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GENERIC AND MOD p KAZHDAN-LUSZTIG THEORY FOR GL_2

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ABSTRACT. Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ and let $\mathbf{G} = GL_{2/F}$. Let $\mathbf{q}$ be an indeterminate and let $\mathcal{H}^{(1)}(\mathbf{q})$ be the generic pro-p Iwahori-Hecke algebra of the p-adic group $\mathbf{G}(F)$. Let $V_{\widehat{\mathbf{G}}}$ be the Vinberg monoid of the dual group $\widehat{\mathbf{G}}$. We establish a generic version for $\mathcal{H}^{(1)}(\mathbf{q})$ of the Kazhdan-Lusztig-Ginzburg spherical representation, the Bernstein map and the Satake isomorphism. We define the flag variety for the monoid $V_{\widehat{\mathbf{G}}}$ and establish the characteristic map in its equivariant K-theory. These generic constructions recover the classical ones after the specialization $\mathbf{q} = q \in \mathbb{C}$. At $\mathbf{q} = q = 0 \in \overline{\mathbb{F}}_q$, the spherical map provides a dual parametrization of all the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}(0)$-modules.

CONTENTS

1. INTRODUCTION

Let F be a non-archimedean local field with ring of integers o_F and residue field $\mathbb{F}_q$. Let $\mathbf{G}$ be a connected split reductive group over F. Let $\mathcal{H}_k = (k[I \setminus \mathbf{G}(F)/I], \star)$ be the Iwahori-Hecke algebra, i.e. the convolution algebra associated to an Iwahori subgroup $I \subset \mathbf{G}(F)$, with coefficients in an algebraically closed field k. On the other hand, let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{G}$ over k, with maximal torus and Borel subgroup $\widehat{\mathbf{T}} \subset \widehat{\mathbf{B}}$ respectively. Let W_0 be the finite Weyl group.

When $k = \mathbb{C}$, the irreducible $\mathcal{H}_{\mathbb{C}}$-modules appear as subquotients of the Grothendieck group $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})_{\mathbb{C}}$ of $\widehat{\mathbf{G}}$-equivariant coherent sheaves on the dual flag variety $\widehat{\mathbf{G}}/\widehat{\mathbf{B}}$. As such they can be parametrized by the isomorphism classes

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of irreducible tame $\widehat{\mathbf{G}}(\mathbb{C})$-representations of the Weil group $\mathcal{W}_F$ of F with unipotent inertial type, thereby realizing a tame part of the local Langlands correspondence (in this setting also called the Deligne-Lusztig conjecture for Hecke modules): Kazhdan-Lusztig [KL87], Ginzburg [CG97]. Their approach to the Deligne-Lusztig conjecture can be divided into two parts: the first part develops the theory of the so-called *spherical representation* leading to a certain dual parametrization of Hecke modules. The second part links these dual data to representations of the group $\mathcal{W}_F$.

The spherical representation is a distinguished faithful action of the Hecke algebra $\mathcal{H}_{\mathbb{C}}$ on a maximal commutative subring $\mathcal{A}_{\mathbb{C}} \subset \mathcal{H}_{\mathbb{C}}$ via $\mathcal{A}_{\mathbb{C}}^{W_0}$-linear operators: elements of the subring $\mathcal{A}_{\mathbb{C}}$ act by multiplication, whereas the standard Hecke operators $T_s \in \mathcal{H}_{\mathbb{C}}$, supported on double cosets indexed by simple reflections $s \in W_0$, act via the classical Demazure operators [D73, D74]. The link with the geometry of the dual group comes then in two steps. First, the classical Bernstein map $\tilde{\theta}$ identifies the ring of functions $\mathbb{C}[\widehat{\mathbf{T}}]$ with $\mathcal{A}_{\mathbb{C}}$, such that the invariants $\mathbb{C}[\widehat{\mathbf{T}}]^{W_0}$ become the center $Z(\mathcal{H}_{\mathbb{C}}) = \mathcal{A}_{\mathbb{C}}^{W_0}$. Second, the characteristic homomorphism $c_{\widehat{\mathbf{G}}}$ of equivariant K-theory identifies the rings $\mathbb{C}[\widehat{\mathbf{T}}]$ and $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})_{\mathbb{C}}$ as algebras over the representation ring $\mathbb{C}[\widehat{\mathbf{T}}]^{W_0} = R(\widehat{\mathbf{G}})_{\mathbb{C}}$.

When $k = \overline{\mathbb{F}}_q$, any irreducible $\widehat{\mathbf{G}}(\overline{\mathbb{F}}_q)$-representation of $\mathcal{W}_F$ is tame, with semisimple inertial type. Dually, one replaces the Iwahori-Hecke algebra by the bigger pro-p-Iwahori-Hecke algebra

$$\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} = (\overline{\mathbb{F}}_q[I^{(1)} \setminus \mathbf{G}(F)/I^{(1)}], \star),$$

where $I^{(1)} \subset I$ is the pro-p-radical of I. The algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ was introduced by Vignéras and its structure theory developed in a series of papers [V04, V05, V06, V14, V15, V16, V17]. More generally, Vignéras introduces and studies a generic version $\mathcal{H}^{(1)}(\mathbf{q})$ of this algebra which is defined over a polynomial ring $\mathbb{Z}[\mathbf{q}]$ in an indeterminate $\mathbf{q}$. The mod p ring $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ is obtained by specialization $\mathbf{q} = q$ followed by extension of scalars from $\mathbb{Z}$ to $\overline{\mathbb{F}}_q$, in short $\mathbf{q} = q = 0$.

The present paper is the first in a series of papers in which we will show that there is a *generic* version of Kazhdan-Lusztig theory, which applies to the generic pro-p Iwahori-Hecke algebra $\mathcal{H}^{(1)}(\mathbf{q})$. On the one hand, it gives back (and actually improves) the classical theory after passing to the direct summand $\mathcal{H}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ and then specializing $\mathbf{q} = q \in \mathbb{C}$. On the other hand, it gives a genuine mod p theory after specializing $\mathbf{q} = q = 0 \in \overline{\mathbb{F}}_q$. Our key observation is that, in the generic setting, *the Langlands dual group* $\widehat{\mathbf{G}}$ *needs to be enlarged to its Vinberg monoid* $V_{\widehat{\mathbf{G}}}$ [V95].

We will work in increasing generality, starting in the present paper with the theory of the spherical representation and the dual parametrization in the simplest case of the group $\mathbf{G} = \mathbf{GL_2}$. Later, for a general split reductive $\mathbf{G}$, we expect that essentially the same constructions will hold, once the appropriate formulation will have been understood (and checked explicitly) here for $\mathbf{GL_2}$. In particular, we expect that the monoid fibration $\mathbf{q} : V_{\widehat{\mathbf{G}}} \to \mathbb{A}^1$ geometrizing the indeterminate $\mathbf{q}$, and the dual parametrization of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules achieved over the 0-fibre $V_{\widehat{\mathbf{G}},0}$, forms a general pattern.

So let $\mathbf{G} = \mathbf{GL_2}$ from now on. Let $k = \overline{\mathbb{F}}_q$ and $\mathbf{q}$ be an indeterminate. Let $\mathbf{T} \subset \mathbf{G}$ be the torus of diagonal matrices. Let $\mathcal{A}^{(1)}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ be the maximal commutative subring[1] and $\mathcal{A}^{(1)}(\mathbf{q})^{W_0} = Z(\mathcal{H}^{(1)}(\mathbf{q}))$ be its ring of invariants. We let $\tilde{\mathbb{Z}} := \mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}]$ and denote by $\tilde{\bullet}$ the base change from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$. The algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ splits as a direct product of subalgebras $\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})$ indexed by the orbits γ of W_0 in the set of characters of the finite torus $\mathbb{T} := \mathbf{T}(\mathbb{F}_q)$. There are regular resp. non-regular components corresponding to $|\gamma| = 2$ resp. $|\gamma| = 1$ and the algebra structure of $\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})$ in these two cases is fundamentally different. We define an analogue of the Demazure operator for the regular components and call it the *Vignéras operator*. Passing to the product over all γ, this allows us to single out a distinguished $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$-linear operator on $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$. Our first main result is the existence of the *generic pro-p spherical representation*:

Theorem A (Cf. Theorems 3.3.1, 4.3.1)**.** *There is a (essentially unique) faithful representation*

$$\tilde{\mathscr{A}}^{(1)}(\mathbf{q}) : \tilde{\mathcal{H}}^{(1)}(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$$

such that

(i) $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})|_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} =$ *the natural inclusion* $\tilde{\mathcal{A}}^{(1)}(\mathbf{q}) \subset \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$
(ii) $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})(T_s) =$ *the Demazure-Vignéras operator on* $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$.

Restricting the representation $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ to the Iwahori component, its base change $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ coincides with the classical spherical representation of Kazhdan-Lusztig and Ginzburg.

We call the left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-module defined by $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ the *generic spherical module* $\tilde{\mathcal{M}}^{(1)}$.

Let $\mathrm{Mat}_{2\times 2}$ be the $\mathbb{Z}$-monoid scheme of 2×2-matrices. The Vinberg monoid $V_{\widehat{\mathbf{G}}}$, as introduced in [V95], in the particular case of $\mathbf{GL_2}$ is the $\mathbb{Z}$-monoid scheme

$$V_{\mathbf{GL_2}} := \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

It implies the striking interpretation of the formal indeterminate $\mathbf{q}$ as a regular function. Indeed, denote by z_2 the canonical coordinate on $\mathbb{G}_m$. Let $\mathbf{q}$ be the homomorphism from $V_{\mathbf{GL_2}}$ to the multiplicative monoid $(\mathbb{A}^1, \cdot)$ defined by $(f, z_2) \mapsto \det(f) z_2^{-1}$:

$$\begin{array}{c} V_{\mathbf{GL_2}} \\ {\scriptstyle \mathbf{q}}\big\downarrow \\ \mathbb{A}^1. \end{array}$$

The fibration $\mathbf{q}$ is trivial over $\mathbb{A}^1 \setminus \{0\}$ with fibre $\mathbf{GL_2}$. The special fibre at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\mathbf{GL_2},0} := \mathbf{q}^{-1}(0) = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m,$$

where $\mathrm{Sing}_{2\times 2}$ represents the singular 2×2-matrices. Let $\mathrm{Diag}_{2\times 2} \subset \mathrm{Mat}_{2\times 2}$ be the submonoid scheme of diagonal 2×2-matrices, and set

$$V_{\widehat{\mathbf{T}}} := \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m \subset V_{\mathbf{GL_2}} = \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

[1]For the choice of the antidominant spherical orientation.

This is a diagonalizable $\mathbb{Z}$-monoid scheme. Restricting the above $\mathbb{A}^1$-fibration to $V_{\widehat{\mathbf{T}}}$ we obtain a fibration, trivial over $\mathbb{A}^1 \setminus \{0\}$ with fibre $\widehat{\mathbf{T}}$. Its special fibre at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0} := \mathbf{q}|_{V_{\widehat{\mathbf{T}}}}^{-1}(0) = \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m,$$

where $\mathrm{SingDiag}_{2\times 2}$ represents the singular diagonal 2×2-matrices. To ease notion, we denote the base change to $\overline{\mathbb{F}}_q$ of these $\mathbb{Z}$-schemes by the same symbols. Let $\mathbb{T}^\vee$ be the finite abelian dual group of $\mathbb{T}$. We let $R(V_{\widehat{\mathbf{T}}}^{(1)})$ be the representation ring of the extended monoid

$$V_{\widehat{\mathbf{T}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}}}.$$

Our second main result is the existence of the *generic pro-p Bernstein isomorphism.*

Theorem B (Cf. Corollary 6.1.2)**.** *There exists a ring isomorphism*

$$\mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}}^{(1)})$$

with the property: Restricting the isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ *to the Iwahori component, its base change* $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ *recovers*[2] *the classical Bernstein isomorphism* $\tilde{\theta}$*.*

The extended monoid $V_{\widehat{\mathbf{T}}}^{(1)}$ has a natural W_0-action and the isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ is equivariant. We call the resulting ring isomorphism

$$\mathscr{S}^{(1)}(\mathbf{q}) := \mathscr{B}^{(1)}(\mathbf{q})^{W_0} : \mathcal{A}^{(1)}(\mathbf{q})^{W_0} \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}}^{(1)})^{W_0}$$

the *generic pro-p-Iwahori Satake isomorphism.* Our terminology is justified by the following. Let $K = \mathbf{G}(o_F)$. Recall that the spherical Hecke algebra of $\mathbf{G}(F)$ with coefficients in any commutative ring R is defined to be the convolution algebra

$$\mathcal{H}_R^{\mathrm{sph}} := (R[K\backslash\mathbf{G}(F)/K], \star)$$

generated by the K-double cosets in $\mathbf{G}(F)$. We define a *generic spherical Hecke algebra* $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ over the ring $\mathbb{Z}[\mathbf{q}]$. Its base change $\mathbb{Z}[\mathbf{q}] \to R$, $\mathbf{q} \mapsto q$ coincides with $\mathcal{H}_R^{\mathrm{sph}}$. Our third main result is the existence of the *generic Satake isomorphism.*

Theorem C (Cf. Theorem 6.2.3)**.** *There exists a ring isomorphism*

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0}$$

with the property: Base change $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ *and specialization* $\mathbf{q} \mapsto q \in \mathbb{C}$ *recovers*[2] *the classical Satake isomorphism between* $\mathcal{H}_{\mathbb{C}}^{\mathrm{sph}}$ *and* $R(\widehat{\mathbf{T}})_{\mathbb{C}}^{W_0}$*.*

We emphasize that the possibility of having a generic Satake isomorphism is conceptually new and of independent interest. Its definition relies on the deep Kazhdan-Lusztig theory for the intersection cohomology on the affine flag manifold. Its proof follows from the classical case by specialization (to an infinite number of points q). The special fibre $\mathscr{S}(0)$ recovers Herzig's mod p Satake isomorphism [H11], by choosing Steinberg coordinates on $V_{\widehat{\mathbf{T}},0}$.

As a corollary we obtain the *generic central elements morphism* as the unique ring homomorphism

$$\mathscr{Z}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow \mathcal{A}(\mathbf{q}) \subset \mathcal{H}(\mathbf{q})$$

[2] By 'recovers' we mean 'coincides up to a renormalization'.

making the diagram

$$\begin{tikzcd} \mathcal{A}(\mathbf{q}) \arrow[r, "\mathscr{B}^{(1)}(\mathbf{q})|_{\mathcal{A}(\mathbf{q})}", "\sim"'] & R(V_{\widehat{\mathbf{T}}}) \\ \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \arrow[u, "\mathscr{Z}(\mathbf{q})"] \arrow[r, "\mathscr{S}(\mathbf{q})", "\sim"'] & R(V_{\widehat{\mathbf{T}}})^{W_0} \arrow[u, hook] \end{tikzcd}$$

commutative. The morphism $\mathscr{Z}(\mathbf{q})$ is injective and has image $Z(\mathcal{H}(\mathbf{q}))$. Base change $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ and specialization $\mathbf{q} \mapsto q \in \mathbb{C}$ recovers[2] Bernstein's classical central elements morphism. Its specialization $\mathbf{q} \mapsto q = 0 \in \overline{\mathbb{F}}_q$ coincides with Ollivier's construction from [O14].

Our fourth main result is the *characteristic homomorphism* in the equivariant K-theory over the Vinberg monoid $V_{\widehat{\mathbf{G}}}$. The monoid $V_{\widehat{\mathbf{G}}}$ carries an action by multiplication on the right from the $\mathbb{Z}$-submonoid scheme

$$V_{\widehat{\mathbf{B}}} := \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m \subset \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m = V_{\widehat{\mathbf{G}}},$$

where $\mathrm{UpTriang}_{2\times 2}$ represents the upper triangular 2×2-matrices. One can construct (virtual) quotients in the context of semigroups and categories of equivariant vector bundles and their K-theory on such quotients, similar to the classical description over a groupoid, and the usual induction functor for vector bundles gives a characteristic homomorphism, which is an isomorphism in the case of monoids [PS20]. Applying this general formalism, the *flag variety* $V_{\widehat{\mathbf{G}}}/V_{\widehat{\mathbf{B}}}$ resp. its extended version $V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}$ is defined as a $\mathbb{Z}$-monoidoid (instead of a groupoid).

Theorem D (Cf. Corollary 5.2.2)**.** *Induction of equivariant vector bundles defines a characteristic isomorphism*

$$c_{V^{(1)}_{\widehat{\mathbf{G}}}} : R(V^{(1)}_{\widehat{\mathbf{T}}}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

The ring isomorphism is $R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0} = R(V^{(1)}_{\widehat{\mathbf{G}}})$-linear and compatible with passage to $\mathbf{q}$-fibres. Over the open complement $\mathbf{q} \neq 0$, its Iwahori-component coincides with the classical characteristic homomorphism $c_{\widehat{\mathbf{G}}}$ between $R(\widehat{\mathbf{T}})$ and $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})$.

We define the *category of Bernstein resp. Satake parameters* $\mathrm{BP}_{\widehat{\mathbf{G}}}$ resp. $\mathrm{SP}_{\widehat{\mathbf{G}}}$ to be the category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ resp. $V^{(1)}_{\widehat{\mathbf{T}}}/W_0$. By Theorem B, restriction of scalars to the subring $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ or $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ defines a functor B resp. P from the category of $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-modules to the categories $\mathrm{BP}_{\widehat{\mathbf{G}}}$ resp. $\mathrm{SP}_{\widehat{\mathbf{G}}}$. For example, the Bernstein resp. Satake parameter of the spherical module $\tilde{\mathcal{M}}^{(1)}$ equals the structure sheaf $\mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}}}}$ resp. the quasi-coherent sheaf corresponding to the $R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0}$-module $K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}})$. We call P the *generic parametrization functor*.

In the other direction, we define the *generic spherical functor* to be the functor $\mathrm{Sph} := (\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet) \circ S^{-1}$ where S is the Satake equivalence between $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$-modules and $\mathrm{SP}_{\widehat{\mathbf{G}}}$. Let $\pi : V^{(1)}_{\widehat{\mathbf{T}}} \to V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ be the projection. The relation between all these functors is expressed by the commutative diagram:

$$\begin{tikzcd} & \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \arrow[d, "B"] \arrow[dr, "P"] & \\ \mathrm{SP}_{\widehat{\mathbf{G}}} \arrow[ur, "\mathrm{Sph}"] \arrow[r, "\pi^*"'] & \mathrm{BP}_{\widehat{\mathbf{G}}} \arrow[r, "\pi_*"'] & \mathrm{SP}_{\widehat{\mathbf{G}}} . \end{tikzcd}$$

This ends our discussion of the theory in the generic setting.

