NAMIKAWA-WEYL GROUPS OF AFFINIZATIONS OF SMOOTH NAKAJIMA QUIVER VARIETIES

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ABSTRACT. We give a description of the Namikawa-Weyl group of affinizations of smooth Nakajima quiver varieties based on combinatorial data of the underlying quiver, and compute some explicit examples. This extends a result of McGerty and Nevins for quiver varieties associated to Dynkin quivers.

CONTENTS

1. INTRODUCTION

1.1. **Poisson deformations of conical symplectic singularity.** We start by recalling the notion of conical symplectic singularity, first defined in [Bea00], and their deformations. Let X be a normal affine Poisson variety.

Definition 1.1. We say X *has symplectic singularities* if its smooth locus X^{reg} is a symplectic variety with symplectic form ω and there is a projective resolution of singularities $\rho : Y \to X$ such that $\rho^*\omega$ extends to a regular 2-form on Y. We say X is *conical* if there is a $\mathbb{C}^*$-action on X that contracts X to a point, and ω has positive weight under this action. We say ρ is a *symplectic resolution of singularities* if $\rho^*\omega$ extends to a symplectic form on Y.

Definition 1.2. Let X be a normal affine Poisson variety equipped with a contracting $\mathbb{C}^*$-action. A *graded Poisson deformation* of X is the data $(\mathcal{X}, B, j)$, where:

(1) $B = \bigoplus_{i \geq 0} B_i$ is a finitely generated positively graded $\mathbb{C}$-algebra, such that $B_0 = \mathbb{C}$.
(2) $\mathcal{X}$ is an affine Poisson variety equipped with a $\mathbb{C}^*$-action over $\mathrm{Spec}(B)$, and the structure morphism $\pi : \mathcal{X} \to \mathrm{Spec}(B)$ is $\mathbb{C}^*$-equivariant and flat.

(3) $j : X \xrightarrow{\sim} \pi^{-1}(0)$ is a $\mathbb{C}^*$-equivariant Poisson isomorphism, where $0 \in \mathrm{Spec}(B)$ corresponds to the maximal ideal $\bigoplus_{i>0} B_i$.

Let X be as in Definition 1.2 and $(\mathcal{X}, B, j), (\mathcal{X}', B', j')$ be two graded Poisson deformations. A *morphism* of graded Poisson deformations from $(\mathcal{X}, B, j)$ to $(\mathcal{X}', B', j')$ consists of $\mathbb{C}^*$-equivariant morphisms $\Phi : \mathcal{X} \to \mathcal{X}'$ and $f : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$, such that the following diagram is Cartesian

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{X}' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ \mathrm{Spec}(B) & \xrightarrow{f} & \mathrm{Spec}(B') \end{array}$$

and moreover, $j' = \Phi|_{\pi^{-1}(0)} \circ j$.

We say a graded Poisson deformation $(\mathcal{X}, B, j)$ is *universal* if for any graded Poisson deformation $(\mathcal{X}', B', j')$, there is a unique morphism of graded Poisson deformations from $(\mathcal{X}', B', j')$ to $(\mathcal{X}, B, j)$.

Let X be a conical symplectic variety, $\rho : Y \to X$ be a symplectic resolution of singularities.

Theorem 1.3 ([Nam11, Theorem 5.5]). *There is a commutative diagram*

(1.1)
$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle\pi_Y} & & \downarrow{\scriptstyle\pi_X} \\ H^2(Y, \mathbb{C}) & \xrightarrow{q} & \mathbb{A}^d \end{array}$$

where $d = \dim H^2(Y, \mathbb{C})$, π_X, π_Y are universal graded Poisson deformations of X and Y respectively, with $\pi_X^{-1}(0) = X, \pi_Y^{-1}(0) = Y$.

We write $\mathfrak{h}_X := H^2(Y, \mathbb{C})$, which is known to depend only on X. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be the codimension 2 symplectic leaves of X. The formal slice S_i to $\mathcal{L}_i$ is a Kleinian singularity, of type A, D or E. Let $\widehat{W}_i$ be the corresponding Weyl group and $\hat{\mathfrak{h}}_i^*$ be the root space. The fundamental group $\pi_1(\mathcal{L}_i)$ acts on $\widehat{W}_i$ and $\hat{\mathfrak{h}}_i^*$ by Dynkin diagram automorphisms. Define $W_i := (\widehat{W}_i)^{\pi_1(\mathcal{L}_i)}$ and $\mathfrak{h}_i := (\hat{\mathfrak{h}}_i^*)^{\pi_1(\mathcal{L}_i)}$. They are the Weyl groups and Cartan space corresponding to the folded Dynkin diagram by the automorphisms $\pi_1(\mathcal{L}_i)$.

Definition 1.4.

(1) The vector space $\mathfrak{h}_X$ is called the *Namikawa-Cartan space* of X.
(2) The direct product

$$W = \prod_i W_i,$$

where i runs over all the codimension 2 symplectic leaves of X, is called the *Namikawa-Weyl group* of X.

The Namikawa-Weyl group is important for the following reason.

Theorem 1.5 ([Nam10, Theorem 1.1]). *The map q in (1.1) is the quotient map of the action of W.*

The Namikawa-Cartan space $\mathfrak{h}_X$ has the following decomposition.

Theorem 1.6 ([Los22, Lemma 2.8]). *There is a vector space isomorphism $H^2(Y, \mathbb{C})$ = $H^2(X^{\mathrm{reg}}, \mathbb{C}) \oplus \bigoplus_i \mathfrak{h}_i$, where i run through the codimension 2 leaves of X.*

Remark 1.7. Let us explain the projection $H^2(Y, \mathbb{C}) \twoheadrightarrow \mathfrak{h}_i$, following [Nam11]. Let $\mathcal{L}_i$ be a codimension 2 symplectic leaf of X, and let $x \in \mathcal{L}_i$. Then, there is an analytic neighbourhood U of x in X, such that

(1) there is a Poisson isomorphism $U \cong S_i \times \Delta^{\dim X - 2}$, where $\Delta^{\dim X - 2}$ is the complex polydisc of dimension $\dim X - 2$;

(2) $\rho^{-1}(U) \cong \tilde{S}_i \times \Delta^{\dim X - 2}$, where $\tilde{S}_i$ is the minimal resolution of the Kleinian singularity S_i.

Take a class $\alpha \in H^2(Y, \mathbb{C})$, restrict it to $\rho^{-1}(U)$, and we get a class $[\alpha]|_i \in H^2(\tilde{S}_i, \mathbb{C})$. The latter is isomorphic to the root space $\hat{\mathfrak{h}}_i^*$. Namikawa showed [Nam11, Proposition 4.2] that $[\alpha]_i$ is invariant under the $\pi_1(\mathcal{L}_i)$ action, i.e. $[\alpha]_i \in \mathfrak{h}_i$. This is the projection of $[\alpha]$ to $\mathfrak{h}_i$.

1.2. Quiver varieties.

We recall the notion of quiver varieties, following [Nak94]. Let Q be a quiver which may contain edge loops, Q_0 be the set of vertices and Q_1 be the set of arrows. For $a \in Q_1$ let $t(a), h(a)$ denote the tail and head of a respectively. For $i, j \in Q_0$ let n_{ij} denote the number of arrows between i, j, regardless of the orientation. To each $i \in Q_0$ we associate a simple root α_i. We identify $\mathbb{Z}_{\geq 0}^{Q_0}$ with the root lattice and define the *Tits form* on $\mathbb{Z}_{\geq 0}^{Q_0}$ by

$$\mathbb{Z}_{\geq 0}^{Q_0} \times \mathbb{Z}_{\geq 0}^{Q_0} \to \mathbb{Z}, ((v_i)_i, (v_i')_i) = v_i v_i' \sum_{i \in Q_0} (2 - n_{ii}) - \sum_{i \neq j} v_i v_j' n_{ij}.$$

Let W_Q be the Weyl group of Q generated by reflections on the root space $\mathbb{Q}^{Q_0} = \mathrm{Span}_{\mathbb{Q}}\{\alpha_i | i \in Q_0\}$ along real simple roots, i.e.

$$s_i : \alpha \mapsto \alpha - (\alpha, \alpha_i)\alpha_i$$

for a vertex i that does not carry an edge loop.

For any quiver Q, let $\overline{Q}$ denote its *double quiver*. By definition, it has the same vertex set as Q, and the set of arrows is $\overline{Q}_1 = \{a, a^* | a \in Q_1\}$, where $t(a^*) = h(a), h(a^*) = t(a)$.

For $v \in \mathbb{Z}_{\geq 0}^{Q_0}$, the following definition will be frequently used:

$$p(v) := 1 - \frac{1}{2}(v, v).$$

Let $v, w \in \mathbb{Z}_{\geq 0}^{Q_0}$; let V_i, W_i be vector spaces with $\dim V_i = v_i$ and $\dim W_i = w_i$. Define the coframed representation space

$$(1.2) \qquad R(Q, v, w) := \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i).$$

We omit w when $w = 0$ and write R for $R(Q, v, w)$ when Q, v, w are clear from the context. The cotangent bundle $T^*R = R \oplus R^*$ carries a natural symplectic vector space structure; it can also be viewed as the representation space $R(\overline{Q}, v, w)$ of the double quiver. The group $G = \mathrm{GL}(v) := \prod_{i \in Q_0} \mathrm{GL}(V_i)$ acts on R naturally; this induces a Hamiltonian G-action on T^*R. Let $\mathfrak{g} := \prod_{i \in Q_0} \mathfrak{gl}(v_i)$ be the Lie algebra of G. We G-equivariantly identify $\mathfrak{g} \cong \mathfrak{g}^*$ via trace pairing. Then, the moment map $\mu : T^*R \to \mathfrak{g}$ for the Hamiltonian G-action on T^*R can be written as follows. Let $(x, y, p, q) = (x_a, y_a, p_i, q_i) \in T^*R$ where

$x_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)}), y_a \in \mathrm{Hom}(V_{h(a)}, V_{t(a)}), q_i \in \mathrm{Hom}(V_i, W_i), p_i \in \mathrm{Hom}(W_i, V_i),$

$a \in Q_1, i \in Q_0$. Then

$$\mu((x_a, y_a, p_i, q_i)) = [x, y] - pq.$$

More precisely, the i-component of the right hand side is

$$\sum_{h(a)=i} x_a y_a - \sum_{t(a)=i} y_a x_a - p_i q_i.$$

Let θ be a character of G; it has the form $\theta(g) = \prod_{i \in Q_0} \det(g_i)^{\theta_i}$, $g_i \in \mathrm{GL}(v_i)$, $\theta_i \in \mathbb{Z}$. We can thus identify $\mathbb{Z}^{Q_0}$ and the character lattice of G, and write $\theta \in \mathbb{Z}^{Q_0}$. We write $\theta > 0$ if $\theta_i > 0$ for all $i \in Q_0$. Let $(T^*R)^{\theta-ss}$ denote the set of θ-semistable points of T^*R. If $\theta_i > 0$ for all i then (x, y, p, q) being semistable is equivalent to $\ker q$ having no non-zero subspace stable under x, y.

We write $\mathfrak{p} := (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$, which is identified with $\mathbb{C}^{Q_0}$. Let $\lambda \in \mathfrak{p}$. The GIT quotient

$$\mathcal{M}_\lambda^\theta(Q, v, w) := (\mu^{-1}(\lambda)^{\theta-ss})//G$$

is called the Nakajima quiver variety. It inherits a Poisson variety structure from the symplectic structure of T^*R. We omit the letter Q when it is clear from context, and omit the letter w when $w = 0$. The variety $\mathcal{M}_\lambda^0(v, w)$ is affine, and we have a projective morphism $\rho : \mathcal{M}_\lambda^\theta(v, w) \to \mathcal{M}_\lambda^0(v, w)$.

For $\theta \in \mathbb{Z}^{Q_0}, \lambda \in \mathfrak{p}$, the pair (θ, λ) is said to be *generic* if there are no positive roots $v' < v$ of Q such that $\lambda \cdot v' = \theta \cdot v' = 0$. We say λ is generic if $(0, \lambda)$ is generic and we say θ is generic if $(\theta, 0)$ is generic. When (θ, λ) is generic, G acts freely on $(T^*R)^{\theta-ss}$ by [Nak98, Section 3.ii], and $\mathcal{M}_\lambda^\theta(v, w)$ is smooth.

We can also define families of Nakajima quiver varieties, i.e.

$$\mathcal{M}_\mathfrak{p}^\theta(v, w) := (\mu^{-1}(\mathfrak{p})^{\theta-ss})//G.$$

It is a scheme over $\mathfrak{p}$, and each $\mathcal{M}_\lambda^\theta(v, w)$ is the fiber over $\lambda \in \mathfrak{p}$.

Let $\Lambda_w = \sum w_i \varpi_i$ where ϖ_i is the fundamental weight corresponding to the simple root α_i. We will frequently use the weight ν defined by

$$\nu = \Lambda_w - \sum_{i \in Q_0} v_i \alpha_i.$$

Assumption 1.8. Throughout the paper, we shall assume $v_i \neq 0$ for all $i \in Q_0$, unless otherwise specified. In fact, if $v_i = 0$ for some i, then we can view $\mathcal{M}_\lambda^\theta(Q, v, w)$ as $\mathcal{M}_\lambda^\theta(Q', v, w')$ where Q' is the subquiver of Q obtained by deleting i and all arrows adjacent to it, and w' is the restriction of w to $\mathbb{Z}_{\geq 0}^{Q_0 \setminus \{i\}}$.

Lemma 1.9 ([BL21, Section 2.1.8]). *For generic θ, $\mathbb{C}[\mathcal{M}_\lambda^\theta(v, w)]$ is a finitely generated algebra independent of θ.*

Lemma 1.9 enables us to make Definition 1.10.

Definition 1.10. For $\lambda \in \mathfrak{p}$, we set $\mathcal{M}_\lambda(v, w) := \mathrm{Spec}(\mathbb{C}[\mathcal{M}_\lambda^\theta(v, w)])$ for generic θ. We say $\mathcal{M}_\lambda(v, w)$ is the *affinization* of $\mathcal{M}_\lambda^\theta(v, w)$.

The proof of [BL21, Proposition 2.3] generalizes to show that, for generic θ, the natural map $p : \mathcal{M}_\lambda^\theta(v, w) \to \mathcal{M}_\lambda(v, w)$ is a symplectic resolution of singularities. Therefore the variety $\mathcal{M}_0(v, w)$ is a conical symplectic singularity, Definition 1.1. The following observation will play an important role.

Example 1.11. The morphism

$$\mathcal{M}_{\mathfrak{p}}(v, w) := \operatorname{Spec} \mathbb{C}[\mathcal{M}_{\mathfrak{p}}^{\theta}(v, w)] \to \mathfrak{p}$$

gives a graded Poisson deformation of $\mathcal{M}_0(v, w)$.

1.3. **Namikawa-Weyl group of quiver varieties.** The goal of this paper is to describe the Namikawa-Weyl group of quiver varieties $\mathcal{M}_0(v, w)$. Let us first record some known special cases. Assume Q is a simply-laced Dynkin quiver and $\nu = \Lambda_w - \sum v_i \alpha_i$ is a dominant weight. In this case it is known that $\mathcal{M}_0(v, w) \cong \mathcal{M}_0^0(v, w)$, see Corollary 2.13. McGerty and Nevins in [MN19] described the Namikawa-Weyl group of $\mathcal{M}_0^0(v, w)$ as follows. Let Φ be the set of roots, $\Phi_\nu = \{\alpha \in \Phi | \langle \nu, \alpha \rangle = 0\}$ and $\Phi_\nu^{\max}$ be the maximal elements of Φ_ν with respect to the usual partial order on Φ.

Proposition 1.12 ([MN19, Lemma 5.1])**.** *The set of codimension 2 symplectic leaves of $\mathcal{M}_0^0(v, w)$ is in bijection with $\Phi_\nu^{\max}$.*

Proposition 1.13 ([MN19, Theorem 5.4])**.** *The Namikawa-Weyl group of $\mathcal{M}_0^0(v, w)$ is W_ν, where W_ν is the Weyl group of the subroot system Φ_ν of Φ.*

The group W_ν coincides with the stabilizer of ν in W_Q.
Theorem 1.14 summarizes the main results of this paper.

Theorem 1.14. *The following are true.*
   (1) *Weyl groups of types $A_n, B_n, D_n, E_6, E_7, E_8, G_2$ can appear as components of the Namikawa-Weyl group of the affinization of some quiver variety $\mathcal{M}_0(v, w)$.*
   (2) *Weyl groups of types $C_n, n \geq 3$, and F_4 cannot appear as components of the Namikawa-Weyl group of the affinization of any quiver variety $\mathcal{M}_0(v, w)$.*
   (3) *Weyl groups of types B_n, G_2 can only appear when the underlying quiver is wild (i.e. not finite type or affine type).*

Part (1) follows from Proposition 1.13, Example 3.7 and Example 3.6. Part (2) is proved in Proposition 3.8 and Proposition 3.9. Part (3) is proved in Proposition 6.10.

