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RESTRICTION OF IRREDUCIBLE UNITARY REPRESENTATIONS OF $\mathrm{Spin}(N,1)$ TO PARABOLIC SUBGROUPS

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Abstract. In this paper, we obtain explicit branching laws for all irreducible unitary representations of $G = \mathrm{Spin}(N,1)$ when restricted to a parabolic subgroup P. The restriction turns out to be a finite direct sum of irreducible unitary representations of P. We also verify Duflo's conjecture for the branching laws of discrete series representations of G with respect to P. That is to show: in the framework of the orbit method, the branching law of a discrete series representation is determined by some geometric behavior of the moment map for the corresponding coadjoint orbit.

1. Introduction

The branching law problem concerning the decomposition of the restriction of irreducible unitary representations to a closed Lie subgroup is an important problem in the representation theory of real Lie groups. In a series of seminal papers [30], [31], [32] Kobayashi initiated the study of discrete decomposability and admissibility for representations when restricted to non-compact subgroups. Let G be a Lie group and let H be a closed Lie subgroup. For an irreducible unitary representation π of G, the restriction of π to H, denoted by $\pi|_H$, is said to be *discretely decomposable* if it is a direct sum of irreducible unitary representations of H. If moreover, all irreducible unitary representations of H have only finite multiplicities in π, then $\pi|_H$ is said to be *admissible*. Kobayashi established criteria for the admissibility for a large class of irreducible unitary representations with respect to reductive subgroups. Based on his work, branching laws for admissible restriction have been studied in many papers including [13], [20], [30], [31], [32], [33], [45], [46], [52]. In this paper we study the case where $G = \mathrm{Spin}(N,1)$ (for $N > 2$) and H is a minimal parabolic subgroup of G, which we denote by P. In the first half of the paper we obtain explicit branching laws for all irreducible unitary representations of G. The formulas are given in §3.4 and §3.5. We find that the restriction is always a finite direct sum of irreducible unitary representations of P.

The orbit method of Kirillov ([26], [27]) and Kostant ([2], [37]) relates the branching problem to the geometry of coadjoint orbits. In the second half of the paper, we study moment maps of coadjoint orbits which are related to branching laws

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via the orbit method. Let π be an irreducible unitary representation of G associated to a G-coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$. It is well known that equipped with the Kirillov-Kostant-Souriau symplectic form, $\mathcal{O}$ becomes a G-Hamiltonian space and hence an H-Hamiltonian space. The corresponding moment map is the natural projection $q\colon \mathcal{O} \to \mathfrak{h}^*$. The orbit method predicts that the branching law of $\pi|_H$ is given in terms of the geometry of the moment map q (see [27]). In fact, when G and H are nilpotent groups, Corwin-Greenleaf [10] proved that the multiplicity of an H-representation associated with an H-coadjoint orbit $\mathcal{O}'$ equals almost everywhere the cardinality of $q^{-1}(\mathcal{O}')/H$. Concerning more general Lie groups, recently Duflo formulated a precise conjecture which describes a connection between the branching law of the restriction to a closed subgroup of discrete series on the representation theory side and the moment map from strongly regular coadjoint orbits on the geometry side. The conjecture is inspired by Heckman's thesis [22] and the "quantization commuting with reduction" program [21]. Note that the orbit correspondence for discrete series of real almost algebraic groups was established previously by Duflo in [12].

Conjecture 1.1. *Let π be a discrete series of a real almost algebraic group G, which is attached to a coadjoint orbit $\mathcal{O}_\pi$ (in the sense of Duflo* [12]*). Let H be a closed almost algebraic subgroup, and let $q\colon \mathcal{O}_\pi \to \mathfrak{h}^*$ be the moment map from $\mathcal{O}_\pi$. Then,*

(i) *$\pi|_H$ is H-admissible (in the sense of Kobayashi) if and only if the moment map $q\colon \mathcal{O}_\pi \to \mathfrak{h}^*$ is weakly proper.*
(ii) *If $\pi|_H$ is H-admissible, then each irreducible H-representation σ which appears in $\pi|_H$ is attached to a strongly regular H-coadjoint orbit $\mathcal{O}'_\sigma$ (in the sense of Duflo* [12]*) contained in $q(\mathcal{O}_\pi)$.*
(iii) *If $\pi|_H$ is H-admissible, then the multiplicity of each such σ can be expressed geometrically in terms of the reduced space $q^{-1}(\mathcal{O}'_\sigma)/H$.*

Let us give some explanations for Conjecture 1.1. The notion of "almost algebraic group" is defined in [12]. Recall that an element $f \in \mathfrak{g}^*$ is called *strongly regular* if f is regular (i.e., the coadjoint orbit containing f is of maximal dimension) and its "reductive factor" $\mathfrak{s}(f) := \{X \in \mathfrak{g}(f) : \mathrm{ad}(X) \text{ is semisimple}\}$ is of maximal dimension among reductive factors of all regular elements in $\mathfrak{g}^*$ (f is regular implies that $\mathfrak{g}(f)$ is commutative). Here, $\mathfrak{g}(f)$ is the Lie algebra of the stabilizer of f in G. Let Υ_{sr} denote the set of strongly regular elements in $\mathfrak{h}^*$. A coadjoint orbit $\mathcal{O}$ is called *strongly regular* if there exists an element $f \in \mathcal{O}$ (then every element in $\mathcal{O}$) which is strongly regular. "Weakly proper" in (i) means that the preimage (for q) of each compact subset which is contained in $q(\mathcal{O}_\pi) \cap \Upsilon_{sr}$ is compact in $\mathcal{O}_\pi$. Note that it is known that the classic properness condition is not sufficient to characterize the H-admissibility when H is not reductive (see [39], [40]). If G is compact, then Duflo's conjecture is a special case of the Spin^c version of *quantization commutes with reduction* principle (see [47]).

In this paper, based on our explicit branching laws and an explicit description of the moment map, we verify Conjecture 1.1 for the restriction to a minimal parabolic subgroup of all discrete series representations of $\mathrm{Spin}(N, 1)$. In our setting the restriction is admissible for any irreducible unitary representation π while the moment map $q\colon \mathcal{O} \to \mathfrak{p}^*$ is weakly proper for any $\mathcal{O}$.

For the proof of our branching laws, a key idea is to construct L^2-models of certain irreducible unitary representations by taking the classical Fourier transform of the non-compact picture (N-picture) of principal series representations of G. Classical Fourier transform was also used in other works studying branching laws, e.g., [48], [49]. The idea of using classical Fourier transform to construct L^2-models of unitary representations appeared in the papers [14], [35], [23], [43], [44]. In order to obtain branching laws for all irreducible unitary representations of G, we employ du Cloux's results [11] on moderate growth smooth Fréchet representations of semi-algebraic groups and Zuckerman translation principle ([56], [28]). Write $\mathcal{C}(G)$ for the category of moderate growth smooth Fréchet representations E of G such that the K-finite part E_K (where $K = \mathrm{Spin}(N)$ is a maximal compact subgroup of G) is a Harish-Chandra module. Write $\mathcal{C}(P)$ (resp. $\mathcal{C}(M')$) for the category of moderate growth smooth Fréchet representations of P (resp. M'), where $M' = \mathrm{Stab}_{MA}(\xi)$ with $0 \neq \xi$ a particular element of $\mathfrak{n}^*$. With du Cloux's results, we define a functor $\Psi\colon \mathcal{C}(P) \to \mathcal{C}(M')$. The functor Ψ obeys certain properties (cf. §3.1). With the Fourier transform method, we are able to identify $\Psi(\pi^{\mathrm{sm}}|_P)$ for the smooth part π^{sm} of all principal series and discrete series with infinitesimal character ρ. Using Zuckerman translation principle ([56], [28]), we extend it to calculate $\Psi(E|_P)$ for any $E \in \mathcal{C}(G)$. When π is an irreducible unitary representation of G, we read off the decomposition of $\pi|_P$ from $\Psi(\pi^{\mathrm{sm}}|_P)$.

On the geometry side, let $\mathcal{O}_f = G/G(f)$ be a regular elliptic coadjoint G-orbit. By parametrizing the double coset space $P\backslash G/G(f)$ we find explicit representatives of P-orbits in $\mathcal{O}_f$. By calculating the Pfaffian and the characteristic polynomial of the related skew-symmetric matrix, we are able to identify the P-class of the moment map image of each representative. With this, we calculate the image and show geometric properties of the moment map.

One might compare our branching laws with Kirillov's conjecture which says that the restriction to a mirabolic subgroup of any irreducible unitary representation of $\mathrm{GL}_n(k)$ (for k an archimedean or non-archimedean local field) is irreducible. Kirillov's conjecture was proved by Bernstein [4] for p-adic groups. It awaited nearly ten years for a breakthrough by Sahi [49] who proved it for tempered representations of $\mathrm{GL}_n(k)$ for k an archimedean local field. It was finally proved by Baruch [3] over archimedean local fields in general through a qualitative approach by studying invariant distributions. The restriction to a mirabolic subgroup of general irreducible unitary representations of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ is determined in [49], [50], [51] and [1]. In the literature, there is another related work by Rossi-Vergne [48] concerning the restriction to a minimal parabolic subgroup of holomorphic (or anti-holomorphic) discrete series of a Hermitian simple Lie group. For $G = \mathrm{SU}(2,1)$, the restriction of discrete series to a minimal parabolic subgroup was studied in [38]. As for the restriction of irreducible unitary representations of $\mathrm{Spin}(N,1)$ ($N \geq 2$) to a minimal parabolic subgroup, we note that the branching law is known in the literature only when $N = 2$ or 3 by Martin [42], and when $N = 4$ by Fabec [15]. On the geometry side, we describe explicitly the moment map image for any coadjoint orbit of $G = \mathrm{Spin}(N,1)$. For the mirabolic subgroup of $\mathrm{GL}_n(\mathbb{R})$ (or $\mathrm{GL}_n(\mathbb{C})$), a similar calculation was done in [41]. Kobayashi [33] studied branching laws for a symmetric pair of holomorphic type and holomorphic discrete series and Kobayashi-Nasrin [34] studied the moment map image of corresponding coadjoint orbits in this setting.

The paper is organized as follows. In §2 we introduce notation used throughout the paper, and we give a classification of irreducible unitary representations of

P. In §3 we obtain branching laws of irreducible unitary representations of G when restricted to P. Section 4 is devoted to the description of the moment map $q\colon \mathcal{O} \to \mathfrak{p}^*$. In §5 we verify Conjecture 1.1 in our setting. In Appendix A, we show that $\pi|_{MN}$ is determined by the K types of π for any unitary representation π of G. In Appendix B, we explain that branching laws shown in this paper are related to a case of Bessel model of the local Gan-Gross-Prasad conjecture ([16], [17]).

2. Preliminaries

2.1. **Notation and conventions.**

Indefinite orthogonal and spin groups of real rank one. Fix a positive integer $m \geq 2$. (Excluding $m = 1$ makes the group M below connected and makes the parametrization of irreducible representations uniform. The case $m = 1$ was treated in [42].) Let $I_{m+1,1}$ be the $(m+2) \times (m+2)$-matrix given as

$$I_{m+1,1} = \begin{pmatrix} I_{m+1} & \\ & -1 \end{pmatrix}.$$

Put

$$\begin{aligned} G_2 &= \mathrm{O}(m+1,1) = \{X \in M_{m+2}(\mathbb{R}) : X I_{m+1,1} X^t = I_{m+1,1}\}, \\ G_3 &= \mathrm{SO}(m+1,1) = \{X \in \mathrm{O}(m+1,1) : \det X = 1\}, \\ G_1 &= \mathrm{SO}_e(m+1,1), \\ G &= \mathrm{Spin}(m+1,1), \end{aligned}$$

where $\mathrm{SO}_e(m+1,1)$ is the identity component of $\mathrm{O}(m+1,1)$ (and of $\mathrm{SO}(m+1,1)$), and $\mathrm{Spin}(m+1,1)$ is a non-trivial 2-fold covering of $\mathrm{SO}_e(m+1,1)$. The Lie algebras of G, G_1, G_2, G_3 are all equal to

$$\mathfrak{g} = \mathfrak{so}(m+1,1) = \{X \in \mathfrak{gl}(m+2,\mathbb{R}) : X I_{m+1,1} + I_{m+1,1} X^t = 0\}.$$

Our results will be stated and proved for G. For concrete matrix calculation we will also work with groups G_1 and G_2. The group G_3 and its representations are used only in Appendix B.

Cartan decomposition. Write

$$\begin{aligned} K &= \mathrm{Spin}(m+1), \\ K_1 &= \{\mathrm{diag}(Y,1) : Y \in \mathrm{SO}(m+1)\}, \\ K_2 &= \{\mathrm{diag}(Y,t) : Y \in \mathrm{O}(m+1), t \in \{\pm 1\}\}, \\ K_3 &= \{\mathrm{diag}(Y,t) : Y \in \mathrm{O}(m+1), t = \det Y\}. \end{aligned}$$

Then, K, K_1, K_2, K_3 are maximal compact subgroups of G, G_1, G_2, G_3 respectively. Their Lie algebras are equal to

$$\mathfrak{k} = \{\mathrm{diag}(Y,0) : Y \in \mathfrak{so}(m+1)\}.$$

Write

$$\mathfrak{s} = \left\{ \begin{pmatrix} 0_{(m+1)\times(m+1)} & \alpha^t \\ \alpha & 0 \end{pmatrix} : \alpha \in M_{1\times(m+1)}(\mathbb{R}) \right\}.$$

Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, which is a Cartan decomposition for $\mathfrak{g}$. The corresponding Cartan involution θ of G_1 (or G_2, G_3) is given by $\theta = \mathrm{Ad}(I_{m+1,1})$.

Restricted roots and Iwasawa decomposition. Put

$$H_0 = \begin{pmatrix} 0_{m\times m} & 0_{m\times 1} & 0_{m\times 1} \\ 0_{1\times m} & 0 & 1 \\ 0_{1\times m} & 1 & 0 \end{pmatrix} \text{ and } \mathfrak{a} = \mathbb{R}\cdot H_0,$$

which is a maximal abelian subspace in $\mathfrak{s}$. Define $\lambda_0 \in \mathfrak{a}^*$ by $\lambda_0(H_0) = 1$. Then, the restricted root system $\Delta(\mathfrak{g},\mathfrak{a})$ consists of two roots $\{\pm\lambda_0\}$. Let λ_0 be a positive restricted root. Then the associated positive nilpotent part is

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0_{m\times m} & -\alpha^t & \alpha^t \\ \alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix} : \alpha \in M_{1\times m}(\mathbb{R}) \right\}.$$

Let ρ' be half the sum of positive roots in $\Delta(\mathfrak{n},\mathfrak{a})$. Then

$$\rho' = \frac{m}{2}\lambda_0 \text{ and } \rho'(H_0) = \frac{m}{2}.$$

One has the *Iwasawa decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Standard parabolic and opposite parabolic subalgebras. Let

$$\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = \{\mathrm{diag}(Y, 0_{2\times 2}) : Y \in \mathfrak{so}(m,\mathbb{R})\}.$$

Write $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$, which is a parabolic subalgebra of $\mathfrak{g}$. We have the opposite nilradical

$$\bar{\mathfrak{n}} = \left\{ \begin{pmatrix} 0_{m\times m} & \alpha^t & \alpha^t \\ -\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix} : \alpha \in M_{1\times m}(\mathbb{R}) \right\}$$

and the opposite parabolic subalgebra $\bar{\mathfrak{p}} = \mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}}$.

Subgroups. Let A (or A_1, A_2, A_3), N (or N_1, N_2, N_3), $\bar{N}$ (or $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$) be connected analytic subgroups of G (or G_1, G_2, G_3) with Lie algebras $\mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}$ respectively. In particular,

$$A_1 = \left\{ \begin{pmatrix} I_m & & \\ & r & s \\ & s & r \end{pmatrix} : r, s \in \mathbb{R},\ r^2 - s^2 = 1,\ r > 0 \right\}.$$

By the two-fold covering $G \twoheadrightarrow G_1$ and the inclusions $G_1 \subset G_2$ and $G_1 \subset G_3$ we identify A, A_2, A_3 with A_1, identify N, N_2, N_3 with N_1, and identify $\bar{N}$, $\bar{N}_2$, $\bar{N}_3$ with $\bar{N}_1$.

Put

$$M = Z_K(\mathfrak{a}), \quad M_1 = Z_{K_1}(\mathfrak{a}), \quad M_2 = Z_{K_2}(\mathfrak{a}), \quad M_3 = Z_{K_3}(\mathfrak{a}).$$

Set

$$P = MAN, \quad P_1 = M_1AN, \quad P_2 = M_2AN, \quad P_3 = M_3AN$$

and

$$\bar{P} = MA\bar{N}, \quad \bar{P}_1 = M_1A\bar{N}, \quad \bar{P}_2 = M_2A\bar{N}, \quad \bar{P}_3 = M_3A\bar{N}.$$

Then, the Lie algebras of M (or M_1, M_2, M_3), P (or P_1, P_2, P_3), $\bar{P}$ (or $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$) are equal to $\mathfrak{m}$, $\mathfrak{p}$, $\bar{\mathfrak{p}}$ respectively. Note that

$$M = \mathrm{Spin}(m),\ M_1 = \mathrm{SO}(m),\ M_2 = \mathrm{O}(m) \times \Delta_2(\mathrm{O}(1))$$

and $M_3 = \mathrm{SO}(m) \times \Delta_2(\mathrm{O}(1))$, where $\Delta_2(\mathrm{O}(1)) = \{\mathrm{diag}(t,t) : t = \pm 1\} \subset \mathrm{O}(2)$.

Nilpotent elements. For a row vector $\alpha \in \mathbb{R}^m$ write

$$X_\alpha = \begin{pmatrix} 0_{m\times m} & -\alpha^t & \alpha^t \\ \alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \quad \bar{X}_\alpha = \begin{pmatrix} 0_{m\times m} & \alpha^t & \alpha^t \\ -\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$

The following Lie brackets will be used later

$$\begin{aligned} &[X_\alpha, \bar{X}_\beta] = \operatorname{diag}(2(\alpha^t\beta - \beta^t\alpha), 0_{2\times 2}) + 2\alpha\beta^t H_0, \\ &[\operatorname{diag}(Y, 0_{2\times 2}),\ X_\alpha] = X_{\alpha Y^t}, \\ &[\operatorname{diag}(Y, 0_{2\times 2}),\ \bar{X}_\beta] = \bar{X}_{\beta Y^t}, \\ &[H_0, X_\alpha] = X_\alpha, \\ &[H_0, \bar{X}_\beta] = -\bar{X}_\beta, \end{aligned} \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}^m$ and $Y \in \mathfrak{so}(m, \mathbb{R})$.

Let $n_\alpha = \exp(X_\alpha)$ and $\bar{n}_\alpha = \exp(\bar{X}_\alpha)$. Then $n_\alpha = I + X_\alpha + \frac{1}{2}X_\alpha^2$ and $\bar{n}_\alpha = I + \bar{X}_\alpha + \frac{1}{2}\bar{X}_\alpha^2$, more concretely,

$$n_\alpha = \begin{pmatrix} I_m & -\alpha^t & \alpha^t \\ \alpha & 1 - \frac{1}{2}|\alpha|^2 & \frac{1}{2}|\alpha|^2 \\ \alpha & -\frac{1}{2}|\alpha|^2 & 1 + \frac{1}{2}|\alpha|^2 \end{pmatrix}$$

and

$$\bar{n}_\alpha = \begin{pmatrix} I_m & \alpha^t & \alpha^t \\ -\alpha & 1 - \frac{1}{2}|\alpha|^2 & -\frac{1}{2}|\alpha|^2 \\ \alpha & \frac{1}{2}|\alpha|^2 & 1 + \frac{1}{2}|\alpha|^2 \end{pmatrix}.$$

Via the maps $\alpha \mapsto X_\alpha$ and $\alpha \mapsto n_\alpha$, one identifies $\mathfrak{n}$ and N with the Euclidean space $\mathbb{R}^m$. We have $\theta(n_\alpha) = \bar{n}_{-\alpha}$.

Invariant bilinear form. For $X, Y \in \mathfrak{g}$, define

$$(X, Y) = \frac{1}{2}\operatorname{tr}(XY). \tag{2.2}$$

Then $(\cdot, \cdot)$ is a non-degenerate symmetric bilinear form on $\mathfrak{g}$, which is invariant under the adjoint action of G, G_1, G_2 and G_3. Define $\iota\colon \mathfrak{g} \to \mathfrak{g}^*$ by

$$\iota(X)(Y) = (X, Y) \ (\forall Y \in \mathfrak{g}). \tag{2.3}$$

Then, ι is an isomorphism of G- (or G_1-, G_2-, G_3-) modules.

