristics. For the Iwahori-Hecke algebras of finite Coxeter groups Geck and Rouquier used Lusztig's asymptotic Hecke algebra to show that adjustment matrices exist whenever the Iwahori-Hecke algebra is defined over a field of "good characteristic", which depends on the root system of the underlying Coxeter group; see [12, Table 1.4 and Theorem 3.6.3]. When $p=1$ we recover the results of [8, §5.6] but, even in the ungraded case, these results appear to be new when $p>2$.

Throughout this section fix $\mathbf{\Lambda}$ as in Section 3.2. If R is an integral domain let $\mathscr{R}^{\mathbf{\Lambda},R}_{p,n}$ be the quiver Hecke algebra over R with weight $\mathbf{\Lambda}$. Similarly, if M is an $\mathscr{R}^{\mathbf{\Lambda},R}_{p,n}$-module then we write $M = M^R$ to emphasize that M is an R-module. By Corollary 5.7, if R contains a primitive pth root of unity and $p \in R^\times$ then $\mathscr{R}^{\mathbf{\Lambda},R}_{p,n}$ is a skew cellular algebra with cell modules $C^{(k),R}_{\boldsymbol{\lambda}}$, for $(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}$. By Theorem 2.17, if R is a field, the graded simple $\mathscr{R}^{\mathbf{\Lambda},R}_{p,n}$-modules are shifts of the non-zero quotients $D^{(k),R}_{\boldsymbol{\lambda}} = C^{(k),R}_{\boldsymbol{\lambda}} / \operatorname{rad} C^{(k),R}_{\boldsymbol{\lambda}}$, for $(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}$.

Fix a field F of characteristic $c > 0$ that contains a primitive pth root of unity ε_F. In particular, this implies that c does not divide p. The subfield $F_{p,c} = \mathbb{Z}/c\mathbb{Z}(\varepsilon_F)$ of F is a splitting field for the p-th cyclotomic polynomial Φ_p over $\mathbb{Z}/c\mathbb{Z}$. By [25, Theorem 2.47], the field $F_{p,c}$ is the finite field with c^r elements, where $r > 0$ is minimal such that $c^r \equiv 1 \pmod p$.

Let $\pi : \mathbb{Z}[\varepsilon] \to F_{p,c}$ be the unique ring homomorphism determined by $\pi(\varepsilon) = \varepsilon_F$. Then $\mathfrak{p} = \ker \pi$ is a prime ideal of $\mathbb{Z}[\varepsilon]$ and the localisation $\mathcal{O} = \mathbb{Z}[\varepsilon]_{\mathfrak{p}}$ of $\mathbb{Z}[\varepsilon]$ at the prime ideal $\mathfrak{p}$ is a discrete valuation ring with maximal ideal $\mathfrak{p}\mathcal{O}$. The next result follows from elementary properties of localisation.

Lemma 5.15. *The residue field $\mathcal{O}/\mathfrak{p}\mathcal{O}$ of $\mathcal{O}$ is isomorphic to $F_{p,c}$.*

Consequently, if $M^{\mathcal{O}}$ is an $\mathscr{R}^{\mathbf{\Lambda},\mathcal{O}}_{p,n}$-module then, by base change, $M^{F_{p,c}} = F_{p,c} \otimes_{\mathcal{O}} M^{\mathcal{O}}$ is an $\mathscr{R}^{\mathbf{\Lambda},F_{p,c}}_{p,n}$-module and $M^F = F \otimes_{F_{p,c}} M^{F_{p,c}} \cong F \otimes_{\mathcal{O}} M^{\mathcal{O}}$ is an $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$-module.

Remark 5.16. By Theorem 2.17 every field is a splitting field for the algebra $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$, so there would be no loss of generality in assuming that $F = F_{p,c}$.

Let $\mathcal{Q} = \operatorname{Frac}(\mathcal{O}) = \mathbb{Q}(\varepsilon)$ be the field of fractions of $\mathcal{O}$. By Definition 2.6, $C^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \cong \mathcal{Q} \otimes_{\mathcal{O}} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ and $C^{(k),F}_{\boldsymbol{\lambda}} \cong F \otimes_{\mathcal{O}} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$, for $(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}$. Now recall from (2.13) that

$$\operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}} = \left\{ y \in C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}} \mid \phi^{(k)}_{\boldsymbol{\lambda}}(x, y) = 0 \text{ for all } x \in C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}} \right\}$$

and that $D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}} = C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}} / \operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ by Definition 2.16.