In addition, we give a description of the Namikawa-Weyl group in Corollary 6.13 when the underline quiver is affine type, which turns out to be similar to Proposition 1.13.

1.4. **Structure of the paper.** In Section 2 we present more results on quiver varieties. In Section 3, we introduce tautological line bundles on quiver varieties, and use them to classify all the possible components of the Namikawa-Weyl group of the affinization of a quiver variety. To carry out the methods in Section 3, we must know certain information of codimension 2 leaves of the quiver varieties (more precisely, Theorem 2.20), and this is usually difficult. Therefore, we take a different approach. In Section 4, we examine the existence of codimension 2 symplectic leaves of deformations of $\mathcal{M}_0(v, w)$ along a subgeneric parameter, and fully classify them in Theorem 4.4. In Section 5, we recall general results about the presence of codimension 2 symplectic leaves in deformations of conical symplectic singularities. In Section 6, we use the results from the previous sections to compute the Namikawa-Weyl groups of some quiver varieties.

2. More on quiver varieties

In this subsection we present more results on quiver varieties. An important goal is to reduce our problem to Assumption 2.18, which largely simplifies subsequent computations.

2.1. Framed vs. non-framed quivers.

The following notations are given in [CB01, Section 1, Remarks]. Given a quiver Q, let Q^∞ be the *extended quiver* with $Q_0^\infty = Q_0 \cup \{\infty\}$; the arrows between vertices in Q_0 are the same as those in Q_1, plus w_i arrows from the vertex i to the vertex ∞. We denote the new simple root associated to the vertex ∞ by α_∞.

Given Q, v, w, we define the extended dimension vector $\tilde{v} \in \mathbb{Z}_{\geq 0}^{Q_0^\infty}$ by $\tilde{v}_\infty = 1$ and $\tilde{v}_i = v_i$ for $i \in Q_0$. It is clear that $R(Q, v, w) = R(Q^\infty, \tilde{v})$ as in (1.2), and $\mathcal{M}_\lambda^\theta(Q, v, w) = \mathcal{M}_{\tilde{\lambda}}^{\tilde{\theta}}(Q^\infty, \tilde{v})$, where $\tilde{\lambda}_\infty = -v \cdot \lambda$, $\tilde{\lambda}_i = \lambda_i$ for $i \in Q_0$; $\tilde{\theta}_\infty = -\theta \cdot v, \tilde{\theta}_i = \theta_i$ for $i \in Q_0$.

By definition, $\langle \nu, v' \rangle = -(\tilde{v}, v')$ for all $v' \in \mathbb{Z}^{Q_0}$. We will use these two notions interchangeably.

2.2. Stratification of affine quiver varieties.

The quiver variety $\mathcal{M}_\lambda^0(v, w)$, which is isomorphic to $\mathcal{M}_{\tilde{\lambda}}^0(\tilde{v})$, has a stratification by symplectic leaves, which we describe below. Let $x \in \mathcal{M}_\lambda^0(v, w)$ and $r \in T^*R$ be a representative of x (with closed G-orbit).

Definition 2.1. Suppose $r = r_0 \oplus r_1^{\oplus n_1} \oplus \cdots \oplus r_k^{\oplus n_k}$, where r_i's are pairwise non-isomorphic simple representations of $\overline{Q^\infty}$, and $\dim(r_0)_\infty = 1$. Write $v^i = \dim r_i \in \mathbb{Z}_{\geq 0}^{Q_0^\infty}$. We say
$$\tau = (v^0, 1; v^1, n_1; v^2, n_2; \ldots; v^k, n_k)$$
is the *representation type* of x.

Proposition 2.2 ([Nak98, Section 3.v])**.** *The Poisson variety $\mathcal{M}_\lambda^0(v, w)$ has finitely many symplectic leaves. The symplectic leaf containing x consists of all elements that has the same representation type as x.*

Let us discuss what dimension vector v^i can appear in a representation type τ. For $\tilde{\lambda} \in \mathbb{C}^{Q_0^\infty}$, define the *deformed preprojective algebra*
$$\Pi^{\tilde{\lambda}} = \mathbb{C}\overline{Q^\infty}/(\sum_{a \in Q_1^\infty} [a, a^*] - \sum_{i \in Q_0^\infty} \tilde{\lambda}_i e_i),$$

where $\mathbb{C}\overline{Q^\infty}$ is the path algebra of the double quiver of Q^∞, and a^* is the opposite arrow to a in the double quiver. For details see [CBH98, Section 2]. An element in $\mu^{-1}(\lambda) \subset T^*R$ is the same as a representation of $\Pi^{\tilde{\lambda}}$. We will frequently use the following results.

Theorem 2.3 ([CB01, Theorem 1.2])**.** *For $v \in \mathbb{Z}_{\geq 0}^{Q_0^\infty}$, the following conditions are equivalent.*

(1) *There is a simple representation of $\Pi^{\tilde{\lambda}}$ (i.e. an element in $\mu^{-1}(\tilde{\lambda})$) with dimension vector v.*

(2) *v is a positive root of Q^∞, $\tilde{\lambda} \cdot v = 0$, and for any decomposition $v = \beta^1 + \cdots + \beta^n$ where $n \geq 2$ and β^i are positive roots of Q^∞ such that $\beta^i \cdot \tilde{\lambda} = 0$, we have $p(v) > \sum_{i=1}^n p(\beta^i)$.*

We write $\Sigma_{\tilde{\lambda}}$ for the set of v such that the conditions in Theorem 2.3 hold.

Theorem 2.4 ([CB01, Theorem 1.3])**.** *Let* $\tau = (v^0, n_0 = 1; v^1, n_1; \ldots; v^k, n_k)$ *be a representation type such that* $\sum_{i=0}^{k} n_i v^i = \tilde{v}$. *The stratum of* $\mathcal{M}_{\lambda}^{0}(\tilde{v})$ *associated to* τ *has dimension*

$$d(\tau) = 2 \sum_{i=0}^{k} p(v^i).$$

Lemma 2.5. *If* $\mathcal{M}_{0}^{\theta}(v, w)$ *is non-empty for generic* θ, *then* $\tilde{v}$ *is a root of* Q^{∞}. *In particular,* $w \neq 0$.

Proof. When λ is generic, the G-action on $\mu^{-1}(\lambda)$ is free, so $\mu^{-1}(\lambda) = \mu^{-1}(\lambda)^{\theta-ss}$ and $\mathcal{M}_{\lambda}^{\theta}(v, w) \cong \mathcal{M}_{\lambda}^{0}(v, w)$. It is non-empty only if $\tilde{v}$ is a root due to Theorem 2.3. Since μ is flat when restricted to the θ-stable locus of T^*R, $\mathcal{M}_{0}^{\theta}(v, w)$ is non-empty only if $\mathcal{M}_{\lambda}^{\theta}(v, w)$ is non-empty. Hence $\tilde{v}$ is a root. If $w = 0$, then $\tilde{v}$ has disconnected support and cannot be a root. $\qquad\square$

The following useful lemma is immediate from the definition of p.

Lemma 2.6. *If* $a, b \in \mathbb{Z}_{\geq 0}^{Q_0}$, *then* $p(a + b) > p(a) + p(b)$ *if and only if* $-(a, b) > 1$.

2.3. **Local structure.** In this subsection we follow [BL21, Section 2.1.6]. Similar results under the hyper-Kähler setting are first given in [Nak94, Section 6].

For a fixed representation type $\tau = (v^0, 1; v^1, n_1; v^2, n_2; \ldots; v^k, n_k)$, we define a new quiver $\underline{Q}$. It has k vertices, in bijection with the k dimension vectors $\{v^1, \ldots, v^k\}$. The number of arrows from i to j is $-(v^i, v^j)$ if $i \neq j$, and $1 - \frac{1}{2}(v^i, v^i) = p(v^i)$ edge loops at each $i \geq 1$.

Let the dimension vector $\underline{v} \in \mathbb{Z}_{\geq 0}^{\underline{Q}_0}$ be defined by $\underline{v}_i = n_i$. Finally, let the framing $\underline{w} \in \mathbb{Z}_{\geq 0}^{\underline{Q}_0}$ be defined by $\underline{w}_i = -(v^0, v^i)$. The property of this construction is that the representation space $T^*R(\underline{Q}, \underline{v}, \underline{w})$ satisfies

$$(2.1) \qquad T^*(\mathfrak{g}.r) \oplus T^*R(\underline{Q}, \underline{v}, \underline{w}) \oplus \mathbb{C}^{2-(v^0, v^0)} \cong T^*R.$$

Let $H = \prod_{i=1}^{k} \mathrm{GL}(n_i)$, so that H is identified with the subgroup of G of automorphisms of a representation with type τ. The isomorphism (2.1) is an isomorphism of symplectic H-modules, with trivial H-action on $\mathbb{C}^{2-(v^0, v^0)}$. Write $R_0 := \mathbb{C}^{2-(v^0, v^0)}$ and $V := T^*R(\underline{Q}, \underline{v}, \underline{w}) \oplus R_0$.

Let $\underline{\mathfrak{p}} := \mathbb{C}^k \cong (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}])^*$, where $\mathfrak{h}$ is the Lie algebra of H, viewed as a Lie subalgebra of $\mathfrak{g}$. For $\theta \in \mathbb{Z}^{Q_0}$, define

$$\underline{\mathcal{M}}_{\underline{\mathfrak{p}}}^{\theta}(\underline{v}, \underline{w}) := \mathcal{M}_{\underline{\mathfrak{p}}}^{\theta}(\underline{Q}, \underline{v}, \underline{w}).$$

Here we abuse notation and write θ for the restriction of θ to H. We define $\underline{\mathcal{M}}_{\mathfrak{p}}^{\theta}(\underline{v}, \underline{w}) := \mathfrak{p} \times_{\underline{\mathfrak{p}}} \underline{\mathcal{M}}_{\underline{\mathfrak{p}}}^{\theta}(\underline{v}, \underline{w})$, via the shifted restriction morphism $\mathfrak{p} \to \underline{\mathfrak{p}}, \eta \mapsto (\eta - \lambda)|_{\mathfrak{h}}$.

Define $\mathcal{M}_{\mathfrak{p}}^{0}(v, w)^{\wedge_x} := \mathrm{Spec}(\mathbb{C}[\mathcal{M}_{\mathfrak{p}}^{0}(v, w)]^{\wedge_x})$ where $\wedge_x$ in the right hand side denotes the completion with respect to the maximal ideal of x. Define $(\underline{\mathcal{M}}_{\mathfrak{p}}^{0}(\underline{v}, \underline{w}) \times R_0)^{\wedge_0}$ similarly. For generic θ, define

$$\mathcal{M}_{\mathfrak{p}}^{\theta}(v, w)^{\wedge_x} := \mathcal{M}_{\mathfrak{p}}^{0}(v, w)^{\wedge_x} \times_{\mathcal{M}_{\mathfrak{p}}^{0}(v, w)} \mathcal{M}_{\mathfrak{p}}^{\theta}(v, w);$$

and define $\underline{\mathcal{M}}_{\mathfrak{p}}^{\theta}(\underline{v}, \underline{w})^{\wedge_0}$ similarly. Theorem 2.7 describes the local structure of x in $M_{\mathfrak{p}}^{\theta}(v, w)$.

Theorem 2.7 ([BL21, Section 2.1.6])**.** *There is a commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}_{\mathfrak{p}}^{\theta}(v,w)^{\wedge_x} & \xrightarrow{\ \sim\ } & (\underline{\mathcal{M}}_{\mathfrak{p}}^{\theta}(\underline{v},\underline{w}) \times R_0)^{\wedge_0} \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho\times\mathrm{id}} \\
\mathcal{M}_{\mathfrak{p}}^{0}(v,w)^{\wedge_x} & \xrightarrow{\ \sim\ } & (\underline{\mathcal{M}}_{\mathfrak{p}}^{0}(\underline{v},\underline{w}) \times R_0)^{\wedge_0}
\end{array} \quad ,$$

where the horizontal morphisms are isomorphisms, and $\hat{\rho} : \underline{\mathcal{M}}_0^{\theta}(\underline{v},\underline{w}) \to \underline{\mathcal{M}}_0^{0}(\underline{v},\underline{w})$ is the natural projective map.

Specializing to $\lambda \in \mathfrak{p}$, we get isomorphisms $\mathcal{M}_\lambda^\theta(v,w)^{\wedge_x} \cong (\underline{\mathcal{M}}_0^\theta(\underline{v},\underline{w}) \times R_0)^{\wedge_0}$, where θ is generic or 0.

Let us record an analytic version of the local structure theorem. Consider the homogeneous bundle $G \times^H ((\mathfrak{g}/\mathfrak{h})^* \oplus V)$. It is isomorphic to the Hamiltonian reduction $(T^*G \times V)///_0 H$, and is therefore symplectic. The natural G action on $G \times^H ((\mathfrak{g}/\mathfrak{h})^* \oplus V)$ is Hamiltonian, and the moment map is given by $\mu_G([g,\alpha,v]) = \mathrm{Ad}\, g(\alpha + \mu_H(v))$, where μ_H is the moment map for the H-action on V.

Let $\pi : T^*R \to T^*R//G$, and $\underline{\pi} : G \times^H ((\mathfrak{g}/\mathfrak{h})^* \oplus V) \to ((\mathfrak{g}/\mathfrak{h})^* \oplus V)//H$ be the categorical quotients by G. Proposition 2.8 is an application of [Los06, Proposition 3].

Proposition 2.8. *There is an analytic neighbourhood U of x in $\mathcal{M}_{\mathfrak{p}}^0(v,w)$, an analytic neighbourhood $\underline{U}$ of 0 in $\underline{\mathcal{M}}_{\mathfrak{p}}^0(\underline{v},\underline{w}) \times R_0$, and a symplectic isomorphism ϕ from U to $\underline{U}$ intertwining the morphisms $\mathcal{M}_{\mathfrak{p}}^0(v,w) \to \mathfrak{p}, \underline{\mathcal{M}}_{\mathfrak{p}}^0(\underline{v},\underline{w}) \to \mathfrak{p}$. It lifts to an isomorphism*

$$\tilde{\phi} : \pi^{-1}(U) \to \underline{\pi}^{-1}(\underline{U})$$

of G-stable analytic neighbourhoods of the orbits $Gr \subset T^*R$ and of $G.[1,0,0] \subset G \times^H ((\mathfrak{g}/\mathfrak{h})^* \oplus V)$; $\tilde{\phi}$ is symplectic, G-equivariant, and intertwines moment maps.

2.4. Maffei's isomorphism.
Suppose (θ,λ) is generic. Then for any $\sigma \in W_Q$, the Weyl group of the quiver Q, we have an isomorphism

$$\mathcal{M}_\lambda^\theta(v,w) \cong \mathcal{M}_{\sigma\lambda}^{\sigma\theta}(\sigma \bullet v, w),$$

where we view σ as an element of W_{Q^∞}, $\sigma \bullet v = \sigma(v + \alpha_\infty) - \alpha_\infty$ (see Section 2.1), $\sigma\theta$ and $\sigma\lambda$ are defined so that $(\sigma\theta) \cdot (\sigma v) = \theta \cdot v$, $(\sigma\lambda) \cdot (\sigma v) = \lambda \cdot v$. Equivalently, the weight ν corresponding to $(\sigma \bullet v, w)$ is $\sigma\nu$, and σ acts on $\mathbb{Z}^{Q_0^\infty}$ by $\tilde{v} \mapsto \sigma\tilde{v}$. These isomorphisms are first proved by Maffei in [Maf02]. We refer to them as Maffei's isomorphisms.

Remark 2.9. We can find $\sigma \in W_Q$ such that $\sigma\tilde{v}$ has minimal height with respect to the simple real roots of Q; equivalently, $(\sigma\tilde{v}, \alpha_i) \leq 0$ for all $i \in Q_0$ (this is automatically true for imaginary simple roots); equivalently, $\nu' = \Lambda_w - \sum_{i \in Q_0} (v')_i \alpha_i$ is dominant. Therefore, we may always assume ν is dominant; equivalently, $(\tilde{v}, \alpha_i) \leq 0$ for all $i \in Q_0$. Note that this does not imply $(\tilde{v}, \alpha_\infty) \leq 0$.

