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ON THE EXTENDED WEIGHT MONOID OF A SPHERICAL HOMOGENEOUS SPACE AND ITS APPLICATIONS TO SPHERICAL FUNCTIONS

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ABSTRACT. Given a connected simply connected semisimple group G and a connected spherical subgroup $K \subseteq G$ we determine the generators of the extended weight monoid of G/K, based on the homogeneous spherical datum of G/K.

Let $H \subseteq G$ be a reductive subgroup and let $P \subseteq H$ be a parabolic subgroup for which G/P is spherical. A triple (G, H, P) with this property is called multiplicity free system and we determine the generators of the extended weight monoid of G/P explicitly in the cases where (G, H) is strictly indecomposable.

The extended weight monoid of G/P describes the induction from H to G of an irreducible H-representation $\pi : H \to \mathrm{GL}(V)$ whose lowest weight is a character of P. The space of regular $\mathrm{End}(V)$-valued functions on G that satisfy $F(h_1gh_2) = \pi(h_1)F(g)\pi(h_2)$ for all $h_1, h_2 \in H$ and all $g \in G$ is a module over the algebra of H-biinvariant regular functions on G. We show that under a mild assumption this module is freely and finitely generated. As a consequence the spherical functions of such a type π can be described as a family of vector-valued orthogonal polynomials with properties similar to Jacobi polynomials.

CONTENTS

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1. Introduction

Let G be a connected simply connected semisimple group over the field of complex numbers $\mathbb{C}$ and let H be a symmetric subgroup. The polynomial algebra $\mathbb{C}[G]^{H\times H}$ of regular H-biinvariant functions can be equipped with a Hermitean inner product in the usual way by integrating over a maximal compact subgroup of G. The matrix coefficients that correspond to irreducible G-modules with an H-fixed vector constitute an orthogonal basis of $\mathbb{C}[G]^{H\times H}$. The latter are called H-spherical G-representations and it is known that the space of H-invariants in an irreducible G-module is at most one-dimensional. The harmonic analysis of the H-biinvariant functions has been an important motivation for the development of the Heckman-Opdam polynomials.

In viewing H-biinvariant matrix coefficients on G as Heckman-Opdam polynomials we need to know which irreducible G-modules have an H-fixed vector. For symmetric spaces this is understood by the Cartan-Helgason theorem. We develop a vast generalization of this theorem in the following context. Instead of looking at symmetric subgroups we look at a spherical subgroup K of the reductive group G and we determine the irreducible G-modules that have a K-stable line. This information is encoded in the extended weight monoid $\widetilde{\Gamma}(G/K)$ associated to the pair (G,K), whose definition we recall below. More precisely, in Theorem 5.5 we give a general combinatorial procedure to derive the generators of $\widetilde{\Gamma}(G/K)$ from the homogeneous spherical datum of G/K. Computations of extended weight monoids are found in the literature for particular spherical subgroups, and carried out with ad hoc methods. Reductive ones are dealt with by Krämer [24] (for G simple) and by Avdeev [2] (for G not simple); the approach is case-by-case using representation theory and explicit computations. The case of K solvable is found in Avdeev and Gorfinkel′'s paper [5] for G semisimple, generalized in Avdeev [4] for G connected reductive.

Computing the extended weight monoid depends on how the character group of K is presented. In [4] Avdeev gave a presentation of the character group of K by generators and relations, where (in the case G is semisimple and simply connected, as in this paper) the generators are precisely the second components of the generators of the extended weight monoid. Here we use the following approach. We choose a parabolic subgroup P of G containing K and minimal with respect to these properties. Then we give the generators of $\widetilde{\Gamma}(G/K)$ as couples (ω,χ') where χ' is a character of P that restricts to the desired character of K.

We apply our techniques to the following situation. Let H be a connected reductive subgroup of G and let P be a parabolic subgroup of H that is spherical in G. Let $\pi: H\to \mathrm{GL}(V)$ be an irreducible representation of H such that V has a P-stable line. Then V occurs with multiplicity at most one in the decomposition of each irreducible G-module into H-modules. The G-modules for which the multiplicity of V is equal to one can be recovered from the extended weight monoid of the pair (G,P). The triples (G,H,P) as above are called multiplicity free systems and they have been classified by He, Nishiyama, Ochiai, and Oshima in [16] and van Pruijssen [31]. We provide the generators of the extended weight monoids of all the pairs (G,P) coming from multiplicity free systems from the classification.

For a multiplicity free system (G, H, P) and an H-module V as above we consider the ring $E^0 = \mathbb{C}[G]^{H\times H}$ of H-biinvariant functions and the E^0-module

$$E = (\mathbb{C}[G] \otimes \operatorname{End}(V))^{H\times H}$$

consisting of regular $\operatorname{End}(V)$-valued functions F on G with $F(h_1gh_2) = \pi(h_1)F(g)\pi(h_2)$ for all $h_1, h_2 \in H$ and all $g \in G$. We show that E is a finitely and freely generated E^0-module under some mild assumptions. In view of the Peter-Weyl Theorem the space E has a natural inner product and a natural orthogonal basis consisting of spherical functions. Moreover, these basis elements are uniquely determined up to scaling as simultaneous eigenfunctions of a commutative algebra of differential operators, see e.g. [31, §6]. The isomorphism $E \cong E^0 \otimes \mathbb{C}^N$ for some N depending on V, of E^0-modules, now provides the space $E^0 \otimes \mathbb{C}^N$ of vector-valued polynomials with an orthogonal basis with interesting properties. In particular, the basis elements are again determined by a commutative algebra of differential operators. The details of this construction can be found in [31, §6]. In loc. cit. this structure was only proved for three examples with ad hoc methods. The results in this paper show that this module structure is much more general and holds true for all the interesting cases, viz. where $N > 1$. In this way we obtain an abundance of families of vector-valued orthogonal polynomials in several variables that have properties similar to the Jacobi polynomials. If we choose H symmetric in G and V a one-dimensional representation, then we recover the Heckman-Opdam polynomials with geometric parameters. If the spherical pair is of rank one, then we recover the results of Heckman and van Pruijssen [17], where the explicit branching rules were used to derive the module structure.

Some examples of families of vector-valued polynomials in several variables associated to multiplicity free systems in this paper have been worked out by Koelink, van Pruijssen and Román [23] and by Koelink and Liu [22], where also connections to classical orthogonal polynomials have been established.

The organization of this paper is as follows. In Section 2 we fix the notation and recall some basic facts about spherical varieties. We reformulate some results of spherical combinatorics in Section 3 to be used in the computations of the extended weight monoid that we discuss in Sections 4 and 5. In Section 6 we prove a sufficient combinatorial condition for a subgroup P of a spherical subgroup H of G to be parabolic. This situation occurs for multiplicity free systems (G, H, P) and we use the results to compute the extended weight monoids of the corresponding pairs (G, P). We start with a typical example in Section 7 where $G = \operatorname{Spin}(2n + 2)$ and $H = \mathbb{C} \times \operatorname{Spin}(2n)$, where two parabolic subgroups of H are large enough to remain spherical in G. The computations of the generators of all other multiplicity free triples are provided in the Appendices while the generators are tabulated in the tables of Section 9. In Section 8 we discuss the module structure of the spaces E.

While this paper was being finalized, Avdeev's paper [1] appeared, where another way to compute the extended weight monoid of spherical homogeneous spaces G/K in general is given. It is based on an explicit description of K as a subgroup of P. There is some overlap with our techniques: in particular, we compute in the same way some generators coming easily from the natural morphism $G/K \to G/P$. For the other generators our approach is different, and uses only the homogeneous spherical datum of G/K.

2. Notations and preliminaries

We fix a simply connected semisimple group G over $\mathbb{C}$. We fix a choice of a Borel subgroup $B_G \subseteq G$ and a maximal torus $T_G \subseteq B_G$, we denote by S_G the corresponding set of simple roots, by B_G^- the opposite Borel subgroup of B_G, and by W_G the Weyl group of G. When no confusion arises, for simplicity they will be also denoted by B, T, S, B^-, W respectively. Given α a simple root of G, we denote the corresponding fundamental dominant weight by ω_α.

In general, if K is an algebraic group, the group of its characters is denoted by $\mathcal{X}(K)$, and the connected component of K containing the neutral element by K°. The commutator of K is denoted by $[K,K]$, the center by $Z(K)$, and all subgroups of K are assumed to be closed. The group $\mathbb{C} \setminus \{0\}$ will be denoted by $\mathbb{C}^\times$.

We recall Definitions 2.1 and 2.2.

Definition 2.1. A subgroup L of a connected reductive group M is called *very reductive* if L is not contained in any proper parabolic subgroup of M.

Definition 2.2. A *spherical G-variety* (or simply a *spherical variety*) is a normal irreducible G-variety having an open B-orbit. If $K \subseteq G$ is a closed subgroup such that G/K is a spherical variety, then K is called a *spherical subgroup* of G.

Let X be a spherical variety. We denote by $\mathbb{C}(X)^{(B)}$ the multiplicative subgroup of non-zero rational functions on X that are eigenvectors for the action of B-translation. The eigenvalues of such eigenvectors form a lattice of characters of B, denoted by $\Xi(X)$. Its rank is by definition the *rank* of X, and we will denote the vector space $\Xi(X) \otimes_\mathbb{Z} \mathbb{Q}$ simply by $\Xi(X)_\mathbb{Q}$. We also associate with X the vector space

$$N(X) = \mathrm{Hom}_\mathbb{Z}(\Xi(X), \mathbb{Q}).$$

The natural pairing between $\Xi(X)$ (or $\Xi(X)_\mathbb{Q}$) and $N(X)$ is denoted by $\langle -, - \rangle$.

By a *discrete valuation* on $\mathbb{C}(X)$ we mean a map $v\colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Q}$ such that $v(fg) = v(f) + v(g)$ for all $f, g \in \mathbb{C}(X) \setminus \{0\}$, such that $v(f+g) \geq \min\{v(f), v(g)\}$ whenever $f, g, f+g \in \mathbb{C}(X) \setminus \{0\}$, such that $v(f) = 0$ if f is constant, and such that the image of v is a discrete (additive) subgroup of $\mathbb{Q}$.

Any discrete valuation v of $\mathbb{C}(X)$ can be restricted to $\mathbb{C}(X)^{(B)}$. Since X is spherical, this yields a well-defined element $\rho(v)$ of $N(X)$, by identifying $\Xi(X)$ with the multiplicative group $\mathbb{C}(X)^{(B)}$ modulo the constant functions. This applies in particular to the valuation associated with any B-stable prime divisor D of X, and in this case we denote simply by $\rho(D)$ (or $\rho_X(D)$) the corresponding element of $N(X)$.

The B-stable but not G-stable prime divisors of X are called *colors*, and their set is denoted by $\Delta(X)$. Given a color $D \in \Delta(X)$ and a simple root $\alpha \in S$, we say that α *moves* D if D is not stable under the minimal parabolic subgroup of G strictly containing B corresponding to α.

There exists a minimal set $\Sigma(X)$ of primitive elements of $\Xi(X)$ such that the set

$$\mathcal{V}(X) = \{\eta \in N(X) \mid \langle \eta, \sigma \rangle \leq 0 \; \forall \sigma \in \Sigma(X)\}$$

is equal to the set of the elements $\rho(\nu)$ for ν any G-stable discrete valuation of $\mathbb{C}(X)$. The elements of $\Sigma(X)$ are called the *spherical roots* of X.

A simple root $\alpha \in S$ can move up to two colors of X, and it moves two colors if and only if $\alpha \in S \cap \Sigma(X)$. The set of simple roots moving no color is denoted by

$S^p(X)$, and the set of colors moved by some simple roots in $S \cap \Sigma(X)$ is denoted by $\mathbf{A}(X)$.

The *homogeneous spherical datum* of X is defined as the quintuple

$$(S^p(X), \Sigma(X), \mathbf{A}(X), \Xi(X), \rho_X \colon \Delta(X) \to N(X)).$$

The pairing $\mathbb{Z}\Delta(X) \times \Xi(X) \to \mathbb{Z}$ induced by $(D, \xi) \mapsto \langle \rho_X(D), \xi \rangle$ for $D \in \Delta(X)$ and $\xi \in \Xi(X)$ is also denoted by $c(-,-)$ (or $c_X(-,-)$), and it is called the *Cartan pairing* of X.

We recall that spherical homogeneous spaces are classified up to G-equivariant isomorphisms by their homogeneous spherical data, which can be defined as purely combinatorial objects satisfying the axioms given by Luna in [26, Section 2].

Many of the spherical subgroups encountered in this paper are *wonderful*, so we recall the definition of this notion.

Definition 2.3. A spherical subgroup K of G is *wonderful* if $\Sigma(G/K)$ is a basis of the lattice $\Xi(G/K)$.

3. Spherical closure and morphisms

In this section we recall some combinatorial constructions and results on equivariant morphisms between spherical homogeneous spaces. Our goal is to prove Proposition 3.7 and Corollary 3.9, which are reformulations of well-known facts. They are formulated with the specific goal of being easy to apply in examples, when computing extended weight monoids of specific spherical homogeneous spaces, so to require as few verifications as possible. For this reason, in this form they are not found in the literature, and we need to recall several facts in order to show how our statements derive from known results.

We first need the following standard definition.

Definition 3.1. Let K be a spherical subgroup of G. The *spherical closure* $\overline{K}$ is defined as the kernel of the action of the normalizer $N_G K$ on $\Delta(G/K)$, induced by the natural action of $N_G K/K$ on G/K by G-equivariant automorphisms. The *spherically closed spherical roots* of G/K, or of any spherical variety X with open G-orbit G/K, are the spherical roots of $G/\overline{K}$. Their set is denoted by $\Sigma^{sc}(X)$.

Remark 3.2.

(1) From its definition it is obvious that $\overline{K}$ contains K, hence $\overline{K}$ is a spherical subgroup of G. Also, for any subgroup J such that $K \subseteq J \subseteq \overline{K}$, the spherical homogeneous spaces G/K and G/J "have the same colors", in the sense that the natural morphism $G/K \to G/J$ induces a bijection $\Delta(G/J) \to \Delta(G/K)$ compatible with the Cartan pairing, and respecting the property of a color of being moved by any given simple root.

(2) The spherically closed spherical roots of a spherical variety X are easily deduced from $\Sigma(X)$. Indeed, the elements of $\Sigma^{sc}(X)$ are equal to the elements of $\Sigma(X)$, except for the fact that any $\sigma \in \Sigma(X)$ is replaced by 2σ, if $\sigma \notin S$ and 2σ is a spherical root of some spherical G-variety. For a proof of this fact see Pezzini and Van Steirteghem [28, Proposition 2.7], where also a more explicit combinatorial description of $\Sigma^{sc}(X)$ can be found.

(3) The spherical closure of any spherical subgroup is wonderful, thanks to a deep result by Knop [20, Corollary 7.6].

We also recall Proposition 3.3, part of Luna's theory of *augmentations*.

Proposition 3.3 ([26, Lemme 6.3.1], see also [27, Section 3])**.** *Let K be a spherical subgroup of G. Then $J \mapsto \Xi(G/J)$ is a bijection between the set of subgroups J of G satisfying $K \subseteq J \subseteq \overline{K}$ and the set of lattices Ξ' such that $\Xi(G/K) \supseteq \Xi' \supseteq \Xi(G/\overline{K}) = \operatorname{span}_{\mathbb{Z}}(\Sigma^{sc}(G/K))$.*

Remark 3.4. In Proposition 3.3, the subgroup J has finite index in $\overline{K}$ if and only if $\Xi(G/\overline{K})$ has finite index in $\Xi(G/J)$. This follows easily from a lemma of Gandini's [15, Lemma 2.4].

We come to equivariant morphisms between spherical varieties. Definitions 3.5 and 3.6 and Proposition 3.7 are well known.

Definition 3.5. A subset $\Delta' \subseteq \Delta(X)$, is called *distinguished* if there is a linear combination of elements of $\rho_X(\Delta')$, with non-negative rational coefficients, that is non-negative on all spherical roots of X. In addition, Δ' is called *parabolic* if there is such a linear combination that is strictly positive on all spherical roots.

Definition 3.6. Let $\varphi\colon X \to Y$ be a dominant G-equivariant morphism between two spherical G-varieties X, Y. We denote by Δ_φ the set of colors of X mapped dominantly to Y.

Proposition 3.7. *Let φ, X, and Y be as in Definition* 3.6*. Then Δ_φ is a distinguished subset of $\Delta(X)$, and we have*

$$\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(Y) = \left(\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(X)\right) \cap \left(\bigcap_{D \in \Delta_\varphi} \ker \rho_X(D) \right). \tag{1}$$

If in addition Y is homogeneous and complete, then Δ_φ is parabolic. Finally, given X homogeneous and a distinguished subset Δ' of $\Delta(X)$, there exist unique (up to G-equivariant isomorphism) Y and φ such that Y is a homogeneous G-variety Y with $\operatorname{span}_{\mathbb{Q}} \Sigma(Y) = \Xi(Y)_{\mathbb{Q}}$, and $\varphi\colon X \to Y$ is a G-equivariant morphism with connected fibers such that $\Delta_\varphi = \Delta'$. If Δ' is parabolic, then Y is complete.

Proof. We show how to reduce the proposition to known results in the literature. First, one may assume X and Y homogeneous by replacing them with their respective open G-orbits, say $X = G/K$ and $Y = G/J$, with $K \subseteq J$ and φ equal to the natural map induced by the inclusion of these subgroups. Also, notice that $\Xi(Y) \subseteq \Xi(X)$ by pulling back B-semiinvariant rational functions from Y to X.

Second, we reduce the proof of equality (1) to the case where φ has connected fibers. Let $\widetilde{J}$ be the union of those connected components of J that intersect K, and consider the factorization of φ given by the natural morphisms $G/K \to G/\widetilde{J} \to G/J$. Observe that $\widetilde{J}^\circ = J^\circ$: this implies by [19, Theorem 4.4 and Proof of Theorem 6.1] that G/J and $G/\widetilde{J}$ have the same spherical roots, up to replacing some elements with positive rational multiples. On the other hand a color of G/K goes dominantly to G/J if and only if it goes dominantly to $G/\widetilde{J}$, therefore equality (1) for the morphism $G/K \to G/\widetilde{J}$ implies the same equality for φ. So, from now on we assume φ has connected fibers.

The subset Δ_φ of $\Delta(X)$ is distinguished because the convex cone generated by $\rho_X(\Delta_\varphi)$ together with $\mathcal{V}(X) \cap N_\varphi$ is a vector subspace of $N(X)$. Here N_φ is the vector subspace of $N(X)$ of the elements that are zero on $\Xi(Y)$, and this result follows from [19, Lemma 4.3, part b)].

In particular N_φ is generated, as a vector space, by $\rho_X(\Delta_\varphi)$ and $N_\varphi \cap \mathcal{V}(X)_{\mathrm{lin}}$, where $\mathcal{V}(X)_{\mathrm{lin}}$ is the linear part of the convex cone $\mathcal{V}(X)$. This implies that

$$\Xi(Y)_{\mathbb{Q}} = U \cap \left(\bigcap_{D\in\Delta_\varphi} \ker \rho_X(D) \right), \tag{2}$$

where U is the common kernel of all elements of $N_\varphi \cap \mathcal{V}(X)_{\mathrm{lin}}$. Notice that

$$U \supseteq \operatorname{span}_{\mathbb{Q}} \Sigma(X). \tag{3}$$

The last assertions of Knop's Theorem 4.4 in [19] also imply that

$$\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(Y) = \Big(\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(X)\Big) \cap \Xi(Y)_{\mathbb{Q}},$$

which yields (1) thanks to (2) and (3).

Vice versa, let X be a spherical homogeneous space and Δ' a distinguished subset of $\Delta(X)$, and denote by N' the vector subspace of $N(X)$ of the elements vanishing on

$$M' = \big(\operatorname{span}_{\mathbb{Q}} \Sigma(X)\big) \cap \left(\bigcap_{D\in\Delta_\varphi} \ker \rho_X(D) \right).$$

Theorem 4.4 in loc. cit. implies that there exist a spherical homogeneous space Y and a dominant G-equivariant morphism $\varphi\colon X \to Y$ with connected fibers, such that $N_\varphi = N'$ (i.e. $M' = \Xi(Y)_{\mathbb{Q}}$), and $\Delta_\varphi = \Delta'$. The same theorem also states that Y and φ are uniquely determined by Δ_φ and N_φ, under the assumptions that Y is homogeneous and that φ is dominant with connected fibers. So Y and φ are uniquely determined by Δ_φ if we fix the subspace $N_\varphi = N'$.

Finally, it is well known that a spherical homogeneous space Y is complete if and only if $\Xi(Y) = \{0\}$, which is equivalent to $N' = N(X)$, which is equivalent to Δ' being parabolic. □

Remark 3.8. Notice that $\Sigma(Y)$ is uniquely determined by the equality (1) and $\Xi(Y)$, because by definition its elements are primitive in the lattice $\Xi(Y)$.

Corollary 3.9. *Let K, J be spherical subgroups of G and set $X = G/K$, $Y = G/J$. Suppose that* $\operatorname{span}_{\mathbb{Q}} \Sigma(Y) = \Xi(Y)_{\mathbb{Q}}$, *and that there exist*

(1) *a distinguished subset $\Delta' \subseteq \Delta(X)$ such that equality* (1) *holds, where we substitute Δ' for Δ_φ, and*
(2) *a bijection $\psi\colon \Delta(Y) \to \Delta(X) \setminus \Delta'$ such that $\rho_Y(\psi(D)) = \rho_X(D)|_{\Xi(Y)_{\mathbb{Q}}}$ for all $D \in \Delta(X) \setminus \Delta'$, and such that any simple root moves D if and only if it moves $\psi(D)$.*

If K is connected, then K is conjugated in G to a subgroup of J.

Proof. By Proposition 3.7 there exist a spherical homogeneous space $\widetilde{Y}$ and a dominant G-equivariant morphism $\varphi\colon X \to \widetilde{Y}$ with connected fibers, such that $\Delta_\varphi = \Delta'$ and $\operatorname{span}_{\mathbb{Q}} \Sigma(Y) = \operatorname{span}_{\mathbb{Q}} \Sigma(\widetilde{Y})$, hence $\Xi(Y)_{\mathbb{Q}} = \Xi(\widetilde{Y})_{\mathbb{Q}}$. Let $\widetilde{J}$ be such that $\widetilde{Y} \cong G/\widetilde{J}$ and the point $eK \in X$ is sent to $e\widetilde{J}$ by φ, i.e. $K \subseteq \widetilde{J}$. Pulling back along φ induces a bijection $\widetilde{\psi}\colon \Delta(\widetilde{Y}) \to \Delta(X) \setminus \Delta'$ with the same above properties stated for ψ, where we replace $\Xi(Y)$ with $\Xi(\widetilde{Y})$ (notice that the inverse image of a color of $\widetilde{Y}$ is a single color of X because φ has connected fibers).

We conclude that $\widetilde{Y}$ and Y have the same homogeneous spherical datum, up to the fact that the lattices $\Xi(Y)$ and $\Xi(\widetilde{Y})$ may be different (but both have maximal rank in the same $\mathbb{Q}$-vector space), and the spherical roots may differ by some rescaling with positive rational numbers. We claim that then the groups $\widetilde{J}^\circ$ and J° are conjugated in G, which would conclude the proof.

Let us show the claim. First we observe that $\Sigma(Y) \cap S = \Sigma(\widetilde{Y}) \cap S$, because they are equal up to rescaling, and we have a bijection between the colors of Y and of $\widetilde{Y}$ respecting the property of being moved by any given simple root. This implies that the simple roots that move two colors each correspond, and these are exactly the simple roots that are spherical roots.

From Remark 3.2, part (2), we conclude that $\Sigma^{sc}(Y) = \Sigma^{sc}(\widetilde{Y})$. By Losev's uniqueness theorem [25, Theorem 1], the spherical closures of J and of $\widetilde{J}$ are conjugated, so we may assume they are equal. Remark 3.4 implies that both J and $\widetilde{J}$ have finite index in $\overline{J}$, which yields the desired equality $J^\circ = \widetilde{J}^\circ$. □

4. The extended weight monoid

Let $K \subseteq G$ be a subgroup. If a regular function $f \in \mathbb{C}[G]$ is simultaneously an eigenvector for the left translation action of B on $\mathbb{C}[G]$ and an eigenvector for the right translation action of K, then we denote by $\omega_f \in \mathcal{X}(B)$ and $\chi_f \in \mathcal{X}(K)$ the B-eigenvalue and the K-eigenvalue respectively.

Definition 4.1. Let K be a spherical subgroup of G. The *extended weight monoid*[1] of $X = G/K$ is the set of couples (ω_f, χ_f), for $f \in \mathbb{C}[G]$ varying in the set of B-eigenvectors for the left translation action and K-eigenvectors for the right translation action. It is denoted by $\widetilde{\Gamma}(X)$.

Consider $X = G/K$ as in Definition 4.1, and $D \in \Delta(X)$. Since G is simply connected, the inverse image $\widetilde{D}$ under the natural projection $G \to G/K$ has a global equation f_D, unique up to a non-zero multiplicative constant. Therefore it is well-defined the element $(\omega_D, \chi_D) = (\omega_{f_D}, \chi_{f_D})$.

We recall the following well-known proposition.

Proposition 4.2. *Let $X = G/K$ be a spherical homogeneous space. Then the monoid $\widetilde{\Gamma}(X)$ is freely generated by the elements (ω_D, χ_D) for D varying in $\Delta(X)$.*

Proof. The weights (ω_D, χ_D) are linearly independent, see Brion [12, Lemma 2.1.1]. We prove they generate the monoid $\widetilde{\Gamma}(X)$, so let $f \in \mathbb{C}[G]$ be a left-B-eigenvector and a right-K-eigenvector. Its divisor is invariant under left translation by B and right translation by K, therefore it is the pull-back along the projection $G \to G/K$ of a B-stable divisor δ on G/K.

Since B is connected, we have that δ is a linear combination of B-stable prime divisors of G/K, i.e. of colors:

$$\delta = \sum_{D \in \Delta(X)} n_D D,$$

where n_D is a non-negative integer for all D. Then the product $F = \prod_{D \in \Delta(X)} f_D^{n_D}$ is a regular function on G having the same divisor of f. It follows that F is a

[1] It is also called the *extended weight semigroup*.

scalar multiple of f, yielding $(\omega_F, \chi_F) = (\omega_f, \chi_f)$. It remains to observe that by construction

$$(\omega_F, \chi_F) = \sum_{D\in\Delta(X)} n_D(\omega_D, \chi_D).$$

□

Lemma 4.3 is an easy generalization of a result of Foschi's [14]. It provides an explicit formula for ω_D for any color D.

Lemma 4.3 ([14])**.** *Let $X = G/K$ be a spherical homogeneous space. Let $D \in \Delta(X)$ and let α be a simple root moving D. Set $\varepsilon = 2$ if $2\alpha \in \Sigma(X)$, or $\varepsilon = 1$ otherwise. Then*

$$\omega_D = \varepsilon \sum_{\beta} \omega_\beta,$$

where β varies in the set of simple roots that move D.

Proof. See [30, Lemma 30.24]. □

5. Computing the extended weight monoid

Let $X = G/K$ be a spherical homogeneous space. In this section we explain how to compute explicitly the extended weight monoid of X, using the homogeneous spherical datum of X. For this we refer to the generators of $\widetilde{\Gamma}(X)$ appearing in Proposition 4.2. The weights ω_D of that proposition are given in Lemma 4.3. Here we will show how to derive the weights χ_D, expressing them as restrictions of certain weights of T, provided we choose K in its conjugacy class in a suitable way.

Let us fix a parabolic subgroup Q of G minimal containing K. Up to conjugating Q and K in G, we may assume that Q contains K and also B_-, and we denote by L_Q the Levi subgroup of Q containing T. We may also assume that K has a Levi subgroup L_K contained in L_Q, then L_K is spherical and very reductive in L_Q, and we have $K^u \subseteq Q^u$ (see e.g. [11, Proposition I.1]).

Let us denote by Δ' the parabolic subset of colors corresponding to the natural map $\pi\colon X \to G/Q$. Then the colors of G/Q are the images $\pi(D)$ of the colors $D \in \Delta(X) \setminus \Delta'$.

The set $\Delta(G/Q)$ is also identified with $S \setminus S^p(G/Q)$, by associating a color $\pi(D) \in \Delta(G/Q)$ with the (unique) simple root α that moves it. Moreover, for all $D \in \Delta(X) \setminus \Delta'$, a simple application of the Peter-Weyl theorem yields

$$(\omega_{\pi(D)}, \chi_{\pi(D)}) = (\omega_\alpha, -\omega_\alpha^Q), \tag{4}$$

where we denote here by $\omega_\alpha^Q\colon Q \to \mathbb{C}^\times$ the extension of $\omega_\alpha\colon T \to \mathbb{C}^\times$ to Q. Denote by ω_α^K the restriction $\omega_\alpha^Q|_K\colon K \to \mathbb{C}^\times$.

Consider now a color $D \in \Delta(X) \setminus \Delta'$: equality (4) yields

$$(\omega_D, \chi_D) = (\omega_\alpha, -\omega_\alpha^K). \tag{5}$$

To determine the weights of the other colors, we use the fact that the spherical roots of X give conditions on these weights, via the Cartan pairing of X.

Lemma 5.1. *For all $\sigma \in \Sigma(X)$ we have*

$$(\sigma, 0) = \sum_{D\in\Delta(X)} c_X(D, \sigma)\,(\omega_D, \chi_D). \tag{6}$$

Proof. The formula is well known. It is proved e.g. by Brion in [12, Section 2.1], under the assumption that K is wonderful. The same argument of loc. cit. yields the formula for any spherical subgroup $K \subseteq G$.

One can also easily reduce the general case to the one of wonderful subgroups, by using the *wonderful closure* $\widehat{K}$ of K, defined in [27]. The relevant properties of $\widehat{K}$ are that $K \subseteq \widehat{K} \subseteq \overline{K}$, that $\Sigma(G/\widehat{K}) = \Sigma(X)$, and that the natural morphism $X \to G/\widehat{K}$ induces a bijection between $\Delta(X)$ and $\Delta(G/\widehat{K})$ compatible with the Cartan pairing. Then the formula of the lemma for G/K is clearly equivalent to the same formula for $G/\widehat{K}$. □

To show that this is enough to determine the weights of all colors, we will need the following known auxiliary results.

