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UNIPOTENT CHARACTER SHEAVES AND STRATA OF A REDUCTIVE GROUP

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ABSTRACT. Let H be a connected reductive group over an algebraically closed field. We define a surjective map from the set $CS(H)$ of unipotent character sheaves on H (up to isomorphism) to the set of strata of H. To do this we use the generalized Springer correspondence. We also give a new parametrization of $CS(H)$ in terms of data coming from bad characteristic.

INTRODUCTION

0.1. Let H be a reductive connected group over $\mathbf{C}$. Let $Pr = \{2, 3, 5, \dots\}$ be the set of prime numbers; let $\overline{Pr} = Pr \cup \{0\}$. For $r \in Pr$ let $\mathbf{k}_r$ be an algebraic closure of a finite field with r elements and let H_r be a reductive connected group over $\mathbf{k}_r$ of the same type as H and with the same Weyl group W (with set of simple reflections $\{s_i; i \in I\}$). We set $\mathbf{k}_0 = \mathbf{C}, H_0 = H$. Let $K_r = \mathbf{C}$ (if $r = 0$) and $K_r = \bar{\mathbf{Q}}_l$ where $l \in Pr - \{r\}$ (if $r \in Pr$). For $r \in \overline{Pr}$ let $CS(H_r)$ be the (finite) set of isomorphism classes of unipotent character sheaves on H_r. These are certain simple perverse K_r-sheaves on H_r, see [L85]. It is known that $CS(H_r)$ is independent of r in a canonical way. Let $Str(H_r)$ be the (finite) set of strata of H_r, see [L15]; these are certain subsets of H_r (unions of conjugacy classes of fixed dimension) which form a partition of H_r. (These subsets are locally closed in H_r, see [C20].)

In this paper we shall define for any $r \in \overline{Pr}$ a surjective map

(a) $$\tau : CS(H_r) \to Str(H_r).$$

In the remainder of this paper (except in 1.1 and 3.3) we shall assume that either H is quasi-simple, that is, H modulo its centre is simple, or that H is a torus. (The general case can be reduced in an obvious way to this case.)

Our definition of the map (a) is based on the generalized Springer correspondence of [L84], especially in bad characteristic.

In 1.11 we use the map (a) to give a new parametrization of $CS(H_r)$ which differs from the known classification [L86] in terms of two-sided cells in the Weyl group. This involves associating to each stratum a very small collection of finite groups which come from unipotent classes in bad characteristic. (It would be interesting to give a definition of these finite groups and of the resulting parametrization which is purely in characteristic 0.)

This can be also viewed as a parametrization of

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(b) $Un(H_r(F_q))$, the set of unipotent representations of the group $H_r(F_q)$ of F_q-rational points of a split form of H_r over a finite subfield F_q of $\mathbf{k}_r$ (with $r \in Pr$ and q a power of r).

Indeed, it is known that

(c) $CS(H_r), Un(H_r(F_q))$ are in a natural $1-1$ correspondence.

We will show elsewhere that similar results hold when H is replaced by a connected component of a disconnected reductive group with identity component H.

A number of results of this paper rely on the wonderful paper [S85] of Nicolas Spaltenstein and also on [LS85].

0.2. **Notation.** Assume that G is a connected reductive group over an algebraically closed field. We denote by Z_G the centre of G and by Z_G^0 its identity component. Let G_{ad} be the adjoint group of G.

For $g \in G$ we denote by $Z_G(g)$ the centralizer of g in G and by $Z_G^0(g)$ its identity component.

If G' is a subgroup of G, we denote by $N_G(G')$ the normalizer of G' in G.

If $\mathcal{W}$ is a Weyl group we denote by $\mathrm{Irr}(\mathcal{W})$ the set of isomorphism classes of irreducible representations of $\mathcal{W}$ over $\mathbf{Q}$.

1. Definition of the map τ

1.1. For $r \in \overline{Pr}$ let $CS^{\emptyset}(H_r)$ be the subset of $CS(H_r)$ consisting of unipotent cuspidal character sheaves.

Let $A \in CS^{\emptyset}(H_r)$. The support of A is the closure in H_r of a single orbit of $Z_{H_r}^0 \times H_r$ acting on H_r by $(z,g) : g_1 \mapsto zgg_1g^{-1}$; this orbit is denoted by σ_A. Let $\delta(A)$ be the dimension of the variety of Borel subgroups of H_r that contain a fixed element $h \in \sigma_A$ (this is independent of the choice of h). We have

$$CS^{\emptyset}(H_r) = \sqcup_{d \in \mathbf{N}} CS_d^{\emptyset}(H_r)$$

where $CS_d^{\emptyset}(H_r) = \{A \in CS^{\emptyset}(H_r); \delta(A) = d\}$.

Lemma 1.2. *For any $d \in \mathbf{N}$ the function $r \mapsto \sharp(CS_d^{\emptyset}(H_r))$ from $\overline{Pr}$ to $\mathbf{N}$ is constant; its value is denoted by $N_d(H) \in \mathbf{N}$.*

This can be deduced from the results in [L86]. (When H_{ad} is of type E_8 or F_4, the results in *loc. cit.* are proved only under the assumption that r is not a bad prime for H. But the same proof works without this assumption, by making use of [S85, p. 336, 337].)

1.3. For $r \in \overline{Pr}$ and $J \subset I$ we fix a Levi subgroup $L_{J,r}$ of a parabolic subgroup of H_r of type J. (For example, $L_{I,r} = H_r$ and $L_{\emptyset,r}$ is a maximal torus.) We say that J is *cuspidal* if for some (or equivalently any) $r \in \overline{Pr}$ we have $CS^{\emptyset}(L_{J,r}) \neq \emptyset$. In this case $L_{J,r}$ is quasi-simple or a torus and J is uniquely determined by the type of $(L_{J,r})_{ad}$. (This follows from the classification of cuspidal character sheaves.) Let W_J be the Weyl group of $L_{J,r}$, viewed as a parabolic subgroup of W.