Then we pass to the special fibre, i.e. we perform the base change $\mathbb{Z}[\mathbf{q}] \to k = \overline{\mathbb{F}}_q$, $\mathbf{q} \mapsto q = 0$. Identifying the k-points of the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ with the skyscraper sheaves on it, the spherical functor Sph induces a map

$$\mathrm{Sph} : \big(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}.$$

Considering the decomposition of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ into its connected components $V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$ indexed by $\gamma \in \mathbb{T}^{\vee}/W_0$, the spherical map decomposes as a disjoint union of maps

$$\mathrm{Sph}^{\gamma} : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\text{-modules}\}.$$

We come to our last main result, the mod p dual parametrization of *all* irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules via the spherical map.

Theorem E (Cf. Theorems 7.4.6 and 7.4.10)**.**

(i) *Let $\gamma \in \mathbb{T}^{\vee}/W_0$ regular. The spherical map induces a bijection*

$$\mathrm{Sph}^{\gamma} : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \xrightarrow{\ \sim\ } \{\textit{simple finite dimensional left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim .$$

The singular locus of the parametrizing k-scheme

$$V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0 \simeq V_{\widehat{\mathbf{T}},0} = \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

is given by $(0,0) \times \mathbb{G}_m \subset V_{\widehat{\mathbf{T}},0}$ in the standard coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence Sph^{γ}.

(ii) *Let $\gamma \in \mathbb{T}^{\vee}/W_0$ be non-regular. Consider the decomposition*

$$V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0 = V_{\widehat{\mathbf{T}},0}/W_0 \simeq \mathbb{A}^1 \times \mathbb{G}_m = D(2)_{\gamma} \cup D(1)_{\gamma},$$

where $D(1)_{\gamma}$ is the closed subscheme defined by the parabola $z_2 = z_1^2$ in the Steinberg coordinates z_1, z_2 and $D(2)_{\gamma}$ is the open complement. The spherical map induces bijections

$$\mathrm{Sph}^{\gamma}(2) : D(2)_{\gamma}(k) \xrightarrow{\ \sim\ } \{\textit{simple 2-dimensional left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim ,$$

$$\mathrm{Sph}^{\gamma}(1) : D(1)_{\gamma}(k) \xrightarrow{\ \sim\ } \{\textit{spherical pairs of characters of } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\}/\sim .$$

The branch locus of the covering

$$V_{\widehat{\mathbf{T}},0} \longrightarrow V_{\widehat{\mathbf{T}},0}/W_0 \simeq V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$$

is contained in $D(2)_{\gamma}$, with equation $z_1 = 0$ in Steinberg coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence $\mathrm{Sph}^{\gamma}(2)$.

In combination with the computation of the Satake parameter $S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})$ in Theorem D, we get that this dual parametrization of mod p Hecke modules is realized in the equivariant K-theory of the dual Vinberg monoid at $\mathbf{q} = 0$, whose Iwahori block is a natural specialization at $\mathbf{q} = 0$ of Kazhdan-Lusztig's parametrization for $\mathbb{C}$-coefficients. This realizes the first part of a mod p semisimple Langlands correspondence. We refer to [PS21, PS23] for the detailed relation between mod p Satake parameters and mod p semisimple Galois representations.

Regarding the strategy of proofs, once the Vinberg monoid is introduced, the generic Satake isomorphism is formulated and the generic spherical module is constructed, everything else follows from Vignéras' structure theory of the generic pro-p-Iwahori Hecke algebra and her classification of the irreducible representations.

Notation. In general, the letter F denotes a locally compact complete non-archimedean field with ring of integers o_F. Let $\mathbb{F}_q$ be its residue field, of characteristic p and cardinality q. We denote by $\mathbf{G}$ the algebraic group $\mathbf{GL_2}$ over F and by $G := \mathbf{G}(F)$ its group of F-rational points. Let $\mathbf{T} \subset \mathbf{G}$ be the torus of diagonal matrices. Finally, $I \subset G$ denotes the upper triangular standard Iwahori subgroup and $I^{(1)} \subset I$ denotes the unique pro-p Sylow subgroup of I.

2. The pro-p-Iwahori-Hecke algebra

2.1. The generic pro-p-Iwahori Hecke algebra.

2.1.1. We denote by $\Phi = \{\pm\alpha\}$ the root system of $(\mathbf{G}, \mathbf{T})$. We let $W_0 = \{1, s = s_\alpha\}$ and $\Lambda = X_*(\mathbf{T}) = \mathbb{Z}\times\mathbb{Z}$ be the *finite Weyl group* of $\mathbf{G}$ and the *lattice of cocharacters* of $\mathbf{T}$ respectively. If $\mathbb{T} = k^\times \times k^\times$ denote the *finite torus* $\mathbf{T}(\mathbb{F}_q)$, then W_0 acts naturally on $\mathbb{T} \times \Lambda$. We choose the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as a lift of s in $\mathbf{G}$; then the *extended Weyl group*, split by this choice,[3] is

$$W^{(1)} = (\mathbb{T} \times \Lambda) \rtimes W_0.$$

It contains the *affine Weyl group* and the *Iwahori-Weyl group*

$$W_{\mathrm{aff}} = \mathbb{Z}(1, -1) \rtimes W_0 \subseteq W = \Lambda \rtimes W_0.$$

The affine Weyl group W_{aff} is a Coxeter group with set of simple reflections $S_{\mathrm{aff}} = \{s_0, s\}$, where $s_0 = (1, -1)s$. Moreover, setting $u = (1, 0)s \in W$ and $\Omega = u^{\mathbb{Z}}$, we have $W = W_{\mathrm{aff}} \rtimes \Omega$. The length function ℓ on W_{aff} can then be inflated to W and $W^{(1)}$.

Definition 2.1.1. Let $\mathbf{q}$ be an indeterminate. The *generic pro-p Iwahori Hecke algebra* is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}^{(1)}(\mathbf{q})$ defined by generators

$$\mathcal{H}^{(1)}(\mathbf{q}) := \bigoplus_{w \in W^{(1)}} \mathbb{Z}[\mathbf{q}]T_w$$

and relations:

- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W^{(1)}$ if $\ell(w) + \ell(w') = \ell(ww')$

[3] Note that a splitting always exists for $\mathbf{GL_n}$, but not for a general split reductive $\mathbf{G}$, cf. [V05, Erratum 1)].

- quadratic relations: $T_{\tilde{s}}^2 = \mathbf{q} + c_s T_{\tilde{s}}$ if $\tilde{s} \in S_{\text{aff}}$, where

$$c_s := \sum_{t \in (1,-1)(k^\times)} T_{t \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}.$$

2.1.2. The identity element is $1 = T_1$. Moreover we set

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

Definition 2.1.2. Let R be any commutative ring. The *pro-p Iwahori Hecke algebra of G with coefficients in R* is defined to be the convolution algebra

$$\mathcal{H}_R^{(1)} := (R[I^{(1)} \backslash G / I^{(1)}], \star)$$

generated by the $I^{(1)}$-double cosets in G.

Theorem 2.1.3 (Vignéras)**.** *Let $\mathbb{Z}[\mathbf{q}] \to R$ be the ring homomorphism mapping $\mathbf{q}$ to q. Then the R-linear map*

$$\mathcal{H}^{(1)}(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} R \longrightarrow \mathcal{H}_R^{(1)}$$

sending T_w, $w \in W^{(1)}$, to the characteristic function of the double coset $I^{(1)} \backslash w / I^{(1)}$, is an isomorphism of R-algebras.

Proof. This is [V16, Thm. 2.2, Prop. 4.4], up to the fact that here our choice of splitting s is different from there. For this reason, in the generic quadratic relations, we need to take the element c_s as defined above instead of the element $\sum_{t \in (1,-1)(k^\times)} T_t$ used in *loc. cit.*; then the relations do specialize to the quadratic relations in $\mathcal{H}_R^{(1)}$, as can be checked by the direct computation of the corresponding convolution products. $\square$

2.2. Idempotents and component algebras.

2.2.1. Recall the finite torus $\mathbb{T} = \mathbf{T}(\mathbb{F}_q)$. Let us consider its group algebra $\tilde{\mathbb{Z}}[\mathbb{T}]$ over the ring

$$\tilde{\mathbb{Z}} := \mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}].$$

As $q-1$ is invertible in $\tilde{\mathbb{Z}}$, so is $|\mathbb{T}| = (q-1)^2$. We denote by $\mathbb{T}^\vee$ the set of characters $\lambda : \mathbb{T} \to \mu_{q-1} \subset \tilde{\mathbb{Z}}$, with its natural W_0-action given by ${}^s\lambda(t_1, t_2) = \lambda(t_2, t_1)$ for $(t_1, t_2) \in \mathbb{T}$. The set of W_0-orbits $\mathbb{T}^\vee / W_0$ has cardinality $\frac{q^2-q}{2}$. Also $W^{(1)}$ acts on $\mathbb{T}^\vee$ through the canonical quotient map $W^{(1)} \to W_0$. Because of the braid relations in $\mathcal{H}^{(1)}(\mathbf{q})$, the rule $t \mapsto T_t$ induces an embedding of $\tilde{\mathbb{Z}}$-*algebras*

$$\tilde{\mathbb{Z}}[\mathbb{T}] \subset \mathcal{H}_{\tilde{\mathbb{Z}}}^{(1)}(\mathbf{q}) := \mathcal{H}^{(1)}(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}.$$

Definition 2.2.1. For all $\lambda \in \mathbb{T}^\vee$ and for $\gamma \in \mathbb{T}^\vee / W_0$, we define

$$\varepsilon_\lambda := |\mathbb{T}|^{-1} \sum_{t \in \mathbb{T}} \lambda^{-1}(t) T_t \text{ and } \varepsilon_\gamma := \sum_{\lambda \in \gamma} \varepsilon_\lambda.$$

Lemma 2.2.2. *The elements ε_λ, $\lambda \in \mathbb{T}^\vee$, are idempotent, pairwise orthogonal and their sum is equal to* 1. *The elements ε_γ, $\gamma \in \mathbb{T}^\vee / W_0$, are idempotent, pairwise* *orthogonal, their sum is equal to* 1 *and they are central in* $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$*. The* $\tilde{\mathbb{Z}}[\mathbf{q}]$*-algebra* $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ *is the direct product of the* $\tilde{\mathbb{Z}}[\mathbf{q}]$*-algebras* $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) := \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})\varepsilon_\gamma$*:*

$$\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \prod_{\gamma\in\mathbb{T}^\vee/W_0} \mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}).$$

In particular, the category of $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$*-modules decomposes into a finite product of the module categories for the individual component rings* $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})\varepsilon_\gamma$*.*

Proof. The elements ε_γ are central because of the relations $T_sT_t = T_{s(t)}T_s$, $T_{s_0}T_t = T_{s_0(t)}T_{s_0}$ and $T_uT_t = T_{s(t)}T_u$ for all $t \in (1,-1)k^\times$. ☐

2.2.2. Following the terminology of [V04], we call $|\gamma| = 2$ a *regular* case and $|\gamma| = 1$ a *non-regular* (or *Iwahori*) case.

2.3. **The Bernstein presentation.** The inverse image in $W^{(1)}$ of any subset of W along the canonical projection $W^{(1)} \to W$ will be denoted with a superscript $^{(1)}$.

Theorem 2.3.1 (Vigneras [V16, Th. 2.10, Cor 5.47])**.** *The* $\mathbb{Z}[\mathbf{q}]$*-algebra* $\mathcal{H}^{(1)}(\mathbf{q})$ *admits the following* Bernstein presentation*:*

$$\mathcal{H}^{(1)}(\mathbf{q}) = \bigoplus_{w\in W^{(1)}} \mathbb{Z}[\mathbf{q}]E(w)$$

satisfying

- *braid relations:* $E(w)E(w') = E(ww')$ *for* $w, w' \in W_0^{(1)}$ *if* $\ell(w)+\ell(w') = \ell(ww')$
- *quadratic relations:* $E(\tilde{s})^2 = \mathbf{q}E(\tilde{s}^2) + c_{\tilde{s}}E(\tilde{s})$ *if* $\tilde{s} = ts \in s^{(1)}$*, where* $c_{\tilde{s}} := T_{s(t)}c_s$ *with* $t \in \mathbb{T}$
- *product formula:* $E(\lambda)E(w) = \mathbf{q}^{\frac{\ell(\lambda)+\ell(w)-\ell(\lambda w)}{2}}E(\lambda w)$ *for* $\lambda \in \Lambda^{(1)}$ *and* $w \in W^{(1)}$
- *Bernstein relations for* $\tilde{s} \in s^{(1)}$ *and* $\lambda \in \Lambda^{(1)}$*: set* $V := \mathbb{R}\Phi^\vee$ *and let*

  $$\nu : \Lambda^{(1)} \to V$$

  be the homomorphism such that $\lambda \in \Lambda^{(1)}$ *acts on* V *by translation by* $\nu(\lambda)$*; then the Bernstein element*

  $$B(\lambda,\tilde{s}) := E(\tilde{s}\lambda\tilde{s}^{-1})E(\tilde{s}) - E(\tilde{s})E(\lambda)$$

  $$\begin{aligned} &= 0 && \textit{if } \lambda \in (\Lambda^s)^{(1)} \\ &= \operatorname{sign}(\alpha\circ\nu(\lambda))\textstyle\sum_{k=0}^{|\alpha\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)c(k,\lambda)E(\mu(k,\lambda)) && \textit{if } \lambda \in \Lambda^{(1)}\setminus(\Lambda^s)^{(1)}, \end{aligned}$$

  where $\mathbf{q}(k,\lambda)c(k,\lambda) \in \mathbb{Z}[\mathbf{q}][\mathbb{T}]$ *and* $\mu(k,\lambda) \in \Lambda^{(1)}$ *are explicit, cf.* [V16, Th. 5.46] *and references therein.*

2.3.1. Let

$$\mathcal{A}(\mathbf{q}) := \bigoplus_{\lambda\in\Lambda}\mathbb{Z}[\mathbf{q}]E(\lambda) \subset \mathcal{A}^{(1)}(\mathbf{q}) := \bigoplus_{\lambda\in\Lambda^{(1)}}\mathbb{Z}[\mathbf{q}]E(\lambda) \subset \mathcal{H}^{(1)}(\mathbf{q}).$$

It follows from the product formula that these are *commutative sub-*$\mathbb{Z}[\mathbf{q}]$*-algebras of* $\mathcal{H}^{(1)}(\mathbf{q})$. Moreover, by definition [V16, 5.22-5.25], we have $E(t) = T_t$ for all $t \in \mathbb{T}$, so that $\mathbb{Z}[\mathbb{T}] \subset \mathcal{A}^{(1)}(\mathbf{q})$. Then, again by the product formula, the commutative algebra $\mathcal{A}^{(1)}(\mathbf{q})$ decomposes as the tensor product of the subalgebras

$$\mathcal{A}^{(1)}(\mathbf{q}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}).$$

Also, after base extension $\mathbb{Z} \to \tilde{\mathbb{Z}}$, we can set $\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) := \mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})\varepsilon_\gamma$, and obtain the decomposition

$$\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \prod_{\gamma \in \mathbb{T}^\vee/W_0} \mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \prod_{\gamma \in \mathbb{T}^\vee/W_0} \mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}).$$

Lemma 2.3.2. *Let X, Y, z_2 be indeterminates. There exists a unique ring homomorphism*

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X,Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}(\mathbf{q})$$

such that

$$X \longmapsto E(1,0), \quad Y \longmapsto E(0,1) \quad \textit{and} \quad z_2 \longmapsto E(1,1).$$

It is an isomorphism. Moreover, for all $\gamma \in \mathbb{T}^\vee/W_0$,

$$\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \begin{cases} (\tilde{\mathbb{Z}}\varepsilon_\lambda \times \tilde{\mathbb{Z}}\varepsilon_\mu) \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) & \textit{if } \gamma = \{\lambda, \mu\} \textit{ is regular,} \\ \tilde{\mathbb{Z}}\varepsilon_\lambda \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) & \textit{if } \gamma = \{\lambda\} \textit{ is non-regular.} \end{cases}$$

Proof. For any $(n_1, n_2) \in \mathbb{Z}^2 = \Lambda$, we have $\ell(n_1, n_2) = |n_1 - n_2|$. Hence it follows from product formula that z_2 is invertible and $XY = \mathbf{q}z_2$, so that we get a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X,Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}(\mathbf{q}).$$

Moreover it maps the $\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}]$-basis

$$\{X^n\}_{n \geq 1} \coprod \{1\} \coprod \{Y^n\}_{n \geq 1}$$

to the $\mathbb{Z}[\mathbf{q}][E(1,1)^{\pm 1}]$-basis

$$\{E(n,0)\}_{n \geq 1} \coprod \{1\} \coprod \{E(0,n)\}_{n \geq 1},$$

and hence is an isomorphism. The rest of the lemma is clear since $\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \tilde{\mathbb{Z}}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q})$ and $\tilde{\mathbb{Z}}[\mathbb{T}] = \prod_{\lambda \in \mathbb{T}^\vee} \tilde{\mathbb{Z}}\varepsilon_\lambda$. $\square$

In the following, we will sometimes view the isomorphism of Lemma 2.3.2 as an identification and write $X = E(1,0), Y = E(0,1)$ and $z_2 = E(1,1)$.