Lemma 5.2. *Let R be a reductive linear algebraic group. If $\mathcal{X}(R)$ is infinite, then R has a non-trivial central torus.*

Proof. If $\mathcal{X}(R)$ is infinite then $\mathcal{X}(R^\circ)$ is infinite too. On the other hand R° is equal to $[R^\circ, R^\circ]\cdot Z(R^\circ)^\circ$, therefore the connected center of R° has positive dimension. It is also a normal subgroup of R, and R/R° acts on $\Omega = \mathcal{X}(Z(R^\circ)^\circ)$ by conjugation. At least one non-trivial character $\omega \in \Omega$ extends to R, and such ω is fixed by the action of R/R°. Since ω is fixed, the vector space $\Omega_{\mathbb{Q}}$ splits as $\Omega_{\mathbb{Q}} = \mathbb{Q}\omega \oplus \Omega'_{\mathbb{Q}}$ for $\Omega' \subset \Omega$ a sublattice of corank 1 stable under R/R°, another consequence of the finiteness of this group. The connected common kernel of all elements of Ω' in $Z(R^\circ)^\circ$ is a one-dimensional torus normal in R, and it is central because ω, which restricts to a non-trivial character of this torus, is fixed by R/R°. □

Lemma 5.3. *Let L be a very reductive subgroup of a connected reductive group M. Then the restriction map $\mathcal{X}(L) \to \mathcal{X}(L \cap Z(M)^\circ)$ has finite kernel.*

Proof. Any character of L trivial on $L\cap Z(M)^\circ$ descends to a character of the image $\overline{L}$ via the quotient $M \to \overline{M} = M/(Z(M)^\circ)$. We claim that the character group of $\overline{L}$ is finite, which would imply the lemma. Notice that $\overline{L}$ is a very reductive subgroup of the semisimple group $\overline{M}$.

Suppose for sake of contradiction that $\mathcal{X}\left(\overline{L}\right)$ is infinite. By Lemma 5.2 we can pick a non-trivial one-parameter subgroup $\xi\colon \mathbb{C}^\times \to Z\left(\overline{L}\right)^\circ$. Then $\overline{L}$ is contained in the set $P(\xi)$ of elements $m \in \overline{M}$ such that the limit

$$\lim_{a\to 0} \xi(a)m\xi(a)^{-1}$$

exists. But $P(\xi)$ is a proper parabolic subgroup of $\overline{M}$ (see [29, Proposition 8.4.5 and its proof])): contradiction. □

We proceed by giving two useful formulae in Lemma 5.4. The first is well known; the second was already implicitly given by Bravi and Pezzini in [9, Lemma 3.2.1], and it turns out to be the core of our procedure for computing extended weight monoids.

Lemma 5.4. *We have*

$$\operatorname{rk}\mathcal{X}(K) = |\Delta(X)| - \operatorname{rk}\Xi(X), \tag{7}$$

$$\operatorname{rk}\Xi(X) - \dim(\operatorname{span}_{\mathbb{Q}} \rho_X(\Delta')) = \operatorname{rk}\mathcal{X}(Q) - \operatorname{rk}\mathcal{X}(K). \tag{8}$$

Proof. The first formula is well known, let us give a proof for convenience. We may assume that BK is open in G. Then we notice that $\Xi(X)$ is isomorphic to the quotient $\mathbb{C}(X)^{(B)}/\mathbb{C}^\times$, by associating to $\lambda \in \Xi(X)$ the class $[f]$ (modulo the constant invertible functions) of a B-eigenvector $f \in \mathbb{C}(X)$ of B-eigenvalue λ. Since BK/K is open in G/K, we may identify $\mathbb{C}(X)^{(B)}/\mathbb{C}^\times$ with the group of characters $\mathcal{X}(B/(B\cap K))$, which is isomorphic to $\mathcal{X}(B)^{B\cap K}$.

By [12, Lemma 2.1.1] (whose proof holds for any spherical subgroup $K \subseteq G$), we have

$$\operatorname{rk}(\mathcal{X}(B) \times_{\mathcal{X}(B\cap K)} \mathcal{X}(K)) = |\Delta(X)|.$$

This, together with the exact sequence

$$0 \to \mathcal{X}(B)^{B\cap K} \to \mathcal{X}(B) \times_{\mathcal{X}(B\cap K)} \mathcal{X}(K) \to \mathcal{X}(K) \to 0$$

(the third map is surjective because all characters of $B \cap K$ extend to characters of B), yields the first formula of the lemma.

Formula (8) follows easily from [9, Lemma 3.2.1] under the assumption $\Xi(X) = \operatorname{span}_{\mathbb{Z}} \Sigma(X)$. The same proof holds without this assumption, let us check the details.

Thanks to Lemma 5.3, and the fact that the restriction map $\mathcal{X}(K) \to \mathcal{X}(L_K)$ is an isomorphism, we may replace $\mathcal{X}(K)$ in formula (8) with $\mathcal{X}(L_K \cap Z(L_Q)^\circ)$. We may also replace $\mathcal{X}(Q)$ by $\mathcal{X}(Z(L_Q)^\circ)$, since $Q = Q^u \cdot [L_Q, L_Q] \cdot Z(L_Q)^\circ$.

Now, the formula follows if we prove that restricting the elements of $\Xi(X)$ to the subtorus $Z(L_Q)^\circ$ of T induces an injective map

$$\varphi\colon \bigcap_{D\in\Delta'} \ker(\rho_X(D)) \to \mathcal{X}(Z(L_Q)^\circ)^{L_K\cap Z(L_Q)^\circ},$$

with finite cokernel. In the case $\Xi(X) = \operatorname{span}_{\mathbb{Z}} \Sigma(X)$, this map is even an isomorphism by [9, Lemma 3.2.1].

Let $\gamma \in \bigcap_{D\in\Delta'} \ker(\rho_X(D))$, and let $f_\gamma \in \mathbb{C}(X)$ be a B-eigenvector of B-eigenvalue γ. Call F_γ the pull-back of f_γ on G. By our assumptions on γ, the function F_γ has no zero nor pole except possibly for the pull-backs of colors of G/Q along the projection $G \to G/Q$.

This implies that F_γ is a B-eigenvector, under the action of G on G by left translation, and a Q-eigenvector under the action of right translation. Using the Peter-Weyl theorem, it is elementary to deduce that the Q-eigenvalue (restricted to T) of F_γ is $-\gamma$. But F_γ is also K-invariant under right translation, therefore L_K-invariant, and so $-\gamma|_{L_K\cap Z(Q)^\circ}$ is trivial. We deduce that the map φ as above is defined.

This argument also shows that γ extends to a character of Q, and if it is trivial on $Z(L_Q)^\circ$ then it trivial on Q thanks to the decomposition $Q = Q^u \cdot [L_Q, L_Q] \cdot Z(L_Q)^\circ$. It follows that φ is injective.

Finally, consider $\chi \in \mathcal{X}(Z(L_Q)^\circ)$, and extend it to a character $\widetilde{\chi}$ of Q. This is, again using the Peter-Weyl theorem, the Q-eigenvalue of some Q-eigenvector $F \in \mathbb{C}(G)$ under right translation, such that F is also a Q_+-eigenvector of weight $-\widetilde{\chi}$ under left translation. Here Q_+ is the parabolic subgroup of G opposite to Q with respect to T.

Suppose that χ (and hence $\widetilde{\chi}$) is trivial on $L_K \cap Z(Q)^\circ$. By Lemma 5.3, a multiple $n\widetilde{\chi}$ for some non-zero $n \in \mathbb{Z}$ is trivial on L_K, and hence on K. We deduce that F^n is fixed under right translation by K, i.e. it descends to a B-eigenvector of $\mathbb{C}(X)$ of B-eigenvalue $\gamma = -n\widetilde{\chi}|_B$, with the property that $-\gamma|_{Z(L_Q)^\circ} = n\chi$. This shows that φ has finite cokernel, and the proof is complete. ∎

Theorem 5.5. *Let $\Delta' \subseteq \Delta(X)$ be a minimal parabolic subset of colors. Then equalities* (5) *for all $D \in \Delta'$, and equalities* (6) *for all $\sigma \in \Sigma(X)$, determine $\chi_D|_{K^\circ}$ uniquely for all $D \in \Delta(X)$.*

Proof. We first prove the theorem under the assumption that $\Xi(X)_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}} \Sigma(X)$.

Let us denote for brevity $\chi_D|_{K^\circ}$ by χ'_D. Equalities (5) determine χ'_D for all $D \in \Delta(X) \smallsetminus \Delta'$, it remains to prove that the elements χ'_E for $E \in \Delta'$ are also uniquely determined. We consider then (6) for σ varying in $\Sigma(X)$ as a system of inhomogeneous linear equations with unknowns χ'_E.

These unknowns take values in $\mathcal{X}(K^\circ)$, which is a free abelian group of finite rank. It is harmless to consider the unknowns as taking values in $\mathcal{X}(K^\circ)_{\mathbb{Q}}$, so that our system of equations has vector unknowns, vector constant terms, and scalar coefficients. To prove that it has a unique solution, we must show that the number of unknowns is equal to the rank of the matrix of the homogeneous system, that is, the matrix

$$A = (c_X(D,\sigma))_{D\in\Delta',\sigma\in\Sigma(X)}\,.$$

We put together the equalities of Lemma 5.4, together with $\operatorname{rk}(\mathcal{X}(Q)) = |\Delta(G/Q)| = |\Delta(X)| - |\Delta'|$ (where the first equality follows from the first equality of Lemma 5.4 applied to the spherical subgroup Q of G), and obtain

$$|\Delta'| = \dim(\operatorname{span}_{\mathbb{Q}} \rho_X(\Delta')). \tag{9}$$

Since $\Sigma(X)$ is a basis of $\Xi_{\mathbb{Q}}$, the right hand side of (9) is equal to the rank of A. Therefore the theorem holds.

To finish the proof, it remains to reduce the general case to the case where $\Xi(X)_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}} \Sigma(X)$, so let K be any spherical subgroup of G. Let us consider again the wonderful closure $\widehat{K}$ of K, and set $Y = G/\widehat{K}$. Thanks to the recalled properties of $\widehat{K}$, in particular the given identification between $\Delta(X)$ and $\Delta(Y)$, the pull-backs to G of the colors of X and of Y are the same. So, for all $D \in \Delta(X)$, one can take the same global equation to define the weight $\chi_D \in \mathcal{X}(K)$ for G/K and the weight for $G/\widehat{K}$, let us denote it by $\widehat{\chi}_D$. This means that $\widehat{\chi}_D|_K = \chi_D$.

The equalities (5) and (6) are the same for X and for Y. The first part of the proof, applied to Y, yields that these equalities determine uniquely $(\omega_D, \widehat{\chi}_D|_{(\widehat{K})^\circ})$ for all D. But K° is contained in $(\widehat{K})^\circ$, hence $(\omega_D, \widehat{\chi}_D|_{K^\circ})$, which is equal to $(\omega_D, \chi_D|_{K^\circ})$, is uniquely determined too. □

We end this section discussing the equalities (6) in case we include into the picture groups that are not simply connected.

Let then G_0 be a quotient of G by a finite central subgroup. As before, let K be a spherical subgroup of G, denote by K_0 the image of K in G_0, and let $\pi\colon G/K \to G_0/K_0$ be the natural induced morphism.

Proposition 5.6. *Suppose that $\Sigma(G_0/K_0) \cap 2S = \varnothing$. Then $D \mapsto \pi(D)$ is a bijection between $\Delta(G/K)$ and $\Delta(G_0/K_0)$ compatible with the Cartan pairing. Moreover, the equalities* (6) *hold for σ varying in $\Sigma(G_0/K_0)$; if one considers them as a system of equalities in the unknowns χ_D taking values in $\mathcal{X}(K^\circ)_{\mathbb{Q}}$, then the system is equivalent to the one where σ varies in $\Sigma(G/K)$.*

Proof. We consider G_0/K_0 naturally as a G-variety, i.e. as $G/\widetilde{K}$ where $\widetilde{K}$ is the inverse image of K_0 in G. Since $\widetilde{K}$ normalizes K, by [19, Theorem 4.4 and Theorem 6.1] the spherical roots of G_0/K_0 and G/K are the same, up to replacing some elements by positive multiples. Moreover, if two corresponding spherical roots are different, then the spherical root of G_0/K_0 is the double of the spherical root of G/K (see e.g. Wasserman's tables in [33]), and we conclude that $\Sigma(G/K) \cap S = \Sigma(G_0/K_0) \cap S$. Then, by the combinatorial description of the set of colors of a spherical variety in [26, Section 2.3], we obtain the first assertion of the proposition. The second assertion follows from the same considerations, because the equalities (6) are linear in σ. □

6. On parabolic subgroups of spherical subgroups

Let H and P be spherical subgroups of G, such that $P \subseteq H$. In this section we prove a sufficient combinatorial condition for P to be a parabolic subgroup of H. Set $X = G/P$ and $Y = G/H$.

Proposition 6.1. *Suppose* $|\Sigma(X)| = \operatorname{rk}\Xi(X)$ *and* $|\Sigma(Y)| = \operatorname{rk}\Xi(Y)$*, and suppose that for all proper subsets* $\Sigma' \subsetneq \Sigma(X)$ *we have*

$$\Sigma(Y) \not\subseteq \operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma'.$$

Then P *is a parabolic subgroup of* H*.*

Proof. Since $|\Sigma(X)| = \operatorname{rk}\Xi(X)$ and $\Sigma(X)$ is linearly independent, the dual cone of $\mathbb{Q}_{\geq 0}\Sigma(X)$ in $\operatorname{Hom}_{\mathbb{Z}}(\Xi(X), \mathbb{Q})$ is strictly convex. By [19, Theorem 3.1] there exists a G-equivariant completion $\overline{X}$ of X (sometimes called the *canonical completion*) with a single closed G-orbit, such that no color of $\overline{X}$ contains any G-orbit, and such that $\rho_X(D)$ generate the convex cone $\mathcal{V}(X)$, for D varying among the G-stable prime divisors of $\overline{X}$. Thanks to Knop [21, Proposition 6.1], for all G-orbits Z of $\overline{X}$ different from X we have $\Sigma(Z) \subsetneq \Sigma(X)$.

The same results can be applied to Y, obtaining a G-equivariant completion $\overline{Y}$ with similar properties. By [19, Theorem 4.1], the natural morphism $\varphi\colon X \to Y$ extends to a G-equivariant morphism $\overline{\varphi}\colon \overline{X} \to \overline{Y}$. Then the fiber $\overline{\varphi}^{-1}(eH)$ over the point $eH \in Y \subseteq \overline{Y}$ is a completion of H/P.

We claim that $\overline{\varphi}^{-1}(eH) = H/P$, which will conclude the proof because it implies that H/P is a complete variety. To show the claim, we proceed by contradiction. Suppose $\overline{\varphi}^{-1}(eH) \supsetneq H/P$, then there exists a G-orbit Z contained in $\overline{X}$ and different from X, such that $\overline{\varphi}(Z) = Y$. For the morphism $\overline{\varphi}|_Z\colon Z \to Y$, the equality (1) takes the form

$$\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(Y) = \left(\operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(Z)\right) \cap \left(\bigcap_{D \in \Delta_{\overline{\varphi}|_Z}} \ker \rho_Z(D) \right).$$

In particular $\Sigma(Y) \subseteq \operatorname{span}_{\mathbb{Q}_{\geq 0}} \Sigma(Z)$, contradicting our hypotheses and concluding the proof. □

7. An example

In this section we give an example of a multiplicity free system (G, H, P) and the computation of the corresponding extended weight monoid. We fix $G = \operatorname{Spin}(2n+2)$ with $n \geq 3$, and we want to consider all possible spherical homogeneous spaces G/P where P is a parabolic subgroup of H, and the latter is a symmetric subgroup of semisimple type D_n. Thanks to Proposition 5.6, it would also be possible to use the classical group $\operatorname{SO}(2n+2)$ instead of its universal cover G, as long as one works in the vector spaces $\mathcal{X}(T)_{\mathbb{Q}}$ and $\mathcal{X}(P)_{\mathbb{Q}}$.

In G we fix B, B^- and T as in the previous sections; we denote the simple roots of G by $\alpha_1, \ldots, \alpha_{n+1}$, numbered as in Bourbaki, with the corresponding fundamental dominant weights denoted by $\omega_1, \ldots, \omega_{n+1}$. For $i \leq j$ we set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, and if $i > j$ then we set $\alpha_{i,j} = 0$.

We take H to be the Levi subgroup containing T of the parabolic subgroup of G containing B and obtained by omitting the first simple root α_1 of G. Then T is a maximal torus of H, and $B \cap H$ is a Borel subgroup of H.

We denote the corresponding simple roots of H by $\beta_1, \ldots, \beta_n$, and by $\varpi_1, \ldots, \varpi_n$ the corresponding fundamental dominant weights. The latter are defined as the elements of $\mathcal{X}(T)_{\mathbb{Q}}$ that have the correct pairing with the simple coroots of H and are zero on the subspace of $\mathcal{X}(T)^*_{\mathbb{Q}}$ corresponding to the connected center of H. Then we have $\beta_i = \alpha_{i+1}$ and $\varpi_i = \omega_{i+1}$ for $i \in \{1, \ldots, n\}$. In addition, we set $\varpi_0 = \omega_1$, which is the restriction to T of a generator of the character group of H.

The parabolic subgroups of H are taken to contain $B^- \cap H$. Given such a subgroup P, we denote by I the set of simple roots of H that are not roots of P.

The subgroup P is spherical in G if and only if I is one of the following (see [16, Table 3]):

(1) $I = \{\beta_{n-1}\}$,
(2) $I = \{\beta_n\}$.

The subgroup P of case (1) is obtained by applying the outer automorphism of G that interchanges α_n and α_{n+1} to the subgroup P of case (2). Also, these two parabolic subgroups of H are conjugated in G (see below). For this reason we only discuss case (2), and we notice that the corresponding P appears as case 53 of [8], with "n" in loc. cit. having the same meaning as here.

This yields

$$\Sigma(G/P) = \{\alpha_1, \sigma_1 = \alpha_{2,n}, \sigma_2 = \alpha_{2,n-1} + \alpha_{n+1}\}$$

and

$$\Delta(G/P) = \{D_1^+, D_1^-, D_2, D_n, D_{n+1}\},$$

where for any $i \in \mathbb{N}$ we denote by D_i a color moved only by the simple root α_i if α_i is not a spherical root, and we denote by D_i^+, D_i^- the two colors moved by the simple root α_i if α_i is also a spherical root.

The Cartan pairing is

	α_1	σ_1	σ_2
D_1^+	1	-1	0
D_1^-	1	0	-1
D_2	-1	1	1
D_n	0	1	-1
D_{n+1}	0	-1	1

The lattice $\Xi(G/P)$ is generated by $\Sigma(G/P)$. For later use, it is useful to notice that $\mathcal{X}(T)/\Xi(G/P)$ is not torsion-free: its torsion subgroup has order 2 and it is generated by the class of the element $\tau = \frac{1}{2}(\sigma_1 - \sigma_2)$. This element takes non-integer values on two of the above colors.

These data for G/P can also be verified directly using our results, let us give the details. It is elementary to check that they satisfy Luna's axioms in [26, Section 2]. So, by the classification of spherical homogeneous spaces, they correspond to a spherical subgroup K of G. The shape of the lattice $\Xi(G/K)$ implies that K is connected. Indeed, the quotient $\Xi(G/K^\circ)/\Xi(G/K)$ is finite, and all elements of $\Xi(G/K^\circ)$ must take anyway integer values on the valuations of the colors of G/K (this is clear just considering the pull-back of divisors from G/K to G/K°). Thanks to the above remark about $\Xi(G/P)$ and the element τ, we conclude that $\Xi(G/K^\circ) = \Xi(G/K)$. Then $K^\circ = K$ by [15, Lemma 2.4].

The set $\{D_n, D_{n+1}\}$ is easily seen to be distinguished, so it corresponds to an inclusion $K \subseteq R$ with R a connected spherical subgroup of G. The spherical roots, the colors and the lattice of G/R are computed using Proposition 3.7, and they turn out to be the same as H (see [10, Case 14]), so we may assume $R = H$. Proposition 6.1 assures that K is conjugated to a parabolic subgroup of H. It remains to decide whether it's our P or the one given by $I = \{\beta_{n-1}\}$, but we notice that the homogeneous spherical datum of G/K remains unchanged if we exchange α_n with α_{n+1}. It follows that these two parabolic subgroups of H are both conjugated to K in G, and we simply assume $K = P$.

We take Q to be the parabolic subgroup of G containing B^- and such that only α_1 and α_{n+1} are not simple roots of its Levi subgroup L_Q (with the same notations as in Section 4). Then Q is a parabolic subgroup of G minimal containing P. By looking at which simple root moves which color, one sees immediately that the colors of G/P mapping not dominantly to G/Q are D_{n+1} and either D_1^+ or D_1^-. Since $\{D_1^+, D_2, D_{n+1}\}$ is not a parabolic subset of colors, we conclude that D_1^- is mapped not dominantly to G/Q.

The above implies that $\chi_{D_1^-}$ and $\chi_{D_{n+1}}$ are the extension to P of the characters resp. $-\omega_1$ and $-\omega_{n+1}$ of T, thanks to the Peter-Weyl theorem. For simplicity we do not distinguish notationally these characters from their extensions, and it is convenient to express them in terms of the above weights of H, obtaining

$$\begin{aligned} \chi_{D_1^-} &= -\varpi_0, \\ \chi_{D_{n+1}} &= -\varpi_n. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield the P-weights of the other colors:

$$\begin{aligned} \chi_{D_1^+} &= \varpi_0, \\ \chi_{D_2} &= 0, \\ \chi_{D_n} &= \varpi_0 - \varpi_n. \end{aligned}$$

Finally, Lemma 4.3 yields

$$\begin{aligned} \omega_{D_1^+} &= \omega_1, \\ \omega_{D_1^-} &= \omega_1, \\ \omega_{D_2} &= \omega_2, \\ \omega_{D_n} &= \omega_n, \\ \omega_{D_{n+1}} &= \omega_{n+1}. \end{aligned}$$

This finishes the computation of $\widetilde{\Gamma}(G/P)$.

8. Application to spherical functions

Let (G, H, P) be a multiplicity free system with G semisimple and simply connected. We choose a maximal torus and a Borel subgroup $T_H \subseteq B_H \subseteq H$ in a compatible way, meaning that $B_H \subseteq B_G$ and $T_H \subseteq T_G$. The opposite Borel subgroups with respect to T_G and T_H are denoted by B_G^- and B_H^- respectively. Furthermore we arrange $B_H^- \subseteq P$. Let $\mathcal{X}(T_G)$ denote the character group of T_G and denote by $\mathcal{X}_+(T_G)$ the monoid of dominant characters with respect to B_G. We recall that they are also called dominant weights and they parametrize the equivalence classes of the irreducible G-representations, by taking the highest weight appearing in the representation. In the same way the dominant characters in $\mathcal{X}_+(T_H) \subseteq \mathcal{X}(T_H)$ parametrize the equivalence classes of irreducible H-representations. Let $\mathcal{X}(P) \subseteq \mathcal{X}(T_H)$ denote the group of characters of T_H that can be extended to a character of P. If $\chi \in \mathcal{X}_+(T_H)$ then we denote the corresponding irreducible H-representation of highest weight χ by $\pi_{H,\chi} : H \to \mathrm{GL}(V_{H,\chi})$.

Let $\chi \in \mathcal{X}_+(T_H) \cap \mathcal{X}(P)$. Following the Borel-Weil Theorem we see that $\mathrm{Ind}_P^H(\chi)$ is an irreducible H-representation of lowest weight $w_0^H(\chi)$, where $w_0^H \in W_H = N_H(T_H)/T_H$ is the longest Weyl group element, and hence $\mathrm{Ind}_P^H(\chi) = \pi_{H,\chi}$. Induction in stages yields $\mathrm{Ind}_H^G(\pi_{H,\chi}) = \mathrm{Ind}_P^G(\chi)$ and the latter space is a G-representation whose decomposition into irreducible G-representations is multiplicity free. Indeed, $\mathrm{Ind}_P^G(\chi)$ is the space of regular sections of a G-line bundle over the space G/P. Since G/P is a spherical variety, the decomposition is multiplicity free, see e.g. [30, Thm.25.1]. This means in particular that $\dim \mathrm{Hom}_H(V_{H,\chi}, V_{G,\lambda}) \leq 1$ for all $\lambda \in \mathcal{X}_+(T_G)$.

If (G, H, P) is a multiplicity free system with G not simply connected and $\chi \in \mathcal{X}_+(T_H) \cap \mathcal{X}(P)$, then $\mathcal{X}(T_G; T_H, \chi)$ can be obtained from $\mathcal{X}(T_{\widetilde{G}}; T_{\widetilde{H}}, \widetilde{\chi})$, where $\psi : \widetilde{G} \to G$ is the simply connected cover, $\widetilde{H}$ is the connected component of $\psi^{-1}(H)$ containing the identity and $\widetilde{\chi}$ is the highest weight of the irreducible representation $\pi_\chi^H \circ (\psi|_{\widetilde{H}})$. Indeed, $\mathcal{X}(T_G; T_H, \chi)$ consists of weights of $\mathcal{X}(T_{\widetilde{G}}; T_{\widetilde{H}}, \widetilde{\chi})$ for which the corresponding irreducible representation descends to an irreducible representation of G. Hence it is not a restriction to assume G simply connected.

A multiplicity free system (G, H, P) is called *strictly indecomposable* if (G, H) is strictly indecomposable, meaning that pair $(G, [H, H])$ is not isogenous to a product $(G_1 \times G_2, H_1 \times H_2)$ with $H_i \subsetneq G_i$ non-trivial. The strictly indecomposable multiplicity free systems (G, H, P) with G semisimple are classified by He, Nishiyama, Ochiai, and Oshima in [16] for (G, H) symmetric and by van Pruijssen in [31] for (G, H) spherical but not symmetric.

In this section we assume that (G, H, P) is a strictly indecomposable multiplicity free system with G simply connected and G, H connected, and $\chi \in \mathcal{X}_+(T_H) \cap \mathcal{X}(P)$. Define

$$E^\chi := (\mathbb{C}[G] \otimes \mathrm{End}(V_{H,\chi}))^{H \times H},$$

where $H \times H$ acts on the space $\mathbb{C}[G]$ by the left and right regular representation and on the space $\mathrm{End}(V_{H,\chi})$ by left and right matrix-multiplication. If $\chi = 0$, then E^0 is the algebra of H-biinvariant functions on G. The space E^χ is a module over E^0. We study the algebra structure of E^0 and the E^0-module structure of E^χ. It turns out that in most of the cases E^0 is a polynomial algebra and that E^χ is freely and finitely generated as an E^0-module.

8.1. **Multiplicity free induction.** To describe the irreducible G-representations $\pi_{G,\lambda}$ that contain $\pi_{H,\chi}$ upon restriction to H we introduce Definition 8.1.

Definition 8.1. With the notation from above,

$$\mathcal{X}_+(T_G; H, \chi) := \{\lambda \in \mathcal{X}_+(T_G) | \dim \mathrm{Hom}_H(V_{H,\chi}, V_{G,\lambda}) = 1\}.$$

We refer to this set as the χ-*well*.

Note that $\lambda \in \mathcal{X}_+(T_G; H, \chi)$ if and only if $(\lambda, -\chi) \in \widetilde{\Gamma}(G/P)$.

8.1.1. *Analysis of the* 0*-well.* The 0-well $\mathcal{X}_+(T_G; H, 0)$ of the spherical pair (G, H) is equal to the weight monoid $\Gamma(G/H)$ consisting of highest weights of irreducible G-submodules of $\mathbb{C}[G/H]$. If H has a zero-dimensional center, then $\Gamma(G/H)$ can be identified with $\widetilde{\Gamma}(G/H)$ and it follows from Proposition 4.2 that the weight monoid of G/H is freely and finitely generated. The generators of $\widetilde{\Gamma}(G/H)$ are recorded in [24, Tabelle 1] for G simple and [2, Table 1] for G semisimple but not simple.

If the center of H is not zero-dimensional, then it follows from inspection of the classification of strictly indecomposable spherical pairs [11, 24] that the center of H is one-dimensional. If (G, H) is symmetric, then $\Gamma(G/H)$ is freely generated [32, Lemme 3.4].

The spherical pairs (G, H) that are left, i.e. the spherical but non-symmetric ones for which H has a one-dimensional center, are recorded in Table 1. In the third column we provide the generators of $\widetilde{\Gamma}(G/H)$, which are taken from [2, 3, 33] or they can be calculated easily from our method.

TABLE 1. The strictly indecomposable spherical pairs that are not symmetric with $\dim(Z(H)) = 1$ and the generators of their extended weight monoids. In the fourth row we employ the convention $\omega'_0 = \omega'_n = 0$.