Roots and weights. Let $n' := \lfloor \frac{m+1}{2} \rfloor$. For $\vec{a} = (a_1, \ldots, a_{n'}) \in \mathbb{R}^{n'}$, let

$$t_{\vec{a}} := \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & a_{n'} & \\ & & & -a_{n'} & 0 & \\ & & & & & 0_{(m+2-2n')\times(m+2-2n')} \end{pmatrix}. \tag{2.4}$$

Then

$$\mathfrak{t} = \{t_{\vec{a}} : a_1, \ldots, a_{n'} \in \mathbb{R}\}$$

is a maximal abelian subalgebra of $\mathfrak{k} = \operatorname{Lie} K$. Write T for the corresponding maximal torus in K. Define $\epsilon_i' \in \mathfrak{t}_{\mathbb{C}}^*$ by

$$\epsilon_i' \colon t_{\vec{a}} \mapsto \mathbf{i} a_i.$$

The root system $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is given by

$$\{\pm\epsilon'_i \pm \epsilon'_j, \pm\epsilon'_k : 1 \le i < j \le n',\ 1 \le k \le n'\} \text{ if } m \text{ is even and}$$
$$\{\pm\epsilon'_i \pm \epsilon'_j : 1 \le i < j \le n'\} \text{ if } m \text{ is odd.}$$

We denote the weight $c_1\epsilon'_1 + \cdots + c_{n'}\epsilon'_{n'} \in \mathfrak{t}^*_{\mathbb{C}}$ by $(c_1, \dots, c_{n'})$. Similar notation will be used for elements in $(\mathfrak{t} \cap \mathfrak{m})^*_{\mathbb{C}}$ and $(\mathfrak{t} \cap \mathfrak{m}')^*_{\mathbb{C}}$, where $\mathfrak{m}'$ denotes the Lie algebra of the group M' defined in (2.6).

Remark 2.1. The bilinear form (2.2) on $\mathfrak{t}$ is given as $(t_{\vec{a}}, t_{\vec{b}}) = -\vec{a} \cdot \vec{b}^t$. Hence by using the isomorphism $\iota\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}^*_{\mathbb{C}}$ defined as (2.3), we have for example $\epsilon'_1 = \mathbf{i} \cdot \iota(t_{(-1,0,\dots,0)})|_{\mathfrak{t}}$.

Define $T_s := (T \cap M) \times A$. Then $T \cap M$ is a maximal torus of M and T_s is a Cartan subgroup of G. Let $n := \lfloor \frac{m+2}{2} \rfloor$. Note that $m = 2n - 2$ and $n = n' + 1$ if m is even; $m = 2n - 1$ and $n = n'$ if m is odd. Define $\epsilon_i \in (\mathfrak{t}_s)^*_{\mathbb{C}}$ by

$$\epsilon_i = \epsilon'_i \text{ on } \mathfrak{t} \cap \mathfrak{m}, \quad \epsilon_i = 0 \text{ on } \mathfrak{a} \text{ for } 1 \le i < n,$$
$$\epsilon_n = 0 \text{ on } \mathfrak{t} \cap \mathfrak{m}, \quad \epsilon_n = \lambda_0 \text{ on } \mathfrak{a}.$$

The root system $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{t}_s)_{\mathbb{C}})$ is given by

$$\{\pm\epsilon_i \pm \epsilon_j : 1 \le i < j \le n\} \text{ if } m \text{ is even and}$$
$$\{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_k : 1 \le i < j \le n,\ 1 \le k \le n\} \text{ if } m \text{ is odd,}$$

where

$$\{\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_k : 1 \le i < j \le n-1,\ 1 \le k \le n-1\}$$

are roots of MA. Choose a positive system

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j : 1 \le i < j \le n\} \text{ if } m \text{ is even and}$$
$$\Delta^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_k : 1 \le i < j \le n,\ 1 \le k \le n\} \text{ if } m \text{ is odd.}$$

Then the corresponding simple roots are $\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}$ for even m and $\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}$ for odd m. A weight of T_s is of the form

$$\gamma = c_1\epsilon_1 + \cdots + c_n\epsilon_n,$$

which we denote by $(c_1, \dots, c_n)$. Put

$$\mu = (c_1, \dots, c_{n-1}) = c_1\epsilon_1 + \cdots + c_{n-1}\epsilon_{n-1},$$

which vanishes on $\mathfrak{a}$ and may be regarded as a weight of $\mathfrak{t} \cap \mathfrak{m}$; put

$$\nu = c_n\epsilon_n,$$

which vanishes on $\mathfrak{t} \cap \mathfrak{m}$ and may be regarded as a weight of $\mathfrak{a}$. Then $\gamma = (\mu, \nu) = \mu + \nu$.

The vector $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ is given as

$$\left(n - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right) \text{ for } m \text{ odd; } (n-1, \dots, 1, 0) \text{ for } m \text{ even.}$$

Principal series representations of G. For a finite-dimensional irreducible complex linear representation (σ, V_σ) of M and a character e^ν of A, form the smoothly induced representation

$$I(\sigma,\nu) = \mathrm{Ind}_{MA\bar{N}}^{G}(\sigma \otimes e^{\nu-\rho'} \otimes \mathbf{1}_{\bar{N}})$$

which consists of smooth functions $h\colon G \to V_\sigma$ with

$$h(gma\bar{n}) = \sigma(m)^{-1} e^{(-\nu+\rho')\log a} h(g)$$

for any $(g, m, a, \bar{n}) \in G \times M \times A \times \bar{N}$. The action of G on $I(\sigma,\nu)$ is given by $(g \cdot h)(x) = h(g^{-1}x)$ for $h \in I(\sigma,\nu)$ and $g, x \in G$. This is called a principal series representation.

If μ is the highest weight of σ, then for simplicity we denote $I(\sigma,\nu)$ also by $I(\mu,\nu)$.

Principal series representations of G_1 and G_2 are similarly defined.

Irreducible finite-dimensional representations. Write F_λ (resp. $V_{K,\lambda}$, $V_{M,\mu}$, $V_{M',\mu}$) for an irreducible finite-dimensional representation of G (resp. K, M, M') with highest weight λ (resp. λ, μ, μ).

2.2. **Irreducible unitary representations of P.** We define unitarily induced representations of P. Through the map

$$Y \mapsto \begin{pmatrix} Y & \\ & I_2 \end{pmatrix},$$

one identifies M_1 with $\mathrm{SO}(m)$. With this identification, the adjoint action of M_1A on $\mathfrak{n}$ is given by

$$\mathrm{Ad}(Y,a)X_\alpha = e^{\lambda_0(\log a)} X_{\alpha Y^t} \ (\forall \alpha \in \mathbb{R}^m,\ \forall Y \in \mathrm{SO}(m),\ \forall a \in A).$$

Moreover, under the identification $\mathfrak{n}^* \cong \mathbb{R}^m$ (via $\xi(X_\alpha) = \xi\alpha^t$), the coadjoint action of M_1A on $\mathfrak{n}^*$ is given by

$$\mathrm{Ad}^*(Y,a)\xi = e^{-\lambda_0(\log a)} \xi Y^t \ (\forall \xi \in \mathbb{R}^m,\ \forall Y \in \mathrm{SO}(m),\ \forall a \in A). \tag{2.5}$$

The coadjoint action of MA is the composition of this action with the covering map $MA \to M_1A$.

Let

$$\xi_0 = (0, \ldots, 0, 1) \in \mathfrak{n}^* \quad \text{and} \quad M_1' = \mathrm{Stab}_{M_1A}\, \xi_0.$$

Then,

$$M_1' = \left\{ \begin{pmatrix} Y & \\ & I_3 \end{pmatrix} : Y \in \mathrm{SO}(m-1) \right\}.$$

For a (not necessarily irreducible) unitary representation (τ, V_τ) of M_1', let

$$I_{P_1,\tau} = \mathrm{Ind}_{M_1' \ltimes N}^{M_1AN}(\tau \otimes e^{\mathbf{i}\xi_0})$$

be a unitarily induced representation. It consists of functions $h\colon M_1AN \to V_\tau$ with

$$h(pm'n) = (\tau \otimes e^{\mathbf{i}\xi_0})(m', n)^{-1} h(p)$$

for all $(p, m', n) \in P_1 \times M_1' \times N$ and $\langle h, h \rangle < \infty$, where

$$\langle h_1, h_2 \rangle := \int_{M_1A/M_1'} \langle h_1(ma), h_2(ma) \rangle_\tau \, \mathrm{d}_l ma$$

for $h_1, h_2 \in \mathrm{Ind}_{M_1'\ltimes N}^{M_1AN}(\tau \otimes e^{\mathbf{i}\xi_0})$. Here $\mathrm{d}_l ma$ is a left M_1A invariant measure on $(M_1A)/M_1'$, and $\langle\cdot,\cdot\rangle_\tau$ denotes an M_1'-invariant inner product on V_τ. The action of P_1 on $I_{P_1,\tau}$ is given by $(p\cdot h)(x) = h(p^{-1}x)$ for $h \in I_{P_1,\tau}$ and $p, x \in P_1$.

Similarly, put

$$M' = \mathrm{Stab}_{MA}\,\xi_0, \quad M_2' = \mathrm{Stab}_{M_2A_2}\,\xi_0, \quad M_3' = \mathrm{Stab}_{M_3A_3}\,\xi_0, \tag{2.6}$$

and define a unitarily induced representation $I_{P,\tau}$ (or $I_{P_2,\tau}$, $I_{P_3,\tau}$) from a unitary representation τ of M' (or M_2', M_3'). One has

$$M' \cong \mathrm{Spin}(m-1), \quad M_2' \cong \mathrm{O}(m-1)\times\mathrm{O}(1), \quad M_3' \cong \mathrm{SO}(m-1)\times\mathrm{O}(1).$$

The classification of irreducible unitary representations of P could be obtained by using Mackey's little group method.

Proposition 2.2 ([55]). *Any infinite-dimensional irreducible unitary representation of P is isomorphic to $I_{P,\tau}$ for a unique (up to isomorphism) irreducible finite-dimensional unitary representation τ of M'.*

Proof. Let π be an irreducible unitary representation of P. If $\pi|_N$ is trivial, then π factors through $P \to MA$ and is finite-dimensional. Assume that $\pi|_N$ is non-trivial, then the support of the spectrum of $\pi|_N$ is not equal to $\{0\}$. As the spectrum of $\pi|_N$ is an MA-stable subset of $\mathfrak{n}^*$ and MA acts transitively on $\mathfrak{n}^* - \{0\}$, ξ_0 is in the support. By Mackey's little group method, one shows that $\pi \cong I_{P,\tau}$ for a unique finite-dimensional irreducible unitary representation τ of M' up to an isomorphism. □

3. Restriction to P of irreducible representations of G

In this section we obtain branching laws of all irreducible unitary representations of G when restricted to P. Although our purpose is studying the decomposition of the restriction of irreducible unitary representations, we find that we have to turn into the category of smooth representations because we would like to use Zuckerman's translation principle. Let us briefly explain our strategy. Based on du Cloux's work, we define and study in §3.1 the functor Ψ from the category of smooth representations of P and that of smooth representations of M'. Lemma 3.6 relates the functor Ψ with the branching laws. In Propositions 3.7 and 3.27, we calculate $\Psi(\pi|_P)$ for π (not necessarily unitary) principal series representations and for π discrete series representations with infinitesimal character ρ, respectively. This will be done using concrete computations of the Fourier transform for the non-compact picture of principal series representations. Since the functor Ψ behaves well under Zuckerman's translation principle (Lemmas 3.3 and 3.4), combining the knowledge of the Grothendieck group of the admissible representations of G which we recall in §3.3, we are able to determine $\Psi(\pi|_P)$ for all irreducible admissible representations of G. The branching laws we want are then obtained in §3.4 and §3.5.

3.1. **Moderate growth smooth representations of G (or P).** Let $\mathcal{C}_K(G)$ denote the category of Harish-Chandra modules, i.e., finitely generated admissible $(\mathfrak{g}, K)$-modules. For a G-representation π, let π_K be the space of K-finite vectors in π. Let $\mathcal{C}(G)$ denote the category of moderate growth, smooth Fréchet G-representations π such that $\pi_K \in \mathcal{C}_K(G)$. The morphisms in $\mathcal{C}(G)$ are defined to be continuous intertwiners with images that are direct summands in the category of Fréchet spaces. The *Casselman-Wallach theorem* asserts that the functor

$$\mathcal{C}(G) \to \mathcal{C}_K(G), \quad \pi \mapsto \pi_K$$

gives an equivalence of abelian categories ([6], [54]). For an object $V \in \mathcal{C}_K(G)$, write $V^{\mathrm{sm}} \in \mathcal{C}(G)$ for a *Casselman-Wallach globalization* of V. Then $(V^{\mathrm{sm}})_K \cong V$.

In [11], du Cloux studied the category of moderate growth, smooth Fréchet representations of a real semialgebraic group. We recall some results of [11] in our setting. Let $\mathcal{C}(P)$ (resp. $\mathcal{C}(M')$) denote the category of moderate growth, smooth Fréchet representations of P (resp. M'). The morphisms are continuous intertwiners.

Let $\mathscr{S}(\mathfrak{n})$ (resp. $\mathscr{S}(\mathfrak{n}^*)$) be the Schwartz space on $\mathfrak{n}$ (resp. $\mathfrak{n}^*$) with the algebra structure by the convolution product (resp. by the usual pointwise multiplication) of functions. The inverse Fourier transform gives an algebra isomorphism $\mathscr{S}(\mathfrak{n}) \xrightarrow{\sim} \mathscr{S}(\mathfrak{n}^*)$. Let $\mathscr{S}(\mathfrak{n}^* - \{0\})$ be the Schwartz space on $\mathfrak{n}^* - \{0\}$. In other words, it consists of $f|_{\mathfrak{n}^*-\{0\}}$ with $f \in \mathscr{S}(\mathfrak{n}^*)$ such that f and its all (higher) derivatives vanish at $0(\in \mathfrak{n}^*)$.

A representation $E \in \mathcal{C}(P)$ can be viewed as a moderate growth, smooth Fréchet representation of N by restriction. Then via exponential map $\mathfrak{n} \cong N$, the Fréchet space E becomes an $\mathscr{S}(\mathfrak{n})$-module and then an $\mathscr{S}(\mathfrak{n}^* - \{0\})$-module by $\mathscr{S}(\mathfrak{n}^* - \{0\}) \subset \mathscr{S}(\mathfrak{n}^*) \cong \mathscr{S}(\mathfrak{n})$.

We shall define a functor

$$\Psi\colon \mathcal{C}(P) \to \mathcal{C}(M')$$

as follows. This functor is given as $E \to E(x_0)$ in [11, Theórème 2.5.8]. Recall that $\xi_0 \in \mathfrak{n}^* - \{0\}$ is defined in §2.1 and the stabilizer of ξ_0 for the coadjoint action of P on $\mathfrak{n}^*$ is $\mathrm{Stab}_P(\xi_0) = M'N$. Define the following algebra by adding the constant function 1, which becomes the unit of the algebra:

$$\tilde{\mathscr{S}}(\mathfrak{n}^* - \{0\}) = \mathbb{C}1 \oplus \mathscr{S}(\mathfrak{n}^* - \{0\}).$$

Define an ideal $\mathfrak{m}_{\xi_0}$ by

$$\mathfrak{m}_{\xi_0} = \{f \in \tilde{\mathscr{S}}(\mathfrak{n}^* - \{0\}) : f(\xi_0) = 0\}.$$

For $E \in \mathcal{C}(P)$, [11, Lemme 2.5.7] shows that the subspace $\mathfrak{m}_{\xi_0} \cdot E$ is closed and stable by the action of $M'N$. Hence the quotient $E/(\mathfrak{m}_{\xi_0} \cdot E)$ is a Fréchet space with a natural $M'N$-action on it. The action of $\tilde{\mathscr{S}}(\mathfrak{n}^* - \{0\})$ on $E/(\mathfrak{m}_{\xi_0} \cdot E)$ factors through the evaluation map $\tilde{\mathscr{S}}(\mathfrak{n}^* - \{0\}) \ni f \mapsto f(\xi_0)$. Hence N acts on $E/(\mathfrak{m}_{\xi_0} \cdot E)$ by $e^{\mathbf{i}\xi_0}$. When we view $E/(\mathfrak{m}_{\xi_0} \cdot E)$ as a representation of M' we write

$$\Psi(E) := E/(\mathfrak{m}_{\xi_0} \cdot E).$$

By [11, Theórème 2.5.8], $\Psi(E) \in \mathcal{C}(M')$ and then Ψ defines a functor $\mathcal{C}(P) \to \mathcal{C}(M')$.

Let F be a finite-dimensional representation of P such that the N-action is trivial. Then it is easy to see that there is a natural isomorphism

$$\Psi(E) \otimes (F|_{M'}) \cong \Psi(E \otimes F). \tag{3.1}$$

Next, we define the induction from M'-representations to P-representations. Let $(\tau, V_\tau) \in \mathcal{C}(M')$. Then $\tau \otimes e^{\mathbf{i}\xi_0}$ is a smooth Fréchet representation of $M'N$. One defines in a natural way the smoothly induced representation $C^\infty \operatorname{Ind}_{M'N}^{P}(\tau \otimes e^{\mathbf{i}\xi_0})$.

Let $\mathscr{S}(P, V_\tau)$ be the space of Schwartz functions on P taking values in V_τ. For $f \in \mathscr{S}(P, V_\tau)$, define $\bar{f} \in C^\infty(P, V_\tau)$ by

$$\bar{f}(g) = \int_{M'N} (\tau \otimes e^{\mathbf{i}\xi_0})(mn) f(gmn)\, \mathrm{d}m\, \mathrm{d}n.$$

Then one has $\bar{f} \in C^\infty \operatorname{Ind}_{M'N}^P(V_\tau)$. Let

$$\mathscr{S} \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0}) = \{\bar{f} : f \in \mathscr{S}(P, V_\tau)\}.$$

Then $\mathscr{S} \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0})$ is a dense subspace of $C^\infty \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0})$ and $\mathscr{S} \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0}) \in \mathcal{C}(P)$.

Let

$$\mathscr{O}_M(P, \tau \otimes e^{\mathbf{i}\xi_0}) = \{f \in C^\infty(P, V_\tau) : h \cdot f \in \mathscr{S}(P, V_\tau)\ (\forall h \in \mathscr{S}(P))\}.$$

Let

$$\mathscr{O}_M \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0})$$

consist of functions $f \in \mathscr{O}_M(P, \tau \otimes e^{\mathbf{i}\xi_0})$ such that

$$f(gmn) = (\tau \otimes e^{\mathbf{i}\xi_0})(mn)^{-1} f(g)\ (\forall (g, mn) \in P \times M'N).$$

This is not a Fréchet space, but P naturally acts on it. Then

$$\mathscr{S} \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0}) \subset \mathscr{O}_M \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0}) \subset C^\infty \operatorname{Ind}_{M'N}^P(\tau \otimes e^{\mathbf{i}\xi_0}).$$

Since $P/(M'N) \cong \mathfrak{n}^* - \{0\}$, these three spaces become $\mathscr{S}(\mathfrak{n}^* - \{0\})$-modules by multiplication.

Since N is nilpotent and M' is compact, the group $M'N$ is unimodular and the restriction to $M'N$ of the modulus character of P is also trivial. Let $E \in \mathcal{C}(P)$. The natural map $E \to \Psi(E)$ is M'-intertwining and corresponds to a P-intertwiner

$$u \colon E \to C^\infty \operatorname{Ind}_{M'N}^P(\Psi(E) \otimes e^{\mathbf{i}\xi_0})$$

by the Frobenius reciprocity. The following is a part of [11, Théorème 2.5.8] applied to the group P.

Fact 3.1. Let $E \in \mathcal{C}(P)$ and let $u \colon E \to C^\infty \operatorname{Ind}_{M'N}^P(\Psi(E) \otimes e^{\mathbf{i}\xi_0})$ be as above. Then

$$\mathscr{S} \operatorname{Ind}_{M'N}^P(\Psi(E) \otimes e^{\mathbf{i}\xi_0}) \subset \operatorname{Im}(u) \subset \mathscr{O}_M \operatorname{Ind}_{M'N}^P(\Psi(E) \otimes e^{\mathbf{i}\xi_0}), \text{ and}$$
$$\operatorname{Ker}(u) = \{v \in E : \mathscr{S}(\mathfrak{n}^* - \{0\}) \cdot v = 0\}.$$

We need several lemmas below.

Lemma 3.2. *Let $E \in \mathcal{C}(P)$ and $W \in \mathcal{C}(M')$. Let*

$$\varphi \colon E \hookrightarrow C^\infty \operatorname{Ind}_{M'N}^P(W \otimes e^{\mathbf{i}\xi_0})$$

be an injective P-intertwining map that is also a homomorphism of $\mathscr{S}(\mathfrak{n}^* - \{0\})$-modules. Then the kernel of the map $\bar{\varphi} \colon E \to W$ given by $\bar{\varphi}(v) = (\varphi(v))(e)$ equals $\mathfrak{m}_{\xi_0} \cdot E$.