2.5. $\mathcal{M}_\lambda(v,w)$ vs. $\mathcal{M}_\lambda^0(v,w)$.
In this subsection we study the relation between affine quiver varieties and affinizations of smooth quiver varieties.

Proposition 2.10 ([BL21, Corollary 2.4, Proposition 2.5])**.** *If the moment map μ is flat, then $\mathcal{M}_\lambda^0(v,w) \cong \mathcal{M}_\lambda(v,w)$.*

Let us record some criteria for flatness of μ.

Theorem 2.11 ([CB01, Theorem 1.1]). *Fix a quiver Q and a dimension vector v (without framing). The following are equivalent.*

(1) *$\mu : T^*R \to \mathfrak{g}$ is flat.*
(2) *$\dim \mu^{-1}(0) = v \cdot v - 1 + 2p(v) = \dim T^*R - \dim G + 1$.*
(3) *For any decomposition $v = \beta_1 + \cdots + \beta_k$, where all β_i are positive roots, $p(v) \geq \sum_{i=1}^k p(\beta_i)$.*
(4) *For any decomposition $v = \beta_1 + \cdots + \beta_k$, where all $\beta_i \in \mathbb{Z}_{\geq 0}^{Q_0}$, $p(v) \geq \sum_{i=1}^k p(\beta_i)$.*

In particular, if $\tilde{v} \in \Sigma_0$, then μ is flat. Moreover, it is clear by Theorem 2.3 that $\tilde{v} \in \Sigma_0$ implies $\tilde{v} \in \Sigma_{\bar{\lambda}}$ for all $\lambda \in \mathfrak{p}$. Thus, we get Corollary 2.12 of Proposition 2.10.

Corollary 2.12. *If $\tilde{v} \in \Sigma_0$, then $\mathcal{M}_\lambda^0(v, w) \cong \mathcal{M}_\lambda(v, w)$ for all $\lambda \in \mathfrak{p}$.*

Corollary 2.13. *If Q is a finite or affine type quiver, and $\nu = \Lambda_w - \sum_{i \in Q_0} v_i \alpha_i$ is dominant, then $\mathcal{M}_0^0(v, w) \cong \mathcal{M}_0(v, w)$.*

Proof. By [BL21, Lemma 2.1], μ is flat in these cases. $\square$

Although it is not always true that $\mathcal{M}_0^0(v, w) \cong \mathcal{M}_0(v, w)$, we have the following result.

Proposition 2.14. *Suppose $\mathcal{M}_0(Q, v, w)$ is non-empty. Then*

$$\mathcal{M}_0(Q, v, w) \cong \mathcal{M}_0^0(Q', v', w')$$

for possibly different Q', v' and w'.

Proof. By Remark 2.9 we may assume $(\tilde{v}, \alpha_i) \leq 0$ for all simple roots α_i. The assumption that $\mathcal{M}_0(Q, v, w)$ is non-empty implies $\tilde{v}$ is a root, so $w \neq 0$.

If $(\tilde{v}, \alpha_\infty) \leq 0$ as well, then $\tilde{v}$ lie in the fundamental domain F_0. If $\tilde{v} \in \Sigma_0$ then μ is flat. Otherwise, $\tilde{v}$ falls in the 3 cases of [CB01, Theorem 8.1]. Case (I) there cannot appear since the multiplicity of α_∞ is 1 in $\tilde{v}$, and in Cases (II) and (III), μ is flat by [Su06, Theorem 1.1]. Therefore, $\mathcal{M}_0^0(Q, v, w) \cong \mathcal{M}_0(Q, v, w)$ by Proposition 2.10.

Suppose now $(\tilde{v}, \alpha_\infty) = 2 + (v, \alpha_\infty) > 0$. Then since $w \neq 0$, we must have $(v, \alpha_\infty) = -1$. Therefore, the vertex ∞ is connected to a unique vertex $i \in Q_0$, and $v_i = 1$.

Therefore, we can view the vertex i as a framing and ∞ as a usual vertex, and apply Maffei's isomorphism for s_∞. Then we see $\mathcal{M}_0(Q, v, w) \cong \mathcal{M}_0(Q', v', w')$, where Q' is obtained from Q by deleting the vertex i, $v_j' = v_j$ for $j \in Q_0'$ and the new framings w_j' equals the number of arrows between i and j. This operation cuts down the size of the underlying quiver. Repeat this process if necessary, and we either get a $\mathcal{M}_0(Q', v', w')$ for which $\tilde{v'} \in F_0$ or get Q' the quiver where all vertices have edge loops. In either case, the moment map μ for (Q', v', w') is flat by [Su06, Theorem 1.1], and we have $\mathcal{M}_0(Q, v, w) \cong \mathcal{M}_0(Q', v', w') \cong \mathcal{M}_0^0(Q', v', w')$. $\square$

Thanks to the above proof, we may always make Assumption 2.15.

Assumption 2.15. We assume $(\tilde{v}, \alpha_i) \leq 0$ for all $i \in Q_0 \cup \{\infty\}$, and that μ is flat. In particular, $\mathcal{M}_\lambda(v, w) = \mathcal{M}_\lambda^0(v, w)$.

2.6. **The canonical decomposition.** Crawley-Boevey introduced a decomposition of affine quiver varieties into a product of simpler subvarieties in [CB02]. More precisely, fix a quiver Q, a dimension vector v, and a parameter $\lambda \in \mathfrak{p}$. Assume there is no framing.

Theorem 2.16 ([CB02, Theorem 1.1, Proposition 1.2]). *There exists a decomposition $v = \sum_{i=1}^{k} m_i v_i$ (called the canonical decomposition of v), where m_i are positive integers and $v_i \in \Sigma_\lambda$, such that any decomposition of v as a sum of elements in Σ_λ is a refinement of this decomposition. We have*

$$\mathcal{M}_\lambda^0(v) = \prod_{i=1}^{k} S^{m_i} \mathcal{M}_\lambda^0(v_i),$$

which we call the canonical decomposition of $\mathcal{M}_\lambda^0(v)$. When v_i is a real root, $\mathcal{M}_\lambda^0(v_i)$ is a point. When v_i is a non-isotropic imaginary root, $m_i = 1$.

We can define the canonical decomposition of $\mathcal{M}_\lambda^0(v, w)$ by identifying it with $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty, \tilde{v})$.

Lemma 2.17. *Assume Assumption 2.15. Suppose v_i appears in the canonical decomposition of $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty, \tilde{v})$. Write $v_i = \sum_{j \in Q_0^\infty} c_{ij} \alpha_j$. Then some $c_{ij} = 1$.*

Proof. If $\tilde{v} \in \Sigma_{\tilde{\lambda}}$ already, then the canonical decomposition is trivial, and note the ∞-component of $\tilde{v}$ is 1. If not, by [CB02, Section 6], we are in one of the following three cases:

  (I) The quiver Q^∞ is an affine type quiver and $\tilde{v} = m\delta$ for some $m \geq 2$, where δ is the minimal imaginary root.
 (II) The quiver Q^∞ decomposes as in [CB02, Lemma 5.3]. More precisely, the vertex set Q_0^∞ is a disjoint union $\mathcal{J} \cup \mathcal{K}$, there is a unique arrow a with one end in $\mathcal{J}$ and the other in $\mathcal{K}$, say connecting $j \in \mathcal{J}$ and $k \in \mathcal{K}$. Moreover, $\tilde{v}_j = \tilde{v}_k = 1$.
(III) The quiver Q^∞ decomposes as in [CB02, Lemma 5.4]. More precisely, the vertex set Q_0^∞ is a disjoint union $\mathcal{J} \cup \mathcal{K}$, there is a unique arrow a with one end in $\mathcal{J}$ and the other in $\mathcal{K}$, say connecting $j \in \mathcal{J}$ and $k \in \mathcal{K}$. Moreover, $\tilde{v}_j = 1$, and the restriction of Q^∞ to $\mathcal{K}$ is an affine type quiver, the restriction of $\tilde{v}$ to $\mathcal{K}$ is $m\delta$ for some $m \geq 2$, where δ is the minimal imaginary root.

Since $\tilde{v}_\infty = 1$, case (I) is impossible. The canonical decomposition of $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty, \tilde{v})$ is obtained as the product of those of $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty|_{\mathcal{J}}, \tilde{v}|_{\mathcal{J}})$ and $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty|_{\mathcal{K}}, \tilde{v}|_{\mathcal{K}})$. In case (II), the j-component of $\tilde{v}|_{\mathcal{J}}$ and the k-component of $\tilde{v}|_{\mathcal{K}}$ are 1; in case (III), the j-component of $\tilde{v}|_{\mathcal{J}}$ is 1, and $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty|_{\mathcal{K}}, \tilde{v}|_{\mathcal{K}}) = S^m \mathcal{M}_{\tilde{\lambda}}^0(Q^\infty|_{\mathcal{K}}, \delta)$, where δ also has a component 1. Inductively, we obtain Lemma 2.17. □

Therefore, we may view each component $\mathcal{M}_{\tilde{\lambda}}^0(Q^\infty, v_i)$ as a *framed* quiver variety. When $\lambda = 0$, it is clear that under Assumption 2.15, the Namikawa-Weyl group of $\mathcal{M}_0(v, w) \cong \mathcal{M}_0^0(v, w)$ is the direct product of those of $S^{m_i} \mathcal{M}_0^0(Q^\infty, v_i)$. This allows us to assume $\tilde{v} \in \Sigma_0$.

In summary, we have reduced the problem of computing the Namikawa-Weyl group of $\mathcal{M}_0(v, w)$ to the following situation.

Assumption 2.18. The dimension vector $\tilde{v} \in \mathbb{Z}_{\geq 0}^{Q_0^\infty}$ is a root, lies in Σ_0, and satisfies $(\tilde{v}, \alpha_i) \leq 0$ for all $i \in Q_0^\infty$. The moment map $\mu : T^*R \to \mathfrak{g}$ is flat, so that $\mathcal{M}_\lambda(v, w) = \mathcal{M}_\lambda^0(v, w)$ for $\lambda \in \mathfrak{p}$.

Remark 2.19. Thanks to Lemma 2.17, we can compute the Namikawa-Weyl group of any affine quiver variety $\mathcal{M}_0^0(v, w)$ once we know how to compute the Namikawa-Weyl group of $\mathcal{M}_0(v, w)$.

2.7. Isotropic decomposition. From the definition, the most straightforward way to compute the Namikawa-Weyl group of a conical symplectic singularity is to find its codimension 2 leaves, determine the corresponding slice Kleinian singularities, and examine the diagram automorphisms induced by the fundamental group action. For affine quiver varieties under Assumption 2.18, the first two steps are done by Theorem 2.20.

Theorem 2.20 ([BS21, Definition 1.18, Theorem 1.20]). *Let $v \in \Sigma_\lambda$ be imaginary. Then the codimension 2 leaves of $\mathcal{M}_\lambda^0(v)$ correspond to the representation types of the form*

$$\tau = (\beta^1, 1; \beta^2, 1; \cdots; \beta^s, 1; \gamma^1, m_1; \cdots; \gamma^t, m_t)$$

such that

(1) *the β^i are imaginary roots, and the γ^i are pairwise distinct real roots;*
(2) *the slice quiver $\underline{Q}$, after removing all the edge loops, is an affine type quiver;*
(3) *the dimension vector $\underline{\tilde{v}} = (1, 1, \cdots, 1; m_1, \cdots, m_t)$ equals the minimal imaginary root of $\underline{Q}$.*

The slice Kleinian singularity has the same type as the slice quiver $\underline{Q}$.

The decomposition $v = \beta^1 + \cdots + \beta^s + m_1\gamma^1 + \cdots \gamma^t m_t$ is called an *isotropic decomposition*. Note that, when $v \in \Sigma_\lambda$ is real, the dimension formula Theorem 2.4 implies $\mathcal{M}_\lambda^0(v)$ is a point.

In [BS21], the action of $\pi_1(\mathcal{L}_i)$ on $\hat{\mathfrak{h}}_i^*$ is not computed, so we cannot directly determine $\mathfrak{h}_X$ and W. In the remaining parts of this paper, we present two methods for computing W, one via tautological line bundles (Section 3) and the other via subgeneric deformations (Section 4).

3. Tautological line bundles

In this section, we recall the notion of tautological line bundles on $\mathcal{M}_0^\theta(v, w)$ and describe the Namikawa-Cartan space of $\mathcal{M}_0^0(v, w)$ in terms of their Chern classes. We also examine what type of Weyl groups can appear as components of the Namikawa-Weyl group of $\mathcal{M}_0^0(v, w)$. We make the assumptions of Assumption 2.18.

3.1. Tautological line bundles. Assume θ is generic so that G acts freely on $(T^*R)^{\theta-ss}$. Let $\chi \in \mathbb{Z}^{Q_0}$ be a character of G.

Definition 3.1. The line bundle over $\mathcal{M}_\mathfrak{p}^\theta(v, w)$, whose total space is the homogeneous bundle

$$\mu^{-1}(\mathfrak{p})^{\theta-ss} \times^G \mathbb{C}_\chi,$$

where G acts on the one dimensional vector space $\mathbb{C}_\chi$ by the character χ, is called a *tautological line bundle*. We denote it by $\mathcal{O}(\chi, \mathcal{M}_\mathfrak{p}^\theta)$.

For any $\lambda \in \mathfrak{p}$, we denote by $\mathcal{O}(\chi, \mathcal{M}_\lambda^\theta)$ the restriction of $\mathcal{O}(\chi, \mathcal{M}_\mathfrak{p}^\theta)$ to $\mathcal{M}_\lambda^\theta(v, w)$; its total space is

$$\mu^{-1}(\lambda)^{\theta-ss} \times^G \mathbb{C}_\chi.$$

Similarly, we can define the line bundle $\mathcal{O}(\chi, \mathcal{M}_{\mathbb{C}\lambda}^\theta)$ on $\mathcal{M}_{\mathbb{C}\lambda}^\theta(v, w)$.

For generic θ, $\mathcal{M}_{\mathfrak{p}}^\theta(v, w) \to \mathfrak{p}$ is a Poisson deformation of $\mathcal{M}_0^\theta(v, w)$. By Theorem 1.3, there is a unique linear map $\kappa : \mathfrak{p} \to H^2(\mathcal{M}_0^\theta(v, w), \mathbb{C})$ such that the pullback of the universal deformation by κ is $\mathcal{M}_{\mathfrak{p}}^\theta(v, w) \to \mathfrak{p}$. Theorem 3.2 relates tautological line bundles with the map κ. Let $c_1(\mathcal{L})$ denote the first Chern class of a line bundle $\mathcal{L}$.

Theorem 3.2 ([Los12, Proposition 3.2.1]). *The map κ coincides with the map*

$$\mathbb{Z}^{Q_0} \to H^2(\mathcal{M}_0^\theta(v, w), \mathbb{C}), \ \chi \mapsto c_1(\mathcal{O}(\chi, \mathcal{M}_0^\theta))$$

extended by complex linearity.

Theorem 3.3 ([MN19, Corollary 3.8]). *The map κ is surjective.*

Let $\chi \in \mathbb{Z}^{Q_0}$ and $\mathcal{L}_i$ be a codimension 2 leaf of $\mathcal{M}_0^0(v, w)$. Let S_i denote the slice Kleinian singularity and $\tilde{S}_i$ its minimal resolution. Proposition 3.4 describes the projection of $\kappa(\chi)$ to the component $\mathfrak{h}_i$ corresponding to $\mathcal{L}_i$.

Proposition 3.4. *Let*

$$\tau = (\beta^1, 1; \beta^2, 1; \cdots ; \beta^s, 1; \gamma^1, m_1; \cdots ; \gamma^t, m_t)$$

be the representation type of the leaf $\mathcal{L}_i$, satisfying Theorem 2.20. Assume the ∞-component of β^1 is 1, and regard the vertex associated to β^1 the extended vertex of the affine type quiver. The projection of $c_1(\mathcal{O}(\chi, \mathcal{M}_0^\theta))$ to $\mathfrak{h}_i$ is given by

$$(\chi \cdot \beta^2, \chi \cdot \beta^3, \cdots, \chi \cdot \gamma^t) \in \mathfrak{h}_i \subset \hat{\mathfrak{h}}_i^* \cong H^2(\tilde{S}_i, \mathbb{C}).$$

Proof. Recall Remark 1.7. Let $x \in \mathcal{L}_i$, and let U be a small enough analytic neighbourhood of x in $\mathcal{M}_0^0(v, w)$ which is isomorphic to $\Delta^{\dim \mathcal{M}_0^0(v,w)-2} \times S_i$ and satisfies Proposition 2.8. We will restrict the line bundle $\mathcal{O}(\chi, \mathcal{M}_0^\theta)$ to $\rho^{-1}(U) \subset \mathcal{M}_0^\theta(v, w)$.