Lemma 5.17. *Let $(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}$. Then $D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ is an $\mathcal{O}$-free graded $\mathscr{R}^{\mathbf{\Lambda},\mathcal{O}}_{p,n}$-module. Moreover, $D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \cong \mathcal{Q} \otimes_{\mathcal{O}} D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ as graded $\mathscr{R}^{\mathbf{\Lambda},\mathcal{Q}}_{p,n}$-modules.*

Proof. Let $\{D^{(k)}_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}_\sigma(\boldsymbol{\lambda}, k)\}$ be the standard basis of the cell module $C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$. Since $\operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ is a pure $\mathcal{O}$-submodule of $C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$, $D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ is free as an $\mathcal{O}$-module. The final claim that $D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \cong \mathcal{O} \otimes_{\mathcal{O}} D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ is immediate because $\operatorname{rad} C^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \cong \mathcal{Q} \otimes_{\mathcal{O}} \operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ since $\mathcal{Q}$ is a field of fractions of $\mathcal{O}$. $\square$

Let $\mathscr{P}^\ell_{\sigma,p,n,0} = \{(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n} \mid D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \neq 0\}$. For convenience, set

$$E^{(k),F}_{\boldsymbol{\lambda}} = F \otimes_{\mathcal{O}} D^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}, \qquad \text{for } (\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}.$$

In general, if $(\boldsymbol{\lambda}, k) \in \mathscr{P}^\ell_{\sigma,p,n}$ then the $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$-modules $D^{(k),F}_{\boldsymbol{\lambda}}$ and $E^{(k),F}_{\boldsymbol{\lambda}}$, and $\operatorname{rad} C^{(k),F}_{\boldsymbol{\lambda}}$ and $F \otimes_{\mathcal{O}} \operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$, are *not* isomorphic. In particular, the $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$-module $E^{(k),F}_{\boldsymbol{\lambda}}$ is not necessarily irreducible.

Suppose that R is a field and let M^R be an $\mathscr{R}^{\mathbf{\Lambda},R}_{p,n}$-module. Recall from (2.20) that if $(\boldsymbol{\mu}, l) \in \mathscr{P}^\ell_{\sigma,p,n,0}$ and then the graded decomposition multiplicity of $D^{(l),R}_{\boldsymbol{\mu}}$

in M^R is

$$[M^R : D_{\boldsymbol{\mu}}^{(l),R}]_t = \sum_{s\in\mathbb{Z}} [M^R : D_{\boldsymbol{\mu}}^{(l),R}\langle s\rangle]\, t^s \quad \in \mathbb{N}[t, t^{-1}].$$

For $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n}$ and $(\boldsymbol{\mu}, l), (\boldsymbol{\nu}, m) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$ define graded decomposition numbers

$$d^{\mathcal{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t) = [C^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} : D^{(l),\mathcal{Q}}_{\boldsymbol{\mu}}]_t,$$
$$d^{F}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t) = [C^{(k),F}_{\boldsymbol{\lambda}} : D^{(l),F}_{\boldsymbol{\mu}}]_t,$$
$$\alpha^{F}_{(\boldsymbol{\mu},l)(\boldsymbol{\nu},m)}(t) = [E^{(l),F}_{\boldsymbol{\mu}} : D^{(m),F}_{\boldsymbol{\nu}}]_t.$$

Let $D_{\mathcal{Q}}(t) = \big(d^{\mathcal{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)\big)$, $D_F(t) = \big(d^{F}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)\big)$ and $A_F(t) = \big(\alpha^{F}_{(\boldsymbol{\mu},l)(\boldsymbol{\nu},m)}(t)\big)$ be the corresponding matrices, where $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n}$ and $(\boldsymbol{\mu}, l), (\boldsymbol{\nu}, m) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$. Then $D_{\mathcal{Q}}(t)$ and $D_F(t)$ are the graded decomposition matrices of $\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$ and $\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$, respectively. The matrix $A_F(t)$ is the **graded adjustment matrix** for these two algebras.