Proof. Take any $f \in \mathfrak{m}_{\xi_0}$ and $v \in E$. Since ϕ is an $\mathscr{S}(\mathfrak{n}^* - \{0\})$-homomorphism, we have $\phi(fv) = f\phi(v)$ and $\bar{\phi}(fv) = f(\xi_0)\bar{\phi}(v)$. Hence, $\operatorname{Ker}(\bar{\varphi}) \supset \mathfrak{m}_{\xi_0} \cdot E$. Then, $\bar{\varphi}$ descends to $\bar{\varphi} \colon \Psi(E) \to W$. The map φ factors as

$$E \xrightarrow{u} C^\infty \operatorname{Ind}_{M'N}^P(\Psi(E) \otimes e^{\mathbf{i}\xi_0}) \to C^\infty \operatorname{Ind}_{M'N}^P(W \otimes e^{\mathbf{i}\xi_0}).$$

By Fact 3.1, $\mathrm{Im}(u) \supset \mathscr{S}\,\mathrm{Ind}_{M'N}^{P}(\Psi(E)\otimes e^{\mathbf{i}\xi_0})$ and then

$$\mathscr{S}\,\mathrm{Ind}_{M'N}^{P}(\Psi(E)\otimes e^{\mathbf{i}\xi_0}) \to C^{\infty}\,\mathrm{Ind}_{M'N}^{P}(W\otimes e^{\mathbf{i}\xi_0})$$

is injective. Therefore, $\bar{\varphi}\colon \Psi(E)\to W$ is also injective. □

Lemma 3.3. *Let $0\to E_1\to E_2\to E_3\to 0$ be a sequence in $\mathcal{C}(P)$ which is exact as vector spaces. Then the induced sequence $0\to\Psi(E_1)\to\Psi(E_2)\to\Psi(E_3)\to 0$ in $\mathcal{C}(M')$ is also exact as vector spaces.*

Proof. It is easy to see that $\Psi(E_1)\to\Psi(E_2)\to\Psi(E_3)\to 0$ is exact.

Assuming $E_1\hookrightarrow E_2$ is an injective homomorphism in $\mathcal{C}(P)$, we will show that $\Psi(E_1)\to\Psi(E_2)$ is injective. By Fact 3.1, we have

$$\begin{aligned} &u_i\colon E_i\to \mathscr{O}_M\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_i)\otimes e^{\mathbf{i}\xi_0}),\\ &\mathrm{Ker}(u_i)=\{v\in E_i : \mathscr{S}(\mathfrak{n}^*-\{0\})\cdot v=0\},\\ &\mathscr{S}\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_i)\otimes e^{\mathbf{i}\xi_0})\subset \mathrm{Im}(u_i)\quad (i=1,2). \end{aligned}$$

By this description of $\mathrm{Ker}(u_i)$, we have $\mathrm{Ker}(u_1)=E_1\cap\mathrm{Ker}(u_2)$ and hence the natural map $\mathrm{Im}(u_1)\to\mathrm{Im}(u_2)$ is injective. By composing

$$\mathscr{S}\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_1)\otimes e^{\mathbf{i}\xi_0})\subset\mathrm{Im}(u_1)\hookrightarrow\mathrm{Im}(u_2)\subset\mathscr{O}_M\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_2)\otimes e^{\mathbf{i}\xi_0}),$$

we obtain an injective map

$$\mathscr{S}\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_1)\otimes e^{\mathbf{i}\xi_0})\hookrightarrow\mathscr{O}_M\,\mathrm{Ind}_{M'N}^{P}(\Psi(E_2)\otimes e^{\mathbf{i}\xi_0}).$$

This is induced from the map $\Psi(E_1)\to\Psi(E_2)$, which must be also injective. □

For $E\in\mathcal{C}(P)$, $\Psi(E)$ is the maximal Hausdorff quotient of E on which $\mathfrak{n}$ acts by $\mathbf{i}\xi_0$ in the following sense. Let F be the linear span of the set

$$\{X\cdot v-\mathbf{i}\xi_0(X)v \mid v\in E,\ X\in\mathfrak{n}\}$$

and let F^{cl} be the closure of F in E. Then F^{cl} is closed by the $M'N$-action and

$$\Psi(E)\cong E/F^{\mathrm{cl}}. \tag{3.2}$$

Take any $f\in\mathfrak{m}_{\xi_0}$ and $v\in E$ and consider the inverse Fourier transform $\mathcal{F}(f)$ of f. Calculate the vector fv by the definition of $\mathscr{S}(\mathfrak{n})$-action on E, i.e. the integration of $\mathcal{F}(f)(n)nv$ over $n\in N$. Since the projection $p\colon E\to E/F^{cl}$ respects the N-action and N acts by $e^{\mathbf{i}\xi_0}$ on E/F^{cl}, we have

$$p(fv)=\int_N \mathcal{F}(f)(n)e^{\mathbf{i}(\xi_0,n)}p(v)dn=f(\xi_0)p(v)=0.$$

Then, the map $E\to E/F^{\mathrm{cl}}$ factors through $\Psi(E)\to E/F^{\mathrm{cl}}$. On the other hand, since $\mathfrak{n}$ acts on $E/(\mathfrak{m}_{\xi_0}\cdot E)$ by $\mathbf{i}\xi_0$, we get a map $E/F^{\mathrm{cl}}\to\Psi(E)$, which is the inverse of the above map. Thus (3.2) follows.

For $V\in\mathcal{C}(G)$, write $V|_P\in\mathcal{C}(P)$ for the representation obtained by restriction of the action of G to P.

Lemma 3.4. *Let $V\in\mathcal{C}(G)$ and let F be a finite-dimensional representation of P. Then there exists an isomorphism of M'-representations:*

$$\Psi(V|_P)\otimes(F|_{M'})\cong\Psi((V\otimes F)|_P).$$

Proof. There exists a filtration $0=F_0\subset F_1\subset\cdots\subset F_k=F$ of P-subrepresentations such that N acts trivially on F_i/F_{i-1}. Then by (3.1) and Lemma 3.3 we get an exact sequence

$$0\to\Psi(V|_P)\otimes F_i/F_{i-1}\to\Psi(V|_P\otimes F_i)\to\Psi(V|_P\otimes F_i)$$

of M'-representations. By Remark 3.5 or Remark 3.11, these M'-representations are finite-dimensional and hence the sequence splits. We have

$$\Psi(V|_P)\otimes(F_i|_{M'})\cong\Psi(V|_P\otimes F_i)$$

inductively and we obtain the conclusion of the lemma. □

Remark 3.5. For $V\in\mathcal{C}(G)$, the dimension of $\Psi(V|_P)$ is finite and $\Psi(V|_P)\cong H_0(\mathfrak{n},V\otimes e^{-\mathbf{i}\xi_0})$, namely, the subspace F defined above (3.2) is closed. This is proved in [7, §8]. The exactness of Ψ for representations of G is also proved there. Vectors in the dual space of $\Psi(V|_P)$ are no other than Whittaker vectors.

We will apply Lemmas 3.2–3.4 to study the restriction of unitary representations. Let V be a non-trivial irreducible unitarizable $(\mathfrak{g},K)$-module. Write $\bar{V}$ for its Hilbert space completion and V^{sm} for the Casselman-Wallach globalization. By Proposition 2.2, an irreducible unitary representation of P is either finite-dimensional or equal to $I_{P,\tau}$ for an irreducible representation τ of M'. In the former case, it factors through $P/N(\cong MA)$. Hence these are parametrized by irreducible unitary representations $\sigma\otimes e^{\nu}$ of MA. Then by general theory, the restriction of $\bar{V}$ to P decomposes into irreducibles as

$$\bar{V}|_P\cong\int^{\oplus}(\sigma\otimes e^{\nu})^{m(\sigma,\nu)}d\mu\oplus\bigoplus_{\tau}(I_{P,\tau})^{m(\tau)}.$$

Here, the first term on the right hand side is a direct integral of irreducible unitary representations and the second term is a Hilbert space direct sum. We will show that actually the first term on the right hand side does not appear and the second term is a finite sum. Since any vector $v\in\int^{\oplus}(\sigma\otimes e^{\nu})^{m(\sigma,\nu)}d\mu$ is N-invariant,

$$V\ni v'\mapsto(v',v)\in\mathbb{C}$$

defines an $\mathfrak{n}$-invariant vector of the algebraic dual space V^*. Since it is known that $H^0(\mathfrak{n},V^*)$ is finite-dimensional [8, Corollary 2.4], $\int^{\oplus}(\sigma\otimes e^{\nu})^{m(\sigma,\nu)}d\mu$ is also finite-dimensional and in particular it only has a discrete spectrum. Suppose that $\sigma\otimes\nu$ appears in $\bar{V}|_P$ as a direct summand. Then by the Frobenius reciprocity, we obtain an intertwining map

$$V\hookrightarrow\mathrm{Ind}_P^G(\sigma\otimes e^{\nu}\otimes\mathbf{1}_N)\big(\cong\mathrm{Ind}_{\bar{P}}^G(\sigma\otimes e^{-\nu}\otimes\mathbf{1}_{\bar{N}})=I(\sigma,-\nu+\rho')\big).$$

Since $\nu\in\mathbf{i}\mathfrak{a}^*$, V is isomorphic to the unique irreducible subrepresentation of $I(\sigma,-\nu+\rho')$. By considering leading exponent of matrix coefficients of V, [9, Theorem 9.1.4] (which in turn is implied by a theorem of Howe-Moore in [25]) implies that $\nu=0$ and V is trivial, which is not the case. Hence, $\int^{\oplus}(\sigma\otimes e^{\nu})^{m(\sigma,\nu)}d\mu=0$.

Let $\bar{\tau}:=\sum_{\tau}^{\oplus}\tau^{m(\tau)}$ be the Hilbert direct sum. Let $\mathrm{Ind}_{M'N}^P(\bar{\tau}\otimes e^{\mathbf{i}\xi_0})$ be the unitarily (L^2-)induced representation. Then

$$\bar{V}|_P\cong\bigoplus_{\tau}(I_{P,\tau})^{m(\tau)}\cong\mathrm{Ind}_{M'N}^P(\bar{\tau}\otimes e^{\mathbf{i}\xi_0}).$$

Let $\bar{\tau}^{\infty}$ be the set of smooth vectors in $\bar{\tau}$ as a representation of M'. Then by the Sobolev embedding theorem (on $P/M'N \cong \mathbb{R}^m - \{0\}$), the smooth vectors in $\bar{V}$ lie in $C^{\infty}\operatorname{Ind}_{M'N}^{P}(\bar{\tau}^{\infty} \otimes e^{\mathbf{i}\xi_0})$. Hence we obtain an injective P-intertwining map

$$V^{\mathrm{sm}} \hookrightarrow C^{\infty}\operatorname{Ind}_{M'N}^{P}(\bar{\tau}^{\infty} \otimes e^{\mathbf{i}\xi_0}).$$

Now we apply Lemma 3.2 and use the denseness of V^{sm} in $\bar{V}$, we conclude that

$$\Psi(V^{\mathrm{sm}}|_P) \cong \bar{\tau}^{\infty}.$$

By Remark 3.5 or Remark 3.11, $\Psi(V^{\mathrm{sm}}|_P)$ is always finite-dimensional. Therefore, $\bar{\tau}^{\infty} = \bar{\tau}$. We thus obtain the following.

Lemma 3.6. *Suppose that V is a non-trivial irreducible unitarizable $(\mathfrak{g}, K)$-module. Then*

$$\bar{V}|_P \cong \operatorname{Ind}_{M'N}^{P}(\Psi(V^{\mathrm{sm}}|_P) \otimes e^{\mathbf{i}\xi_0}).$$

3.2. **Restrictions of principal series representations.** We defined the functor $\Psi\colon \mathcal{C}(P) \to \mathcal{C}(M')$ in the previous subsection. In this subsection we calculate $\Psi(\pi|_P)$ for a (not necessarily unitary) principal series representation π.

Proposition 3.7. *Let $I(\sigma,\nu) = \operatorname{Ind}_{\bar{P}}^{G}(V_\sigma \otimes e^{\nu-\rho'} \otimes \mathbf{1}_{\bar{N}})$ be a principal series representation. Then*

$$\Psi(I(\sigma,\nu)|_P) \cong V_\sigma|_{M'}.$$

Remark 3.8. Proposition 3.7 is proved in [7, Lemma 8.5] by using the Bruhat filtration. However, we include another proof below based on Fourier transform. This argument is of independent interest, especially it will be used for some concrete Fourier computation for discrete series in §3.6.

The main idea in our proof is to consider the restriction of the functions $f \in I(\sigma,\nu)$ to N and take inverse Fourier transform.

For $f \in I(\sigma,\nu)$, let $f_N = f|_N$. We have the map

$$I(\sigma,\nu) \to C^{\infty}(N, V_\sigma), \quad f \mapsto f_N.$$

The action of $P = MAN$ on $I(\sigma,\nu)$ is compatible with the following P-action on $C^{\infty}(N, V_\sigma)$: for $F \in C^{\infty}(N, V_\sigma)$ and $n \in N$,

$$\begin{aligned}
&(n' \cdot F)(n) = F(n'^{-1}n) \quad (n' \in N);\\
&(a \cdot F)(n) = e^{(\nu-\rho')\log a} F(a^{-1}na) \quad (a \in A);\\
&(m_0 \cdot F)(n) = \sigma(m_0) F(m_0^{-1} n m_0) \quad (m_0 \in M).
\end{aligned}$$

Next, define the inverse Fourier transform of $f_N \in C^{\infty}(N, V_\sigma)$. If f_N is L^1, then its inverse Fourier transform is defined as a function on $\mathfrak{n}^*$ as

$$\widehat{f_N}(\xi) = \mathcal{F}(f_N)(\xi) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{\mathbf{i}(\xi,x)} f(n_x)\,\mathrm{d}x. \tag{3.3}$$

In general, for $f \in I(\sigma,\nu)$, the function f_N on $N \cong \mathbb{R}^m$ is of at most polynomial growth at infinity. Hence f_N is a tempered distribution. The map (3.3) extends for tempered distributions and we obtain $\widehat{f_N}(\xi)$ as distributions on $\mathfrak{n}^*$. The action of G on the Fourier transformed picture is defined as

$$g(\widehat{f_N}) = \widehat{(gf)_N}$$

for $f \in I(\sigma,\nu)$ and $g \in G$. Then the P-action is given as follows: for $f \in I(\sigma,\nu)$ and $\xi \in \mathfrak{n}^*$,

$$\begin{aligned}
(n_x \cdot \widehat{f_N})(\xi) &= e^{\mathbf{i}(\xi,x)}\widehat{f_N}(\xi) \quad (x \in \mathbb{R}^m);\\
(a \cdot \widehat{f_N})(\xi) &= e^{(\nu+\rho')\log a}\widehat{f_N}(\mathrm{Ad}^*(a^{-1})\xi) \quad (a \in A);\\
(m_0 \cdot \widehat{f_N})(\xi) &= \sigma(m_0)\widehat{f_N}(\mathrm{Ad}^*(m_0^{-1})\xi) \quad (m_0 \in M).
\end{aligned}$$

To study the behavior of $\widehat{f_N}$, we need some preparation. For a row vector $0 \neq x \in \mathbb{R}^m$, write

$$r_x = I_m - \frac{2}{|x|^2}x^t x \in \mathrm{O}(m), \tag{3.4}$$

which is a reflection. The action of r_x on $\mathbb{R}^m$ is given by

$$r_x(y) = y - \frac{2yx^t}{|x|^2}x \quad (\forall y \in \mathbb{R}^m).$$

Let r_x also denote the element

$$\mathrm{diag}(r_x, I_2) \in G_2 = \mathrm{O}(m+1,1).$$

For $x \in \mathbb{R}^m$, write

$$s_x = \begin{pmatrix} I_m - \frac{2x^t x}{1+|x|^2} & -\frac{2x^t}{1+|x|^2} & 0 \\ \frac{2x}{1+|x|^2} & \frac{1-|x|^2}{1+|x|^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(m+1,1). \tag{3.5}$$

By a direct calculation, one shows the following opposite Iwasawa decomposition for elements in N:

$$n_x = s_x \exp(-\log(1+|x|^2)H_0)\bar{n}_{\frac{x}{1+|x|^2}}. \tag{3.6}$$

By the Bruhat decomposition,

$$G_2 = NM_2A\bar{N} \sqcup sM_2A\bar{N}.$$

If we write

$$s = \mathrm{diag}(I_m, -1, 1) \in \mathrm{O}(m+1,1), \tag{3.7}$$

$sn_x \in NM_2A\bar{N}$ for any $0 \neq x \in \mathbb{R}^m$. By direct calculation one shows the following decomposition. For $0 \neq x \in \mathbb{R}^m$, we have

$$sn_x = n_{\frac{x}{|x|^2}} r_x e^{-(2\log|x|)H_0}\bar{n}_{\frac{x}{|x|^2}}, \tag{3.8}$$

where r_x is as in (3.4).

Lemma 3.9. *Let $f \in I(\sigma,\nu)$. Then the restriction of $\widehat{f_N}$ to $\mathfrak{n}^* - \{0\}$ is a C^∞-function.*

Proof. We first prove a similar claim for the group $G_2 = \mathrm{O}(m+1,1)$. Let $I(\sigma,\nu)$ be a principal series representation of G_2 for an irreducible representation σ of M_2 and take $f \in I(\sigma,\nu)$. To prove that $\widehat{f_N}|_{\mathfrak{n}^*-\{0\}}$ is a smooth function, we need to see the behavior of $f(n_x)$ as $x \to \infty$. This is equivalent to the behavior of $f(sn_x)$ near $x = 0$, where $s = \mathrm{diag}(I_m,-1,1)$. Put $F(x) := f(sn_x)$ for $x \in \mathbb{R}^n$. By (3.8),

$$F(x) = f\big(n_{\frac{x}{|x|^2}} r_x e^{-(2\log|x|)H_0}\bar{n}_{\frac{x}{|x|^2}}\big) = |x|^{2(\nu-\rho')(H_0)}\sigma(r_x)f\big(n_{\frac{x}{|x|^2}}\big).$$

Since F is smooth, $|f(n_x)|$ is bounded by $C|x|^{2(-\nu+\rho')(H_0)}$ as $x \to \infty$ for some constant $C > 0$.

The G-action on $I(\sigma,\nu)$ differentiates to the $\mathfrak{g}$-action. Take $X_y \in \mathfrak{n}$ for $y \in \mathbb{R}^m$ and consider the function $X_y \cdot f \in I(\sigma,\nu)$. We have

$$(X_y \cdot f)(sn_x) = \frac{d}{dt}\Big|_{t=0} f(n_{ty}^{-1} s n_x).$$

By (3.8) again,

$$n_{ty}^{-1} s n_x = n_{-ty+\frac{x}{|x|^2}} r_x e^{-(2\log|x|)H_0} \bar{n}_{\frac{x}{|x|^2}}.$$

Putting $z := -ty + \frac{x}{|x|^2}$, we have

$$\begin{aligned} & n_{-ty+\frac{x}{|x|^2}} r_x e^{-(2\log|x|)H_0} \bar{n}_{\frac{x}{|x|^2}} \\ &= s n_{\frac{z}{|z|^2}} r_z e^{-(2\log|z|)H_0} \bar{n}_{\frac{z}{|z|^2}} r_x e^{-(2\log|x|)H_0} \bar{n}_{\frac{x}{|x|^2}} \\ &\in s n_{\frac{z}{|z|^2}} r_z r_x e^{-(2\log|z|+2\log|x|)H_0} \bar{N}. \end{aligned}$$

Hence

$$(X_y \cdot f)(sn_x) = \frac{d}{dt}\Big|_{t=0} |z|^{2(\nu-\rho')(H_0)} |x|^{2(\nu-\rho')(H_0)} \sigma(r_z r_x) F\Big(\frac{z}{|z|^2}\Big).$$

Note that $r_z = r_{-t|x|^2 y + x}$. We calculate

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} |z|^{2(\nu-\rho')(H_0)} |x|^{2(\nu-\rho')(H_0)} = -2(\nu-\rho')(H_0)(y,x), \\ &\frac{d}{dt}\Big|_{t=0} r_z r_x = 2(y^t x - x^t y), \\ &\frac{d}{dt}\Big|_{t=0} F\Big(\frac{z}{|z|^2}\Big) = \big(2(x,y)x - y|x|^2\big)(\nabla_y F)(x). \end{aligned}$$

Combining above equations, we see that if $F(x)$ vanishes at $x = 0$ of order k, then $(X_y \cdot f)(sn_x)$ vanishes at $x = 0$ of order $k+1$. Hence $(X_y \cdot f)(n_x)$ is bounded by $C|x|^{2(-\nu+\rho')(H_0)-1}$ for some C. By repeating this, $(X_y^l \cdot f)(n_x)$ is bounded by $C|x|^{2(-\nu+\rho')(H_0)-l}$ for $l > 0$. Then for any $k > 0$, if l is sufficiently large, then $P(x)(X_y^l \cdot f)(n_x)$ is in L^1 for any polynomial $P(x)$ of degree k. Therefore, its inverse Fourier transform is continuous. By

$$\mathcal{F}(x_j h) = -\mathbf{i}\partial_{\xi_j}\mathcal{F}(h), \quad \mathcal{F}(\partial_{x_j} h) = -\mathbf{i}\xi_j \mathcal{F}(h), \tag{3.9}$$

we have

$$\mathcal{F}(P(x)(X_y^l \cdot f)(n_x)) = P(-\mathbf{i}\partial_\xi)(-\mathbf{i}\xi, y)^l \cdot \widehat{f_N}(\xi).$$

Hence $\widehat{f_N}(\xi)$ is C^k in $(\xi, y) \neq 0$. Since k and y are arbitrary, we proved that $\widehat{f_N}$ is C^∞ on $\mathfrak{n}^* - \{0\}$.