Let $\underline{U}$ be an analytic neighbourhood of $0 \in \underline{\mathcal{M}}_0^0(\underline{v}, \underline{w}) \times R_0$ satisfying Proposition 2.8. Write

$$\pi_0^\theta : \mu^{-1}(0)^{\theta-ss} \to \mathcal{M}_0^\theta(v, w),$$
$$\underline{\pi}_0^\theta : G \times^H \mu_H^{-1}(0)^{\theta-ss} \to \underline{\mathcal{M}}_0^\theta(\underline{v}, \underline{w}) \times R_0$$

for the projections. We see

$$(\pi_0^\theta)^{-1}\rho^{-1}(U) \cong G \times^H (\underline{\pi}_0^\theta)^{-1}\underline{\rho}^{-1}(\underline{U}).$$

The total space of the restriction of $\mathcal{O}(\chi, \mathcal{M}_0^\theta)$ to $\rho^{-1}(U)$ is

$$(3.1) \qquad \begin{aligned} (\pi_0^\theta)^{-1}\rho^{-1}(U) \times^G \mathbb{C}_\chi &\cong (G \times^H (\underline{\pi}_0^\theta)^{-1}\underline{\rho}^{-1}(\underline{U})) \times^G \mathbb{C}_\chi \\ &= (\underline{\pi}_0^\theta)^{-1}\underline{\rho}^{-1}(\underline{U}) \times^H \mathbb{C}_{\chi|_H}. \end{aligned}$$

Remark 1.7 implies that the $\mathfrak{h}_i$-component of $\kappa(\chi)$ is the Chern class of the tautological line bundle on $\underline{\mathcal{M}}_0^\theta(\underline{v}, \underline{w})$ associated to the character $\chi|_H$ of H, which is exactly $(\chi \cdot \beta^2, \chi \cdot \beta^3, \cdots, \chi \cdot \gamma^t)$. $\qquad \square$

Therefore, given a codimension 2 leaf $\mathcal{L}_i$ and the corresponding representation type τ, we can determine $\mathfrak{h}_i$ thanks to Theorem 3.3 and Theorem 3.2.

3.2. **Examples.** We give examples of $\mathcal{M}_0^0(v, w)$ whose Namikawa-Weyl groups have types B_n and G_2, and show that type $C_n, n \geq 3$ and F_4 Weyl groups cannot appear as a component of the Namikawa-Weyl group.

Example 3.5. By Proposition 1.13, any A, D or E type Weyl group can be the Namikawa-Weyl group of some $\mathcal{M}_0^0(v, w)$.

Example 3.6 is suggested to the author by Hiraku Nakajima, and it appears in [BFN19, 5(v)].

**Example 3.6.** Consider the following quiver:
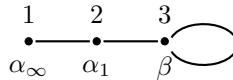
where we view α_∞ as a framing, so that $\tilde{v} = \alpha_\infty + 2\alpha_1 + 3\beta$. It is not hard to see $\tilde{v} \in \Sigma_0$, and the only codimension 2 leaf corresponds to the representation type

$$\tau = (\alpha_\infty, 1; \alpha_1, 2; \beta, 1; \beta, 1; \beta, 1).$$

For any $x \in \mathcal{L}_\tau$, the (extended) slice quiver $\underline{Q}^\infty$ is the type $\hat{D}_4$ quiver, and the dimension vector $\underline{\tilde{v}}$ is precisely the minimal imaginary root type $\hat{D}_4$. The image of $\kappa : \mathfrak{p} \to \hat{\mathfrak{h}}^*$ is exactly the subspace of fixed points of the degree 3 diagram automorphism, by Proposition 3.4. Therefore, the Namikawa-Weyl group of $\mathcal{M}_0^0(\tilde{v})$ is the Weyl group of type G_2.

Example 3.7. For any $k \geq 0$, consider the following quiver:
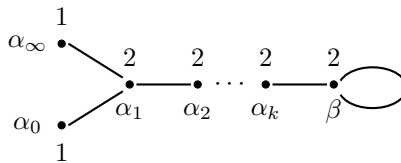
We view α_∞ as a framing so $v = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_k) + 2\beta$. A direct computation shows that $\Sigma_0 = \{\alpha_i, \beta, \tilde{v}\}$, $i \in \{0, 1, \cdots, k, \infty\}$, and there is only one codimension 2 leaf of $\mathcal{M}_0^0(v)$, corresponding to the representation type

$$\tau = (\alpha_\infty, 1; \alpha_0, 1; \alpha_1, 2; \cdots, \alpha_k, 2; \beta, 1; \beta, 1).$$

The slice quiver $\underline{Q}^\infty$ has type $\hat{D}_{k+3}$, and it is easy to see the image of κ is the subspace stable under the order 2 diagram folding. Therefore, the Namikawa-Weyl group of $\mathcal{M}_0^0(\tilde{v})$ is the Weyl group of type B_{k+2}.

Proposition 3.8. F_4 *cannot appear as a component of the Namikawa-Weyl group.*

Proof. Assume F_4 Weyl group appears. Recall that F_4 is obtained by folding the type E_6 diagram. Therefore, by Theorem 2.20, the slice quiver $\underline{Q}^\infty$ is $\hat{E}_6$, and the dimension vector $\underline{\tilde{v}}$ is the minimal imaginary root δ. Under a suitable labeling, the root δ takes the form $(1, 2, 3, 2, 1, 2, 1)$, and we require the image of κ to be pointwise fixed by the order 2 folding. In particular, the first two "2" in the expression of δ must both correspond to the same real root in $\Sigma_0(Q^\infty, \tilde{v})$, contradicting Theorem 2.20. $\qquad\square$

Proposition 3.9. *For $n \geq 3$, C_n cannot appear as a component of the Namikawa-Weyl group.*

Proof. Suppose $\mathcal{L}_i$ is a codimension 2 leaf such that $\mathfrak{h}_i$ has type C_n. Then $\hat{\mathfrak{h}}_i^*$ has type A_{2n-1} or A_{2n}. Suppose first $\hat{\mathfrak{h}}_i^*$ has type A_{2n-1}. Then the slice quiver $\underline{Q}^\infty$ is $\hat{A}_{2n-1}$, and dimension vector $\tilde{\underline{v}}$ is the minimal imaginary root δ. Let the vertices of $\hat{A}_{2n-1}$ be denoted by $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \beta, \gamma_{n-1}, \cdots, \gamma_1$, where α_0 corresponds to the extended vertex, α_j, γ_j and β are elements of Σ_0, and the representation type of $\mathcal{L}_i$ is
$$\tau = (\alpha_0, 1; \alpha_1, 1; \cdots; \gamma_1, 1).$$
The diagram automorphism identifies α_i and γ_i. Since $\mathfrak{h}_i$ has type C_n, $\alpha_j = \gamma_j$ and they are all imaginary roots. In particular, $2\alpha_j$ are roots.

We claim that $\mathcal{L}_i$ cannot have codimension 2. Let us compute $\dim \mathcal{L}_i$ first:

$$\dim \mathcal{L}_i = d(\tau) = 2 - (\alpha_0, \alpha_0) + 2 - (\beta, \beta) + 2\sum_{j \geq 1}(2 - (\alpha_j, \alpha_j))$$

(3.2)

$$= 4n - ((\alpha_0, \alpha_0) + (\beta, \beta) + 2\sum_{j=1}^{n-1}(\alpha_i, \alpha_i)).$$

On the other hand, by assumption $\tilde{v} = \alpha_0 + \beta + 2\sum \alpha_j \in \Sigma_0$. Therefore,

$$\dim \mathcal{M}_0^0(Q, v, w) = 2p(\tilde{v})$$

(3.3)

$$= 2 - ((\alpha_0, \alpha_0) + (\beta, \beta) + 4\sum_{i=1}^{n-1}(\alpha_i, \alpha_i))$$

$$+ (\alpha_0, 2\alpha_1) + (2\alpha_1, \alpha_0 + 2\alpha_2)$$

$$+ (2\alpha_2, 2\alpha_1 + 2\alpha_3)$$

$$\vdots$$

$$+ (2\alpha_{n-1}, 2\alpha_{n-2} + \beta) + (\beta, 2\alpha_{n-1})$$

$$= 2 - ((\alpha_0, \alpha_0) + (\beta, \beta) + 4\sum_{i=1}^{n-1}(\alpha_i, \alpha_i)) + 8(n - 1).$$

Therefore, $\mathcal{L}_i$ has codimension 2 if and only if $n = 2$ and $(\alpha_1, \alpha_1) = 0$ (this is exactly the situation of Example 3.7 when $k = 0$).

The proof for the case where $\underline{Q}^\infty$ has type $\hat{A}_{2n}$ is completely analogous. We conclude that C_n for $n \geq 3$ cannot appear as a Namikawa-Weyl group component of $\mathcal{M}_0^0(v, w)$. $\qquad\square$

4. SUBGENERIC DEFORMATIONS

It is not easy to classify codimension 2 leaves of $\mathcal{M}_0^0(v, w)$ (equivalently, find all the isotropic decompositions Theorem 2.20), although once this is done, it is straightforward to calculate the Namikawa-Weyl group by methods in Section 3. We will take a different approach to determine the Namikawa-Weyl groups, one via subgeneric deformation. We will see the relation between subgeneric deformations and isotropic decompositions in Corollary 6.5.

In this subsection, we examine the deformations $\mathcal{M}_\lambda^0(v, w)$ of $\mathcal{M}_0^0(v, w)$ where λ is *subgeneric*, i.e. very close to being generic. In particular, we examine when

such a deformation has a codimension 2 symplectic leaf. We continue to assume Assumption 2.18.

4.1. The set $\Sigma_{\tilde{\lambda}}$ for subgeneric λ.

Definition 4.1. Let $\lambda \in \mathfrak{p}$ and $v^1 \leq v$ be a dimension vector in $\mathbb{Z}_{\geq 0}^{Q_0}$.

    (1) We say v^1 is *indecomposable with respect to* λ if there is no proper decomposition $v^1 = \sum \beta_i$ where all β_i are positive roots such that $\beta_i \cdot \lambda = 0$.

    (2) We say λ is *subgeneric for* v if there is a unique (up to scalar) positive root v^1 of Q such that $v^1 < v$ and $\lambda \cdot v^1 = 0$. We just say λ is *subgeneric* when v is clear from context.

Recall the definitions of $\tilde{v}, \tilde{\lambda}$ in Section 2.1. If λ is subgeneric and v^1 is indecomposable with respect to λ, then $v^1 \in \Sigma_{\tilde{\lambda}}$ if we view it as an element of $\mathbb{C}^{Q_0^{\infty}}$ whose ∞-component is 0. Lemmas 4.2 and 4.3 are used in the proof of Theorem 4.4.

Lemma 4.2. *Let λ be subgeneric and v^1 be the indecomposable root with respect to λ. Let $n \in \mathbb{Z}_{>0}$.*

    (1) *If v^1 is a real root, then $nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if $n = 1$.*

    (2) *If v^1 is an isotropic imaginary root, i.e. $(v^1, v^1) = 0$, then $nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if $n = 1$.*

    (3) *If v^1 is a non-isotropic imaginary root, i.e. $(v^1, v^1) \leq -2$, then $nv^1 \in \Sigma_{\tilde{\lambda}}$ for all n.*

Proof. The assumption on v^1 guarantees that for $n \geq 2$, any decomposition of nv^1 in (2) of Theorem 2.3 has the form $nv^1 = \sum_{i=1}^{k} n_i v^1$ where $\sum_{i=1}^{k} n_i = n$.

The first two parts follow easily from Theorem 2.3. Let v^1 be a non-isotropic imaginary root. We have

$$p(nv^1) = 1 - \frac{n^2}{2}(v^1, v^1), \quad \sum_{i=1}^{k} p(n_i v^1) = k - \sum_{i=1}^{k} \frac{n_i^2}{2}(v^1, v^1).$$

Since $(v^1, v^1) \leq -2$, it is clear that, for $k \geq 2$ and $\sum_{i=1}^{k} n_i = n$,

$$(4.1) \qquad 1 - \frac{n^2}{2}(v^1, v^1) > k - \sum_{i=1}^{k} \frac{n_i^2}{2}(v^1, v^1),$$

i.e. $p(nv^1) > \sum p(n_i v^1)$, which proves part (3) of the lemma. $\qquad \square$

If λ is subgeneric for v and v^1 is the indecomposable root with respect to λ, then any $u \in \Sigma_{\tilde{\lambda}}$ satisfying $u \leq \tilde{v}$ have the form $u = nv^1$ or $u = \tilde{v} - nv^1$ for some $n \geq 0$. Lemma 4.3 gives some conditions for $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$.

Lemma 4.3. *Assume $\mathcal{M}_0^{\theta}(v, w)$ is non-empty for generic θ. Let $\lambda \in \mathfrak{p}$ be subgeneric for v, and v^1 be the unique indecomposable positive root such that $v^1 \leq v$ and $v^1 \cdot \lambda = 0$ (so that $v^1 \in \Sigma_{\tilde{\lambda}}$). Let $n \in \mathbb{Z}_{\geq 0}$.*

    (1) *Suppose v^1 is a real root. Then $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if $\tilde{v} - nv^1$ is a root and $(\tilde{v} - nv^1, v^1) \leq 0$.*

    (2) *Suppose v^1 is an isotropic imaginary root. If $(\tilde{v}, v^1) = 0$ or -1, then $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if n is the largest integer such that $\tilde{v} - nv^1$ is a root. If $(\tilde{v}, v^1) \leq -2$, then $\tilde{v} \in \Sigma_{\tilde{\lambda}}$.*

(3) *Suppose v^1 is a non-isotropic imaginary root, and let N be the largest integer such that $\tilde{v} - Nv^1$ is a root. Then $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if $n = N$ or $p(\tilde{v} - nv^1) > p(\tilde{v} - Nv^1) + p((N-n)v^1)$.*

Proof. (1) Let v^1 be a real root. Assume $\tilde{v} - nv^1$ is a root, and $(\tilde{v} - nv^1, v^1) = c > 0$. Let s be the reflection along the real root v^1. Then $s(\tilde{v} - nv^1) = \tilde{v} - (n+c)v^1$ is a root. But since $(\tilde{v} - nv^1, \tilde{v} - nv^1) = (s(\tilde{v} - nv^1), s(\tilde{v} - nv^1))$, and $p(v^1) = 0$, we have

$$p(\tilde{v} - (n+c)v^1) + cp(v^1) = p(\tilde{v} - nv^1).$$

Therefore $\tilde{v} - nv^1 \notin \Sigma_{\tilde{\lambda}}$.

Conversely, suppose $(\tilde{v} - nv^1, v^1) \leq 0$ and $\tilde{v} - nv^1$ is a root. Let $(\tilde{v} - nv^1 - lv^1) + v^1 + \cdots + v^1$ be a decomposition of $\tilde{v} - nv^1$, $l > 0$, and note that any decomposition in (2) of Theorem 2.3 has this form. We have

$$p(\tilde{v} - nv^1 - lv^1) + lp(v^1) = p(\tilde{v} - nv^1) + (\tilde{v} - nv^1, lv^1) - \frac{1}{2}l^2(v^1, v^1) + 0$$
$$= p(\tilde{v} - nv^1) + l(\tilde{v} - nv^1, v^1) - l^2$$
$$< p(\tilde{v} - nv^1).$$

Therefore $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$.

(2) Let v^1 be an isotropic imaginary root. Suppose $(\tilde{v}, v^1) = 0$ or -1. If n is the largest integer such that $\tilde{v} - nv^1$ is a root then clearly $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$, since condition (2) of Theorem 2.3 is automatic. Conversely, suppose there is $m > n$ such that $\tilde{v} - mv^1$ is a root. Then

$$p(\tilde{v} - nv^1) = p(\tilde{v} - mv^1 + (m-n)v^1)$$
$$= p(\tilde{v} - mv^1) - (m-n)(\tilde{v} - mv^1, v^1) - 0$$
$$\leq p(\tilde{v} - mv^1) + (m-n)$$
$$= p(\tilde{v} - mv^1) + (m-n)p(v^1).$$

Therefore, $\tilde{v} - nv^1 \notin \Sigma_{\tilde{\lambda}}$. This proves the first half of (2).

Now assume $(\tilde{v}, v^1) \leq -2$. For any $n \geq 1$ and any decomposition $\tilde{v} = (\tilde{v} - nv^1) + n_1v^1 + \ldots + n_kv^1$ where $\sum_{i=1}^{k} n_i = n$,

$$p(\tilde{v} - nv^1) + \sum_i p(n_iv^1) = p(\tilde{v}) + n(\tilde{v}, v^1) + k < p(\tilde{v}).$$

Since $\tilde{v}$ is a root by Lemma 2.5, $\tilde{v} \in \Sigma_{\tilde{\lambda}}$.