Let $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$ and $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$ be the Grothendieck groups of finitely generated graded $\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$- and $\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$-modules, respectively. Both Grothendieck groups are free $\mathbb{Z}[t, t^{-1}]$-modules where t acts by grading shift. If M is a module for one of these algebras let $[M]$ be the image of M in the corresponding Grothendieck group.

Lemma 5.18. *There is a unique abelian group homomorphism* $d^{\mathcal{Q}}_F : \operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n} \longrightarrow \operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$ *such that* $d^{\mathcal{Q}}_F[C^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}}] = [C^{(k),F}_{\boldsymbol{\lambda}}]$ *and* $d^{\mathcal{Q}}_F[D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}}] = [E^{(k),F}_{\boldsymbol{\lambda}}]$, *for* $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n}$.

Proof. Since $\mathcal{O}$ is a discrete valuation ring, for any $\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$-module $M^{\mathcal{Q}}$ there exists an $\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{O}}_{p,n}$-module $M^{\mathcal{O}}$, a full $\mathcal{O}$-lattice, such that $M^{\mathcal{Q}} \cong \mathcal{Q} \otimes_{\mathcal{O}} M^{\mathcal{O}}$. Define $M^F = F \otimes_{\mathcal{O}} M^{\mathcal{O}}$. The choice of $\mathcal{O}$-lattice is not unique but $[M^F]$ is independent of the choice of $\mathcal{O}$-lattice; compare with [10, Proposition 16.16]. Hence, we define $d^{\mathcal{Q}}_F[M^{\mathcal{Q}}] = [M^F]$. By Definition 2.6, if $(\boldsymbol{\nu}, m) \in \mathscr{P}^{\ell}_{\sigma,p,n}$ then $d^{\mathcal{Q}}_F[C^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}}] = [C^{(k),F}_{\boldsymbol{\lambda}}]$. Moreover, we have $d^{\mathcal{Q}}_F[D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}}] = [E^{(k),F}_{\boldsymbol{\lambda}}]$ because $E^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \cong D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}}$ by Lemma 5.17. Finally, this establishes the uniqueness of $d^{\mathcal{Q}}_F$ since $\{D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \mid (\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}\}$ is a basis of $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$. ∎

Let $\overline{} : \mathbb{Z}[t, t^{-1}] \longrightarrow \mathbb{Z}[t, t^{-1}]$ be the unique $\mathbb{Z}$-linear map such that $\overline{t^k} = t^{-k}$, for $k \in \mathbb{Z}$. Observe that $\dim_t M^* = \overline{\dim_t M}$ if M is a graded module. Hence, we can extend $\overline{}$ to a map of the Grothendieck $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},\mathcal{Q}}_{p,n}$ and $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$ by setting $\overline{[M]} = [M^*]$.

Proposition 5.19. *Let* $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n}$ *and* $(\boldsymbol{\mu}, l) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$. *Then*

(a) $\alpha^{F}_{(\boldsymbol{\mu},l)(\boldsymbol{\mu},l)}(t) = 1$,

(b) $\alpha^{F}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t) \neq 0$ *only if* $(\boldsymbol{\lambda}, k) \trianglerighteq_{\boldsymbol{\rho}} (\boldsymbol{\mu}, l)$,

(c) $\alpha^{F}_{(\boldsymbol{\lambda},-k)(\boldsymbol{\mu},-l)}(t) = \overline{\alpha^{F}_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)}$.

Moreover, $D^{(l),F}_{\boldsymbol{\mu}} \neq 0$ *if and only if* $(\boldsymbol{\mu}, l) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$.

Proof. By construction, $F \otimes_{\mathcal{O}} \operatorname{rad} C^{(k),\mathcal{O}}_{\boldsymbol{\lambda}}$ is an $\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$-submodule of $\operatorname{rad} C^{(k),F}_{\boldsymbol{\lambda}}$, so parts (a) and (b) are immediate from Proposition 2.21. For part (c), in $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$

we have

$$[E_{\boldsymbol{\lambda}}^{(k),F}] = \sum_{(\boldsymbol{\mu},l)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} \alpha^F_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)[D_{\boldsymbol{\mu}}^{(l),F}].$$