To prove the claim for $G = \mathrm{Spin}(m+1,1)$, fix $m_0 \in M_2$ such that $m_0 s \in M_1 = \mathrm{SO}(m)$ and use a lift of $m_0 s$ in $M = \mathrm{Spin}(m)$ instead of s in the above argument. □

Recall $\xi_0 = (0,\dots,0,1) \in \mathfrak{n}^*$. For $h \in C^\infty(\mathfrak{n}^* - \{0\}, V_\sigma)$, define a function $h_{at,\nu}$ on P by

$$h_{at,\nu}(p) = (p^{-1} \cdot h)(\xi_0) \quad (p \in P).$$

More concretely,

$$\begin{aligned}&h_{at,\nu}(p)\\&= e^{-\mathbf{i}(\xi_0,x)}e^{(-\nu-\rho')\log a}\sigma(m_0)^{-1}h(\mathrm{Ad}^*(m_0a)\xi_0)\\&= e^{-\mathbf{i}(\xi_0,x)}|\,\mathrm{Ad}^*(m_0a)(\xi_0)|^{\frac{2\nu(H_0)+m}{2}}\sigma(m_0)^{-1}h(\mathrm{Ad}^*(m_0a)\xi_0)\end{aligned}$$

for $p = m_0an_x \in P$. We call $h_{at,\nu}$ the anti-trivialization of h. The term 'anti-trivialization' comes from: h is a function on $\mathfrak{n}^* - \{0\}$, i.e., a section of the trivial bundle on $P/M'N \cong \mathfrak{n}^* - \{0\}$, and $h_{at,\nu}$ is a section of the vector bundle $P \times_{M'N} (\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$ on $P/M'N$.

Lemma 3.10. *The image of the map $C^\infty(\mathfrak{n}^* - \{0\}, V_\sigma) \ni h \mapsto h_{at,\nu}$ is equal to the representation space of the smoothly induced representation $C^\infty\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$. The map $h \mapsto h_{at,\nu}$ respects the actions of P and $\mathscr{S}(\mathfrak{n}^* - \{0\})$.*

Proof. For any $m' \in M'$, $n_x \in N$ and $p \in P$, we have

$$\begin{aligned}&h_{at,\nu}(pm'n_x)\\&= (n_x^{-1}(m')^{-1}p^{-1}\cdot h)(\xi_0)\\&= e^{-\mathbf{i}(\xi_0,x)}\sigma(m')^{-1}(p^{-1}\cdot h)(\mathrm{Ad}^*(m')\xi_0)\\&= (\sigma \otimes e^{\mathbf{i}\xi_0})(m', n_x)^{-1}h_{at,\nu}(p),\end{aligned}$$

where we used $\mathrm{Ad}^*(m')\xi_0 = \xi_0$. This shows that $h_{at,\nu}$ is a section of the vector bundle $P \times_{M'N} (\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$.

It directly follows from the definition of $h_{at,\nu}$ that the map $h \mapsto h_{at,\nu}$ respects the P-actions. The actions of $\mathscr{S}(\mathfrak{n}^* - \{0\})$ on $C^\infty\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$ and $C^\infty(\mathfrak{n}^* - \{0\}, V_\sigma)$ are given by multiplications. Hence the map $h \mapsto h_{at,\nu}$ is a $\mathscr{S}(\mathfrak{n}^* - \{0\})$-homomorphism.

The inverse map

$$C^\infty\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0}) \to C^\infty(\mathfrak{n}^* - \{0\}, V_\sigma), \quad h' \mapsto h'_{t,\nu}$$

is given as follows: for any $\xi \in \mathfrak{n}^* - \{0\}$, choose $m_0a \in MA$ such that $\xi = \mathrm{Ad}^*(m_0a)\xi_0$ and define

$$h'_{t,\nu}(\xi) = |\xi|^{-\frac{m+2\nu(H_0)}{2}}\sigma(m_0)h'(m_0a).$$

It is easy to see that the maps $h \mapsto h_{at,\nu}$ and $h' \mapsto h'_{t,\nu}$ are inverse to each other. □

Proof of Proposition 3.7. By Lemmas 3.9 and 3.10, we obtain a map

$$\varphi\colon I(\sigma, \nu) \to C^\infty\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0}), \quad f \mapsto (\widehat{f_N})_{at,\nu}$$

which respects the actions of P and $\mathscr{S}(\mathfrak{n}^* - \{0\})$. If $f \in \mathrm{Ker}\,\varphi$, then the support of $\widehat{f_N}$ is contained in $\{0\}$, or equivalently, f_N is a polynomial. Hence $\mathscr{S}(\mathfrak{n}^* - \{0\}) \cdot (\mathrm{Ker}\,\varphi) = 0$ and $\Psi(I(\sigma,\nu)|_P) \cong \Psi(I(\sigma,\nu)|_P/\mathrm{Ker}\,\varphi)$. Then by Lemma 3.2, there exists an injective map $\Psi(I(\sigma,\nu)|_P) \to \sigma|_{M'}$. To show the surjectivity, take any vector $v \in \sigma|_{M'}$ and take a function $h \in \mathscr{S}\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$ (or a function $h \in C^\infty\,\mathrm{Ind}_{M'N}^P(\sigma|_{M'} \otimes e^{\mathbf{i}\xi_0})$ which is compactly supported modulo $M'N$) such that $h(e) = v$. Then there exists $f \in I(\sigma,\nu)$ such that $(\widehat{f_N})_{at,\nu} = h$, which implies that the map $\Psi(I(\sigma,\nu)|_P) \to \sigma|_{M'}$ is surjective. □

Remark 3.11. By Casselman's subrepresentation theorem, any moderate growth, irreducible admissible smooth Fréchet representation π of G is a subrepresentation of a principal series representation $I(\sigma,\nu)$. Then by Lemma 3.3 and Proposition 3.7, $\Psi(\pi|_P) \subset \Psi(I(\sigma,\nu)|_P) \cong \sigma|_{M'}$. In particular, $\Psi(\pi|_P)$ is finite-dimensional.

Let $K(G)$ (resp. $K(M')$) be the Grothendieck group of the category of Harish-Chandra modules (resp. the category of finite-dimensional representations of M'). By Lemma 3.3, $\mathcal{C}(G) \ni \pi \mapsto \Psi(\pi|_P)$ induces a homomorphism $\Psi\colon K(G) \to K(M')$.

3.3. **Classification of irreducible representations of G.** In this subsection we recall the classification of irreducible admissible representations $\pi \in \mathcal{C}(G)$.

Suppose first $G = \mathrm{Spin}(2n,1)$. The infinitesimal character γ of π is conjugate to

$$(\mu + \rho_M, \nu) = \Big(a_1 + n - \frac{3}{2}, \cdots, a_{n-1} + \frac{1}{2}, a\Big),$$

where $\mu = (a_1, \ldots, a_{n-1})$ and $\nu = a\lambda_0$. We have $a_1 \geq \cdots \geq a_{n-1} \geq 0$; and $a_1, \ldots, a_{n-1}$ are all integers or all half-integers. The weight γ is integral if and only if $a - (a_j + \frac{1}{2}) \in \mathbb{Z}$. The singularity of integral γ has several possibilities:

(1) If $a \neq a_j + n - j - \frac{1}{2}$ for $1 \leq j \leq n-1$ and $a \neq 0$, then γ is regular. Write Λ_0 for the set of integral regular dominant weights.

(2) If $a = a_j + n - j - \frac{1}{2}$ for some $1 \leq j \leq n-1$, then up to conjugation

$$\gamma = \Big(a_1 + n - \frac{3}{2}, \ldots, a_j + n - j - \frac{1}{2}, a_j + n - j - \frac{1}{2}, \ldots, a_{n-1} + \frac{1}{2}\Big),$$

and $\alpha_j = \epsilon_j - \epsilon_{j+1}$ is the only simple root orthogonal to γ. Write Λ_j for the set of such integral dominant weights.

(3) If $a = 0$, then

$$\gamma = \Big(a_1 + n - \frac{3}{2}, \ldots, a_{n-1} + \frac{1}{2}, 0\Big).$$

a_j $(1 \leq j \leq n-1)$ are half-integers, and $\alpha_n = \epsilon_n$ is the only simple root orthogonal to γ. Write Λ_n for the set of such integral dominant weights.

To describe irreducible representations with the infinitesimal character γ, we introduce several notation for every type of γ.

For a weight

$$\gamma = \Big(a_1 + n - \frac{1}{2}, \cdots, a_{n-1} + \frac{3}{2}, a_n + \frac{1}{2}\Big) \in \Lambda_0$$

with $a_1 \geq \cdots \geq a_{n-1} \geq a_n \geq 0$, let

$$\mu_j = (a_1 + 1, \cdots, a_j + 1, a_{j+2}, \cdots, a_n) \text{ and } \nu_j = \Big(a_{j+1} + n - \frac{1}{2} - j\Big)\lambda_0$$

for $0 \leq j \leq n-1$. Put

$$I_j^{\pm}(\gamma) = I(\mu_j, \pm\nu_j) = \mathrm{Ind}_{MA\bar{N}}^G(V_{M,\mu_j} \otimes e^{\pm\nu_j - \rho'} \otimes \mathbf{1}_{\bar{N}}).$$

For each j, there are non-zero intertwining operators

$$J_j^{+}(\gamma)\colon I_j^{+}(\gamma) \to I_j^{-}(\gamma) \text{ and } J_j^{-}(\gamma)\colon I_j^{-}(\gamma) \to I_j^{+}(\gamma).$$

Write $\pi_j(\gamma)$ (resp. $\pi'_j(\gamma)$) for the image of $J_j^{-}(\gamma)$ (resp. $J_j^{+}(\gamma)$). Put

$$\lambda^+ = (a_1 + 1, \ldots, a_{n-1} + 1, a_n + 1) \text{ and } \lambda^- = (a_1 + 1, \ldots, a_{n-1} + 1, -(a_n + 1)).$$

Write $\pi^+(\gamma)$ for the discrete series with the lowest K-type V_{K,λ^+}, and write $\pi^-(\gamma)$ for the discrete series with the lowest K-type V_{K,λ^-}.

Let $1 \leq j \leq n-1$. For a weight

$$\gamma = \Big(a_1 + n - \frac{3}{2}, \ldots, a_j + n - j - \frac{1}{2}, a_j + n - j - \frac{1}{2}, \ldots, a_{n-1} + \frac{1}{2}\Big) \in \Lambda_j,$$

write

$$\mu = (a_1, \ldots, a_{n-1}) \text{ and } \nu = \Big(a_j + n - j - \frac{1}{2}\Big)\lambda_0.$$

Put

$$\pi(\gamma) = I(\mu, \nu).$$

For a weight

$$\gamma = \Big(a_1 + n - \frac{3}{2}, a_2 + n - \frac{5}{2}, \ldots, a_{n-1} + \frac{1}{2}, 0\Big) \in \Lambda_n,$$

write

$$\mu = (a_1, \ldots, a_{n-1}) \text{ and } I(\gamma) = I(\mu, 0).$$

By Schmid's identity [28, Theorem 12.34] $I(\gamma)$ is the direct sum of two limits of discrete series [28, Theorem 12.26]. Write $\pi^+(\gamma)$ (resp. $\pi^-(\gamma)$) for a limit of discrete series with the lowest K-type V_{K,λ^+} (resp. V_{K,λ^-}), where

$$\lambda^+ = \Big(a_1, a_2, \ldots, a_{n-1}, \frac{1}{2}\Big) \text{ and } \lambda^- = \Big(a_1, a_2, \ldots, a_{n-1}, -\frac{1}{2}\Big).$$

For a non-integral weight

$$\gamma = \Big(a_1 + n - \frac{3}{2}, \cdots, a_{n-1} + \frac{1}{2}, a\Big),$$

write

$$\mu = (a_1, \ldots, a_{n-1}) \text{ and } \nu = a\lambda_0.$$

Put

$$\pi(\gamma) = I(\mu, \nu).$$

Note that $I(\mu, \nu) \cong I(\mu, -\nu)$.

Using the above notation, the Langlands classification of irreducible representations of G is given as follows. In Fact 3.12, an irreducible representation of G means an irreducible, moderate growth, smooth Fréchet representation.

Fact 3.12.

(1) For $\gamma \in \Lambda_0$, any irreducible representation of G with infinitesimal character γ is equivalent to one of

$$\{\pi_0(\gamma), \ldots, \pi_{n-1}(\gamma), \pi^+(\gamma), \pi^-(\gamma)\}.$$

When $0 \leq j \leq n-2$, $\pi_{j+1}(\gamma) \cong \pi'_j(\gamma)$; $\pi_0(\gamma)$ is a finite-dimensional module; and $\pi'_{n-1}(\gamma) \cong \pi^+(\gamma) \oplus \pi^-(\gamma)$.

(2) For $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$, any irreducible representation of G with infinitesimal character γ is equivalent to $\pi(\gamma)$.

(3) For $\gamma \in \Lambda_n$, any irreducible representation of G with infinitesimal character γ is equivalent to $\pi^+(\gamma)$ or $\pi^-(\gamma)$.

(4) For a non-integral weight γ, any irreducible representation of G with infinitesimal character γ is equivalent to $\pi(\gamma)$.

Among these representations, unitarizable ones are given as follows [24].

Fact 3.13.

(1) For $\gamma \in \Lambda_0$, $\pi^+(\gamma)$ and $\pi^-(\gamma)$ are unitarizable (discrete series). $\pi_j(\gamma)$ is unitarizable if and only if $a_i = 0$ for any $j < i \leq n$.
(2) For $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$, $\pi(\gamma)$ is unitarizable if and only if $a_i = 0$ for any $j \leq i \leq n-1$.
(3) For $\gamma \in \Lambda_n$, $\pi^+(\gamma)$ and $\pi^-(\gamma)$ are unitarizable (limit of discrete series).

Fact 3.14. For a non-integral weight γ, $\pi(\gamma)$ is unitarizable if and only if at least one of the following two conditions holds.

(1) $a \in \mathbf{i}\mathbb{R}$ (unitary principal series).
(2) $a \in \mathbb{R}$, $|a| < n - \frac{1}{2}$, $a_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$ and $a_j = 0$ for any $n - |a| - \frac{1}{2} < j \leq n-1$ (complementary series).

Remark 3.15. The unitarizable representations $\pi(\gamma)$ for $\gamma \in \Lambda_j$ in Fact 3.13(2) can be regarded as a complementary series and also as $A_{\mathfrak{q}}(\lambda)$ as we see below.

The unitarizable $(\mathfrak{g}, K)$-modules with integral infinitesimal character are isomorphic to Vogan-Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$. General references for $A_{\mathfrak{q}}(\lambda)$ are e.g. [29], [53]. For $0 \leq j \leq n-1$ let $\mathfrak{q}_j$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that the real form of the Levi component of $\mathfrak{q}_j$ is isomorphic to $\mathfrak{u}(1)^j + \mathfrak{so}(2(n-j), 1)$. For the normalization of parameters, we follow the book of Knapp-Vogan [29]. In particular, $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$.

Remark 3.16. The parameter $\lambda = (a_1, \ldots, a_j, 0, \ldots, 0)$ for $\mathfrak{q}_j$ is in the good range if and only if $a_1 \geq a_2 \geq \cdots \geq a_j \geq 0$. It is in the weakly fair range if and only if $a_i + 1 \geq a_{i+1}$ for $1 \leq i \leq j-1$ and $a_j \geq -n+j$. When λ is in the weakly fair range, $A_{\mathfrak{q}}(\lambda)$ is non-zero if and only if $a_1 \geq \cdots \geq a_j$ and $a_{j-1} \geq -1$.

Let $\gamma \in \Lambda_0$ and $0 \leq j \leq n-1$ such that $\pi_j(\gamma)$ is unitarizable. Then

$$\pi_j(\gamma)_K \cong A_{\mathfrak{q}_j}(\lambda),$$

where $\lambda = (a_1, \ldots, a_j, 0, \ldots, 0)$.

Let $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$ and $1 \leq i \leq j$. Assume that $a_i = \cdots = a_n = 0$. Then

$$\pi(\gamma)_K \cong A_{\mathfrak{q}_i}(\lambda),$$

where $\lambda = (a_1 - 1, \ldots, a_{i-1} - 1, i - j - 1, 0, \ldots, 0)$.

Suppose next that m is even and then $m = 2n-2$ and $G = \mathrm{Spin}(2n-1, 1)$. This case is similar to and simpler than the previous case. The infinitesimal character γ of $\pi \in \mathcal{C}(G)$ is conjugate to

$$(\mu + \rho_M, \nu) = (a_1 + n - 2, a_2 + n - 3, \cdots, a_{n-1}, a),$$

where $\mu = (a_1, \ldots, a_{n-1})$ and $\nu = a\lambda_0$. We have $a_1 \geq \cdots \geq a_{n-1} \geq 0$; and $a_1, \ldots, a_{n-1}$ are all integers or all half-integers. The weight γ is integral if and only if $a - a_j \in \mathbb{Z}$. The singularity of integral γ has the following possibilities:

(1) If $a \neq a_j + n - j - 1$ for $1 \leq j \leq n-1$, then γ is regular. Write Λ_0 for the set of integral regular dominant weights.
(2) If $a = a_j + n - j - 1$ for some $1 \leq j \leq n-1$, then up to conjugation

$$\gamma = (a_1 + n - 2, \ldots, a_j + n - j - 1, a_j + n - j - 1, \ldots, a_{n-1}).$$

Write Λ_j for the set of such integral dominant weights.

We introduce several notation for every type of γ.

For a weight

$$\gamma = (a_1 + n - 1, a_2 + n - 2, \cdots, a_{n-1} + 1, a_n) \in \Lambda_0$$

with $a_1 \geq \cdots \geq a_{n-1} \geq a_n \geq 0$, let

$$\mu_j = (a_1 + 1, \cdots, a_j + 1, a_{j+2}, \cdots, a_n) \text{ and } \nu_j = (a_{j+1} + n - j - 1)\lambda_0$$

for $0 \leq j \leq n-1$. Put

$$I_j^{\pm}(\gamma) = I(\mu_j, \pm\nu_j).$$

For each j, there are non-zero intertwining operators

$$J_j^+(\gamma)\colon I_j^+(\gamma) \to I_j^-(\gamma) \text{ and } J_j^-(\gamma)\colon I_j^-(\gamma) \to I_j^+(\gamma).$$

Write $\pi_j(\gamma)$ (resp. $\pi'_j(\gamma)$) for the image of $J_j^-(\gamma)$ (resp. $J_j^+(\gamma)$).

Let $1 \leq j \leq n-1$. For a weight

$$\gamma = (a_1 + n - 2, \ldots, a_j + n - j - 1, a_j + n - j - 1, \ldots, a_{n-1}) \in \Lambda_j,$$

write

$$\mu = (a_1, \ldots, a_{n-1}) \text{ and } \nu = (a_j + n - j - 1)\lambda_0.$$

Put

$$\pi(\gamma) = I(\mu, \nu).$$

For a non-integral weight

$$\gamma = (a_1 + n - 2, a_2 + n - 3, \cdots, a_{n-1}, a),$$

write

$$\mu = (a_1, \ldots, a_{n-1}) \text{ and } \nu = a\lambda_0.$$

Put

$$\pi(\gamma) = I(\mu, \nu).$$

Note that $I(\mu, \nu) \cong I(\mu, -\nu)$.

Using these notation, the Langlands classification is given as follows.

Fact 3.17.

(1) For $\gamma \in \Lambda_0$, any irreducible representation of G with infinitesimal character γ is equivalent to one of

$$\{\pi_0(\gamma), \ldots, \pi_{n-1}(\gamma)\}.$$

When $0 \leq j \leq n-2$, $\pi_{j+1}(\gamma) \cong \pi'_j(\gamma)$; $\pi_{n-1}(\gamma) \cong \pi'_{n-1}(\gamma)$; and $\pi_0(\gamma)$ is a finite-dimensional module. If $a_n = 0$, then $\pi_{n-1}(\gamma)$ is tempered.

(2) For $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$, any irreducible representation of G with infinitesimal character γ is equivalent to $\pi(\gamma)$.

(3) For a non-integral weight γ, any irreducible representation of G with infinitesimal character γ is equivalent to $\pi(\gamma)$.

Among these representations, unitarizable ones are given as follows [24].

Fact 3.18.

(1) For $\gamma \in \Lambda_0$, $\pi_j(\gamma)$ is unitarizable if and only if $a_i = 0$ for any $j < i \leq n$.

(2) For $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$, $\pi(\gamma)$ is unitarizable if and only if $a_i = 0$ for any $j \leq i \leq n-1$.

Fact 3.19. For a non-integral weight γ, $\pi(\gamma)$ is unitarizable if and only if at least one of the following two conditions holds.

(1) $a \in \mathbf{i}\mathbb{R}$ (unitary principal series).
(2) $a \in \mathbb{R}$, $|a| < n-1$, $a_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$ and $a_j = 0$ for any $n - |a| - 1 < j \leq n-1$ (complementary series).

For $0 \leq j \leq n-1$, let $\mathfrak{q}_j$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that the real form of the Levi component of $\mathfrak{q}_j$ is isomorphic to $\mathfrak{u}(1)^j + \mathfrak{so}(2(n-j)-1,1)$.

Remarks 3.15 and 3.16 are valid without change of words.

Let $\gamma \in \Lambda_0$ and $0 \leq j \leq n-1$ such that $\pi_j(\gamma)$ is unitarizable. Then

$$\pi_j(\gamma)_K \cong A_{\mathfrak{q}_j}(\lambda),$$

where $\lambda = (a_1, \ldots, a_j, 0, \ldots, 0)$.