(3) It is clear from definition that $\tilde{v} - Nv^1 \in \Sigma_{\tilde{\lambda}}$, and if $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ then $p(\tilde{v} - nv^1) > p(\tilde{v} - Nv^1) + p((N-n)v^1)$.

Conversely, suppose $n < N$ and $p(\tilde{v} - nv^1) > p(\tilde{v} - Nv^1) + p((N-n)v^1)$. By Lemma 2.6, this inequality holds if and only if $-(N-n)(\tilde{v} - Nv^1, v^1) > 1$, which implies $(\tilde{v} - Nv^1, v^1) \leq -1$.

Let $N > m > n$ be such that $\tilde{v} - mv^1$ is a root. Note that

$$(\tilde{v} - mv^1, v^1) = (\tilde{v} - Nv^1, v^1) + (N-m)(v^1, v^1) < (\tilde{v} - Nv^1, v^1) \leq -1$$

and therefore $-(m-n)(\tilde{v} - mv^1, v^1) > 1$. So $p(\tilde{v} - nv^1) > p(\tilde{v} - mv^1) + p((m-n)v^1)$ by Lemma 2.6. Thanks to Inequality (4.1), we conclude $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$.

$\square$

4.2. **Codimension 2 roots.** In this subsection we investigate for which subgeneric $\lambda \in \mathfrak{p}$ the variety $\mathcal{M}_\lambda^0(v, w)$ has a codimension 2 symplectic leaf. Let $v^1 \in \Sigma_\lambda$ be the minimal positive root such that $\lambda \cdot v^1 = 0$.

Theorem 4.4. *The affine variety $\mathcal{M}_\lambda^0(v, w) = \mathcal{M}_{\tilde{\lambda}}^0(\tilde{v})$ has a codimension 2 symplectic leaf if and only if one of the following conditions hold:*

(1) v^1 *is a real root,* $\langle \nu, v^1 \rangle = 0$, *and* $\tilde{v} - v^1$ *is a root of Q^∞.*
(2) v^1 *is an isotropic imaginary root,* $\langle \nu, v^1 \rangle = 2$, *and* $\tilde{v} - v^1$ *is a root of Q^∞.*
(3) v^1 *is a non-isotropic imaginary root,* $\tilde{v} - nv^1$ *is a root for $n = 1$ or 2, $(\tilde{v} - nv^1, nv^1) = -2$, and $\tilde{v} - mv^1$ is not a root for any $m > n$.*

Definition 4.5. We say an indecomposable root v^1 of Q is a *codimension 2 root* if it satisfies one of the conditions in Theorem 4.4. We say it is of type (i) if it falls in Case (i) in Theorem 4.4, $i = 1, 2, 3$.

Lemma 4.6 and Corollary 4.7 on imaginary roots are useful. It was proved for quivers without loops, but the proof works for quivers with loops as well, thanks to [Kac80, Lemma 5, Lemma 6].

Lemma 4.6 ([Mor80, Theorem 2]). *Let Q be a quiver and $\alpha = \sum n_i \alpha_i \neq 0, n_i \in \mathbb{Z}_{\geq 0}$. Then α is an imaginary root if and only if for any $\sigma \in W_Q$, we have $\sigma(\alpha) > 0$ and $\mathrm{Supp}(\alpha)$ is connected.*

Corollary 4.7. *Suppose v is a positive root, α is a positive imaginary root, and $(v, \alpha) \leq -1$, then $v + \alpha$ is a positive imaginary root.*

Proof. If v is real, then the v-string through α contains $\alpha + v$ since $(v, \alpha) \leq -1$. Since $(v + \alpha, v + \alpha) = (v, v) + 2(v, \alpha) + (\alpha, \alpha) \leq 0$, $v + \alpha$ must be an imaginary root.

If v is imaginary, then for any $w \in W_Q$, $w.\alpha$ and $w.v$ are positive with connected support. Since $(w.\alpha, w.v) = (\alpha, v) \neq 0$, we conclude $w.(v + \alpha)$ has connected support. Therefore, $v + \alpha$ is an imaginary root by Corollary 4.7. $\square$

Proof of Theorem 4.4. We use Theorem 2.4 to search for codimension 2 strata. Recall, Theorem 2.4, that $d(\tau)$ denotes the dimension of the stratum associated to a representation type τ. Moreover, by the assumption $\tilde{v} \in \Sigma_0$, we see $\tilde{v} \in \Sigma_{\tilde{\lambda}}$ for all $\lambda \in \mathfrak{p}$. Therefore, $\dim \mathcal{M}_\lambda^0(v, w) = 2p(\tilde{v})$.

(1) If v^1 is real, then by Lemma 4.2, any possible representation type has the form $\tau_n = (\tilde{v} - nv^1, 1; v^1, n)$. We have
$$d(\tau_n) = 2(p(\tilde{v} - nv^1) + p(v^1)) = 2(p(\tilde{v}) + n(\tilde{v}, v^1) - n^2).$$
Note this formula holds for $n = 0$ as well. Then $\dim \mathcal{M}_\lambda^0(v, w) - d(\tau_n) = 2n^2 - 2n(\tilde{v}, v^1)$. If the leaf associated to τ_n has codimension 2 if and only if $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ and $n(n - (\tilde{v}, v^1)) = 1$, i.e.
$$n = 1, (\tilde{v}, v^1) = 0.$$
Conversely, if $(\tilde{v}, v^1) = 0$ and $\tilde{v} - v^1$ is a root, then by Lemma 2.5 and Lemma 4.3, $\tilde{v} - v^1 \in \Sigma_{\tilde{\lambda}}$. This proves case (1).

(2) Suppose v^1 is an isotropic imaginary root, i.e. $(v^1, v^1) = 0$. By Lemma 4.2, $nv^1 \in \Sigma_{\tilde{\lambda}}$ if and only if $n = 1$. Therefore, any valid representation type has the form $\tau = (\tilde{v} - nv^1, 1; v^1, n_1; \ldots; v^1, n_k)$ where $n = n_1 + \cdots + n_k$. By Theorem 2.4, the dimension of the stratum corresponding to τ is

(4.2) $$d(\tau) = 2(p(\tilde{v} - nv^1) + k) = 2(p(\tilde{v}) + n(\tilde{v}, v^1) + k).$$

Therefore, for fixed n, the dimension of stratum is maximal when $k = n$ and all $n_k = 1$; in this case, $d(\tau) = 2p(\tilde{v}) + 2n(1 + (\tilde{v}, v^1))$. The dimension difference is

$$\dim \mathcal{M}_\lambda^0(v, w) - d(\tau) = -2n(1 + (\tilde{v}, v^1)).$$

The difference is 2 if and only if

$$n = 1, (\tilde{v}, v^1) = -2.$$

This proves part (2) of the theorem.

(3) Suppose $v^1 \in \Sigma_{\tilde{\lambda}}$ is a non-isotropic imaginary root, i.e. $(v^1, v^1) \leq -2$. By Lemma 4.2, $nv^1 \in \Sigma_{\tilde{\lambda}}$ for all integers $n \geq 1$. Any representation type has the form $\tau = (\tilde{v} - nv^1, 1; n_1v^1, m_1; \ldots; n_kv^1, m_k)$ where $\tilde{v} - nv^1 \in \Sigma_{\tilde{\lambda}}$ and $\sum_i m_i n_i = n$. The dimension of the corresponding stratum is

$$d(\tau) = 2(p(\tilde{v} - nv^1) + \sum_i p(n_iv^1)) = 2p(\tilde{v} - nv^1) + 2k - \sum_i n_i^2(v^1, v^1).$$

For fixed n, the dimension is maximal when $k = 1$. For $n \geq 1$, we denote

$$\tau_n = (\tilde{v} - nv^1, 1; nv^1, 1),$$

so that

(4.3) $$d(\tau_n) = 2(p(\tilde{v}) + n(\tilde{v}, v^1) + 1 - n^2(v^1, v^1)).$$

The dimension difference is

$$\dim \mathcal{M}_\lambda^0(v, w) - d(\tau_n) = -2(n(\tilde{v}, v^1) + 1 - n^2(v^1, v^1)).$$

The difference is 2 if and only if $n(\tilde{v} - nv^1, v^1) = -2$. Therefore, either $n = 1$ and $(\tilde{v} - v^1, v^1) = -2$; or $n = 2$ and $(\tilde{v} - 2v^1, 2v^1) = -2$.

In either case, if $\tilde{v} - mv^1$ is a root for some $m > n$, then it is easy to see $d(\tau_m) > d(\tau_n)$, contradicting that $\tilde{v} \in \Sigma_{\tilde{\lambda}}$. This proves part (3) of the theorem.

$\square$

Example 4.8. We give an example of each of the above cases.

(1) Let Q be the type A_1 quiver, with one vertex and no loops. Let $v = 1, w = 2$. Let α be the simple root. The variety $\mathcal{M}_0(v, w) = \mathrm{Spec}(\mathbb{C}[T^*\mathbb{P}^1]) \cong \mathcal{N}$, the nilpotent cone of $\mathfrak{sl}_2$. We can take $v^1 = \alpha$. This is an example of case (1).

(2) Let Q be the type $\hat{A}_2$ quiver, $v = \delta$ and w be 2 at the extending vertex, 0 at the other two vertices. Then $v^1 = \delta$ is an example of case (2).

(3) Let Q be the following quiver, $v = \alpha + 2\beta$ and the framing be 1 over β.
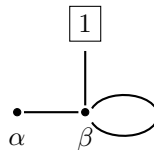
Then, $v^1 = \alpha + 2\beta$ is an example of case (3) with $n = 1$. Note that this is a special case of Example 3.7 for $k = 0$.

(4) Let Q be the quiver with 1 vertex and 2 edge loops; let $v = 2, w = 1$. Then v is an example of case (3) with $n = 2$.

Remark 4.9. Theorem 4.4 is an example of Theorem 2.20. In each case, the slice quiver $\underline{Q}^\infty$ (without edge loops) for a codimension 2 leaf is type $\hat{A}_1$, and the dimension vector $\tilde{\underline{v}}$ is $(1,1)$, the minimal imaginary root. Moreover, it is clear from the proof that in each case $\mathcal{M}^0_\lambda(v,w)$ has exactly one codimension 2 leaf.

4.3. **Restriction of tautological line bundles.** Let v^1 be a codimension 2 root of Q, λ a subgeneric parameter such that $\lambda \cdot v^1 = 0$. The variety $\mathcal{M}^0_\lambda(v,w) \cong \mathcal{M}_\lambda(v,w)$ has a unique codimension 2 symplectic leaf $\mathcal{L}$. Let $x \in \mathcal{L}$ and consider the natural map $\rho : \mathcal{M}^\theta_\lambda(v,w) \to \mathcal{M}^0_\lambda(v,w)$.

Lemma 4.10. *Assume $\theta \cdot v^1 > 0$. Then the fiber $\rho^{-1}(x)$ is identified with $\mathbb{P}^1$, and the restriction of $\mathcal{O}(\chi, \mathcal{M}^\theta_\lambda)$ to $\rho^{-1}(x)$ has degree $\chi \cdot v^1$.*

Proof. The quiver $\underline{Q}$ defined with respect to $x \in \mathcal{L}$ in Section 2.3 can be viewed as a type A_1 quiver, with dimension vector 1 and framing 2. Therefore, the fiber $\rho^{-1}(x) \cong \underline{\rho}^{-1}(0) \cong \mathbb{P}^1$ is the zero section of $T^*\mathbb{P}^1$. Thanks to Proposition 2.8, we can pick analytic neighbourhoods $U, \underline{U}$ as in the proof of Proposition 3.4, so that the restriction of $\mathcal{O}(\chi, \mathcal{M}^\theta_\lambda)$ to $U \cong \underline{U}$ is

$$(R_0 \times \mu_H^{-1}(0)^{\theta-ss} \times^H \mathbb{C}_{\chi|_H})|_{\underline{U}}.$$

We have $H \cong \mathbb{C}^*$ acting on $\mathbb{C}$ by the character $\chi \cdot v^1$. Therefore, the restriction of $\mu_H^{-1}(0)^{\theta-ss} \times^H \mathbb{C}_{\chi|_H}$ to $\underline{\rho}^{-1}(0) \cong \mathbb{P}^1$ is the line bundle $\mathcal{O}(\chi \cdot v^1)$. $\square$

We remark that if we assumed $\theta \cdot v^1 < 0$ instead, then the restriction of $\mathcal{O}(\chi, \mathcal{M}^\theta_\lambda)$ to $\rho^{-1}(x)$ has degree $-\chi \cdot v^1$.

5. Codimension 2 leaves in deformations of conical symplectic singularities

Let X be a conical symplectic singularity and $\pi : \mathcal{X} \to \mathfrak{h}_X/W$ be the universal graded Poisson deformation. Let $\lambda \in \mathfrak{h}_X$ and $\bar{\lambda}$ be its image in $\mathfrak{h}_X/W$. We investigate when does $\mathcal{X}_\lambda = \pi^{-1}(\bar{\lambda})$ have a codimension 2 symplectic leaf.

5.1. **Deformation of Kleinian singularities.** In this subsection we classify the deformation parameters of a Kleinian singularity that gives codimension 2 symplectic leaves in the deformed variety. We follow [CBH98].

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. Let Q be the corresponding McKay quiver, which is an affine type quiver. Its vertex set Q_0 is in bijection with $\{N_i\}$, the set of simple representations of Γ. Let δ be the minimal positive imaginary root of this quiver. Then the quiver variety $\mathcal{M}^0_0(Q, \delta)$ is isomorphic to $X = \mathbb{C}^2/\Gamma$.

Let $\lambda \in (\mathbb{C}\Gamma)^\Gamma$ (where Γ acts on $\mathbb{C}\Gamma$ by conjugation). We can identify λ with $(\mathrm{tr}\,\lambda|_{N_i})$ and view it as an element of $\mathfrak{p}$. Define the algebra $\mathscr{I}^\lambda = (\mathbb{C}\langle x, y\rangle \# \Gamma)/(xy - yx - \lambda)$ where $\#$ denotes the skew group algebra. Then define $\mathscr{O}^\lambda = e\mathscr{I}^\lambda e$ where $e = \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \gamma$.

Theorem 5.1 ([CBH98, Theorem 0.2]). *Suppose $\lambda \cdot \delta = 0$, then the algebra $\mathscr{O}^\lambda$ is commutative, and $\mathrm{Spec}(\mathscr{O}^\lambda) = \mathcal{M}^0_\lambda(Q, \delta)$.*

We say a root α of Q is a *Dynkin root* if $N_0 \notin \mathrm{Supp}(\alpha)$.

Theorem 5.2 ([CBH98, Theorem 0.4]). *If $\lambda \cdot \delta = 0$, then the global dimension of $\mathscr{O}^\lambda$ is finite if and only if $\lambda \cdot \alpha \neq 0$ for any Dynkin root α of Q.*

Corollary 5.3 now follows from the fact that a Noetherian local ring is regular if and only if its global dimension is finite.

Corollary 5.3. *Suppose $\lambda \cdot \delta = 0$, then $\mathcal{M}_\lambda^0(Q, \delta)$ is smooth if and only if $\lambda \cdot \alpha \neq 0$ for any Dynkin root α of Q.*

Let $\mathfrak{g}$ be the simple Lie algebra whose Dynkin diagram is Q_{fin} (that is Q without the extended vertex), $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and W be the Weyl group of $\mathfrak{g}$. Slodowy showed in [Slo80, Section 1.5, Theorem 1] that $\pi : S \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}//G \cong \mathfrak{h}/W$ is the universal graded Poisson deformation of $\mathbb{C}[X]$, where S is the *Slodowy slice* to a subregular nilpotent orbit of $\mathfrak{g}$.

Let $\mu : T^*R(Q, \delta, 0) \to \mathfrak{g}$ be the moment map. We have an identification $\mathfrak{h} = \{\lambda \in \mathfrak{p} | \lambda \cdot \delta = 0\}$. By remarks at the end of [CBH98, Section 8], the map $\mu : \mu^{-1}(\mathfrak{h}) \to \mathfrak{h}$ is obtained from $\pi : S \to \mathfrak{h}/W$ by pulling back through the quotient $\mathfrak{h} \to \mathfrak{h}/W$. Therefore, by Corollary 5.3, we have

Corollary 5.4. *Suppose $\lambda \in \mathfrak{h}$, and $\bar{\lambda}$ is its image in $\mathfrak{h}/W$. Then $\pi^{-1}(\bar{\lambda})$ is not smooth (i.e. has a codimension 2 leaf) if and only if $\lambda \cdot \alpha = 0$ for some Dynkin root of Q.*

5.2. **General conical symplectic singularities.** In this subsection we describe the conditions on $\lambda \in \mathfrak{h}_X$ for $\mathcal{X}_\lambda = \pi^{-1}(\bar{\lambda})$ to have a codimension 2 symplectic leaf.