	G	H	generators of $\widetilde{\Gamma}(G/H)$
1	$\mathrm{SL}(2n+1)$ $n \geq 2$	$\mathrm{Sp}(2n) \times \mathbb{C}^\times$	$(\omega_{2i-1}, -\frac{n+1-i}{n}\chi)$, $1 \leq i \leq n$ $(\omega_{2j}, -\frac{j}{n}\chi)$, $1 \leq j \leq n$
2	$\mathrm{Sp}(2n)$	$\mathrm{Sp}(2n-2) \times \mathbb{C}^\times$	(ω_1, ω_1), $(\omega_1, -\omega_1)$, $(\omega_2, 0)$
3	$\mathrm{SO}(2n+1)$	$\mathrm{GL}(n)$	$(\omega_i, 0)$, $1 \leq i \leq n-1$ (ω_n, χ), $(\omega_n, -\chi)$
4	$\mathrm{Spin}(10)$	$\mathrm{Spin}(7) \times \mathbb{C}^\times$	$(\omega_1, 2\chi)$, $(\omega_1, -2\chi)$ $(\omega_2, 0)$, (ω_4, χ), $(\omega_5, -\chi)$
5	$\mathrm{SL}(n+1) \times \mathrm{SL}(n)$	$\mathrm{SL}(n) \times \mathbb{C}^\times$	$(\omega_i + \omega'_{n+1-i}, (n+1-i)\chi)$, $1 \leq i \leq n$ $(\omega_j + \omega'_{n-j}, -j\chi)$, $1 \leq j \leq n$
6	$\mathrm{SL}(n) \times \mathrm{Sp}(2m)$ $n \geq 3$, $m \geq 1$	$\mathbb{C}^\times \cdot (\mathrm{SL}(n-2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2m-2))$	$(\omega_{n-2}, 2\chi)$, $(\omega_{n-1} + \omega'_1, \chi)$, $(\omega_1 + \omega'_1, -\chi)$, $(\omega_2, -2\chi)$ $(\omega_1 + \omega_{n-1}, 0)$ $(n \geq 4)$ $(\omega'_2, 0)$ $m \geq 2$

Lemma 8.2. *The pairs* $(\mathrm{Sp}(2n), \mathrm{Sp}(2n-2) \times \mathbb{C}^\times)$ *and* $(\mathrm{SO}(2n+1), \mathrm{GL}(n))$ *are the only pairs* (G, H) *in Table* 1 *whose weight monoid* $\Gamma(G/H)$ *is freely generated.*

Proof. First note that the weight monoid of the pair $(\mathrm{Sp}(2n), \mathrm{Sp}(2n-2) \times \mathbb{C}^\times)$ is freely generated by $2\omega_1$ and ω_2 and the weight monoid of the pair $(\mathrm{SO}(2n+1), \mathrm{GL}(n))$ is freely generated by $\omega_1, \ldots, \omega_{n-1}, 2\omega_2$. For the pair in the first line we have

$$\mathcal{X}_+(T_G; H, 0) = \left\{ \sum_{i=1}^{2n} a_i \omega_i \,\middle|\, \sum_{j=1}^{n} (n-j+1) a_{2j-1} = \sum_{j=1}^{n} j a_{2j} \right\}.$$

The elements that are sums of two or three fundamental weights are

- $\omega_i + \omega_{2n+1-i}, 1 \leq i \leq n$,
- $\omega_{2i-1} + \omega_{2k} + \omega_{2\ell}$ for $1 \leq i, k, \ell \leq n$ with $k + \ell + i = n + 1$,
- $\omega_{2k-1} + \omega_{2\ell-1} + \omega_{2i}$ with $1 \leq i, k, \ell \leq n$ with $k + \ell + i = 2(n+1)$,

and unless $n = 1$ there are strictly more than $2n$ of these elements. Hence, for $n > 1$, the weight monoid is not freely generated. Indeed, if it were, then one of these elements would not be a generator. But none of these elements can be expressed as a linear combination of the others with coefficients in $\mathbb{Z}_{\geq 0}$, a contradiction. The arguments for lines 3–6 are similar and the details are left for the reader. ☐

Remark 8.3. Note that for a strictly indecomposable spherical pair (G, H) the weight monoid $\Gamma(G/H)$ is freely generated if and only if there are at most two colors on G/H with non-trivial H-weight. For the symmetric pairs this follows from [32, Lemme 3.4], for the non-symmetric ones this follows from Lemma 8.2. In this case the H-characters of the weights of these two colors are opposite.

Furthermore, note that whenever G/H has more than two colors with non-trivial H-weight, there are no non-trivial parabolic subgroups $P \subseteq H$ that remain spherical in G.

8.1.2. *Analysis of a more general χ-well.* In this paragraph we assume that $\Gamma(G/H)$ is freely generated. If $\lambda \in \mathcal{X}_+(T_G; H, \chi)$ and $\sigma \in \mathcal{X}_+(T_G; H, 0)$, then $\lambda + \sigma \in \mathcal{X}_+(T_G; H, \chi)$ by means of the Cartan projection $V_{G,\lambda} \otimes V_{G,\sigma} \to V_{G,\lambda+\sigma}$, which is surjective and G-equivariant. This suggests that the χ-well has certain minimal elements.

Definition 8.4. Let

$$\mathcal{B}_+(T_G; H, \chi) = \{\lambda \in \mathcal{X}_+(T_G; H, \chi) | \forall \sigma \in \mathcal{X}_+(T_G; H, 0) : \lambda - \sigma \notin \mathcal{X}_+(T_G; H, \chi)\},$$

which we call the *bottom* of the χ-well.

Example 8.5. Consider the spherical pair $(\mathsf{G}_2, \mathrm{SL}(3))$. The parabolic subgroup $P_{\{\alpha_1\}} \subset \mathrm{SL}(3)$ remains spherical in G_2 and the extended weight monoid $\widetilde{\Gamma}(\mathsf{G}_2/P_{\{\alpha_1\}})$ is generated by $(\omega_1, 0), (\omega_1, -\varpi_1), (\omega_2, -\varpi_1)$. Consider the dominant weight $3\varpi_1$ for $\mathrm{SL}(3)$. The $3\varpi_1$-well is given by

$$a\omega_1 + b\omega_1 + c\omega_2 \quad \text{with } a, b, c \in \mathbb{Z}_{\geq 0} \text{ and } b + c = 3.$$

The bottom of the $3\varpi_1$-well is given by

$$3\omega_1, 2\omega_1 + \omega_2, \omega_1 + 2\omega_2, 3\omega_2.$$

The $3\varpi_1$-well is illustrated in Figure 1. The roots of $\mathrm{SL}(3)$ are the long roots of G_2.

If the center of H is one-dimensional, then there are two colors D_0 and D_1 of G/H whose H-characters are non-trivial and opposite by Remark 8.3. Let (ω_{D_0}, μ) and $(\omega_{D_1}, -\mu)$ be the corresponding generators of $\widetilde{\Gamma}(G/H)$. Let $r = \mathrm{rk}(G/H)$ denote the spherical rank. If $r > 1$, then the other colors of G/H are denoted by $D_2, \ldots, D_r$ and the corresponding generators of $\widetilde{\Gamma}(G/H)$ are denoted by $(\omega_{D_i}, 0)$ with $2 \leq i \leq r$.

If the center of H is zero-dimensional, then the colors of G/H are denoted by $D_1, \ldots, D_r$. The corresponding generators of $\widetilde{\Gamma}(G/H)$ are denoted by $(\omega_{D_i}, 0)$ with $1 \leq i \leq r$.

In both cases the colors of G/H can be pulled back to G/P where they correspond to generators of $\widetilde{\Gamma}(G/P)$ given by the same weights. We denote the other colors of G/P by $E_1, \ldots, E_s$ and the corresponding generators by $(\omega_{E_j}, \chi_{E_j})$ with $1 \leq j \leq s$.

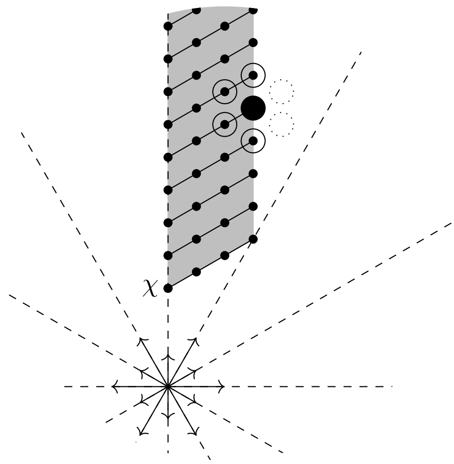

FIGURE 1. The $\chi = 3\varpi_1$-well for the pair $(\mathsf{G}_2, \mathrm{SL}(3))$. The nodes are explained in Example 8.13.

For an element $\lambda \in \mathcal{X}_+(T_G; H, \chi)$ we write

$$(\lambda, -\chi) = a_0(\omega_{D_0}, \mu) + a_1(\omega_{D_1}, -\mu) + \sum_{i=2}^{r} a_i(\omega_{D_i}, 0) + \sum_{j=1}^{s} b_j(\omega_{E_i}, \chi_{E_j}) \tag{10}$$

if H has a one dimensional center and

$$(\lambda, -\chi) = \sum_{i=1}^{r} a_i(\omega_{D_i}, 0) + \sum_{j=1}^{s} b_j(\omega_{E_i}, \chi_{E_j}) \tag{11}$$

if H has a zero-dimensional center, where the coefficients a_i, b_j are in $\mathbb{Z}_{\geq 0}$.

Lemma 8.6. *An element $\lambda \in \mathcal{X}_+(T_G; H, \chi)$ is contained in $\mathcal{B}_+(T_G; H, \chi)$ if and only if $\min(a_0, a_1) = a_2 = \cdots = a_r = 0$ in case $\dim Z(H) = 1$ or $a_1 = \cdots = a_r = 0$ in case $\dim Z(H) = 0$.*

Proof. If $\dim Z(H) = 0$, then the statement is clear from the definition of the bottom of the χ-well. Suppose that $\dim Z(H) = 1$ and $\lambda \in \mathcal{B}_+(T_G; H, \chi)$. If $a_i \neq 0$ for some $2 \leq i \leq r$, then $\lambda - \omega_{D_i} \in \mathcal{X}_+(T_G; H, \chi)$, a contradiction to $\lambda \in \mathcal{B}_+(T_G; H, \chi)$. If $\min(a_0, a_1) > 0$, then $\lambda - (\omega_{D_0} + \omega_{D_1}) \in \mathcal{X}_+(T_G; H, \chi)$, again a contradiction to $\lambda \in \mathcal{B}_+(T_G; H, \chi)$. The other direction is clear. □

Proposition 8.7. *The χ-well can be written as*

$$\mathcal{X}_+(T_G; H, \chi) = \mathcal{B}_+(T_G; H, \chi) + \mathcal{X}_+(T_G; H, 0)$$

and the induced decomposition of any element is unique.

Proof. The equality of the two sets is clear, the one inclusion follows from Definition 8.4, while the other follows from the Cartan projection. To show uniqueness we write $\lambda = b(\lambda) + s(\lambda)$ with $b(\lambda) \in \mathcal{B}_+(T_G; H, \chi)$ and $s(\lambda) \in \mathcal{X}_+(T_G; H, 0)$. By Lemma 8.6 the coefficients of $(b(\lambda), -\chi)$ and $(s(\lambda), 0)$ are uniquely determined by the coefficients of $(\lambda, -\chi)$. Hence the decomposition is unique. □

The action of H on the annihilator $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$ of $\mathfrak{h}$ has a generic stabilizer $H_* \subseteq H$ which is reductive, see [18, Kor.8.2]. The decomposition of the restriction of $\pi_{H,\chi}$

to H_* is multiplicity free because $\pi_{H,\chi}$ is realized in the space of sections of a homogeneous line bundle over H/P which is H_*-spherical by [31, Proposition 2.4]. Moreover, we can choose T_G in such a way that $T_{H_*} = T_G \cap H_*$ is a maximal torus of H_* and the weights $\lambda \in \Gamma(G/H)$ vanish on T_{H_*}. The reason is that H_* is contained in a Levi subgroup of the parabolic subgroup adapted to G/H (i.e. corresponding to the subset of simple roots $S^p(G/H)$), but it also contains the commutator subgroup of this Levi subgroup. Note that we can also choose a compatible Borel subgroup B_{H_*} of H_* that is contained in a Borel subgroup of G. This implies that restriction to T_{H_*} induces a map $\mathcal{X}_+(T_G) \to \mathcal{X}_+(T_{H_*}), \lambda \mapsto \lambda_*$.

As explained in [31, §3], the restriction of a weight $\lambda \in \mathcal{X}_+(T_G; H, \chi)$ to T_{H_*} is the highest weight of an irreducible representation of H_* that is contained in the restriction of $\pi_{H,\chi}$ to H_*. In loc. cit. it is shown that all irreducible H_*-representations in the decomposition of $\pi_{H,\chi}$ can be obtained in this way. For later reference we record the following result.

Lemma 8.8. *The elements in $\mathcal{B}_+(T_G; H, \chi)$ are in one-to-one-correspondence with the H_*-types in the restriction $\pi_{H,\chi}|_{H_*}$, via the map $\lambda \mapsto \lambda_*$.*

Proof. It remains to show that the map $\mathcal{B}_+(T_G; H, \chi) \to \mathcal{X}_+(T_{H_*})$ is injective. Let $\lambda, \lambda' \in \mathcal{B}_+(T_G; H, \chi)$ with $\lambda_* = \lambda'_*$. Then $\lambda - \lambda' \in \mathbb{Z}\Gamma(G/H) \subseteq \Xi(G/H)$ because $(\lambda, -\chi) - (\lambda', -\chi) = (\lambda - \lambda', 0)$. If $\lambda - \lambda' \neq 0$, then one of the coefficients a_i or a'_i of $(\lambda, -\chi)$ or $(\lambda', -\chi)$ in (10) of (11) is non-zero. By Lemma 8.6 this can only happen when $\dim(Z(H)) = 1$. Without loss of generality we assume that $a_0 > 0$ and $a_1 = 0$. Since $\lambda - \lambda' \in \mathbb{Z}\Gamma(G/H)$, we must have $a'_0 - a'_1 = a_0$ to make sure that the second component is zero. At the same time $\min(a'_0, a'_1) = 0$ by Lemma 8.6. It follows that $a_1 = 0$ and $a'_0 = a_0$, i.e. $\lambda = \lambda'$. □

This result shows in particular that $\mathcal{B}_+(T_G; H, \chi)$ is a finite set.

8.2. **Spherical functions.** We proceed to investigate the algebra structure of E^0 and the E^0-module structure of E^χ. We recall the definition of a spherical function in our framework.

Definition 8.9. Let $\lambda \in \mathcal{X}_+(T_G; H, \chi)$. Let $j^\chi_\lambda : V_{H,\chi} \to V_{G,\lambda}$ and $p^\chi_\lambda : V_{G,\lambda} \to V_{H,\chi}$ be an H-equivariant inclusion and projection. Note that both maps are uniquely determined up to scaling due to the multiplicity one property. We fix $j^\chi_\lambda, p^\chi_\lambda$ with $p^\chi_\lambda \circ j^\chi_\lambda = \mathrm{Id}_{V_{H,\chi}}$. The *spherical function* of type χ associated to λ is defined by $\Phi^\chi_\lambda : G \to \mathrm{End}(V_{H,\chi}) : g \mapsto p^\chi_\lambda \circ \pi_{G,\lambda}(g) \circ j^\chi_\lambda$. If $\chi = 0$, then the spherical function of type 0 associated to $\lambda \in \mathcal{X}(T_G; H, 0)$ is called a *zonal spherical function* associated to λ and it is denoted by ϕ_λ.

For later reference we introduce the following notation. If $K \subseteq G$ is a subgroup and $\mu : K \to \mathbb{C}^\times$, then we denote by

$$(V_{G,\lambda})^{(K)}_{(\mu)} := \{v \in V_{G,\lambda} : kv = \mu(k)v\}$$

the weight space of K of weight μ. We denote $(V_{G,\lambda})^K := (V_{G,\lambda})^{(K)}_{(0)}$, the space of K-invariants.

If $v \in V^G_\lambda$ and $\eta \in (V^G_\lambda)^*$ then $m^\lambda_{v\otimes\eta} \in \mathbb{C}[G]$ denotes the matrix coefficient defined by $m^\lambda_{v\otimes\eta}(g) = \eta(g^{-1}v)$.

Lemma 8.10. *Let* $\lambda \in \mathcal{X}_+(T_G; H, 0)$ *and* $\lambda' \in \mathcal{X}_+(T_G; H, \chi)$. *Let* $m^{\lambda}_{v\otimes\eta}, m^{\lambda'}_{v'\otimes\eta'} \in \mathbb{C}[G]$ *with* $v \in (V_{G,\lambda})^H$, $\eta \in ((V_{G,\lambda})^*)^H$, $v' \in V_{G,\lambda'}$ *and* $\eta' \in ((V_{G,\lambda'})^*)^{(P)}_{(-\chi)}$. *Then*

$$m^{\lambda}_{v\otimes\eta} m^{\lambda'}_{v'\otimes\eta'} = \sum_{\beta\in\mathbb{Z}_{\geq 0}\Sigma(G/P)} c^{\lambda,\lambda'}_{\lambda+\lambda'-\beta} m^{\lambda+\lambda'-\beta}_{v''\otimes\eta''}$$

for some $v'' \in V_{G,\lambda+\lambda'-\beta}$ *and* $\eta'' \in ((V_{G,\lambda+\lambda'-\beta})^*)^{(P)}_{(-\chi)}$. *Moreover,* $c^{\lambda,\lambda'}_{\lambda+\lambda'} \neq 0$.

Proof. Let $M, M' \subseteq \mathbb{C}[G]$ be the irreducible G-modules that contain the matrix coefficients $m^{\lambda}_{v\otimes\eta}, m^{\lambda'}_{v'\otimes\eta'}$ respectively. If $M'' \subseteq M \cdot M'$ is an irreducible submodule of highest weight λ'', then $\lambda + \lambda' - \lambda''$ is a linear combination of spherical roots of G/P with coefficients in $\mathbb{Z}_{\geq 0}$. The non-vanishing follows from the fact that the Cartan projection $V_{G,\lambda} \otimes V_{G,\lambda'} \to V_{G,\lambda+\lambda'}$ is surjective and that simple tensors are not in the kernel. □

8.2.1. *Zonal spherical functions.* The zonal spherical function associated to $\lambda \in \Gamma(G/H)$ can be written as a matrix coefficient. Indeed, let $v^H \in V_{G,\lambda}$ and $\eta^H \in (V_{G,\lambda})^*$ be two non-trivial H-fixed vectors for which $m^{\lambda}_{v^H\otimes\eta^H}(e) = 1$. Then $m^{\lambda}_{v^H\otimes\eta^H} = \phi_\lambda$.

Let $\lambda_1, \ldots, \lambda_s$ be elements of $\Gamma(G/H)$ and let $\phi_{\lambda_1}, \ldots, \phi_{\lambda_s}$ be the corresponding zonal spherical functions. The elements $\lambda_1, \ldots, \lambda_s$ generate $\Gamma(G/H)$ as a monoid if and only if the zonal spherical functions $\phi_{\lambda_1}, \ldots, \phi_{\lambda_s}$ generate E^0 as an algebra.

To see this, suppose $\Gamma(G/H)$ is generated by $\lambda_1, \ldots, \lambda_s$. If $\lambda \in \Gamma(G/H)$, then Lemma 8.10 implies

$$\phi_\lambda \phi_{\lambda_i} = \sum_{\beta\in\mathbb{Z}_{\geq 0}\Sigma(G/H)} c_{\lambda,\lambda_i,\beta} \phi_{\lambda+\lambda_i-\beta}$$

with $c_{\lambda,\lambda_i,0} \neq 0$. Note that $\lambda + \lambda_i - \beta \leq \lambda + \lambda_i$, where $\leq$ denotes the standard partial ordering on $\mathcal{X}(T_G)$. Since there are only finitely many elements $\mu \in \Gamma(G/H)$ with $\mu \leq \lambda + \lambda_i$, an induction argument shows that every zonal spherical function can be expressed as a polynomial of the zonal spherical functions ϕ_{λ_i} with $1 \leq i \leq s$. Conversely, if the $\phi_{\lambda_1}, \ldots, \phi_{\lambda_s}$ generate E^0, then it is clear that $\Gamma(G/H)$ is generated by $\lambda_1, \ldots, \lambda_s$.

Lemma 8.11. *The algebra* E^0 *is freely generated by* $\phi_{\lambda_1}, \ldots, \phi_{\lambda_r}$ *if and only if* $\Gamma(G/H)$ *is freely generated by* $\lambda_1, \ldots, \lambda_r$.

Proof. Suppose that E^0 is not freely generated. Consider the map $\zeta : \mathbb{C}[z_1, \ldots, z_r] \to E^0$ given by $z^a \mapsto \prod_i \phi^{a_i}_{\lambda_i}$ and let $p(z) = \sum_{a\in\mathbb{Z}^r_{\geq 0}} c_a z^a$ with $\zeta(p) = 0$, where we use multi-index notation. Let $C(p) = \{a \in \mathbb{Z}^r_{\geq 0} : c_a \neq 0\}$. For $a \in \mathbb{Z}^r_{\geq 0}$ let $\lambda(a) = \sum_{i=1}^r a_i\lambda_i$. If $C(p) = \emptyset$, then $p = 0$. Suppose $p \neq 0$ so that $C(p) \neq \emptyset$. The set $C'(p) = \{(\lambda(a), a) : a \in C(p)\}$ is non-empty. Let $(\lambda(a), a) \in C'(p)$ with $\lambda(a)$ maximal in the partial ordering $\leq$. Then $\zeta(p) = c\phi_{\lambda(a)} +$ other terms, with $c \neq 0$. Since $\zeta(p) = 0$ and the spherical functions are linearly independent, there must be $a' \in C(p)$ different from a with $(\lambda(a'), a') \in C'(p)$ and $\lambda(a') = \lambda(a)$, because $c\phi_{\lambda(a)}$ must be canceled. It follows that $\Gamma(G/H)$ is not free.

To prove the converse implication, suppose that the generators $\lambda_1, \ldots, \lambda_r$ are not free. Then there exist two disjoint subsets of indices $1 \leq i_1 < \cdots < i_s \leq r$ and

$1 \leq j_1 < \cdots < j_t \leq r$ and coefficients $a_{i_1}, \ldots, a_{i_s}, b_{j_1}, \ldots, b_{j_t}$ for which

$$\sum_{k=1}^{s} a_{i_k}\lambda_{i_k} = \sum_{\ell=1}^{s} b_{j_\ell}\lambda_{j_\ell}$$

and for which $\lambda = \sum_{k=1}^{s} a_{i_k}\lambda_{i_k}$ is minimal with respect to $\leq$. Let $p(z_1,\ldots,z_r) = \prod_{k=1}^{s} z_{i_k}^{a_{i_k}}$ and $q(z_1,\ldots,z_r) = \prod_{\ell=1}^{t} z_{j_\ell}^{b_{j_\ell}}$. Then

$$p(\phi_{\lambda_1},\ldots,\phi_{\lambda_r}) = \sum_{\lambda'\leq\lambda} c(\lambda',\lambda)\phi_{\lambda'}, \quad q(\phi_{\lambda_1},\ldots,\phi_{\lambda_r}) = \sum_{\lambda'\leq\lambda} d(\lambda',\lambda)\phi_{\lambda'}$$

with $c(\lambda,\lambda) \neq 0$ and $d(\lambda,\lambda) \neq 0$. Hence

$$\zeta(d(\lambda,\lambda)p - c(\lambda,\lambda)q) = \sum_{\lambda'<\lambda} e(\lambda',\lambda)\phi_{\lambda'}$$

for some coefficients $e(\lambda,\lambda')$. Let $r_{\lambda'} \in \mathbb{C}[z_1,\ldots,z_r]$ be so that $\zeta(r_{\lambda'}) = \phi_{\lambda'}$ for $\lambda' < \lambda$. Then for all monomials z^f with $f \in C(r_{\lambda'})$ we have $\lambda(f) \leq \lambda'$. In particular we note that the monomials p and q do not occur in any of the polynomials $r_{\lambda'}$. Hence the polynomial $d(\lambda,\lambda)p - c(\lambda,\lambda)q - \sum_{\lambda'<\lambda} e(\lambda',\lambda)r_{\lambda'}$ is non-trivial and it is mapped to 0 by ζ. □

We conclude that under our assumptions on (G,H,P), the algebra E^0 is a polynomial algebra if and if $\Gamma(G/H)$ is freely generated.

8.2.2. *Spherical functions of type χ.* We retain the assumptions on (G,H,P) and χ. Moreover, in this paragraph we assume that E^0 is a polynomial algebra, i.e. the weight monoid of G/H is freely generated.

Theorem 8.12. *The space E^χ is freely and finitely generated as an E^0-module by the spherical functions Φ^χ_λ with $\lambda \in \mathcal{B}_+(T_G;H,\chi)$.*

Proof. Let $\lambda \in \mathcal{X}_+(T_G;H,\chi)$ and let $\lambda_i \in \Gamma(G/H)$ be a generator. Upon writing the spherical function Φ^χ_λ in coordinates subject to a weight basis of $V_{H,\chi}$, it will have a matrix coefficient $m^\lambda_{v\otimes\eta}$ among its entries, where v and η are P-eigenvectors. It follows from Lemma 8.10 that

$$\phi_{\lambda_i}\Phi^\chi_\lambda = \sum_{\beta\in\mathbb{Z}_{\geq 0}\Sigma(G/P)} c^{\lambda,\lambda_i}_{\lambda+\lambda_i-\beta}(\chi)\Phi^\chi_{\lambda+\lambda_i-\beta}$$

with $c^{\lambda,\lambda_i}_{\lambda+\lambda_i}(\chi) \neq 0$. Since the number of $\lambda' \in \mathcal{X}_+(T_G;H,\chi)$ with $\lambda' \leq \lambda$ is finite, an induction argument shows that Φ^χ_λ can be expressed as an E^0-linear combination of the spherical functions Φ^χ_b with $b \in \mathcal{B}_+(T_G;H,\chi)$. This shows that E^χ is finitely generated as an E^0-module. Let $p_b \in E^0$ with $b \in \mathcal{B}_+(T_G;H,\chi)$ be a family of polynomials and consider

$$\sum_{b\in\mathcal{B}_+(T_G;H,\chi)} p_b\Phi^\chi_b = \sum_{\lambda\in\mathcal{X}_+(T_G;H,\chi)} c(\lambda)\Phi^\chi_\lambda. \tag{12}$$

Similarly write $p_b\Phi^\chi_b = \sum_{\lambda\in\mathcal{X}_+(T_G;H,\chi)} c_b(\lambda)\Phi^\chi_\lambda$. The polynomial p_b can be written as a linear combination of zonal spherical functions, say with spherical weights in the set $P(p_b)$. If $s \in P(p_b)$ is maximal, then $c_b(b+s) \neq 0$ by Lemma 8.10. Let $\max(P(p_b))$ denote the set of maximal elements in $P(p_b)$. Let $\lambda \in \cup_b(b + \max(P(p_b)))$ be maximal and write $\lambda = b(\lambda) + s(\lambda)$ according to Proposition 8.7.

We claim that $c_b(\lambda) = 0$ if $b \neq b(\lambda)$. Suppose that $c_b(\lambda) \neq 0$. Then $\lambda = b + s - \beta$ for some $s \in P(p_b)$ and $\beta \in \mathbb{Z}_{\geq 0}\Sigma(G/P)$, by Lemma 8.10. If $s \in P(p_b)$ is not maximal, then there is an element $s' \in P(p_b)$ with $s' = s + \gamma$, for γ a linear combination of roots with non-negative integer coefficients. Hence $\lambda = b + s' - \gamma - \beta$. If $\gamma + \beta = 0$ then $b = b(\lambda)$ by Proposition 8.7, contradicting the assumption that $b \neq b(\lambda)$. Hence $\gamma + \beta \neq 0$. But this implies $\lambda < b + s'$ while at the same time $b + s' \in \cup_b(b + \max(P(p_b)))$. This contradicts λ being maximal and we conclude that $c_b(\lambda) = 0$.

These arguments also show that if $p_b \neq 0$ for some $b \in \mathcal{B}_+(T_G; H, \chi)$, then there is a non-zero coefficient $c(\lambda)$ in (12). Conversely, if (12) is equal to zero, then all p_b must be equal to zero. This shows that E^χ is freely generated over E^0. □

Example 8.13. The algebra of SL(3)-biinvariant functions on G_2 is $E^0 = \mathbb{C}[\phi_{\omega_1}]$. Let $\lambda = 4\omega_1 + 3\omega_2$, the black node in Figure 1. The function $\phi_{\omega_1}\Phi^{3\varpi_1}_\lambda$ can be expressed as a linear combination of the spherical functions

$$\Phi^{3\varpi_1}_{\lambda+\omega_1}, \Phi^{3\varpi_1}_{\lambda+2\omega_1-\omega_2}, \Phi^{3\varpi_1}_{\lambda+\omega_1-\omega_2} \quad \text{and} \quad \Phi^{3\varpi_1}_{\lambda-\omega_1}.$$

Conversely, the spherical function $\Phi^{3\varpi_1}_{\lambda+\omega_1}$ can be expressed as an E^0-linear combination of spherical functions of type $3\varpi_1$ associated to dominant weights $< \lambda + \omega_1$. This illustrates the induction argument of Theorem 8.12.

9. Tables

In Tables 2–4 we gather all the indecomposable multiplicity free systems (G, H, P) with $P \subset H$ a proper parabolic subgroup, together with the generators of the extended weight monoid $\widetilde{\Gamma}(G/P)$.

The data in Tables 2–4 are all derived with lengthy but elementary applications of our results of Sections 5, as done in the example of Section 7. The detailed computations are in Appendices B and C.

For the notations, see Appendices A, B, and C. As usual, P is indicated by the missing simple root(s).