Let us now fix a cuspidal J and $A' \in CS^{\emptyset}(L_{J,r})$. The induced object $\mathrm{ind}(A')$ is a well defined semisimple perverse sheaf on H_r (see [L85, §4]); it is in fact a direct sum of character sheaves on H_r. By arguments in [L84, §3, §4], $\mathrm{End}(\mathrm{ind}(A'))$ has a canonical decomposition as a direct sum of lines $\oplus_w \mathcal{L}_w$ with w running through $N_{H_r}(L_{J,r})/L_{J,r} = N_W(W_J)/W_J$ such that $\mathcal{L}_w \mathcal{L}_{w'} = \mathcal{L}_{ww'}$ for any w, w' in $N_W(W_J)/W_J$. One can verify that there is a unique $A \in CS(H_r)$ such that A is a summand with multiplicity one of $\mathrm{ind}(A')$ and the value of the a-function of W on the two-sided cell of W attached to A is equal to the value of the a-function of W_J on the two-sided cell of W_J attached to A'. Now the summand A of $\mathrm{ind}(A')$ is stable under each $\mathcal{L}_w$ and we can choose uniquely a nonzero vector $t_w \in \mathcal{L}_w$ which acts on A as identity. We have $t_w t_{w'} = t_{ww'}$ for any w, w' in $N_W(W_J)/W_J$. We see that $\mathrm{End}(\mathrm{ind}(A'))$ is canonically the group algebra of $N_W(W_J)/W_J$ (which is known to be a Weyl group). For any $E' \in \mathrm{Irr}(N_W(W_J)/W_J)$ let $A'[E']$ be the perverse sheaf $\mathrm{Hom}_{N_W(W_J)/W_J}(E', \mathrm{ind}(A'))$ on H_r. This is an object of $CS(H_r)$.

1.4. Let $CS'(H_r)$ be the set of triples (J, E', A') where J is a cuspidal subset of I, $E' \in \mathrm{Irr}(N_W(W_J)/W_J)$ and $A' \in CS^{\emptyset}(L_{J,r})$. We have a bijection

(a) $$CS'(H_r) \xrightarrow{\sim} CS(H_r)$$

given by $(J, E', A') \mapsto A'[E']$.

1.5. Let $r \in \overline{Pr}$. Let $\mathcal{U}(H_r)$ be the set of unipotent classes in H_r; for $\gamma \in \mathcal{U}(H_r)$ the Springer correspondence (defined for any r in [L84]) associates to γ and the constant local system K_r on γ an element $e_r(\gamma) \in \mathrm{Irr}(W)$. Thus we have a well defined (injective) map $e_r : \mathcal{U}(H_r) \to \mathrm{Irr}(W)$, whose image is denoted by $\mathrm{Irr}_r(W)$.

Let $CS^{\emptyset}(H_r)^{un}$ be the set of all $A \in CS^{\emptyset}(H_r)$ such that $\sigma_A = Z^0_{H_r}\gamma_A$ where $\gamma_A \in \mathcal{U}(H_r)$. Let

$$CS'(H_r)^{un} = \{(J, E', A') \in CS'(H_r); A' \in CS^{\emptyset}(L_{J,r})^{un}\}.$$

We define a map

(a) $$\tilde{e}_r : CS'(H_r)^{un} \to \mathrm{Irr}_r(W)$$

as follows. Let $(J, E', A') \in CS'(H_r)^{un}$. Then the unipotent class $\gamma_{A'}$ of $L_{J,r}$ is defined; the restriction of A' to $\gamma_{A'}$ is (up to a shift) a cuspidal local system. Now the generalized Springer correspondence [L84] associates to this cuspidal local system and to E' a unipotent class γ of H_r and an irreducible local system on it. By definition, we have $\tilde{e}_r(J, E', A') = e_r(\gamma)$.

1.6. Let

$$\mathrm{Irr}_*(W) = \cup_{r \in Pr} \mathrm{Irr}_r(W) = \cup_{r \in \overline{Pr}} \mathrm{Irr}_r(W).$$

Let $r \in \overline{Pr}$. In [L15] a bijection

(a) $$Str(H_r) \to \mathrm{Irr}_*(W)$$

is defined. Using this and 1.4(a), we see that defining τ in 0.1(a) is the same as defining a map

$$\underline{\tau}_r : CS'(H_r) \to \mathrm{Irr}_*(W).$$

Lemma 1.7. *Let $d \in \mathbf{N}$ be such that $N_d(H) > 0$ (see 1.2). Let $X = \{r \in Pr; CS^{\emptyset}_d(H_r) \subset CS^{\emptyset}(H_r)^{un}\}$. Then one of the following holds.*

(i) *X consists of a single element r_0.*

(ii) *$X = Pr$ and $d \geq 1$. (In this case H_{ad} is of type E_8, F_4 or G_2, d is $16, 4, 1$ respectively and $N_d(H) = 1$.)*

(iii) *$X = Pr$ and $d = 0$. (In this case H is a torus.)*

(iv) *$X = \emptyset$. (In this case $d = 0$ and H_{ad} is of type E_8, F_4 or G_2.)*

This follows from 3.2.

1.8. Let $r \in \overline{Pr}$. We will now define the map $\underline{\tau}_r : CS'(H_r) \to \mathrm{Irr}_*(W)$. In the case where $I = \emptyset$, this map is the bijection between two sets with one element. Assume now that $I \neq \emptyset$. Let $(J, E', A') \in CS'(H_r)$. We want to define $\underline{\tau}_r(J, E', A')$.

Let X be as in Lemma 1.7 for $L_{J,r}$ instead of H_r and for $d \in \mathbf{N}$ defined by $A' \in CS_d^{\emptyset}(L_{J,r})$.

Assume first that $J \neq I, J \neq \emptyset$. Then X is not as in 1.7(ii),(iii),(iv), hence it is as in 1.7(i). Let $r_0 \in Pr$ be such that $X = \{r_0\}$. We set $\underline{\tau}_r(J, E', A') = \tilde{e}_{r_0}(J, E', A')$.

Next we assume that $J = I$ and X is as in 1.7(i). Let $r_0 \in Pr$ be such that $X = \{r_0\}$. We set $\underline{\tau}_r(J, E', A') = \tilde{e}_{r_0}(J, E', A')$.

Next we assume that $J = I$ and d, X are as in 1.7(ii). We have $E' = 1$. We set $\underline{\tau}_r(I, 1, A') = \tilde{e}_{r'}(I, 1, A')$ where $r' \in \overline{Pr}$ (this is independent of r' by results in [S85]).

If $J = I$ then d cannot be as in 1.7(iii) since this would imply $I = \emptyset$, contrary to our assumption.

Assume now that $J = I$ and X is as in 1.7(iv). We have $E' = 1$. We set $\underline{\tau}_r(I, 1, A') =$ unit representation.

Finally, assume that $J = \emptyset$. Then A' is the constant sheaf K_r. For any $r' \in Pr$ we set $\tilde{e}_{r'}(\emptyset, E', K_r) = E'_{r'}$. If $E'_{r'}$ is independent of r', then $\underline{\tau}_r(\emptyset, E', K_r)$ is defined to be this constant value of $E'_{r'}$. If $E'_{r'}$ is not independent of r', then there is a unique $r_0 \in Pr$ such that $E'_{r'}$ is constant for $r' \in Pr - \{r_0\}$. (This is an issue only in exceptional types where it can be checked from the tables in [S85].) We then set $\underline{\tau}_r(\emptyset, E', K_r) = E'_{r_0}$. This completes the definition of $\underline{\tau}_r$ hence also that of τ in 0.1(a).