Let $\mathcal{X}_{\mathbb{C}\lambda} = \pi^{-1}(\mathbb{C}\lambda)$. We have a graded Poisson morphism $\pi|_{\mathcal{X}_{\mathbb{C}\lambda}} : \mathcal{X}_{\mathbb{C}\lambda} \to \mathbb{C}\lambda = \text{Spec}\,\mathbb{C}[t]$. Therefore, each fiber of $\pi|_{\mathcal{X}_{\mathbb{C}\lambda}}$ is a Poisson subvariety. Let $Z \subset \mathcal{X}_\lambda$ be a closed Poisson subvariety of dimension d. Let $Y = \overline{\mathbb{C}^* \cdot Z}$ and consider $Y \cap \mathcal{X}_0$. The following result is standard.

Proposition 5.5. $Y \cap \mathcal{X}_0$ *is a d-dimensional Poisson subvariety of X.*

Corollary 5.6. *If $Z \subset \mathcal{X}_\lambda$ is a codimension 2 symplectic leaf, then $Y \cap \mathcal{X}_0$ is the union of the closures of some codimension 2 symplectic leaves of X.*

Proposition 5.7. *If $\lambda \in \mathfrak{h}_X$, then $\mathcal{X}_\lambda$ has a codimension 2 symplectic leaf if and only if λ lies in a root hyperplane of the Namikawa-Weyl group W.*

Proof. We may assume X has at least one codimension 2 symplectic leaf. Otherwise, the proposition is vacuous by Theorem 1.6.

Let $\mathcal{L}_i$ be a codimension 2 symplectic leaf of X and Σ_i be the slice Kleinian singularity at some $x \in \mathcal{L}_i$. Let λ_i be the projection of λ to the Cartan space of Σ_i in the decomposition in Theorem 1.6, viewed as an element of $\hat{\mathfrak{h}}_i$. Let $\bar{\lambda}_i \in \hat{\mathfrak{h}}_i / \widehat{W}_i$ be its image in the quotient. Let $\pi_i : S_i \to \hat{\mathfrak{h}}_i / \widehat{W}_i$ be the universal graded Poisson deformation of the Kleinian singularities Σ_i, and let $S_{\mathbb{C}\lambda_i} = \pi_i^{-1}(\mathbb{C}\bar{\lambda}_i) \subset S_i$.

Let $\mathfrak{m}_x$ be the maximal ideal of $\mathbb{C}[\mathcal{X}_{\mathbb{C}\lambda}]$ corresponding to x, and define

$$(\mathcal{X}_{\mathbb{C}\lambda})^{\wedge_x} = \text{Spec}((\mathscr{O}_{\mathcal{X}_{\mathbb{C}\lambda}, x})^{\wedge_{\mathfrak{m}_x}}).$$

Let V be the tangent space $T_x \mathcal{L}_i$, and consider $V \times S_{\mathbb{C}\lambda_i}$, viewed as a scheme over $\mathbb{C}\lambda_i$. Let $\mathfrak{m}_0$ be the maximal ideal corresponding to $0 \in V \times S_{\mathbb{C}\lambda_i}$. Let

$$(V \times S_{\mathbb{C}\lambda_i})^{\wedge_0} = \text{Spec}(\mathbb{C}[V \times S_{\mathbb{C}\lambda_i}]^{\wedge_{\mathfrak{m}_0}}).$$

Then $(\mathcal{X}_{\mathbb{C}\lambda})^{\wedge_x} \cong (V \times S_{\mathbb{C}\lambda_i})^{\wedge_0}$ (see [Los22, Section 2.3]).

Consider the completion of the sheaf of relative differentials $(\Omega_{\mathcal{X}_{\mathbb{C}\lambda}/\mathbb{C}\lambda})^{\wedge_x}$ (it is different from $\Omega_{(\mathcal{X}_{\mathbb{C}\lambda})^{\wedge_x}/\mathbb{C}\lambda}$), viewed as a sheaf on $(\mathcal{X}_{\mathbb{C}\lambda})^{\wedge_x}$. It only depends on the completed algebra $(\mathcal{O}_{\mathcal{X}_{\mathbb{C}\lambda},x})^{\wedge_{\mathfrak{m}_x}}$. Similarly, we define the completion of the module of relative differentials $(\Omega_{(V \times S_{\mathbb{C}\lambda_i})/\mathbb{C}\lambda})^{\wedge_0}$, which only depends on $(V \times S_{\mathbb{C}\lambda_i})^{\wedge_0}$. Therefore, we have

$$(5.1) \qquad (\Omega_{(V \times S_{\mathbb{C}\lambda_i})/\mathbb{C}\lambda})^{\wedge_0} \cong (\Omega_{\mathcal{X}_{\mathbb{C}\lambda}/\mathbb{C}\lambda})^{\wedge_x}.$$

By Corollary 5.6, $\mathcal{X}_\lambda$ has a codimension 2 symplectic leaf Z if and only if $\pi_\lambda : \mathcal{X}_{\mathbb{C}\lambda} \to \mathbb{C}\lambda$ is non-smooth along a codimension 2 closed Poisson subvariety (i.e. $\overline{\mathbb{C}^* \cdot Z}$), which intersects X at a union of closure of codimension 2 leaves. Equivalently, for some codimension 2 leaf $\mathcal{L}_i$ of X, there is a point $x \in \mathcal{L}_i$, and a (non-closed) point $\mathfrak{p}$ of $\mathcal{X}_{\mathbb{C}\lambda}$ whose closure contains x, such that $(\Omega_{\mathcal{X}_{\mathbb{C}\lambda}/\mathbb{C}\lambda})^{\wedge_x}$ has rank larger than $\dim X$ at $\mathfrak{p}$ (viewed as a point in $(\mathcal{X}_{\mathbb{C}\lambda})^{\wedge_x}$). By (5.1), this is equivalent to that $(\Omega_{(V \times S_{\mathbb{C}\lambda_i})/\mathbb{C}\lambda})^{\wedge_0}$ has rank larger than $\dim X$ at some non-closed point $\mathfrak{q} \in V \times S_{\mathbb{C}\lambda_i}$ whose closure contains 0. This happens if and only if $V \times S_{\mathbb{C}\lambda_i}$ has a codimension 2 symplectic leaf, which is further equivalent to λ_i lying on a root hyperplane of the Cartan space of S_i, by Corollary 5.4. Since $\pi_1(\mathcal{L}_i)$ act by diagram automorphisms, by restricting to $\mathfrak{h}_i = (\hat{\mathfrak{h}}_i^*)^{\pi_1(\mathcal{L}_i)}$, we see λ lies in a root hyperplane of W_i. $\square$

6. Namikawa-Weyl group via codimension 2 roots

In this section, we continue to make the assumptions of Assumption 2.18, unless otherwise stated.

6.1. Linear relations between codimension 2 roots.

We combine the results of Section 4 and Section 5 to describe the Namikawa-Weyl group of the affinization of a smooth quiver variety. Recall the map $\kappa : \mathfrak{p} \to H^2(\mathcal{M}_0^\theta(v,w),\mathbb{C})$ defined in Section 3.1.

Write $X = \mathcal{M}_0^0(v,w)$. For any positive root v^1 of Q, let $H_{v^1} \subset \mathfrak{p}$ be the hyperplane $\{\lambda | \lambda \cdot v^1 = 0\}$.

Proposition 6.1. *The map κ gives a bijection between the set*

$$\{H_{v^1} | v^1 \text{ is a codimension 2 root of } Q\}$$

and the set of root hyperplanes of $\mathfrak{h}_X$.

Proof. Since a generic element in H_{v^1} produces a deformation of $\mathcal{M}_0(v,w)$ which contains a codimension 2 leaf, $\kappa(H_{v^1})$ must be a proper subspace of $\mathfrak{h}_X$. Since κ is surjective by Theorem 3.3, $\kappa(H_{v^1})$ is in fact a hyperplane. By Proposition 5.7, $\kappa(H_{v^1})$ is a root hyperplane of $\mathfrak{h}_X$.

Conversely, for any root hyperplane H of $\mathfrak{h}_X$, the preimage $\kappa^{-1}(H)$ is a hyperplane of $\mathfrak{p}$ since κ is surjective. The fiber over any $h \in H$ has a codimension 2 symplectic leaf, so by Theorem 4.4, $\kappa^{-1}(H)$ must be of the form H_{v^1}. It is clear that κ and κ^{-1} give mutually inverse bijections on the set of hyperplanes as desired. $\square$

Recall the simply laced Cartan space $\hat{\mathfrak{h}}_i^* \cong \mathbb{C}^{\underline{Q}_0}$ where $\underline{Q}$ is the slice quiver associated to the leaf $\mathcal{L}_i$; it is a finite type quiver by Theorem 2.20. Note there is a unique Dynkin diagram automorphism σ such that $\mathfrak{h}_i = (\hat{\mathfrak{h}}_i^*)^\sigma$. If α is a root of $\mathfrak{h}_i$, then there exists a (unique up to σ translation) positive root $\underline{\alpha}$ in $\hat{\mathfrak{h}}_i^*$, such that

$\alpha = \sum_{j=1}^{m(\alpha)} \sigma^j \underline{\alpha}$, where $m(\alpha)$ is the size of the σ-orbit of $\underline{\alpha}$. View $\underline{\alpha}$ as an element of $\mathbb{Z}_{\geq 0}^{Q_0}$.

Lemma 6.2. *Suppose the hyperplane of $\mathfrak{p}$ cut out by v^1 maps to the root hyperplane of α in $\mathfrak{h}_i$ under the map κ, and let $\underline{\alpha}$ be defined as above. Then for any $\chi \in \mathbb{Z}^{Q_0}$, $\chi \cdot v^1 = \kappa_i(\chi) \cdot \underline{\alpha}$.*

Proof. Let $\mathcal{L}_i$ be the codimension 2 leaf of $\mathcal{M}_0(v, w)$ corresponding to $\mathfrak{h}_i$, and $x \in \mathcal{L}_i$. Let S denote the slice Kleinian singularity. Let λ be a subgeneric parameter annihilated by v^1. Let λ_i be the image of λ under the natural restriction $\mathfrak{p} \to \underline{\mathfrak{p}}$. We know by Theorem 2.20 that $\underline{Q}^\infty$ is an affine type quiver and the dimension vector is the minimal imaginary root δ. For each $c \in \mathbb{C}^*$, $\underline{\mathcal{M}}_{c\lambda_i}^0(\delta)$ has $m(\alpha)$ codimension 2 leaves, corresponding to the representation types $(\delta - \sigma^j(\underline{\alpha}), 1; \sigma^j(\underline{\alpha}), 1)$ for $1 \leq j \leq m(\alpha)$.

Pick analytic open neighbourhoods U of $x \in \mathcal{M}_{\mathbb{C}\lambda}^0(v, w)$, $\underline{U}$ of $0 \in \underline{\mathcal{M}}_{\mathbb{C}\lambda_i}^0(\delta) \times R_0$ and an isomorphism $\phi : U \to \underline{U}$ that satisfy Proposition 2.8. Pick $y \in U$ that lies in the unique codimension 2 leaf of $\mathcal{M}_{\varepsilon\lambda}^0(v, w)$ for some small $\varepsilon \neq 0$. Then, $y' := \phi(y)$ lies in one of the $m(\alpha)$ codimension 2 leaves of $\underline{\mathcal{M}}_{\varepsilon\lambda_i}^0(\delta) \times R_0$. By Lemma 4.10, for $\theta > 0$, the restriction of the line bundle $\mathcal{O}(\chi, \mathcal{M}_{\mathfrak{p}}^\theta(v, w))$ to $\rho^{-1}(y)$ has degree $\chi \cdot v^1$. Thanks to Remark 1.7 and Theorem 3.2, ϕ identifies the line bundles

$$\mathcal{O}(\chi, \mathcal{M}_{\mathfrak{p}}^\theta(v, w))|_U \cong \mathcal{O}(\kappa_i(\chi), \underline{\mathcal{M}}_{\underline{\mathfrak{p}}}^\theta(\delta)) \boxtimes \mathbb{C}[R_0]|_{\underline{U}}.$$

Lemma 4.10 then shows the restriction to $(\rho \times \mathrm{id})^{-1}(y') \cong \mathbb{P}^1 \times R_0$ has degree $\underline{\alpha} \cdot \kappa_i(\chi)$. Therefore, $\chi \cdot v^1 = \kappa_i(\chi) \cdot \underline{\alpha}$. $\square$

Corollary 6.3 enables us to recover the linear relations of roots of $\mathfrak{h}_i$ from linear relations of codimension 2 roots of Q.

Corollary 6.3. *Suppose v^j are codimension 2 roots of Q corresponding to the root α_j of $\mathfrak{h}_i$, for $j = 1, 2, 3$. Then the following conditions are equivalent:*

(i) $\alpha_1 = \alpha_2 + \alpha_3$ *in* $\mathfrak{h}_i$;
(ii) $m(\alpha_1)v^1 = m(\alpha_2)v^2 + m(\alpha_3)v^3$ *in* $\mathbb{Z}_{\geq 0}^{Q_0}$.

Proof. We have

$$\alpha_1 = \alpha_2 + \alpha_3$$
$$\iff \sum_{j=1}^{m(\alpha_1)} \sigma^j \underline{\alpha_1} = \sum_{j=1}^{m(\alpha_2)} \sigma^j \underline{\alpha_2} + \sum_{j=1}^{m(\alpha_3)} \sigma^j \underline{\alpha_3}$$
$$\iff \sum_{j=1}^{m(\alpha_1)} \sigma^j \underline{\alpha_1} \cdot \kappa_i(\chi) = \left(\sum_{j=1}^{m(\alpha_2)} \sigma^j \underline{\alpha_2} + \sum_{j=1}^{m(\alpha_3)} \sigma^j \underline{\alpha_3} \right) \cdot \kappa_i(\chi), \; \forall \chi \in \mathbb{Z}^{Q_0}$$
$$\iff m(\alpha_1)\underline{\alpha_1} \cdot \kappa_i(\chi) = (m(\alpha_2)\underline{\alpha_2} + m(\alpha_3)\underline{\alpha_3}) \cdot \kappa_i(\chi), \; \forall \chi \in \mathbb{Z}^{Q_0}$$
$$\iff m(\alpha_1)v^1 = m(\alpha_2)v^2 + m(\alpha_3)v^3.$$

The first equivalence above holds by the definition of $\underline{\alpha}$; the second equivalence holds since κ is surjective; the third equivalence holds since $\kappa_i(\chi)$ is invariant under σ-action; the last equivalence holds by Lemma 6.2. $\square$

Remark 6.4. It is classical that if $\mathfrak{h}_i$ is type B, C or F_4 (we have seen the latter two cases are impossible) then $m(\alpha) = 2$ when α is a long root, and $m(\alpha) = 1$ when α is a short root. If $\mathfrak{h}_i$ is type G_2 then $m(\alpha) = 3$ when α is a long root, and $m(\alpha) = 1$ when α is a short root. If $\mathfrak{h}_i$ is simply-laced then all $m(\alpha) = 1$.

Therefore, thanks to Corollary 6.3, by examining the root strings of multiply-laced Dynkin diagrams, we see that only the following relations for codimension 2 roots v^j, $j \in \{1, 2, 3\}$, are possible (up to reordering):

(a) $v^1 + v^2 = v^3$
(b) $v^1 + 2v^2 = v^3$
(c) $v^1 + v^2 = 2v^3$
(d) $v^1 + 3v^2 = v^3$
(e) $v^1 + v^2 = 3v^3$

In the above list, if the coefficient before v^j is greater than 1, then v^j corresponds to a long root, and the other two roots in the relation are short roots.

Therefore, we can recover the Namikawa-Weyl group of $\mathcal{M}_0(v, w)$ by finding all linear relations of codimension 2 roots of Q of the forms (a)-(e).

We say a codimension 2 root v^1 is *simple* if it does not satisfy any relation $m_1 v^1 = m_2 v^2 + m_3 v^3$ for codimension 2 roots v^2, v^3 and $m_i \in \{1, 2, 3\}$. Corollary 6.5, which follows directly from Corollary 6.3, relates the isotropic decomposition Theorem 2.20 and codimension 2 roots. It enables us to find all the codimension 2 leaves of $\mathcal{M}_0(v, w)$.