2.3.2. The rule $E(\lambda) \mapsto E(w(\lambda))$ defines an action of the finite Weyl group $W_0 = \{1, s\}$ on $\mathcal{A}^{(1)}(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra homomorphisms. By [V05, Th. 4] (see also [V14, Th. 1.3]), the subring of W_0-invariants is equal to the center of $\mathcal{H}^{(1)}(\mathbf{q})$, and the same is true after the scalar extension $\mathbb{Z} \to \tilde{\mathbb{Z}}$. Now the action on $\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ stabilizes each component $\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ and then the resulting subring of W_0-invariants is the center of $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. In terms of the description of $\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ given in Lemma 2.3.2, this translates into :

Lemma 2.3.3. *Let $\gamma \in \mathbb{T}^\vee/W_0$.*

- *If $\gamma = \{\lambda, \mu\}$ is regular, then the map*
$$\begin{aligned} \mathcal{A}_{\tilde{\mathbb{Z}}}(\mathbf{q}) &\longrightarrow \mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})^{W_0} = Z(\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})), \\ a &\longmapsto a\varepsilon_\lambda + s(a)\varepsilon_\mu \end{aligned}$$
is an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras. It depends on the choice of order (λ, μ) on the set γ.
- *If $\gamma = \{\lambda\}$ is non-regular, then*
$$Z(\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})) = \mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})^{W_0} = \tilde{\mathbb{Z}}[\mathbf{q}][z_2^{\pm 1}, z_1]\varepsilon_\lambda$$
with $z_1 := X + Y$.

2.3.3. One can express $X, Y, z_2 \in \mathcal{A}^{(1)}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ in terms of the distinguished elements 2.1.2. This is an application of [V16, Ex. 5.30]. We find:
$$(1,0) = s_0 u = us \in \Lambda \Rightarrow X := E(1,0) = (S_0 - c_{s_0})U = U(S - c_s),$$
$$(0,1) = su \in \Lambda \Rightarrow Y := E(0,1) = SU,$$
$$(1,1) = u^2 \in \Lambda \Rightarrow z_2 := E(1,1) = U^2.$$
Also
$$z_1 := X + Y = U(S - c_s) + SU.$$

3. The generic regular spherical representation

3.1. **The generic regular Iwahori-Hecke algebras.** Let $\gamma = \{\lambda, \mu\} \in \mathbb{T}^\vee/W_0$ be a regular orbit. We define a model $\mathcal{H}_2(\mathbf{q})$ over $\mathbb{Z}$ for the component algebra $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. The algebra $\mathcal{H}_2(\mathbf{q})$ itself will not depend on γ.

3.1.1. By construction, the $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebra $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ admits the following presentation:
$$\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = (\tilde{\mathbb{Z}}\varepsilon_\lambda \times \tilde{\mathbb{Z}}\varepsilon_\mu) \otimes'_{\mathbb{Z}} \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}]T_w,$$
where $\otimes'_{\mathbb{Z}}$ is the tensor product $\otimes_{\mathbb{Z}}$ of $\mathbb{Z}$-modules, whose algebra structure is *twisted* by the W-action on $\{\lambda, \mu\}$ through the quotient map $W \to W_0$, together with the orthogonality relation $\varepsilon_\lambda \varepsilon_\mu = 0$ and the
- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_{\tilde{s}}^2 = \mathbf{q}$ if $\tilde{s} \in S_{\mathrm{aff}}$.

Definition 3.1.1. Let $\mathbf{q}$ be an indeterminate. The *generic second Iwahori-Hecke algebra* is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}_2(\mathbf{q})$ defined by generators
$$\mathcal{H}_2(\mathbf{q}) := (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}]T_w,$$
where $\otimes'_{\mathbb{Z}}$ is the tensor product $\otimes_{\mathbb{Z}}$ of $\mathbb{Z}$-modules, whose algebra structure is twisted by the W-action on $\{1, 2\}$ through the quotient map $W \to W_0 = \mathfrak{S}_2$, together with $\varepsilon_1\varepsilon_2 = 0$, and the relations:
- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_{\tilde{s}}^2 = \mathbf{q}$ if $\tilde{s} \in S_{\mathrm{aff}}$.

3.1.2. The identity element of $\mathcal{H}_2(\mathbf{q})$ is $1 = T_1$. Moreover we set in $\mathcal{H}_2(\mathbf{q})$

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

Then one checks that

$$\mathcal{H}_2(\mathbf{q}) = (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = \mathbf{q}, \quad U^2 S = SU^2$$

is a presentation of $\mathcal{H}_2(\mathbf{q})$ (where S and U do *not* commute). Note that the element U^2 is invertible in $\mathcal{H}_2(\mathbf{q})$.

3.1.3. Choosing the ordering (λ, μ) on the set $\gamma = \{\lambda, \mu\}$ and mapping $\varepsilon_1 \mapsto \varepsilon_\lambda, \varepsilon_2 \mapsto \varepsilon_\mu$ defines an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras

$$\mathcal{H}_2(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}} \xrightarrow{\sim} \mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}),$$

such that $S \otimes 1 \mapsto S\varepsilon_\gamma$, $U \otimes 1 \mapsto U\varepsilon_\gamma$ and $S_0 \otimes 1 \mapsto S_0\varepsilon_\gamma$.

3.1.4. We identify two important commutative subrings of $\mathcal{H}_2(\mathbf{q})$. We define $\mathcal{A}_2(\mathbf{q}) \subset \mathcal{H}_2(\mathbf{q})$ to be the $\mathbb{Z}[\mathbf{q}]$-subalgebra generated by the elements ε_1, ε_2, US, SU and $U^{\pm 2}$. Let X, Y and z_2 be indeterminates. Then there is a unique $(\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$(\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}_2(\mathbf{q})$$

such that $X \mapsto US, Y \mapsto SU, z_2 \mapsto U^2$, and it is an isomorphism. In particular, $\mathcal{A}_2(\mathbf{q})$ is a *commutative* subalgebra of $\mathcal{H}_2(\mathbf{q})$. The isomorphism 3.1.3 identifies $\mathcal{A}_2(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ with $\mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q})$. Moreover, permuting ε_1 and ε_2, and X and Y, extends to an action of $W_0 = \mathfrak{S}_2$ on $\mathcal{A}_2(\mathbf{q})$ by homomorphisms of $\mathbb{Z}[\mathbf{q}]$-algebras, whose invariants are the center $Z(\mathcal{H}_2(\mathbf{q}))$ of $\mathcal{H}_2(\mathbf{q})$, and the map

$$\begin{aligned} \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) &\longrightarrow \mathcal{A}_2(\mathbf{q})^{W_0} = Z(\mathcal{H}_2(\mathbf{q})), \\ a &\longmapsto a\varepsilon_1 + s(a)\varepsilon_2 \end{aligned}$$

is an isomorphism of $\mathbb{Z}[\mathbf{q}]$-algebras. This is a consequence of Sections 3.1.3 and 2.3.3, and Lemmas 2.3.2 and 2.3.3. In the following, we will sometimes view the above isomorphisms as identifications. In particular, we will write $X = US, Y = SU$ and $z_2 = U^2$.

3.2. **The Vignéras operator.** In this subsection and the following, we will investigate the structure of the $Z(\mathcal{H}_2(\mathbf{q}))$-algebra $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ of $Z(\mathcal{H}_2(\mathbf{q}))$-linear endomorphisms of $\mathcal{A}_2(\mathbf{q})$. Recall from the preceding subsection that $Z(\mathcal{H}_2(\mathbf{q})) = \mathcal{A}_2(\mathbf{q})^s$ is the subring of invariants of the commutative ring $\mathcal{A}_2(\mathbf{q})$.

Lemma 3.2.1. *We have*

$$\mathcal{A}_2(\mathbf{q}) = \mathcal{A}_2(\mathbf{q})^s\varepsilon_1 \oplus \mathcal{A}_2(\mathbf{q})^s\varepsilon_2$$

as $\mathcal{A}_2(\mathbf{q})^s$-modules.

Proof. This is immediate from the two isomorphisms in 3.1.4. □

According to Lemma 3.2.1, we may use the $\mathcal{A}_2(\mathbf{q})^s$-basis $\varepsilon_1, \varepsilon_2$ to identify $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ with the algebra of 2×2-matrices over

$$\mathcal{A}_2(\mathbf{q})^s = \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2).$$

Definition 3.2.2. The endomorphism of $\mathcal{A}_2(\mathbf{q})$ corresponding to the matrix

$$V_s(\mathbf{q}) := \begin{pmatrix} 0 & Y\varepsilon_1 + X\varepsilon_2 \\ z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & 0 \end{pmatrix}$$

will be called *the Vignéras operator on* $\mathcal{A}_2(\mathbf{q})$.

Lemma 3.2.3. *We have* $V_s(\mathbf{q})^2 = \mathbf{q}$.

Proof. This is a short calculation. □

3.3. **The generic regular spherical representation.** In Theorem 3.3.1 we define the generic regular spherical representation of the algebra $\mathcal{H}_2(\mathbf{q})$ on the $Z(\mathcal{H}_2(\mathbf{q}))$-module $\mathcal{A}_2(\mathbf{q})$. Note that the commutative ring $\mathcal{A}_2(\mathbf{q})$ is naturally a subring

$$\mathcal{A}_2(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q})),$$

an element $a \in \mathcal{A}_2(\mathbf{q})$ acting by multiplication $b \mapsto ab$ on $\mathcal{A}_2(\mathbf{q})$.

Theorem 3.3.1. *There exists a unique* $\mathbb{Z}[\mathbf{q}]$*-algebra homomorphism*

$$\mathscr{A}_2(\mathbf{q}) : \mathcal{H}_2(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$$

such that

(i) $\mathscr{A}_2(\mathbf{q})|_{\mathcal{A}_2(\mathbf{q})} =$ *the natural inclusion* $\mathcal{A}_2(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$
(ii) $\mathscr{A}_2(\mathbf{q})(S) = V_s(\mathbf{q})$.

Proof. Recall that $\mathcal{H}_2(\mathbf{q}) = (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ with the relations $S^2 = \mathbf{q}$ and $U^2 S = SU^2$. In particular $\mathscr{A}_2(\mathbf{q})(S) := V_s(\mathbf{q})$ is well-defined thanks to 3.2.3. Now let us consider the question of finding the restriction of $\mathscr{A}_2(\mathbf{q})$ to the subalgebra $\mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$. As the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{A}_2(\mathbf{q}) \cap \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ is generated by

$$z_2 = U^2, \quad X = US \quad \text{and} \quad Y = SU,$$

such a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism exists if and only if there exists

$$\mathscr{A}_2(\mathbf{q})(U) \in \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$$

satisfying

(1) $\mathscr{A}_2(\mathbf{q})(U)^2 = \mathscr{A}_2(\mathbf{q})(U^2) = \mathscr{A}_2(\mathbf{q})(z_2) = z_2\,\mathrm{Id}$ (in particular $\mathscr{A}_2(\mathbf{q})(U)$ is invertible)
(2) $\mathscr{A}_2(\mathbf{q})(U)V_s(\mathbf{q}) =$ multiplication by X
(3) $V_s(\mathbf{q})\mathscr{A}_2(\mathbf{q})(U) =$ multiplication by Y.

As before we use the $Z(\mathcal{H}_2(\mathbf{q}))$-basis $\varepsilon_1, \varepsilon_2$ of $\mathcal{A}_2(\mathbf{q})$ to identify $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ with the algebra of 2×2-matrices over the ring $Z(\mathcal{H}_2(\mathbf{q})) = \mathcal{A}_2(\mathbf{q})^s$. Then, by definition,

$$V_s(\mathbf{q}) = \begin{pmatrix} 0 & Y\varepsilon_1 + X\varepsilon_2 \\ z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & 0 \end{pmatrix}.$$

Moreover, the multiplications by X and by Y on $\mathcal{A}_2(\mathbf{q})$ correspond then to the matrices

$$\begin{pmatrix} X\varepsilon_1 + Y\varepsilon_2 & 0 \\ 0 & Y\varepsilon_1 + X\varepsilon_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y\varepsilon_1 + X\varepsilon_2 & 0 \\ 0 & X\varepsilon_1 + Y\varepsilon_2 \end{pmatrix}.$$

Now, writing

$$\mathscr{A}_2(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

we have:

$$\mathscr{A}_2(\mathbf{q})(U)^2 = z_2 \operatorname{Id} \Longleftrightarrow \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix},$$

$$\mathscr{A}_2(\mathbf{q})(U)V_s(\mathbf{q}) = \text{ multiplication by } X$$

$$\Longleftrightarrow \begin{pmatrix} cz_2^{-1}(X\varepsilon_1+Y\varepsilon_2) & a(Y\varepsilon_1+X\varepsilon_2) \\ dz_2^{-1}(X\varepsilon_1+Y\varepsilon_2) & b(Y\varepsilon_1+X\varepsilon_2) \end{pmatrix} = \begin{pmatrix} X\varepsilon_1+Y\varepsilon_2 & 0 \\ 0 & Y\varepsilon_1+X\varepsilon_2 \end{pmatrix}$$

and

$$V_s(\mathbf{q})\mathscr{A}_2(\mathbf{q})(U) = \text{ multiplication by } Y$$

$$\Longleftrightarrow \begin{pmatrix} b(Y\varepsilon_1+X\varepsilon_2) & d(Y\varepsilon_1+X\varepsilon_2) \\ az_2^{-1}(X\varepsilon_1+Y\varepsilon_2) & cz_2^{-1}(X\varepsilon_1+Y\varepsilon_2) \end{pmatrix} = \begin{pmatrix} Y\varepsilon_1+X\varepsilon_2 & 0 \\ 0 & X\varepsilon_1+Y\varepsilon_2 \end{pmatrix}.$$

Each of the two last systems admits a unique solution, namely

$$\mathscr{A}_2(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix},$$

which is also a solution of the first one. Moreover, the determinant

$$ad - bc = -z_2$$

is invertible.

Finally, $\mathcal{A}_2(\mathbf{q})$ is generated by $\mathcal{A}_2(\mathbf{q})\cap\mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ together with ε_1 and ε_2. The latter are assigned to map to the projectors

$$\text{multiplication by } \varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{multiplication by } \varepsilon_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus it only remains to check that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathscr{A}_2(\mathbf{q})(S) = \mathscr{A}_2(\mathbf{q})(S) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathscr{A}_2(\mathbf{q})(S) = \mathscr{A}_2(\mathbf{q})(S) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and similarly with $\mathscr{A}_2(\mathbf{q})(U)$ in place of $\mathscr{A}_2(\mathbf{q})(S)$, which is straightforward. □

Remark 3.3.2. The map $\mathscr{A}_2(\mathbf{q})$, together with the fact that it is an isomorphism (see below), is a rewriting of a theorem of Vignéras, namely [V04, Cor. 2.3]. In loc. cit., the algebra $\mathcal{H}_2(\mathbf{q})$ is identified with the algebra of 2×2-matrices over the ring $\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2)$. In our approach, we have replaced the *abstract rank* 2 *module* underlying the standard representation of this matrix algebra, by the *subring* $\mathcal{A}_2(\mathbf{q})$ *of* $\mathcal{H}_2(\mathbf{q})$ with $\{\varepsilon_1, \varepsilon_2\}$ for the canonical basis.

Proposition 3.3.3. *The homomorphism* $\mathscr{A}_2(\mathbf{q})$ *is an isomorphism.*

Proof. It follows from 3.1.2 and 3.1.4 that the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}_2(\mathbf{q})$ is generated by the elements

$$\varepsilon_1,\ \varepsilon_2,\ S,\ U,\ SU$$

as a module over its center $Z(\mathcal{H}_2(\mathbf{q}))$. Moreover, as $SU^2 = U^2S =: z_2S$ and $SU =: Y$, we have

$$S = z_2^{-1}YU = z_2^{-1}Y(\varepsilon_1U + \varepsilon_2U) = z_2^{-1}(Y\varepsilon_1 + X\varepsilon_2)\varepsilon_1U + z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2)\varepsilon_2U,$$

$$U = \varepsilon_1U + \varepsilon_2U \quad \text{and} \quad SU = (Y\varepsilon_1 + X\varepsilon_2)\varepsilon_1 + (X\varepsilon_1 + Y\varepsilon_2)\varepsilon_2.$$

Consequently $\mathcal{H}_2(\mathbf{q})$ is generated as a $Z(\mathcal{H}_2(\mathbf{q}))$-module by the elements

$$\varepsilon_1,\ \varepsilon_2,\ z_2^{-1}\varepsilon_1 U,\ \varepsilon_2 U.$$

Since

$$\mathscr{A}_2(\mathbf{q})(U) := \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix},$$

these four elements are mapped by $\mathscr{A}_2(\mathbf{q})$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As $\mathscr{A}_2(\mathbf{q})$ identifies $Z(\mathcal{H}_2(\mathbf{q})) \subset \mathcal{H}_2(\mathbf{q})$ with the center of the matrix algebra

$$\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q})) = \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(Z(\mathcal{H}_2(\mathbf{q}))\varepsilon_1 \oplus Z(\mathcal{H}_2(\mathbf{q}))\varepsilon_2),$$

it follows that the elements $\varepsilon_1,\ \varepsilon_2,\ z_2^{-1}\varepsilon_1 U,\ \varepsilon_2 U$ are linearly independent over $Z(\mathcal{H}_2(\mathbf{q}))$ and that $\mathscr{A}_2(\mathbf{q})$ is an isomorphism. □

We record Corollary 3.3.4 of the proof.

Corollary 3.3.4. *The ring $\mathcal{H}_2(\mathbf{q})$ is a free $Z(\mathcal{H}_2(\mathbf{q}))$-module on the basis $\varepsilon_1, \varepsilon_2, z_2^{-1}\varepsilon_1 U, \varepsilon_2 U$.*

3.3.1. We end this section by noting an equivariance property of $\mathscr{A}_2(\mathbf{q})$. As already noticed, the finite Weyl group W_0 acts on $\mathcal{A}_2(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra automorphisms, and the action is clearly faithful. Moreover $\mathcal{A}_2(\mathbf{q})^{W_0} = Z(\mathcal{H}_2(q))$. Hence W_0 can be viewed as a subgroup of $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$, and we can let it act on $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ by conjugation.

Lemma 3.3.5. *The embedding $\mathscr{A}_2(\mathbf{q})|_{\mathcal{A}_2(\mathbf{q})}$ is W_0-equivariant.*

Proof. Indeed, for all $a, b \in \mathcal{A}_2(\mathbf{q})$ and $w \in W_0$, we have

$$\mathscr{A}_2(\mathbf{q})(w(a))(b) = w(a)b = w(aw^{-1}(b)) = (waw^{-1})(b) = (w\mathscr{A}_2(\mathbf{q})(a)w^{-1})(b).$$

□

4. The generic non-regular spherical representation

4.1. **The generic non-regular Iwahori-Hecke algebras.** Let $\gamma = \{\lambda\} \in \mathbb{T}^\vee/W_0$ be a non-regular orbit. As in the regular case, we define a model $\mathcal{H}_1(\mathbf{q})$ over $\mathbb{Z}$ for the component algebra $\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. The algebra $\mathcal{H}_1(\mathbf{q})$ will not depend on γ.