1.9. Let $r \in \overline{Pr}$. If H_{ad} is of classical type or of type E_6, E_7 or F_4, then for any $E' \in \mathrm{Irr}(W)$, $E'_{r'}$ is constant for $r' \in Pr - \{2\}$. It follows that

$$\underline{\tau}_r(\emptyset, E', K_r) = E'_2.$$

Hence if $E' \in \mathrm{Irr}_2(W)$, then $\underline{\tau}_r(\emptyset, E', K_r) = E'$. If H_{ad} is of type G_2, then for any $E' \in \mathrm{Irr}(W)$, $E'_{r'}$ is constant for $r' \in Pr - \{3\}$. It follows that

$$\underline{\tau}_r(\emptyset, E', K_r) = E'_3.$$

Hence if $E' \in \mathrm{Irr}_3(W)$, then $\underline{\tau}_r(\emptyset, E', K_r) = E'$. We see that if H_{ad} is not of type E_8, then $\underline{\tau}_r(\emptyset, E', K_r) = E'$ for $E' \in \mathrm{Irr}_*(W)$. The same holds if H_{ad} is of type E_8 (we use the tables in [S85]).

Note that $\mathrm{Irr}_*(W)$ can be viewed as a subset of $CS'(H_r)$ by $E' \mapsto (\emptyset, E', K_r)$. The results above show that $\underline{\tau}_r$ can be viewed as a retraction of $CS'(H_r)$ onto its subset $\mathrm{Irr}_*(W)$. In particular, $\underline{\tau}_r$ is surjective.

1.10. Let $E \in \mathrm{Irr}_*(W)$. Let $\overline{Pr}(E) = \{r' \in \overline{Pr}; E \in \mathrm{Irr}_{r'}(W)\}$. For $r' \in \overline{Pr}(E)$ we denote by γ_E the unique element of $\mathcal{U}_{H_{r'}}$ such that $e_{r'}(\gamma_E) = E$ (see 1.5). We set $\mathcal{A}_{r',E} = Z_{(H_{r'})_{ad}}(u)/Z^0_{(H_{r'})_{ad}}(u)$ where u is in the image of γ_E under $H_{r'} \to (H_{r'})_{ad}$; this finite group is well defined up to isomorphism. If $\overline{Pr}(E) = \overline{\mathrm{Pr}}$ we define $\overline{Pr}'(E) = \{r' \in \overline{Pr}; \mathcal{A}_{r',E} \cong \mathcal{A}_{0,E}\}$; this is a subset of $\overline{Pr}$ with finite complement.

We define a finite collection $c(E)$ of finite groups as follows.

If $\overline{Pr}(E) = \overline{\mathrm{Pr}} = \overline{Pr}'(E)$, then $c(E)$ consists of $\mathcal{A}_{0,E}$.

If $\overline{Pr}(E) = \overline{\mathrm{Pr}} \neq \overline{Pr}'(E)$, then $c(E)$ consists of $\{\mathcal{A}_{r',E}; r' \in \overline{\mathrm{Pr}} - \overline{Pr}'(E)\}$; one can verify that for $r' \neq r''$ in $\overline{\mathrm{Pr}} - \overline{Pr}'(E)$, we have $\mathcal{A}_{r',E} \not\cong \mathcal{A}_{r'',E}$.

If $\overline{Pr}(E) \neq \overline{\mathrm{Pr}}$, then $\overline{Pr}(E)$ consists of a single element $r'_0 \in \overline{Pr}$ (we have necessarily $r'_0 \neq 0$); then $c(E)$ consists of $\mathcal{A}_{r'_0,E}$.

If H is a torus, then $c(E)$ consists of $\{1\}$. If H_{ad} is of type A, B, C or D, then $c(E)$ consists of a single group and this is a product of cyclic groups of order 2. If H_{ad} is of exceptional type then $c(E)$ consists of one of the following groups:

(a) $$1,\ \mathcal{C}_2,\ \mathcal{C}_2 \times \mathcal{C}_2,\ S_3,\ \Delta_8,\ S_3 \times \mathcal{C}_2,\ S_5$$

or one of the pair of groups:

(b) $$(\mathcal{C}_2, \mathcal{C}_3),\ (\mathcal{C}_4, \mathcal{C}_3),\ (\mathcal{C}_2 \times \mathcal{C}_2, \mathcal{C}_2 \times \mathcal{C}_3)$$

or the triple of groups:

(c) $$(\mathcal{C}_4, \mathcal{C}_3, \mathcal{C}_5).$$

(See the tables in §2.) Here $\mathcal{C}_m$ denotes a cyclic group of oder m, S_m denotes the symmetric group in m letters, Δ_8 denotes a dihedral group of order 8.

We now define a finite set $c(E)^*$ as follows. If $c(E)$ consists of a single group Γ then $c(E)^* = \hat{\Gamma}$. (For a finite group Γ we denote by $\hat{\Gamma}$ the set of isomorphism classes of irreducible representations of Γ over K_r.)

If $c(E)$ consists of two groups Γ, Γ' (see (b)), then $\Gamma'' = \mathcal{A}_{0,E}$ is well defined and is a quotient of both Γ, Γ'. (We have $\Gamma'' = 1, 1, S_2$ respectively in the three cases in (b).) Hence we can regard $\hat{\Gamma}''$ as a subset of $\hat{\Gamma}$ and also as a subset of $\hat{\Gamma}'$. We define $c(E)^* = (\hat{\Gamma} - \hat{\Gamma}'') \sqcup (\hat{\Gamma}' - \hat{\Gamma}'') \sqcup \hat{\Gamma}''$.

If $c(E)$ consists of three groups (see (c)) we define $c(E)^* = \sqcup_{m \in [1,6]} \hat{\mathcal{C}}^!_m$ where $\hat{\mathcal{C}}^!_m$ consists of the faithful irreducible representations of $\mathcal{C}_m$. (This case occurs only when H_{ad} is of type E_8 and when $E = 1$. The fact that $\hat{\mathcal{C}}^!_6$ enters in the definition should be connected to the fact that 6 appears as a coefficient of the highest root of H.)

The following theorem can be deduced from the definitions using the results in §2.

Theorem 1.11. *Let $r \in \overline{Pr}$. There exists a bijection*

$$CS(H_r) \xrightarrow{\sim} \sqcup_{E \in \mathrm{Irr}_*(W)} c(E)^*$$

which makes the following diagram commutative:

$$\begin{array}{ccc} CS(H_r) & \xrightarrow{\ \sim\ } & \sqcup_{E \in \mathrm{Irr}_*(W)} c(E)^* \\ {\scriptstyle\tau}\downarrow & & \downarrow \\ Str(H_r) & \xrightarrow{\ \sim\ } & \mathrm{Irr}_*(W) \end{array}$$

(The left vertical map is as in 0.1(a)*; the right vertical map is the obvious one; the lower horizontal map is as in* 1.6(a)*.)*

2. Examples

2.1. Assume that H_{ad} is of type A_{n-1}, $n \geq 2$. We have

$$CS'(H_r) = \{(\emptyset, E', K_r); E' \in \mathrm{Irr}(W)\}, \quad \mathrm{Irr}_*(W) = \mathrm{Irr}(W).$$

In this case $\underline{\tau}_r$ is the bijection $(\emptyset, E', K_r) \mapsto E'$.