Taking duals and using Proposition 2.19 this becomes

$$\overline{[E_{\boldsymbol{\lambda}}^{(k),F}]} = \sum_{(\boldsymbol{\mu},l)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} \overline{\alpha^F_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)}[D_{\boldsymbol{\mu}}^{(-l),F}].$$

On the other hand, $\overline{[E_{\boldsymbol{\lambda}}^{(k),F}]} = \overline{d_F^{\mathscr{Q}}[D_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}}]} = d_F^{\mathscr{Q}}\overline{[D_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}}]} = d_F^{\mathscr{Q}}[D_{\boldsymbol{\lambda}}^{(-k),\mathscr{Q}}] = [E_{\boldsymbol{\lambda}}^{(-k),F}]$, where the second last equality comes from the same argument used in the proof of [10, 16.16] and Proposition 2.19, and the last equality from Lemma 5.18. This proves (c).

Finally, we prove that $D_{\boldsymbol{\lambda}}^{(k),F} \neq 0$ if and only if $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$. First, observe that if $D_{\boldsymbol{\lambda}}^{(k),F} \neq 0$ then $\operatorname{rad} C_{\boldsymbol{\lambda}}^{(k),F} \neq C_{\boldsymbol{\lambda}}^{(k),F}$, implying that $\operatorname{rad} C_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}} \neq C_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}}$. Hence, $D_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}} \neq 0$ and $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$. Conversely, if $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$ then $D_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}} \neq 0$ so that $E_{\boldsymbol{\lambda}}^{(k),F} \neq 0$. Hence, $E_{\boldsymbol{\lambda}}^{(k),F} \neq 0$ by part (a). ∎

We can now prove the main result of this section.

Theorem 5.20. *Suppose that F is a field of characteristic c and that F contains a primitive pth root of unity ε_F. Let $(\boldsymbol{\lambda}, k) \in \mathscr{P}^{\ell}_n$ and $(\boldsymbol{\mu}, l) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}$. Then*

$$d^F_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t) = \sum_{(\boldsymbol{\nu},m)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^{\mathscr{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\nu},m)}(t)\alpha^F_{(\boldsymbol{\nu},m)(\boldsymbol{\mu},l)}(t).$$

That is, $D_F(t) = D_{\mathscr{Q}}(t)A_F(t)$.

Proof. Using Lemma 5.18, and Proposition 2.21 twice,

$$\begin{aligned}
&\sum_{(\boldsymbol{\mu},l)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^F_{(\boldsymbol{\lambda},k)(\boldsymbol{\mu},l)}(t)[D_{\boldsymbol{\mu}}^{(l),F}] \\
&\qquad = [C_{\boldsymbol{\lambda}}^{(k),F}] = d_F^{\mathscr{Q}}[C_{\boldsymbol{\lambda}}^{(k),\mathscr{Q}}] \\
&\qquad = d_F^{\mathscr{Q}}\Big(\sum_{(\boldsymbol{\nu},m)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^{\mathscr{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\nu},m)}(t)[E_{\boldsymbol{\nu}}^{(m),\mathscr{Q}}]\Big) \\
&\qquad = \sum_{(\boldsymbol{\nu},m)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^{\mathscr{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\nu},m)}(t)[E_{\boldsymbol{\nu}}^{(m),F}] \\
&\qquad = \sum_{(\boldsymbol{\nu},m)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^{\mathscr{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\nu},m)}(t) \sum_{(\boldsymbol{\mu},l)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} \alpha^F_{(\boldsymbol{\nu},m)(\boldsymbol{\mu},l)}(t)[D_{\boldsymbol{\mu}}^{(l),F}] \\
&\qquad = \sum_{(\boldsymbol{\mu},l)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} \Big(\sum_{(\boldsymbol{\nu},m)\in\mathscr{P}^{\ell}_{\sigma,p,n,0}} d^{\mathscr{Q}}_{(\boldsymbol{\lambda},k)(\boldsymbol{\nu},m)}(t)\alpha^F_{(\boldsymbol{\nu},m)(\boldsymbol{\mu},l)}(t)\Big)[D_{\boldsymbol{\mu}}^{(l),F}].
\end{aligned}$$