Let $\gamma \in \Lambda_j$ $(1 \leq j \leq n-1)$ and $1 \leq i \leq j$. Assume that $a_i = \cdots = a_n = 0$. Then

$$\pi(\gamma)_K \cong A_{\mathfrak{q}_i}(\lambda),$$

where $\lambda = (a_1 - 1, \ldots, a_{i-1} - 1, i - j - 1, 0, \ldots, 0)$.

3.4. **Branching laws for $G = \mathrm{Spin}(2n,1)$.** Let $G = \mathrm{Spin}(2n,1)$. In this subsection, we deduce the branching law of $\pi|_P$ for all irreducible unitary representations π of G. A similar result for the group $G = \mathrm{Spin}(2n-1,1)$ will be given in the next subsection.

By Fact 3.13, many of irreducible unitary representations of G are the completion of principal series representations. This is the case if the infinitesimal character γ lies in Λ_j $(1 \leq j \leq n-1)$ or γ is not integral.

Theorem 3.20. *Suppose that an irreducible unitary representation π of Spin$(2n,1)$ is isomorphic to the completion of a principal series representation $I(\mu,\nu)$, where $\mu = (a_1, \ldots, a_{n-1})$ and $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$. Then*

$$\pi|_P \cong \bigoplus_{\tau} \mathrm{Ind}_{M'N}^{P}(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1, \ldots, b_{n-1})$ runs over tuples such that

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_{n-1} \geq |b_{n-1}|$$

and $b_i - a_1 \in \mathbb{Z}$.

Proof. By Lemma 3.6 and Proposition 3.7, the theorem follows from the well-known branching law from $M = \mathrm{Spin}(2n-1)$ to $M' = \mathrm{Spin}(2n-2)$ (see e.g. [19, Theorem 8.1.3]). □

Next, let $\gamma \in \Lambda_0$, namely, γ is a regular integral weight. Recall that in §3.3 we defined $\pi_j(\gamma)$ to be the image of the intertwining operator $J_j^-(\gamma)\colon I_j^-(\gamma) \to I_j^+(\gamma)$. We give branching laws for $\pi_j(\gamma)$ for $1 \leq j \leq n-1$ when it is unitarizable.

Theorem 3.21. *Let $1 \leq j \leq n-1$ and let*

$$\gamma = \Big(a_1 + n - \frac{1}{2}, \ldots, a_j + n - j + \frac{1}{2}, n - j - \frac{1}{2}, \ldots, \frac{1}{2}\Big),$$

where $a_1 \geq \cdots \geq a_j \geq 0$ are integers. Then

$$\bar{\pi}_j(\gamma)|_P \cong \bigoplus_{\tau} \mathrm{Ind}_{M'N}^{P}(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1, \ldots, b_{j-1}, 0, \ldots, 0)$ runs over tuples of integers such that

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq b_2 \geq \cdots \geq a_{j-1} + 1 \geq b_{j-1} \geq a_j + 1.$$

Proof. Let $0 \leq i < j$. It is known that $[\pi_i(\gamma)] + [\pi_{i+1}(\gamma)] = [I_i^+(\gamma)]$ in the Grothendieck group $K(G)$. Hence by Proposition 3.7

$$\Psi([\pi_i(\gamma)]) + \Psi([\pi_{i+1}(\gamma)]) = [V_{M,\mu_i}|_{M'}],$$

where $\mu_i = (a_1{+}1, \dots, a_i{+}1, a_{i+2}, \dots, a_j, 0, \dots, 0)$. Since $\pi_0(\gamma)$ is finite-dimensional, $\Psi([\pi_0(\gamma)]) = 0$. Then by induction on i, we have

$$\Psi([\pi_i(\gamma)]) = \bigoplus_\tau [V_{M',\tau}],$$

where $\tau = (b_1, \dots, b_{j-1}, 0, \dots, 0)$ runs over tuples of integers such that

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq \cdots \geq b_{i-1} \geq a_i + 1, \text{ and}$$
$$a_{i+1} \geq b_i \geq a_{i+2} \geq b_{i+1} \geq \cdots \geq a_j \geq |b_{j-1}|.$$

Hence the theorem follows from Lemma 3.6. □

We have the following formula for $A_{\mathfrak{q}}(\lambda)$ by Theorems 3.20 and 3.21. For $0 \leq j \leq n-1$ let $\mathfrak{q}_j$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that the real form of the Levi component of $\mathfrak{q}_j$ is isomorphic to $\mathfrak{u}(1)^j + \mathfrak{so}(2(n-j),1)$. For a weakly fair parameter $\lambda = (a_1, \dots, a_j, 0, \dots, 0)$, we have

$$\overline{A_{\mathfrak{q}_j}(\lambda)}|_P \cong \bigoplus_\tau \operatorname{Ind}_{M'N}^P(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1, \dots, b_{j-1}, 0, \dots, 0)$ runs over tuples of integers such that

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq b_2 \geq \cdots \geq a_{j-1} + 1 \geq b_{j-1} \geq \max\{a_j + 1, 0\}.$$

The remaining representations are (limit of) discrete series representations. The proof of Theorem 3.22 is based on the translation principle and the branching law of two special discrete series $\pi^+(\rho)$ and $\pi^+(\rho)$. The branching laws of $\pi^\pm(\rho)|_P$ are Proposition 3.27, which will be proved at the end of this section.

Theorem 3.22. *Let*

$$\gamma = \Big(a_1 + n - \frac{1}{2}, a_2 + n - \frac{3}{2}, \dots, a_n + \frac{1}{2}\Big),$$

where $a_1 \geq \cdots \geq a_n \geq -\frac{1}{2}$ *are all integers or all half-integers. Then*

$$\bar{\pi}^\pm(\gamma)|_P \cong \bigoplus_\tau \operatorname{Ind}_{M'N}^P(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1, \dots, b_{n-1})$ *runs over tuples such that*

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq \cdots \geq b_{n-2} \geq a_{n-1} + 1 \geq \mp b_{n-1} \geq a_n + 1$$

and $b_i - a_1 \in \mathbb{Z}$.

Proof. By Lemma 3.6, it suffices to calculate $\Psi([\pi^\pm(\gamma)])$ for $\gamma \in \Lambda_0 \sqcup \Lambda_n$.

The same argument as in the proof of Theorem 3.21 yields

$$\Psi([\pi^+(\gamma)]) + \Psi([\pi^-(\gamma)]) = \Psi([\pi'_{n-1}(\gamma)]) = \bigoplus_\tau [V_{M',\tau}], \tag{3.10}$$

where $\tau = (b_1, \dots, b_{n-1})$ runs over tuples of integers such that

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq \cdots \geq b_{n-2} \geq a_{n-2} + 1 \geq |b_{n-1}| \geq a_{n-1} + 1.$$

Therefore, it suffices to show that the two modules $\Psi([\pi^+(\gamma)])$ and $\Psi([\pi^-(\gamma)])$ are separated by the sign of b_{n-1}.

First, prove the statement for $\gamma \in \Lambda_0$ by induction on $|\gamma|$. When $\gamma = \rho$, the conclusion follows from Proposition 3.27. Let $\gamma \notin \Lambda_0 - \{\rho\}$ and assume that the conclusion holds for weights in Λ_0 having norm strictly smaller than $|\gamma|$. Write ω_k for the k-th fundamental weight, namely,

$$\omega_k = (\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{n-k}) \text{ for } 1 \leq k \leq n-1 \text{ and } \omega_n = \underbrace{\Big(\frac{1}{2},\ldots,\frac{1}{2}\Big)}_{n}.$$

Then, one finds $\gamma' \in \Lambda_0$ and a fundamental weight ω_k such that $\gamma = \gamma' + \omega_k$. By the Zuckerman translation principle ([53], [56]), $\pi^{\pm}(\gamma)$ occurs as a composition factor of $\pi^{\pm}(\gamma') \otimes F_{\omega_k}$. Hence by Lemma 3.4, if an irreducible M'-representation $V_{M',\mu}$ with $\mu = (b_1,\ldots,b_{n-1})$ occurs in $\Psi([\pi^+(\gamma)])$, then it also occurs in $\Psi([\pi^+(\gamma')]) \otimes [F_{\omega_k}|_{M'}]$. For any irreducible $V_{M',\mu'}$ in $\Psi([\pi^+(\gamma')])$ with $\mu' = (b'_1,\ldots,b'_{n-1})$, one has $b'_{n-1} \leq -1$ by induction hypothesis and for any weight μ'' appearing in $F_{\omega_k}|_{M'}$ with $\mu'' = (b''_1,\ldots,b''_{n-1})$, we have $b''_{n-1} \in \{1,-1,\frac{1}{2},-\frac{1}{2}\}$. Hence $b_{n-1} \leq 0$. Therefore, we get $b_{n-1} \leq -1$ from (3.10). The statement for $\pi^-(\gamma)$ is similarly proved.

Next, suppose that $\gamma \in \Lambda_n$. Let $\gamma' = \gamma + \omega_n \in \Lambda_0$. Then again by the translation principle, $\pi^{\pm}(\gamma)$ occurs as a composition factor of $\pi^{\pm}(\gamma') \otimes F_{\omega_n}$. Then by using the result for $\Psi([\pi^{\pm}(\gamma')])$ proved above, the statement for $\Psi([\pi^{\pm}(\gamma)])$ is similarly obtained. □

3.5. **Branching laws for $G = \mathrm{Spin}(2n-1,1)$.** Let $G = \mathrm{Spin}(2n-1,1)$. Branching laws for the restriction to P are similar to the previous case where $G = \mathrm{Spin}(2n,1)$.

Theorem 3.23. *Suppose that an irreducible unitary representation π of Spin$(2n-1,1)$ is isomorphic to the completion of a principal series representation $I(\mu,\nu)$, where $\mu = (a_1,\ldots,a_{n-1})$ and $a_1 \geq \cdots \geq a_{n-2} \geq |a_{n-1}|$. Then*

$$\pi|_P \cong \bigoplus_{\tau} \mathrm{Ind}_{M'N}^{P}(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1,\ldots,b_{n-2})$ runs over tuples such that

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_{n-1} \geq b_{n-2} \geq |a_{n-1}|$$

and $b_i - a_1 \in \mathbb{Z}$.

Theorem 3.24. *Let*

$$\gamma = (a_1+n-1,\ldots,a_j+n-j,n-j-1,\ldots,0),$$

where $a_1 \geq \cdots \geq a_j \geq 0$ are integers and let $1 \leq j \leq n-1$. Then

$$\bar{\pi}_j(\gamma)|_P \cong \bigoplus_{\tau} \mathrm{Ind}_{M'N}^{P}(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1,\ldots,b_{j-1},0,\ldots,0)$ runs over tuples of integers such that

$$a_1+1 \geq b_1 \geq a_2+1 \geq b_2 \geq \cdots \geq a_{j-1}+1 \geq b_{j-1} \geq a_j+1.$$

For $0 \leq j \leq n-1$ let $\mathfrak{q}_j$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that the real form of the Levi component of $\mathfrak{q}_j$ is isomorphic to $\mathfrak{u}(1)^j + \mathfrak{so}(2(n-j)-1,1)$. For a weakly fair parameter $\lambda = (a_1,\ldots,a_j,0,\ldots,0)$, we have

$$\overline{A_{\mathfrak{q}_j}(\lambda)}|_P \cong \bigoplus_{\tau} \mathrm{Ind}_{M'N}^{P}(V_{M',\tau} \otimes e^{\mathbf{i}\xi_0}),$$

where $\tau = (b_1, \dots, b_{j-1}, 0, \dots, 0)$ runs over tuples of integers such that

$$a_1 + 1 \geq b_1 \geq a_2 + 1 \geq b_2 \geq \cdots \geq a_{j-1} + 1 \geq b_{j-1} \geq \max\{a_j + 1, 0\}.$$

3.6. **Fourier transform for discrete series representations.** Let $G = \mathrm{Spin}(2n,1)$. Write $\pi^+(\rho)$ for the discrete series with lowest K-type $V_{K,(\underbrace{1,\dots,1}_{n})}$; and write $\pi^-(\rho)$ for the discrete series with lowest K-type $V_{K,(\underbrace{1,\dots,1}_{n-1},-1)}$. Then $\pi'_{n-1}(\rho) \cong \pi^+(\rho) \oplus \pi^-(\rho)$. Write $\bar{\pi}^+(\rho)$, $\bar{\pi}^-(\rho)$ for the Hilbert completion of $\pi^+(\rho)$, $\pi^-(\rho)$, respectively. By Theorem 3.21, we have

$$\bar{\pi}^+(\rho)|_P \oplus \bar{\pi}^-(\rho)|_P \cong \mathrm{Ind}_{M'N}^{MAN}\Big(\bigwedge^{n-1} \mathbb{C}^{2n-2} \otimes e^{\mathbf{i}\xi_0}\Big). \tag{3.11}$$

The restriction $\bigwedge^{n-1} \mathbb{C}^{2n-2}|_{M'}$ is the direct sum of two finite-dimensional irreducible representations of $M' = \mathrm{Spin}(2n-2)$ with highest weights

$$\mu^+ = (\underbrace{1,\dots,1}_{n-1}) \text{ and } \mu^- = (\underbrace{1,\dots,1}_{n-2},-1),$$

respectively. After (3.11), we need to determine whether $\bar{\pi}^-(\rho)|_P$ is isomorphic to $\mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^+} \otimes e^{\mathbf{i}\xi_0})$ or $\mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^-} \otimes e^{\mathbf{i}\xi_0})$. In order to do this, we calculate the Fourier transform of a specific K-type function in $\pi^-(\rho)$. For f in a small K-type, the explicit form of $f|_N$ was given in Kobayashi-Speh [36, §8.2]. We follow their description and then we calculate its Fourier transform.

One has

$$\bigwedge^{n} \mathbb{C}^{2n} \cong V_{K,\lambda^+} \oplus V_{K,\lambda^-}, \text{ where } \lambda^+ = (\underbrace{1,\dots,1}_{n}) \text{ and } \lambda^- = (\underbrace{1,\dots,1}_{n-1},-1).$$

Note that V_{K,λ^+} is the lowest K-type of $\pi^+(\rho)$, and V_{K,λ^-} is the lowest K-type of $\pi^-(\rho)$. Via this isomorphism the group $\mathrm{O}(2n)$ naturally acts on $V_{K,\lambda^+} \oplus V_{K,\lambda^-}$.

Put $V = \mathbb{C}^{2n}$ and $V' = \mathbb{C}^{2n-1}$. Let $\{e_j : 1 \leq j \leq 2n\}$ be the standard orthonormal basis of V. Then, $V = V' \oplus \mathbb{C}e_{2n}$ and this decomposition induces

$$\bigwedge^{n} V = \bigwedge^{n} V' \oplus \Big(\bigwedge^{n-1} V' \wedge \mathbb{C}e_{2n}\Big).$$

Let $p\colon V \to V'$ be the projection along $\mathbb{C}e_{2n}$. Then it induces the projection $\bigwedge^n V \to \bigwedge^n V'$ along $\bigwedge^{n-1} V' \wedge \mathbb{C}e_{2n}$, still denoted by p. Note that p is M-equivariant. For each $u \in \bigwedge^n V$, define

$$f_u(ka\bar{n}) = e^{(\rho'+\nu_{n-1})\log a} p(k^{-1}u), \quad ka\bar{n} \in KA\bar{N},$$

which belongs to $\mathrm{Ind}_{MA\bar{N}}^{G}(\bigwedge^n \mathbb{C}^{2n-1} \otimes e^{-\nu_{n-1}-\rho'} \otimes \mathbf{1}_{\bar{N}}) \cong I_{n-1}(-\nu_{n-1})$. This isomorphism is induced by $\bigwedge^n V'|_{M'} \cong \bigwedge^{n-1} V'|_{M'}$. By (3.6), we have

$$f_u(n_x) = (1+|x|^2)^{-n} p(s_x^{-1}u),$$

where s_x is as in (3.5).

Write $v_j = e_{2j-1} + \mathbf{i}e_{2j}$ for each $1 \leq j \leq n$. Put

$$u^+ = v_1 \wedge \cdots \wedge v_n \text{ and } u^- = v_1 \wedge \cdots \wedge v_{n-1} \wedge (e_{2n-1} - \mathbf{i}e_{2n}).$$

Then u^+ (resp. u^-) is a non-zero vector in $\bigwedge^n \mathbb{C}^{2n}$ with highest weight λ^+ (resp. λ^-). Then f_{u^-} gives a function in $\pi^-(\rho) \subset \mathrm{Ind}_{MA\bar{N}}^{G}(\bigwedge^n \mathbb{C}^{2n-1} \otimes e^{-\nu_{n-1}-\rho'} \otimes \mathbf{1}_{\bar{N}})$, corresponding to a highest weight vector of the lowest K-type of $\pi^-(\rho)$. Below we calculate the inverse Fourier transform of f_{u^-}.

Put $y = (x, 1) \in \mathbb{R}^{2n}$. Let r'_x denote both

$$I_{2n} - \frac{2y^t y}{|y|^2} \in \mathrm{O}(2n) \text{ and } \mathrm{diag}\Big(I_{2n} - \frac{2y^t y}{|y|^2}, 1\Big) \in \mathrm{O}(2n, 1).$$

Note that $s_x = s r'_x$ (see (3.5) and (3.7)). Then

$$p(s_x^{-1} u^-) = p(r'_x s u^-) = p(r'_x u^+).$$

Set $x_{2n} = 1$ for notational convenience, but we keep $|x|^2 = x_1^2 + \cdots + x_{2n-1}^2$.

Lemma 3.25. *One has*

$$\begin{aligned} &r'_x u^+ \\ &= u^+ + \sum_{1\le k\le n} (-1)^k \frac{2(x_{2k-1} + \mathbf{i} x_{2k})}{1+|x|^2} (x_{2k-1} e_{2k-1} + x_{2k} e_{2k}) \\ &\qquad \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_n \\ &\quad + \sum_{1\le k<j\le n} (-1)^{j-k} \frac{2\mathbf{i}(x_{2k-1} + \mathbf{i} x_{2k})(x_{2j-1} + \mathbf{i} x_{2j})}{1+|x|^2} e_{2j-1} \wedge e_{2j} \\ &\qquad \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_n \\ &\quad + \sum_{1\le j<k\le n} (-1)^{j-k-1} \frac{2\mathbf{i}(x_{2k-1} + \mathbf{i} x_{2k})(x_{2j-1} + \mathbf{i} x_{2j})}{1+|x|^2} e_{2j-1} \wedge e_{2j} \\ &\qquad \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_n. \end{aligned} \tag{3.12}$$

Proof. We have

$$r'_x u^+ = r'_x v_1 \wedge \cdots \wedge r'_x v_n.$$

Since $r'_x v_k - v_k$ is proportional to y,

$$r'_x u^+ = u^+ + \sum_{k=1}^{n} (-1)^{k-1} (r'_x v_k - v_k) \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_n.$$

Then we calculate

$$\begin{aligned} r'_x v_k - v_k &= -\frac{2(v_k, y)}{|y|^2} y \\ &= -\sum_{i=1}^{n} \frac{2(x_{2k-1} + \mathbf{i} x_{2k})}{1+|x|^2} (x_{2i-1} e_{2i-1} + x_{2i} e_{2i}). \end{aligned}$$

The term for $i = k$ corresponds to the second term of the right hand side of (3.12). Then the lemma follows from

$$(x_{2j-1} e_{2j-1} + x_{2j} e_{2j}) \wedge v_j = \mathbf{i}(x_{2j-1} + \mathbf{i} x_{2j}) e_{2j-1} \wedge e_{2j}. \qquad \square$$

To calculate the inverse Fourier transform $\mathcal{F}((f_{u^-})_N)$ we need some formulas.

$$\mathcal{F}(1+|x|^2)^{-n} = \frac{2^{\frac{1}{2}-n} \pi^{\frac{1}{2}}}{(n-1)!} e^{-|\xi|}, \tag{3.13}$$

$$\mathcal{F}(1+|x|^2)^{-(n+1)} = \frac{2^{-\frac{1}{2}-n} \pi^{\frac{1}{2}}}{n!} (1+|\xi|) e^{-|\xi|}. \tag{3.14}$$

First, the Fourier transform of the function of one variable $\sqrt{\frac{\pi}{2}}e^{-|\xi|}$ is equal to $(1+x^2)^{-1}$. By the Fourier inversion formula, we get $\mathcal{F}(1+x^2)^{-1} = \sqrt{\frac{\pi}{2}}e^{-|\xi|}$. Using $(1+x^2)^{-2} = (1+\frac{x}{2}\frac{d}{dx})(1+x^2)^{-1}$, we see that $\mathcal{F}(1+x^2)^{-2} = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+|\xi|)e^{-|\xi|}$. For $m \geq 2$, write Ω_m for the volume of the $(m-1)$-dimensional sphere. It is well known that $\Omega_m = 2\pi^{\frac{m}{2}}/\Gamma(\frac{m}{2})$. Letting $x = (x_1, \sqrt{1+x_1^2}z)$ with $z \in \mathbb{R}^{2n-2}$, we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^{2n-1}} (1+|x|^2)^{-n} e^{\mathbf{i}(\xi,x)}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{2n-1}} (1+|x|^2)^{-n} e^{\mathbf{i}x_1|\xi|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{2n-2}} \int_{\mathbb{R}} (1+x_1^2)^{-1} e^{\mathbf{i}x_1\xi}(1+|z|^2)^{-n}\,\mathrm{d}x_1\,\mathrm{d}z \\
&= \sqrt{2\pi}\mathcal{F}(1+x_1^2)^{-1} \times \Omega_{2n-2} \int_0^\infty (1+r^2)^{-n} r^{2n-3}\,\mathrm{d}r \\
&= \pi e^{-|\xi|} \times \frac{2\pi^{n-1}}{\Gamma(n-1)} \times \frac{1}{2}\int_0^\infty (1+s)^{-n}s^{n-2}\,\mathrm{d}s = \frac{\pi^n e^{-|\xi|}}{(n-1)!}.
\end{aligned}$$

For the first equation, we used a rotation on coordinates to assume $\xi = (|\xi|, 0, \ldots, 0)$. Therefore, we obtain (3.13). Equation (3.14) is similarly proved using $\mathcal{F}(1+x^2)^{-2} = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+|\xi|)e^{-|\xi|}$.