2.2. Assume that H_{ad} is of type D_n, $n \geq 4$ or B_n, $n \geq 3$, or C_n, $n \geq 2$. If H_{ad} is of type D_n, let $CS''(H)$ be the set of pairs (J, E') where J is either $\emptyset$ (so that $N_W(W_J)/W_J = W$) or J is such that W_J is of type D_{4k^2} for some $k \geq 1$ with $4k^2 \leq n$ (so that $N_W(W_J)/W_J$ is a Weyl group of type B_{n-4k^2}) and $E' \in \mathrm{Irr}(N_W(W_J)/W_J)$. (We use the convention that a Weyl group of type B_0 is $\{1\}$.)

If H_{ad} is of type B_n or C_n, let $CS''(H)$ be the set of pairs (J, E') where J is either $\emptyset$ (so that $N_W(W_J)/W_J = W$) or J is such that W_J is of type $B_{k(k+1)}$ for some $k \geq 1$ with $k(k+1) \leq n$ (so that $N_W(W_J)/W_J$ is a Weyl group of type $B_{n-k(k+1)}$) and $E' \in \mathrm{Irr}(N_W(W_J)/W_J)$.

In any case we have a bijection $CS'(H_r) \xrightarrow{\sim} CS''(H)$ given by $(J, E', A') \mapsto (J, E')$. Moreover we have $\mathrm{Irr}_*(W) = \mathrm{Irr}_2(W)$. Hence the map $\underline{\tau}_r$ can be viewed as a map

$$CS''(H) \to Irr_2(W). \tag{a}$$

Now $CS''(H)$ can also be viewed as the set of pairs consisting of a cuspidal J and a cuspidal local system on a unipotent class in $L_{J,2}$. The generalized Springer correspondence [L84] attaches to such a pair a unipotent class in H_2 and an irreducible local system on it.

By forgetting this last local system and by identifying $\mathcal{U}(H_2)$ with $\mathrm{Irr}_2(W)$ via e_2 (see 1.5), we obtain a map $CS''(H_r) \to \mathrm{Irr}_2(W)$ which, on the one hand, is explicitly computed in [LS85] in terms of certain types of symbols and, on the other hand, it coincides with the map (a).

2.3. In 2.4-2.8 we describe the map $\underline{\tau}_r$ in terms of tables in the case where H_{ad} is of type G_2, F_4, E_6, E_7 or E_8. The tables are computed using results in [S85] with one indeterminacy in type E_8 being removed by [H22].

In each case the table consists of a sequence of rows. There is one row for each $E \in \mathrm{Irr}_*(W)$; it is written as $()'$.......$()''$ where $()'$ represents the fibre of $\underline{\tau}_r$ over E and $()''$ is a sequence of finite groups of which the boxed ones describe $c(E)$.

The elements of $()'$ are written as symbols $(J, E', d)_{\sharp=n}$. Such a symbol stands for the n triples (J, E', A') in $CS'(H_r)$ with J, E' fixed and A' running through the set $CS_d^{\emptyset}(L_{J,r})$ (assumed to have $n \geq 1$ elements). When $n = 1$ we omit the subscript $\sharp = n$. We specify J by indicating the type of W_J. (For example, in the table for E_8 in 2.8, the row of 8_1 contains an item $(E_6, \epsilon_c, 0)_\sharp$ which stands for two objects; in the triple $(E_6, \epsilon_c, 0)$, E_6 represents a subset of type E_6 of the simple reflections, ϵ_c is a certain representation of a Weyl group of type G_2 and 0 represents the dimension of a certain variety.) When $J = \emptyset$ we must have $d = 0, n = 1$ and we write E' instead of (J, E', d). Note that the first entry in $()'$ is E itself.

The groups in $()''$ are as follows. If $\overline{Pr}(E) = \overline{Pr} = \overline{Pr}'(E)$ then $()''$ consists of the single group in $c(E)$ put inside a box.

If $\overline{Pr}(E) = \overline{Pr} \neq \overline{Pr}'(E)$ then $()''$ is $\Gamma, \Gamma', (\Gamma'')$ where $\Gamma = \mathcal{A}_{2,E}, \Gamma' = \mathcal{A}_{3,E}, \Gamma'' = \mathcal{A}_{0,E}$; the boxed entries Γ or Γ'' or both represent the set $c(E)$; an exception is when $E = 1$ in type E_8: in this case $()''$ is $\Gamma, \Gamma', \Gamma'', (1)$ where $\Gamma = \mathcal{A}_{2,E} = \mathcal{C}_4, \Gamma' = \mathcal{A}_{3,E} = \mathcal{C}_3, \Gamma'' = \mathcal{A}_{5,E} = \mathcal{C}_5, \mathcal{A}_{0,E} = 1$ and $c(E)$ consists of $\Gamma, \Gamma', \Gamma''$ (all boxed).

If $\overline{Pr}(E) \neq \overline{Pr}$ then $\overline{Pr} - \overline{Pr}(E) = \{r'_0\}$ where $r'_0 \in \{2, 3\}$. If $r'_0 = 2$ then $()''$ is $\Gamma, -, (-)$ where $\Gamma = \mathcal{A}_{2,E}$ and $c(E)$ consists of Γ (it is boxed); if $r'_0 = 3$ then $()''$ is $-, \Gamma', (-)$ where $\Gamma' = \mathcal{A}_{3,E}$ and $c(E)$ consists of Γ' (it is boxed).

If H_{ad} is of type G_2, we have $J = \emptyset$ or $W_J = W$ with $N_W(W_J)/W_J = \{1\}$.

If H_{ad} is of type F_4, we have $J = \emptyset$ or W_J of type B_2 with $N_W(W_J)/W_J$ of type B_2 or $W_J = W$ with $N_W(W_J)/W_J = \{1\}$.

If H_{ad} is of type E_6, we have $J = \emptyset$ or W_J of type D_4 with $N_W(W_J)/W_J$ of type A_2 or $W_J = W$ with $N_W(W_J)/W_J = \{1\}$.

If H_{ad} is of type E_7, we have $J = \emptyset$ or W_J of type D_4 with $N_W(W_J)/W_J$ of type B_3 or W_J of type E_6 with $N_W(W_J)/W_J$ of type A_1 or $W_J = W$ with $N_W(W_J)/W_J = \{1\}$.