Table 2. (G, H) symmetric

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
B.1.1	$\mathrm{SL}(2n)$	$\mathrm{Sp}(2n)$	$\{\beta_1\}$	$(\omega_{2i}, 0)$, $1 \le i \le n-1$ $(\omega_{2j-1}, -\varpi_1), 1 \le j \le n$
B.1.2	$\mathrm{SL}(6)$	$\mathrm{Sp}(6)$	$\{\beta_3\}$	$(\omega_1 + \omega_3 + \omega_5, -\varpi_3)$, $(\omega_1 + \omega_4, -\varpi_3)$, $(\omega_2 + \omega_5, -\varpi_3)$, $(\omega_2, 0), (\omega_3, -\varpi_3), (\omega_4, 0)$
B.1.3	$\mathrm{SL}(4)$	$\mathrm{Sp}(4)$	$\{\beta_1, \beta_2\}$	$(\omega_1 + \omega_3, -\varpi_2), (\omega_1, -\varpi_1), (\omega_2, 0)$ $(\omega_2, -\varpi_2), (\omega_3, -\varpi_1)$
B.2.1	$\mathrm{SL}(q+3)$ $1 \le q$	$\mathrm{S}(\mathrm{L}(2) \times \mathrm{L}(q+1))$	$\{\beta_1\}$	$(\omega_1 + \omega_{q+2}, 0), (\omega_1, -\varpi_1), (\omega_2, -\varpi_2)$, $(\omega_{q+1}, \varpi_2), (\omega_{q+2}, \varpi_2 - \varpi_1)$
B.2.2	$\mathrm{SL}(p+q+2)$ $2 \le p \le q$ $n = p+q+1$	$\mathrm{S}(\mathrm{L}(p+1) \times \mathrm{L}(q+1))$	$\{\beta_1\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p$, $(\omega_j + \omega_{n+2-j}, -\varpi_1)$, $2 \le j \le p$, $(\omega_{q+2}, -\varpi_1 + \varpi_{p+1}), (\omega_1, -\varpi_1)$, $(\omega_{q+1}, \varpi_{p+1}), (\omega_{p+1}, -\varpi_{p+1})$
B.2.2	$\mathrm{SL}(p+q+2)$ $2 \le p \le q$ $n = p+q+1$	$\mathrm{S}(\mathrm{L}(p+1) \times \mathrm{L}(q+1))$	$\{\beta_p\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p$, $(\omega_j + \omega_{n-j}, -\varpi_p + \varpi_{p+1})$, $1 \le j \le p-1$, $(\omega_p, -\varpi_p), (\omega_n, -\varpi_p + \varpi_{p+1})$, $(\omega_{p+1}, -\varpi_{p+1}), (\omega_{q+1}, \varpi_{p+1})$
B.2.3	$\mathrm{SL}(p+q+2)$ $1 \le p < q$ $n = p+q+1$	$\mathrm{S}(\mathrm{L}(p+1) \times \mathrm{L}(q+1))$	$\{\beta_{p+1}\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p$, $(\omega_j + \omega_{n+2-j}, \varpi_{p+1} - \varpi_{p+2})$, $2 \le j \le p+1$, $(\omega_1, \varpi_{p+1} - \varpi_{p+2}), (\omega_{p+1}, -\varpi_{p+1})$, $(\omega_{p+2}, -\varpi_{p+2}), (\omega_{q+1}, \varpi_{p+1})$

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Table 2 – *Continued from previous page*

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
B.2.3	$\mathrm{SL}(p+q+2)$ $1 \le p < q$ $n = p+q+1$	$\mathrm{S}(\mathrm{L}(p+1) \times \mathrm{L}(q+1))$	$\{\beta_{p+q}\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p,$ $(\omega_j + \omega_{n-j}, -\varpi_n),$ $1 \le j \le p,$ $(\omega_n, -\varpi_n), (\omega_{q+1}, \varpi_{p+1}),$ $(\omega_q, \varpi_{p+1} - \varpi_n), (\omega_{p+1}, -\varpi_{p+1})$
B.2.4	$\mathrm{SL}(q+3)$ $1 < q$ $n = q+2$	$\mathrm{S}(\mathrm{L}(2) \times \mathrm{L}(q+1))$	$\{\beta_i'\}$ $2 \le i \le q$	$(\omega_1 + \omega_{i+1}, -\varpi_{i+2}), (\omega_1 + \omega_n, 0),$ $(\omega_2, -\varpi_2), (\omega_i, \varpi_2 - \varpi_{i+2}),$ $(\omega_{i+1} + \omega_n, \varpi_2 - \varpi_{i+2}),$ $(\omega_{i+1}, -\varpi_{i+2}), (\omega_{n-1}, \varpi_2)$
B.2.5	$\mathrm{SL}(q+2)$ $1 \le q$ $n = q+1$	$\mathrm{S}(\mathrm{L}(1) \times \mathrm{L}(q+1))$	S_H	$(\omega_i, -\varpi_i), 1 \le i \le n,$ $(\omega_j, \varpi_1 - \varpi_{j+1}), 1 \le j \le n-1,$ (ω_n, ϖ_1)
B.3	$\mathrm{SO}(2n+2)$	$\mathrm{SO}(2) \times \mathrm{SO}(2n)$	$\{\beta_{n-1}\}$	$(\omega_1, \varpi_0), (\omega_1, -\varpi_0), (\omega_2, 0),$ $(\omega_{n+1}, \varpi_0 - \varpi_{n-1}), (\omega_n, -\varpi_{n-1})$
B.3	$\mathrm{SO}(2n+2)$	$\mathrm{SO}(2) \times \mathrm{SO}(2n)$	$\{\beta_n\}$	$(\omega_1, \varpi_0), (\omega_1, -\varpi_0), (\omega_2, 0),$ $(\omega_n, \varpi_0 - \varpi_n), (\omega_{n+1}, -\varpi_n)$
B.4	$\mathrm{SO}(2n+1)$ $n \ge 3$	$\mathrm{SO}(2n)$	S_H	$(\omega_i, -\varpi_i), i \in \{1, \ldots, n-2, n\},$ $(\omega_{n-1}, -\varpi_{n-1} - \varpi_n), (\omega_1, 0)$ $(\omega_j, -\varpi_{j-1}), 2 \le j \le n$
B.5	$\mathrm{SO}(2n+2)$ $n \ge 3$	$\mathrm{SO}(2n+1)$	S_H	$(\omega_i, -\varpi_i), 1 \le i \le n,$ $(\omega_{n+1}, -\varpi_n), (\omega_1, 0),$ $(\omega_j, -\varpi_{j-1}), 2 \le j \le n$
B.6	$\mathrm{SO}(2n+2)$ $n \ge 2$ even	$\mathrm{GL}(n+1)$	$\{\beta_1\}$	$(\omega_{2i-1}, -\varpi_1), i = 1, \ldots, \frac{n}{2},$ $(\omega_{2j}, 0), i = 1, \ldots, \frac{n}{2} - 1,$ $(\omega_n, \frac{1}{2}\varpi_{n+1}), (\omega_{n+1}, -\frac{1}{2}\varpi_{n+1}),$ $(\omega_{n+1}, \frac{1}{2}\varpi_{n+1} - \varpi_1)$

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Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
B.6	$\mathrm{SO}(2n+2)$ $n \geq 3$ odd	$\mathrm{GL}(n+1)$	$\{\beta_1\}$	$(\omega_{2i-1}, -\varpi_1), i = 1, \dots, \frac{n-1}{2}$, $(\omega_{2j}, 0), i = 1, \dots, \frac{n-1}{2}$, $(\omega_n, \frac{1}{2}\varpi_{n+1} - \varpi_1), (\omega_{n+1}, -\frac{1}{2}\varpi_{n+1})$, $(\omega_{n+1}, \frac{1}{2}\varpi_{n+1})$
B.6	$\mathrm{SO}(2n+2)$ $n \geq 2$ even	$\mathrm{GL}(n+1)$	$\{\beta_n\}$	$(\omega_{2i-1}, -\varpi_n), i = 1, \dots, \frac{n}{2}$, $(\omega_{2j}, 0), i = 1, \dots, \frac{n}{2} - 1$, $(\omega_n, -\frac{1}{2}\varpi_{n+1}), (\omega_{n+1}, \frac{1}{2}\varpi_{n+1})$, $(\omega_{n+1}, -\frac{1}{2}\varpi_{n+1} - \varpi_n)$
B.6	$\mathrm{SO}(2n+2)$ $n \geq 3$ odd	$\mathrm{GL}(n+1)$	$\{\beta_2\}$	$(\omega_{2i-1}, -\varpi_n), i = 1, \dots, \frac{n-1}{2}$, $(\omega_{2j}, 0), i = 1, \dots, \frac{n-1}{2}$, $(\omega_n, -\frac{1}{2}\varpi_{n+1} - \varpi_n), (\omega_{n+1}, \frac{1}{2}\varpi_{n+1})$, $(\omega_{n+1}, -\frac{1}{2}\varpi_{n+1})$
B.7.1	$\mathrm{Sp}(2p+2q)$ $p > q$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2q)$	$\{\beta_1\}$	$(\omega_{2i}, 0), 1 \leq i \leq q$, $(\omega_{2j-1}, -\varpi_1), 1 \leq i \leq q+1$
B.7.1	$\mathrm{Sp}(2p+2q)$ $p \leq q$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2q)$	$\{\beta_1\}$	$(\omega_{2i}, 0), 1 \leq i \leq p$, $(\omega_{2j-1}, -\varpi_1), 1 \leq i \leq p$
B.7.2	$\mathrm{Sp}(6+2q)$ $q \geq 3$	$\mathrm{Sp}(6) \times \mathrm{Sp}(2q)$	$\{\beta_3\}$	$(\omega_1 + \omega_4, -\varpi_3), (\omega_1 + \omega_3 + \omega_5, -\varpi_3)$, $(\omega_3, -\varpi_3), (\omega_4, 0), (\omega_6, 0)$, $(\omega_2 + \omega_5, -\varpi_3), (\omega_2, 0)$
B.7.2	$\mathrm{Sp}(10)$	$\mathrm{Sp}(6) \times \mathrm{Sp}(4)$	$\{\beta_3\}$	$(\omega_1 + \omega_4, -\varpi_3), (\omega_1 + \omega_3 + \omega_5, -\varpi_3)$, $(\omega_2 + \omega_5, -\varpi_3), (\omega_2, 0), (\omega_3, -\varpi_3), (\omega_4, 0)$
B.7.2	$\mathrm{Sp}(4+2q)$	$\mathrm{Sp}(4) \times \mathrm{Sp}(2q)$	$\{\beta_2\}$	$(\omega_1 + \omega_3, -\varpi_2), (\omega_2, 0)$, $(\omega_2, -\varpi_2), (\omega_4, 0)$
B.7.2	$\mathrm{Sp}(2p+4)$ $p \geq 4$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(4)$	$\{\beta_2\}$	$(\omega_1 + \omega_{q+1}, -\varpi_p), (\omega_1 + \omega_3 + \omega_{p+2}, -\varpi_p)$, $(\omega_2 + \varpi_{p+2}, -\varpi_p), (\omega_2, 0)$, $(\omega_3 + \omega_{p+1}, -\varpi_p), (\omega_4, 0), (\omega_p, -\varpi_p)$

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Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
B.7.2	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_2\}$	$(\omega_1+\omega_{p+1}, -\varpi_p), (\omega_2, 0), (\omega_p, -\varpi_p)$
B.7.3	$\mathrm{Sp}(2p+2)$ $p \geq 3$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i\}$ $2 \leq i \leq p-1$	$(\omega_1+\omega_{i+1}, -\varpi_i), (\omega_2, 0),$ $(\omega_i, -\varpi_i), (\omega_{i+2}, -\varpi_i)$
B.7.4	$\mathrm{Sp}(4+2q)$ $q \geq 2$	$\mathrm{Sp}(4) \times \mathrm{Sp}(2q)$	$\{\beta_1, \beta_2\}$	$(\omega_1+\omega_3, -\omega_2), (\omega_1, -\varpi_1), (\omega_2, -\varpi_2)$ $(\omega_2, 0), (\omega_3, -\varpi_1), (\omega_4, 0)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 < i < j-1 < p-1$	$(\omega_1+\omega_{i+1}, -\varpi_i), (\omega_1+\omega_{j+1}, -\varpi_j), (\omega_2, 0),$ $(\omega_i, -\varpi_i), (\omega_{i+1}+\omega_{j+1}, -\varpi_i-\varpi_j),$ $(\omega_{i+2}, -\varpi_i), (\omega_j, -\varpi_j), (\omega_{j+2}, -\varpi_j)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 = i < j-1 < p-1$	$(\omega_1+\omega_{j+1}, -\varpi_1), (\omega_1, -\varpi_1), (\omega_2, 0),$ $(\omega_2+\omega_{j+1}, -\varpi_1-\varpi_j), (\omega_3, -\varpi_1),$ $(\omega_j, -\varpi_j), (\omega_{j+2}, -\varpi_j)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 < i = j-1 < p-1$	$(\omega_1+\omega_{j+1}, -\varpi_j), (\omega_1+\omega_i, -\varpi_j), (\omega_2, 0),$ $(\omega_i, -\varpi_i), (\omega_j, -\varpi_j),$ $(\omega_{j+1}, -\varpi_i), (\omega_{j+2}, -\varpi_j)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 < i < j-1 = p-1$	$(\omega_1+\omega_{i+1}, -\varpi_i), (\omega_1+\omega_{p+1}, -\varpi_p), (\omega_2, 0),$ $((\omega_i, -\varpi_i)), (\omega_{i+1}+\omega_{p+1}, -\varpi_i-\varpi_p),$ $(\omega_{i+2}, -\varpi_i), (\omega_p, -\varpi_p)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 = i = j-1 < p-1$	$(\omega_1+\omega_3, -\varpi_2), (\omega_1, -\varpi_1), (\omega_2, -0),$ $(\omega_2, -\varpi_2), (\omega_3, -\varpi_1), (\omega_4, -\varpi_2)$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 < i = j-1 = p-1$	$(\omega_1+\omega_p, -\varpi_{p-1}), (\omega_1+\omega_{p+1}, -\varpi_p),$ $(\omega_2, 0), (\omega_{p-1}, -\varpi_{p-1}),$ $(\omega_p, -\varpi_p), (\omega_{p+1}, -\varpi_{p-1})$
B.7.5	$\mathrm{Sp}(2p+2)$ $p \geq 2$	$\mathrm{Sp}(2p) \times \mathrm{Sp}(2)$	$\{\beta_i, \beta_j\}$ $1 = i = j-1 = p-1$	$(\omega_1+\omega_3, -\varpi_2), (\omega_1, -\varpi_1), (\omega_2, -\varpi_2),$ $(\omega_2, 0), (\omega_3, -\varpi_1)$
B.7.6	$\mathrm{Sp}(2+2q)$	$\mathrm{Sp}(2) \times \mathrm{Sp}(2q)$	$\{\beta_1, \beta_i'\}$	$(\omega_1+\omega_{i+1}, -\varpi_i'), (\omega_1, -\varpi_1), (\omega_2, 0),$

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Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
	$q \geq 1$		$1 < i < q$	$(\omega_i, -\varpi'_i), (\omega_{i+1}, -\varpi'_i - \varpi_1), (\omega_{i+2}, -\varpi'_i)$
B.7.6	$\mathrm{Sp}(2+2q)$ $q \geq 1$	$\mathrm{Sp}(2) \times \mathrm{Sp}(2q)$	$\{\beta_1, \beta'_i\}$ $1 = i < q$	$(\omega_1, -\varpi_1), (\omega_1, -\varpi'_1), (\omega_2, 0),$ $(\omega_2, -\varpi'_1 - \varpi_1), (\omega_3, -\varpi'_1)$
B.7.6	$\mathrm{Sp}(2+2q)$ $q \geq 1$	$\mathrm{Sp}(2) \times \mathrm{Sp}(2q)$	$\{\beta_1, \beta'_i\}$ $1 < i = q$	$(\omega_1 + \omega_{p+1}, -\varpi'_q), (\omega_1, -\varpi_1), (\omega_2, 0),$ $(\omega_p, -\varpi'_q), (\omega_{p+1}, -\varpi'_q - \varpi_1)$
B.7.6	$\mathrm{Sp}(2+2q)$ $q \geq 1$	$\mathrm{Sp}(2) \times \mathrm{Sp}(2q)$	$\{\beta_1, \beta'_i\}$ $1 = i = q$	$(\omega_1, -\varpi_1), (\omega_1, -\varpi'_1),$ $(\omega_2, 0), (\omega_2, -\varpi'_1 - \varpi_1)$
B.8.1	F_4	$\mathrm{Spin}(9)$	$\{\beta_1, \beta_2\}$	$(\omega_1, -\varpi_2), (\omega_2, -\varpi_2), (\omega_3, -\varpi_2)$ $(\omega_3, -\varpi_1), (\omega_4, -\varpi_1), (\omega_4, 0)$
B.8.2	F_4	$\mathrm{Spin}(9)$	$\{\beta_3\}$	$(\omega_1 + \omega_4, -\varpi_3), (\omega_1 + \omega_3, -\varpi_3), (\omega_2, -\varpi_3)$ $(\omega_3, -\varpi_3), (\omega_4, 0)$
B.8.3	F_4	$\mathrm{Spin}(9)$	$\{\beta_4\}$	$(\omega_1, -\varpi_4), (\omega_3, -\varpi_4),$ $(\omega_4, -\varpi_4), (\omega_4, 0)$
B.9	E_6	$\mathrm{SO}(10) \times \mathbb{C}^\times$	$\{\beta_1\}$	$(\omega_1, -\varpi_1 + 2\epsilon), (\omega_1, -4\epsilon), (\omega_2, 0),$ $(\varpi_6, -\varpi_1 - 2\epsilon), (\omega_6, 4\epsilon),$ $(\omega_5, -\varpi_1 + 2\epsilon), (\omega_3, -\varpi_1 - 2\epsilon)$
B.10	E_6	F_4	$\{\beta_1\}$	$(\omega_1, 0), (\omega_2, -\varpi_1), (\omega_3, -\varpi_1)$ $(\omega_4, -\varpi_1), (\omega_5, -\varpi_1), (\omega_6, 0)$
B.11	$\mathrm{SL}(n) \times \mathrm{SL}(n)$	$\mathrm{diag}(\mathrm{SL}(n))$	$\{\beta_1\}$	$(\omega_i + \omega'_{n-i}, 0), 1 \leq i \leq n-1,$ $(\omega_i + \omega'_{n+1-i}, -\varpi_1), 1 \leq i \leq n$
B.11	$\mathrm{SL}(n) \times \mathrm{SL}(n)$	$\mathrm{diag}(\mathrm{SL}(n))$	$\{\beta_{n-1}\}$	$(\omega_i + \omega'_{n-i}, 0), 1 \leq i \leq n-1,$ $(\omega_{n+1-i} + \omega'_i, -\varpi_{n-1}), 1 \leq i \leq n$

Table 3. (G, H) spherical, not symmetric, G simple

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
C.1	$\mathrm{SL}(p+q+2)$ $2 \le p < q$ $n = p+q+1$	$\mathrm{SL}(p+1) \times \mathrm{SL}(q+1)$	$\{\beta_1\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p,$ $(\omega_j + \omega_{n+2-j}, -\varpi_1), 2 \le j \le p,$ $(\omega_{q+2}, -\varpi_1), (\omega_1, -\varpi_1),$ $(\omega_{q+1}, 0), (\omega_{p+1}, 0)$
C.1	$\mathrm{SL}(p+q+2)$ $2 \le p < q$ $n = p+q+1$	$\mathrm{SL}(p+1) \times \mathrm{SL}(q+1)$	$\{\beta_p\}$	$(\omega_i + \omega_{n+1-i}, 0), 1 \le i \le p,$ $(\omega_j + \omega_{n-j}, -\varpi_p), 1 \le j \le p-1,$ $(\omega_p, -\varpi_p), (\omega_n, -\varpi_p),$ $(\omega_{p+1}, 0), (\omega_{q+1}, 0)$
C.1	$\mathrm{SL}(q+3)$ $q > 1$	$\mathrm{SL}(2) \times \mathrm{SL}(q+1)$	$\{\beta_1\}$	$(\omega_1, 0), (\omega_1, -\varpi_1), (\omega_2, 0),$ $(\omega_{q+1}, 0), (\omega_{q+2}, -\varpi_1)$
C.1	$\mathrm{SL}(q+3)$ $q \ge 4$ $n = q+2$	$\mathrm{SL}(2) \times \mathrm{SL}(q+1)$	$\{\beta_i'\}$ $2 \le i \le q-2$	$(\omega_1 + \omega_i, -\varpi_{i+1}), (\omega_1 + \omega_n, 0),$ $(\omega_2, 0), (\omega_{i-1}, -\varpi_{i+1}),$ $(\omega_i + \omega_n, -\varpi_{i+n}),$ $(\omega_{i+1}, -\varpi_{i+1}), (\omega_{n-1}, 0)$
C.1	$\mathrm{SL}(q+2)$ $q \ge 1$ $n = q+1$	$\mathrm{SL}(q+1)$	$S_H \setminus \{\beta_k\}$ $1 \le k \le q$	$(\omega_i, -\varpi_i), 2 \le i \le n$, but $i \ne k$ $(\omega_j, -\varpi_{j+1}), 1 \le j \le n-1$, but $j \ne k$, $(\omega_1, 0), (\omega_n, 0)$
C.2	$\mathrm{SO}(4n+2)$ $n \ge 2$	$\mathrm{SL}(2n+1)$	$\{\beta_1\}$	$(\omega_{2i-1}, -\varpi_1), 1 \le i \le n,$ $(\omega_{2j}, 0), 1 \le j \le n-1,$ $(\omega_{2n}, 0), (\omega_{2n+1}, 0), (\omega_{2n+1}, -\varpi_1)$
C.2	$\mathrm{SO}(4n+2)$ $n \ge 2$	$\mathrm{SL}(2n+1)$	$\{\beta_{2n}\}$	$(\omega_{2i-1}, -\varpi_{2n}), 1 \le i \le n,$ $(\omega_{2j}, 0), 1 \le j \le n-1,$ $(\omega_{2n}, 0), (\omega_{2n+1}, 0), (\omega_{2n+1}, -\varpi_{2n})$
C.3	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$	$\{\beta_1\}$	$(\omega_1, 0), (\omega_2, -\varpi_1), (\omega_3, -\varpi_1)$ $(\omega_4, 0), (\omega_4, -\varpi_1)$

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Table 3 – *Continued from previous page*

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
C.4.1	Spin(7)	G_2	$\{\beta_1\}$	$(\omega_1, -\varpi_1), (\omega_2, -\varpi_1)$ $(\omega_3, 0), (\omega_3, -\varpi_1)$
C.4.2	Spin(7)	G_2	$\{\beta_2\}$	$(\omega_1 + \omega_2, -\varpi_2), (\omega_2, -\varpi_2)$ $(\omega_1 + \omega_3, -\varpi_2), (\omega_3, 0)$
C.5	G_2	SL(3)	$\{\beta_1\}$	$(\omega_1, 0), (\omega_1, -\varpi_1), (\omega_2, -\varpi_1)$
C.5	G_2	SL(3)	$\{\beta_2\}$	$(\omega_1, 0), (\omega_1, -\varpi_2), (\omega_2, -\varpi_2)$

Table 4. (G, H) spherical, not symmetric, G not simple

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
C.6.1	$\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$ $n > 1, m > 2$	$\mathrm{Sp}(2m-2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2n-2)$	$\{\beta_{m-1}\}$	$(\omega_1 + \omega_m, -\varpi_{m-1}), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0), (\omega_{m-1}, -\varpi_{m-1})$, $(\omega_m + \omega_1', -\varpi_{m-1}), (\omega_2', 0)$
C.6.1	$\mathrm{Sp}(2m) \times \mathrm{Sp}(2)$ $m > 2$	$\mathrm{Sp}(2m-2) \times \mathrm{SL}(2)$	$\{\beta_{m-1}\}$	$(\omega_1 + \omega_m, -\varpi_{m-1}), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0), (\omega_{m-1}, -\varpi_{m-1})$, $(\omega_m + \omega_1', -\varpi_{m-1})$
C.6.1	$\mathrm{Sp}(4) \times \mathrm{Sp}(2n)$ $n \geq 2$	$\mathrm{Sp}(2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2n-2)$	$\{\beta_1\}$	$(\omega_1, -\varpi_1), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0)$, $(\omega_2 + \omega_1', -\varpi_1), (\omega_2', 0)$
C.6.1	$\mathrm{Sp}(4) \times \mathrm{Sp}(2)$	$\mathrm{Sp}(2) \times \mathrm{SL}(2)$	$\{\beta_1\}$	$(\omega_1, -\varpi_1), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0), (\omega_2 + \omega_1', -\varpi_1)$
C.6.1	$\mathrm{Sp}(2m) \times \mathrm{Sp}(2n)$ $1 < i < m-1$ $n > 1$	$\mathrm{Sp}(2m-2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2n-2)$	$\{\beta_i\}$	$(\omega_1 + \omega_{i+1}, -\varpi_i), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0), (\omega_i, -\varpi_i), (\omega_2', 0)$, $(\omega_{i+1} + \omega_1', -\varpi_i), (\omega_{i+2}, -\varpi_i)$
C.6.1	$\mathrm{Sp}(2m) \times \mathrm{Sp}(2)$ $1 < i < m-1$	$\mathrm{Sp}(2m-2) \times \mathrm{SL}(2)$	$\{\beta_i\}$	$(\omega_1 + \omega_{i+1}, -\varpi_i), (\omega_1 + \omega_1', 0)$, $(\omega_2, 0), (\omega_i, -\varpi_i)$,

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Table 4 – *Continued from previous page*

Section no.	G	H	P	Generators $\widetilde{\Gamma}(G/P)$
				$(\omega_{i+1}+\omega'_1,-\varpi_i),(\omega_{i+2},-\varpi_i)$
C.6.1	$\mathrm{Sp}(2m)\times\mathrm{Sp}(2n)$ $m\geq 3, n>1$	$\mathrm{Sp}(2m-2)\times\mathrm{SL}(2)\times\mathrm{Sp}(2n-2)$	$\{\beta_1\}$	$(\omega_1+\omega'_1,0),(\omega_1,-\varpi_1)$ $(\omega_2,0),(\omega_2+\omega'_1,-\varpi_1),$ $(\omega_3,-\varpi_1),(\omega'_2,0)$
C.6.1	$\mathrm{Sp}(2m)\times\mathrm{Sp}(2)$ $m\geq 3$	$\mathrm{Sp}(2m-2)\times\mathrm{SL}(2)$	$\{\beta_1\}$	$(\omega_1+\omega'_1,0),(\omega_1,-\varpi_1)$ $(\omega_2,0),(\omega_2+\omega'_1,-\varpi_1),$ $(\omega_3,-\varpi_1)$
C.6.2	$\mathrm{Sp}(2m)\times\mathrm{Sp}(2n)$ $n>1, m>1$	$\mathrm{Sp}(2m-2)\times\mathrm{SL}(2)\times\mathrm{Sp}(2n-2)$	$\{\beta'_1\}$	$(\omega_1+\omega'_1,0),$ $(\omega_1,-\varpi'_1),(\omega'_1,-\varpi'_1),$ $(\omega_2,0),(\omega'_2,0)$
C.6.2	$\mathrm{Sp}(2m)\times\mathrm{Sp}(2n)$ $n>1, m>1$	$\mathrm{Sp}(2m-2)\times\mathrm{SL}(2)\times\mathrm{Sp}(2n-2)$	$\{\beta'_1\}$	$(\omega_1+\omega'_1,0),$ $(\omega_1,-\varpi'_1),(\omega'_1,-\varpi'_1),$ $(\omega_2,0),(\omega'_2,0)$
C.6.2	$\mathrm{Sp}(2m)\times\mathrm{Sp}(2)$ $m>1=n$	$\mathrm{Sp}(2m-2)\times\mathrm{SL}(2)$	$\{\beta'_1\}$	$(\omega_1+\omega'_1,0),(\omega_2,0)$ $(\omega_1,-\varpi'_1),(\omega'_1,-\varpi'_1)$

Remark 9.1. We take the opportunity to fix a mistake in [31]. In Remark 4.4(4) of loc. cit. it is claimed that for the case $\mathrm{SL}(n)/\mathrm{S}(\mathrm{L}(n-2)\times\mathrm{L}(2))$, an irreducible representation of $\mathrm{SL}(n)$ cannot contain two irreducible $\mathrm{S}(\mathrm{L}(n-2)\times\mathrm{L}(2))$-modules of highest weight $p\omega_{n-1}$ and $q\omega_{n-1}$ with $p\neq q$, upon restriction. This is not true. Indeed, the extended weight semigroup $\widetilde{\Gamma}(\mathrm{SL}(n)/P)$, where $P\subset\mathrm{S}(\mathrm{L}(n-2)\times\mathrm{L}(2))$ is the parabolic subgroup obtained by leaving out the last simple root, has $(\omega_{n-1},0)$ and $(\omega_{n-1},-\varpi_{n-1})$ among its generators, see Section B.2.4. Hence, if λ is the highest weight of an irreducible $\mathrm{SL}(n)$-representation that contains $p\varpi_{n-1}$, then $\lambda+\omega_{n-1}$ is the highest weight of an irreducible $\mathrm{SL}(n)$-representation that contains irreducible representations of $\mathrm{S}(\mathrm{L}(n-2)\times\mathrm{L}(2))$ of highest weight $p\varpi_{n-1}$ and $(p+1)\varpi_{n-1}$.

The induction argument of loc. cit. is still valid but it has to be applied more carefully. The result is that the subgroup $\mathrm{GL}(n-2)\times\mathrm{SL}(2)\times\mathrm{Sp}(2m-2)\subset\mathrm{SL}(n)\times\mathrm{Sp}(2m)$ does not admit a proper parabolic subgroup that remains spherical in $\mathrm{SL}(n)\times\mathrm{Sp}(2m)$. This is also in line with Remark 8.3.

Appendix A. Preliminaries on the cases

In the next two sections we consider spherical homogeneous spaces of the form G/H, where G is semisimple simply connected and H is a connected reductive proper subgroup, such that H has a proper parabolic subgroup P that is spherical in G. In other words (G,H,P) is a multiplicity free system. For such triples, we compute the extended weight monoid of G/P. The choice of H in its conjugacy class is often relevant for our computations and will be specified in each case.

We use many of the notations of Section 7, let us recall them here and add some further ones. We denote by α_1, α_2, $\ldots$ the simple roots of G, numbered as in Bourbaki. The corresponding fundamental dominant weights will be denoted by ω_1, ω_2, $\ldots$. For $i\leq j$ we set $\alpha_{i,j}=\alpha_i+\cdots+\alpha_j$, and if $i>j$ then we set $\alpha_{i,j}=0$. For convenience in some formulae, if $G=\mathrm{SL}(n)$ we also set $\omega_n=0$.

If $G\subseteq\mathrm{GL}(n)$ is a classical group, it will be defined in such a way that B (resp. T) can be taken to be the set of upper triangular (resp. diagonal) matrices in G.

If G is the universal cover of a classical group, to simplify notations, we will implicitly replace G with the classical group G_0 and the subgroups H, P with their images H_0, P_0 in G_0. Thanks to Proposition 5.6, it will be enough to notice that no element of $2S$ appears among the spherical roots of G_0/P_0, to assure that our computations carried out for G_0 are equivalent to those for G.

Unless otherwise stated, we denote by $S_H=\{\beta_1,\beta_2,\ldots,\beta'_1,\beta'_2,\ldots\}$ the simple roots of H, grouped according to the various simple normal subgroups of H and numbered as in Bourbaki, corresponding as in Section 1 to a choice of a Borel subgroup B_H and a maximal torus $T_H\subseteq B_H$ of H. We recall that we take $B_H\subseteq B$ and $T_H\subseteq T$, and P containing the opposite Borel subgroup B_H^- of B_H with respect to T_H.

We denote by ϖ_1, ϖ_2, $\ldots$, ϖ'_1, ϖ'_2, $\ldots$ the corresponding fundamental dominant weights, which we define as the elements of $\mathcal{X}(T_H)_{\mathbb{Q}}$ that have the correct pairing with the simple coroots of H and are zero on the subspace corresponding to the connected center of H. We call them the *fundamental weights* of H.

We will often restrict characters of groups to subgroups, or extend them when possible to characters of larger groups. To simplify notations, we will denote with the same symbol the original character and the restriction or extension, if no confusion arises.

We denote by I the set of simple roots of H such that the Levi subgroup of P containing T_H has set of simple roots $S_H \smallsetminus I$.