Then by (3.9), we obtain the following.

$$\mathcal{F}(x_j(1+|x|^2)^{-(n+1)}) = \mathbf{i}\frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!}\xi_j e^{-|\xi|}, \tag{3.15}$$

$$\mathcal{F}(x_j^2(1+|x|^2)^{-(n+1)}) = \frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!}\Big(1-\frac{\xi_j^2}{|\xi|}\Big)e^{-|\xi|}, \tag{3.16}$$

$$\mathcal{F}(x_j x_k(1+|x|^2)^{-(n+1)}) = -\frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!}\frac{\xi_j\xi_k}{|\xi|}e^{-|\xi|} \quad (j \neq k). \tag{3.17}$$

Lemma 3.26. *We have*

$$\mathcal{F}(f_{u^-})_N = \frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!}e^{-|\xi|}(|\xi|(1-r_\xi)u + 2u' \wedge \xi)$$

at $0 \neq \xi \in \mathbb{R}^{2n-1}$*, where*

$$\begin{aligned}
u &= v_1 \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \in \bigwedge^n V' \text{ and} \\
u' &= v_1 \wedge \cdots \wedge v_{n-1} \in \bigwedge^{n-1} V'.
\end{aligned}$$

Proof. By Lemma 3.25, we have

$$
\begin{aligned}
f_{u^-}(n_x) &= (1+|x|)^{-n} v_1 \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ \sum_{1\le k\le n-1} (-1)^k 2(1+|x|)^{-n-1}(x_{2k-1}+\mathbf{i}x_{2k})(x_{2k-1}e_{2k-1}+x_{2k}e_{2k}) \\
&\qquad \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ (-1)^n 2(1+|x|)^{-n-1}(x_{2n-1}+\mathbf{i})x_{2n-1}e_{2n-1} \wedge v_1 \wedge \cdots \wedge v_{n-1} \\
&+ \sum_{1\le k<j\le n-1} (-1)^{j-k} 2(1+|x|)^{-n-1}\mathbf{i}(x_{2k-1}+\mathbf{i}x_{2k})(x_{2j-1}+\mathbf{i}x_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ \sum_{1\le j<k\le n-1} (-1)^{j-k-1} 2(1+|x|)^{-n-1}\mathbf{i}(x_{2k-1}+\mathbf{i}x_{2k})(x_{2j-1}+\mathbf{i}x_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ \sum_{1\le j\le n-1} (-1)^{j-n-1} 2(1+|x|)^{-n-1}\mathbf{i}(x_{2n-1}+\mathbf{i})(x_{2j-1}+\mathbf{i}x_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_{n-1}.
\end{aligned}
$$

Then by (3.13) – (3.17),

$$
\begin{aligned}
&\Big(\frac{2^{\frac12-n}\pi^{\frac12}}{n!}e^{-|\xi|}\Big)^{-1}\mathcal{F}((f_{u^-})_N) = n v_1 \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} + \sum_{1\le k\le n-1}(-1)^k \\
&\Big(\Big(1-\frac{\xi_{2k-1}^2}{|\xi|}-\mathbf{i}\frac{\xi_{2k-1}\xi_{2k}}{|\xi|}\Big)e_{2k-1}+\Big(\mathbf{i}-\mathbf{i}\frac{\xi_{2k}^2}{|\xi|}-\frac{\xi_{2k-1}\xi_{2k}}{|\xi|}\Big)e_{2k}\Big) \\
&\qquad \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ (-1)^n\Big(1-\frac{\xi_{2n-1}^2}{|\xi|}-\xi_{2n-1}\Big)e_{2n-1} \wedge v_1 \wedge \cdots \wedge v_{n-1} \\
&+ \sum_{1\le k<j\le n-1}(-1)^{j-k-1}\mathbf{i}|\xi|^{-1}(\xi_{2k-1}+\mathbf{i}\xi_{2k})(\xi_{2j-1}+\mathbf{i}\xi_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ \sum_{1\le j<k\le n-1}(-1)^{j-k}\mathbf{i}|\xi|^{-1}(\xi_{2k-1}+\mathbf{i}\xi_{2k})(\xi_{2j-1}+\mathbf{i}\xi_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge \hat{v_k} \wedge \cdots \wedge v_{n-1} \wedge e_{2n-1} \\
&+ \sum_{1\le j\le n-1}(-1)^{j-n}\mathbf{i}(1+|\xi|^{-1}\xi_{2n-1})(\xi_{2j-1}+\mathbf{i}\xi_{2j}) \\
&\quad e_{2j-1}\wedge e_{2j} \wedge v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_{n-1}.
\end{aligned}
$$

Similarly to Lemma 3.25, we have

$$\begin{aligned} r_\xi(u) = u &+ \sum_{1\le k\le n-1} (-1)^k \frac{2(\xi_{2k-1}+\mathbf{i}\xi_{2k})}{|\xi|^2}(\xi_{2k-1}e_{2k-1}+\xi_{2k}e_{2k}) \\ &\quad \wedge v_1\wedge\cdots\wedge\hat{v_k}\wedge\cdots\wedge v_{n-1}\wedge e_{2n-1} \\ &+ (-1)^n\frac{2\xi_{2n-1}}{|\xi|^2}\xi_{2n-1}e_{2n-1}\wedge v_1\wedge\cdots\wedge v_{n-1} \\ &+ \sum_{1\le k<j\le n-1}(-1)^{j-k}\frac{2\mathbf{i}(\xi_{2k-1}+\mathbf{i}\xi_{2k})(\xi_{2j-1}+\mathbf{i}\xi_{2j})}{|\xi|^2}e_{2j-1}\wedge e_{2j} \\ &\quad \wedge v_1\wedge\cdots\wedge\hat{v_k}\wedge\cdots\wedge\hat{v_j}\wedge\cdots\wedge v_{n-1}\wedge e_{2n-1} \\ &+ \sum_{1\le j<k\le n-1}(-1)^{j-k-1}\frac{2\mathbf{i}(\xi_{2k-1}+\mathbf{i}\xi_{2k})(\xi_{2j-1}+\mathbf{i}\xi_{2j})}{|\xi|^2}e_{2j-1}\wedge e_{2j} \\ &\quad \wedge v_1\wedge\cdots\wedge\hat{v_j}\wedge\cdots\wedge\hat{v_k}\wedge\cdots\wedge v_{n-1}\wedge e_{2n-1} \\ &+ \sum_{1\le j\le n-1}(-1)^{j-n-1}\frac{2\mathbf{i}\xi_{2n-1}(\xi_{2j-1}+\mathbf{i}\xi_{2j})}{|\xi|^2}e_{2j-1}\wedge e_{2j} \\ &\quad \wedge v_1\wedge\cdots\wedge\hat{v_j}\wedge\cdots\wedge v_{n-1}. \end{aligned}$$

The lemma follows from these equations. □

Proposition 3.27. *Let $\bar{\pi}^{\pm}(\rho)$ be discrete series representation of infinitesimal character ρ defined in §3.3. Then*

$$\bar{\pi}^{+}(\rho)|_P \cong \mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^-}\otimes e^{\mathbf{i}\xi_0})$$

and

$$\bar{\pi}^{-}(\rho)|_P \cong \mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^+}\otimes e^{\mathbf{i}\xi_0}).$$

Proof. Let $h := \mathcal{F}((f_{u^-})_N)$. By Lemma 3.26,

$$h(\xi) = \frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!}e^{-|\xi|}(|\xi|(1-r_\xi)u + 2u'\wedge\xi)$$

for $\xi\neq 0$. Evaluating at $\xi=\xi_0=e_{2n-1}$, we have

$$h(e_{2n-1}) = \frac{2^{-\frac{1}{2}-n}\pi^{\frac{1}{2}}}{n!e}\cdot(4u).$$

Hence

$$h_{at,\nu}(e) = h(\xi_0) = cu$$

for a constant $c\neq 0$. This is a highest weight vector for M' with weight μ^+ in the representation $\bigwedge^n V'|_{M'}\cong\bigwedge^{n-1}V'|_{M'}$. Hence the inverse Fourier transform of f_{u^-} must lie in $\mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^+}\otimes e^{\mathbf{i}\xi_0})$. Therefore,

$$\bar{\pi}^{-}(\rho)|_P \cong \mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^+}\otimes e^{\mathbf{i}\xi_0})$$

and then

$$\bar{\pi}^{+}(\rho)|_P \cong \mathrm{Ind}_{M'N}^{MAN}(V_{M',\mu^-}\otimes e^{\mathbf{i}\xi_0}).$$ □

4. Moment map for regular elliptic coadjoint orbits

In this section we calculate the projection of G-regular elliptic coadjoint orbits with respect to the natural map $\mathfrak{g}^* \to \mathfrak{p}^*$. Before we start the calculation, we emphasize that the method used below actually allows us to describe explicitly the projection of all G-orbits (no matter whether elliptic or not, regular or not). However, since original Duflo's conjecture concerns discrete series representations which are associated to regular elliptic orbits, we only treat these orbits in the paper.

In this section we suppose $G = \mathrm{Spin}(2n, 1)$.

4.1. **Classification of coadjoint orbits in $\mathfrak{p}^*$.** In this subsection we describe all coadjoint P-orbits in $\mathfrak{p}^*$. Write $L = MA$ and $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$. Then, $P = L \ltimes N$ is a Levi decomposition of P. Similarly, write $L_1 = M_1 A \subset P_1$ and then $P_1 = L_1 \ltimes N$. There are exact sequences of P-modules

$$0 \to \mathfrak{n} \to \mathfrak{p} \to \mathfrak{l} \to 0 \ \text{ and } \ 0 \to \mathfrak{l}^* \to \mathfrak{p}^* \to \mathfrak{n}^* \to 0.$$

We regard $\mathfrak{l}^*$ as a subspace of $\mathfrak{p}^*$.

Inspired by Bernstein-Zelevinsky depth for representations of general linear group over a p-adic field ([5], [49]), we define a notion of "depth" for coadjoint P-orbits in $\mathfrak{p}^*$.

Definition 4.1. We say a coadjoint P-orbit $\mathcal{O}$ in $\mathfrak{p}^*$ has *depth zero* if it is contained in $\mathfrak{l}^*$, otherwise we say $\mathcal{O}$ has depth one.

Note that the action of P on $\mathfrak{p}$ (or $\mathfrak{p}^*$) factors through P_1. We have

$$L_1 = M_1 A \cong \mathrm{SO}(2n-1) \times \mathbb{R}_{>0}, \quad \mathfrak{n}^* \cong \mathbb{R}^{2n-1},$$

and L_1 acts on $\mathfrak{n}^*$ as in (2.5). Thus, there are only two L-orbits $\{0\}$ and $\mathfrak{n}^* - \{0\}$ in $\mathfrak{n}^*$. Then if a coadjoint P-orbit $\mathcal{O}$ has depth zero (resp. depth one), $\mathcal{O}$ maps to $\{0\}$ (resp. $\mathfrak{n}^* - \{0\}$) by the projection $\mathfrak{p}^* \to \mathfrak{n}^*$.

Fix a vector $\xi(\neq 0) \in \mathfrak{n}^*$. Write $P(\xi) = \mathrm{Stab}_P(\xi)$ and $L(\xi) = \mathrm{Stab}_L(\xi)$. Then $P(\xi) = L(\xi) \ltimes N$ and $\mathfrak{l}(\xi) \cong \mathfrak{so}(2n-2)$. Since the bilinear form (2.2) is non-degenerate on $\mathfrak{l}(\xi)$, we have a decomposition $\mathfrak{l} = \mathfrak{l}(\xi) \oplus \mathfrak{l}(\xi)^{\perp}$, where $\mathfrak{l}(\xi)^{\perp}$ is the orthogonal complement of $\mathfrak{l}(\xi)$ in $\mathfrak{l}$. We note that $\mathfrak{l}(\xi) \subset \mathfrak{m}$ and $\mathfrak{l}(\xi)^{\perp} \supset \mathfrak{a}$. Then we have decompositions

$$\mathfrak{p} = \mathfrak{l}(\xi) \oplus \mathfrak{l}(\xi)^{\perp} \oplus \mathfrak{n} \ \text{ and } \ \mathfrak{p}^* = \mathfrak{l}(\xi)^* \oplus (\mathfrak{l}(\xi)^{\perp})^* \oplus \mathfrak{n}^*.$$

Proposition 4.2. *We define a map $\mathfrak{l}(\xi)^* \to \mathfrak{p}^* - \mathfrak{l}^*$ by $\eta \mapsto (\eta, 0, \xi) \in \mathfrak{l}(\xi)^* \oplus (\mathfrak{l}(\xi)^{\perp})^* \oplus \mathfrak{n}^*$. Then it descends to a bijection*

$$\mathfrak{l}(\xi)^* / \mathrm{Ad}^*(L(\xi)) \xrightarrow{\sim} (\mathfrak{p}^* - \mathfrak{l}^*) / \mathrm{Ad}^*(P),$$

namely, the coadjoint $L(\xi)$-orbits correspond bijectively to the coadjoint P-orbits which have depth one.

Proof. As $\mathfrak{n}$ is abelian, it acts trivially on $\mathfrak{n}^*$. Thus, $\mathrm{ad}^*(\mathfrak{n})(\mathfrak{p}^*) \subset \mathfrak{l}^*$. Let $\tilde{\xi} = (0, 0, \xi) \in \mathfrak{l}(\xi)^* \oplus (\mathfrak{l}(\xi)^{\perp})^* \oplus \mathfrak{n}^* = \mathfrak{p}^*$. We first claim that $(\mathfrak{l}(\xi)^{\perp})^* = \mathrm{ad}^*(\mathfrak{n})\tilde{\xi}$. For $X \in \mathfrak{l}$ and $Y \in \mathfrak{n}$,

$$(\mathrm{ad}^*(X)\xi)(Y) = (\mathrm{ad}^*(X)\tilde{\xi})(Y) = -\tilde{\xi}([X, Y]) = -(\mathrm{ad}^*(Y)\tilde{\xi})(X).$$

Consequently, $X \in \mathfrak{l}(\xi)$ if and only if $(\mathrm{ad}^*(\mathfrak{n})(\tilde{\xi}))(X) = 0$. Thus, the null space of $\mathfrak{l}(\xi)$ in $\mathfrak{l}^*$ is $\mathrm{ad}^*(\mathfrak{n})\tilde{\xi}$ and the claim is proved.

It is easy to see that the map

$$\mathfrak{l}(\xi)^*/\operatorname{Ad}^*(L(\xi)) \to (\mathfrak{p}^* - \mathfrak{l}^*)/\operatorname{Ad}^*(P)$$

in the proposition is well-defined. To show that it is injective, take two elements $\eta, \eta' \in (\mathfrak{l}(\xi))^*$ such that $\operatorname{Ad}^*(p)(\eta, 0, \xi) = (\eta', 0, \xi)$ for some $p \in P$. Write $p = ln \in LN$ and $n = \exp(X)$. Then $\operatorname{Ad}^*(n)(\eta, 0, \xi) = (\eta, \operatorname{ad}^*(X)\xi, \xi)$. Hence $\operatorname{Ad}^*(l)(\eta, \operatorname{ad}^*(X)\xi, \xi) = (\eta', 0, \xi)$, which implies $l \in L(\xi)$ and $\eta = \eta'$.

To show the surjectivity, take a coadjoint P-orbit $\mathcal{O}$ in $\mathfrak{p}^* - \mathfrak{l}^*$. Since L acts transitively on $\mathfrak{n}^* - \{0\}$, we can find an element of the form (η, ζ, ξ) in $\mathcal{O}$. By $(\mathfrak{l}(\xi)^{\perp})^* = \operatorname{ad}^*(\mathfrak{n})\tilde{\xi}$, there exists $X \in \mathfrak{n}$ such that $\operatorname{ad}^*(X)\tilde{\xi} = -\zeta$. Then $\operatorname{Ad}^*(\exp(X))(\eta, \zeta, \xi) = (\eta, 0, \xi)$. This proves the surjectivity. ☐

In what follows, we carry out an explicit matrix calculation for the correspondence in Proposition 4.2. Define $\operatorname{pr}\colon \mathfrak{g} \to \mathfrak{p}^*$ by

$$\operatorname{pr}(X)(Y) = (X, Y) \ (\forall Y \in \mathfrak{p}). \tag{4.1}$$

Then, the kernel of pr is $\mathfrak{n}$, and pr gives a P-module isomorphism $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{p}^*$. Since $\bar{\mathfrak{p}}$ is a complement of $\mathfrak{n}$ in $\mathfrak{g}$, we have an isomorphism $\operatorname{pr}\colon \bar{\mathfrak{p}} \xrightarrow{\sim} \mathfrak{p}^*$.

For $Y \in \mathfrak{so}(2n-1)$, $\beta \in \mathbb{R}^{2n-1}$ and $a \in \mathbb{R}$, put

$$X_{Y,\beta,a} = \begin{pmatrix} Y & \beta^t & \beta^t \\ -\beta & 0 & a \\ \beta & a & 0 \end{pmatrix} \in \overline{\mathfrak{p}}.$$

Then, for any $X = \begin{pmatrix} Y & \beta_1^t & \beta_2^t \\ -\beta_1 & 0 & a \\ \beta_2 & a & 0 \end{pmatrix} \in \mathfrak{g}$, one has $\operatorname{pr}(X) = \operatorname{pr}(X_{Y, \frac{\beta_1+\beta_2}{2}, a})$; for any $f \in \mathfrak{p}^*$, there exists a unique triple

$$(Y, \beta, a) \in \mathfrak{so}(2n-1) \times \mathbb{R}^{2n-1} \times \mathbb{R}$$

such that $f = \operatorname{pr}(X_{Y,\beta,a})$.

Write $\psi_n\colon \mathfrak{p}^* \to \mathfrak{n}^*$ for the natural projection.