If H_{ad} is of type E_8, we have $J = \emptyset$ or W_J of type D_4 with $N_W(W_J)/W_J$ of type F_4 or W_J of type E_6 with $N_W(W_J)/W_J$ of type G_2 or W_J of type E_7 with $N_W(W_J)/W_J$ of type A_1 or $W_J = W$ with $N_W(W_J)/W_J = \{1\}$.

The notation for the elements of $\mathrm{Irr}(W)$ or $\mathrm{Irr}(N_W(W_J)/W_J)$ is taken from [S85] with one note of caution. In the case where H_{ad} is of type E_8 and W_J is of type E_6, the two 2-dimensional irreducible representations θ', θ'' of $N_W(W_J)/W_J$ which appear in the generalized Springer correspondence with $r = 3$ are identified in [S85] only up to order. This indeterminacy is removed in [H22] which shows that θ' is the reflection representation.

Table 2.4. Table for G_2

ϵ	....	$\boxed{1}$
ϵ_l	....	$-$, $\boxed{1}$, $(-)$
ϵ_c	....	$\boxed{1}$
θ''	....	$\boxed{1}$
θ', $(G_2, 1, 1)$	....	S_3, $\boxed{\mathcal{C}_2}$, (S_3)
1, $(G_2, 1, 0)_{\sharp=3}$	....	$\boxed{\mathcal{C}_2, \mathcal{C}_3}$, (1)

Table 2.5. Table for F_4

$\chi_{1,4}$	....	$\boxed{1}$
$\chi_{2,4}$	....	$\boxed{1}$
$\chi_{2,2}$	....	$\boxed{1}$, $-$, $(-)$
$\chi_{4,4}$	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
$\chi_{9,4}$	....	$\boxed{1}$
$\chi_{8,4}$, $\chi_{1,2}$	....	$\boxed{\mathcal{C}_2}$
$\chi_{8,2}$, $\chi_{1,3}$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
χ_4, $(B_2, \epsilon, 0)$	....	$\boxed{\mathcal{C}_2}$, $-$, $(-)$
$\chi_{4,3}$	....	$\boxed{1}$, $-$, $(-)$
$\chi_{4,2}$	....	$\boxed{1}$
$\chi_{9,3}$	....	$\boxed{1}$, $-$, $(-)$

TABLE 2.5. (Continued from previous page)

$\chi_{9,2}$	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
$\chi_{6,1}$	....	$\boxed{1}$
χ_{16}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
χ_{12}, $\chi_{6,2}$, $(F_4,1,4)$	....	$\boxed{S_3}$, S_4, (S_4)
$\chi_{8,3}$, $(B_2,\epsilon_l,0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
$\chi_{8,1}$, $(B_2,\epsilon_c,0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
$\chi_{9,1}$, $\chi_{2,1}$, $\chi_{2,3}$, $(B_2,\theta,0)$, $(F_4,1,2)$	....	$\boxed{\Delta_8}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
$\chi_{4,1}$, $(F_4,1,1)$	....	$\boxed{\mathcal{C}_2}$
$\chi_{1,1}$, $(B_2,1,0)$, $(F_4,1,0)_{\sharp=4}$	....	$\boxed{\mathcal{C}_4,\mathcal{C}_3}$, (1)

TABLE 2.6. Table for E_6

1_{36}	....	$\boxed{1}$
6_{25}	....	$\boxed{1}$
20_{20}	....	$\boxed{1}$
15_{16}	....	$\boxed{1}$
30_{15}, 15_{17}	....	$\boxed{\mathcal{C}_2}$
64_{13}	....	$\boxed{1}$
24_{12}	....	$\boxed{1}$
60_{11}	....	$\boxed{1}$
81_{10}	....	$\boxed{1}$
10_9	....	$\boxed{1}$
60_8	....	$\boxed{1}$
80_7, 90_8, 20_{10}	....	$\boxed{S_3}$
81_6	....	$\boxed{1}$
24_6, $(D_4,\epsilon,0)$	....	$\boxed{S_2}$, 1, (1)
60_5	....	$\boxed{1}$
64_4	....	$\boxed{1}$
15_4	....	$\boxed{1}$
30_3, 15_5	....	$\boxed{\mathcal{C}_2}$

Table 2.6. (Continued from previous page)

20_2, $(D_4,\phi,0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
6_1	....	$\boxed{1}$
1_0, $(D_4,1,0)$, $(E_6,1,0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_2,\mathcal{C}_3}$, (1)

Table 2.7. Table for E_7

1_{63}	....	$\boxed{1}$
7_{46}	....	$\boxed{1}$
27_{37}	....	$\boxed{1}$
21_{36}	....	$\boxed{1}$
35_{31}	....	$\boxed{1}$
56_{30}, 21_{33}	....	$\boxed{\mathcal{C}_2}$
15_{28}	....	$\boxed{1}$
120_{25}, 105_{28}	....	$\boxed{\mathcal{C}_2}$
189_{22}	....	$\boxed{1}$
105_{21}	....	$\boxed{1}$
168_{21}	....	$\boxed{1}$
210_{21}	....	$\boxed{1}$
189_{20}	....	$\boxed{1}$
70_{18}	....	$\boxed{1}$
280_{17}	....	$\boxed{1}$
315_{16}, 280_{18}, 35_{22}	....	$\boxed{S_3}$
216_{16}	....	$\boxed{1}$
405_{15}, 189_{17}	....	$\boxed{\mathcal{C}_2}$
105_{15}, $(D_4,(0,1^3),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
84_{15}	....	$\boxed{1}$, $-$, $(-)$
378_{14}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
210_{13}	....	$\boxed{1}$
420_{13}, 336_{14}	....	$\boxed{\mathcal{C}_2}$
84_{12}, $(D_4,(1^3,0),0)$	....	$\boxed{\mathcal{C}_2}$

Table 2.7. (Continued from previous page)

105_{12}	....	$\boxed{1}$
512_{11}, 512_{12}	....	$\boxed{\mathcal{C}_2}$
210_{10}	....	$\boxed{1}$
420_{10}, 336_{11}	....	$\boxed{\mathcal{C}_2}$
378_9	....	$\boxed{1}$
216_9	....	$\boxed{1}$
70_9	....	$\boxed{1}$
280_8	....	$\boxed{1}$
405_8, 189_{10}	....	$\boxed{\mathcal{C}_2}$
189_7, $(D_4,(1,1^2),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
315_7, 280_9, 35_{13}	....	$\boxed{S_3}$
168_6, $(D_4,(1^2,1),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
210_6, $(D_4,(0,21),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
105_6, 15_7	....	$\boxed{\mathcal{C}_2}$, 1, (1)
189_5	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
35_4, $(D_4,(21,0),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
120_4, 105_5	....	$\boxed{\mathcal{C}_2}$
21_3, $(D_4,(1,2),0)$, $(E_6,1,0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_2,\mathcal{C}_3}$, (1)
56_3, 21_6	....	$\boxed{\mathcal{C}_2}$
27_2, $(D_4,(2,1),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
7_1, $(D_4,(0,3),0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
1_0, $(D_4,(3,0),0)$, $(E_6,1,0)_{\sharp=2}$, $(E_7,1,0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_4,\mathcal{C}_3}$, (1)