4.1.1. By construction, the $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebra $\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q})$ admits the following presentation:

$$\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}] T_w \varepsilon_\lambda,$$

with

- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_{\tilde{s}}^2 = \mathbf{q} + (q-1)T_{\tilde{s}}$ if $\tilde{s} \in S_{\mathrm{aff}}$.

Definition 4.1.1. Let $\mathbf{q}$ be an indeterminate. The *generic Iwahori-Hecke algebra* is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}_1(\mathbf{q})$ defined by generators

$$\mathcal{H}_1(\mathbf{q}) := \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}] T_w$$

and relations:

- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_{\tilde{s}}^2 = \mathbf{q} + (\mathbf{q}-1)T_{\tilde{s}}$ if $\tilde{s} \in S_{\text{aff}}$.

4.1.2. The identity element of $\mathcal{H}_1(\mathbf{q})$ is $1 = T_1$. Moreover we set in $\mathcal{H}_1(\mathbf{q})$

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

Then one checks that

$$\mathcal{H}_1(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = \mathbf{q} + (\mathbf{q}-1)S, \quad U^2 S = SU^2$$

is a presentation of $\mathcal{H}_1(\mathbf{q})$. Note that the element U^2 is invertible in $\mathcal{H}_1(\mathbf{q})$.

4.1.3. Sending 1 to ε_γ defines an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras

$$\mathcal{H}_1(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}} \xrightarrow{\sim} \mathcal{H}_{\tilde{\mathbb{Z}}}^{\gamma}(\mathbf{q}),$$

such that $S \otimes 1 \mapsto S\varepsilon_\gamma$, $U \otimes 1 \mapsto U\varepsilon_\gamma$ and $S_0 \otimes 1 \mapsto S_0\varepsilon_\gamma$.

4.1.4. We define $\mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ to be the $\mathbb{Z}[\mathbf{q}]$-subalgebra generated by the elements $(S_0 - (\mathbf{q}-1))U$, SU and $U^{\pm 2}$. Let X, Y and z_2 be indeterminates. Then there is a unique $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}_1(\mathbf{q})$$

such that $X \mapsto (S_0 - (\mathbf{q}-1))U$, $Y \mapsto SU$, $z_2 \mapsto U^2$, and it is an isomorphism. In particular, $\mathcal{A}_1(\mathbf{q})$ is a *commutative* subalgebra of $\mathcal{H}_1(\mathbf{q})$. The isomorphism 4.1.3 identifies $\mathcal{A}_1(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ with $\mathcal{A}_{\tilde{\mathbb{Z}}}^{\gamma}(\mathbf{q})$. Moreover, permuting X and Y extends to an action of $W_0 = \mathfrak{S}_2$ on $\mathcal{A}_1(\mathbf{q})$ by homomorphisms of $\mathbb{Z}[\mathbf{q}]$-algebras, whose invariants are the center $Z(\mathcal{H}_1(\mathbf{q}))$ of $\mathcal{H}_1(\mathbf{q})$ and

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][z_1] \xrightarrow{\sim} \mathcal{A}_1(\mathbf{q})^{W_0} = Z(\mathcal{H}_1(\mathbf{q}))$$

with $z_1 := X + Y$. This is a consequence of 4.1.3, 2.3.3, 2.3.2 and 2.3.3. In the following, we will sometimes view the above isomorphisms as identifications. In particular, we will write

$$X = (S_0 - (\mathbf{q}-1))U = U(S - (\mathbf{q}-1)), \quad Y = SU \quad \text{and} \quad z_2 = U^2 \quad \text{in} \quad \mathcal{H}_1(\mathbf{q}).$$

4.1.5. It is well-known that the generic Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$ is a $\mathbf{q}$-deformation of the group ring $\mathbb{Z}[W]$ of the Iwahori-Weyl group $W = \Lambda \rtimes W_0$. More precisely, specializing the chain of inclusions $\mathcal{A}_1(\mathbf{q})^{W_0} \subset \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ at $\mathbf{q} = 1$ yields the chain of inclusions $\mathbb{Z}[\Lambda]^{W_0} \subset \mathbb{Z}[\Lambda] \subset \mathbb{Z}[W]$.

4.2. **The Kazhdan-Lusztig-Ginzburg operator.** As in the regular case, we will study the $Z(\mathcal{H}_1(\mathbf{q}))$-algebra $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ of $Z(\mathcal{H}_1(\mathbf{q}))$-linear endomorphisms of $\mathcal{A}_1(\mathbf{q})$. Recall that $Z(\mathcal{H}_1(\mathbf{q})) = \mathcal{A}_1(\mathbf{q})^s$ is the subring of invariants of the commutative ring $\mathcal{A}_1(\mathbf{q})$.

Lemma 4.2.1. *We have*

$$\mathcal{A}_1(\mathbf{q}) = \mathcal{A}_1(\mathbf{q})^s X \oplus \mathcal{A}_1(\mathbf{q})^s = \mathcal{A}_1(\mathbf{q})^s \oplus \mathcal{A}_1(\mathbf{q})^s Y$$

as $\mathcal{A}_1(\mathbf{q})^s$-modules.

Proof. Applying s, the two decompositions are equivalent; so it suffices to check that $\mathbb{Z}[z_2^{\pm 1}][X,Y]$ is free of rank 2 with basis $1, Y$ over the subring of symmetric polynomials $\mathbb{Z}[z_2^{\pm 1}][X+Y, XY]$. First if $P = QY$ with P and Q symmetric, then applying s we get $P = QX$ and hence $Q(X-Y) = 0$ which implies $P = Q = 0$. It remains to check that any monomial X^iY^j, $i,j \in \mathbb{N}$, belongs to

$$\mathbb{Z}[z_2^{\pm 1}][X+Y, XY] + \mathbb{Z}[z_2^{\pm 1}][X+Y, XY]Y.$$

As $X = (X+Y) - Y$ and $Y^2 = -XY + (X+Y)Y$, the latter is stable under multiplication by X and Y; as it contains 1, the result follows. □

Remark 4.2.2. The basis $\{1, Y\}$ specializes at $\mathbf{q} = 1$ to the so-called *Pittie-Steinberg basis* [St75] of $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}[\Lambda]^{W_0}$.

Definition 4.2.3. We let

$$\begin{aligned} D_s &:= \text{ projector on } \mathcal{A}_1(\mathbf{q})^s Y \text{ along } \mathcal{A}_1(\mathbf{q})^s, \\ D'_s &:= \text{ projector on } \mathcal{A}_1(\mathbf{q})^s \text{ along } \mathcal{A}_1(\mathbf{q})^s X, \\ D_s(\mathbf{q}) &:= D_s - \mathbf{q}D'_s. \end{aligned}$$

Remark 4.2.4. The operators D_s and D'_s specialize at $\mathbf{q} = 1$ to the *Demazure operators* on $\mathbb{Z}[\Lambda]$, as introduced in [D73, D74].

Lemma 4.2.5. *We have*

$$D_s(\mathbf{q})^2 = (1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q}.$$

Proof. Noting that $Y = z_1 - X$, we have

$$D_s(\mathbf{q})^2(1) = D_s(\mathbf{q})(-\mathbf{q}) = \mathbf{q}^2 = (1-\mathbf{q})(-\mathbf{q}) + \mathbf{q} = ((1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q})(1)$$

and

$$\begin{aligned} D_s(\mathbf{q})^2(Y) &= D_s(\mathbf{q})(Y - \mathbf{q}z_1) \\ &= Y - \mathbf{q}z_1 - \mathbf{q}z_1(-\mathbf{q}) \\ &= (1-\mathbf{q})(Y - \mathbf{q}z_1) + \mathbf{q}Y \\ &= ((1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q})(Y). \end{aligned}$$

□

4.3. **The generic non-regular spherical representation.** We define the generic non-regular spherical representation of the algebra $\mathcal{H}_1(\mathbf{q})$ on the $Z(\mathcal{H}_1(\mathbf{q}))$-module $\mathcal{A}_1(\mathbf{q})$. The commutative ring $\mathcal{A}_1(\mathbf{q})$ is naturally a subring

$$\mathcal{A}_1(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q})),$$

an element $a \in \mathcal{A}_1(\mathbf{q})$ acting by multiplication $b \mapsto ab$ on $\mathcal{A}_1(\mathbf{q})$.

Theorem 4.3.1. *There exists a unique* $\mathbb{Z}[\mathbf{q}]$*-algebra homomorphism*

$$\mathscr{A}_1(\mathbf{q}) : \mathcal{H}_1(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$$

such that

(i) $\mathscr{A}_1(\mathbf{q})|_{\mathcal{A}_1(\mathbf{q})} =$ *the natural inclusion* $\mathcal{A}_1(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$
(ii) $\mathscr{A}_1(\mathbf{q})(S) = -D_s(\mathbf{q})$.

Proof. Recall that $\mathcal{H}_1(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ with the relations $S^2 = (\mathbf{q}-1)S + \mathbf{q}$ and $U^2S = SU^2$. In particular $\mathscr{A}_1(\mathbf{q})(S) := -D_s(\mathbf{q})$ is well-defined thanks to 4.2.5. On the other hand, the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{A}_1(\mathbf{q})$ is generated by

$$z_2 = U^2, \quad X = US + (1-\mathbf{q})U \quad \text{and} \quad Y = SU.$$

Consequently, there exists a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism $\mathscr{A}_1(\mathbf{q})$ as in the statement of the theorem if and only if there exists

$$\mathscr{A}_1(\mathbf{q})(U) \in \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$$

satisfying

(1) $\mathscr{A}_1(\mathbf{q})(U)^2 = \mathscr{A}_1(\mathbf{q})(U^2) = \mathscr{A}_1(\mathbf{q})(z_2) = z_2 \,\mathrm{Id}$ (in particular $\mathscr{A}_1(\mathbf{q})(U)$ is invertible)
(2) $\mathscr{A}_1(\mathbf{q})(U)(-D_s(\mathbf{q})) + (1-\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) =$ multiplication by X
(3) $-D_s(\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) =$ multiplication by Y.

Let us use the $Z(\mathcal{H}_1(\mathbf{q}))$-basis $1, Y$ of $\mathcal{A}_1(\mathbf{q})$ to identify $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ with the algebra of 2×2-matrices over the ring $Z(\mathcal{H}_1(\mathbf{q})) = \mathcal{A}_1(\mathbf{q})^s$. Then, by definition,

$$-D_s(\mathbf{q}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \mathbf{q}\begin{pmatrix} 1 & z_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{q} & \mathbf{q}z_1 \\ 0 & -1 \end{pmatrix}.$$

Moreover, as $X = z_1 - Y$, $XY = \mathbf{q}z_2$ and $Y^2 = -XY + (X+Y)Y = -\mathbf{q}z_2 + z_1Y$, the multiplications by X and by Y on $\mathcal{A}_1(\mathbf{q})$ get identified with the matrices

$$\begin{pmatrix} z_1 & \mathbf{q}z_2 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix}.$$

Now, writing

$$\mathscr{A}_1(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

we have:

$$\mathscr{A}_1(\mathbf{q})(U)^2 = z_2\,\mathrm{Id} \iff \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix},$$

$$\mathscr{A}_1(\mathbf{q})(U)(-D_s(\mathbf{q})) + (1-\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) = \text{ multiplication by } X$$

$$\iff \begin{pmatrix} a & \mathbf{q}(az_1 - c) \\ b & \mathbf{q}(bz_1 - d) \end{pmatrix} = \begin{pmatrix} z_1 & \mathbf{q}z_2 \\ -1 & 0 \end{pmatrix}$$

and

$$-D_s(\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) = \text{ multiplication by } Y$$

$$\iff \begin{pmatrix} \mathbf{q}(a + z_1 b) & \mathbf{q}(c + z_1 d) \\ -b & -d \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix}.$$

Each of the two last systems admits a unique solution, namely

$$\mathscr{A}_1(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} z_1 & z_1^2 - z_2 \\ -1 & -z_1 \end{pmatrix},$$

which is also a solution of the first one. Moreover, the determinant

$$ad - bc = -z_1^2 + (z_1^2 - z_2) = -z_2$$

is invertible. □

4.3.1. The relation between our generic non-regular representation $\mathscr{A}_1(\mathbf{q})$ and the theory of Kazhdan-Lusztig [KL87], and Ginzburg [CG97], is the following. Introducing a square root $\mathbf{q}^{\frac{1}{2}}$ of $\mathbf{q}$ and extending scalars along $\mathbb{Z}[\mathbf{q}] \subset \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$, we obtain the Hecke algebra $\mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}})$ together with its commutative subalgebra $\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}})$. The latter contains the elements $\tilde{\theta}_\lambda$, $\lambda \in \Lambda$, introduced by Bernstein and Lusztig, which are defined as follows: writing $\lambda = \lambda_1 - \lambda_2$ with λ_1, λ_2 antidominant, one has

$$\tilde{\theta}_\lambda := \tilde{T}_{e^{\lambda_1}} \tilde{T}_{e^{\lambda_2}}^{-1} := \mathbf{q}^{-\frac{\ell(\lambda_1)}{2}} \mathbf{q}^{\frac{\ell(\lambda_2)}{2}} T_{e^{\lambda_1}} T_{e^{\lambda_2}}^{-1}.$$

They are related to the Bernstein basis $\{E(w),\ w \in W\}$ of $\mathcal{H}_1(\mathbf{q})$ introduced by Vignéras (which is analogous to the Bernstein basis of $\mathcal{H}^{(1)}(\mathbf{q})$ which we have recalled in 2.3.1) by the formula:

$$\forall \lambda \in \Lambda,\ \forall w \in W_0, \quad E(e^\lambda w) = \mathbf{q}^{\frac{\ell(e^\lambda w) - \ell(w)}{2}} \tilde{\theta}_\lambda T_w \quad \in \mathcal{H}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}).$$

In particular $E(e^\lambda) = \mathbf{q}^{\frac{\ell(e^\lambda)}{2}} \tilde{\theta}_\lambda$, and by the product formula (analogous to the product formula for $\mathcal{H}^{(1)}(\mathbf{q})$, cf. 2.3.1), the $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$-linear isomorphism

$$\begin{aligned} \tilde{\theta} : \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda] &\xrightarrow{\sim} \mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}}), \\ e^\lambda &\longmapsto \tilde{\theta}_\lambda \end{aligned}$$

is in fact multiplicative, i.e. it is an isomorphism of $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$-algebras.

Consequently, if we base change our action map $\mathscr{A}_1(\mathbf{q})$ to $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$, we get a representation

$$\mathscr{A}_1(\mathbf{q}^{\pm\frac{1}{2}}) : \mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}))}(\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}})) \simeq \mathrm{End}_{\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]^{W_0}}(\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]),$$

which coincides with the natural inclusion $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda] \subset \mathrm{End}_{\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]^{W_0}}(\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda])$ when restricted to $\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}}) \simeq \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]$, and which sends S to the opposite $-D_s(\mathbf{q})$ of the $\mathbf{q}$-deformed Demazure operator. Hence, modulo our choice of antidominant orientation, this is the spherical representation defined by Kazhdan-Lusztig [KL87, Lem. 3.9] and Ginzburg [CG97, 7.6].[4]

In particular, $\mathscr{A}_1(1)$ is the usual action of the Iwahori-Weyl group $W = \Lambda \rtimes W_0$ on Λ, and $\mathscr{A}_1(0)$ can be thought of as a degeneration of the latter.

Proposition 4.3.2. *The homomorphism $\mathscr{A}_1(\mathbf{q})$ is injective.*

Proof. It follows from 4.1.2 and 4.1.4 that the ring $\mathcal{H}_1(\mathbf{q})$ is generated by the elements

$$1,\ S,\ U,\ SU$$

as a module over its center $Z(\mathcal{H}_1(\mathbf{q})) = \mathbb{Z}[\mathbf{q}][z_1, z_2^{\pm 1}]$. As the latter is mapped isomorphically to the center of the matrix algebra $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ by $\mathscr{A}_1(\mathbf{q})$, it suffices to check that the images

$$1,\ \mathscr{A}_1(\mathbf{q})(S),\ \mathscr{A}_1(\mathbf{q})(U),\ \mathscr{A}_1(\mathbf{q})(SU)$$

[4] Moreover, it can be checked, in analogy to loc. cit., that the $\mathcal{H}_1(\mathbf{q})$-module $\mathscr{A}_1(\mathbf{q})$ is isomorphic to the induction of the trivial character of the finite Hecke (sub)algebra $\mathbb{Z}[\mathbf{q}][S]$. But we will not make use of this in the following.

of $1, S, U, SU$ by $\mathscr{A}_1(\mathbf{q})$ are free over $Z(\mathcal{H}_1(\mathbf{q}))$. So let $\alpha, \beta, \gamma, \delta \in Z(\mathcal{H}_1(\mathbf{q}))$ (which is an integral domain) be such that

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} \mathbf{q} & \mathbf{q}z_1 \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} z_1 & z_1^2 - z_2 \\ -1 & -z_1 \end{pmatrix} + \delta \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix} = 0.$$

Then

$$\begin{cases} \alpha + \beta\mathbf{q} + \gamma z_1 & = 0, \\ -\gamma + \delta & = 0, \\ \beta\mathbf{q}z_1 + \gamma(z_1^2 - z_2) - \delta\mathbf{q}z_2 & = 0, \\ \alpha - \beta + (\delta - \gamma)z_1 & = 0. \end{cases}$$

We obtain $\delta = \gamma$, $\alpha = \beta$ and

$$\begin{cases} \alpha(1 + \mathbf{q}) + \gamma z_1 & = 0, \\ \alpha\mathbf{q}z_1 + \gamma(z_1^2 - z_2 - \mathbf{q}z_2) & = 0. \end{cases}$$

The latter system has determinant

$$(1 + \mathbf{q})(z_1^2 - z_2 - \mathbf{q}z_2) - \mathbf{q}z_1^2 = z_1^2 - z_2 - 2\mathbf{q}z_2 - \mathbf{q}^2 z_2$$

which is non-zero (its specialization at $\mathbf{q} = 0$ is equal to $z_1^2 - z_2 \neq 0$), whence $\alpha = \gamma = 0 = \beta = \delta$. $\square$

We record Corollaries 4.3.3 and 4.3.4 of the proof.