Since $\{[D_{\boldsymbol{\mu}}^{(l),F}] \mid (\boldsymbol{\mu}, l) \in \mathscr{P}^{\ell}_{\sigma,p,n,0}\}$ is a basis of $\operatorname{Rep}\mathscr{R}^{\boldsymbol{\Lambda},F}_{p,n}$, the result follows by comparing the coefficient of $[D_{\boldsymbol{\mu}}^{(l),F}]$ on both sides of this equation. ∎

Let $\mathcal{F}$ be a field of characteristic zero that contains a primitive pth root of unity ε and let $\mathscr{H}^{\mathcal{F}}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$ be a Hecke algebra of type $G(\ell, p, n)$ over $\mathcal{F}$. Similarly, let $\mathscr{H}^{F}_{p,n}(q, \mathbf{Q}^{\vee\varepsilon})$ be the corresponding Hecke algebra over the field F, which contains a primitive pth root of unity ε_F. By Theorem 3.14, $\mathscr{H}^F_{p,n}(q,\mathbf{Q}^{\vee\varepsilon}) \cong \mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$, so $D_F = D_F(1)$ is the decomposition matrix of $\mathscr{H}^F_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$, where $D_F(1)$ is the graded decomposition matrix of $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$ evaluated at $t=1$. By Theorem 2.17(b), every field is a splitting field for $\mathscr{R}^{\mathbf{\Lambda},\mathcal{Q}}_{p,n}$, and $\mathscr{R}^{\mathbf{\Lambda},\mathcal{F}}_{p,n} \cong \mathcal{F}\otimes_{\mathcal{Q}}\mathscr{R}^{\mathbf{\Lambda},\mathcal{Q}}_{p,n}$, so $D_{\mathcal{Q}}(t)$ is the graded decomposition matrix of $\mathscr{R}^{\mathbf{\Lambda},\mathcal{F}}_{p,n}$. On the other hand, $\mathscr{H}^{\mathcal{F}}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon}) \cong \mathscr{R}^{\mathbf{\Lambda},\mathcal{F}}_{p,n}$ by Theorem 3.14, so $D_{\mathcal{F}} = D_{\mathcal{Q}}(1)$ is the decomposition matrix of $\mathscr{H}^{\mathcal{F}}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$. Finally, set $A_F = A_F(1)$. Hence, by Theorem 5.20 we obtain the following.

Corollary 5.21. *Suppose that $\mathcal{F}$ is a field of characteristic 0 containing a primitive pth root of unity ε and that F is a field of characteristic $c>0$ that contains a primitive pth root of unity ε_F. Then the decomposition matrix D_F of $\mathscr{H}^F_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$ factorises as $D_F = D_{\mathcal{F}}A_F$.*

Remark 5.22. This section does not really use the machinery of skew cellular algebras. Rather the results in this section follow from the fact that the skew cellular basis of Corollary 5.7 is defined over the ring $\mathbb{Z}[\varepsilon]$, which makes it easy to apply standard modular reduction arguments. The existence of adjustment matrices usually requires a delicate choice of modular system. The beauty of using KLR algebras is that we can work over $\mathbb{Z}[\varepsilon]$, which makes this result almost trivial.

5.4. **Graded simple modules.** Let F be a field. The algebra $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$ is a skew cellular algebra by Corollary 5.7, so $\{D^{(k),F}_{\boldsymbol{\lambda}}\langle s\rangle \mid (\boldsymbol{\lambda},k)\in\mathscr{P}^\ell_{\sigma,p,n,0} \text{ and } s\in\mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple modules by Theorem 2.17 (and Proposition 5.19). The aim of this section is to explicitly describe the set $\mathscr{P}^\ell_{\sigma,p,n,0}$.

By definition, $\mathscr{P}^\ell_{\sigma,p,n,0} = \{(\boldsymbol{\lambda},k)\in\mathscr{P}^\ell_{\sigma,p,n} \mid D^{(k),F}_{\boldsymbol{\lambda}}\neq 0\}$, where $\mathscr{P}^\ell_{\sigma,p,n} = \mathscr{P}^\ell_{\sigma,n}\times\mathbb{Z}/\mathsf{o}_{\boldsymbol{\lambda}}\mathbb{Z}$ and $\mathscr{P}^\ell_{\sigma,n}$ is a fixed set of representatives in $\mathscr{P}^\ell_n$ under the action of $\sigma_{\mathscr{P}}$.