Table 2.8. Table for E_8

1_{120}	....	$\boxed{1}$
8_{91}	....	$\boxed{1}$
35_{74}	....	$\boxed{1}$
84_{64}	....	$\boxed{1}$
112_{63}, 28_{68}	....	$\boxed{\mathcal{C}_2}$

TABLE 2.8. (Continued from previous page)

50_{56}	....	$\boxed{1}$
210_{52}, 160_{55}	....	$\boxed{\mathcal{C}_2}$
560_{47}	....	$\boxed{1}$
567_{46}	....	$\boxed{1}$
400_{43}	....	$\boxed{1}$
700_{42}, 300_{44}	....	$\boxed{\mathcal{C}_2}$
448_{39}	....	$\boxed{1}$
1344_{38}	....	$\boxed{1}$
1400_{37}, 1008_{39}, 56_{49}	....	$\boxed{S_3}$
175_{36}	....	$\boxed{1}$
525_{36}, $(D_4, \chi_{1,4}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
1050_{34}	....	$\boxed{1}$
1400_{32}, 1575_{34}, 350_{38}	....	$\boxed{S_3}$
972_{32}	....	$\boxed{1}$, $-$, $(-)$
3240_{31}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
2268_{30}, 1296_{33}	....	$\boxed{\mathcal{C}_2}$
1400_{29}	....	$\boxed{1}$
2240_{28}, 840_{31}	....	$\boxed{\mathcal{C}_2}$
700_{28}, $(D_4, \chi_{2,2}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
840_{26}	....	$\boxed{1}$
4096_{26}, 4096_{27}	....	$\boxed{\mathcal{C}_2}$
2800_{25}, 2100_{28}	....	$\boxed{\mathcal{C}_2}$
4200_{24}, 3360_{25}	....	$\boxed{\mathcal{C}_2}$
168_{24}, $(D_4, \chi_{1,3}, 0)$	....	$\boxed{\mathcal{C}_2}$, $-$, $(-)$
4536_{23}	....	$\boxed{1}$
2835_{22}	....	$\boxed{1}$
6075_{22}	....	$\boxed{1}$
3200_{32}	....	$\boxed{1}$
4200_{21}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
5600_{21}, 2400_{23}	....	$\boxed{\mathcal{C}_2}$

Table 2.8. (Continued from previous page)

420_{20}	....	$\boxed{1}$
2100_{20}, $(D_4, \chi_{4,4}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
1344_{19}	....	$\boxed{1}$
2016_{19}	....	$\boxed{1}$
3150_{18}, 1134_{20}	....	$\boxed{\mathcal{C}_2}$
4200_{18}, 2688_{20}	....	$\boxed{\mathcal{C}_2}$
7168_{17}, 5600_{19}, 448_{25}	....	$\boxed{S_3}$
3200_{16}, $(D_4, \chi_{8,2}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
4480_{16}, 5670_{18}, 4536_{18}, 1400_{20}, 1680_{22}, 70_{32}, $(E_8, 1, 16)$	....	$\boxed{S_5}$
5600_{15}, 2400_{17}, $(D_4, \chi_{9,4}, 0)$, $(D_4, \chi_{2,4}, 0)$	....	$\boxed{\mathcal{C}_2 \times \mathcal{C}_2}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
4200_{15}, 700_{16}	....	$\boxed{\mathcal{C}_2}$, 1, (1)
2835_{14}	....	$\boxed{1}$
6075_{14}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
840_{14}, $(D_4, \chi_{4,3}, 0)$	....	$\boxed{\mathcal{C}_2}$, $-$, $(-)$
4536_{13}	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
2800_{13}, 2100_{16}	....	$\boxed{\mathcal{C}_2}$
972_{12}, $(D_4, \chi_{9,3}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
4200_{12}, 3360_{13}	....	$\boxed{\mathcal{C}_2}$
525_{12}, $(D_4, \chi_{8,4}, 0)$, $(E_6, \epsilon, 0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_2, \mathcal{C}_3}$, (1)
175_{12}	....	$-$, $\boxed{1}$, $(-)$
1400_{11}	....	$\boxed{1}$
4096_{11}, 4096_{12}	....	$\boxed{\mathcal{C}_2}$
2268_{10}, 1296_{13}	....	$\boxed{\mathcal{C}_2}$
2240_{10}, 840_{13}	....	S_3, $\boxed{\mathcal{C}_2}$, (S_3)
1050_{10}, $(D_4, \chi_4, 0)$	....	$\boxed{\mathcal{C}_2}$, $-$, $(-)$
3240_9	....	$\boxed{1}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
448_9, $(D_4, \chi_{6,1}, 0)$, $(E_6, \epsilon_l, 0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_2, \mathcal{C}_3}$, (1)
1344_8, $(D_4, \chi_{16}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
1400_8, 1575_{10}, 350_{14}	....	$\boxed{S_3}$

TABLE 2.8. (Continued from previous page)

1400_7, 1008_9, 56_{19}, $(D_4, \chi_{12}, 0)$, $(D_4, \chi_{6,2}, 0)$, $(E_8, 1, 7)$	....	$\boxed{S_3 \times \mathcal{C}_2}$, S_3, (S_3)
400_7, $(D_4, \chi_{2,3}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
700_6, 300_8, 50_9, $(D_4, \chi_{8,3}, 0)$, $(E_8, 1, 6)$	....	$\boxed{\Delta_8}$, $\mathcal{C}_2$, $(\mathcal{C}_2)$
567_6, $(D_4, \chi_{9,2}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
560_5, $(D_4, \chi_{4,2}, 0)$	....	$\boxed{\mathcal{C}_2}$
210_4, 160_7	....	$\boxed{\mathcal{C}_2}$
84_4, $(D_4, \chi_{9,1}, 0)$, $(E_6, \theta'', 0)_{\sharp=2}$, $(E_7, 1, 0)_{\sharp=2}$	....	$\boxed{\mathcal{C}_4, \mathcal{C}_3}$, (1)
112_3, 28_8, $(D_4, \chi_{8,1}, 0)$, $(D_4, \chi_{1,2}, 0)$, $(E_6, \theta', 0)_{\sharp=2}$, $(E_8, 1, 3)_{\sharp=2}$	....	$\boxed{\mathcal{C}_2 \times \mathcal{C}_2, \mathcal{C}_2 \times \mathcal{C}_3}$, $(\mathcal{C}_2)$
35_2, $(D_4, \chi_{4,1}, 0)$	....	$\boxed{\mathcal{C}_2}$, 1, (1)
8_1, $(D_4, \chi_{2,1}, 0)$, $(E_6, \epsilon_c, 0)_{\sharp=2}$, $(E_8, 1, 1)_{\sharp=2}$	....	$\boxed{\mathcal{C}_4, \mathcal{C}_3}$, (1)
1_0, $(D_4, \chi_{1,1}, 0)$, $(E_6, 1, 0)_{\sharp=2}$, $(E_7, 1, 0)_{\sharp=2}$, $(E_8, 1, 0)_{\sharp=6}$	....	$\boxed{\mathcal{C}_4, \mathcal{C}_3, \mathcal{C}_5}$, (1)

3. COMPLEMENTS

3.1. The restriction of the map τ in 0.1(a) to $CS^{\emptyset}(H_r)$ has an alternative definition. Namely, for $A \in CS^{\emptyset}(H_r)$, there is a unique stratum $X \in Str(H_r)$ such that $\sigma_A \subset X$ (notation of 1.1); we have $\tau(A) = X$.