It will also be possible to choose a parabolic subgroup Q of G, containing the Borel subgroup B^- opposite to B with respect to T, and minimal among the parabolic subgroups of G containing a G-conjugate of P. This will enable us to apply the results of Section 5 to the subgroup P. As before, we will denote by L_Q the Levi subgroup of Q containing T. Notice that P is connected, therefore Theorem 5.5 yields the characters χ_D for all $D \in \Delta(G/P)$.

If $J \subseteq S$, then we will also use the notation Q_J instead of simply Q, where Q_J is the parabolic subgroup of G containing B_- and such that its Levi subgroup has simple roots $S \smallsetminus J$.

If a simple root α_i moves only one color, then the latter will be denoted by D_i or E_i. If α_i moves two colors, they will be denoted by D_i^+ and D_i^-, or by D_i and E_i.

If it doesn't create ambiguities, we will allow the abuse of notation of denoting in the same way elements of the Weyl groups of reductive groups and a choice of representatives in the normalizer of the chosen maximal torus.

Finally we record the following observations.

Remark A.1. Let (G, H, P) be a strictly indecomposable multiplicity free system and let P^{op} be opposite to P with respect to the maximal torus $T_H \subseteq H$. Then (G, H, P^{op}) is also a strictly indecomposable multiplicity free system and the generators of $\widetilde{\Gamma}(G/P)$ are related to those of $\widetilde{\Gamma}(G/P^{\mathrm{op}})$ by $(\omega_D, \chi_D) \leftrightarrow (\omega_D^*, \chi_D^*)$ where ω_D^* is the highest weight of the dual of $\pi^G_{\omega_D}$ and χ_D^* is the lowest weight of the dual representation of $\pi^H_{\chi_D}$.

Remark A.2. Let (G, H, P_1) and (G, H, P_2) be multiplicity free systems with $P_1 \subseteq P_2$. Then $\widetilde{\Gamma}(G/P_2)$ is the subset of $\widetilde{\Gamma}(G/P_1)$ of the couples (λ, ω) such that ω extends to a weight of P_2. For this reason, during the computations in Appendices B and C, sometimes we will only discuss those parabolic subgroups $P \subseteq H$ that are minimal such that (G, H, P) is a multiplicity free system. We cannot just deal everywhere only with minimal cases though, because some computations for non-minimal subgroups will be necessary to complete computations for the minimal ones.

Remark A.3. Let (G, H, P) be a multiplicity free system, and let $\widetilde{H} \subseteq H$ be a connected reductive subgroup containing the commutator (H, H). In this case it is harmless to assume that H is the product of $\widetilde{H}$ and a subtorus $\widetilde{T}$ of T. The root systems of H and $\widetilde{K}$ are naturally identified, so that P corresponds to a parabolic subgroup $\widetilde{P}$ of $\widetilde{H}$ contained in P and such that $\widetilde{P} \cdot \widetilde{T} = P$. Suppose now that $(G, \widetilde{H}, \widetilde{P})$ is a multiplicity free system, and notice that then the natural map $G/\widetilde{P} \to G/P$ induces a bijection between the respective sets of colors, since $\widetilde{T} \subseteq B$. Proposition 4.2 implies that restriction of weights from P to $\widetilde{P}$ induces an isomorphism $\widetilde{\Gamma}(G/P) \to \widetilde{\Gamma}(G/\widetilde{P})$.

Appendix B. Symmetric cases

B.1. $\mathrm{SL}(2n)/\mathrm{Sp}(2n)$ **with** $n \geq 2$. We define $\mathrm{Sp}(2n)$ to be the stabilizer of the skew-symmetric bilinear form given by the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

In this way $B_H = B \cap \mathrm{Sp}(2n)$ is a Borel subgroup of $\mathrm{Sp}(2n)$ and $T_H = T \cap \mathrm{Sp}(2n) \subseteq B_H$ is a maximal torus of $\mathrm{Sp}(2n)$ contained in B_H. The simple roots of $\mathrm{Sp}(2n)$ are given by $\beta_i(t) = t_i t_{i+1}^{-1}$ with $1 \leq i \leq n-1$ and $\beta_n(t) = t_n^2$, where $t = (t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \in T_H$.

The group H is simply connected, and the fundamental dominant weights ϖ_1, $\dots$, ϖ_n have the property that ϖ_i is the restriction of ω_i and also of ω_{2n-i} to T_H, for all $i \in \{1, \dots, n\}$.

We have three possibilities for I:

(1) $I = \{\beta_1\}$ with any $n \geq 3$,
(2) $I = \{\beta_3\}$ with $n = 3$,
(3) $I =$ any subset of roots of H with $n = 2$.

B.1.1. $I = \{\beta_1\}$ *with any* $n \geq 3$. The subgroup P appears in [8] as case 6. The parameter "n" appearing in loc. cit. is equal here to $2n-1$. This gives

$$\Sigma(G/P) = \{\alpha_1 + \alpha_2, \dots, \alpha_{2n-2} + \alpha_{2n-1}\}$$

and $\Delta(G/P) = \{D_1, \dots, D_{2n-1}\}$, so that α_i moves D_i for all $i \in \{1, \dots, 2n-1\}$, and the Cartan pairing is given by $\rho(D_i) = \alpha_i^\vee|_{\Xi(G/P)}$.

We can take Q to be the parabolic subgroup such that L_Q has simple roots α_2, $\dots$, α_{2n-2}. Then Q is minimal for containing P and G/Q has two colors whose inverse images in G/P are D_1 and D_{2n-1}. Since $\omega_1^P = \omega_{2n-1}^P = \varpi_1$ we have

$$\chi_{D_1} = \chi_{D_{2n-1}} = -\varpi_1.$$

Using the Cartan pairing one deduces from equalities (6) of Lemma 5.1 the following system of equations:

$$\begin{cases} 0 &= -\varpi_1 + \chi_{D_2} - \chi_{D_3}, \\ 0 &= \varpi_1 + \chi_{D_2} + \chi_{D_3} - \chi_{D_4}, \\ 0 &= -\chi_{D_2} + \chi_{D_3} + \chi_{D_4} - \chi_{D_5}, \\ & \vdots \\ 0 &= -\chi_{D_{2n-5}} + \chi_{D_{2n-4}} + \chi_{D_{2n-3}} - \chi_{D_{2n-2}}, \\ 0 &= -\chi_{D_{2n-4}} + \chi_{D_{2n-3}} + \chi_{D_{2n-2}} + \varpi_1, \\ 0 &= -\chi_{D_{2n-3}} + \chi_{D_{2n-2}} - \varpi_1 \end{cases}$$

from which we obtain

$$\begin{aligned} \chi_{D_2} &= \dots = \chi_{D_{2n-2}} = 0, \\ \chi_{D_1} &= \dots = \chi_{D_{2n-1}} = -\varpi_1. \end{aligned}$$

B.1.2. $I = \{\beta_3\}$ *with* $n = 3$. The subgroup P appears in [8] as case 27, giving

$$\Sigma(G/P) = S,$$

and $\Delta(G/P) = \{D_1^+ = D_3^+ = D_5^+, D_1^- = D_4^-, D_2^+, D_2^- = D_5^-, D_3^-, D_4^+\}$. We can take Q such that only α_3 is not a simple root of L_Q. Then Q is minimal for containing P. The inverse image in G/P of the unique color of G/Q is the unique color of G/P moved only by α_3, i.e. D_3^-. Then

$$\chi_{D_3^-} = -\omega_3^P = -\varpi_3.$$

The Cartan pairing of G/P is

	α_1	α_2	α_3	α_4	α_5
D_1^+	1	−1	1	−1	1
D_1^-	1	0	−1	1	−1
D_2^+	0	1	0	0	−1
D_2^-	−1	1	−1	0	1
D_3^-	−1	0	1	0	−1
D_4^+	−1	0	0	1	0

which yields, thanks to the equalities (6) of Lemma 5.1,

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_1^-} = \chi_{D_2^-} &= -\varpi_3, \\ \chi_{D_2^+} = \chi_{D_4^+} &= 0. \end{aligned}$$

B.1.3. $I =$ *any subset of roots of* H, $n = 2$. We discuss only the minimal case $I = \{\beta_1, \beta_2\}$; the subgroup P corresponding to $I = \{\beta_1\}$ is found in [33] as the first case 5 in Table A, and the one corresponding to $I = \{\beta_2\}$ is found in [33] as the second case 7 in Table A.

We claim that the subgroup $P = B_H$ corresponding to $I = S_H$ has lattice $\Xi(G/B_H) = \mathcal{X}(T_G)$, the following spherical roots

$$\Sigma(G/B_H) = S$$

and colors $\Delta(G/B_H) = \{D_1^+ = D_3^+, D_1, D_2^+, D_2^-, D_3^-\}$ with Cartan pairing

	α_1	α_2	α_3
D_1^+	1	−1	1
D_1^-	1	0	−1
D_2^+	0	1	0
D_2^-	−1	1	−1
D_3^-	−1	0	1

To prove this, we recall that G/H has one spherical root $\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$, which generates $\Xi(G/H)$, and one color D_2' with valuation coinciding with $\alpha_2^\vee$ on the lattice of G/H (see [33, Case 5A in Table 1]). Using Corollary 3.9 and Proposition 6.1 one checks that a subgroup P corresponding to the above data is indeed conjugated in G to a parabolic subgroup of H, so we may assume $P \supseteq B_H$. Formula (8) assures that its character group has rank 2, hence $P = B_H$.

The subset of colors $\{D_1^+, D_2^+\}$ is the only one that corresponds to a G-equivariant morphism $G/B_H \to G/B_G$, therefore it corresponds to the inclusion $B_H \subseteq B_G$, and the colors D_1^-, D_2^-, D_3^- are the inverse images of the three colors of G/B_G. We conclude

$$\begin{array}{rcrcl} \chi_{D_1^-} &=& -\omega_1^{B_H} &=& -\varpi_1, \\ \chi_{D_2^-} &=& -\omega_2^{B_H} &=& -\varpi_2, \\ \chi_{D_3^-} &=& -\omega_3^{B_H} &=& -\varpi_1. \end{array}$$

The equalities (6) of Lemma 5.1 yield

$$\chi_{D_1^+} = -\varpi_2,$$
$$\chi_{D_2^+} = 0.$$

B.2. $\mathrm{SL}(p+q+2)/\mathrm{S}(\mathrm{L}(p+1)\times\mathrm{L}(q+1))$ **with** $p+q \geq 1$ **and** $0 \leq p \leq q$**.** Set $n = p+q+1$. We take the subgroup H to be the matrices with blocks on the diagonal of sizes resp. $p+1$ and $q+1$, and zeros elsewhere. In this way $B_H = B\cap H$ is a Borel subgroup of H with maximal torus $T_H = T$. The simple roots of H are $\beta_i = \alpha_i$ for $1 \leq i \leq p$ and $\beta'_j = \beta_{p+1+j} = \alpha_{p+1+j}$ for $1 \leq j \leq q$. We set $\varpi_i = \omega_i$ for $1 \leq i \leq n$, and $\varpi_{n+1} = 0$. Then the ϖ_i's are the fundamental weights of H, except for ϖ_{n+1}, and also except for ϖ_{p+1}, which is the restriction to T of a generator of the character group of H. The dominant integral weights of H are of the form $\sum_{k=1}^n b_k\omega_k$ with $b_i \in \mathbb{Z}_{\geq 0}$ for all i except for $b_{p+1} \in \mathbb{Z}$.

We have the following possibilities for I:

(1) $I = \{\beta_1\}$ with $p = 1 \leq q$.
(2) $I = \{\beta_1\}$ with $2 \leq p \leq q$.
(3) $I = \{\beta_p\}$ with $2 \leq p \leq q$.
(4) $I = \{\beta'_1\}$ with $1 \leq p < q$.
(5) $I = \{\beta'_q\}$ with $1 \leq p < q$.
(6) $I = \{\beta'_i\}$ with $p = 1, q \geq 3$ and $i \in \{2, \dots, q-1\}$.
(7) I= any subset of simple roots of H, with $p = 0$.

B.2.1. $I = \{\beta_1\}$ *with* $p = 1 \leq q$*, i.e.* $n = q+2$*.* The subgroup P is conjugated to the subgroup $\widetilde{P}$ that appears in [8] as case 9; the parameter "p" of loc. cit. is equal here to q. We have $gPg^{-1} = \widetilde{P}$ with $g = w_0^G s_1$, where $w_0^G \in W_G$ is the longest element.

We have

$$\Sigma(G/P) = \{\alpha_1, \alpha_{2,q+1}, \alpha_{q+2}\}$$

and $\Delta(G/P) = \{D_1^+ = D_{q+2}^+, D_1^-, D_2, E_{q+1}, D_{q+2}^-\}$ with Cartan pairing

	α_1	$\alpha_{2,q+1}$	α_{q+2}
D_1^+	1	-1	1
D_1^-	1	0	-1
D_2	-1	1	0
E_{q+1}	0	1	-1
D_{q+2}^-	-1	0	1

We can take Q so that only α_1 and α_2 are not roots of L_Q, and the subset of colors corresponding to the inclusion $P \subseteq Q$ is $\{D_1^+, E_{q+1}, D_{q+2}^-\}$. This is obvious if $q > 1$, just by looking at which color is moved by which simple root. If $q = 1$ the only other possibility $\{D_1^+, D_2, D_{q+2}^-\}$ is excluded because it is not distinguished.

So the inverse images of the colors of G/Q are D_1^- and D_2, which yields

$$\begin{array}{rcccl} \chi_{D_1^-} & = & -\omega_1^P & = & -\varpi_1, \\ \chi_{D_2} & = & -\omega_2^P & = & -\varpi_2. \end{array}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= 0, \\ \chi_{E_{q+1}} &= \varpi_2, \\ \chi_{D_{q+2}^-} &= \varpi_2 - \varpi_1. \end{aligned}$$

B.2.2. *$I = \{\beta_1\}$ or $I = \{\beta_p\}$ with $2 \le p \le q$.* In view of Remark A.1 it is enough to consider only case (3), i.e. $I = \{\beta_p\}$. The subgroup P is conjugated to the subgroup $\widetilde{P}$ that appears in [8] as case 11. The parameters "p" and "q" of loc. cit. are equal here respectively to p and $q+1-p$. We have $gPg^{-1} = \widetilde{P}$ with $g = w_0^G w_0^H$.

The set of spherical roots and the set of colors are given by

$$\begin{aligned} \Sigma(G/P) = \{&\alpha_1, \dots, \alpha_p, \alpha_{p+1,q+1}, \alpha_{q+2}, \dots, \alpha_n\}, \\ \Delta(G/P) = \{&D_1^+ = D_n^+, D_2^+ = D_{n-1}^+, \dots, D_p^+ = D_{q+2}^+, \\ &D_1^- = D_{n-1}^-, \dots, D_{p-1}^- = D_{q+2}^-, D_p^-, D_n^-, \\ &D_{p+1}, E_{q+1}\}. \end{aligned}$$

The Cartan pairing is given by the following matrix:

	α_1	α_2	α_3	α_4	$\dots$	α_{p-1}	α_p	$\alpha_{p+1,q+1}$	α_{q+2}	α_{q+3}	$\dots$	α_{p+q-1}	α_{p+q}	α_{p+q+1}
D_1^+	1	0	0	0	$\dots$	0	0	0	0	0	$\dots$	0	−1	1
D_1^-	1	−1	0	0	$\dots$	0	0	0	0	0	$\dots$	0	1	−1
D_2^+	−1	1	0	0	$\dots$	0	0	0	0	0	$\dots$	−1	1	0
D_2^-	0	1	−1	0	$\dots$	0	0	0	0	0	$\dots$	1	−1	0
D_3^+	0	−1	1	0	$\dots$	0	0	0	0	0	$\dots$	1	0	0
D_3^-	0	0	1	−1	$\dots$	0	0	0	0	0	$\dots$	−1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
D_{p-1}^+	0	0	0	0	$\dots$	1	0	0	−1	1	$\dots$	0	0	0
D_{p-1}^-	0	0	0	0	$\dots$	1	−1	0	1	−1	$\dots$	0	0	0
D_p^+	0	0	0	0	$\dots$	−1	1	−1	1	0	$\dots$	0	0	0
D_p^-	0	0	0	0	$\dots$	0	1	0	−1	0	$\dots$	0	0	0
D_{p+1}	0	0	0	0	$\dots$	0	−1	1	0	0	$\dots$	0	0	0
E_{q+1}	0	0	0	0	$\dots$	0	0	1	−1	0	$\dots$	0	0	0
D_n^-	−1	0	0	0	$\dots$	0	0	0	0	0	$\dots$	0	0	1

We can take Q so that only α_p and α_{p+1} are not simple roots of L_Q. Then Q is minimal parabolic containing P. The inverse images in G/P of the two colors of G/Q are D_p^- and D_{p+1}. This is obvious if $p < q$, since these are the only colors of G/P that are moved only by resp. α_p and α_{p+1}. If $p = q$, then one excludes the other possibility, namely D_p^- and E_{p+1}, checking that the set $\Delta(G/P) \smallsetminus \{D_p^-, E_{p+1}\}$ is not parabolic. This can be seen by looking just at the two spherical roots α_{p+1} and α_p: the first implies $a_{p+1} > a_p^+$ (where $a_i^{(\pm)}$ is the coefficient of $D_i^{(\pm)}$ in a linear combination that is positive on all spherical roots) while the second implies $a_p^+ - a_{p+1} - a_{p-1}^- > 0$, but this yields a contradiction. Therefore

$$\begin{array}{rcccl} \chi_{D_p^-} & = & -\omega_p^P & = & -\varpi_p, \\ \chi_{D_{p+1}} & = & -\omega_{p+1}^P & = & -\varpi_{p+1}. \end{array}$$

We obtain

$$\begin{array}{ccccccccccc} \chi_{D_1^-} & = & \chi_{D_2^-} & = & \dots & = & \chi_{D_{p-1}^-} & = & \chi_{D_n^-} & = & -\varpi_p + \varpi_{p+1}, \\ & & \chi_{D_1^+} & = & \chi_{D_2^+} & = & \dots & = & \chi_{D_p^+} & = & 0, \\ & & & & & & & & \chi_{E_{q+1}} & = & \varpi_{p+1}, \end{array}$$

which is easily checked to satisfy equalities (6) of Lemma 5.1.

B.2.3. *$I = \{\beta_1'\}$ or $I = \{\beta_q'\}$ with $1 \le p < q$.* In view of Remark A.1 it is enough to consider only case (4), i.e. $I = \{\beta_1'\}$. The subgroup P is conjugated to the subgroup $\widetilde{P}$ that appears in [8] as case 12, the parameters "p" and "q" of loc. cit. are equal here respectively to $p+1$ and $q-p$. We have $gPg^{-1} = \widetilde{P}$ with $g = w_0^G w_0^H$, where $w_0^G \in W_G$ and $w_0^H \in W_H$ are the longest Weyl group elements.

This gives

$$\Sigma(G/P) = \{\alpha_1, \dots, \alpha_{p+1}, \alpha_{p+2,q+1}, \alpha_{q+2}, \dots, \alpha_n\}$$

and

$$\begin{aligned} \Delta(G/P) = \{&D_1^+, D_2^+ = D_n^+, \dots, D_{p+1}^+ = D_{q+2}^+, \\ &D_1^- = D_n^-, \dots, D_p^- = D_{q+2}^-, D_{p+1}^-, \\ &D_{p+2}, E_{q+1}\}. \end{aligned}$$

The Cartan pairing is

	α_1	α_2	α_3	α_4	$\dots$	α_p	α_{p+1}	$\alpha_{p+2,q+1}$	α_{q+2}	α_{q+3}	$\dots$	α_{p+q-1}	α_{p+q}	α_{p+q+1}
D_1^+	1	0	0	0	$\dots$	0	0	0	0	0	$\dots$	0	0	−1
D_1^-	1	−1	0	0	$\dots$	0	0	0	0	0	$\dots$	0	0	1
D_2^+	−1	1	0	0	$\dots$	0	0	0	0	0	$\dots$	0	−1	1
D_2^-	0	1	−1	0	$\dots$	0	0	0	0	0	$\dots$	0	1	−1
D_3^+	0	−1	1	0	$\dots$	0	0	0	0	0	$\dots$	−1	1	0
D_3^-	0	0	1	−1	$\dots$	0	0	0	0	0	$\dots$	1	−1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
D_p^+	0	0	0	0	$\dots$	1	0	0	−1	1	$\dots$	0	0	0
D_p^-	0	0	0	0	$\dots$	1	−1	0	1	−1	$\dots$	0	0	0
D_{p+1}^+	0	0	0	0	$\dots$	−1	1	−1	1	0	$\dots$	0	0	0
D_{p+1}^-	0	0	0	0	$\dots$	0	1	0	−1	0	$\dots$	0	0	0
D_{p+2}	0	0	0	0	$\dots$	0	−1	1	0	0	$\dots$	0	0	0
E_{q+1}	0	0	0	0	$\dots$	0	0	1	−1	0	$\dots$	0	0	0

We can take Q so that only α_{p+1} and α_{p+2} are not simple roots of L_Q. The inverse images in G/P of the two colors of G/Q are D_{p+1}^- and D_{p+2} (if $q = 1$ the other possibility D_{p+1}^+ and E_{q+1} is excluded as before). Therefore

$$\begin{array}{ccccc} \chi_{D_{p+1}^-} & = & -\omega_{p+1}^P & = & -\varpi_{p+1}, \\ \chi_{D_{p+2}} & = & -\omega_{p+2}^P & = & -\varpi_{p+2}. \end{array}$$

We obtain

$$\begin{array}{ccccccccc} \chi_{D_1^-} & = & \chi_{D_2^-} & = & \dots & = & \chi_{D_p^-} & = & 0, \\ \chi_{D_1^+} & = & \chi_{D_2^+} & = & \dots & = & \chi_{D_{p+1}^+} & = & \varpi_{p+1} - \varpi_{p+2}, \\ & & & & & & \chi_{E_{q+1}} & = & \varpi_{p+1}, \end{array}$$

which is easily checked to satisfy equalities (6) of Lemma 5.1.

B.2.4. $I = \{\beta_i'\}$ *with* $p = 1, q \geq 3$, $i \in \{2, \dots, q-1\}$. A conjugate $\widetilde{P}$ of P appears as case 7 of [8]; the parameters "p" and "q" of loc. cit. are equal here respectively to $i-1$ and $q-i$. Denoting by $\widetilde{H} = \mathrm{S}(\mathrm{L}(i+2) \times \mathrm{L}(q+1-i))$ and by $g = w_0^{\widetilde{H}}$ the longest element of its Weyl group, then $\widetilde{P} = gPg^{-1}$. This yields

$$\Sigma(G/P) = \{\alpha_1, \alpha_{2,i}, \alpha_{i+1}, \alpha_{i+2,n-1}, \alpha_n\}$$

and

$$\Delta(G/P) = \{D_1^+ = D_{i+1}^+, D_1^- = D_n^+, D_2, E_i, D_{i+1}^- = D_n^-, D_{i+2}, E_{n-1}\},$$

with Cartan pairing

	α_1	$\alpha_{2,i}$	α_{i+1}	$\alpha_{i+2,n-1}$	α_n
D_1^+	1	-1	1	0	-1
D_1^-	1	0	-1	0	1
D_2	-1	1	0	0	0
E_i	0	1	-1	0	0
D_{i+1}^-	-1	0	1	-1	1
D_{i+2}	0	0	-1	1	0
E_{n-1}	0	0	0	1	-1

We can take Q so that only α_2 and α_{i+2} are not simple roots of L_Q, and the inverse images in G/P of the two colors of G/Q are D_2 and D_{i+2}. This is shown easily if $i > 2$ or if $i < q-1$, as above: one excludes the other possibilities by looking at which color is moved by which simple root. Let us postpone the discussion of the case $i = 2 = q-1$.

This yields

$$\begin{aligned} \chi_{D_2} &= -\omega_2^P &= -\varpi_2, \\ \chi_{D_{i+2}} &= -\omega_{i+2}^P &= -\varpi_{i+2} \end{aligned}$$

which implies

$$\begin{aligned} \chi_{E_i} = \chi_{D_{i+1}^-} &= \varpi_2 - \varpi_{i+2}, \\ \chi_{D_1^+} &= -\varpi_{i+2}, \\ \chi_{D_1^-} &= 0, \\ \chi_{E_{n-1}} &= \varpi_2. \end{aligned}$$

If $i = 2 = q-1$, the computation above is the same, but there is another possible parabolic subset of colors, namely $\{E_2, E_4\}$. It does correspond to a G-equivariant morphism $G/P \to G/Q$, which is not the one induced by the inclusion $P \subseteq Q$. It is rather obtained from the latter by twisting it with a G-equivariant automorphism of G/P. However, it is not necessary to prove these claims: for us, it is enough to notice that the resulting generators of the extended weight monoid of G/P turn up to be the same (except for the "labeling" with the colors), as the reader may easily check, if one takes E_2 and E_4 instead of our choice D_2 and D_{i+2}.

B.2.5. $p = 0$ *and* $I =$ *any subset of simple roots of* H. If $I = S_H$ then $P = B_H^-$: the spherical roots, the colors, the Cartan pairing, and also the extended weight monoid of G/P were computed in this case in [4]. Thanks to Remark A.2, the other cases of the subset I follow from this one. However, for completeness, let us recompute the extended weight monoid of G/B_H^- in a direct manner and see how the other cases of I can be derived from our computation.

We do this by finding global equations in G of the pull-back of the colors of G/B_H^-. For this task it is more convenient to first replace G with $\mathrm{GL}(n+1)$, and replace the groups B and B_H^- with their inverse images in $\mathrm{GL}(n+1)$.

Consider the regular functions $a_i, b_i \in \mathbb{C}[\mathrm{GL}(n+1)]$, for $i \in \{1, \ldots, n\}$ defined as follows. For $g \in \mathrm{GL}(n+1)$, we set $a_i(g)$ to be the determinant of the lower right minor of size $n+1-i$, and we set $b_i(g)$ to be the determinant of the minor of g obtained by taking the last $n+1-i$ rows, the first column and the last $n-i$ columns.

The functions a_i and b_i are left B-semiinvariant and right B_H^--semiinvariant, and they are given by irreducible polynomials. Their zero sets are moved only by α_i. Therefore the images of their zero sets in G/B_H^-, denoted resp. by D_i^+ and D_i^-, are the two colors moved by α_i. Since no other colors can be moved by any simple root of G, we conclude

$$\Delta(G/B_H^-) = \{D_1^+, D_1^-, \ldots, D_n^+, D_n^-\}.$$

We get back to $G = \mathrm{SL}(n+1)$, where one easily deduces

$$\chi_{D_i^+} = -\omega_i^{B_H^-} = -\varpi_i$$

and

$$\chi_{D_i^-} = \omega_1^{B_H^-} - \omega_{i+1}^{B_H^-} = \varpi_1 - \varpi_{i+1}$$

for all i.

We can now compute the extended weight monoid of G/P, with P corresponding to any I as above: it is enough to determine which colors of G/B_H^- are mapped to colors of G/P under the natural map $G/B_H^- \to G/P$ (i.e. are not mapped dominantly), which is elementary to accomplish given the global equations on G we have given.

We obtain that D_1^+ is mapped to a color of G/P, and for $i \in \{2, \ldots, n\}$ the color D_i^+ is mapped to a color of G/P if and only if $\beta_{i-1} \in I$. Moreover D_n^- is mapped to a color of G/P, and for $i \in \{1, \ldots, n-1\}$ the color D_i^- is mapped to a color of G/P if and only if $\beta_i \in I$.

B.3. $\mathrm{SO}(2n+2)/(\mathrm{SO}(2n) \times \mathrm{SO}(2))$ **with** $n \geq 3$**.** This case is discussed in Section 7.

B.4. $\mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ **with** $n \geq 3$**.** Here I is any subset of simple roots of H. We discuss the case $P = B_H^-$, and deal with the general case using Remark A.2.

It is useful to define $G = \mathrm{SO}(2n+1)$ to be the stabilizer of the symmetric bilinear form given by $(e_i, e_{2n-j+2}) = \delta_{i,j}$ for all $i, j \in \{1, \ldots, 2n+1\}$, where $(e_1, \ldots, e_{2n+1})$ is the standard basis of $\mathbb{C}^{2n+1}$. We take $H = \mathrm{SO}(2n)$ to be the stabilizer in G of e_{n+1}. We recall that, with this choice, the subgroups B, T, B_H, B_H^-, and $T_H = T$ can be taken as in Appendix A. We have $\alpha_1 = \beta_1, \ldots, \alpha_{n-1} = \beta_{n-1}$, $\alpha_{n-1} + 2\alpha_n = \beta_n$, and $\omega_1 = \varpi_1, \ldots, \omega_{n-2} = \varpi_{n-2}$, $\omega_{n-1} = \varpi_{n-1} + \varpi_n$, $\omega_n = \varpi_n$.

The case $P = B_H^-$ belongs again to the class of subgroups studied in [4]. Here, for convenience, we determine the extended weight monoid of G/P more directly, by finding global equations on G of its colors.

For $i \in \{1, \ldots, n-1\}$, set $h = 2n+1-i$ and let $f_i(A)$ be the determinant of the $h \times h$-minor in the lower right corner of the matrix $A \in G$. Then $f_i \in \mathbb{C}[G]$ is B-semiinvariant for the left translation and B^--semiinvariant for the right translation on G, and the B-weight is the i-th fundamental dominant weight. It follows that f_i is a global equation of the (pull-back on G of the) color D_i of G/B^- moved by

α_i. One defines similarly f_n and D_n, and for the same reason the pull-back of f_n to $\mathrm{Spin}(2n+1)$ is the square of a global equation for D_n.

The pull-back of these colors from G/B^- to G/B_H^- via the inclusion $B_H^- \subseteq B^-$ produces n colors of G/B_H^-, denoted by $D_1^+, \ldots, D_n^+$, where for all i the color D_i^+ is moved exactly by α_i. Their B-weights are evidently the fundamental dominant weights, and the B_--weights are the opposites. To summarize, we have

$$\chi_{D_i^+} = -\omega_i^{B_H^-} \quad \forall i \in \{1, \ldots, n\},$$

i.e.

$$\begin{aligned} \chi_{D_i^+} &= -\varpi_i \qquad \forall i \in \{1, \ldots, n-2, n\}, \\ \chi_{D_{n-1}^+} &= -\varpi_{n-1} - \varpi_n. \end{aligned}$$

We claim that G/B_H^- has other n colors $D_1^-, \ldots, D_n^-$. For $j \in \{1, \ldots, n\}$, set $k = 2n+1-j$ and denote by $F_j(A)$ the determinant of the $k \times k$-minor involving the last k rows of the matrix $A \in G$ and the columns j through n and $n+2$ through $2n+1$. The function $F_j \in \mathbb{C}[G]$ is B-semiinvariant under left translation and B_H^--semiinvariant under right translation. If $j < n$ then its B-weight is ω_j, and the B-weight of F_n is $2\omega_n$.