Lemma 4.3. *For $0 \neq \beta \in \mathbb{R}^{2n-1}$, put $\xi = \psi_n(\operatorname{pr}(X_{0,\beta,0})) \in \mathfrak{n}^*$. In order that $\operatorname{pr}(X_{Y,0,a}) \in \mathfrak{l}^*(\xi)$ it is necessary and sufficient that $a = 0$ and $Y\beta^t = 0$.*

Proof. We have

$$\operatorname{pr}(X_{Y,0,a}) \in \mathfrak{l}^*(\xi) \Leftrightarrow (\operatorname{pr}(X_{Y,0,a}), \operatorname{ad}^*(\mathfrak{n})(\tilde{\xi})) = 0 \Leftrightarrow ([X_{Y,0,a}, X_{0,\beta,0}], \mathfrak{n}) = 0.$$

Since $[X_{Y,0,a}, X_{0,\beta,0}] = X_{0,\beta Y^t - a\beta, 0} \in \bar{\mathfrak{n}}$ by (2.1), $\operatorname{pr}(X_{Y,0,a}) \in \mathfrak{l}^*(\xi)$ is equivalent to $\beta Y^t - a\beta = 0$. If $\beta Y^t - a\beta = 0$, then $(\beta Y^t - a\beta)\beta^t = -a\beta\beta^t = 0$. Hence $a = 0$ and $\beta Y^t = 0$. The lemma follows. ☐

Lemma 4.4. *For a general triple $(Y, \beta, a) \in \mathfrak{so}(2n-1) \times \mathbb{R}^{2n-1} \times \mathbb{R}$ with $\beta \neq 0$, put $\xi = \psi_n(\operatorname{pr}(X_{Y,\beta,a})) \in \mathfrak{n}^*$. Then, there exists a unique $\gamma \in \mathbb{R}^{2n-1}$ such that $\operatorname{Ad}^*(n_\gamma)(\operatorname{pr}(X_{Y,\beta,a})) \in \mathfrak{l}^*(\xi) + \tilde{\xi}$. Moreover,*

$$\operatorname{Ad}^*(n_\gamma)(\operatorname{pr}(X_{Y,\beta,a})) = \operatorname{pr}(X_{Y - \frac{1}{|\beta|^2}(Y\beta^t\beta - \beta^t\beta Y^t), \beta, 0}).$$

Proof. For $\gamma \in \mathbb{R}^{2n-1}$, we calculate by using (2.1)

$$\operatorname{pr}(\operatorname{Ad}(n_\gamma)(X_{Y,\beta,a})) = \operatorname{pr}(X_{Y + 2\gamma^t\beta - 2\beta^t\gamma,\, \beta,\, a + 2\gamma\beta^t}).$$

By Lemma 4.3, in order that $\mathrm{Ad}^*(n_\gamma)(\mathrm{pr}(X_{Y,\beta,a})) \in \mathfrak{l}(\xi)^* + \tilde{\xi}$, it is necessary and sufficient that $a+2\gamma\beta^t = 0$ and $\beta(Y+2\gamma^t\beta-2\beta^t\gamma)^t = 0$. From these two equations, one solves that

$$\gamma = -\frac{1}{2|\beta|^2}(\beta Y^t + a\beta).$$

Then we have

$$\mathrm{Ad}^*(n_\gamma)(X_{Y,\beta,a}) = X_{Y-\frac{1}{|\beta|^2}(Y\beta^t\beta-\beta^t\beta Y^t),\,\beta,\,0}. \qquad \square$$

Lemma 4.5. *Assume that $\beta \neq 0$ and $Y\beta^t = 0$. Then, the orbit $\mathrm{Ad}^*(P)\,\mathrm{pr}(X_{Y,\beta,0})$ is determined by the class of $Z_{Y,\beta}$ with respect to the conjugation action of $\mathrm{SO}(2n)$, where*

$$Z_{Y,\beta} = \begin{pmatrix} Y & \frac{\beta^t}{|\beta|} \\ -\frac{\beta}{|\beta|} & 0 \end{pmatrix}.$$

Proof. Put

$$H' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, there exists $(W, a) \in \mathrm{SO}(2n-1) \times \mathbb{R}_{>0}$ such that

$$WYW^{-1} = \mathrm{diag}(x_1H', \ldots, x_{n-1}H', 0)$$

and $a\beta W^t = (\underbrace{0, \ldots, 0}_{2n-2}, 1)$, where $x_1 \geq x_2 \geq \cdots \geq x_{n-2} \geq |x_{n-1}|$. By Proposition 4.2, the orbit $\mathrm{Ad}^*(P)\,\mathrm{pr}(X_{Y,\beta,0})$ is determined by the tuple $(x_1, \ldots, x_{n-1})$. Since $Z_{Y,\beta}$ is conjugate to

$$\mathrm{diag}(x_1H', \ldots, x_{n-1}H', H'),$$

the class of $Z_{Y,\beta}$ with respect to the conjugation action of $\mathrm{SO}(2n)$ is also determined by the tuple $(x_1, \ldots, x_{n-1})$. Hence, the conclusion of the lemma follows. $\square$

Remark 4.6. It is known that the $\mathrm{SO}(2n)$-conjugacy class of $Z_{Y,\beta}$ is determined by its singular values and the sign of its Pfaffian (which can be 1, -1 or 0). Here, the singular values of $Z_{Y,\beta}$ mean the square roots of eigenvalues of $(Z_{Y,\beta})^tZ_{Y,\beta}$. From the proof of Lemma 4.5, we see that singular values of $Z_{Y,\beta}$ are

$$\{x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, 1, 1\}$$

and the singular values of Y are

$$\{x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}, 0\}.$$

Therefore, the class of $Z_{Y,\beta}$ is also determined by the singular values of Y and the Pfaffian of $Z_{Y,\beta}$. The sign of the Pfaffian of $Z_{Y,\beta}$ is equal to the sign of x_{n-1}.

Remark 4.7. It is easy to see that if a coadjoint orbit in $\mathfrak{p}^*$ is strongly regular (see §1), then it has depth one. In the above notation the orbit $\mathrm{Ad}^*(P)\,\mathrm{pr}(X_{Y,\beta,0})$ is strongly regular if and only if $x_1 > \cdots > x_{n-2} > |x_{n-1}| > 0$.

4.2. P**-orbits in** $\mathcal{O}_f$**.** For $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0$, write $\vec{a} = (a_1, a_2, \dots, a_n)$. As in (2.4), put

$$t_{\vec{a}} = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & a_n & \\ & & & -a_n & 0 & \\ & & & & & 0 \end{pmatrix}.$$

Under the isomorphism $\iota \colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ in (2.3), we set

$$f = f_{\vec{a}} = \iota(t_{\vec{a}}),$$

which is an elliptic element in $\mathfrak{g}^*$. Moreover, each elliptic coadjoint orbit in $\mathfrak{g}^*$ contains $f_{\vec{a}}$ for a unique vector $\vec{a}$.

In this case, it is not hard to see that f is regular if and only if

$$a_1 > a_2 > \cdots > a_{n-1} > |a_n| > 0.$$

Let $G(f)$ be the stabilizer of f in G. Then, $G(f) = T$, where T is the preimage in G of the maximal torus

$$T_1 = \left\{ \begin{pmatrix} y_1 & z_1 & & & & \\ -z_1 & y_1 & & & & \\ & & \ddots & & & \\ & & & y_n & z_n & \\ & & & -z_n & y_n & \\ & & & & & 1 \end{pmatrix} : y_1^2 + z_1^2 = \cdots = y_n^2 + z_n^2 = 1 \right\}$$

of G_1. Let $\mathcal{O}_f = G \cdot f$. Under the isomorphism $\mathcal{O}_f \cong G/G(f)$, parametrization of P-orbits in $\mathcal{O}_f$ is equivalent to parametrization of double cosets in $P\backslash G/G(f)$. Since the map

$$P\backslash G/G(f) \ni PgG(f) \mapsto G(f)g^{-1}P \in G(f)\backslash G/P$$

is an isomorphism, it is also equivalent to parametrizing $G(f)$-orbits in G/P. Write

$$X_n = \{\vec{x} = (x_1, \dots, x_{2n}, x_0) : x_0^2 = \sum_{i=1}^{2n} x_i^2 \text{ and } x_0 > 0\}/\sim .$$

Here, for two vectors $\vec{x}$ and $\vec{x'}$, we define

$$\vec{x} \sim \vec{x'} \Leftrightarrow \exists s > 0 \text{ such that } \vec{x'} = s\vec{x}.$$

As a manifold, $X_n \cong S^{2n-1}$. The group G acts on X_n transitively as

$$g \cdot [\vec{x}] = [\vec{x} g_1^t], \quad G \ni g \mapsto g_1 \in G_1.$$

Let

$$v_0 = [(0, \dots, 0, 1, 1)].$$

As $\mathrm{Stab}_G(v_0) = P$, $X_n \cong G/P$. Therefore, parametrization of $G(f)$-orbits in G/P is equivalent to parametrization of T-orbits in X_n.

Let

$$B = \Big\{ \vec{b} = (b_1, \dots, b_n) : b_1, \dots, b_n \geq 0, \ \sum_{i=1}^{n-1} b_i^2 = 1 - 2b_n \Big\}.$$

Then, $0 \leq b_n \leq \frac{1}{2}$ for any $\vec{b} \in B$. Write

$$\alpha = \alpha_{\vec{b}} = (0, b_1, 0, b_2, \ldots, 0, b_{n-1}, 0) \text{ and } \bar{X}_{\vec{b}} = \begin{pmatrix} 0_{2n-1} & \alpha^t & \alpha^t \\ -\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$

Put

$$\bar{n}_{\vec{b}} = \exp(\bar{X}_{\vec{b}}) = \begin{pmatrix} I_{2n-1} & \alpha^t & \alpha^t \\ -\alpha & 1 - \frac{1}{2}|\alpha|^2 & -\frac{1}{2}|\alpha|^2 \\ \alpha & \frac{1}{2}|\alpha|^2 & 1 + \frac{1}{2}|\alpha|^2 \end{pmatrix} \in \bar{N}.$$

Then,

$$\bar{n}_{\vec{b}}^{-1} \cdot v_0 = [(0, -b_1, 0, -b_2, \ldots, 0, -b_{n-1}, 0, b_n, 1 - b_n)].$$

Lemma 4.8. *The map $B \to T\backslash X_n$ defined by*

$$\vec{b} \mapsto (\bar{n}_{\vec{b}})^{-1} \cdot v_0$$

is a bijection.

Proof. Identify the image of T in G_1 with $\mathrm{U}(1)^n$. Then, T acts on X_n by

$$\begin{aligned} &(y_1 + z_1\mathbf{i}, \ldots, y_n + z_n\mathbf{i}) \cdot [(x_1, \ldots, x_{2n}, x_0)] \\ &= [(y_1x_1 + z_1x_2, -z_1x_1 + y_1x_2, \ldots, y_nx_{2n-1} + z_nx_{2n}, -z_nx_{2n-1} + y_nx_{2n}, x_0)]. \end{aligned}$$

Then each T-orbit in X_n has a unique representative of the form

$$[(0, -b_1, \ldots, 0, -b_{n-1}, 0, b_n, 1 - b_n)],$$

where $b_i \geq 0$ $(1 \leq i \leq n)$. Moreover, the equation

$$\sum_{i=1}^{2n} x_i^2 = x_0^2$$

leads to the equation

$$\sum_{i=1}^{n-1} b_i^2 = 1 - 2b_n.$$

By this, the map $\vec{b} \mapsto (\bar{n}_{\vec{b}})^{-1} \cdot v_0$ is a bijection. □

One direct consequence of Lemma 4.8 is the following

Lemma 4.9. *Each P-orbit in $\mathcal{O}_f = G \cdot f$ contains some $\bar{n}_{\vec{b}} \cdot f$ for a unique tuple $\vec{b} \in B$.*

4.3. **The moment map $\mathcal{O}_f \to \mathfrak{p}^*$.** Recall that in §4.1, we obtained an explicit parametrization of P-coadjoint orbits in $\mathfrak{p}^*$.

In this subsection, we use the results in §4.1 to calculate the image of the moment map $q \colon \mathcal{O}_f \to \mathfrak{p}^*$. Here, the moment map q is defined by the composition of the inclusion $\mathcal{O}_f \hookrightarrow \mathfrak{g}^*$ and the natural projection $\mathfrak{g}^* \to \mathfrak{p}^*$.

Recall that the map pr was defined in (4.1). Let

$$H' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 4.10. *We have* $q(\bar{n}_{\vec{b}} \cdot f) = \mathrm{pr}(X_{Y,\beta,0})$, *where*

$$\beta = (-a_1b_1, 0, \dots, -a_{n-1}b_{n-1}, 0, a_nb_n) \neq 0,$$
$$Y = \mathrm{diag}(a_1H', \dots, a_{n-1}H', 0) + (\beta')^t\alpha - \alpha^t\beta', \text{ and } \beta' = (\underbrace{0, \dots, 0}_{2n-2}, a_n).$$

Proof. Write $Y' = \mathrm{diag}(a_1H', \dots, a_{n-1}H', 0)$. Then following notation in §2.1 we have $t_{\vec{a}} = \mathrm{diag}(Y', 0_{2\times 2}) - X_{\frac{\beta'}{2}} + \bar{X}_{\frac{\beta'}{2}}$. Using (2.1) we calculate

$$\begin{aligned}\mathrm{Ad}(\bar{n}_{\vec{b}})(t_{\vec{a}}) &= t_{\vec{a}} + [\bar{X}_\alpha, t_{\vec{a}}] + \frac{1}{2}[[\bar{X}_\alpha, [\bar{X}_\alpha, t_{\vec{a}}]] \\ &= \big(\mathrm{diag}(Y', 0_{2\times 2}) - X_{\frac{\beta'}{2}} + \bar{X}_{\frac{\beta'}{2}}\big) \\ &\quad - \big(\bar{X}_{\alpha(Y')^t} - \mathrm{diag}((\beta')^t\alpha - \alpha^t\beta', 0_{2\times 2})\big) - \frac{1}{2}\bar{X}_{\alpha\alpha^t\beta' - \alpha(\beta')^t\alpha}.\end{aligned}$$

Hence the lemma follows from

$$Y' + (\beta')^t\alpha - \alpha^t\beta' = Y \text{ and } \frac{\beta'}{2} - \alpha(Y')^t - \frac{1}{2}(\alpha\alpha^t\beta' - \alpha(\beta')^t\alpha) = \beta. \qquad \square$$

Put

$$Y_{\vec{b}} = Y - \frac{1}{|\beta|^2}(Y\beta^t\beta - (Y\beta^t\beta)^t), \quad Z_{\vec{b}} = \begin{pmatrix} Y_{\vec{b}} & \frac{\beta^t}{|\beta|} \\ -\frac{\beta}{|\beta|} & 0 \end{pmatrix}.$$

By Lemmas 4.3, 4.4 and 4.5 and Remark 4.6, the P-conjugacy class of $q(\bar{n}_{\vec{b}} \cdot f)$ is determined by the sign of the Pfaffian of $Z_{\vec{b}}$ and singular values of $Y_{\vec{b}}$. Put

$$\gamma_1 = (a_1b_1, \dots, a_{n-1}b_{n-1}, -a_nb_n),$$
$$\gamma_2 = ((a_1^2 - a_n^2b_n)b_1, \dots, (a_{n-1}^2 - a_n^2b_n)b_{n-1}, 0).$$

For a permutation σ on $\{1, 2, \dots, 2n\}$, let $Q_\sigma = (x_{ij})_{1\le i,j\le 2n}$ be the permutation matrix corresponding to σ, that is, $x_{i,j} = 1$ if $j = \sigma(i)$; and $x_{i,j} = 0$ if $j \neq \sigma(i)$.

Lemma 4.11. *Let* σ *be the permutation*

$$\sigma(i) = \begin{cases} 2i-1 & (1 \le i \le n) \\ 2(i-n) & (n+1 \le i \le 2n) \end{cases}.$$

Then

$$Q_\sigma Z_{\vec{b}} Q_\sigma^{-1} = \begin{pmatrix} 0_n & Z \\ -Z^t & 0_n \end{pmatrix},$$

where

$$Z = \begin{pmatrix} a_1 & \dots & 0 & \frac{-a_1b_1}{|\beta|} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_{n-1} & \frac{-a_{n-1}b_{n-1}}{|\beta|} \\ a_nb_1 & \dots & a_nb_{n-1} & \frac{a_nb_n}{|\beta|} \end{pmatrix} - \frac{\gamma_1^t\gamma_2}{|\beta|^2}. \tag{4.2}$$

Proof. By calculation

$$Y\beta^t = (0, (a_1^2 - a_n^2b_n)b_1, \dots, 0, (a_{n-1}^2 - a_n^2b_n)b_{n-1}, 0)^t.$$

By inputting the forms of Y, β, $Y\beta^t$ in

$$Z_{\vec{b}} = \begin{pmatrix} Y - \frac{1}{|\beta|^2}(Y\beta^t)\beta + \frac{1}{|\beta|^2}\beta^t(Y\beta^t)^t & \frac{\beta^t}{|\beta|} \\ -\frac{\beta}{|\beta|} & 0 \end{pmatrix},$$

we get the form of $Z_{\vec{b}}$. It is easy to see that $Q_\sigma Z_{\vec{b}} Q_\sigma^{-1}$ is of the block diagonal form as in the lemma. □

Lemma 4.12. *The Pfaffian of $Z_{\vec{b}}$ is equal to*

$$\frac{1-b_n}{|\beta|}\prod_{1\le i\le n} a_i.$$

Proof. By Lemma 4.11, the Pfaffian of $Z_{\vec{b}}$ is equal to $\det Z$. Since γ_1^t is proportional to the right most column of the first matrix in the right hand side of (4.2) and the last entry of γ_2 is zero, the term $\frac{1}{|\beta|^2}\gamma_1^t\gamma_2$ makes no contribution to $\det Z$. Therefore,

$$\begin{aligned}\det Z &= \det\begin{pmatrix} a_1 & \dots & 0 & \frac{-a_1b_1}{|\beta|} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & a_{n-1} & \frac{-a_{n-1}b_{n-1}}{|\beta|} \\ a_nb_1 & \dots & a_nb_{n-1} & \frac{a_nb_n}{|\beta|}\end{pmatrix} \\ &= \frac{1-b_n}{|\beta|}\prod_{1\le i\le n} a_i.\end{aligned}$$ □

Write Z' for the $n\times(n-1)$ matrix obtained from Z by removing the last column. Put

$$h_{\vec{b}}(x)=\det(xI_{n-1}-(Z')^tZ').$$

Then we claim that the singular values of $Y_{\vec{b}}$ are square roots of zeros of $h_{\vec{b}}(x)$. Indeed, let $W\in \mathrm{SO}(n)$ be a matrix such that the right most column of WZ is $(0,0,\dots,0,1)^t$. Then $WZ=\mathrm{diag}(Z_0',1)$ for some $(n-1)\times(n-1)$ matrix Z_0'. Hence the eigenvalues of Z^tZ are the eigenvalues of $(Z')^tZ'=(Z_0')^tZ_0'$ plus 1. Since the eigenvalues of Z^tZ are the same as those of $(Z_{\vec{b}})^tZ_{\vec{b}}$ and they are the singular values of $Y_{\vec{b}}$ plus 1 (see Remark 4.6), the claim follows.

Proposition 4.13. *We have*

$$h_{\vec{b}}(x)=\sum_{1\le i\le n}\frac{a_i^2b_i^2}{|\beta|^2}\prod_{1\le j\le n,\, j\ne i}(x-a_j^2). \tag{4.3}$$

For any $1\le i\le n$, a_i^2 is a root of $h_{\vec{b}}(x)$ if and only if $b_i=0$.

Proof. Put

$$\begin{aligned}\gamma_3&=(a_nb_1,\dots,a_nb_{n-1}),\\ \gamma_4&=\frac{1}{|\beta|}\big((a_1^2-a_n^2b_n)b_1,\dots,(a_{n-1}^2-a_n^2b_n)b_{n-1}\big).\end{aligned}$$

By a direct calculation we see that

$$(Z')^tZ'=\mathrm{diag}(a_1^2,\dots,a_{n-1}^2)+\gamma_3^t\gamma_3-\gamma_4^t\gamma_4.$$

From this we calculate that

$$h_{\vec{b}}(a_i^2)=\frac{a_i^2b_i^2}{|\beta|^2}\prod_{1\le j\le n,\, j\ne i}(a_i^2-a_j^2)$$

for $1 \leq i \leq n-1$. Since $h_{\vec{b}}(x)$ is a monic polynomial of degree $n-1$, we get

$$h_{\vec{b}}(x) = \sum_{1\leq i\leq n} \frac{a_i^2 b_i^2}{|\beta|^2} \prod_{1\leq j\leq n,\, j\neq i} (x-a_j^2). \qquad \square$$

Corollary 4.14. *The polynomial $h_{\vec{b}}(x)$ has $n-1$ positive roots, which lie in the intervals*

$$[a_n^2, a_{n-1}^2], \ldots, [a_2^2, a_1^2],$$

respectively.

Proof. First assume that none of b_i is zero. Then by Proposition 4.13, $h_{\vec{b}}(a_i^2)$ and $h_{\vec{b}}(a_{i+1}^2)$ have different signs. Thus, $h_{\vec{b}}(x)$ has a zero in (a_{i+1}^2, a_i^2) for each $1 \leq i \leq n-1$. Hence, the $n-1$ zeros of $h_{\vec{b}}(x)$ lie in the intervals

$$(a_n^2, a_{n-1}^2), \ldots, (a_2^2, a_1^2),$$

respectively. Therefore, $h_{\vec{b}}(x)$ has no double zeros.

In general, among $\{b_1, \ldots, b_n\}$ let $b_{i_1}, \ldots, b_{i_l}$ with $1 \leq i_1 < \cdots < i_l \leq n$ be all non-zero members. Write $I = \{i_1, \ldots, i_l\}$ and $J = \{1, \ldots, n\} - \{i_1, \ldots, i_l\}$. By Proposition 4.13,

$$h_{\vec{b}}(x) = \Big(\sum_{1\leq j\leq l} \frac{a_{i_j}^2 b_{i_j}^2}{|\beta|^2} \prod_{1\leq k\leq l,\, k\neq j} (x-a_{i_k}^2) \Big) \prod_{i\in J} (x-a_i^2).$$

Thus, a_i^2 $(i \in J)$ are zeros of $h_{\vec{b}}(x)$. By a similar argument as above, one shows that other $l-1$ zeros of $h_{\vec{b}}(x)$ lie in the intervals

$$(a_{i_l}^2, a_{i_{l-1}}^2), \ldots, (a_{i_2}^2, a_{i_1}^2),$$

respectively. This shows that: $h_{\vec{b}}(x)$ has $n-1$ positive roots, which lie in the intervals

$$[a_n^2, a_{n-1}^2], \ldots, [a_2^2, a_1^2],$$

respectively. $\square$

By Corollary 4.14, $h_{\vec{b}}(x)$ has at most double zero, and the only possible double zeros are $a_2^2, \ldots, a_{n-1}^2$; for each $2 \leq i \leq n-2$, a_i^2 and a_{i+1}^2 cannot be both double zeros. By (4.3), in order that a_i^2 for $2 \leq i \leq n-1$ is a double zero of $h_{\vec{b}}(x)$ it is necessary and sufficient that $b_i = 0$ and

$$\sum_{1\leq k\leq n,\, k\neq i} \frac{a_k^2 b_k^2}{|\beta|^2} \prod_{1\leq j\leq n,\, j\neq i,k} (a_i^2 - a_j^2) = 0.$$

Let $x_1 \geq \cdots \geq x_{n-1} \geq 0$ be square roots of zeros of $h_{\vec{b}}(x)$. By Corollary 4.14, $a_{i+1} \leq x_i \leq a_i$ for each $1 \leq i \leq n-2$, and $|a_n| \leq x_{n-1} \leq a_{n-1}$. Write

$$\vec{x} = (x_1, \ldots, x_{n-1}).$$

Corollary 4.15. *The map $\vec{b} \mapsto \vec{x}$ give a bijection from B to*

$$[a_2, a_1] \times \cdots \times [a_{n-1}, a_{n-2}] \times [|a_n|, a_{n-1}].$$

Proof. For $\vec{b} = (b_1, \ldots, b_n) \in B$ and $\vec{b'} = (b'_1, \ldots, b'_n) \in B$, suppose $h_{\vec{b}}(x)$ and $h_{\vec{b'}}(x)$ have the same zeros. Then, $h_{\vec{b}} = h_{\vec{b'}}$. Thus, $h_{\vec{b}}(a_i^2) = h_{\vec{b'}}(a_i^2)$ for each $1 \leq i \leq n$. By Proposition 4.13, this implies that $b_i = b'_i$ for each i. Thus, $\vec{b} = \vec{b'}$. This shows the injectivity.