3.2. The results in this subsection can be used to verify 1.7. In the examples below we assume that H is semisimple, $H \neq \{1\}$, and for $A \in CS^{\emptyset}(H_r)$ we denote by s the semisimple part of an element of σ_A. We describe the structure of $Z^0_{H_r}(s)$ in various cases. We also specify the value of X in 1.7.

If H is of type C_n with $n = k(k+1), k \geq 1$ then:

if $r \neq 2$ then $Z^0_{H_r}(s)$ is of type $C_{n/2} \times C_{n/2}$; if $r = 2$ then $Z^0_{H_r}(s) = H_r$. Thus X is as in 1.7(i).

If H is of type B_n with $n = k(k+1), k \geq 1$ then:

if $r \neq 2$ then $Z^0_{H_r}(s)$ is of type $B_a \times D_b$ where $(2a+1, 2b) = ((k+1)^2, k^2)$ if k is even and $(2a+1, 2b) = (k^2, (k+1)^2)$ if k is odd; if $r = 2$ then $Z^0_{H_r}(s) = H_r$. Thus X is as in 1.7(i).

If H is of type D_n with $n = 4k^2, k \geq 1$ then:

if $r \neq 2$ then $Z^0_{H_r}(s)$ is of type $D_{2k^2} \times D_{2k^2}$; if $r = 2$ then $Z^0_{H_r}(s) = H_r$. Thus X is as in 1.7(i).

If H is of type G_2 and $A \in CS^{\emptyset}_d(H_r)$ then:

if $d = 1$ then $Z^0_{H_r}(s) = H_r$ (thus X is as in 1.7(ii)); if $d = 0$ and $r \notin \{2, 3\}$ then $Z^0_{H_r}(s)$ is of type A_2 for two values of A and of type $A_1 \times A_1$ for the third value of A; if $d = 0$ and $r = 2$ then $Z^0_{H_r}(s)$ is of type A_2 for two values of A and is H_r for the third value of A; if $d=0$ and $r=3$ then $Z^0_{H_r}(s)$ is H_r for two values of A and is of type $A_1\times A_1$ for the third value of A. Thus X is as in 1.7(iv).

If H is of type F_4 and $A\in CS^{\emptyset}_d(H_r)$ then:

if $d=4$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(ii));

if $d=2$, $r\neq 2$, then $Z^0_{H_r}(s)$ is of type B_4; if $d=2$, $r=2$, then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=1$, $r\neq 2$, then $Z^0_{H_r}(s)$ is of type $C_3\times A_1$; if $d=1$, $r=2$, then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=0$, $r\notin\{2,3\}$, then $Z^0_{H_r}(s)$ is of type $A_2\times A_2$ for two values of A and of type $A_3\times A_1$ for the other two values of A; if $d=0$, $r=2$, then $Z^0_{H_r}(s)$ is of type $A_2\times A_2$ for two values of A and is H_r for the other two values of A; if $d=0$, $r=3$, then $Z^0_{H_r}(s)$ is of type $A_3\times A_1$ for two values of A and is H_r for the other two values of A. Thus X is as in 1.7(iv).

If H is of type E_6 then:

if $r\neq 3$, then $Z^0_{H_r}(s)$ is of type $A_2\times A_2\times A_2$; if $r=3$, then $Z^0_{H_r}(s)=H_r$. Thus X is as in 1.7(i).

If H is of type E_7 then $d=0$ and:

if $r\neq 2$, then $Z^0_{H_r}(s)$ is of type $A_3\times A_3\times A_1$; if $r=2$, then $Z^0_{H_r}(s)=H_r$. Thus X is as in 1.7(i).

If H is of type E_8 and $A\in CS^{\emptyset}_d(H_r)$ then:

if $d=16$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(ii)).

if $d=7$ and $r\neq 2$ then $Z^0_{H_r}(s)$ is of type $E_7\times A_1$; if $d=7$ and $r=2$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=6$ and $r\neq 2$ then $Z^0_{H_r}(s)$ is of type D_8; if $d=6$ and $r=2$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=3$ and $r\neq 3$ then $Z^0_{H_r}(s)$ is of type $E_6\times A_2$; if $d=3$ and $r=3$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=1$ and $r\neq 2$ then $Z^0_{H_r}(s)$ is of type $D_5\times A_3$; if $d=1$ and $r=2$ then $Z^0_{H_r}(s)=H_r$ (thus X is as in 1.7(i));

if $d=0$ and $r\notin\{2,3,5\}$ then $Z^0_{H_r}(s)$ is of type $A_4\times A_4$ for four values of A and of type $A_5\times A_2\times A_1$ for two values of A; if $d=0$ and $r=5$ then $Z^0_{H_r}(s)$ is H_r for four values of A and of type $A_5\times A_2\times A_1$ for two values of A; if $d=0$ and $r=3$ then $Z^0_{H_r}(s)$ is of type $A_4\times A_4$ for four values of A and of type $E_7\times A_1$ for two values of A; if $d=0$ and $r=2$ then $Z^0_{H_r}(s)$ is of type $A_4\times A_4$ for four values of A and of type $E_6\times A_2$ for two values of A. Thus X is as in 1.7(iv).

The results in this section (for H of type E_8 and $d=0$) contradict the statement (f) on p. 351 in [Sh95]. (Indeed, if $r=2$ there is no semisimple $s\in H_2$ with $Z_{H_2}(s)$ of type $A_5\times A_2\times A_1$.)