For $\boldsymbol{\lambda}\in\mathscr{P}^\ell_n$ let $C^F_{\boldsymbol{\lambda}}$ be the corresponding cell module and $D^F_{\boldsymbol{\lambda}} = C^F_{\boldsymbol{\lambda}}/\operatorname{rad}C^F_{\boldsymbol{\lambda}}$ be a graded simple $\mathscr{R}^{\mathbf{\Lambda},F}_n$-module (or zero). By Corollary 5.12, $\sigma^{\mathbf{\Lambda}}_n$ is ε-splittable, so we can apply Theorem 2.48 to deduce the following:

Lemma 5.23. *Suppose that F is a field. Then $\mathscr{P}^\ell_{\sigma,p,n,0} = \{(\boldsymbol{\lambda},k)\in\mathscr{P}^\ell_{\sigma,p,n} \mid D^F_{\boldsymbol{\lambda}}\neq 0\}$.*

The simple $\mathscr{R}^{\mathbf{\Lambda},F}_n$-modules have been independently classified by Bowman [3] and Kerschl [21]. To state their result we first need some new notation.

Let $<_{\mathsf{x}}$ be the total order on the set of nodes $\mathscr{N}^\ell_n$ where $A<_{\mathsf{x}}B$ if $\mathsf{x}_{\boldsymbol{\rho}}(A)<\mathsf{x}_{\boldsymbol{\rho}}(B)$. Extending notation from Section 3.4, for $i\in\mathcal{I}$ let

$$\mathcal{A}_i(\boldsymbol{\lambda}) = \{A\in\mathcal{A}(\boldsymbol{\lambda}) \mid \operatorname{res}(A)=i\} \quad\text{and}\quad \mathcal{R}_i(\boldsymbol{\lambda}) = \{A\in\mathcal{R}(\boldsymbol{\lambda}) \mid \operatorname{res}(A)=i\}$$

be the sets of addable and removable i-nodes of $\boldsymbol{\lambda}\in\mathscr{P}^\ell_n$. If $A\in\mathcal{A}_i(\boldsymbol{\lambda})\cup\mathcal{R}_i(\boldsymbol{\lambda})$ define

$$d_A(\boldsymbol{\lambda}) = \#\{B\in\mathcal{A}_i(\boldsymbol{\lambda}) \mid A<_{\mathsf{x}}B\} - \#\{B\in\mathcal{R}_i(\boldsymbol{\lambda}) \mid A<_{\mathsf{x}}B\}.$$

Given a second node C with $A<_{\mathsf{x}}C$ set

$$d^C_A(\boldsymbol{\lambda}) = \#\{B\in\mathcal{A}_i(\boldsymbol{\lambda}) \mid A<_{\mathsf{x}}B<_{\mathsf{x}}C\} - \#\{B\in\mathcal{R}_i(\boldsymbol{\lambda}) \mid A<_{\mathsf{x}}B<_{\mathsf{x}}C\}.$$

A **good i-node** of $\boldsymbol{\lambda}$ is a removable i-node $A\in\mathcal{R}_i(\boldsymbol{\lambda})$ that is minimal node with respect to $<_{\mathsf{x}}$ such that $d_A(\boldsymbol{\lambda})\le 0$ and $d^C_A(\boldsymbol{\lambda})<0$ whenever $A<_{\mathsf{x}}C$ and $C\in\mathcal{R}_i(\boldsymbol{\lambda})$.

Definition 5.24. If $n > 0$ then the set of **$\boldsymbol{\rho}$-Uglov ℓ-partitions** of n is

$$\mathscr{U}^{\mathbf{\Lambda}}_{\boldsymbol{\rho},n} = \{\boldsymbol{\lambda} \in \mathscr{P}^{\ell}_{n} \mid \boldsymbol{\lambda} = \boldsymbol{\mu} \cup \{A\} \text{ where } A \text{ is a good } i\text{-node of } \boldsymbol{\lambda} \text{ for some } i \in \mathcal{I}\},$$

where $\mathscr{U}^{\mathbf{\Lambda}}_{\boldsymbol{\rho},0} = \mathscr{P}^{\ell}_{0}$.