From Lemma 4.3, it follows that $F_1, \ldots, F_{n-1}$ are global equations of $n-1$ colors $D_1^-, \ldots, D_{n-1}^-$ of G/B_H^-, where D_j^- is moved exactly by α_j. Since a simple root can move up to 2 colors, there is at most one other color in G/B_H^-, and in this case it is moved by α_n. The divisor of the function F_n has irreducible components that are colors moved only by α_n, and it does not coincide (as a set) with D_n^+. So the additional color moved by α_n exists and F_n vanishes on it, let us denote it by D_n^-. Given the B-weight of F_n, its divisor is either $D_n^+ + D_n^-$ or $2D_n^-$. The second possibility is excluded because the B_H^--weight of F_n is not divisible by 2.

This yields

$$\begin{aligned} \chi_{D_1^-} &= 0, \\ \chi_{D_i^-} &= -\omega_{i-1}^{B_H^-} \qquad \forall i \in \{2, \ldots, n-1\}, \\ \chi_{D_n^-} &= -\omega_{n-1}^{B_H^-} + \omega_n^{B_H^-}, \end{aligned}$$

i.e.

$$\begin{aligned} \chi_{D_1^-} &= 0, \\ \chi_{D_i^-} &= -\varpi_{i-1} \qquad \forall i \in \{2, \ldots, n\}. \end{aligned}$$

Since B_H^- is minimal among parabolic subgroups of H, we may skip the other possibilities for P (see Remark A.2).

B.5. $\mathrm{SO}(2n+2)/\mathrm{SO}(2n+1)$ **with** $n \geq 3$**.** Here I is any subset of simple roots of H. We discuss the case $P = B_H^-$, and deal with the general case using Remark A.2.

It is useful to define $G = \mathrm{SO}(2n+2)$ to be the stabilizer of the symmetric bilinear form given by $(e_i, e_{2n-j+3}) = \delta_{i,j}$ for all $i, j \in \{1, \ldots, 2n+2\}$, where $(e_1, \ldots, e_{2n+2})$ is the standard basis of $\mathbb{C}^{2n+2}$. We take $H = \mathrm{SO}(2n+1)$ to be the stabilizer in G of $e_{n+1} - e_{n+2}$. We recall that, with this choice, the subgroups B, T, B_H, B_H^-, and T_H can be taken as in Appendix A. The simple roots of G are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\ldots$, $\alpha_n = \varepsilon_n - \varepsilon_{n+1}$, $\alpha_{n+1} = \varepsilon_n + \varepsilon_{n+1}$, where ε_i is the character of T given by taking the i-th entry on the diagonal. We have $\alpha_i|_{T_H} = \beta_i$ and $\omega_i|_{T_H} = \varpi_i$ for all $i \in \{1, \dots, n\}$, and $\alpha_{n+1}|_{T_H} = \beta_n$, $\omega_{n+1}|_{T_H} = \varpi_n$, since ε_{n+1} is trivial on T_H.

To discuss the case $P = B_H^-$ we use the results on the extended weight monoid in [4]. Following loc. cit., the first step is to compute the *active roots* of B_H^-, i.e. the positive roots α of G such that the root space $\mathfrak{g}_{-\alpha}$ of the Lie algebra $\mathfrak{g}$ of G is not contained in the Lie algebra of B_H^-. It is elementary to compute the active roots in this case, and they are

$$\begin{array}{rr} \alpha_n, & \alpha_{n+1}, \\ \alpha_{n-1} + \alpha_n, & \alpha_{n-1} + \alpha_{n+1}, \\ \alpha_{n-2} + \alpha_{n-1} + \alpha_n, & \alpha_{n-2} + \alpha_{n-1} + \alpha_{n+1}, \\ \vdots & \vdots \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_n, & \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + \alpha_{n+1}. \end{array}$$

Notice that active roots on the same line in the above list have the same restriction to T_H. Denote by τ the restriction of characters from T to T_H, and set $\varphi_i = \tau(\alpha_i + \cdots + \alpha_n)$. Then we observe that $\tau(\alpha_i) = \varphi_i - \varphi_{i+1}$ for all $i \in \{1, \dots, n-1\}$, and $\tau(\alpha_n) = \tau(\alpha_{n+1}) = \varphi_n$.

By the results of loc. cit., we have the following:

$$\Sigma(G/B_H^-) = S$$

and

$$\Delta(G/B_H^-) = \left\{D_i^+, D_i^- \mid i \in \{1, \dots, n+1\}\right\},$$

where $D_n^- = D_{n+1}^-$. The Cartan pairing is as follows (all the other values that are not explicitly given are zero):

(1) $\langle \rho(D_1^+), \alpha_1 \rangle = \langle \rho(D_1^-), \alpha_1 \rangle = 1$, and $\langle \rho(D_1^+), \alpha_2 \rangle = -1$;
(2) for all $i \in \{2, \dots n-2\}$ we have $\langle \rho(D_i^+), \alpha_i \rangle = \langle \rho(D_i^-), \alpha_i \rangle = 1$, and $\langle \rho(D_i^+), \alpha_{i+1} \rangle = \langle \rho(D_i^-), \alpha_{i-1} \rangle - 1$;
(3) we have

	α_{n-2}	α_{n-1}	α_n	α_{n+1}
D_{n-1}^+	0	1	-1	-1
D_{n-1}^-	-1	1	0	0
D_n^+	0	0	1	-1
D_n^-	0	-1	1	1
D_{n+1}^+	0	0	-1	1

Again using loc. cit., or applying our usual method, we obtain

$$\begin{aligned} \chi_{D_i^+} &= -\omega_i^{B_H^-} \quad \forall i \in \{1, \dots, n+1\}, \\ \chi_{D_1^-} &= 0, \\ \chi_{D_j^-} &= -\omega_{j-1}^{B_H^-} \quad \forall j \in \{2, \dots, n\}, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} \chi_{D_i^+} &= -\varpi_i \quad \forall i \in \{1, \dots, n\}, \\ \chi_{D_{n+1}^+} &= -\varpi_n, \\ \chi_{D_1^-} &= 0, \\ \chi_{D_j^-} &= -\varpi_{j-1} \quad \forall j \in \{2, \dots, n\}. \end{aligned}$$

Since B_H^- is minimal among parabolic subgroups of H, we may skip the other possibilities for P (see Remark A.2).

B.6. $\mathrm{SO}(2n+2)/\mathrm{GL}(n+1)$ **with** $n \geq 3$. The subgroup H is a Levi subgroup of the parabolic subgroup of G by omitting the one but last root α_n of G. We have the following possibilities for I:

(1) $I = \{\beta_1\}$,
(2) $I = \{\beta_n\}$.

The two cases are related by the observation in Remark A.1. For this reason we only discuss case (1), and we notice that a conjugate of the corresponding P appears as case 55 of [8]: an element conjugating one subgroup to the other is the longest element of the Weyl group of the Levi subgroup of G having all simple roots of G except for α_1.

We denote as usual by $\varpi_1, \ldots, \varpi_n$ the fundamental weights of H. Noticing that $T = T_H$, we have $\varpi_i = \omega_i$ for all $i \in \{1, \ldots, n-1\}$, and $\varpi_n = \omega_n + \omega_{n+1}$. We also have $\frac{1}{2}\varpi_{n+1} = \omega_{n+1}$, whose double is a generator of the character group of H. This yields

$$\Sigma(G/P) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{n-1} + \alpha_n, \alpha_{n+1}\}$$

and

$$\Delta(G/P) = \{D_1, \ldots, D_n, D_{n+1}^+, D_{n+1}^-\},$$

with Cartan pairing

	$\alpha_1+\alpha_2$	$\alpha_2+\alpha_3$	$\alpha_3+\alpha_4$	$\alpha_4+\alpha_5$	$\ldots$	$\alpha_{n-3}+\alpha_{n-2}$	$\alpha_{n-2}+\alpha_{n-1}$	$\alpha_{n-1}+\alpha_n$	α_{n+1}
D_1	1	-1	0	0	$\ldots$	0	0	0	0
D_2	1	1	-1	0	$\ldots$	0	0	0	0
D_3	-1	1	1	-1	$\ldots$	0	0	0	0
$\vdots$									
D_{n-2}	0	0	0	0	$\ldots$	1	1	-1	0
D_{n-1}	0	0	0	0	$\ldots$	-1	1	1	-1
D_n	0	0	0		$\ldots$	0	-1	1	0
D_{n+1}^+	0	0	0	0	$\ldots$	0	0	-1	1
D_{n+1}^-	0	0	0	0	$\ldots$	0	-1	0	1

We can take Q to be such that only α_1 and α_{n+1} are not simple roots of L_Q. Then Q is minimal containing P. The colors of G/P mapping not dominantly to G/Q are D_1 and either D_{n+1}^+ or D_{n+1}^-. We claim that D_{n+1}^- is not mapped dominantly.

To prove the claim, we distinguish the two cases of n even and n odd.

If n is even, we consider the natural projection $G/P \to G/H$. The homogeneous space G/H appears as case 16 of [10], where the index "p" of loc. cit. is equal here to $n+1$. From loc. cit. we see that G/H has a unique color $\widetilde{D}_{n+1}$ moved by α_{n+1}, and it takes value -1 on the spherical root $\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1}$ of G/H. We deduce that the inverse image of $\widetilde{D}_{n+1}$ in G/P is D_{n+1}^-.

In order to apply this fact to the morphism $G/P \to G/Q$, we consider the parabolic subgroup Q_0 of G containing B^- and with Levi subgroup containing T having all simple roots of G except for α_{n+1}. Then Q_0 contains H, and G/Q_0 has a unique color moved by α_{n+1}. The diagram

$$\begin{array}{ccc} G/P & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ G/Q & \longrightarrow & G/Q_0 \end{array}$$

commutes, where the maps are all induced by the inclusions of the respective subgroups. Denote by E_{n+1} the color of G/Q moved by α_{n+1}.

Since the unique color of G/Q_0 has inverse image $\widetilde{D}_{n+1}$ in G/H and E_{n+1} in G/Q, it follows that the inverse images in G/P of $\widetilde{D}_{n+1}$ and of E_{n+1} coincide. In other words D^-_{n+1} is mapped to E_{n+1}, whence our claim.

We discuss now the case n odd. We use a similar argument, but with another subgroup $K \subseteq G$ instead of H. Namely, we consider $K = Q_1^u L_Q$, where Q_1 is the parabolic subgroup of G containing B^- and with Levi subgroup containing T having all simple roots of G except for α_1. Notice that $Q_1 \supset Q$, hence $Q_1^u \subseteq Q^u$ and $L_{Q_1} \supset L_Q$.

Our choice implies that G/K is the parabolic induction of the homogeneous space L_{Q_1}/L_Q by means of the parabolic subgroup Q_1 of G. For details on parabolic induction we refer to [26, Section 3.4]. Notice that L_{Q_1}/L_Q is isomorphic to $\mathrm{PSO}(2n)/\mathrm{PGL}(n)$, and it is identified with the subset Q_1/K of G/K.

The consequence is that G/K has the same spherical roots as L_{Q_1}/L_Q, and the colors of G/K that are moved by simple roots of L_{Q_1} correspond bijectively (via the assignment $D \mapsto D \cap (Q_1/K)$) to the colors of L_{Q_1}/L_Q, in such a way to preserve the Cartan pairing and the property for a color to be moved by a simple root.

The homogeneous space $\mathrm{PSO}(2n)/\mathrm{PGL}(n)$ appears in [10], as case 16 if $n \geq 5$ (with p of loc. cit. being equal to n), and as case 3 (with p and q loc. cit. being equal to resp. 1 and 3) if $n = 3$. In both cases, taking into account its isomorphism with L_{Q_1}/L_Q, the simple root α_{n+1} moves only one color, which takes the value 1 on the spherical root $\alpha_{n-1} + \alpha_n + \alpha_{n+1}$.

We conclude that G/K has only one color $\widetilde{D}_{n+1}$ moved by α_{n+1}, and it takes value 1 on $\alpha_{n-1} + \alpha_n + \alpha_{n+1}$. This enables us to conclude the above claim, using the diagram

$$\begin{array}{ccc} G/P & \longrightarrow & G/K \\ \downarrow & & \downarrow \\ G/Q & \longrightarrow & G/Q_1 \end{array}$$

and by reasoning exactly as in the case n even.

The claim, and equalities (6) of Lemma 5.1, yields

$$\begin{array}{ccccccccccc} \chi_{D_1} & = & \chi_{D_3} & = & \dots & = & \chi_{D_{n-1}} & = & -\omega_1^P & = & -\varpi_1, \\ & & \chi_{D_2} & = & \chi_{D_4} & = & \dots & = & \chi_{D_{n-2}} & = & 0, \\ & & & & & & \chi_{D_n} & = & \omega_{n+1}^P & = & \frac{1}{2}\varpi_{n+1}, \\ & & & & & & \chi_{D^-_{n+1}} & = & -\omega_{n+1}^P & = & -\frac{1}{2}\varpi_{n+1}, \\ & & & & & & \chi_{D^+_{n+1}} & = & \omega_{n+1}^P - \omega_1^P & = & \frac{1}{2}\varpi_{n+1} - \varpi_1 \end{array}$$

if n is even, and

$$\begin{array}{ccccccccccc} \chi_{D_1} & = & \chi_{D_3} & = & \dots & = & \chi_{D_{n-2}} & = & -\omega_1^P & = & -\varpi_1, \\ & & \chi_{D_2} & = & \chi_{D_4} & = & \dots & = & \chi_{D_{n-1}} & = & 0, \\ & & & & & & \chi_{D_n} & = & \omega_{n+1}^P - \omega_1^P & = & \frac{1}{2}\varpi_{n+1} - \varpi_1, \\ & & & & & & \chi_{D_{n+1}^-} & = & -\omega_{n+1}^P & = & -\frac{1}{2}\varpi_{n+1}, \\ & & & & & & \chi_{D_{n+1}^+} & = & \omega_{n+1}^P & = & \frac{1}{2}\varpi_{n+1}, \end{array}$$

if n is odd.

B.7. $\mathrm{Sp}(2p+2q)/(\mathrm{Sp}(2p)\times\mathrm{Sp}(2q))$ **with** $1 \le p$**,** $1 \le q$**.** Set $m = \min\{p, q\}$. Notice that in the paper [16] the assumption $p \le q$ is made, while here we drop it for convenience in the notations.

We define the subgroup H as the stabilizer of the subspace generated by the first p and the last p vectors of the canonical basis of $\mathbb{C}^{2p+2q}$. With this choice, the simple roots of H are $\beta_i = \alpha_i$ for $i \in \{1,\dots,p-1\}$, $\beta_p = 2\alpha_{p,p+q-1} + \alpha_{p+q}$, and $\beta'_j = \alpha_{p+j}$ for $j \in \{1,\dots,q\}$. We also have $\varpi_i = \omega_i$ for all $i \in \{1,\dots,p\}$, and $\varpi'_j + \varpi_p = \omega_{j+p}$ for all $j \in \{1,\dots,q\}$.

It is useful to recall the spherical roots of G/H, taken from [10, Symmetric case number 10]:

$$\Sigma(G/H) = \{\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3 + 2\alpha_4 + \alpha_5, \dots, \alpha_{2m-3} + 2\alpha_{2m-2} + \alpha_{2m-1}, \\ \alpha_{2m-1} + 2\alpha_{2m} + \dots + 2\alpha_{n-1} + \alpha_n\}.$$

We have the following possibilities for the set I.

(1) $I = \{\beta_1\}$,
(2) $I = \{\beta_p\}$ assuming $p \ge 2$, and we have $p \le 3$ or $q \le 2$,
(3) $I = \{\beta_i\}$ for all $i \in \{2,\dots,p-1\}$, with $p \ge 3$ and $q = 1$,
(4) $I = \{\beta_1, \beta_2\}$ with $p = 2$ and $q \ge 2$,
(5) $I = \{\beta_i, \beta_j\}$ for any $i, j \in \{1,\dots,p\}$ such that $i < j$, with $p \ge 2$ and $q = 1$,
(6) $I = \{\beta_1, \beta'_i\}$ for any $i \in \{1,\dots,q\}$, with $p = 1$.

It is useful to distinguish in our notations the different subgroups P arising in each case. For this reason, we denote them by $P_{(1)}, \dots, P_{(6)}$. If required by the context, we will also add to our notation the indices i, j appearing in the above list, writing e.g. $P_{(3)}(i)$ and $P_{(5)}(i,j)$ instead of simply $P_{(3)}$ and $P_{(5)}$.

Moreover, while discussing each case, we will introduce the usual parabolic subgroup Q, the Levi subgroup L of P containing T, and also another auxiliary subgroup K of G. As for P, we denote them resp. by $Q_{(1)}, \dots, Q_{(6)}$, $L_{(1)}, \dots, L_{(6)}$, and $K_{(1)}, \dots, K_{(6)}$.

For $i \in \{(1),\dots,(6)\}$, let us choose a parabolic subgroup Q_i minimal containing B_- and P_i. We only specify which simple roots are not simple roots of L_{Q_i}:

- α_1 for $Q_{(1)}$,
- α_p for $Q_{(2)}$,
- α_i for $Q_{(3)}$,
- α_1 and α_2 for $Q_{(4)}$,
- α_i and α_j for $Q_{(5)}$,
- α_1 and α_{i+1} for $Q_{(6)}$,

It is elementary to check that, for all i, the subgroup L_i is a very reductive subgroup of L_{Q_i}. This implies that Q_i is a minimal parabolic subgroup of G containing P_i.

We discuss now the single cases.

B.7.1. $I = \{\beta_1\}$. We claim that the spherical roots and the colors of $G/P_{(1)}$ are as follows. If $p > q$ (notice that then $2q < p+q$), we have

$$\Sigma(G/P_{(1)}) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{2q-2} + \alpha_{2q-1}, \alpha_{2q-1} + \alpha_{2q}, \\ \alpha_{2q} + 2\alpha_{2q+1,p+q-1} + \alpha_{p+q}\}$$

and

$$\Delta(G/P_{(1)}) = \{D_1, D_2, \ldots, D_{2q+1}\}$$

with Cartan pairing $\rho(D_i) = \alpha_i^\vee|_{\Xi(G/P_{(1)})}$ for all i.

If instead $p \leq q$ (and then $2p-1 < p+q$), we have

$$\Sigma(G/P_{(1)}) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{2p-2} + \alpha_{2p-1}, \alpha_{2p-1} + 2\alpha_{2p,p+q-1} + \alpha_{p+q}\}$$

and

$$\Delta(G/P_{(1)}) = \{D_1, D_2, \ldots, D_{2p}\}$$

with Cartan pairing $\rho(D_i) = \alpha_i^\vee|_{\Xi(G/P_{(1)})}$ for all i.

To prove the claim, let us denote resp. by Σ and Δ the sets at the right hand side of the above equalities. By the classification of spherical homogeneous spaces, there exists a spherical homogeneous space $G/K_{(1)}$ such that $\Sigma = \Sigma(G/K_{(1)})$, $\Delta = \Delta(G/K_{(1)})$, $\mathbb{Z}\Sigma = \Xi(G/K_{(1)})$, and the Cartan pairing is as specified. By Corollary 3.9, the subgroup $K_{(1)}$ is conjugated in G to a subgroup of H, so we may assume $K_{(1)} \subseteq H$.

It is elementary to check the hypotheses of Proposition 6.1 in this case, and conclude that $K_{(1)}$ is a parabolic subgroup of H. Also, one checks that it is a maximal proper subgroup of H, again by Corollary 3.9. Hence $K_{(1)}$ must be conjugated in G to $P_{(1)}$, $P_{(2)}$, or $P_{(3)}$ which are the maximal proper subgroups of H in our list.

One checks that there is no parabolic subset of Δ that could correspond to a G-equivariant morphism $G/K_{(1)} \to G/Q_{(2)}$ nor to a G-equivariant morphism $G/K_{(1)} \to G/Q_{(3)}$. Hence $K_{(1)}$ is conjugated to $P_{(1)}$ in G.

There is only one color of $G/P_{(1)}$ mapping not dominantly to $G/Q_{(1)}$, namely D_1, yielding

$$\chi_{D_1} = -\omega_1^{P_{(1)}} = -\varpi_1.$$

The equalities (6) of Lemma 5.1 yield

$$\begin{array}{ccccccccc} \chi_{D_2} & = & \chi_{D_4} & = & \cdots & = & \chi_{D_{2q}} & = & 0, \\ \chi_{D_3} & = & \chi_{D_5} & = & \cdots & = & \chi_{D_{2q+1}} & = & -\varpi_1 \end{array}$$

if $p > q$, and

$$\begin{array}{ccccccccc} \chi_{D_2} & = & \chi_{D_4} & = & \cdots & = & \chi_{D_{2p}} & = & 0, \\ \chi_{D_3} & = & \chi_{D_5} & = & \cdots & = & \chi_{D_{2p-1}} & = & -\varpi_1 \end{array}$$

if $p \leq q$.

B.7.2. *$I = \{\beta_p\}$ assuming $p \geq 2$, and we have $p \leq 3$ or $q \leq 2$.* We claim that G/P has spherical roots and colors as described below, where all possible values of p and q are discussed. In each case, the proof of this claim goes as before: the objects we indicate correspond in each case to a maximal proper parabolic subgroup of H, and there is no G-equivariant morphism from the homogeneous space to $G/Q_{(1)}$ or to $G/Q_{(3)}$. The claim follows.

For $p = 3$ and $q \geq 3$:

$$\Sigma(G/P_{(2)}) = \{\alpha_1, \ldots, \alpha_5, \sigma = \alpha_5 + 2\alpha_{6,q+2} + \alpha_{q+3}\}$$

and

$$\Delta(G/P_{(2)}) = \{D_1^+ = D_4^+, D_1^- = D_3^- = D_5^-, D_2^+ = D_5^+, D_2^-, D_3^+, D_4^-, D_6\}$$

with Cartan pairing

	α_1	α_2	α_3	α_4	α_5	σ
D_1^+	1	0	−1	1	−1	0
D_1^-	1	−1	1	−1	1	0
D_2^+	−1	1	−1	0	1	0
D_2^-	0	1	0	0	−1	0
D_3^+	−1	0	1	0	−1	0
D_4^-	−1	0	0	1	0	−1
D_6	0	0	0	0	−1	1

There is only one color of $G/P_{(2)}$ mapping not dominantly to $G/Q_{(2)}$, namely D_3^+, yielding

$$\chi_{D_3^+} = -\omega_3^{P_{(2)}} = -\varpi_3.$$

The equalities (6) of Lemma 5.1 yield

$$\begin{array}{ccccccc} \chi_{D_1^+} & = & \chi_{D_1^-} & = & \chi_{D_2^+} & = & -\varpi_3, \\ \chi_{D_2^-} & = & \chi_{D_4^-} & = & \chi_{D_6} & = & 0. \end{array}$$

For $p = 3$ and $q = 2$ we have

$$\Sigma(G/P_{(2)}) = S$$

and

$$\Delta(G/P_{(2)}) = \{D_1^+ = D_4^+, D_1^- = D_3^- = D_5^-, D_2^+ = D_5^+, D_2^-, D_3^+, D_4^-\}$$

with Cartan pairing

	α_1	α_2	α_3	α_4	α_5
D_1^+	1	0	−1	1	−1
D_1^-	1	−1	1	−1	1
D_2^+	−1	1	−1	0	1
D_2^-	0	1	0	0	−1
D_3^+	−1	0	1	0	−1
D_4^-	−1	0	0	1	−1

There is only one color of $G/P_{(2)}$ mapping not dominantly to $G/Q_{(2)}$, namely D_3^+. This yields

$$\chi_{D_3^+} = -\omega_3^{P_{(2)}} = -\varpi_3.$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= \chi_{D_1^-} = \chi_{D_2^+} = -\varpi_3, \\ \chi_{D_2^-} &= \chi_{D_4^-} = 0. \end{aligned}$$

For $p = 2$ and $q \geq 2$ it holds

$$\Sigma(G/P_{(2)}) = \{\alpha_1 + \alpha_3, \alpha_2, \sigma = \alpha_3 + 2\alpha_{4,q+1} + \alpha_{q+2}\}$$

and

$$\Delta(G/P_{(2)}) = \{D_1 = D_3, D_2^+, D_2^-, D_4\}$$

with Cartan pairing

	$\alpha_1 + \alpha_3$	α_2	σ
D_1	2	-1	0
D_2^+	0	1	-1
D_2^-	-2	1	0
D_4	-1	0	1

There is only one color of $G/P_{(2)}$ mapping not dominantly to $G/Q_{(2)}$, namely D_2^-. The only other possibility would be D_2^+, excluded by the fact that the other colors do not form a parabolic subset. This yields

$$\chi_{D_2^-} = -\omega_2^{P_{(2)}} = -\varpi_2.$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_2^+} = \chi_{D_4} &= 0, \\ \chi_{D_1} &= -\varpi_2. \end{aligned}$$

For $q = 2$ and $p \geq 4$ we have

$$\Sigma(G/P_{(2)}) = \{\alpha_1, \alpha_2, \alpha_3, \sigma = \alpha_{4,p}, \alpha_{p+1}, \alpha_{p+2}\}$$

and

$$\Delta(G/P_{(2)}) = \{D_1^+ = D_{q+1}^+, D_1^- = D_3^- = D_{p+2}^-, D_2^+ = D_{p+2}^+, D_2^-, D_3^+ = D_{p+1}^-, D_4, E_p\}$$

with Cartan pairing

	α_1	α_2	α_3	σ	α_{p+1}	α_{p+2}
D_1^+	1	0	-1	0	1	-1
D_1^-	1	-1	1	0	-1	1
D_2^+	-1	1	-1	0	0	1
D_2^-	0	1	0	0	0	-1
D_3^+	-1	0	1	-1	1	-1
D_4	0	0	-1	1	0	0
E_p	0	0	0	1	-1	0

There is only one color of $G/P_{(2)}$ mapping not dominantly to $G/Q_{(2)}$, namely E_p. If $p = 4$ we would have the other possibility D_4, excluded by the fact that the other colors do not form a parabolic subset. This yields

$$\chi_{E_p} = -\omega_p^{P_{(2)}} = -\varpi_p.$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_1^-} &= \chi_{D_2^+} = \chi_{D_3^+} = -\varpi_p, \\ \chi_{D_2^-} &= \chi_{D_4} = 0. \end{aligned}$$

For $q = 1$ and $p \geq 2$, the subgroup $P_{(2)}$ is found in [33] as case 4' of Table C. After loc. cit. we have

$$\Sigma(G/P_{(2)}) = \{\alpha_1 + \alpha_{p+1}, \alpha_{2,p}\}$$

and

$$\Delta(G/P_{(2)}) = \{D_1 = D_{p+1}, D_2, E_p\}$$

with Cartan pairing

	$\alpha_1 + \alpha_{p+1}$	$\alpha_{2,p}$
D_1	2	-1
D_2	-1	1
E_p	-2	1

There is only one color of $G/P_{(2)}$ mapping not dominantly to $G/Q_{(2)}$, namely E_p. For $p = 2$ we would have the other possibility D_2, excluded by the fact that the other colors do not form a parabolic subset. This yields

$$\chi_{E_p} = -\omega_p^{P_{(2)}} = -\varpi_p.$$

The equalities (6) of Lemma 5.1 yield

$$\chi_{D_1} = -\varpi_p,$$
$$\chi_{D_2} = 0.$$

B.7.3. $I = \{\beta_i\}$ *with* $p \geq 3$, $q = 1$, *and* $i \in \{2, \ldots, p-1\}$. Using the same argument of the previous two cases, one shows that

$$\Sigma(G/P_{(3)}) = \{\sigma = \alpha_1 + \alpha_{i+1}, \alpha_{2,i}, \gamma = \alpha_{i+1} + 2\alpha_{i+2,p+1} + \alpha_{p+1}\}$$

and

$$\Delta(G/P_{(3)}) = \{D_1 = D_{i+1}, D_2, E_i, D_{i+2}\}$$

with Cartan pairing

	σ	$\alpha_{2,i}$	γ
D_1	2	-1	0
D_2	-1	1	0
E_i	-1	1	-1
D_{i+2}	-1	0	1

The only color of $G/P_{(3)}$ not mapped dominantly to $G/Q_{(3)}$ is E_i. For $i = 2$ we would have the other possibility of D_2, excluded by the fact that the other colors do not form a parabolic subset. Therefore we have

$$\chi_{E_i} = -\omega_i^{P_{(3)}} = -\varpi_i.$$

The equalities (6) of Lemma 5.1 yield

$$\chi_{D_2} = 0,$$
$$\chi_{D_{i+2}} = \chi_{D_1} = -\varpi_i.$$

B.7.4. $I = \{\beta_1, \beta_2\}$ *with* $p = 2$ *and* $q \geq 2$. We claim that

$$\Sigma(G/P_{(4)}) = \{\alpha_1, \alpha_2, \alpha_3, \sigma = \alpha_3 + 2\alpha_{4,q+1} + \alpha_{q+2}\}$$

and

$$\Delta(G/P_{(4)}) = \{D_1^+ = D_3^+, D_1^-, D_2^+, D_2^-, D_3^-, D_4\}$$

with Cartan pairing

	α_1	α_2	α_3	σ
D_1^+	1	-1	1	0
D_1^-	1	0	-1	0
D_2^+	-1	1	-1	0
D_2^-	0	1	0	-1
D_3^-	-1	0	1	0
D_4	0	0	-1	1

To show the claim, we define as usual $K_{(4)}$ to be a subgroup of G having the above spherical roots and colors. With the same techniques used before, one shows that we can take $K_{(4)}$ inside H, in which case it is a non-maximal proper parabolic subgroup of H. Moreover, using the same arguments, one shows that $G/K_{(4)}$ admits G-equivariant morphisms to $G/P_{(1)}$ and to $G/P_{(2)}$, but not to $G/P_{(3)}(i)$ for any i. This proves the claim.

The colors D_1^-, D_2^+ are not mapped dominantly to G/Q (the other possibility given by D_1^-, D_2^- is excluded because the other colors do not form a parabolic subset).