Corollary 6.5. *If a root appears in an isotropic decomposition, then it is a multiple of a simple codimension 2 root. Conversely, if v^1 is a simple codimension 2 root, then a multiple of v^1 appears in an isotropic decomposition.*

The following is a consequence of Proposition 6.1 and Corollary 6.5.

Corollary 6.6. *If v^1 is a simple codimension 2 root, then there is a unique isotropic decomposition in which a multiple of v^1 appears. In particular, all the roots appearing in the representation types of different codimension 2 leaves are distinct.*

Let us recall the four types of codimension 2 roots of Q, given by Theorem 4.4:

(1) v^1 is a real root, $\langle \nu, v^1 \rangle = 0$, and $\tilde{v} - v^1$ is a root of Q^∞.
(2) v^1 is an isotropic imaginary root, $\langle \nu, v^1 \rangle = 2$, and $\tilde{v} - v^1$ is a root of Q^∞.
(3) v^1 is a non-isotropic imaginary root, $\tilde{v} - nv^1$ is a root for $n = 1$ or 2, $(\tilde{v} - nv^1, nv^1) = -2$, and $\tilde{v} - mv^1$ is not a root for any $m > n$.

The relations in Remark 6.4 are not arbitrary. For $i, j, k \in \{1, 2, 3\}$, if α, β, γ are type $(i), (j), (k)$ codimension 2 roots respectively, and there exists $m, n, p \in \{1, 2, 3\}$ such that $m\alpha + n\beta = p\gamma$, then we write $(i) + (j) = (k)$; otherwise we write $(i) + (j) \neq (k)$.

Theorem 6.7. *The following (symmetric) table lists all possibilities for $(i) + (j) = (k)$.*

+	(1)	(2)	(3)
(1)	(1)		
(2)	(2),(3)	(3)	
(3)	(2),(3)	(3)	(3)

Proof. We first prove all statements $(i) + (j) \neq (k)$, then give an example for each valid $(i) + (j) = (k)$.

(a) $(1) + (1) \neq (i)$, $(i) + (j) \neq (1)$, $(1) + (i) \neq (1)$ for $i, j \in \{2, 3\}$.

If α is a type (2) or (3) codimension 2 root then $(\tilde{v}, \alpha) < 0$, but if α is type (1) then $(\tilde{v}, \alpha) = 0$.

(b) $(2) + (2) \neq (2)$.

Suppose α, β, γ are linearly independent imaginary roots and $m, n, p \in \{1, 2, 3\}$ such that $m\alpha + n\beta = p\gamma$ as in Remark 6.4. Take the Tit's form of both sides, we see $(\alpha, \beta) = 0$. Let $\sigma \in W_Q$ be such that $\sigma\alpha = \delta$ for some affine Dynkin diagram, then $(\sigma\alpha, \sigma\beta) = (\delta, \sigma\beta) = 0$. Since $\mathrm{Supp}(m\sigma\alpha + n\sigma\beta) = \mathrm{Supp}(p\sigma\gamma)$ is connected, we must have $\mathrm{Supp}(\sigma\beta) \subset \mathrm{Supp}(\delta)$. By Lemma 4.6, $\sigma\beta$ is a positive isotropic imaginary root, so we must have $\sigma\beta = \delta$, which implies $\alpha = \beta$, contradiction.

(c) $(3) + (2) \neq (2)$ and $(3) + (3) \neq 2$.

In fact, if α is a positive non-isotropic imaginary root, β is a positive imaginary root, and $m, n > 0$ such that $m\alpha + n\beta$ is an isotropic imaginary root, then $(m\alpha + n\beta, m\alpha + n\beta) = 0$, so $(\alpha, \beta) > 0$. But if $\sigma \in W_Q$ is such that $(\sigma\alpha, \alpha_i) \leq 0$ for all simple roots α_i, we would have $(\sigma\alpha, \sigma\beta) > 0$, implying $\sigma\beta < 0$. This contradicts that β is an imaginary root.

We now provide examples for all $(i) + (j) = (k)$ in the table.

(d) $(1) + (1) = (1)$, see Proposition 1.13.

(e) $(1) + (2) = (2)$. Consider the following quiver
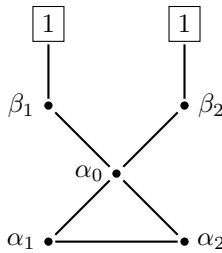
We take $v = \alpha_0 + \alpha_1 + \alpha_2 + \beta_1 + \beta_2$, and w be 1 over the vertices (corresponding to the simple roots) β_1, β_2, and 0 elsewhere, as is indicated by the squares. Let $\delta = \alpha_0 + \alpha_1 + \alpha_2$ which is easily checked to be a root of type (2). The simple real roots β_1 and β_2 are both of type (1). The root $\delta + \beta_1$ is of type (2). This gives examples of $(1) + (2) = (2)$.

(f) $(1) + (2) = (3)$, $(1) + (3) = (2)$.

Consider the quiver in part (3) of Example 4.8. In the notations there, α is a type (1) codimension 2 root, β and $\alpha + \beta$ are type (2), while $\alpha + 2\beta$ is type (3). The relations

$$\alpha + 2\beta = \alpha + 2\beta, \alpha + (\alpha + 2\beta) = 2(\alpha + \beta)$$

giving examples of $(1) + (2) = (3)$ and $(1) + (3) = (2)$ respectively.

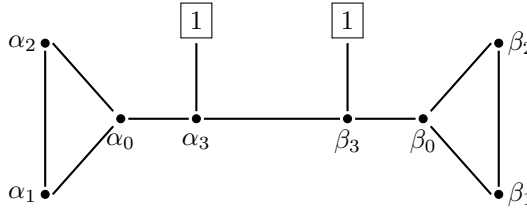(g) $(1) + (3) = (3)$, $(2) + (2) = (3)$ and $(3) + (3) = (3)$.

Consider the quiver and the dimension vector in Example 3.6, i.e. $v = 2\alpha_1 + 3\beta$, w is 1 over α_1 and 0 elsewhere. We see α_1 is a type (1) codimension

2 root, $\alpha_1 + 2\beta$, $\alpha_1 + 3\beta$ and $2\alpha_1 + 3\beta$ are type (3) roots, β and $\alpha_1 + \beta$ are type (2). We have relations

$$(\alpha_1) + (\alpha_1 + 3\beta) = 2\alpha_1 + 3\beta;$$
$$(\beta) + (\alpha_1 + \beta) = \alpha_1 + 2\beta;$$
$$(\alpha_1 + 3\beta) + (2\alpha_1 + 3\beta) = 3(\alpha_1 + 2\beta)$$

which are examples of $(1) + (3) = (3), (2) + (2) = (3)$ and $(3) + (3) = (3)$.

(h) $(2) + (3) = (3)$. Consider the following quiver.

Write $\delta_1 = \alpha_0 + \alpha_1 + \alpha_2$ and $\delta_2 = \beta_0 + \beta_1 + \beta_2$. Let $v = \delta_1 + \alpha_3 + \beta_3 + 2\delta_2$, and w be 1 over α_3, β_3, 0 elsewhere. Let $\alpha = \delta_1 + \alpha_3$ so that α is an isotropic imaginary root, and $(\tilde{v}, \alpha) = -2$. Since $\tilde{v} - \alpha = \alpha_\infty + \beta_3 + 2\delta_2$ is an imaginary root, we see α is a type (2) root. Let $\beta = 2\delta_2 + \beta_3$ so that β is a non-isotropic imaginary root, and $(\tilde{v} - \beta, \beta) = (\alpha_\infty + \alpha_3 + \delta_1, 2\delta_2 + \beta_3) = -2$. Also, $\tilde{v} - \beta = \alpha_\infty + \delta_1 + \alpha_3$ is an imaginary root, and $\tilde{v} \not\geq n\beta$ for $n \geq 2$. Therefore, β is a type (3) root.

Note v is a non-isotropic imaginary root. $(\tilde{v} - \gamma, \gamma) = (\alpha_\infty, \gamma) = -2$, $\tilde{v} - v = \alpha_\infty$ is a root, and clearly $\tilde{v} - nv$ is not a root for $n \geq 2$. Therefore, v is a type (3) root and $v = \alpha + \beta$ gives an example of $(2) + (3) = (3)$.

□

Remark 6.8. We finish this subsection with a remark on generalizing our method to quiver varieties $\mathcal{M}_0^0(v)$ without framing. An important ingredient of our approaches is the surjectivity of κ, Theorem 3.3. For non-framed quiver variety, the source of κ is restricted to $\{\lambda \in \mathfrak{p} | \lambda \cdot v = 0\}$. However, it is not known to our knowledge whether similar results hold when all components of the dimension vector v are larger than 1, even if it is indivisible, which means v is not a non-trivial integral multiple of a smaller root. For more details, see [MN18, Lemma 4.11].

If v is not indivisible, then κ need not be surjective. In fact, let Q be the quiver with a single vertex and 2 edge loops, and let $v = 2$. Then the set $\{\lambda \in \mathfrak{p} | \lambda \cdot v = 0\}$ is just $\{0\}$. On the other hand, it is known that (e.g. [BS21, Section 5]) $\mathcal{M}_0^0(v) \cong \mathcal{N} \times \mathbb{C}^4$ where $\mathcal{N} \subset \mathfrak{sp}(4)$ is the closure of the nilpotent orbit $\{B \in \mathfrak{sp}(4) | B^2 = 0, \text{rank } B = 2\}$. The variety $\mathcal{N}$ has a symplectic resolution given by $T^*(G/P)$ where $P \subset \text{Sp}(4)$ is the stabilizer of a Lagrangian subspace. Therefore, $T^*(G/P) \times \mathbb{C}^4 \to \mathcal{M}_0^0(v)$ is a symplectic resolution of singularities. The base of universal deformation is $H^2(T^*(G/P) \times \mathbb{C}^4, \mathbb{C}) \cong H^2(G/P, \mathbb{C}) \neq 0$, e.g. by the Bruhat decomposition. Therefore, κ is not surjective.

6.2. **Examples.** We now apply the idea of subgeneric deformation to compute some Namikawa-Weyl groups. Let us first consider Example 3.6 again.

Example 6.9. Let Q, v, w be as in Example 3.6. Then the codimension 2 roots are listed below: α_1, type (1); β and $\alpha_1 + \beta$, type (2); $\alpha_1 + 2\beta$, $\alpha_1 + 3\beta$ and $2\alpha_1 + 3\beta$, type (3). The linear relations between them satisfy the G_2 Dynkin diagram:
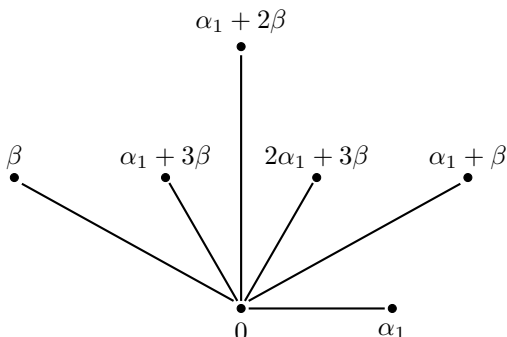
Here, linear relations are normalized according to short/long roots: if a long root γ is involved, we replace γ by 3γ. For example, $\alpha_1 + (2\alpha_1 + 3\beta) = 3(\alpha_1 + \beta)$; the left hand side is the sum of short roots, and the right hand side is a normalized long root. Therefore, $\mathcal{M}_0^0(v)$ has only one codimension 2 root, and its Namikawa-Weyl group is the Weyl group of type G_2.

Proposition 6.10. *Suppose Q is a Dynkin quiver or an affine quiver. Then the Namikawa-Weyl group of $\mathcal{M}_0(v, w)$ is a direct product of type A, D or E Weyl groups.*

Proof. For Dynkin or affine Q, type (3) codimension roots cannot appear. Suppose a codimension 2 root α of type (1) or (2). We show all α-root strings have length at most 2. This will imply all the components of the Namikawa-Weyl group are of simply-laced type.

If not, then there exist $m, n, p, m', n', p' \in \{1, 2, 3\}$ and roots β, γ, γ' of type (1), (2) such that $m\alpha + n\beta = p\gamma, m'\alpha + n'\gamma = p'\gamma'$. Since Q is simply laced Dynkin or affine, it is impossible that all of $\alpha, \beta, \gamma, \gamma'$ are real. By Theorem 6.7, the only possibility is α is type (1) and β, γ, γ' are type (2). Pair both sides of $m\alpha + n\beta = p\gamma$ with $\tilde{v}$, we see $n = p = 1$. Then $(m\alpha + \beta, m\alpha + \beta) = 0$ implies $(\alpha, \beta) = -m$. Similarly, $m'\alpha + n'\gamma = p'\gamma'$ implies $(\alpha, \gamma) = -m'$. But $(\alpha, \gamma) = (\alpha, \beta + m\alpha) = m > 0$, contradiction. $\square$

We recover McGerty and Nevins' results, Proposition 1.13, from Corollary 6.3. We need a combinatorial lemma, whose proof we postpone to the end of this section.

Lemma 6.11. *Suppose Q is a type A, D or E Dynkin or affine type quiver, the dimension vector v and framing w are such that $\nu = \Lambda_w - \sum v_i \alpha_i$ is dominant, and $w \neq 0$. Then for any positive real root α such that $\langle \nu, \alpha \rangle = 0$, $\tilde{v} - \alpha$ is a root of Q^∞.*

The above statement does not hold for wild quivers. For example, consider the following quiver with dimension vector and framing indicated by the numbers.
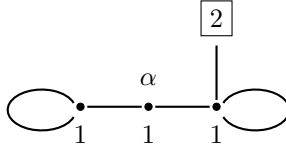
The only simple real root α, associated to the vertex without an edge loop, satisfies $\langle \nu, \alpha \rangle = 0$, but $\tilde{v} - \alpha$ does not have connected support and cannot be a root.

Corollary 6.12 ([MN19, Theorem 5.4]). *Suppose Q is a Dynkin quiver and v, w are dimension vectors and framing such that ν is dominant. Let W_ν denote the Weyl group of the subroot system Φ_ν. Then the Namikawa-Weyl group of $\mathcal{M}_0^0(v, w)$ is W_ν.*

Here, we do not a priori assume Assumption 2.18; instead, we deduce it from our weaker assumptions.

Proof. Recall by Corollary 2.13 that μ is flat and $\mathcal{M}_0^0(v, w) \cong \mathcal{M}_0(v, w)$. Suppose, for a contradiction, that $(\tilde{v}, \alpha_\infty) > 0$. We have seen in the proof of Proposition 2.14 that $(\alpha_\infty, v) = -1$ in this case, and v is a (real) root supported on Q. Then $(\tilde{v}, v) = 1$, contradicting the assumption that ν is dominant. Therefore, $(\tilde{v}, \alpha_i) \leq 0$ for all $i \in Q_0^\infty$, i.e. Assumption 2.18 is satisfied.

Only type (1) codimension 2 roots can appear. Let β be a maximal root of Q with $\langle \nu, \beta \rangle = 0$. By Lemma 6.11, all positive roots $v^1 \leq \beta$ satisfy conditions of a type (1) codimension 2 root. The linear relations between these codimension 2 roots are exactly the linear relations between the subroot system generated by $\mathrm{Supp}(\beta)$. Let W_β denote the Weyl group of this root system. Repeat for all such maximal β, we see the Weyl group $W = \prod_{\beta \in \Phi_\nu^{\max}} W_\beta$, where $\Phi_\nu^{\max}$ denotes the set of maximal positive roots of Q vanishing at ν. Equivalently, $W = W_\nu$. $\square$

We can generalize the above result to affine type quivers.

Let Q be affine type. By the construction of Proposition 2.14 and the canonical decomposition Theorem 2.16, we see that for any v, w, we can reduce the computation of the Namikawa-Weyl group of $\mathcal{M}_0(v, w)$ to that of

(1) $\mathcal{M}_0^0(Q', v', w')$, where Q' is a finite type quiver and $(v')^\sim \in \Sigma_0$, or
(2) $\mathcal{M}_0^0(Q, v', w')$, so that $\tilde{v} \in \Sigma_0$.

Therefore, we may assume Assumption 2.18. Let δ be the minimal positive imaginary root of Q. Let $W_{Q_{fin}}$ denote the Weyl group of Q_{fin}, the Dynkin quiver corresponding to Q. Let $(\Phi_\nu^{\max})'$ be the set of maximal positive real roots β of Q such that $\langle \nu, \beta \rangle = 0$ and $\mathrm{Supp}(\beta) \subsetneq Q_0$. Let W_β denote the Weyl group corresponding to $\mathrm{Supp}(\beta)$ which is a Dynkin diagram. Write W be the Namikawa-Weyl group of $\mathcal{M}_0^0(v, w)$.