Corollary 4.3.3. *The ring $\mathcal{H}_1(\mathbf{q})$ is a free $Z(\mathcal{H}_1(\mathbf{q}))$-module on the basis $1, S, U, SU$.*

Corollary 4.3.4. *The homomorphism $\mathscr{A}_1(0)$ is injective.*

4.3.2. We end this section by noting an equivariance property of $\mathscr{A}_1(\mathbf{q})$. As already noticed, the finite Weyl group W_0 acts on $\mathcal{A}_1(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra automorphisms, and the action is clearly faithful. Moreover $\mathcal{A}_1(\mathbf{q})^{W_0} = Z(\mathcal{H}_1(q))$. Hence W_0 can be viewed as a subgroup of $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$, and we can let it act on $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ by conjugation.

Lemma 4.3.5. *The embedding $\mathscr{A}_1(\mathbf{q})|_{\mathcal{A}_1(\mathbf{q})}$ is W_0-equivariant.*

Proof. Indeed, for all $a, b \in \mathcal{A}_1(\mathbf{q})$ and $w \in W_0$, we have

$$\mathscr{A}_1(\mathbf{q})(w(a))(b) = w(a)b = w(aw^{-1}(b)) = (waw^{-1})(b) = (w\mathscr{A}_1(\mathbf{q})(a)w^{-1})(b).$$

$\square$

5. K-theory of the dual flag variety

5.1. The Vinberg monoid of the dual group $\widehat{\mathbf{G}} = \mathbf{GL_2}$.

5.1.1. The Langlands dual group over $k := \overline{\mathbb{F}}_q$ of the connected reductive algebraic group GL_2 over F is $\widehat{\mathbf{G}} = \mathbf{GL_2}$. We recall the k-monoid scheme introduced by Vinberg in [V95], in the particular case of $\mathbf{GL_2}$. It is in fact defined over $\mathbb{Z}$, as the group $\mathbf{GL_2}$. In the following, all the fibre products are taken over the base ring $\mathbb{Z}$.

Definition 5.1.1. Let $\mathrm{Mat}_{2\times 2}$ be the $\mathbb{Z}$-monoid scheme of 2×2-matrices (with usual matrix multiplication as operation). The *Vinberg monoid for* $\mathbf{GL_2}$ is the $\mathbb{Z}$-monoid scheme

$$V_{\mathbf{GL_2}} := \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

5.1.2. The group $\mathbf{GL_2} \times \mathbb{G}_m$ is recovered from the monoid $V_{\mathbf{GL_2}}$ as its group of units. The group $\mathbf{GL_2}$ itself is recovered as follows. Denote by z_2 the canonical coordinate on $\mathbb{G}_m$. Then *let* $\mathbf{q}$ *be the homomorphism from* $V_{\mathbf{GL_2}}$ *to the multiplicative monoid* $(\mathbb{A}^1, \cdot)$ *defined by* $(f, z_2) \mapsto \det(f) z_2^{-1}$:

$$\begin{tikzcd} V_{\mathbf{GL_2}} \arrow[d, "\mathbf{q}"'] \\ \mathbb{A}^1. \end{tikzcd}$$

Then $\mathbf{GL_2}$ is recovered as the fibre at $\mathbf{q} = 1$, canonically:

$$\mathbf{q}^{-1}(1) = \{(f, z_2) : \det(f) = z_2\} \xrightarrow{\sim} \mathbf{GL_2}, \quad (f, z_2) \mapsto f.$$

The fibre at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\mathbf{GL_2},0} := \mathbf{q}^{-1}(0) = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m,$$

where $\mathrm{Sing}_{2\times2}$ represents the singular 2×2-matrices. Note that it has no identity element, i.e. it is a semigroup which is not a monoid.

5.1.3. Let $\mathrm{Diag}_{2\times2} \subset \mathrm{Mat}_{2\times2}$ be the submonoid scheme of diagonal 2×2-matrices, and set

$$V_{\widehat{\mathbf{T}}} := \mathrm{Diag}_{2\times2} \times \mathbb{G}_m \subset V_{\mathbf{GL_2}} = \mathrm{Mat}_{2\times2} \times \mathbb{G}_m.$$

This is a diagonalizable $\mathbb{Z}$-monoid scheme with character monoid

$$\mathbb{X}^\bullet(V_{\widehat{\mathbf{T}}}) = \mathbb{N}(1,0) \oplus \mathbb{N}(0,1) \oplus \mathbb{Z} \subset \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) \oplus \mathbb{Z} = \Lambda \oplus \mathbb{Z} = \mathbb{X}^\bullet(\widehat{\mathbf{T}}) \oplus \mathbb{X}^\bullet(\mathbb{G}_m).$$

In particular, setting $X := e^{(1,0)}$ and $Y := e^{(0,1)}$ in the group ring $\mathbb{Z}[\Lambda]$, we have

$$\widehat{\mathbf{T}} = \mathrm{Spec}(\mathbb{Z}[X^{\pm1}, Y^{\pm1}]) \subset \mathrm{Spec}(\mathbb{Z}[z_2^{\pm1}][X, Y]) = V_{\widehat{\mathbf{T}}}.$$

Again, this closed subgroup is recovered as the fibre at $\mathbf{q} = 1$ of the fibration $\mathbf{q}|_{V_{\widehat{\mathbf{T}}}} : V_{\widehat{\mathbf{T}}} \to \mathbb{A}^1$, and the fibre at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme $\mathrm{SingDiag}_{2\times2} \times \mathbb{G}_m$ where $\mathrm{SingDiag}_{2\times2}$ represents the singular diagonal 2×2-matrices:

$$\begin{tikzcd} \widehat{\mathbf{T}} \arrow[r, hook] \arrow[d] & V_{\widehat{\mathbf{T}}} \arrow[d, "\mathbf{q}"'] & \mathrm{SingDiag}_{2\times2} \times \mathbb{G}_m \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}) \arrow[r, hook, "1"] & \mathbb{A}^1 & \mathrm{Spec}(\mathbb{Z}). \arrow[l, hook', "0"'] \end{tikzcd}$$

In terms of equations, the $\mathbb{A}^1$-family

$$\mathbf{q} : V_{\widehat{\mathbf{T}}} = \mathrm{Diag}_{2\times2} \times \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[z_2^{\pm1}][X, Y]) \longrightarrow \mathbb{A}^1$$

is given by the formula $\mathbf{q}(\mathrm{diag}(x, y), z_2) = \det(\mathrm{diag}(x, y)) z_2^{-1} = xy z_2^{-1}$. Hence, after fixing $z_2 \in \mathbb{G}_m$, the fibre over a point $\mathbf{q} \in \mathbb{A}^1$ is the hyperbola $xy = \mathbf{q} z_2$, which is non-degenerate if $\mathbf{q} \neq 0$, and is the union of the two coordinate axes if $\mathbf{q} = 0$.

5.2. **The associated flag variety and its equivariant K-theory.**

5.2.1. Let $\widehat{\mathbf{B}} \subset \mathbf{GL_2}$ be the Borel subgroup of upper triangular matrices, let $\mathrm{UpTriang}_{2\times 2}$ be the $\mathbb{Z}$-monoid scheme representing the upper triangular 2×2-matrices, and set

$$V_{\widehat{\mathbf{B}}} := \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m \subset \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m =: V_{\mathbf{GL_2}}.$$

Then we can apply to this inclusion of $\mathbb{Z}$-monoid schemes the general formalism developed in [PS20]. In particular, the *flag variety* $V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}$ is defined as a $\mathbb{Z}$-monoidoid. Moreover, after base changing along $\mathbb{Z} \to k$, we have defined a ring $K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}})$ of $V_{\mathbf{GL_2}}$-equivariant K-theory on the flag variety, together with an induction isomorphism

$$\mathcal{I}nd_{V_{\widehat{\mathbf{B}}}}^{V_{\mathbf{GL_2}}} : R(V_{\widehat{\mathbf{B}}}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}})$$

from the ring $R(V_{\widehat{\mathbf{B}}})$ of right representations of the k-monoid scheme $V_{\widehat{\mathbf{B}}}$ on finite dimensional k-vector spaces.

5.2.2. Now, we have the inclusion of monoids

$$V_{\widehat{\mathbf{T}}} = \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m \subset V_{\widehat{\mathbf{B}}} = \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m,$$

which admits the retraction

$$\begin{aligned} V_{\widehat{\mathbf{B}}} &\longrightarrow V_{\widehat{\mathbf{T}}} \\ \left(\begin{pmatrix} x & c \\ 0 & y \end{pmatrix}, z_2 \right) &\longmapsto \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, z_2 \right). \end{aligned}$$

Let $\mathrm{Rep}(V_{\widehat{\mathbf{T}}})$ be the category of representations of the commutative k-monoid scheme $V_{\widehat{\mathbf{T}}}$ on finite dimensional k-vector spaces. The above preceding inclusion and retraction define a *restriction functor* and an *inflation functor*

$$\mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}}}) \rightleftarrows \mathrm{Rep}(V_{\widehat{\mathbf{T}}}) : \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}}.$$

These functors are exact and compatible with the tensors products and units.

Lemma 5.2.1. *The ring homomorphisms*

$$\mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : R(V_{\widehat{\mathbf{B}}}) \rightleftarrows R(V_{\widehat{\mathbf{T}}}) : \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}}$$

are isomorphisms, which are inverse one to the other.

Proof. We have $\mathrm{Res}_{V_{\widehat{\mathbf{T}}}} \circ \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} = \mathrm{Id}$ by construction. Conversely, let M be an object of $\mathrm{Rep}(V_{\widehat{\mathbf{B}}})$. The solvable subgroup $\widehat{\mathbf{B}} \times \mathbb{G}_m \subset V_{\widehat{\mathbf{B}}}$ stabilizes a line $L \subseteq M$. As $\widehat{\mathbf{B}} \times \mathbb{G}_m$ is dense in $V_{\widehat{\mathbf{B}}}$, the line L is automatically $V_{\widehat{\mathbf{B}}}$-stable. Moreover the unipotent radical $\widehat{\mathbf{U}} \subset \widehat{\mathbf{B}}$ acts trivially on L, so that $\widehat{\mathbf{B}} \times \mathbb{G}_m$ acts on L through the quotient $\widehat{\mathbf{T}} \times \mathbb{G}_m$. Hence, by density again, $V_{\widehat{\mathbf{B}}}$ acts on L through the retraction $V_{\widehat{\mathbf{B}}} \to V_{\widehat{\mathbf{T}}}$. This shows that any irreducible M is a character inflated from a character of $V_{\widehat{\mathbf{T}}}$. In particular, the map $R(V_{\widehat{\mathbf{T}}}) \to R(V_{\widehat{\mathbf{B}}})$ is surjective and hence bijective. $\square$

Corollary 5.2.2. *We have a ring isomorphism*

$$c_{V_{\mathbf{GL_2}}} := \mathcal{I}nd_{V_{\widehat{\mathbf{B}}}}^{V_{\mathbf{GL_2}}} \circ \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : \mathbb{Z}[X, Y, z_2^{\pm 1}] \cong R(V_{\widehat{\mathbf{T}}}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}})$$

that we call the characteristic isomorphism in the equivariant K-theory of the flag variety $V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}$.

5.2.3. We have a commutative diagram *specialization at* $\mathbf{q} = 1$

$$\begin{array}{ccc} \mathbb{Z}[X, Y, z_2^{\pm 1}] & \xrightarrow[\sim]{c_{V_{\mathbf{GL_2}}}} & K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}) \\ \downarrow & & \downarrow \\ \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] & \xrightarrow[\sim]{c_{\mathbf{GL_2}}} & K^{\mathbf{GL_2}}(\mathbf{GL_2}/\widehat{\mathbf{B}}). \end{array}$$

The vertical map on the left-hand side is given by specialization $\mathbf{q} = 1$, i.e. by the surjection

$$\mathbb{Z}[X, Y, z_2^{\pm 1}] = \mathbb{Z}[\mathbf{q}][X, Y, z_2^{\pm 1}]/(XY - \mathbf{q}z_2) \\ \longrightarrow \mathbb{Z}[X, Y, z_2^{\pm 1}]/(XY - z_2) = \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}].$$

The vertical map on the right-hand side is given by restricting equivariant vector bundles to the 1-fibre of $\mathbf{q} : V_{\mathbf{GL_2}} \to \mathbb{A}^1$, thereby recovering the classical theory.

5.2.4. Let $\mathrm{Rep}(V_{\mathbf{GL_2}})$ be the category of right representations of the k-monoid scheme $V_{\mathbf{GL_2}}$ on finite dimensional k-vector spaces. The inclusion $V_{\widehat{\mathbf{B}}} \subset V_{\mathbf{GL_2}}$ defines a restriction functor

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{B}}}} : \mathrm{Rep}(V_{\mathbf{GL_2}}) \longrightarrow \mathrm{Rep}(V_{\widehat{\mathbf{B}}}),$$

whose composition with $\mathrm{Res}^{V_{\widehat{\mathbf{B}}}}_{V_{\widehat{\mathbf{T}}}}$ is the restriction from $V_{\mathbf{GL_2}}$ to $V_{\widehat{\mathbf{T}}}$:

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{T}}}} = \mathrm{Res}^{V_{\widehat{\mathbf{B}}}}_{V_{\widehat{\mathbf{T}}}} \circ \mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{B}}}} : \mathrm{Rep}(V_{\mathbf{GL_2}}) \longrightarrow \mathrm{Rep}(V_{\widehat{\mathbf{T}}}).$$

These restriction functors are exact and compatible with the tensors products and units.

5.2.5. The action of the Weyl group W_0 on $\mathbb{X}^\bullet(\widehat{\mathbf{T}}) \oplus \mathbb{X}^\bullet(\mathbb{G}_m)$ (trivial on $\mathbb{X}^\bullet(\mathbb{G}_m)$) stabilizes $\mathbb{X}^\bullet(V_{\widehat{\mathbf{T}}})$. Consequently W_0 acts on $V_{\widehat{\mathbf{T}}}$ and the inclusion $\widehat{\mathbf{T}} \subset V_{\widehat{\mathbf{T}}}$ is W_0-equivariant. Explicitly, $W_0 = \{1, s\}$ and s acts on $V_{\widehat{\mathbf{T}}} = \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m$ by permuting the two diagonal entries and trivially on the $\mathbb{G}_m$-factor.

Lemma 5.2.3. *The ring homomorphism*

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{T}}}} : R(V_{\mathbf{GL_2}}) \longrightarrow R(V_{\widehat{\mathbf{T}}})$$

is injective, with image the subring $R(V_{\widehat{\mathbf{T}}})^{W_0} \subset R(V_{\widehat{\mathbf{T}}})$ *of* W_0*-invariants. The resulting ring isomorphism*

$$\chi_{V_{\mathbf{GL_2}}} : R(V_{\mathbf{GL_2}}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0}$$

is the character isomorphism of $V_{\mathbf{GL_2}}$.

Proof. This is a general result on the representation theory of $V_{\widehat{\mathbf{G}}}$. Note that in the case of $\widehat{\mathbf{G}} = \mathbf{GL_2}$, we have

$$R(V_{\widehat{\mathbf{T}}})^{W_0} = \mathbb{Z}[X + Y, XYz_2^{-1} =: \mathbf{q}, z_2^{\pm 1}] \subset \mathbb{Z}[X, Y, z_2^{\pm 1}] = R(V_{\widehat{\mathbf{T}}}).$$

□

6. Dual parametrization of generic Hecke modules

We keep all the notations introduced in the preceding section. In particular, $k = \overline{\mathbb{F}}_q$.

6.1. **The generic Bernstein isomorphism.** Recall from 2.3.1 the subring $\mathcal{A}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ and the remarkable Bernstein basis elements $E(1,0)$, $E(0,1)$ and $E(1,1)$. Also recall from 5.1.3 the representation ring $R(V_{\widehat{\mathbf{T}}}) = \mathbb{Z}[X, Y, z_2^{\pm 1}]$ of the diagonalizable k-submonoid scheme $V_{\widehat{\mathbf{T}}} \subset V_{\widehat{\mathbf{G}}}$ of the Vinberg k-monoid scheme of the Langlands dual k-group $\widehat{\mathbf{G}} = \mathbf{GL_2}$ of $GL_{2,F}$.

Theorem 6.1.1. *There exists a unique ring homomorphism*

$$\mathscr{B}(\mathbf{q}) : \mathcal{A}(\mathbf{q}) \longrightarrow R(V_{\widehat{\mathbf{T}}})$$

such that

$$\mathscr{B}(\mathbf{q})(E(1,0)) = X, \quad \mathscr{B}(\mathbf{q})(E(0,1)) = Y,$$
$$\mathscr{B}(\mathbf{q})(E(1,1)) = z_2 \quad \textit{and} \quad \mathscr{B}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}.$$

It is an isomorphism.

Proof. This is a reformulation of the first part of 2.3.2. □

6.1.1. Then recall from 2.3.1 the subring $\mathcal{A}^{(1)}(\mathbf{q}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ where $\mathbb{T}$ is the finite abelian group $\mathbf{T}(\mathbb{F}_q)$. Let $\mathbb{T}^\vee$ be the finite abelian dual group of $\mathbb{T}$. As $\mathbb{T}^\vee$ has order prime to p, it defines a constant finite diagonalizable k-group scheme, whose group of characters is $\mathbb{T}$, and hence whose representation ring $R(\mathbb{T}^\vee)$ identifies with $\mathbb{Z}[\mathbb{T}]$: $t \in \mathbb{T} \subset \mathbb{Z}[\mathbb{T}]$ corresponds to the character ev_t of $\mathbb{T}^\vee$ given by evaluation at t. Set

$$V_{\widehat{\mathbf{T}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}}}.$$

Corollary 6.1.2. *There exists a unique ring homomorphism*

$$\mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \longrightarrow R(V_{\widehat{\mathbf{T}}}^{(1)})$$

such that

$$\mathscr{B}^{(1)}(\mathbf{q})(E(1,0)) = X, \quad \mathscr{B}^{(1)}(\mathbf{q})(E(0,1)) = Y, \mathscr{B}^{(1)}(\mathbf{q})(E(1,1)) = z_2,$$
$$\mathscr{B}^{(1)}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}$$
$$\textit{and} \quad \forall t \in \mathbb{T},\ \mathscr{B}^{(1)}(\mathbf{q})(T_t) = \mathrm{ev}_t\,.$$

It is an isomorphism that we call the generic (pro-p) Bernstein isomorphism.