This gives

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^P &= -\varpi_1, \\ \chi_{D_2^+} &= -\omega_2^P &= -\varpi_2. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= -\varpi_2, \\ \chi_{D_3^-} &= -\varpi_1, \\ \chi_{D_2^-} = \chi_{D_4} &= 0. \end{aligned}$$

B.7.5. $I = \{\beta_i, \beta_j\}$ *with* $q = 1$ *and* $1 \leq i < j \leq p$. We first assume $1 < i < j - 1 < p - 1$, we will discuss separately the other possibilities.

We claim that, with this condition on i and j, we have

$$\Sigma(G/P_{(5)}) = \{\alpha_1, \sigma = \alpha_{2,i}, \alpha_{i+1}, \gamma = \alpha_{i+2,j}, \alpha_{j+1}, \eta = \alpha_{j+1} + 2\alpha_{j+2,p} + \alpha_{p+1}\}$$

and

$$\Delta(G/P_{(5)}) = \{D_1^+ = D_{i+1}^+, D_1^- = D_{j+1}^+, D_2, E_i, D_{i+1}^- = D_{j+1}^-, D_{i+2}, E_j, D_{j+2}\}$$

with Cartan pairing

	α_1	σ	α_{i+1}	γ	α_{j+1}	η
D_1^+	1	−1	1	0	−1	0
D_1^-	1	0	−1	0	1	0
D_2	−1	1	0	0	0	0
E_i	0	1	−1	0	0	0
D_{i+1}^-	−1	0	1	−1	1	0
D_{i+2}	0	0	−1	1	0	0
E_j	0	0	0	1	−1	−1
D_{j+2}	0	0	0	0	−1	1

The claim is shown as usual: we denote by $K_{(5)}(i,j)$ a subgroup of G corresponding to the above spherical roots and colors. The standard technique shows that $K_{(5)}(i,j)$ can be taken to be a non-maximal proper parabolic subgroup of H, and that it is conjugated in G to a subgroup of $P_{(3)}(k)$ if and only if k is equal to i or j. This shows that $K_{(5)}(i,j)$ is conjugated to $P_{(5)} = P_{(5)}(i,j)$.

As usual one checks that E_i and E_j are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$, yielding

$$\begin{aligned}\chi_{E_i} &= -\omega_i^{P_{(5)}} = -\varpi_i,\\ \chi_{E_j} &= -\omega_j^{P_{(5)}} = -\varpi_j.\end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned}\chi_{D_1^+} = \chi_{D_{i+2}} &= -\varpi_i,\\ \chi_{D_1^-} = \chi_{D_{j+2}} &= -\varpi_j,\\ \chi_{D_2} &= 0,\\ \chi_{D_{i+1}^-} &= -\varpi_i - \varpi_j.\end{aligned}$$

We continue with case (5) and make now the assumption $1 = i < j-1 < p-1$. We claim that then

$$\Sigma(G/P_{(5)}) = \{\alpha_1, \alpha_2, \gamma = \alpha_{3,j}, \alpha_{j+1}, \eta = \alpha_{j+1} + 2\alpha_{j+2,p} + \alpha_{p+1}\}$$

and

$$\Delta(G/P_{(5)}) = \{D_1^+ = D_{j+1}^+, D_1^-, D_2^+, D_2^- = D_{j+1}^-, D_3, E_j, D_{j+2}\}$$

with Cartan pairing

	α_1	α_2	γ	α_{j+1}	η
D_1^+	1	−1	0	1	0
D_1^-	1	0	0	−1	0
D_2^+	0	1	0	−1	0
D_2^-	−1	1	−1	1	0
D_3	0	−1	1	0	0
E_j	0	0	1	−1	−1
D_{j+2}	0	0	0	−1	1

The claim is shown exactly as above, where we have discussed the assumption $1 < i < j-1 < p-1$. Here D_1^- and E_j are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

For $j=3$ there is another possibility for these two colors, namely D_1^- and D_3. But $P_{(5)}(1,3)$ can be chosen in such a way that $P_{(5)}(1,3) \subseteq P_{(3)}(3) \subseteq Q_{(3)}(3)$, and with $P_{(5)}(1,3)$ containing $P_{(3)}(3) \cap B_-$. In this case we have the commutative diagram

$$\begin{array}{ccc} G/P_{(5)}(1,3) & \longrightarrow & G/P_{(3)}(3) \\ \downarrow & & \downarrow \\ G/Q_{(5)}(1,3) & \longrightarrow & G/Q_{(3)}(3) \end{array} \tag{13}$$

where the morphisms are induced by the above inclusions of subgroups, and it shows that E_3 cannot be mapped dominantly to $G/Q_{(5)}(1,3)$ because it is not mapped dominantly to $G/Q_{(3)}(3)$.

So we have

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_{(5)}} &= -\varpi_1, \\ \chi_{E_j} &= -\omega_j^{P_{(5)}} &= -\varpi_j. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_{j+2}} &= -\varpi_j, \\ \chi_{D_3} &= -\varpi_1, \\ \chi_{D_2^+} &= 0, \\ \chi_{D_2^-} &= -\varpi_1 - \varpi_j. \end{aligned}$$

We continue with case (5), with the assumption $1 < i = j-1 < p-1$. We claim that then

$$\Sigma(G/P_{(5)}) = \{\alpha_1, \gamma = \alpha_{2,i}, \alpha_j, \alpha_{j+1}, \eta = \alpha_{j+1} + 2\alpha_{j+2,p} + \alpha_{p+1}\}$$

and

$$\Delta(G/P_{(5)}) = \{D_1^+ = D_{j+1}^+, D_1^- = D_j^-, D_2, E_i, D_j^+, D_{j+1}^-, D_{j+2}\}$$

with Cartan pairing

	α_1	γ	α_j	α_{j+1}	η
D_1^+	1	0	-1	1	0
D_1^-	1	-1	1	-1	0
D_2	-1	1	0	0	0
E_i	0	1	-1	0	0
D_j^+	-1	0	1	0	-1
D_{j+1}^-	-1	0	0	1	0
D_{j+2}	0	0	0	-1	1

The claim is shown as above. Here E_i and D_j^+ are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

For $i=2$ there is another possibility for these two colors, namely D_2 and D_j^+, excluded by the fact that the other colors do not form a parabolic subset.

So we have

$$\begin{aligned} \chi_{E_i} &= -\omega_i^{P_{(5)}} &= -\varpi_i, \\ \chi_{D_j^+} &= -\omega_j^{P_{(5)}} &= -\varpi_j. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= \chi_{D_{j+2}} = -\varpi_j, \\ \chi_{D_1^-} &= \chi_{D_{j+1}^-} = -\varpi_i, \\ & \chi_{D_2^+} = 0. \end{aligned}$$

We continue with case (5), with the assumption $1 < i < j-1 = p-1$. We claim that then

$$\Sigma(G/P_{(5)}) = \{\alpha_1, \sigma = \alpha_{2,i}, \alpha_{i+1}, \gamma = \alpha_{i+2,j}, \alpha_{p+1}\}$$

and

$$\Delta(G/P_{(5)}) = \{D_1^+ = D_{i+1}^+, D_1^- = D_{p+1}^+, D_2, E_i, D_{i+1}^- = D_{p+1}^-, D_{i+2}, E_p\}$$

with Cartan pairing

	α_1	σ	α_{i+1}	γ	α_{p+1}
D_1^+	1	−1	1	0	−1
D_1^-	1	0	−1	0	1
D_2	−1	1	0	0	0
E_i	0	1	−1	0	0
D_{i+1}^-	−1	0	1	−1	1
D_{i+2}	0	0	−1	1	0
E_p	0	0	0	1	−2

The claim is shown as above. With the same techniques as above, one shows that E_i and E_p are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

So we have

$$\begin{aligned} \chi_{E_i} &= -\omega_i^{P_{(5)}} = -\varpi_i, \\ \chi_{E_p} &= -\omega_p^{P_{(5)}} = -\varpi_p. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= \chi_{D_{i+2}} = -\varpi_i, \\ & \chi_{D_1^-} = -\varpi_p, \\ & \chi_{D_2} = 0, \\ & \chi_{D_{i+1}^-} = -\varpi_i - \varpi_p. \end{aligned}$$

We continue with case (5), with the assumption $1 = i = j-1 < p-1$. We claim that then

$$\Sigma(G/P) = \{\alpha_1, \alpha_2, \alpha_3, \sigma = \alpha_3 + 2\alpha_{4,p+q-1} + \alpha_{p+q}\}$$

and

$$\Delta(G/P) = \{D_1^+ = D_3^+, D_1^-, D_2^+, D_2^-, D_3^-, D_4\}$$

with Cartan pairing

	α_1	α_2	α_3	σ
D_1^+	1	−1	1	0
D_1^-	1	0	−1	0
D_2^+	0	1	−1	0
D_2^-	−1	1	0	−1
D_3^-	−1	0	1	0
D_4	0	0	−1	1

The claim is shown as above. With the same techniques as above, one shows that D_1^- and D_2^- are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

So we have

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_{(5)}} &= -\varpi_1, \\ \chi_{D_2^-} &= -\omega_2^{P_{(5)}} &= -\varpi_2. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_4} &= -\varpi_2, \\ \chi_{D_2^+} &= 0, \\ \chi_{D_3^-} &= -\varpi_1. \end{aligned}$$

We continue with case (5), with the assumption $1 < i = j-1 = p-1$. We claim that then

$$\Sigma(G/P) = \{\alpha_1, \sigma = \alpha_{2,p-1}, \alpha_p, \alpha_{p+1}\}$$

and

$$\Delta(G/P) = \{D_1^+ = D_p^+, D_1^- = D_{p+1}^-, D_2, E_{p-1}, D_p^-, D_{p+1}^+\}$$

with Cartan pairing

	α_1	σ	α_p	α_{p+1}
D_1^+	1	−1	1	−1
D_1^-	1	0	−1	1
D_2	−1	1	0	0
E_{p-1}	0	1	−1	0
D_p^-	−1	0	1	−1
D_{p+1}^+	−1	0	0	1

The claim is shown as above. With the same techniques as above, one shows that E_{p-1} and D_p^- are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

So we have

$$\begin{aligned} \chi_{E_{p-1}} &= -\omega_{p-1}^{P_{(5)}} &= -\varpi_{p-1}, \\ \chi_{D_p^-} &= -\omega_p^{P_{(5)}} &= -\varpi_p. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_{p+1}^+} &= -\varpi_{p-1}, \\ \chi_{D_2} &= 0, \\ \chi_{D_1^-} &= -\varpi_p. \end{aligned}$$

We finish case (5), discussing the last remaining assumption $1 = i = j-1 = p-1$. We claim that then

$$\Sigma(G/P) = S$$

and

$$\Delta(G/P) = \{D_1^+ = D_3^+, D_1^-, D_2^+, D_2^-, D_3^-\}$$

with Cartan pairing

	α_1	α_2	α_3
D_1^+	1	−1	1
D_1^-	1	0	−1
D_2^+	−1	1	−1
D_2^-	0	1	−1
D_3^-	−1	0	1

The claim is shown as above. With the same techniques as above, one shows that D_1^- and D_2^+ are the colors of $G/P_{(5)}$ not mapped dominantly to $G/Q_{(5)}$.

So we have

$$\begin{array}{rcccl} \chi_{D_1^-} & = & -\omega_1^{P_{(5)}} & = & -\varpi_1, \\ \chi_{D_2^+} & = & -\omega_2^{P_{(5)}} & = & -\varpi_2. \end{array}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} &= -\varpi_2, \\ \chi_{D_2^-} &= 0, \\ \chi_{D_3^-} &= -\varpi_1. \end{aligned}$$

B.7.6. *$I = \{\beta_1, \beta_i'\}$ for any $i \in \{1, \ldots, q\}$, with $p = 1$.* Let us first suppose $1 < i < q$, we will deal later with the other possibilities. We claim that, with this condition on i, we have

$$\Sigma(G/P_{(6)}) = \{\alpha_1, \sigma = \alpha_{2,i}, \alpha_{i+1}, \gamma = \alpha_{i+1} + 2\alpha_{i+2,q} + \alpha_{q+1}\}$$

and

$$\Delta(G/P_{(6)}) = \{D_1^+ = D_{i+1}^+, D_1^-, D_2, E_i, D_{i+1}^-, D_{i+2}\}$$

with Cartan pairing

	α_1	σ	α_{i+1}	γ
D_1^+	1	−1	1	0
D_1^-	1	0	−1	0
D_2	−1	1	0	0
E_i	0	1	−1	−1
D_{i+1}^-	−1	0	1	0
D_{i+2}	0	0	−1	1

The claim is shown as for the previous cases. With the same techniques as above, one shows that D_1^- and D_{i+1}^- are the colors of $G/P_{(6)}$ not mapped dominantly to $G/Q_{(6)}$.

So we have

$$\begin{array}{rcccl} \chi_{D_1^-} & = & -\omega_1^{P_{(6)}} & = & -\varpi_1, \\ \chi_{D_{i+1}^-} & = & -\omega_{i+1}^{P_{(6)}} & = & -\varpi_i' - \varpi_1. \end{array}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{array}{rcccccl} \chi_{D_1^+} & = & \chi_{E_i} & = & \chi_{D_{i+2}} & = & -\varpi_i', \\ & & & & \chi_{D_2} & = & 0. \end{array}$$

We continue with case (6), with the assumption $1 = i < q$. We claim that then

$$\Sigma(G/P_{(6)}) = \{\alpha_1, \alpha_2, \gamma = \alpha_2 + 2\alpha_{3,q} + \alpha_{q+1}\}$$

and

$$\Delta(G/P_{(6)}) = \{D_1^+, D_1^-, D_2^+, D_2^-, D_3\}$$

with Cartan pairing

	α_1	α_2	γ
D_1^+	1	−1	0
D_1^-	1	0	−1
D_2^+	0	1	0
D_2^-	−1	1	0
D_3	0	−1	1

The claim is shown as for the previous cases. We can take $Q_{(6)}$ so that α_1 and α_2 are the only simple roots that are not simple roots of $L_{Q_{(6)}}$. With the same techniques as above, one shows that D_1^+ and D_2^- are the colors of $G/P_{(6)}$ not mapped dominantly to $G/Q_{(6)}$. To exclude other possibilities, one uses the inclusions $P_{(6)} \subseteq P_{(1)} \subseteq Q_{(1)}$ and the corresponding diagram similar to (13).

So we have

$$\begin{aligned} \chi_{D_1^+} &= -\omega_1^{P_{(6)}} &&= -\varpi_1, \\ \chi_{D_2^-} &= -\omega_2^{P_{(6)}} &&= -\varpi_1' - \varpi_1. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^-} = \chi_{D_3} &= -\varpi_1', \\ \chi_{D_2^+} &= 0. \end{aligned}$$

We continue with case (6), with the assumption $1 < i = q$. We claim that then

$$\Sigma(G/P_{(6)}) = \{\alpha_1, \sigma = \alpha_{2,q}, \alpha_{q+1}\}$$

and

$$\Delta(G/P_{(6)}) = \{D_1^+ = D_{p+1}^+, D_1^-, D_2, E_p, D_{q+1}^-\}$$

with Cartan pairing

	α_1	σ	α_{q+1}
D_1^+	1	−1	1
D_1^-	1	0	−1
D_2	−1	1	0
E_p	0	1	−2
D_{p+1}^-	−1	0	1

The claim is shown as for the previous cases. With the same techniques as above, one shows that D_1^- and D_{q+1}^- are the colors of $G/P_{(6)}$ not mapped dominantly to $G/Q_{(6)}$.

So we have

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_{(6)}} &&= -\varpi_1, \\ \chi_{D_{q+1}^-} &= -\omega_{q+1}^{P_{(6)}} &&= -\varpi_q' - \varpi_1. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{E_p} &= -\varpi_q', \\ \chi_{D_2} &= 0. \end{aligned}$$

We finish case (6), discussing the assumption $1 = i = q$. The subgroup $P_{(6)}$ of Sp(4) appears then as the first occurrence of case 8 of [33, Table C], which gives

$$\Sigma(G/P_{(6)}) = S$$

and

$$\Delta(G/P_{(6)}) = \{D_1^+, D_1^-, D_2^+, D_2^-\}$$

with Cartan pairing

	α_1	α_2
D_1^+	1	-1
D_1^-	1	-1
D_2^+	0	1
D_2^-	-1	1

Here there are two possible choices of colors of $G/P_{(6)}$ not mapped dominantly to $G/Q_{(6)}$, namely $\{D_1^+, D_2^-\}$ and $\{D_1^-, D_2^-\}$. They are exchanged by the non-trivial G-equivariant automorphism of $G/P_{(6)}$, so (as one can easily check directly) they produce the same computation for the extended weight monoid, except for the labeling of the generators by the colors.

Let us choose $\{D_1^+, D_2^-\}$, which yields

$$\begin{aligned} \chi_{D_1^+} &= -\omega_1^{P_{(6)}} &&= -\varpi_1, \\ \chi_{D_2^-} &= -\omega_2^{P_{(6)}} &&= -\varpi_1' - \varpi_1. \end{aligned}$$

The equalities (6) of Lemma 5.1 yield

$$\begin{aligned} \chi_{D_1^-} &= -\varpi_1', \\ \chi_{D_2^+} &= 0. \end{aligned}$$

B.8. $\mathsf{F}_4/\operatorname{Spin}(9)$**.** Let G denote the connected semisimple group of Dynkin type F_4. The subgroup $H \subseteq \mathsf{F}_4$ has maximal torus T and simple roots $\beta_1 = \epsilon_1 - \epsilon_2 = 2\alpha_4 + \alpha_2 + 2\alpha_3, \beta_2 = \alpha_1, \beta_3 = \alpha_2$ and $\beta_4 = \alpha_3$ and is isomorphic to Spin(9). With this choice, we have $\omega_1 = \varpi_2$, $\omega_2 = \varpi_1 + \varpi_3$, $\omega_3 = \varpi_1 + \varpi_4$, $\omega_4 = \varpi_1$.

Assuming I is minimal, we have the following possibilities:

(1) $I = \{\beta_1, \beta_2\}$,
(2) $I = \{\beta_3\}$,
(3) $I = \{\beta_4\}$.

Note that case (1) gives rise to the two subcases $I = \{\beta_1\}$ and $I = \{\beta_2\}$. Let us denote the corresponding parabolic subgroups by P_{12}, P_1, P_2, P_3 and P_4. These subgroups turn out to be wonderful and their Luna diagrams can be found in [7, §3.2]. Additionally, one can check that the spherical roots and the colors given below are correct as usual: applying Corollary 3.9 and Proposition 6.1 to check that they correspond to minimal parabolic subgroups of H that are spherical in G, and using dimensions to check that the given data correspond to the correct parabolic subgroup.

B.8.1. $I = \{\beta_1, \beta_2\}$. We have

$$\Sigma(G/P_{12}) = \{\sigma_1 = \alpha_1 + \alpha_2, \sigma_2 = \alpha_2 + \alpha_3, \sigma_3 = \alpha_3, \sigma_4 = \alpha_4\}$$

and

$$\Delta(G/P_{12}) = \{D_1, D_2, D_3^\pm, D_4^\pm\}$$

with Cartan pairing

	σ_1	σ_2	σ_3	σ_4
D_1	1	−1	0	0
D_2	1	1	−1	0
D_3^+	0	0	1	−1
D_3^-	−2	0	1	0
D_4^+	0	0	−1	1
D_4^-	0	−1	0	1

Let Q_{12} be the parabolic subgroup so that α_2 and α_3 are the simple roots of $L_{Q_{12}}$. Then Q_{12} is minimal for containing P_{12}. The pull-backs of the two colors of G/Q_{12} to G/P_{12} are D_1 and one of $\{D_4^\pm\}$. Since $\{D_2, D_3^\pm, D_4^-\}$ is parabolic and $\{D_2, D_3^\pm, D_4^+\}$ is not, it follows that D_4^+ is pulled back from G/Q_{12}. Hence

$$\begin{array}{rcccl} \chi_{D_1} & = & -\omega_1^{P_{12}} & = & -\varpi_2, \\ \chi_{D_4^-} & = & -\omega_4^{P_{12}} & = & -\varpi_1. \end{array}$$

We recover as usual the other weights:

$$\chi_{D_2} = -\varpi_2,$$
$$\chi_{D_3^+} = -\varpi_1,$$
$$\chi_{D_3^-} = -\varpi_2,$$
$$\chi_{D_4^-} = 0.$$

Subcase ($I = \{\beta_1\}$). Here we have $\Sigma(G/P_1) = \{\sigma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3, \sigma_2 = \alpha_4\}$ and $\Delta(G/P_1) = \{D_3, D_4^\pm\}$. The inverse images of the colors D_3, D_4^+, D_4^- of G/P_1 under the natural projection $G/P_{12} \to G/P_1$ are the colors D_3^+, D_4^- and D_4^+ of G/P_{12} respectively.

Subcase ($I = \{\beta_2\}$). Here we have $\Sigma(G/P_1) = \{\sigma_1 = \alpha_1 + \alpha_2, \sigma_2 = \alpha_2 + \alpha_3, \sigma_3 = \alpha_3 + \alpha_4\}$ and $\Delta(G/P_1) = \{D_1, D_2, D_3, D_4\}$. The inverse images of the colors D_1, D_2, D_3, D_4 of G/P_2 under the natural projection $G/P_{12} \to G/P_2$ are the colors D_1, D_2, D_3^- and D_4^- of G/P_{12} respectively.

B.8.2. $I = \{\beta_3\}$. We have

$$\Sigma(G/P_3) = \{\sigma_1 = \alpha_1, \sigma_2 = \alpha_2 + \alpha_3, \sigma_3 = \alpha_3, \sigma_4 = \alpha_4\}$$

and

$$\Delta(G/P_3) = \{D_1^+ = D_4^+, D_1^- = D_3^-, D_2, D_3^+, D_4^-\}$$

with Cartan pairing

	σ_1	σ_2	σ_3	σ_4
D_1^+	1	−1	−1	1
D_1^-	1	0	1	−1
D_2	−1	1	−1	0
D_3^+	−1	0	1	0
D_4^-	−1	0	0	1

A parabolic subset must contain an element of each of the sets $\{D_1^+, D_1^-\}, \{D_2\}$, $\{D_1^-, D_3\}$ and $\{D_1^+, D_4^-\}$. Moreover, the colors that are moved by more than one root have to be in the parabolic subset, since colors on a (partial) flag variety are moved by at most one root. It follows that a parabolic subset must contain one of the sets $\{D_1^+, D_1^-, D_2\}$ as a subset. Suppose that $a_1^+ D_1^+ + a_1^- D_1^- + a_2 D_2$ is strictly positive on every spherical root. Then $a_1^+ + a_2 < a_1^-$ and $a_1^- < a_1^+$, which implies $a_2 < 0$, and we conclude that $\{D_1^+, D_1^-, D_2\}$ itself is not parabolic.

Suppose that $a_1^+ D_1^+ + a_1^- D_1^- + a_2 D_2 + a_3^+ D_3^+$ is strictly positive on all spherical roots. Then $a_1^+ + a_1^- - a_2 - a_3^+ > 0$, $a_1^+ < a_2$, $-a_1^+ + a_1^- - a_2 + a_3^+ > 0$ and $a_1^+ > a_1^-$, from which $a_1^- > a_2$ and $a_1^- < a_2$, a contradiction.

Note that $4D_1^+ + 14D_1^- + 6D_2 + 11D_4^-$ is strictly positive on each spherical root. We conclude that $\{D_1^+, D_1^-, D_2, D_4^-\}$ is the minimal parabolic subset of $\Delta(G/P_3)$.

First we choose another system of positive roots for G: $\alpha_1' = \epsilon_2 - \epsilon_3, \alpha_2' = \epsilon_1 - \epsilon_2, \alpha_3' = -\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4), \alpha_4' = \epsilon_4$. Note that $\epsilon_3 - \epsilon_4 = 2\alpha_3' + \alpha_2'$, which implies that this choice of positive roots is compatible with the standard choice of positive roots of $\mathfrak{so}_9$. Also note that $s_{\epsilon_4} \circ s_{\frac{1}{2}(-\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)}(\alpha_i) = \alpha_i'$ for $i = 1, 2, 3, 4$.

It is clear that $P_3 \subseteq Q_{\{\alpha_1', \alpha_2', \alpha_4'\}}$ and that $Q_{\{\alpha_1', \alpha_2', \alpha_4'\}}$ is minimal for this inclusion. Hence we found the conjugate $\widetilde{P}_3$ that is contained in Q_3. It follows that the P-weight of D_3^+ is $(-\omega_3')^P = -\varpi_3$. Hence

$$\chi_{D_3^+} = -\varpi_3$$

and the equations

$$\begin{aligned} \chi_{D_1^+} + \chi_{D_1^-} - \chi_{D_2} + \chi_{D_3^+} - \chi_{D_4^-} &= 0, \\ -\chi_{D_1^+} + \chi_{D_2} &= 0, \\ -\chi_{D_1^+} + \chi_{D_1^-} - \chi_{D_2} + \chi_{D_3^+} &= 0, \\ \chi_{D_1^+} - \chi_{D_1^-} + \chi_{D_4^-} &= 0 \end{aligned}$$

yield

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_1^-} = \chi_{D_2} &= -\varpi_3, \\ \chi_{D_4^-} &= 0. \end{aligned}$$

B.8.3. $I = \{\beta_4\}$. We have

$$\Sigma(G/P_4) = \{\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3, \sigma_2 = \alpha_2 + 2\alpha_3 + \alpha_4, \sigma_3 = \alpha_4\}$$

and

$$\Delta(G/P_3) = \{D_1, D_3, D_4^+, D_4^-\}$$

with Cartan pairing

	σ_1	σ_2	σ_3
D_1	1	-1	0
D_3	0	1	-1
D_4^+	0	0	1
D_4^-	-1	0	1

A parabolic subset must contain D_1, D_3 and an element of $\{D_4^+, D_4^-\}$. Suppose that $a_1 D_1 + a_3 D_3 + a_4^- D_4^-$ is strictly positive on the spherical roots. This implies $a_1 > a_4^-$, $a_1 < a_3$ and $a_3 < a_4^-$, from which $a_4^- < a_4^-$, a contradiction.

Since $D_1 + 2D_3 + 3D_4^+$ is strictly positive on each spherical root, the minimal parabolic subset is given by $\{D_1, D_3, D_4^+\}$. Let Q_4 denote the parabolic subgroup such that only α_4 is not a simple root of L_{Q_4}. Then Q_4 is minimal for containing a G-conjugate $\widetilde{P}_4$ of P_4. To find this conjugate we argue as follows.

First we look at a conjugate of B_4 which has the following system of positive roots: $\beta_1' = \epsilon_3 + \epsilon_4, \beta_2 = \epsilon_2 - \epsilon_3, \beta_3' = \epsilon_3 - \epsilon_4, \beta_4' = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$. Note that this system is compatible with our choice of positive system of $\mathfrak{f}_4$. We claim that $P_{\{\beta_1',\beta_2',\beta_3'\}} \subseteq Q_4$ regularly. The Levi of P has Lie algebra $\mathfrak{so}_6 + \mathbb{C}$ which is embedded in the Lie algebra of the Levi of Q, which is $\mathfrak{so}_7 + \mathbb{C}$. A weight of the nilpotent radical of $P_{\{\beta_1',\beta_2',\beta_3'\}}$ has the short root β_4' in its support. Hence it has α_4 in its support, which implies that it is a weight of the nilpotent radical of Q. Since $(\beta_4')^\vee = \alpha_4^\vee + \alpha_3^\vee$, we see that $\omega_4((\beta_i')^\vee) = \delta_{4i}$. Hence $\omega_4^P = \varpi_4'$. It follows that

$$\chi_{D_4^-} = -\varpi_4.$$

The equations

$$\chi_{D_1} - \chi_{D_4^-} = 0, \quad -\chi_{D_1} + \chi_{D_2} = 0, \quad -\chi_{D_3} + \chi_{D_4^+} + \chi_{D_4^-} = 0$$

imply

$$\chi_{D_1} = \chi_{D_3} = -\varpi_4,$$
$$\chi_{D_4^+} = 0.$$

B.9. $\mathsf{E}_6/(\mathrm{Spin}(10)\times\mathbb{C}^\times)$. Let G denote the connected and simply connected semisimple group of Dynkin type E_6. Let $H \subseteq G$ be the Levi component with simple roots

$$\beta_1 = \alpha_6, \beta_2 = \alpha_5, \beta_3 = \alpha_4, \beta_4 = \alpha_3, \beta_5 = \alpha_2.$$

Then H is isogenous to $\mathbb{C}^\times \times \mathrm{Spin}(10)$ and we have

$$\varpi_1 = -\frac{1}{2}\omega_1 + \omega_6,$$
$$\varpi_2 = -\omega_1 + \omega_5,$$
$$\varpi_3 = -\frac{3}{2}\omega_1 + \omega_4,$$
$$\varpi_4 = -\frac{5}{4}\omega_1 + \omega_3,$$
$$\varpi_5 = -\frac{3}{4}\omega_1 + \omega_2.$$

We denote by ϵ the extension of a generator of $\mathcal{X}(\mathbb{C}^\times)$ to a character of the universal cover of H, chosen in such a way that $\omega_1 = 4\epsilon$, and we have

$$\omega_1 = 4\epsilon,$$
$$\omega_2 = \varpi_5 + 3\epsilon,$$
$$\omega_3 = \varpi_4 + 5\epsilon,$$
$$\omega_4 = \varpi_3 + 6\epsilon,$$
$$\omega_5 = \varpi_2 + 4\epsilon,$$
$$\omega_6 = \varpi_1 + 2\epsilon.$$

There is one possibility for I, namely $I = \{\beta_1\}$. Let $P \subseteq H$ denote the corresponding parabolic subgroup, i.e. $P = P_{\{\beta_2,\beta_3,\beta_4,\beta_5\}}$. Then L_P is isogenous to $\mathbb{C}^\times \times \mathrm{Spin}(8) \times \mathbb{C}^\times$. Consider the Luna diagram for that appears as number 42 in [6] and denote its corresponding subgroup with $P' \subseteq G$. We have

$$\Delta(G/P') = \{D_1^+, D_1^-, D_3, D_2, D_5, D_6^+, D_6^-\}$$

and[2]

$$\Sigma(G/P') = \{\alpha_1, \alpha_{234}, \alpha_{345}, \alpha_{245}, \alpha_6\}$$

with Cartan pairing

	σ_1	σ_2	σ_3	σ_4	σ_5
D_1^+	1	0	−1	0	0
D_1^-	1	−1	0	0	0
D_3	−1	1	1	−1	0
D_2	0	1	−1	1	0
D_5	0	−1	1	1	−1
D_6^+	0	0	−1	0	1
D_6^-	0	0	0	−1	1

Note that $\Sigma(G/H) = \{\alpha_{13456}, 2\alpha_2 + 2\alpha_4 + \alpha_3 + \alpha_5\}$, see e.g. [10, Case 18]. The subset $\{D_1^+, D_3, D_5, D_6^+\} \subseteq \Sigma(G/P')$ is distinguished and it gives a map to G/H. It follows that P' is conjugate to a subgroup of H. An application of Proposition 6.1 readily implies that P' is conjugate to a parabolic subgroup of H. It follows from the classification that P and P' are conjugates.