The singular values $x_1, \dots, x_{n-1}$ give a polynomial

$$p(x) = \prod_{1 \le i \le n-1} (x - x_i^2).$$

Since $(-1)^{i-1} f(a_i^2) \ge 0$, we can write

$$p(x) = \sum_{1 \le i \le n} c_i \prod_{1 \le j \le n,\, j \ne i} (x - a_j^2),$$

for some $c_i \ge 0$ with $\sum_{i=1}^{n} c_i = 1$. Hence there exists a unique tuple $\vec{b} \in B$ such that $h_{\vec{b}}(x) = p(x)$. This shows the surjectivity. □

Proposition 4.16. *The image of the moment map $q(\mathcal{O}_f)$ consists of all depth one coadjoint orbits of P with the sign of the Pfaffian equal to the sign of a_n, and with singular values $(x_1, \dots, x_{n-1})$ such that*

$$a_1 \ge x_1 \ge a_2 \ge x_2 \ge \cdots \ge a_{n-1} \ge x_{n-1} \ge |a_n|.$$

Moreover, q maps different P-orbits in $\mathcal{O}_f$ to different P-orbits in $\mathfrak{p}^*$.

Proof. By the form of $q(\bar{n}_{\vec{b}} \cdot f)$ in Lemma 4.10, we have $\beta \ne 0$. Thus, the P-orbit containing $q(\bar{n}_{\vec{b}} \cdot f)$ has depth one. By Lemma 4.12, the Pfaffian of $Z_{\vec{b}}$ has the same sign as the sign of a_n. The other statements follow from Corollary 4.15. □

Lemma 4.17. *For any compact set $\Omega \subset \mathfrak{p}^* - \mathfrak{l}^*$, $q^{-1}(\Omega)$ is compact.*

Proof. Write a general element in $q^{-1}(\Omega)$ as

$$f' = an'm\bar{n}_{\vec{b}} \cdot f,$$

where $a \in A$, $n' \in N$, $m \in M$, and $\vec{b} \in B$. Recall that $f \in \mathfrak{g}^*$ is the elliptic element which was fixed at the beginning of §4.2. Then

$$q(f') = an'm \cdot q(\bar{n}_{\vec{b}} \cdot f) \in \Omega.$$

Note that M and B are compact. Hence, m and $\vec{b}$ are bounded. Write

$$m \cdot q(\bar{n}_{\vec{b}} \cdot f) = \eta_1 + \phi_n(\xi_1),$$

where $m^{-1} \cdot \xi_1$ is given by the vector β as in Lemma 4.10 and $\phi_n \colon \mathfrak{n}^* \to \mathfrak{p}^*$ is defined by the extension by zero on $\mathfrak{l}$. Then

$$\frac{1}{2}|a_n| \le |a_n||\vec{b}| \le |\xi_1| = |\beta| \le |a_1||\vec{b}| \le |a_1|,$$

where we used $\frac{1}{2} \le |\vec{b}| = 1 - b_n \le 1$. Write $an'm \cdot q(\bar{n}_{\vec{b}} \cdot f) = \eta + \phi_n(\xi)$, where $\eta \in \mathfrak{l}^*$ and $\xi \in \mathfrak{n}^*$. By the compactness of $\Omega \subset \mathfrak{p}^* - \mathfrak{l}^*$, $|\eta|$ is bounded from above, and $|\xi|$ is bounded from both above and below. We have

$$an' \cdot (\eta_1 + \phi_n(\xi_1)) = \eta + \phi_n(\xi).$$

Write $n' = \exp(X)$ and $\xi_1 = \mathrm{pr}(Y)$, where $X \in \mathfrak{n}$ and $Y \in \bar{\mathfrak{n}}$. Then,

$$an' \cdot (\eta_1 + \phi_n(\xi_1)) = \eta_1 + \mathrm{pr}([X, Y]) + e^{-\lambda_0 \log a} \phi_n(\xi_1).$$

Thus, $\eta = \eta_1 + \mathrm{pr}([X, Y])$ and $\xi = e^{-\lambda_0 \log a} \xi_1$. Now, $|\xi_1|, |Y|, |\xi|$ are bounded both from above and below, and $|\eta|, |\eta_1|$ are bounded from above. Thus, $\log a$ is bounded both from above and below, and $|X|$ is bounded from above. This shows the compactness of $q^{-1}(\Omega)$. □

Proposition 4.18. *The moment map $q\colon \mathcal{O}_f = G\cdot f \to \mathfrak{p}^*$ has image in the set of depth one elements. For any $g \in G$, the reduced space $q^{-1}(q(g\cdot f))/\operatorname{Stab}_P(q(g\cdot f))$ is a singleton. The moment map q is weakly proper, but not proper.*

Proof. The first and the second statements follow from Proposition 4.16. By Lemma 4.17, we see that q is weakly proper. Since the closure of every depth one orbit contains a depth zero orbit, $q(\mathcal{O}_f)$ is not closed in $\mathfrak{p}^*$. Hence $q\colon \mathcal{O}_f \to \mathfrak{p}^*$ is not proper. □

5. Verification of Duflo's conjecture for $G = \operatorname{Spin}(N, 1)$

The orbit method is based on the concept that unitary representations of Lie groups could be attached to coadjoint orbit, and that algebraic properties of representations could be reflected by geometric properties of coadjoint orbits. Duflo's conjecture (Conjecture 1.1) gives a connection between branching laws of unitary representations and geometry of moment map of coadjoint orbits. In this section we verify Conjecture 1.1 in our setting.

5.1. **Discrete series representations.** We recall how to associate coadjoint orbits to discrete series representations.

Suppose that $G = \operatorname{Spin}(2n, 1)$. Put

$$H' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let

$$\gamma = \Big(a_1 + n - \frac{1}{2}, a_2 + n - \frac{3}{2}, \dots, a_n + \frac{1}{2}\Big) \in \Lambda_0$$

be a regular integral weight so that $a_1, \dots, a_n$ are all integers or all half-integers and $a_1 \geq \cdots \geq a_n \geq 0$. Let $\pi^+(\gamma)$ (resp. $\pi^-(\gamma)$) be a discrete series representation of G with infinitesimal character γ and the lowest K-type V_{K,λ^+} (resp. V_{K,λ^-}), where

$$\lambda^+ = (a_1 + 1, \cdots, a_n + 1) \text{ and } \lambda^- = (a_1 + 1, \cdots, a_{n-1} + 1, -(a_n + 1)).$$

In view of Remark 2.1, the orbit $\mathcal{O}$ associated to $\pi^+(\gamma)$ is $G \cdot \iota(t_{-\gamma})$, where

$$t_{-\gamma} = -\operatorname{diag}\Big(\Big(a_1 + n - \frac{1}{2}\Big)H', \Big(a_2 + n - \frac{3}{2}\Big)H', \dots, \Big(a_n + \frac{1}{2}\Big)H', 0\Big).$$

Putting

$$\vec{a}' = (a_1', \dots, a_n') = \Big(a_1 + n - \frac{1}{2}, a_2 + n - \frac{3}{2}, \dots, (-1)^n\Big(a_n + \frac{1}{2}\Big)\Big),$$

$t_{-\gamma}$ is G-conjugate to $t_{\vec{a}'}$. Hence $\mathcal{O} = G \cdot \iota(t_{\vec{a}'})$. Similarly, the orbit associated to $\pi^-(\gamma)$ is the coadjoint G-orbit through

$$\iota\Big(\operatorname{diag}\Big(\Big(a_1 + n - \frac{1}{2}\Big)H', \Big(a_2 + n - \frac{3}{2}\Big)H', \dots, (-1)^{n-1}\Big(a_n + \frac{1}{2}\Big)H', 0\Big)\Big).$$

For P-representations, let $V_{M',\mu}$ be an irreducible representation of M' with highest weight $\mu = (b_1, \dots, b_{n-1})$. Let $I_{P,V_{M',\mu}} = \operatorname{Ind}_{M'N}^P(V_{M',\mu} \otimes e^{\mathbf{i}\xi_0})$ be the unitarily induced representation of P. Then the corresponding orbit is $P \cdot \operatorname{pr}(Z_{Y,\beta})$ in the notation of §4.1 such that the singular values of Y are

$$(x_1, \dots, x_{n-1}) = (b_1 + n - 2,\, b_2 + n - 3, \dots, b_{n-2} + 1, |b_{n-1}|)$$

and the sign of the Pfaffian of $Z_{Y,\beta}$ equals the sign of $(-1)^{n-1} b_{n-1}$.

Theorem 5.1. *Let P be a minimal parabolic subgroup P of $G = \mathrm{Spin}(2n, 1)$. Let π be a discrete series representation of G, which is associated to a regular elliptic coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. Write $q\colon \mathcal{O} \to \mathfrak{p}^*$ for the moment map.*

1. *The restriction of $\bar{\pi}$ to P decomposes into a finite direct sum of irreducible unitarily induced representations of P from $M'N$, and this decomposition is multiplicity-free.*
2. *Let τ be an irreducible unitarily induced representation of P which is associated to a coadjoint orbit $\mathcal{O}' \subset \mathfrak{p}^*$. Assume that $Z(G)$, the center of G, acts by the same scalar on π and on τ. Then for τ to appear in $\pi|_P$ it is necessary and sufficient that $\mathcal{O}' \subset q(\mathcal{O})$.*
3. *The moment map $q\colon \mathcal{O} \to \mathfrak{p}^*$ is weakly proper, but not proper.*
4. *For every coadjoint P-orbit $\mathcal{O}' \subset \mathfrak{p}^*$, the reduced space $q^{-1}(\mathcal{O}')/P$ is a singleton.*

Proof. (1) follows from Theorem 3.22 for discrete series. (3) and (4) follow from Proposition 4.18. It remains to show (2), that is, to compare the restriction of discrete series representations and the image of moment map of corresponding coadjoint orbits.

For $a_1 \geq a_2 \geq \cdots \geq a_n \geq -\frac{1}{2}$,

$$\gamma = \Big(a_1 + n - \frac{1}{2}, a_2 + n - \frac{3}{2}, \ldots, a_n + \frac{1}{2}\Big) \in \Lambda_0 \cup \Lambda_n,$$

the restriction of the (limit of) discrete series $\pi^+(\gamma)$ is given by Theorem 3.22:

$$\bar{\pi}^+(\gamma)|_P = \bigoplus_{\mu} I_{P,V_{M',\mu}},$$

where $\mu = (b_1, \ldots, b_{n-1})$ runs over tuples such that

$$a_1 + 1 \geq b_1 \geq a_2 \geq \cdots \geq b_{n-2} \geq a_{n-1} + 1 \geq -b_{n-1} \geq a_n + 1 \tag{5.1}$$

and $b_i - a_1 \in \mathbb{Z}$. On the other hand, the moment map image of the corresponding orbit $\mathcal{O}$ was studied in §4.3. Let $\mathcal{O}'$ be a coadjoint P-orbit which corresponds to a unitary representation $\tau = I_{P,V_{M',\mu}}$ with $\mu = (b_1, \ldots, b_{n-1})$. Then the singular values for the P-orbit $\mathcal{O}'$ are

$$(x_1, \ldots, x_{n-1}) = (b_1 + n - 1, \ldots, b_{n-2} + 1, |b_{n-1}|)$$

and the sign of the Pfaffian equals $\mathrm{sgn}(-1)^{n-1} b_{n-1}$. Assume that the center $Z(G)$ acts on π and τ by the same scalar, which is equivalent to $b_i - a_i \in \mathbb{Z}$. Then by Proposition 4.16, $\mathcal{O}' \subset q(\mathcal{O})$ if and only if

$$a_1 + n - \frac{1}{2} \geq x_1 \geq a_2 + n - \frac{3}{2} \geq x_2 \geq \cdots \geq a_{n-1} + \frac{3}{2} \geq x_{n-1} \geq a_n + \frac{1}{2}$$

and the sign of Pfaffian equals $(-1)^n$. Under our assumption $b_1 - a_i \in \mathbb{Z}$, this is equivalent to (5.1). Therefore, Statement (2) for $\pi = \pi^+(\gamma)$ is proved. The case of $\pi = \pi^-(\gamma)$ is similar. ☐

Appendix A. Obtaining $\bar{\pi}|_{MN}$ from $\bar{\pi}|_K$

By the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$, we have $\mathfrak{k} = \mathfrak{m} \oplus \{X + \theta(X) : X \in \mathfrak{n}\}$. By this, we can view $\mathfrak{m} + \mathfrak{n}$ as the limit $\lim_{t \to +\infty} \mathrm{Ad}(\exp(tH_0))(\mathfrak{k})$. On the group level, we can view MN as the limit $\lim_{t \to +\infty} \exp(tH_0) K \exp(-tH_0)$. From this viewpoint, we may expect that $\bar{\pi}|_{MN}$

is determined by $\bar{\pi}|_K$ for any unitarizable irreducible representation π of G. Here we observe the relationship between two restrictions. The writing of this section is motivated by a question of Professor David Vogan.

As in §2, write $I_{P,\tau} = \mathrm{Ind}_{M'N}^{P}(\tau \otimes e^{\mathbf{i}\xi_0})$ for a unitarily induced representation of P where τ is a finite-dimensional unitary representation of M'. Then, $I_{P,\tau}$ is irreducible when τ is so. For any finite-dimensional unitary representation τ of M' and any $0 \neq t \in \mathbb{R}$, write $I_{t,\tau} = \mathrm{Ind}_{M'N}^{MN}(\tau \otimes e^{\mathbf{i}t\xi_0})$ for a unitarily induced representation of MN. Then, $I_{t,\tau}$ is irreducible when τ is so.

Using Mackey's theory for unitarily induced representations and considering the action of MN on $P/M'N$, Lemma A.1 follows easily.

Lemma A.1. *We have*

$$I_{P,\tau}|_{MN} \cong \int_{t>0}^{\oplus} I_{t,\tau}\,\mathrm{d}t.$$

Corollary A.2. *Let π be an infinite-dimensional irreducible unitarizable representation of $G = \mathrm{Spin}(m+1,1)$. Then $\bar{\pi}|_P$ and $\bar{\pi}|_{MN}$ determine each other.*

Proof. We have shown that $\bar{\pi}|_P$ is a finite direct sum of $I_{P,\tau}$. Then, the conclusion follows as the spectra of $I_{P,\tau}|_{MN}, I_{P,\tau'}|_{MN}$ are disjoint whenever $\tau \ncong \tau'$. □

In §3.1, we constructed a homomorphism

$$\Psi\colon K(G) \to K(M').$$

Write $\widehat{K}$ for the set of isomorphism classes of finite-dimensional irreducible representations of K and write $\mathbb{Z}^{\widehat{K}}$ for the abelian group of functions $\widehat{K} \to \mathbb{Z}$ with addition given by point-wise addition. Taking the multiplicities of irreducible representations of K appearing in $\pi|_K$ ($\pi \in \mathcal{C}(G)$), we obtain a homomorphism

$$m\colon K(G) \to \mathbb{Z}^{\widehat{K}}.$$

Write $\mathbb{Z}(K)$ for the quotient group of $\mathbb{Z}^{\widehat{K}}$ by the subgroup of functions $f\colon \widehat{K} \to \mathbb{Z}$ such that $\sharp\{[\sigma] \in \widehat{K} : f([\sigma]) \neq 0\}$ is finite. Let

$$p\colon \mathbb{Z}^{\widehat{K}} \to \mathbb{Z}(K)$$

be the quotient map.

As in §2, put $n = \lfloor \frac{m+2}{2} \rfloor$ and $n' = \lfloor \frac{m+1}{2} \rfloor$. Then,

$$n = \begin{cases} n' & \text{if } m \text{ is odd;} \\ n'+1 & \text{if } m \text{ is even.} \end{cases}$$

The ranks of $K = \mathrm{Spin}(m+1)$, $M = \mathrm{Spin}(m)$, $M' = \mathrm{Spin}(m-1)$ are equal to n', $n-1$, $n'-1$, respectively. For a highest weight $\vec{b} = (b_1, \dots, b_{n'-1})$ of M', write $V_{M',\vec{b}}$ for an irreducible representation of M' with highest weight $\vec{b}$. Then, $[V_{M',\vec{b}}]$ is a basis of $K(M')$. Let

$$\phi\colon K(M') \to \mathbb{Z}(K)$$

be defined by

$$\phi([V_{M',\vec{b}}]) = \sum_{k \geq 0} [V_{K,(k+b_1,b_1,\dots,b_{n'-2},(-1)^m b_{n'-1})}].$$

Proposition A.3. *We have $\phi \circ \Psi = p \circ m$.*

Proof. When m is even, $K(G)$ is generated by induced representations $I(\sigma,\nu)$ and finite-dimensional representations. Then, the conclusion follows from Proposition 3.7 and branching laws for the pair $M \subset K$ (giving K types of induced representations).

When m is odd, $K(G)$ is generated by induced representations $I(\sigma,\nu)$, (limits of) discrete series and finite-dimensional representations. Then, the conclusion follows from Proposition 3.7, branching laws for the pair $M \subset K$, Theorem 3.22 and Blattner's formula (giving K types of discrete series and limits of discrete series). □

Corollary A.4. *For any $\pi \in \mathcal{C}(G)$, $\Psi(\pi)$ is determined by $\pi_K|_K$.*

Proof. Note that $\Psi(\pi)$ is a finite direct sum of finite-dimensional irreducible unitary representations of M'. Then, the conclusion follows from Proposition A.3 directly. □

Corollary A.5. *Let π be a unitarizable irreducible representation of G. Then $\bar{\pi}|_{MN}$ is determined by $\bar{\pi}|_K$.*

Proof. When π is a unitarizable irreducible representation, $\bar{\pi}|_K$ and $\pi_K|_K$ determine each other. By Corollary A.4, $\Psi(\pi)$ is determined by $\pi_K|_K$. By Lemma 3.6, $\bar{\pi}|_P$ and $\Psi(\pi)$ determine each other. By Corollary A.2, $\bar{\pi}|_P$ and $\bar{\pi}|_{MN}$ determine each other. Then, the conclusion of the corollary follows. □

Appendix B. A case of Bessel model and relation with the local GGP conjecture

Results in this paper determine branching laws arising in a special case of the Bessel model in the local Gan-Gross-Prasad conjecture ([16, Conjecture 17.1], [17, Conjecture 6.1]). Precisely to say, as in [16, §2], take a vector space V over $\mathbb{R}$ with a non-degenerate symmetric bilinear form of signature $(m+1,1)$. Let W be a codimension 3 non-degenerate subspace of V such that its orthogonal complement $W^{\perp}$ contains an isotropic subspace of dimension 1. Then, we can determine branching laws in the Bessel model of the local GGP conjecture associated to the pair (V,W). Note that Bessel models were studied by Gomez-Wallach in more general setting [18].

Take $G_3 = \mathrm{SO}(m+1,1)$ and let $P_3 = M_3AN$ be a standard minimal parabolic subgroup. Put $H_3 = M_3' \ltimes N$. Define categories $\mathcal{C}(G_3)$, $\mathcal{C}(P_3)$, $\mathcal{C}(M_3)$ similarly as that for $\mathcal{C}(G)$, $\mathcal{C}(P)$, $\mathcal{C}(M)$ in §2. For any $\pi \in \mathcal{C}(P_3)$, as in §2, define $\Psi(\pi) = \pi/\mathfrak{m}_{\xi_0}\cdot\pi$. Then $\Psi(\pi) \in \mathcal{C}(M_3')$ and Ψ defines a functor $\mathcal{C}(P_3) \to \mathcal{C}(M_3')$. Then, for any $\pi \in \mathcal{C}(G_3)$ we have

$$\Psi(\pi) \cong \bigoplus_{\tau \in \widehat{M_3'}} n_\tau(\pi)\tau,$$

where $n_\tau(\pi) = \dim \mathrm{Hom}_{H_3}(\pi, \tau \otimes e^{\mathbf{i}\xi_0})$. By an analogue of Proposition 3.7 for the pair $P_3 \subset G_3$ and Casselman's subrepresentation theorem, it follows that: when $\pi \in \mathcal{C}(G_3)$ is irreducible, $n_\tau(\pi) = 0$ or 1 (the multiplicity one theorem) and there are only finitely many $\tau \in \widehat{M_3'}$ such that $n_\tau(\pi) = 1$. Moreover, $\Psi(\pi)$ are all calculated with the Langlands parameter of π by results in this paper. Hence, we know $n_\tau(\pi) = 1$ for exactly which pairs (π,τ). We shall remark that local root numbers are used for predicting branching laws in the local Gan-Gross-Prasad conjecture. However, we stick to Langlands parameters for the description of branching laws.

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