6.1.2. Also, setting $V_{\widehat{\mathbf{B}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{B}}}$, we have from 5.2.1 the ring isomorphism

$$\mathrm{Infl}_{V_{\widehat{\mathbf{T}}}^{(1)}}^{V_{\widehat{\mathbf{B}}}^{(1)}} = \mathrm{Id}_{\mathbb{Z}[\mathbb{T}]} \otimes_{\mathbb{Z}} \mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : R(V_{\widehat{\mathbf{T}}}^{(1)}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} R(V_{\widehat{\mathbf{T}}})$$
$$\xrightarrow{\ \sim\ } R(V_{\widehat{\mathbf{B}}}^{(1)}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} R(V_{\widehat{\mathbf{B}}}),$$

and setting $V_{\widehat{\mathbf{G}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{G}}}$, we have from [PS20, 2.5.2], the ring isomorphism

$$\mathcal{I}nd_{V_{\widehat{\mathbf{B}}}^{(1)}}^{V_{\widehat{\mathbf{G}}}^{(1)}} : R(V_{\widehat{\mathbf{B}}}^{(1)}) \xrightarrow{\ \sim\ } K^{V_{\widehat{\mathbf{G}}}^{(1)}}(V_{\widehat{\mathbf{G}}}^{(1)}/V_{\widehat{\mathbf{B}}}^{(1)});$$

hence by composition we get the *characteristic isomorphism*

$$c_{V^{(1)}_{\widehat{\mathbf{G}}}} : R(V^{(1)}_{\widehat{\mathbf{T}}}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

Whence a ring isomorphism

$$c_{V^{(1)}_{\widehat{\mathbf{G}}}} \circ \mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

6.1.3. The representation ring $R(V_{\widehat{\mathbf{T}}})$ is canonically isomorphic to the ring $\mathbb{Z}[V_{\widehat{\mathbf{T}}}]$ of regular functions of $V_{\widehat{\mathbf{T}}}$ considered now as a diagonalizable monoid scheme over $\mathbb{Z}$. Also recall from 2.2.1 the ring extension $\mathbb{Z} \subset \tilde{\mathbb{Z}}$, and denote by $\tilde{\bullet}$ the base change functor from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$. For example, we will from now on write $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ instead of $\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. We have the constant finite diagonalizable $\tilde{\mathbb{Z}}$-group scheme $\mathbb{T}^\vee$, whose group of characters is $\mathbb{T}$, and whose ring of regular functions is

$$\tilde{\mathbb{Z}}[\mathbb{T}] = \prod_{\lambda \in \mathbb{T}^\vee} \tilde{\mathbb{Z}}\varepsilon_\lambda.$$

Hence applying the functor Spec to $\tilde{\mathscr{B}}^{(1)}(\mathbf{q})$, we obtain the commutative diagram of $\tilde{\mathbb{Z}}$-schemes

$$\begin{tikzcd}
\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \arrow[dr, "\pi_0 \times \mathbf{q}"'] & & V^{(1)}_{\widehat{\mathbf{T}}} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}}} \arrow[ll, "\mathrm{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q}))"', "\sim"] \arrow[dl, "\mathrm{Id} \times \mathbf{q}"] \\
& (\mathbb{A}^1)^{(1)} := \mathbb{T}^\vee \times \mathbb{A}^1 &
\end{tikzcd}$$

where $\pi_0 : \mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \to \mathbb{T}^\vee$ is the decomposition of $\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$ into its connected components. In particular, for each $\lambda \in \mathbb{T}^\vee$, we have the subring $\tilde{\mathcal{A}}^\lambda(\mathbf{q}) = \tilde{\mathcal{A}}^{(1)}(\mathbf{q})\varepsilon_\lambda$ of $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ and the isomorphism

$$\mathrm{Spec}(\tilde{\mathcal{A}}^\lambda(\mathbf{q})) \xleftarrow[\sim]{\mathrm{Spec}(\tilde{\mathscr{B}}^\lambda(\mathbf{q}))} \{\lambda\} \times V_{\widehat{\mathbf{T}}}$$

of $\tilde{\mathbb{Z}}$-schemes over $\{\lambda\} \times \mathbb{A}^1$. In turn, each of these isomorphisms admits a model over $\mathbb{Z}$, obtained by applying Spec to the ring isomorphism in 4.1.4

$$\mathscr{B}_1(\mathbf{q}) : \mathcal{A}_1(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}}).$$

6.2. **The generic Satake isomorphism.** Recall part of our notation: $\mathbf{G}$ is the algebraic group $\mathbf{GL_2}$ (which is defined over $\mathbb{Z}$), F is a local field and $G := \mathbf{G}(F)$. We have denoted by o_F the ring of integers of F. Now we set $K := \mathbf{G}(o_F)$.

Definition 6.2.1. Let R be any commutative ring. The *spherical Hecke algebra of G with coefficients in R* is defined to be the convolution algebra

$$\mathcal{H}^{\mathrm{sph}}_R := (R[K\backslash G/K], \star)$$

generated by the K-double cosets in G.

6.2.1. By the work of Kazhdan and Lusztig, the R-algebra $\mathcal{H}_R^{\mathrm{sph}}$ depends on F only through the cardinality q of its residue field. Indeed, choose a uniformizer $\varpi \in o_F$. For a dominant cocharacter $\lambda \in \Lambda^+$ of $\mathbf{T}$, let $\mathbb{1}_\lambda$ be the characteristic function of the double coset $K\lambda(\varpi)K$. Then $(\mathbb{1}_\lambda)_{\lambda\in\Lambda^+}$ is an R-basis of $\mathcal{H}_R^{\mathrm{sph}}$. Moreover, for all $\lambda, \mu, \nu \in \Lambda^+$, there exist polynomials

$$N_{\lambda,\mu;\nu}(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}]$$

depending only on the triple (λ, μ, ν), such that

$$\mathbb{1}_\lambda \star \mathbb{1}_\mu = \sum_{\nu\in\Lambda^+} N_{\lambda,\mu;\nu}(q)\mathbb{1}_\nu,$$

where $N_{\lambda,\mu;\nu}(q)$ is the image under $\mathbb{Z} \to R$ of the value of $N_{\lambda,\mu;\nu}(\mathbf{q})$ at $\mathbf{q} = q$. These polynomials are uniquely determined by this property since when the non-archimedean local field F varies (already over its unramified extensions), the corresponding integers q form an infinite set. Their existence can be deduced from the theory of the spherical algebra with coefficients in $\mathbb{C}$, as $\mathcal{H}_R^{\mathrm{sph}} = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}}$ and $\mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}} \subset \mathcal{H}_{\mathbb{C}}^{\mathrm{sph}}$ (e.g. using arguments similar to those in the proof of 6.2.3 below).

Definition 6.2.2. Let $\mathbf{q}$ be an indeterminate. The *generic spherical Hecke algebra* is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ defined by generators

$$\mathcal{H}^{\mathrm{sph}}(\mathbf{q}) := \oplus_{\lambda\in\Lambda^+}\mathbb{Z}[\mathbf{q}]T_\lambda$$

and relations:

$$T_\lambda T_\mu = \sum_{\nu\in\Lambda^+} N_{\lambda,\mu;\nu}(\mathbf{q})T_\nu \quad \text{for all } \lambda, \mu \in \Lambda^+.$$

Theorem 6.2.3. *There exists a unique ring homomorphism*

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow R(V_{\widehat{\mathbf{T}}})$$

such that

$$\mathscr{S}(\mathbf{q})(T_{(1,0)}) = X + Y, \quad \mathscr{S}(\mathbf{q})(T_{(1,1)}) = z_2 \quad \textit{and} \quad \mathscr{S}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}.$$

It is an isomorphism onto the subring $R(V_{\widehat{\mathbf{T}}})^{W_0}$ of W_0-invariants

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0} \subset R(V_{\widehat{\mathbf{T}}}).$$

In particular, the algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ is commutative.

Proof. Let

$$\mathscr{S}_{\mathrm{cl}} : \mathcal{H}_{\mathbb{C}}^{\mathrm{sph}} \xrightarrow{\sim} \mathbb{C}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$$

be the 'classical' isomorphism constructed by Satake [Sat63]. We use [Gr98] as a reference.

For $\lambda \in \Lambda^+$, let $\chi_\lambda \in \mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$ be the character of the irreducible representation of $\widehat{\mathbf{G}}$ of highest weight λ. Then $(\chi_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$. Set $f_\lambda := \mathscr{S}_{\mathrm{cl}}^{-1}(q^{\langle\rho,\lambda\rangle}\chi_\lambda)$, where $2\rho = \alpha := (1, -1)$. Then for each $\lambda, \mu \in \Lambda^+$, there exist polynomials $d_{\lambda,\mu}(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}]$ such that

$$f_\lambda = \mathbb{1}_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(q)\mathbb{1}_\mu \in \mathcal{H}_{\mathbb{C}}^{\mathrm{sph}},$$

where $d_{\lambda,\mu}(q) \in \mathbb{Z}$ is the value of $d_{\lambda,\mu}(\mathbf{q})$ at $\mathbf{q} = q$; the polynomial $d_{\lambda,\mu}(\mathbf{q})$ depends only on the couple (λ, μ), in particular it is uniquely determined by this property. As $(\mathbb{1}_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}$-basis of $\mathcal{H}^{\mathrm{sph}}_{\mathbb{Z}}$, so is $(f_\lambda)_{\lambda\in\Lambda^+}$. Then let us set

$$f_\lambda(\mathbf{q}) := T_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(\mathbf{q}) T_\mu \in \mathcal{H}^{\mathrm{sph}}(\mathbf{q}).$$

As $(T_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}[\mathbf{q}]$-basis of $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$, so is $(f_\lambda(\mathbf{q}))_{\lambda\in\Lambda^+}$.

Next consider the following $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$-linear map:

$$\begin{aligned} \mathscr{S}_{\mathrm{cl}}(\mathbf{q}) : \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) &\longrightarrow \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})] = \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}][\mathbb{X}^\bullet(\widehat{\mathbf{T}})], \\ 1 \otimes f_\lambda(\mathbf{q}) &\longmapsto \mathbf{q}^{\langle\rho,\lambda\rangle} \chi_\lambda. \end{aligned}$$

We claim that it is a ring homomorphism. Indeed, for $h_1(\mathbf{q}), h_2(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$, we need to check the identity

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_1(\mathbf{q})h_2(\mathbf{q})) = \mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_1(\mathbf{q}))\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_2(\mathbf{q})) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}][\mathbb{X}^\bullet(\widehat{\mathbf{T}})].$$

Projecting in the $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$-basis $\mathbb{X}^\bullet(\widehat{\mathbf{T}})$, the latter corresponds to (a finite number of) identities in the ring $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$ of polynomials in the variable $\mathbf{q}^{\frac{1}{2}}$. Now, by construction and because $\mathscr{S}_{\mathrm{cl}}$ is a ring homomorphism, the desired identities hold after specializing $\mathbf{q}$ to any power of a prime number; hence they hold in $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$. Also note that $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ maps $1 = T_{(0,0)}$ to $1 = \chi_{(0,0)}$ by definition.

It can also be seen that $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ is injective using a specialization argument: if $h(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ satisfies $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h(\mathbf{q})) = 0$, then the coordinates of $h(\mathbf{q})$ (in the basis $(1 \otimes f_\lambda(\mathbf{q}))_{\lambda\in\Lambda^+}$ say, one can also use the basis $(1 \otimes T_\lambda)_{\lambda\in\Lambda^+}$) are polynomials in the variable $\mathbf{q}^{\frac{1}{2}}$ which must vanish for an infinite number of values of $\mathbf{q}$, and hence they are identically zero.

Let us describe the image of $\mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \subset \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ under the ring embedding $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$. By construction, we have

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) = \bigoplus_{\lambda\in\Lambda^+} \mathbb{Z}[\mathbf{q}]\mathbf{q}^{\langle\rho,\lambda\rangle}\chi_\lambda.$$

Explicitly,

$$\Lambda^+ = \mathbb{N}(1,0) \oplus \mathbb{Z}(1,1) \subset \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) = \Lambda,$$

so that

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) = \left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}[\mathbf{q}]\mathbf{q}^{\frac{n}{2}}\chi_{(n,0)}\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_{(1,1)}^{\pm 1}].$$

On the other hand, recall that the ring of symmetric polynomials in the two variables $e^{(1,0)}$ and $e^{(0,1)}$ is a graded ring generated the two characters $\chi_{(1,0)} = e^{(1,0)} + e^{(0,1)}$ and $\chi_{(1,1)} = e^{(1,0)}e^{(0,1)}$:

$$\mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s_n = \mathbb{Z}[\chi_{(1,0)}, \chi_{(1,1)}].$$

As $\chi_{(1,0)}$ is homogeneous of degree 1 and $\chi_{(1,1)}$ is homogeneous of degree 2, this implies that

$$\mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s_n = \bigoplus_{\substack{(a,b)\in\mathbb{N}^2 \\ a+2b=n}} \mathbb{Z}\chi_{(1,0)}^a \chi_{(1,1)}^b.$$

Now if $a+2b=n$, then $\mathbf{q}^{\frac{n}{2}}\chi_{(1,0)}^a\chi_{(1,1)}^b = (\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)})^a(\mathbf{q}\chi_{(1,1)})^b$. As the symmetric polynomial $\chi_{(n,0)}$ is homogeneous of degree n, we get the inclusion

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) \subset \mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}, \mathbf{q}\chi_{(1,1)}] \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_{(1,1)}^{\pm 1}] = \mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}, \chi_{(1,1)}^{\pm 1}].$$

Since by definition of $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ we have $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(1,0)}(\mathbf{q})) = \mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}$, $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(1,1)}(\mathbf{q})) = \chi_{(1,1)}$ and $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(-1,-1)}(\mathbf{q})) = \chi_{(-1,-1)} = \chi_{(1,1)}^{-1}$, this inclusion is an equality. We have thus obtained the $\mathbb{Z}[\mathbf{q}]$-algebra isomorphism:

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})} : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} \mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}, \chi_{(1,1)}^{\pm 1}].$$

Also note that $T_{(1,0)} \mapsto \mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}$ and $T_{(1,1)} \mapsto \chi_{(1,1)}$ since $T_{(1,0)} = f_{(1,0)}(\mathbf{q})$ and $T_{(1,1)} = f_{(1,1)}(\mathbf{q})$.

Finally, recall that $V_{\widehat{\mathbf{T}}}$ being the diagonalizable k-monoid scheme

$$\mathrm{Spec}(k[X, Y, z_2^{\pm 1}]),$$

we have

$$R(V_{\widehat{\mathbf{T}}})^{W_0} = \mathbb{Z}[X, Y, z_2^{\pm 1}]^{W_0} = \mathbb{Z}[X+Y, XY, z_2^{\pm 1}] = \mathbb{Z}[X+Y, XYz_2^{-1}, z_2^{\pm 1}].$$

Hence we can define a ring isomorphism

$$\iota : \mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}, \chi_{(1,1)}^{\pm 1}] \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0}$$

by $\iota(\mathbf{q}) := XYz_2^{-1}$, $\iota(\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}) = X+Y$ and $\iota(\chi_{(1,1)}) = z_2$. Composing, we get the desired isomorphism

$$\mathscr{S}(\mathbf{q}) := \iota \circ \mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})} : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0}.$$

Note that $\mathscr{S}(\mathbf{q})(T_{(1,0)}) = X+Y$, $\mathscr{S}(\mathbf{q})(T_{(1,1)}) = z_2$, $\mathscr{S}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}$, and that $\mathscr{S}(\mathbf{q})$ is uniquely determined by these assignments since the ring $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ is the polynomial ring in the variables $\mathbf{q}$, $T_{(1,0)}$ and $T_{(1,1)}^{\pm 1}$, thanks to the isomorphism $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})}$. $\square$

Remark 6.2.4. The choice of the isomorphism ι in the preceding proof may seem *ad hoc*. However, it is natural from the point of view of the Vinberg fibration $\mathbf{q} : V_{\widehat{\mathbf{T}}} \to \mathbb{A}^1$.

First, as pointed out by Herzig in [H11, §1.2], one can make the classical complex Satake transform $\mathscr{S}_{\mathrm{cl}}$ integral, by removing the factor $\delta^{\frac{1}{2}}$ from its definition, where δ is the modulus character of the Borel subgroup. Doing so produces a ring embedding

$$\mathcal{S}' : \mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}} \hookrightarrow \mathbb{Z}[\mathbb{X}^{\bullet}(\widehat{\mathbf{T}})].$$

The image of $\mathcal{S}'$ is not contained in the subring $\mathbb{Z}[\mathbb{X}^{\bullet}(\widehat{\mathbf{T}})]^{W_0}$ of W_0-invariants. In fact,

$$\mathcal{S}'(T_{(1,0)}) = qe^{(1,0)} + e^{(0,1)} \quad \text{and} \quad \mathcal{S}'(T_{(1,1)}) = e^{(1,1)},$$

so that

$$\mathcal{S}' : \mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}} \xrightarrow{\sim} \mathbb{Z}[(qe^{(1,0)} + e^{(0,1)}), e^{\pm(1,1)}] \subset \mathbb{Z}[\mathbb{X}^{\bullet}(\widehat{\mathbf{T}})].$$

Now,

$$\mathbb{Z}[\mathbb{X}^{\bullet}(\widehat{\mathbf{T}})] = \mathbb{Z}[\widehat{\mathbf{T}}] = \mathbb{Z}[V_{\widehat{\mathbf{T}},1}],$$

where $\widehat{\mathbf{T}} \cong V_{\widehat{\mathbf{T}},1}$ is the fibre *at* 1 of the fibration $\mathbf{q} : V_{\widehat{\mathbf{T}}} \to \mathbb{A}^1$ considered over $\mathbb{Z}$. But the algebra $\mathcal{H}^{\mathrm{sph}}_{\mathbb{Z}}$ is the specialization *at* q of the generic algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$. From this perspective, the morphism $\mathcal{S}'$ is unnatural, since it mixes a 1-fibre with a q-fibre. To restore the $\mathbf{q}$-compatibility, one must consider the composition of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{S}'$ with the isomorphism

$$\begin{aligned}\mathbb{Q}[V_{\widehat{\mathbf{T}},1}] = \mathbb{Q}[X, Y, z_2^{\pm 1}]/(XY - z_2) \xrightarrow{\sim} \mathbb{Q}[V_{\widehat{\mathbf{T}},q}] &= \mathbb{Q}[X, Y, z_2^{\pm 1}]/(XY - qz_2),\\ X &\mapsto q^{-1}X,\\ Y &\mapsto Y,\\ z_2 &\mapsto z_2.\end{aligned}$$

But then one obtains the formulas

$$\begin{aligned}&\mathcal{H}^{\mathrm{sph}}_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}[V_{\widehat{\mathbf{T}},q}] = \mathbb{Q}[X, Y, z_2^{\pm 1}]/(XY - qz_2),\\ &T_{(1,0)} \mapsto X + Y,\\ &T_{(1,1)} \mapsto z_2.\end{aligned}$$

This composed map is defined over $\mathbb{Z}$, it sends $\mathcal{H}^{\mathrm{sph}}_{\mathbb{Z}}$ onto the subring $\mathbb{Z}[V_{\widehat{\mathbf{T}},q}]^{W_0}$ of W_0-invariants, and its integral model is precisely the specialization $\mathbf{q} = q$ of the isomorphism $\mathscr{S}(\mathbf{q})$ from 6.2.3.