For our particular choice of $\mathsf{x}_{\boldsymbol{\rho}}$-coordinate function, a special case of the results of Bowman [3] and Kerschl [21] is the following:

Theorem 5.25 (Bowman [3, Theorem B] and Kerschl [21, Main Theorem]). *Suppose that F is a field. Then $\{D_{\boldsymbol{\lambda}}\langle s\rangle \mid \boldsymbol{\lambda} \in \mathscr{U}^{\mathbf{\Lambda}}_{\boldsymbol{\rho},n}$ and $s \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple $\mathscr{R}^{\mathbf{\Lambda},F}_{n}$-modules.*

Hence, by Lemma 5.23 we obtain:

Corollary 5.26. *Suppose that F is a field. Then $\mathscr{P}^{\ell}_{\sigma,p,n,0} = \{(\boldsymbol{\lambda},k) \in \mathscr{P}^{\ell}_{\sigma,p,n} \mid \boldsymbol{\lambda} \in \mathscr{U}^{\mathbf{\Lambda}}_{\boldsymbol{\rho},n}\}$. That is, $\{D^{(k)}_{\boldsymbol{\lambda}}\langle s\rangle \mid (\boldsymbol{\lambda},k) \in \mathscr{P}^{\ell}_{\sigma,p,n}, \boldsymbol{\lambda} \in \mathscr{U}^{\mathbf{\Lambda}}_{\boldsymbol{\rho},n}$ and $s \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic graded simple $\mathscr{R}^{\mathbf{\Lambda},F}_{p,n}$-modules.*

Index of notation

Symbol	Description	Page
α	An $\mathcal{I}$-composition of n	531
$[\alpha]$	The σ-orbit of α	548
$\mathbb{A}^{\boldsymbol{\rho}}_{n}$	A rational Cherednik algebra	539
$\mathbb{A}^{\boldsymbol{\rho}}_{\alpha}$	A block of $\mathbb{A}^{\boldsymbol{\rho}}_{n}$	539
$\mathbb{A}^{\boldsymbol{\rho}}_{n}(\boldsymbol{\omega})$	The $\boldsymbol{\omega}$-weight space of $\mathbb{A}^{\boldsymbol{\rho}}_{n}$	542
$\mathbb{A}^{\boldsymbol{\rho}}_{[\alpha]}$	$= \bigoplus_{\beta\in[\alpha]} \mathbb{A}^{\boldsymbol{\rho}}_{\beta}$	548
$B^{\mathrm{reg}}_{\mathfrak{t}}$	A regular diagram indexed by $\mathfrak{t}$	553
$B^{\mathrm{sing}}_{\mathfrak{t}}$	A singular diagram indexed by $\mathfrak{t}$	554
$B_{\mathfrak{t}}$	The diagram $B^{\mathrm{reg}}_{\mathfrak{t}}B^{\mathrm{sing}}_{\mathfrak{t}}$	555
$B^{[k]}_{\boldsymbol{\lambda}}$	The diagram $B^{[k],\mathrm{sing}}_{\boldsymbol{\lambda}}B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}$	557
$B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda},\mathfrak{t}}$	The diagram $B^{[k],\mathrm{reg}}_{\boldsymbol{\lambda}}B^{\mathrm{reg}}_{\mathfrak{t}} \in \mathbb{A}^{\boldsymbol{\rho}}_{n}(\widehat{\boldsymbol{\nu}}^{k}_{\mathfrak{t}}, \widehat{\boldsymbol{\nu}}_{\mathfrak{t}})$	559
$\mathscr{C}^{\ell}_{n}$	The set of $\mathcal{I}$-compositions of n	531
$C_{\mathfrak{s}\mathfrak{t}}$	Cellular basis element of $\mathbb{A}^{\boldsymbol{\rho}}_{n}$	542
ε	A fixed pth root of unity	529
$\mathsf{E}_{\boldsymbol{\omega}}$	An idempotent in $\mathbb{A}^{\boldsymbol{\rho}}_{n}$	542
$\mathscr{H}_{n}(q,\mathbf{Q}^{\vee\varepsilon})$	Hecke algebra of type $G(\ell,1,n)$	529
$\mathscr{H}_{p,n}(q,\mathbf{Q}^{\vee\varepsilon})$	Hecke algebra of type $G(\ell,p,n)$	530
$\mathcal{I}$	The vertex set $\{\varepsilon^{j}q^{i} \mid i,j \in \mathbb{Z}\}$ for the quiver Γ	531
$\boldsymbol{\kappa}$	The multicharge $\boldsymbol{\kappa} = (\rho_1,\dots,\rho_d)$	533
$\boldsymbol{\lambda}$	An ℓ-partition of n	534
$n,p,d = \frac{\ell}{p}$	The parameters for $\mathscr{R}^{\mathbf{\Lambda}}_{p,n}$	529
$\mathscr{N}^{\ell}_{n}$	The set of nodes	534
o_{λ}	The size of the orbit of $\lambda \in \mathcal{P}$ under the action of $\sigma_{\mathcal{P}}$	518
$\mathcal{P}$	A poset	511
$\mathscr{P}^{\ell}_{n}$	The set of ℓ-partitions of n	534
$\mathscr{P}^{\ell}_{\sigma,n}$	A set of σ-orbit representatives in $\mathscr{P}^{\ell}_{n}$	556
$\mathscr{P}^{\ell}_{\sigma,p,n}$	The set $\mathscr{P}^{\ell}_{\sigma,n} \times \mathbb{Z}/\mathsf{o}_{\boldsymbol{\lambda}}\mathbb{Z}$	564
$\mathscr{P}^{\ell}_{\sigma,p,n,0}$	The set $\{(\boldsymbol{\lambda},k) \in \mathscr{P}^{\ell}_{\sigma,p,n} \mid D^{(k),\mathcal{Q}}_{\boldsymbol{\lambda}} \neq 0\}$	566
q	The Hecke parameter	530