3.3. Let H^* be a connected reductive group over $\mathbf{C}$ of type dual to that of H. Let $\mathbf{CS}(H)$ be the set of isomorphism classes of (not necessarily unipotent) character sheaves on H. We now assume that $Z_H=Z^0_H$. It is known that we can identify $\mathbf{CS}(H)$ with $\sqcup_s CS(Z_{H^*}(s)^*)$ where s runs over the semisimple elements of finite order of H^* up to conjugacy. Using 0.1(a), we obtain a surjective map $\mathbf{CS}(H)\to\sqcup_s Str(Z_{H^*}(s)^*)$. From [L15] we can identify $Str(Z_{H^*}(s)^*)=Str(Z_{H^*}(s))$. Hence we obtain a surjective map

$$\mathbf{CS}(H)\to\sqcup_s Str(Z_{H^*}(s)).$$

3.4. Let $\mathcal{X}'_H$ be the set of numbers which appear as coefficients of the highest root of H; let $\mathcal{X}_H = \mathcal{X}'_H \cup \{1\}$. Note that $\mathcal{X}_H$ consists of the numbers $1, 2, \dots, z_H$ where $z_H = 1$ for H of type A, $z_H = 2$ for H of type B, C, D, $z_H = 3$ for H of type G_2 or E_6, $z_H = 4$ for H of type F_4 or E_7, $z_H = 6$ for H of type E_8.

We note the following property (which can be verified from the results in §2).

(a) *The fibre of* τ *in* 0.1(a) *at the stratum consisting of regular elements in* H_r *(or equivalently at* $E = 1 \in \mathrm{Irr}_*(W)$*) is in bijection with the set*

(b) $$\sqcup_{m \in \mathbf{N}; 1 \le m \le z_H} \rho_m$$

where ρ_m *is the set of primitive* m*th roots of* 1 *in* K_r*.*

It is remarkable that the set (b) appears also in a quite different situation. Let $r, q, H_r(F_q)$ be as in 0.1(b). We can view $H_r(F_q)$ as a fixed point set of a Frobenius map $F : H_r \to H_r$. For any $w \in W$ let X_w be the variety attached to H_r, F, w in [DL76]. Now F acts on the cohomology with compact support $H^i_c(X_w)$ of X_w and in particular on $H^{|w|}_c(X_w)$. (We denote by $|w|$ the length of w.) Let w be a Coxeter element of minimal length in W. From [L76] it is known that the F-action on $H^{|w|}_c(X_w)$ is semisimple and that the eigenspaces are irreducible (unipotent) representations of $H_r(F_q)$. These unipotent representations are in bijection with the character sheaves in the fibre of τ at $E = 1$. (We use the usual bijection $Un(H_r(F_q)) \leftrightarrow CS(H_r)$ composed with the involution of $Un(H_r(F_q))$ which interchanges "small" representations with "big" representations.) The eigenvalues of the F-action are listed in [L76, p. 146, 147]. It turns out that

(c) these eigenvalues are exactly the roots of 1 in (b) times integral powers of $q^{1/2}$.

3.5. Let $cl(W)$ be the set of conjugacy classes in W. In [L15, §4], a surjective map $\Phi : cl(W) \to \mathrm{Irr}_*(W)$ is defined. In [L15, 4.10] a map $\mathrm{Irr}_*(W) \to cl(W)$, $E \to C_E$, is described; it is such that $\Phi(C_E) = E$ for all $E \in \mathrm{Irr}_*(W)$, hence its image $cl_*(W) \subset cl(W)$ is such that Φ restricts to a bijection $cl_*(W) \xrightarrow{\sim} \mathrm{Irr}_*(W)$. This allows us to identify the sets $cl_*(W), \mathrm{Irr}_*(W) = Str(H_r)$, so that $\tau : CS(H_r) \to Str(H_r)$ becomes a surjective map

(a) $$\tau' : Un(H_r(F_q)) \to cl_*(W)$$

(with $r, q, H_r(F_q)$ as in 0.1(b), see 0.1(c)). Let $Un^\emptyset(H_r(F_q))$ be the subset of $Un(H_r(F_q))$ consisting of unipotent cuspidal representations.

One can verify that the restriction of τ' to $Un^\emptyset(H_r(F_q))$ coincides with the map $\rho \mapsto C_\rho$ in [L02, 2.17]. From [L02] we see that

(b) for any $\rho \in Un^\emptyset(H_r(F_q))$, ρ appears with multiplicity 1 in $H^{|w|}_c(X_w)$ (notation of 3.4) where w is an element of minimal length in $\tau'(\rho)$.

3.6. In this subsection we assume that H_r is the symplectic group with W of type B_2. The simple reflections s_1, s_2 satisfy $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$. We have $\mathrm{Irr}_*(W) = \mathrm{Irr}(W) = \mathrm{Irr}_2(W)$; this set consists of $1, \rho, \epsilon_1, \epsilon_2, \epsilon$ where ρ is the reflection representation, ϵ is the sign representation and ϵ_1, ϵ_2 are the one-dimensional representations other than $1, \epsilon$. We have $cl(W) = cl_*(W)$. (The numbering of s_1, s_2 and of ϵ_1, ϵ_2 is chosen so that (a),(b) below hold.)

The conjugacy classes in W are $(1), (s_1), (s_2), (s_1s_2), (s_1s_2s_1s_2)$ where (w) is the conjugacy class of $w \in W$. The bijection $\mathrm{Irr}_*(W) \to cl_*(W)$ is given by

(a) $$1 \mapsto (s_1s_2),\ \rho \mapsto (s_1s_2s_1s_2),\ \epsilon_1 \mapsto (s_1),\ \epsilon_2 \mapsto (s_2),\ \epsilon \mapsto (1).$$

There are five strata; they are indexed by the elements of $\mathrm{Irr}_*(W)$; we denote them by $\sigma(1), \sigma(\rho), \sigma(\epsilon_1), \sigma(\epsilon_2), \sigma(\epsilon)$. Here

(b) $\sigma(1)$ is the union of all conjugacy classes of dimension 8; $\sigma(\rho)$ is the union of all conjugacy classes of dimension 6; $\sigma(\epsilon_1)$ is a conjugacy class of dimension 4 (a semisimple one if $r \neq 2$ and a unipotent one if $r = 2$; $\sigma(\epsilon_2)$ is a union of one (if $r = 2$) or two (if $r \neq 2$) conjugacy classes of dimension 4; $\sigma(\epsilon)$ is the centre of H_r.

$CS(H_r)$ consists of six objects: $\bar{\mathbf{Q}}_l[1]$, $\bar{\mathbf{Q}}_l[\rho]$, $\bar{\mathbf{Q}}_l[\epsilon_1]$, $\bar{\mathbf{Q}}_l[\epsilon_2]$, $\bar{\mathbf{Q}}_l[\epsilon]$ and A (an object of $CS^{\emptyset}(H_r)$). The map τ is $\bar{\mathbf{Q}}_l[1] \mapsto 1$, $\bar{\mathbf{Q}}_l[\rho] \mapsto \rho$, $\bar{\mathbf{Q}}_l[\epsilon_1] \mapsto \epsilon_1$, $\bar{\mathbf{Q}}_l[\epsilon_2] \mapsto \epsilon_2$, $\bar{\mathbf{Q}}_l[\epsilon] \mapsto \epsilon$, $A \mapsto 1$.

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