Definition 6.2.5. We call

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\ \sim\ } R(V_{\widehat{\mathbf{T}}})^{W_0}$$

the *generic Satake isomorphism.*

6.2.2. Composing with the inverse of the character isomorphism $\chi^{-1}_{V_{\widehat{\mathbf{G}}}} : R(V_{\widehat{\mathbf{T}}})^{W_0} \xrightarrow{\sim} R(V_{\widehat{\mathbf{G}}})$ from 5.2.3, we arrive at an isomorphism

$$\chi^{-1}_{V_{\widehat{\mathbf{G}}}} \circ \mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\ \sim\ } R(V_{\widehat{\mathbf{G}}}).$$

6.2.3. Next, recall the generic Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$ 4.1.1, and the commutative subring $\mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ 4.1.4 together with the isomorphism $\mathscr{B}_1(\mathbf{q})$ in 6.1.3.

Definition 6.2.6. The *generic central elements morphism* is the unique ring homomorphism

$$\mathscr{Z}_1(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$$

making the diagram

$$\begin{tikzcd} \mathcal{A}_1(\mathbf{q}) \arrow[r, "\mathscr{B}_1(\mathbf{q})", "\sim"'] & R(V_{\widehat{\mathbf{T}}}) \\ \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \arrow[u, "\mathscr{Z}_1(\mathbf{q})"] \arrow[r, "\mathscr{S}(\mathbf{q})", "\sim"'] & R(V_{\widehat{\mathbf{T}}})^{W_0} \arrow[u, hook] \end{tikzcd}$$

commutative.

6.2.4. By construction, the morphism $\mathscr{Z}_1(\mathbf{q})$ is injective, and is uniquely determined by the following equalities in $\mathcal{A}_1(\mathbf{q})$:

$$\mathscr{Z}_1(\mathbf{q})(T_{(1,0)}) = z_1, \quad \mathscr{Z}_1(\mathbf{q})(T_{(1,1)}) = z_2 \quad \text{and} \quad \mathscr{Z}_1(\mathbf{q})(\mathbf{q}) = \mathbf{q}.$$

Moreover the group W_0 acts on the ring $\mathcal{A}_1(\mathbf{q})$ and the invariant subring $\mathcal{A}_1(\mathbf{q})^{W_0}$ is equal to the center $Z(\mathcal{H}_1(\mathbf{q})) \subset \mathcal{H}_1(\mathbf{q})$. As the isomorphism $\mathscr{B}_1(\mathbf{q})$ is W_0-equivariant by construction, we obtain that the image of $\mathscr{Z}_1(\mathbf{q})$ indeed is equal to the *center of the generic Iwahori-Hecke algebra* $\mathcal{H}_1(\mathbf{q})$:

$$\mathscr{Z}_1(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} Z(\mathcal{H}_1(\mathbf{q})) \subset \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q}).$$

6.2.5. Under the identification $R(V_{\widehat{\mathbf{T}}}) = \mathbb{Z}[V_{\widehat{\mathbf{T}}}]$ of 6.1.3, the elements $\mathscr{S}(\mathbf{q})(T_{(1,0)}) = X + Y$, $\mathscr{S}(\mathbf{q})(\mathbf{q}) = \mathbf{q}$, $\mathscr{S}(\mathbf{q})(T_{(1,1)}) = z_2$, correspond to the *Steinberg choice of coordinates* z_1, $\mathbf{q}$, z_2 on the affine $\mathbb{Z}$-scheme $V_{\widehat{\mathbf{T}}}/W_0 = \mathrm{Spec}(\mathbb{Z}[V_{\widehat{\mathbf{T}}}]^{W_0})$. On the other hand, the *Trace of representations morphism* $\mathrm{Tr} : R(V_{\widehat{\mathbf{G}}}) \to \mathbb{Z}[V_{\widehat{\mathbf{G}}}]^{\widehat{\mathbf{G}}}$ fits into the commutative diagram

$$\begin{tikzcd}
R(V_{\widehat{\mathbf{T}}})^{W_0} \arrow[d, equal] & R(V_{\widehat{\mathbf{G}}}) \arrow[l, "\chi_{V_{\widehat{\mathbf{G}}}}"', "\sim"] \arrow[d, "\mathrm{Tr}"] \\
\mathbb{Z}[V_{\widehat{\mathbf{T}}}]^{W_0} & \mathbb{Z}[V_{\widehat{\mathbf{G}}}]^{\widehat{\mathbf{G}}} \arrow[l, "\mathrm{Ch}"', "\sim"]
\end{tikzcd}$$

where $\chi_{V_{\widehat{\mathbf{G}}}}$ is the character isomorphism of 5.2.3, and Ch is the *Chevalley isomorphism* which is constructed for the Vinberg monoid $V_{\widehat{\mathbf{G}}}$ by Bouthier in [Bo15, Prop. 1.7]. So we have the following commutative diagram of $\mathbb{Z}$-schemes

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{A}_1(\mathbf{q})) \arrow[d, two heads, "\mathrm{Spec}(\mathscr{Z}_1(\mathbf{q}))"'] & & V_{\widehat{\mathbf{T}}} \arrow[ll, "\mathrm{Spec}(\mathscr{B}_1(\mathbf{q}))"', "\sim"] \arrow[r, hook] \arrow[d, two heads] & V_{\widehat{\mathbf{G}}} \arrow[d, two heads] \\
\mathrm{Spec}(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) \arrow[dr, "\sim", "(T_{(1,0)},\mathbf{q},T_{(1,1)})"'] & & V_{\widehat{\mathbf{T}}}/W_0 \arrow[ll, "\mathrm{Spec}(\mathscr{S}(\mathbf{q}))"', "\sim"] \arrow[r, "\mathrm{Spec}(\mathrm{Ch})", "\sim"'] \arrow[dl, "\sim"', "(z_1,\mathbf{q},z_2)"] & V_{\widehat{\mathbf{G}}}//\widehat{\mathbf{G}} \\
& \mathbb{A}^2 \times \mathbb{G}_m. & &
\end{tikzcd}$$

Note that for $\widehat{\mathbf{G}} = \mathbf{GL_2}$, the composed *Chevalley-Steinberg map* $V_{\widehat{\mathbf{G}}} \to \mathbb{A}^2 \times \mathbb{G}_m$ is given explicitly by attaching to a 2×2 matrix its characteristic polynomial (when $z_2 = 1$).

6.2.6. We have recalled that for the generic pro-p-Iwahori-Hecke algebra $\mathcal{H}^{(1)}(\mathbf{q})$ too, the center can be described in terms of W_0-invariants, namely $Z(\mathcal{H}^{(1)}(\mathbf{q})) = \mathcal{A}^{(1)}(\mathbf{q})^{W_0}$, cf. 2.3.2. As the generic Bernstein isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ is W_0-equivariant by construction, cf. 6.1.2, we can make Definition 6.2.7.

Definition 6.2.7. We call

$$\mathscr{S}^{(1)}(\mathbf{q}) := \mathscr{B}^{(1)}(\mathbf{q})^{W_0} : \mathcal{A}^{(1)}(\mathbf{q})^{W_0} \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}}^{(1)})^{W_0}$$

the *generic pro-p-Iwahori Satake isomorphism.*

6.2.7. Note that with $V^{\gamma}_{\widehat{\mathbf{T}}} := \coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}}}$ we have $V^{(1)}_{\widehat{\mathbf{T}}} = \mathbb{T}^{\vee}\times V_{\widehat{\mathbf{T}}} = \coprod_{\gamma\in\mathbb{T}^{\vee}/W_0} V^{\gamma}_{\widehat{\mathbf{T}}}$ and the W_0-action on this scheme respects these γ-components. We obtain the decomposition into connected components

$$V^{(1)}_{\widehat{\mathbf{T}}}/W_0 = \coprod_{\gamma\in\mathbb{T}^{\vee}/W_0} \Big(\coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}}}\Big)/W_0 = \coprod_{\gamma\in\mathbb{T}^{\vee}/W_0} V^{\gamma}_{\widehat{\mathbf{T}}}/W_0.$$

If γ is regular, then $V^{\gamma}_{\widehat{\mathbf{T}}}/W_0 \simeq V_{\widehat{\mathbf{T}}}$, the isomorphism depending on a choice of order on the set γ, cf. 2.3.3. Hence, passing to $\tilde{\mathbb{Z}}$ as in 6.1.3, with $\tilde{\mathcal{H}}^{(1)}(\mathbf{q}) := \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$, we obtain the following commutative diagram of $\tilde{\mathbb{Z}}$-schemes.

$$\begin{tikzcd}
\operatorname{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \arrow[d, two heads] & & V^{(1)}_{\widehat{\mathbf{T}}} \arrow[ll, "\operatorname{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q}))"', "\sim"] \arrow[d, two heads] \\
\operatorname{Spec}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \arrow[d, "\wr"] & & V^{(1)}_{\widehat{\mathbf{T}}}/W_0 \arrow[ll, "\operatorname{Spec}(\tilde{\mathscr{S}}^{(1)}(\mathbf{q}))"', "\sim"] \arrow[d, "\wr"', "2.3.3"] \\
(\mathbb{A}^2\times\mathbb{G}_m)^{\mathbb{T}^{\vee}/W_0} & & \coprod_{(\mathbb{T}^{\vee}/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}}} \coprod_{(\mathbb{T}^{\vee}/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}}}/W_0, \arrow[ll, "\sim"]
\end{tikzcd}$$

where the bottom isomorphism of the diagram is given by the standard coordinates (x, y, z_2) on the regular components and by the Steinberg coordinates $(z_1, \mathbf{q}, z_2)$ on the non-regular components.

6.3. **The generic parametrization.** We keep the notation $\mathbb{Z}\subset\tilde{\mathbb{Z}}$ for the ring extension of 2.2.1. Then we have defined the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ in 6.1.3, and we have considered in 6.2.7 its quotient by the natural W_0-action. Also recall that $\widehat{\mathbf{G}} = \mathbf{GL_2}$ is the Langlands dual k-group of $GL_{2,F}$.

Definition 6.3.1. The category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ will be called the *category of Satake parameters*, and denoted by $\mathrm{SP}_{\widehat{\mathbf{G}}}$:

$$\mathrm{SP}_{\widehat{\mathbf{G}}} := \operatorname{QCoh}\big(V^{(1)}_{\widehat{\mathbf{T}}}/W_0\big).$$

For $\gamma\in\mathbb{T}^{\vee}/W_0$, we also define $\mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}}} := \operatorname{QCoh}\big(V^{\gamma}_{\widehat{\mathbf{T}}}/W_0\big)$, where as above $V^{\gamma}_{\widehat{\mathbf{T}}} = \coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}}}$.

6.3.1. Now, over $\tilde{\mathbb{Z}}$, we have the isomorphism

$$i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})} := \operatorname{Spec}(\tilde{\mathscr{S}}^{(1)}(\mathbf{q})) : V^{(1)}_{\widehat{\mathbf{T}}}/W_0 \xrightarrow{\ \sim\ } \operatorname{Spec}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})))$$

from the scheme $V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ to the spectrum of the center $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ of the generic pro-p-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$, cf. 6.2.7.

Corollary 6.3.2. *The category of modules over $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ is equivalent to the category of Satake parameters:*

$$S := (i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})^* : \operatorname{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \mathrm{SP}_{\widehat{\mathbf{G}}} : (i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})_*.$$

The equivalence S will be referred to as the functor of Satake parameters.[5] *The quasi-inverse $(i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})_*$ will be denoted by S^{-1}.*

6.3.2. Still from 6.2.7, these categories decompose as products over $\mathbb{T}^\vee/W_0$ (considered as a finite set), compatibly with the equivalences: for all $\gamma \in \mathbb{T}^\vee/W_0$,

$$S^\gamma := (i_{\tilde{\mathscr{S}}^\gamma(\mathbf{q})})^* : \mathrm{Mod}(Z(\tilde{\mathcal{H}}^\gamma(\mathbf{q}))) \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \mathrm{SP}^\gamma_{\widehat{\mathbf{G}}} : (i_{\tilde{\mathscr{S}}^\gamma(\mathbf{q})})_*,$$

where

$$\mathrm{SP}^\gamma_{\widehat{\mathbf{G}}} \simeq \begin{cases} \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}) & \text{if } \gamma \text{ is regular,} \\ \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}/W_0) & \text{if } \gamma \text{ is non-regular.} \end{cases}$$

In the regular case, the latter isomorphism depends on a choice of order on the set γ.

6.3.3. In particular, we have the trivial orbit $\gamma := \{1\}$. The corresponding component $\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})$ of $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ is canonically isomorphic to the $\tilde{\mathbb{Z}}$-base change of the generic non-regular Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$. Hence from 6.2.4 we have an isomorphism

$$\tilde{\mathscr{Z}}_1(\mathbf{q}) : \tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\ \sim\ } Z(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) \subset \tilde{\mathcal{A}}^{\{1\}}(\mathbf{q}) \subset \tilde{\mathcal{H}}^{\{1\}}(\mathbf{q}) \subset \tilde{\mathcal{H}}^{(1)}(\mathbf{q}).$$

Using these identifications, the equivalence S^γ for $\gamma := \{1\}$ can be rewritten as

$$S^{\{1\}} : \mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})) \xrightarrow{\ \sim\ } \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}}\,.$$

Definition 6.3.3. The category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ will be called the *category of Bernstein parameters*, and denoted by $\mathrm{BP}_{\widehat{\mathbf{G}}}$:

$$\mathrm{BP}_{\widehat{\mathbf{G}}} := \mathrm{QCoh}\left(V^{(1)}_{\widehat{\mathbf{T}}}\right).$$

6.3.4. Over $\tilde{\mathbb{Z}}$, we have the isomorphism

$$i_{\tilde{\mathscr{B}}^{(1)}(\mathbf{q})} := \mathrm{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q})) : V^{(1)}_{\widehat{\mathbf{T}}} \xrightarrow{\ \sim\ } \mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$$

from the scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ to the spectrum of the commutative subring $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ of the generic pro-p-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$, cf. 6.1.3. Also we have the *restriction functor*

$$\mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} : \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \longrightarrow \mathrm{Mod}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \cong \mathrm{QCoh}(\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})))$$

from the category of left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-modules to the one of $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$-modules, equivalently of quasi-coherent modules on $\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$.

Definition 6.3.4. The *functor of Bernstein parameters* is the composed functor

$$B := (i_{\tilde{\mathscr{B}}^{(1)}(\mathbf{q})})^* \circ \mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} : \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \longrightarrow \mathrm{BP}_{\widehat{\mathbf{G}}}\,.$$

[5]We hope that there is only little risk of confusing the notation S with the Hecke operator introduced in 2.1.2.

6.3.5. Still from 6.1.3, the category $\mathrm{BP}_{\widehat{\mathbf{G}}}$ decomposes as a product over the finite group $\mathbb{T}^\vee$:

$$\mathrm{BP}_{\widehat{\mathbf{G}}} \cong \prod_{\lambda\in\mathbb{T}^\vee} \mathrm{BP}^{\lambda}_{\widehat{\mathbf{G}}}, \quad \text{where} \quad \forall\lambda\in\mathbb{T}^\vee,\ \mathrm{BP}^{\lambda}_{\widehat{\mathbf{G}}} \simeq \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}).$$

6.3.6. Denoting by $\pi : V^{(1)}_{\widehat{\mathbf{T}}} \to V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ the canonical projection, the compatibility between the functors S and B of Satake and Bernstein parameters is expressed by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \arrow[r, "B"] \arrow[d, "\mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}"'] & \mathrm{BP}_{\widehat{\mathbf{G}}} \arrow[d, "\pi_*"] \\
\mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \arrow[r, "S", "\sim"'] & \mathrm{SP}_{\widehat{\mathbf{G}}}\,.
\end{tikzcd}$$

Definition 6.3.5. The *generic parametrization functor* is the functor

$$P := S \circ \mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} = \pi_* \circ B :$$

$$\begin{tikzcd}
\mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \arrow[d] \\
\mathrm{SP}_{\widehat{\mathbf{G}}}\,.
\end{tikzcd}$$

6.3.7. It follows from the definitions that for all $\gamma\in\mathbb{T}^\vee/W_0$, the fibre of P over the direct factor $\mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}}} \subset \mathrm{SP}_{\widehat{\mathbf{G}}}$ is the direct factor $\mathrm{Mod}(\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})) \subset \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$:

$$P^{-1}(\mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}}}) = \mathrm{Mod}(\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})) \subset \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$

Accordingly the parametrization functor P decomposes as the product over the finite set $\mathbb{T}^\vee/W_0$ of functors

$$P^{\gamma} : \mathrm{Mod}(\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})) \longrightarrow \mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}}}\,.$$

6.3.8. In the case of the trivial orbit $\gamma := \{1\}$, it follows from 6.3.3 that $P^{\{1\}}$ factors as

$$\begin{tikzcd}
\mathrm{Mod}(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) \arrow[d, "\mathrm{Res}^{\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})}_{\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})}"'] \arrow[dr, "P^{\{1\}}"] & \\
\mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})) \arrow[r, "S^{\{1\}}", "\sim"'] & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}}\,.
\end{tikzcd}$$

6.4. **The generic spherical module.** Recall the generic regular and non-regular spherical representations $\mathscr{A}_2(\mathbf{q})$ 3.3.1 and $\mathscr{A}_1(\mathbf{q})$ 4.3.1 of $\mathcal{H}_2(\mathbf{q})$ and $\mathcal{H}_1(\mathbf{q})$. Thanks to 3.1.3 and 4.1.3, they are models over $\mathbb{Z}$ of representations $\tilde{\mathscr{A}}^{\gamma}(\mathbf{q})$ of the regular and non-regular components $\tilde{\mathscr{A}}^{\gamma}(\mathbf{q})$, $\gamma\in\mathbb{T}^\vee/W_0$, of the generic pro-$p$-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ over $\tilde{\mathbb{Z}}$, cf. 2.2.2 and 2.3.1. Taking the product over $\mathbb{T}^\vee/W_0$ of these representations, we obtain a representation

$$\tilde{\mathscr{A}}^{(1)}(\mathbf{q}) : \tilde{\mathcal{H}}^{(1)}(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})).$$

By construction, the representation $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ depends on a choice of order on each regular orbit γ.