Corollary 6.13. *The group W takes the following form.*

(1) *If $\langle \nu, \delta \rangle = 2$ and $\tilde{v} - \delta$ is a root, then $W = \prod_{\beta \in (\Phi_\nu^{\max})'} W_\beta \times \mathbb{Z}/2\mathbb{Z}$.*
(2) *Otherwise, $W = \prod_{\beta \in (\Phi_\nu^{\max})'} W_\beta$*

Proof. We first show $(\tilde{v}, \delta) \leq -2$. In fact, suppose $(\tilde{v}, \delta) \geq -1$. Then since $(\alpha_i, \delta) = 0$ for all $i \in Q_0$, we conclude $(\alpha_\infty, v) \geq -1$. But then $(\alpha_\infty, \tilde{v}) \geq 1$, contradicting Assumption 2.18.

Only type (1) and (2) codimension 2 roots can appear, and a type (2) root must be δ. Suppose v^1 is type (1). Then since ν is dominant, $\langle \nu, \delta \rangle \geq 2$ and $\langle \nu, v^1 \rangle = 0$, we have $\mathrm{Supp}(v^1) \subsetneq Q_0$, i.e. the support is Dynkin. Let β be maximal among such roots. Then by Lemma 6.11, every real root $\alpha \leq \beta$ is a type (1) codimension 2 root. These real roots give subroot systems supported on $\mathrm{Supp}\,\beta$ and corresponding Weyl groups W_β.

Suppose $v^1 = \delta$ is a type 2 codimension 2 root, which implies $\tilde{v} - \delta$ is a root and $(\tilde{v}, \delta) = -2$. By Theorem 6.7, δ is not a linear combination of other (real) codimension 2 roots. Therefore, it contributes a $\mathbb{Z}/2\mathbb{Z}$ component in the Namikawa-Weyl group. $\qquad\square$

Let us finish by examining a related problem, i.e. the Namikawa-Weyl group of symplectic quotient singularities, see [BC20].

Example 6.14. Let Q be affine type, $v = m\delta$ and w be 1 over the extended vertex and 0 elsewhere. Recall μ is flat by Corollary 2.13, so $\mathcal{M}_0(v, w) = \mathcal{M}_0^0(v, w)$. It is classical that $\mathcal{M}_0^0(v, w)$ is the symplectic quotient singularity $\mathbb{C}^{2m}/(\Gamma \rtimes \mathfrak{S}_m)$, where Γ is the finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ corresponding to Q under McKay correspondence and $\mathfrak{S}_m$ is the symmetric group; see e.g. [BC20, Section 1].

In this situation, Assumption 2.18 is not satisfied. Nevertheless, it is easy to write down the canonical decomposition of $\mathcal{M}_0^0(v, w)$:

$$\mathcal{M}_0^0(v, w) = S^m \mathcal{M}_0^0(\delta) \times pt.$$

Clearly, $\mathcal{M}_0^0(v, w)$ has 2 codimension 2 leaves. Let $\alpha_0, \cdots, \alpha_n$ be the simple real roots of Q and write $\delta = \sum_{i=0}^n c_i \alpha_i$. Then, the two codimension 2 leaves correspond to representation types

$$\tau_1 = (\alpha_\infty, 1; \alpha_0, c_0; \cdots; \alpha_n, c_n; \delta, m-1)$$

and

$$\tau_2 = (\alpha_\infty, 1; \delta, 1; \delta, m-1).$$

The first leaf gives a Namikawa-Weyl group component of type $W_{Q_{fin}}$, and the second leaf gives a $\mathbb{Z}/2\mathbb{Z}$. Therefore, the Namikawa-Weyl group of $\mathbb{C}^{2m}/(\Gamma \rtimes \mathfrak{S}_m)$ is $W_{Q_{fin}} \times \mathbb{Z}/2\mathbb{Z}$. This is exactly [BC20, Proposition 2.2].

We now prove Lemma 6.11.

Proof of Lemma 6.11. Suppose β is a maximal element among

$$\{\beta | \langle \nu, \beta \rangle = 0, \beta \text{ is a positive root}\}.$$

First we show that β must be a real root. In fact, if β is imaginary (for affine type Q), then $(\beta, v) = 0$, so $\langle \Lambda_w, \beta \rangle = 0$. But $\mathrm{Supp}(\beta) = Q_0$ and $w \neq 0$, contradiction. Note that if $\beta = \sum_i m_i \alpha_i$ where α_i are simple roots and $m_i \in \mathbb{Z}_{>0}$, then $(\tilde{v}, \alpha_i) = 0$ since ν is dominant. Note also that if $i \in \mathrm{Supp}\,\beta$, then $(\alpha_i, \beta) \geq 0$; otherwise $s_i(\beta) > \beta$ and $(\tilde{v}, s_i\beta) = 0$, contradicting the maximality of β.

Step 1. We show $\tilde{v} - \beta$ has connected support.

If not, let $v_1, v_2, \ldots, v_r$ be the restrictions of $\tilde{v} - \beta$ to the connected components of $\mathrm{Supp}(\tilde{v} - \beta)$; that is, $\sum_{i=1}^r v_i = \tilde{v} - \beta$, and $\mathrm{Supp}(\tilde{v} - \beta) = \bigsqcup_{i=1}^r \mathrm{Supp}(v_i)$. Exactly one of $\mathrm{Supp}(v_i)$, say $\mathrm{Supp}(v_1)$, contains the vertex ∞. For $i \geq 2$, $\mathrm{Supp}(v_i)$ is a proper subdiagram of Q (since it is not connected to ∞), and is therefore a Dynkin diagram.

Let $i \in \mathrm{Supp}(v_2)$, α_i be the simple root attached to i, and consider $(\tilde{v} - \beta, \alpha_i) = (v_2, \alpha_i)$; the equality holds since $\mathrm{Supp}(v_2)$ is a connected component of $\mathrm{Supp}(\tilde{v} - \beta)$. We have the following possibilities for i.

(1) $i \in \mathrm{Supp}(\beta)$.

In this case $(\tilde{v}, \alpha_i) = 0$ and $(v_2, \alpha_i) = (\tilde{v} - \beta, \alpha_i) = -(\beta, \alpha_i) \leq 0$ by maximality of β.

(2) $i \notin \mathrm{Supp}(\beta)$ and is not connected to $\mathrm{Supp}(\beta)$ by an edge.

Then $(\alpha_i, \beta) = 0$, and $(v_2, \alpha_i) = (\tilde{v} - \beta, \alpha_i) = (\tilde{v}, \alpha_i) \leq 0$ since ν is dominant.

(3) i is connected to $\mathrm{Supp}(\beta)$ by an edge and $(\alpha_i, \beta) = -1$.

By maximality of β, we know that $(\tilde{v}, s_i\beta) = (\tilde{v}, \beta + \alpha_i) < 0$; therefore $(\tilde{v}, \alpha_i) \leq -1$. Then $(v_2, \alpha_i) = (\tilde{v} - \beta, \alpha_i) \leq 0$.

(4) i is connected to $\mathrm{Supp}(\beta)$ by an edge, and $(\alpha_i, \beta) \leq -2$.

We still have $(\tilde{v}, \alpha_i) \leq -1$. Note that both β and $s_i\beta = \beta - (\alpha_i, \beta)\alpha_i$ are roots, so $\beta + \alpha_i$ is also a root. Now consider $(\alpha_i + \beta, \alpha_i + \beta) = 4 + 2(\alpha_i, \beta)$. Since Q is of Dynkin or affine type, we must have $(\alpha_i, \beta) = -2$. In this case, if Q is affine type, then $\alpha_i + \beta = \delta$, the minimal positive imaginary root.

Suppose for all $i \in \mathrm{Supp}(v_2)$, i falls in case (1), (2) or (3) above. Then $(v_2, \alpha_i) \leq 0$ for all $i \in \mathrm{Supp}(v_2)$. This contradicts that $\mathrm{Supp}(v_2)$ is a Dynkin diagram.

Therefore, let $i \in \mathrm{Supp}(v_2)$ fall in case (4) and $(v_2, \alpha_i) = (\tilde{v} - \beta, \alpha_i) > 0$. Since $(\tilde{v}, \alpha_i) \leq -1$ and $(\beta, \alpha_i) = -2$, we must have $(\tilde{v} - \beta, \alpha_i) = (v_2, \alpha_i) = 1$. Therefore, for all $i \neq i' \in \mathrm{Supp}(v_2)$, $(v_2, \alpha_{i'}) = (\tilde{v} - \beta, \alpha_{i'}) = (\tilde{v}, \alpha_{i'}) - (\beta, \alpha_{i'}) = (\tilde{v}, \alpha_{i'}) - (\delta - \alpha_i, \alpha_{i'}) = (\tilde{v}, \alpha_{i'}) + (\alpha_i, \alpha_{i'}) \leq 0$.

We conclude that $\mathrm{Supp}(v_2)$ is a proper subdiagram of Q that contains an extended vertex. By checking the position of extended vertices of all affine type quivers, we see $\mathrm{Supp}(v_2)$ must be of type A_n, D_n, E_6 or E_7. In terms of [OV12, Table 2] (and compare to [OV12, Table 6]), i may correspond to

(a) any column of the Cartan matrix if $\mathrm{Supp}(v_2)$ is type A_n;
(b) the first column or the last two columns of the Cartan matrix if $\mathrm{Supp}(v_2)$ is type D_n;
(c) the first or the fifth column of the Cartan matrix if $\mathrm{Supp}(v_2)$ is type E_6;
(d) the first column of the Cartan matrix if $\mathrm{Supp}(v_2)$ is type E_7.

By inspection, we see there is no positive integral vector satisfying $(v_2, \alpha_i) = 1, (v_2, \alpha_j) \leq 0$ for all $j \neq i$. Therefore, the component v_2 of $v - \beta$ cannot exist, and we conclude $\tilde{v} - \beta$ has connect support.

Step 2. We show $\tilde{v} - \beta$ is a root. We have the following cases.

(I) For all $i \in \mathrm{Supp}(v - \beta)$, $(\tilde{v} - \beta, \alpha_i) \leq 0$. There are 2 subcases.

(i) $(\tilde{v} - \beta, \alpha_\infty) \leq 0$. In this case, $(\tilde{v} - \beta, \alpha_j) \leq 0$ for all $j \in Q_0^\infty$, and $\tilde{v} - \beta$ is an imaginary root.

(ii) $(\tilde{v} - \beta, \alpha_\infty) \geq 1$. In this case, note that $(\tilde{v} - \beta, \alpha_\infty) = (\alpha_\infty + v - \beta, \alpha_\infty) = 2 + (v - \beta, \alpha_\infty) \leq 1$, so $(\tilde{v} - \beta, \alpha_\infty) = 1$. Thus, there is a unique edge connecting the vertex ∞ to some other vertex in $\mathrm{Supp}(\tilde{v} - \beta)$; we call this vertex i_1. The multiplicity of α_{i_1} in $\tilde{v} - \beta$ is 1. See Figure 1 for a sketch; there the numbers below the vertices are the multiplicities of the corresponding simple roots in $\tilde{v} - \beta$. Therefore, $(\tilde{v} - \beta, \alpha_{i_1}) \leq 1$.
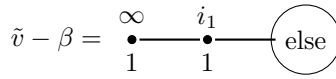
$$\tilde{v} - \beta =$$

FIGURE 1. The dimension vector $\tilde{v} - \beta$.

If $(\tilde{v} - \beta, \alpha_{i_1}) \leq -1$, then $(s_\infty(\tilde{v} - \beta), \alpha_j) = (\tilde{v} - \beta - \alpha_\infty, \alpha_j) \leq 0$ for all $j \in Q_0^\infty$, and we get an imaginary root.

If $(\tilde{v} - \beta, \alpha_{i_1}) = 1$ then $\tilde{v} - \beta = \alpha_1 + \alpha_\infty$ is a real root.

Finally, if $(\tilde{v} - \beta, \alpha_{i_1}) = 0$ then there is a unique vertex $i_2 \in \text{Supp}(\tilde{v} - \beta)$ such that i_1 is connected only to ∞ and i_2 by a single edge, and the multiplicity of α_{i_2} in $\tilde{v} - \beta$ is 1. In Figure 1, i_2 belongs to the "else" part. Apply the previous arguments to $s_\infty(\tilde{v} - \beta) = v - \beta$ (now α_{i_1} plays the role of α_∞ before). After finitely many iterations we see $s_{i_k} s_{i_{k-1}} \ldots s_\infty(\tilde{v} - \beta)$ is a root for a sequence of vertices i_k, and therefore $\tilde{v} - \beta$ is a root.

(II) For some $i \in \text{Supp}(v - \beta)$, $(\tilde{v} - \beta, \alpha_i) \geq 1$. Since $(\tilde{v}, \alpha_i) \leq 0$, we have $(\alpha_i, \beta) \leq -1$. Then the vertex i is connected to β; so by maximality of β we have $(\tilde{v}, \alpha_i) \leq -1$, so $(\alpha_i, \beta) \leq -2$. In other words, i must fall in case (4) in Step 1. In particular, Q is affine type, $i \notin \text{Supp}(\beta)$, $(\tilde{v}, \alpha_i) = -1$ and $\delta = \alpha_i + \beta$. Therefore

$$(\tilde{v}, \delta) = (\tilde{v}, \alpha_i + \beta) = (\alpha_\infty, \delta) = -1,$$

so the framing w has the form $w_{i'} = 1$ for some i' such that multiplicity of $\alpha_{i'}$ in δ is 1, and $w_j = 0$ for all other j.

We claim $i' = i$. Otherwise, $i' \in \text{Supp}(\beta)$, and $(v, \alpha_{i'}) = -(\alpha_\infty, \alpha_{i'}) = 1$. Also, $(v, \alpha_i) = (\tilde{v}, \alpha_i) = -1$, and $(v, \alpha_j) = 0$ for $Q_0 \ni j \neq i, i'$. The set for all such $v = (v_k)|_{k \in Q_0}$, i.e. the integral solution set of the linear system

$$\begin{cases} (v, \alpha_{i'}) = 1, \\ (v, \alpha_i) = -1, \\ (v, \alpha_j) = 0 \qquad \text{for all } j \in Q_0, j \neq i, i' \end{cases}$$

is the set of integral elements in the affine line $v' + \mathbb{C}\delta$, where v' is any special solution.

To find a special solution $v' = (v'_k)|_{k \in Q_0}$, we may let $v'_i = 0$. Then $\text{Supp}(v') \subset \text{Supp}(\beta)$, a Dynkin diagram, and the problem is reduced to solving the system

$(\star')$
$$\begin{cases} v_i = 0, \\ (v, \alpha_{i'}) = 1, \\ (v, \alpha_j) = 0 \qquad \text{for all } j \in \text{Supp}(\beta), j \neq i' \end{cases}$$

which has a unique solution. Recall that the multiplicity of $\alpha_{i'}$ in δ is 1. In terms of [OV12, Table 2], i' may correspond to any column of the Cartan matrix if Q is type A_n; i' may correspond to the first or last 2 columns of the Cartan matrix if Q is type D_n; i' may correspond to the first or the fifth column of the Cartan matrix if Q is type E_6; i' may correspond to the first column of the Cartan matrix if Q is type E_7; and Q cannot be type E_8. By inspection, we see v' cannot be integral in any case. But the

multiplicity of α_i in δ is 1, and therefore an integral solution for v has the form $v' + n\delta$. We conclude there is no integral solution for v.

Therefore $w_i = 1, w_j = 0$ for $j \neq 0$. Then for $j \in \operatorname{Supp}(\beta)$, $(v, \alpha_j) = (\tilde{v}, \alpha_j) = 0$; and $(v, \alpha_i) = (\tilde{v}, \alpha_i) + 1 = 0$. Hence $v = n\delta$ for some positive integer n, and it is clear that $\tilde{v} - \beta$ is a real root.

This finishes *Step* 2 of the proof.

Step 3. We show that for any positive real root $\alpha \leq \beta$, $\tilde{v} - \alpha$ is a root.

In fact, there is a sequence of simple reflections $s_{i_1}, \ldots, s_{i_m}$ where each $i_k \in \operatorname{Supp}(\beta)$ such that $\alpha = s_{i_1} s_{i_2} \ldots s_{i_m} \beta$. Since $(\tilde{v}, \alpha_{i_k}) = 0$, we have $s_{i_1} s_{i_2} \ldots s_{i_m} (\tilde{v} - \beta) = \tilde{v} - \alpha$. Hence $\tilde{v} - \alpha$ is also a root. This finishes the proof of the lemma. $\square$

Acknowledgments

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