Let $Q \subseteq G$ be the parabolic subgroup for which L_Q has simple roots $\{\alpha_1, \alpha_6\}$. Then $\{D_1^+, D_3, D_2, D_5, D_6^-\} \subseteq \Delta(G/P)$ is a distinguished subset that gives a map $G/P \to G/Q$. This means that D_1^- and D_6^+ come from G/Q. Hence

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^P &&= -4\epsilon, \\ \chi_{D_6^+} &= -\omega_6^P &&= -\varpi_1 - 2\epsilon. \end{aligned}$$

We find

$$\begin{aligned} \chi_{D_1^+} = \chi_{D_5} &= -\varpi_1 + 2\epsilon, \\ \chi_{D_2} &= 0, \\ \chi_{D_3} &= -\varpi_1 - 2\epsilon, \\ \chi_{D_6^-} &= 4\epsilon. \end{aligned}$$

B.10. $\mathsf{E}_6/\mathsf{F}_4$**.** Let G denote the connected semisimple group of Dynkin type E_6. Let $H \subseteq G$ be the connected subgroup with simple roots

$$\beta_1 = \alpha_2, \beta_2 = \alpha_4, \beta_3 = \frac{1}{2}(\alpha_3 + \alpha_5), \beta_4 = \frac{1}{2}(\alpha_1 + \alpha_6).$$

Then $H \subseteq G$ is a subgroup of Dynkin type F_4, which is unique up to conjugacy [13, Thm. 11.1]. Moreover, we have $\omega_2 = \varpi_1$ which follows from the expression of the fundamental weights in terms of the simple roots.

There is one possibility for I, $I = \{\beta_1\}$. Let $P \subseteq H$ denote the corresponding parabolic subgroup, i.e. $P = P_{\{\beta_2,\beta_3,\beta_4\}}$. Then $L_P = \mathrm{Sp}(6) \times \mathbb{C}^\times$ and $\mathfrak{p}^u = \mathbb{C}^{15}$. Let $Q \subseteq G$ be the parabolic subgroup for which L_Q has simple roots $S \setminus \{\alpha_2\}$. It is clear that $L_P \subseteq L_Q$ and $P^u \subseteq Q^u$. These observations suggest that the Luna diagram of G/P is given by [6, Case 7]. One can also verify this by applying as usual Corollary 3.9 and Proposition 6.1. We have

$$\Sigma(G/P) = \{\sigma_1 = \alpha_1 + \alpha_3, \sigma_2 = \alpha_2 + \alpha_4, \sigma_3 = \alpha_3 + \alpha_4, \sigma_4 = \alpha_4 + \alpha_5, \sigma_5 = \alpha_5 + \alpha_6\}$$

and

$$\Delta(G/P) = \{D_1, D_2, D_3, D_4, D_5, D_6\}$$

[2]Here we denote by α_{ijk} the sum $\alpha_i + \alpha_j + \alpha_k$. Similar notations will be used throughout this case.

with Cartan matrix

	σ_1	σ_2	σ_3	σ_4	σ_5
D_1	1	0	−1	0	0
D_2	0	1	−1	−1	0
D_3	1	−1	1	−1	0
D_4	−1	1	1	1	−1
D_5	0	−1	−1	1	1
D_6	0	0	0	−1	1

Our discussion above implies that

$$\chi_{D_2} = -\omega_2^P = -\varpi_1.$$

We find

$$\begin{aligned} \chi_{D_2} = \chi_{D_3} = \chi_{D_4} &= \chi_{D_5} = -\varpi_1, \\ \chi_{D_1} &= \chi_{D_6} = 0. \end{aligned}$$

B.11. $\mathrm{SL}(n) \times \mathrm{SL}(n)/\operatorname{diag}(\mathrm{SL}(n))$**.** There are two possibilities: $I = \{\beta_1\}$ or $I = \{\beta_{n-1}\}$. The generators for the case $I = \{\beta_1\}$ have been calculated in [23, Lemma 5.2]. The second case is related to the first by Remark A.2.

Appendix C. Non-symmetric cases

C.1. $\mathrm{SL}(p+q+2)/(\mathrm{SL}(p+1)\times\mathrm{SL}(q+1))$ **with** $p > q \geq 0$**.** There are four different cases, listed below. In each case one can derive the generators immediately from the indicated subcases of Case B.2 by Remark A.3.

(1) $2 \leq p < q$, $I = \{\beta_1\}$ or $I = \{\beta_p\}$, use Case B.2.2,
(2) $p = 1 < q$, $I = \{\beta\}$, use Case B.2.1,
(3) $p = 1, q \geq 4$, $I = \{\beta'_j\}$, $2 \leq j \leq q-2$, use Case B.2.4,
(4) $p = 0, q \geq 1$, $I = S_H \smallsetminus \{\beta_j\}$, $1 \leq j \leq q$, use Case B.2.5.

C.2. $\mathrm{SO}(4n+2)/\mathrm{SL}(2n+1)$ **with** $n \geq 1$**.** We have the following possibilities for I: $I = \{\beta_1\}$ or $I = \{\beta_{2n}\}$. This case derives immediately from Case (B.6), by Remark A.3.

C.3. $\mathrm{Spin}(9)/\mathrm{Spin}(7)$**.** Let $G = \mathrm{Spin}(9)$ and let H be the connected semisimple subgroup whose simple roots are $\beta_1 = \epsilon_3 + \epsilon_4, \beta_2 = \alpha_2$ and $\beta_3 = \frac{1}{2}(\alpha_1 + \alpha_3)$. Then H is isomorphic to Spin(7) and the isomorphism is given by the representation $\varpi_3 + 1$.

There is only one possibility for I, namely $I = \{\beta_1\}$. Using as usual Corollary 3.9 and Proposition 6.1, one checks that

$$\Sigma(G/P) = \{\sigma_1 = \alpha_1 + \alpha_2, \sigma_2 = \alpha_2 + \alpha_3, \sigma_3 = \alpha_3 + \alpha_4, \sigma_4 = \alpha_4\}$$

and

$$\Delta(G/P) = \{D_1, D_2, D_3, D_4^+, D_4^-\}$$

with Cartan pairing

	σ_1	σ_2	σ_3	σ_4
D_1	1	−1	0	0
D_2	1	1	−1	0
D_3	−1	1	1	−1
D_4^+	0	0	0	1
D_4^-	0	−2	0	1

Consider the parabolic subgroup $Q = Q_{\{\alpha_1,\alpha_2,\alpha_3\}}$ of G. We have $P \subseteq Q$ and Q is minimal for this inclusion. Indeed, the corresponding set of colors corresponding to the map $G/P \to G/Q$ is either $\{D_1, D_2, D_3, D_4^+\}$ or $\{D_1, D_2, D_3, D_4^-\}$. We note that the latter is not a parabolic subset of $\Delta(G/P)$, that the former is, and in fact that it is minimal.

Using that $\omega_4 = \beta_1 + \beta_2 + \beta_3 = \varpi_1$, we get

$$\chi_{D_4^-} = -\omega_4^P = -\varpi_1,$$

and we obtain

$$\begin{array}{rcccl} \chi_{D_1} & = & \chi_{D_4^+} & = & 0, \\ \chi_{D_2} & = & \chi_{D_3} & = & -\varpi_1. \end{array}$$

C.4. Spin(7)/G_2. Let $G = \mathrm{Spin}(7)$ and let $H \subseteq G$ be the connected subgroup with simple roots $\beta_1 = \frac{1}{3}(\alpha_1 + 2\alpha_3)$ and $\beta_2 = \alpha_2$. Then H is simply connected of Dynkin type G_2. In fact, H has an irreducible 7-dimensional representation $H \to \mathrm{SO}(7)$ of highest weight ϖ_1 and its lift $H \to \mathrm{Spin}(7)$ is the embedding we described above.

We have the following possibilities for I:

(1) $I = \{\beta_1\}$,
(2) $I = \{\beta_2\}$.

The Luna diagrams for the respective homogeneous spaces, denoted here by G/P_1 and G/P_2, appear in [7, §3.4]. We will prove in a moment which diagram corresponds to which case, and we will use the observation that $Q_{\{\alpha_1,\alpha_3\}}$ has Levi subgroup of semisimple type A_1, and the unipotent radical $Q^u_{\{\alpha_1,\alpha_3\}}$ does not contain the irreducible SL(2)-module of dimension 4.

C.4.1. $I = \{\beta_1\}$. We have

$$\Sigma(G/P_1) = \{\sigma_1 = \alpha_1 + \alpha_2, \sigma_2 = \alpha_2 + \alpha_3, \sigma_3 = \alpha_3\}$$

and

$$\Delta(G/P_1) = \{D_1, D_2, D_3^+, D_3^-\}$$

with Cartan pairing

	σ_1	σ_2	σ_3
D_1	1	−1	0
D_2	1	1	−1
D_3^+	0	0	1
D_3^-	−2	0	1

We observe that $\{D_2, D_3^+\}$ is the minimal parabolic subset, as there are no other parabolic subsets with two or less elements. This implies that the colors D_1, D_3^- are pull-backs of colors on G/Q via $G/P_1 \to G/Q$ which is given by the inclusion $P_1 = P_{\{\beta_1\}} \subseteq Q_{\{\alpha_1,\alpha_3\}} = Q$.

This also shows that these spherical roots and colors correspond to P_1 and not P_2. Indeed, the unipotent radical of P_2 contains the 4-dimensional irreducible SL(2)-module, whereas $Q_{\{\alpha_1,\alpha_2\}}$ doesn't.

We deduce

$$\begin{array}{rcccl} \chi_{D_1} & = & -\omega_1^P & = & -\varpi_1, \\ \chi_{D_3^-} & = & -\omega_3^P & = & -\varpi_1, \end{array}$$

and this yields

$$\chi_{D_2} = -\varpi_1,$$
$$\chi_{D_3^+} = 0.$$

C.4.2. $I = \{\beta_2\}$. We have

$$\Sigma(G/P_2) = \{\sigma_1 = \alpha_1, \sigma_2 = \alpha_2, \sigma_3 = \alpha_3\}$$

and

$$\Delta(G/P_2) = \{D_1^+ = D_2^+, D_1^- = D_3^-, D_2^-, D_3^+\}$$

with Cartan pairing

	σ_1	σ_2	σ_3
D_1^+	1	1	-1
D_1^-	1	-2	1
D_2^-	-2	1	0
D_3^+	-1	0	1

Since D_1^+ and D_1^- are moved by two roots, they must be in the parabolic subset. Suppose that $\{D_1^+, D_1^-, D_2^-\}$ is parabolic. Then there exist non-negative coefficients a_1^+, a_1^-, a_2^- such that $a_1^+ D_1^+ + a_1^- D_1^- + a_2^- D_2^-$ is strictly positive on each spherical root. In particular, $a_1^- > a_1^+$. The first two spherical roots yield $a_1^+ + a_1^- - 2a_2^- > 0$ and $a_1^+ - 2a_1^- + a_2^- > 0$, which implies $a_1^+ < a_1^-$, a contradiction.

Since $3D_1^+ + D_1^- + 3D_3^+$ is strictly positive on each spherical root, we see that $\{D_1^+, D_1^-, D_3^+\}$ is a minimal parabolic subset of colors. This means that $Q = Q_{\{\alpha_2\}}$ contains P_2 up to conjugation, and that D_2^- comes from G/Q.

Hence

$$\chi_{D_2^-} = -\omega_2^P = -\varpi_2,$$

and we obtain

$$\chi_{D_1^+} = \chi_{D_1^-} = -\varpi_2,$$
$$\chi_{D_3^+} = 0.$$

C.5. $\mathsf{G}_2/\operatorname{SL}(3)$**.** Let G be the connected group of Dynkin type G_2 and let $H \subseteq G$ be the connected subgroup whose roots are the long roots of G. Set $\beta_2 = \alpha_2$ and $\beta_1 = s_{\omega_2}(\beta_1)$. Then $H = \operatorname{SL}(3)$. There are two possibilities for I,

(1) $I = \{\beta_1\}$,
(2) $I = \{\beta_2\}$.

The corresponding subgroups P_1 and P_2 are conjugated by a simple reflection of W_G. The Luna diagram of G/P_1 occurs as case G3 in [33]. Denoting by α_1 the short simple root, and by α_2 the long one, one checks with the usual argument that

$$\Sigma(G/P_1) = \{\sigma_1 = \alpha_1, \sigma_2 = \alpha_1 + \alpha_2\}$$

and

$$\Delta(G/P_1) = \{D_1^+, D_1^-, D_2\}$$

with Cartan pairing

	σ_1	σ_2
D_1^+	1	0
D_1^-	1	-1
D_2	-1	1

The minimal parabolic subset is $\{D_1^+, D_2\}$ which implies that D_1^- is a pull-back from the color of $G/Q_{\{\alpha_2\}}$, where $P_1 = P_{\{\beta_2\}} \subseteq Q_{\{\alpha_2\}}$ is regular. It follows that

$$\chi_{D_1^-} = -\omega_1^{P_1} = -\varpi_1$$

and the equations yield

$$\begin{aligned}\chi_{D_2} &= -\varpi_1,\\ \chi_{D_1^+} &= 0.\end{aligned}$$

C.6. $(\mathrm{Sp}(2m) \times \mathrm{Sp}(2n))/(\mathrm{Sp}(2m-2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2n-2))$ **with** $m, n \geq 1$ **and** $\min\{m, n\} > 1$**.** We have the following possibilities for I:

(1) $I = \{\beta_i\}$ with $m \geq 2$ and $i \in \{1, \ldots, m-1\}$,
(2) $I = \{\beta'_1\}$.

There is also a third possibility $I = \{\beta''_j\}$ with $n > 1$ and $j \in \{1, \ldots, n-1\}$, but it can be skipped here because it falls under the analysis of the first case by just swapping m with n.

We recall the following data for G/H, taken from [10, Case 42]. If $n, m > 1$ we have

$$\Sigma(G/H) = \{\alpha_1 + \alpha'_1, \alpha_1 + 2\alpha_{2,m-1} + \alpha_m, \alpha'_1 + 2\alpha'_{2,n-1} + \alpha'_n\}$$

and

$$\Delta(G/H) = \{D_1 = D_{1'}, D_2, D_{2'}\}$$

with Cartan pairing given by the restriction to $\Xi(G/H)$ of respectively $\alpha_1^\vee$, $\alpha_2^\vee$, $(\alpha'_2)^\vee$. If $m = 1$ or $n = 1$, then the second (resp. third) spherical root and the second (resp. third) color do not appear.

C.6.1. $I = \{\beta_i\}$ *with* $m > 1$ *and* $i \in \{1, \ldots, m-1\}$. Let G act naturally on $\mathbb{C}^{2m} \oplus \mathbb{C}^{2n}$, and let $V \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ be the subspace generated by the m-th and the $(m+1)$-th elements of the canonical basis of $\mathbb{C}^{2m}$, and by the n-th and the $(n+1)$-th elements of the canonical basis of $\mathbb{C}^{2n}$. We choose H to be inside the stabilizer of V in G, so that H contains the diagonal subgroup of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ (embedded in $\mathrm{GL}(V)$ in the obvious way).

We recall that, with this choice (and up to restricting weights from T_G to T_H), we have $\omega_j = \varpi_j$ for all $j \in \{1, \ldots, m-1\}$, and $\omega_m = \varpi_{m-1} + \varpi'_1$.

As usual, we choose P so that it contains the intersection $H \cap B_-$. We use notations similar to Section B.7: the parabolic subgroup P will be denoted also by P_1 or $P_1(i)$, whereas that of case (2) will be denoted also by P_2. With similar notation we consider the subgroup $Q_1(i)$, which is such that its Levi subgroup has all simple roots except for α_i.

Let us discuss how one checks that $G/P_1(i)$ has the spherical roots and colors indicated below. The usual procedure assures that the given data correspond to parabolic subgroups of H. The inclusion of $P_1(i)$ in $Q_1(i)$ is then enough to conclude that the data we give for $P_1(i)$ does not correspond to $P_1(j)$ for any $j \neq i$.

It remains to exclude the possibility that the data given here corresponds to P_2. This is done by noticing that P_2 is contained in a proper parabolic subgroup of G containing the first factor $\mathrm{Sp}(2m)$ of G, something that is not compatible with the data given here.

We analyze now all possible values of i, m, and n.

Suppose $i = m-1$, $m > 2$ and $n > 1$. Then

$$\Sigma(G/P_1(m-1)) \\ = \{\alpha_1, \sigma_1 = \alpha_{2,m-1}, \alpha_m, \alpha'_1, \sigma_2 = \alpha'_1 + 2\alpha_{2,n-1} + \alpha_n\}$$

and

$$\Delta(G/P_1(m-1)) = \{D_1^+ = D_m^+, D_1^- = D_{1'}^-, D_2, E_{m-1}, D_m^- = D_{1'}^+, D_{2'}\}$$

with Cartan pairing

	α_1	σ_1	α_m	α'_1	σ_2
D_1^+	1	−1	1	−1	0
D_1^-	1	0	−1	1	0
D_2	−1	1	0	0	0
E_{m-1}	0	1	−2	0	0
D_m^-	−1	0	1	1	0
$D_{2'}$	0	0	0	−1	1

There is only one color not mapped dominantly to $G/Q_1(m-1)$, namely E_{m-1}. Indeed, if $m > 3$, it's the only color moved by α_{m-1}. If $m = 3$ the other possibility given by D_2 does not correspond to a parabolic subset of colors. Thus

$$\chi_{E_{m-1}} = -\omega_{m-1}^{P_1(m-1)} = -\varpi_{m-1},$$

and this yields

$$\begin{aligned} \chi_{D_1^+} &= \chi_{D_m^-} = -\varpi_{m-1}, \\ \chi_{D_1^-} = \chi_{D_2} &= \chi_{D_{2'}} = 0. \end{aligned}$$

Suppose $i = m-1$, $m > 2$, and $n = 1$. Then

$$\Sigma(G/P_1(m-1)) = \{\alpha_1, \sigma = \alpha_{2,m-1}, \alpha_m, \alpha'_1\}$$

and

$$\Delta(G/P_1(m-1)) = \{D_1^+ = D_m^+, D_1^- = D_{1'}^+, D_2, E_{m-1}, D_m^- = D_{1'}^-\}$$

with Cartan pairing

	α_1	σ	α_m	α'_1
D_1^+	1	−1	1	−1
D_1^-	1	0	−1	1
D_2	−1	1	0	0
E_{m-1}	0	1	−2	0
D_m^-	−1	0	1	1

We proceed exactly as in the previous case, obtaining

$$\chi_{E_{m-1}} = -\omega_{m-1}^{P_1(m-1)} = -\varpi_{m-1},$$

and

$$\begin{aligned} \chi_{D_1^+} &= \chi_{D_m^-} = -\varpi_{m-1}, \\ \chi_{D_1^-} &= \chi_{D_2} = 0. \end{aligned}$$

Suppose $i = 1$, $m = 2$ and $n \geq 2$. Then

$$\Sigma(G/P_1(1)) = \{\alpha_1, \alpha_2, \alpha'_1, \sigma = \alpha'_1 + 2\alpha'_{2,n-1} + \alpha'_n\}$$

and

$$\Delta(G/P_1(1)) = \{D_1^+, D_1^- = D_{1'}^-, D_2^+, D_2^- = D_{1'}^+, D_{2'}\}$$

with Cartan pairing

	α_1	α_2	α'_1	σ
D_1^+	1	−1	−1	0
D_1^-	1	−1	1	0
D_2^+	0	1	−1	0
D_2^-	−1	1	1	0
$D_{2'}$	0	0	−1	1

The only color not mapped dominantly to $G/Q_1(1)$ is D_1^+, which gives

$$\chi_{D_1^+} = -\omega_1^{P_1(1)} = -\varpi_1,$$

and

$$\begin{aligned} && \chi_{D_2^-} &= -\varpi_1, \\ \chi_{D_1^-} = \chi_{D_2^+} &= \chi_{D_{2'}} &&= 0. \end{aligned}$$

Suppose $i = 1$, $m = 2$, and $n = 1$. Then

$$\Sigma(G/P_1(1)) = \{\alpha_1, \alpha_2, \alpha'_1\}$$

and

$$\Delta(G/P_1(1)) = \{D_1^+, D_1^- = D_{1'}^-, D_2^+, D_2^- = D_{1'}^+\}$$

with Cartan pairing

	α_1	α_2	α'_1
D_1^+	1	−1	−1
D_1^-	1	−1	1
D_2^+	0	1	−1
D_2^-	−1	1	1

We proceed exactly as in the previous case, obtaining

$$\chi_{D_1^+} = -\omega_1^{P_1(1)} = -\varpi_1,$$

and

$$\begin{aligned} && \chi_{D_2^-} &= -\varpi_1, \\ \chi_{D_1^-} &= \chi_{D_2^+} &&= 0. \end{aligned}$$

Suppose $1 < i < m-1$ and $n > 1$. Then

$$\Sigma(G/P_1(i)) \\ = \{\alpha_1, \sigma_1 = \alpha_{2,i}, \alpha_{i+1}, \sigma_2 = \alpha_{i+1} + 2\alpha_{i+2,m-1} + \alpha_m, \alpha'_1, \sigma_3 = \alpha'_1 + 2\alpha'_{2,n-1} + \alpha'_n\}$$

and

$$\Delta(G/P_1(i)) = \{D_1^+ = D_{i+1}^+, D_1^- = D_{1'}^-, D_2, E_i, D_{i+1}^- = D_{1'}^+, D_{i+2}, D_{2'}\}$$

with Cartan pairing

	α_1	σ_1	α_{i+1}	σ_2	α'_1	σ_3
D_1^+	1	−1	1	0	−1	0
D_1^-	1	0	−1	0	1	0
D_2	−1	1	0	0	0	0
E_i	0	1	−1	−1	0	0
D_{i+1}^-	−1	0	1	0	1	0
D_{i+2}	0	0	−1	1	0	0
$D_{2'}$	0	0	0	0	−1	1

The only color not mapped dominantly to $G/Q_1(i)$ is E_i. This is obvious if $i>2$; if $i=2$ then the other possibility D_2 is excluded because the other colors do not form a parabolic subset.

We deduce
$$\chi_{E_i} = -\omega_i^{P_1(i)} = -\varpi_i,$$
and
$$\begin{array}{ccccccc} \chi_{D_1^+} & = & \chi_{D_{i+1}^-} & = & \chi_{D_{i+2}} & = & -\varpi_i, \\ \chi_{D_1^-} & = & \chi_{D_2} & = & \chi_{D_{2'}} & = & 0. \end{array}$$

Suppose $1<i<m-1$ and $n=1$. Then
$$\Sigma(G/P_1(i)) = \{\alpha_1, \sigma_1 = \alpha_{2,i}, \alpha_{i+1}, \sigma_2 = \alpha_{i+1} + 2\alpha_{i+2,m-1} + \alpha_m, \alpha'_1\}$$
and
$$\Delta(G/P_1(i)) = \{D_1^+ = D_{i+1}^+, D_1^- = D_{1'}^-, D_2, E_i, D_{i+1}^- = D_{1'}^+, D_{i+2}\}$$
with Cartan pairing

	α_1	σ_1	α_{i+1}	σ_2	α'_1
D_1^+	1	−1	1	0	−1
D_1^-	1	0	−1	0	1
D_2	−1	1	0	0	0
E_i	0	1	−1	−1	0
D_{i+1}^-	−1	0	1	0	1
D_{i+2}	0	0	−1	1	0

We proceed exactly as in the previous case, obtaining
$$\chi_{E_i} = -\omega_i^{P_1(i)} = -\varpi_i,$$
and
$$\begin{array}{ccccccc} \chi_{D_1^+} & = & \chi_{D_{i+1}^-} & = & \chi_{D_{i+2}} & = & -\varpi_i, \\ & & \chi_{D_1^-} & = & \chi_{D_2} & = & 0. \end{array}$$

Suppose $i=1$, $m\geq 3$, and $n>1$. Then
$$\Sigma(G/P_1(i)) = \{\alpha_1, \alpha_2, \sigma_1 = \alpha_2 + 2\alpha_{3,m-1} + \alpha_m, \alpha'_1, \sigma_2 = \alpha'_1 + 2\alpha'_{2,n-1} + \alpha'_n\}$$
and
$$\Delta(G/P_1(i)) = \{D_1^+ = D_{1'}^+, D_1^-, D_2^+, D_2^- = D_{1'}^-, D_3, D_{2'}\}$$
with Cartan pairing

	α_1	α_2	σ_1	α'_1	σ_2
D_1^+	1	−1	0	1	0
D_1^-	1	0	−1	−1	0
D_2^+	0	1	0	−1	0
D_2^-	−1	1	0	1	0
D_3	0	−1	1	0	0
$D_{2'}$	0	0	0	−1	1

The only color not mapped dominantly to $G/Q_1(i)$ is D_1^-. This yields
$$\chi_{D_1^-} = -\omega_1^{P_1(i)} = -\varpi_1,$$
and
$$\begin{array}{ccccccc} & & \chi_{D_2^-} & = & \chi_{D_3} & = & -\varpi_1, \\ \chi_{D_1^+} & = & \chi_{D_2^+} & = & \chi_{D_{2'}} & = & 0. \end{array}$$

It remains the case $i = 1$, $m \geq 3$, and $n = 1$. Then

$$\Sigma(G/P_1(i)) = \{\alpha_1, \alpha_2, \sigma = \alpha_2 + 2\alpha_{3,m-1} + \alpha_m, \alpha'_1\}$$

and

$$\Delta(G/P_1(i)) = \{D_1^+ = D_{1'}^+, D_1^-, D_2^+, D_2^- = D_{1'}^-, D_3\}$$

with Cartan pairing

	α_1	α_2	σ	α'_1
D_1^+	1	−1	0	1
D_1^-	1	0	−1	−1
D_2^+	0	1	0	−1
D_2^-	−1	1	0	1
D_3	0	−1	1	0

We proceed exactly as in the previous case, obtaining

$$\chi_{D_1^-} = -\omega_1^{P_1(i)} = -\varpi_1,$$

and

$$\begin{aligned} \chi_{D_2^-} &= \chi_{D_3} &= -\varpi_1, \\ \chi_{D_1^+} &= \chi_{D_2^+} &= 0. \end{aligned}$$

C.6.2. $I = \{\beta'_1\}$. We choose now a different embedding of H into G. Let $V' \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ be the subspace generated by the first and last elements of the canonical basis of $\mathbb{C}^{2m}$, and by the first and last elements of the canonical basis of $\mathbb{C}^{2n}$. We choose H to be inside the stabilizer of V' in G, so that H contains the diagonal subgroup of $\mathrm{SL}(2) \times \mathrm{SL}(2)$ (embedded in $\mathrm{GL}(V')$ in the obvious way).

We recall that, with this choice and up to restricting weights from T_G to T_H, we have $\omega_1 = \omega'_1 = \varpi'_1$, $\omega_j = \varpi'_1 + \varpi_{j-1}$ for all $j \in \{2, \ldots, m\}$, and $\omega'_j = \varpi'_1 + \varpi''_{j-1}$ for all $j \in \{2, \ldots, n\}$.

In this case the subgroup Q_2 is such that its Levi subgroup has all simple roots except for α_1 and α'_1. To check that G/P_2 has the spherical roots and colors indicated below, one proceeds as for $G/P_1(i)$.

Suppose that $m, n > 1$. Then we have

$$\Sigma(G/P) = \{\alpha_1, \sigma = \alpha_1 + 2\alpha_{2,m-1} + \alpha_m, \alpha'_1, \sigma' = \alpha'_1 + 2\alpha'_{2,n-1} + \alpha'_n\}$$

and

$$\Delta(G/P) = \{D_1^+ = D_{1'}^+, D_1^-, D_{1'}^-, D_2, D_{2'}\}$$

with Cartan pairing

	α_1	σ	α'_1	σ'
D_1^+	1	0	1	0
D_1^-	1	0	−1	0
$D_{1'}^-$	−1	0	1	0
D_2	−1	1	0	0
$D_{2'}$	0	0	−1	1

The colors not mapped dominantly to G/Q_2 are D_1^- and $D_{1'}^-$. Therefore

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_2} &= -\varpi'_1, \\ \chi_{D_{1'}^-} &= -(\omega'_1)^{P_2} &= -\varpi'_1, \end{aligned}$$

and

$$\chi_{D_1^+} = \chi_{D_2} = \chi_{D_{2'}} = 0.$$

Suppose now $m > n = 1$. Then we have

$$\Sigma(G/P) = \{\alpha_1, \sigma = \alpha_1 + 2\alpha_{2,m-1} + \alpha_m, \alpha_1'\}$$

and

$$\Delta(G/P) = \{D_1^+ = D_{1'}^+, D_1^-, D_{1'}^-, D_2\}$$

with Cartan pairing

	α_1	σ	α_1'
D_1^+	1	0	1
D_1^-	1	0	-1
$D_{1'}^-$	-1	0	1
D_2	-1	1	0

We proceed exactly as in the previous case, obtaining

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_2} &= -\varpi_1', \\ \chi_{D_{1'}^-} &= -(\omega_1')^{P_2} &= -\varpi_1', \end{aligned}$$

and

$$\chi_{D_1^+} = \chi_{D_2} = 0.$$

If $n > m = 1$, then the computation is the same as the previous one (with n and m swapped), and we get

$$\Delta(G/P) = \{D_1^+ = D_{1'}^+, D_1^-, D_{1'}^-, D_{2'}\}$$

with

$$\begin{aligned} \chi_{D_1^-} &= -\omega_1^{P_2} &= -\varpi_1', \\ \chi_{D_{1'}^-} &= -(\omega_1')^{P_2} &= -\varpi_1', \end{aligned}$$

and

$$\chi_{D_1^+} = \chi_{D_{2'}} = 0.$$

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