Symbol	Description	Page
$\mathbf{Q}^{\vee\varepsilon}$	The cyclotomic parameters of $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$	529
$\operatorname{res}\gamma$	The residue of $\gamma \in \mathscr{N}_n^{\ell}$	536
$\mathscr{R}_n^{\mathbf{\Lambda}}$	The quiver Hecke algebra of type $G(\ell, 1, n)$	531
$\mathscr{R}_\alpha^{\mathbf{\Lambda}}$	A block of $\mathscr{R}_n^{\mathbf{\Lambda}}$	531
$\mathscr{R}_{p,n}^{\mathbf{\Lambda}}$	The quiver Hecke algebra of type $G(\ell, p, n)$	533
σ	An automorphism of $\mathscr{H}_n(q, \mathbf{Q}^{\vee\varepsilon})$ or $\mathscr{R}_n^{\mathbf{\Lambda}}$	530
$\sigma_{\mathcal{P}}$	A poset automorphism of $\mathcal{P}$	516
$\sigma_n^{\mathbf{\Lambda}}$	An automorphism of $\mathscr{R}_n^{\mathbf{\Lambda}}$	532
$\sigma_{\mathscr{P}}$	An automorphism of $\mathscr{P}_n^{\ell}$	546
σ_{Std}	An automorphism of $\mathrm{Std}(\mathscr{P}_n^{\ell})$	546
$\mathrm{Std}(\boldsymbol{\lambda})$	The set of standard tableaux of shape $\boldsymbol{\lambda}$	541
$\mathrm{Std}_\sigma(\boldsymbol{\lambda})$	A set of σ-orbit representatives of standard tableaux of shape $\boldsymbol{\lambda}$	555
$\mathfrak{s}, \mathfrak{t}$	Standard tableaux	541
$\widehat{\boldsymbol{\nu}}$	A generalised partition	536
$\widehat{\boldsymbol{\nu}}_{\mathfrak{t}}$	A generalised partition constructed from $\mathfrak{t}$	553
$\mathscr{W}_{\boldsymbol{\rho}}$	The set of (isotopy classes of) Webster diagrams	538
$\boldsymbol{\omega}$	The ℓ-partition $(1^n\mid\dots\mid 0) \in \mathscr{P}_n^{\ell}$	541
$\mathsf{x}_{\boldsymbol{\rho}}$	The loading function $\mathsf{x}_{\boldsymbol{\rho}} : \mathscr{N}_n^{\ell} \longrightarrow \mathbb{Q}$	534
$\rhd_{\boldsymbol{\rho}}$	The $\boldsymbol{\rho}$-dominance order on ℓ-partitions	537

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