Definition 6.4.1. We call $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ the *generic spherical representation*, and the corresponding left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-module $\tilde{\mathcal{M}}^{(1)}$ the *generic spherical module.*

Proposition 6.4.2.

(1) *The generic spherical representation is faithful.*
(2) *The Bernstein parameter of the spherical module is the structural sheaf:*

$$B(\mathcal{M}^{(1)}) = \mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}}}}.$$

(3) *The Satake parameter of the spherical module is the $\tilde{R}(V^{(1)}_{\widehat{\mathbf{G}}})$-module of $V^{(1)}_{\widehat{\mathbf{G}}}$-equivariant K-theory of the flag variety of $V^{(1)}_{\widehat{\mathbf{G}}}$:*

$$\tilde{c}_{V^{(1)}_{\widehat{\mathbf{G}}}} : S(\mathcal{M}^{(1)}) \xrightarrow{\sim} \tilde{K}^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

Proof. Part (1) follows from 3.3.3 and 4.3.2, part (2) from the property (i) in 3.3.1 and 4.3.1, and part (3) from the characteristic isomorphism in 6.1.2. $\square$

6.4.1. Now, being a left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-module, the spherical module $\tilde{\mathcal{M}}^{(1)}$ defines a functor

$$\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet : \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \longrightarrow \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$

On the other hand, recall the canonical projection $\pi : V^{(1)}_{\widehat{\mathbf{T}}} \to V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ from 6.3.6. Then point (2) of 6.4.2 has the following consequence.

Corollary 6.4.3. *The diagram*

$$\begin{tikzcd} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \arrow[r, "B"] & \mathrm{BP}_{\widehat{\mathbf{G}}} \\ \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \arrow[u, "\tilde{\mathcal{M}}^{(1)}\otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}\bullet"] \arrow[r, "S", "\sim"'] & \mathrm{SP}_{\widehat{\mathbf{G}}} \arrow[u, "\pi^*"'] \end{tikzcd}$$

is commutative.

Definition 6.4.4. The *generic spherical functor* is the functor

$$\mathrm{Sph} := (\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet) \circ S^{-1} :$$

$$\mathrm{SP}_{\widehat{\mathbf{G}}} \longrightarrow \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$

Corollary 6.4.5. *The diagram*

$$\begin{tikzcd} & & \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \arrow[d, "P"] \\ \mathrm{SP}_{\widehat{\mathbf{G}}} \arrow[urr, "\mathrm{Sph}"] \arrow[r, "\pi^*"'] & \mathrm{BP}_{\widehat{\mathbf{G}}} \arrow[r, "\pi_*"'] & \mathrm{SP}_{\widehat{\mathbf{G}}} \end{tikzcd}$$

is commutative.

Proof. One has $P \circ \mathrm{Sph} = \pi_* \circ (B \circ \mathrm{Sph}) = \pi_* \circ \pi^*$ by the preceding corollary. $\square$

6.4.2. By construction, the spherical functor Sph decomposes as a product of functors Sph^γ for $\gamma \in \mathbb{T}^\vee/W_0$, and accordingly the previous diagram decomposes over $\mathbb{T}^\vee/W_0$.

6.4.3. In particular for $\gamma = \{1\}$ we have the commutative diagram

$$\begin{tikzcd} & & \mathrm{Mod}(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) \arrow[d, "P^{\{1\}}"] \arrow[dr, "\mathrm{Res}^{\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})}_{\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})}"] & \\ \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}} \arrow[urr, "\mathrm{Sph}^{\{1\}}"] \arrow[r, "\pi^*"'] & \mathrm{BP}^{\{1\}}_{\widehat{\mathbf{G}}} \arrow[r, "\pi_*"'] & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}} & \mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})). \arrow[l, "\sim", "S^{\{1\}}"'] \end{tikzcd}$$

7. The theory at $\mathbf{q} = q = 0$

We keep all the notations introduced in the preceding section. In particular, $k = \overline{\mathbb{F}}_q$.

7.1. K-theory of the dual flag variety at $\mathbf{q} = 0$.

7.1.1. Recall from 5.1 the k-semigroup scheme

$$V_{\mathbf{GL_2},0} = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m,$$

which can even be defined over $\mathbb{Z}$, and which is obtained as the 0-fibre of

$$\begin{tikzcd} V_{\mathbf{GL_2}} \arrow[d, "\mathbf{q}"] \\ \mathbb{A}^1. \end{tikzcd}$$

7.1.2. It admits

$$V_{\widehat{\mathbf{T}},0} = \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

as a commutative subsemigroup scheme. The latter has the following structure: it is the pinching of the monoids

$$\mathbb{A}^1_X \times \mathbb{G}_m := \mathrm{Spec}(k[X, z_2^{\pm 1}]) \quad \text{and} \quad \mathbb{A}^1_Y \times \mathbb{G}_m := \mathrm{Spec}(k[Y, z_2^{\pm 1}])$$

along the sections $X = 0$ and $Y = 0$. The categories of representations of these monoids on finite dimensional k-vector spaces are semisimple, with corresponding representation rings

$$R(\mathbb{A}^1_X \times \mathbb{G}_m) = \mathbb{Z}[X, z_2^{\pm 1}] \quad \text{and} \quad R(\mathbb{A}^1_Y \times \mathbb{G}_m) = \mathbb{Z}[Y, z_2^{\pm 1}].$$

There are three remarkable elements in $V_{\widehat{\mathbf{T}},0}$, namely

$$\varepsilon_X := (\mathrm{diag}(1,0), 1), \quad \varepsilon_Y := (\mathrm{diag}(0,1), 1) \quad \text{and} \quad \varepsilon_0 := (\mathrm{diag}(0,0), 1).$$

They are idempotents. Now let M be a finite dimensional k-representation of $V_{\widehat{\mathbf{T}},0}$. The idempotents act on M as projectors, and as the semigroup $V_{\widehat{\mathbf{T}},0}$ is commutative, the k-vector space M decomposes as a direct sum

$$M = \bigoplus_{(\lambda_X,\lambda_Y,\lambda_0)\in\{0,1\}^3} M(\lambda_X, \lambda_Y, \lambda_0),$$

where

$$M(\lambda_X, \lambda_Y, \lambda_0) = \{m \in M \mid m\varepsilon_X = \lambda_X m,\ m\varepsilon_Y = \lambda_Y m,\ m\varepsilon_0 = \lambda_0 m\}.$$

Moreover, since $V_{\widehat{\mathbf{T}},0}$ is commutative, each of these subspaces is in fact a subrepresentation of M.

As $\varepsilon_X\varepsilon_Y = \varepsilon_0 \in V_{\widehat{\mathbf{T}},0}$, we have $M(\lambda_X,\lambda_Y,\lambda_0) \neq 0 \implies \lambda_X\lambda_Y = \lambda_0$. Consequently

$$M = M(1,0,0) \bigoplus M(0,1,0) \bigoplus M(1,1,1) \bigoplus M(0,0,0).$$

The restriction $\mathrm{Res}^{V_{\widehat{\mathbf{T}},0}}_{\mathbb{A}^1_X} M(1,0,0)$ is a representation of the monoid $\mathbb{A}^1_X$ where 0 acts by 0, and $\mathrm{Res}^{V_{\widehat{\mathbf{T}},0}}_{\mathbb{A}^1_Y} M(1,0,0)$ is the null representation. Hence, if for $n>0$ we still denote by X^n the character of $V_{\widehat{\mathbf{T}},0}$ which restricts to the character X^n of $\mathbb{A}^1_X \times \mathbb{G}_m$ and the null map of $\mathbb{A}^1_Y \times \mathbb{G}_m$, then $M(1,0,0)$ decomposes as a sum of weight spaces

$$M(1,0,0) = \oplus_{n>0} M(X^n) := \oplus_{n>0,m\in\mathbb{Z}} M(X^n z_2^m).$$

Similarly

$$M(0,1,0) = \oplus_{n>0} M(Y^n) := \oplus_{n>0,m\in\mathbb{Z}} M(Y^n z_2^m).$$

Finally, $V_{\widehat{\mathbf{T}},0}$ acts through the projection $V_{\widehat{\mathbf{T}},0} \to \mathbb{G}_m$ on

$$M(1,1,1) =: M(1) = \oplus_{m\in\mathbb{Z}} M(z_2^m),$$

and by 0 on

$$M(0,0,0) =: M(0).$$

Thus we have obtained the following

Lemma 7.1.1. *The category* $\mathrm{Rep}(V_{\widehat{\mathbf{T}},0})$ *is semisimple, and there is a ring isomorphism*

$$R(V_{\widehat{\mathbf{T}},0}) \cong \big(\mathbb{Z}[X,Y,z_2^{\pm 1}]/(XY)\big) \times \mathbb{Z}.$$

7.1.3. Next let

$$V_{\widehat{\mathbf{B}},0} = \mathrm{SingUpTriang}_{2\times 2} \times \mathbb{G}_m \subset V_{\mathbf{GL_2},0} = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m$$

be the subsemigroup scheme of singular upper triangular 2×2-matrices. It contains $V_{\widehat{\mathbf{T}},0}$, and the inclusion $V_{\widehat{\mathbf{T}},0} \subset V_{\widehat{\mathbf{B}},0}$ admits a retraction $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$, namely the specialization at $\mathbf{q}=0$ of the retraction 5.2.2.

Let M be an object of $\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})$. Write

$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} M = M(1,0,0) \oplus M(0,1,0) \oplus M(1) \oplus M(0).$$

For a subspace $N \subset M$, consider the following property:

(P_N) *the subspace $N \subset M$ is a subrepresentation, and $V_{\widehat{\mathbf{B}},0}$ acts on N through the retraction of k-semigroup schemes $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$.*

Let us show that $(\mathrm{P}_{M(0,1,0)})$ is true. Indeed for $m \in M(0,1,0) = \oplus_{n>0} M(Y^n)$, we have

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = (m\varepsilon_Y)\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m\varepsilon_0 = 0 = m\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = (m\varepsilon_Y)\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

Next assume $M(0,1,0) = 0$, and let us show that in this case $(\mathrm{P}_{M(0)})$ is true. Indeed for $m \in M(0)$, we have

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m\Big(\varepsilon_X \begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\Big) = (m\varepsilon_X)\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = 0,$$

and if we decompose

$$m' := m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m'_{(1,0,0)} + m'_1 + m'_0 \in M(1,0,0) \oplus M(1) \oplus M(0),$$

then by applying ε_X on the right we see that $0 = m'_{(1,0,0)} + m'_1$ so that $m' \in M(0)$ and hence

$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\Big(\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\varepsilon_Y\Big) = m'\varepsilon_Y = 0.$$

Next assume $M(0,1,0) = M(0) = 0$, and let us show that in this case $(\mathrm{P}_{M(1,0,0)})$ is true. Indeed, let $m \in M(1,0,0) = \oplus_{n>0} M(X^n)$. Then for any $c \in k$,

$$m' := m\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

satisfies $m'\varepsilon_X = 0$, $m'\varepsilon_Y = m'$, $m'\varepsilon_0 = 0$, i.e. $m' \in M(0,1,0)$, and hence is equal to 0 by our assumption. It follows that

$$\begin{aligned} m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = (m\varepsilon_X)\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\Big(\varepsilon_X \begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\Big) &= m\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \\ &= 0 = m\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}. \end{aligned}$$

On the other hand, if we decompose

$$m' := m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m'_{(1,0,0)} + m'_1 \in M(1,0,0) \oplus M(1),$$

then by applying ε_0 on the right we find $0 = m'_1$, i.e. $m' \in M(1,0,0)$ and hence

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m' = m'\varepsilon_X = m\Big(\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\varepsilon_X\Big) = m\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally assume $M(0,1,0) = M(0) = M(1,0,0) = 0$, and let us show that in this case $(\mathrm{P}_{M(1)})$ is true, i.e. that $V_{\widehat{\mathbf{B}},0}$ acts through the projection $V_{\widehat{\mathbf{B}},0} \to \mathbb{G}_m$ on $M = M(1)$. Indeed for any m we have

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = \Big(m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\Big)\varepsilon_0 = m\Big(\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\varepsilon_0\Big) = m\varepsilon_0 = m$$

and

$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = \Big(m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\Big)\varepsilon_0 = m\Big(\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\varepsilon_0\Big) = m\varepsilon_0 = m.$$

It follows from the preceding discussion that the irreducible representations of $V_{\widehat{\mathbf{B}},0}$ are the characters, which are inflated from those of $V_{\widehat{\mathbf{T}},0}$ through the retraction $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$. As a consequence, considering the *restriction* and *inflation* functors

$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \rightleftarrows \mathrm{Rep}(V_{\widehat{\mathbf{T}},0}) : \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}},$$

which are exact and compatible with tensor products and units, we get:

Lemma 7.1.2. *The ring homomorphisms*

$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : R(V_{\widehat{\mathbf{B}},0}) \rightleftarrows R(V_{\widehat{\mathbf{T}},0}) : \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}}$$

are isomorphisms, which are inverse one to the other.

7.1.4. Finally, note that $\varepsilon_0 \in V_{\mathbf{GL_2}}(k)$ belongs to all the left $V_{\mathbf{GL_2}}(k)$-cosets in $V_{\mathbf{GL_2}}(k)$. Hence, by [PS20, 2.4.3], the category $\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})$ is equivalent to the one of induced vector bundles on the semigroupoid flag variety $V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}$:

$$\mathcal{I}nd^{V_{\mathbf{GL_2},0}}_{V_{\widehat{\mathbf{B}},0}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \xrightarrow{\sim} \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}) \subset \mathcal{C}^{V_{\mathbf{GL_2},0}}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

Corollary 7.1.3. *We have a ring isomorphism*

$$\mathcal{I}nd^{V_{\mathbf{GL_2},0}}_{V_{\widehat{\mathbf{B}},0}} \circ \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : R(V_{\widehat{\mathbf{T}},0}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

Definition 7.1.4. We call *relevant* the full subcategory

$$\mathrm{Rep}(V_{\widehat{\mathbf{T}},0})^{\mathrm{rel}} \subset \mathrm{Rep}(V_{\widehat{\mathbf{T}},0})$$

whose objects M satisfy $M(0) = 0$. Correspondingly, we have relevant full subcategories

$$\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}} \subset \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \quad \text{and} \quad \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}} \subset \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

Corollary 7.1.5. *We have a ring isomorphism*

$$c_{V_{\mathbf{GL_2},0}} := \mathbb{Z}[X, Y, z_2^{\pm 1}]/(XY) \cong R(V_{\widehat{\mathbf{T}},0})^{\mathrm{rel}} \xrightarrow{\sim} K^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}$$

that we call the characteristic isomorphism in the equivariant K-theory of the flag variety $V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}$.

7.1.5. We have a commutative diagram *specialization at* $\mathbf{q} = 0$

$$\begin{tikzcd}
\mathbb{Z}[X, Y, z_2^{\pm 1}] \arrow[r, "c_{V_{\mathbf{GL_2}}}", "\sim"'] \arrow[d, two heads] & K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}) \arrow[d, two heads] \\
\mathbb{Z}[X, Y, z_2^{\pm 1}]/(XY) \arrow[r, "c_{V_{\mathbf{GL_2},0}}", "\sim"'] & K^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}},
\end{tikzcd}$$

where the vertical right-hand side map is given by restricting equivariant vector bundles to the 0-fibre of $\mathbf{q} : V_{\mathbf{GL_2}} \to \mathbb{A}^1$.

7.2. The mod p Satake and Bernstein isomorphisms.

Notation 7.2.1. In the sequel, we will denote by $(\bullet)_{\overline{\mathbb{F}}_q}$ the *specialization at* $\mathbf{q} = q = 0$, i.e. the base change functor along the ring morphism

$$\begin{aligned} \mathbb{Z}[\mathbf{q}] &\longrightarrow \overline{\mathbb{F}}_q =: k, \\ \mathbf{q} &\longmapsto 0. \end{aligned}$$

Also we fix an embedding $\mu_{q-1} \subset \overline{\mathbb{F}}_q^{\times}$, so that the above morphism factors through the inclusion $\mathbb{Z}[\mathbf{q}] \subset \tilde{\mathbb{Z}}[\mathbf{q}]$, where $\mathbb{Z} \subset \tilde{\mathbb{Z}}$ is the ring extension considered in 2.2.1.

7.2.1. **The mod p Satake and pro-p-Iwahori Satake isomorphisms.** Specializing 6.2.5, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{S}_{\overline{\mathbb{F}}_q} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}]^{W_0} = \big(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\big)^{W_0}.$$

In [H11], Herzig constructed an isomorphism

$$\mathscr{S}_{\mathrm{Her}} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[\mathbb{X}^\bullet(\widehat{\mathbf{T}})_-] = \overline{\mathbb{F}}_q[e^{(0,1)}, e^{\pm(1,1)}]$$

(this is $\overline{\mathbb{F}}_q \otimes_{\mathbb{Z}} \mathcal{S}'$, with the notation $\mathcal{S}'$ from 6.2.4). They are related by the Steinberg choice of coordinates $z_1 := X+Y$ and z_2 on the quotient $V_{\widehat{\mathbf{T}},0}/W_0$, cf. 6.2.5, i.e. by the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \arrow[rr, "\mathscr{S}_{\overline{\mathbb{F}}_q}", "\sim"'] \arrow[dr, "\sim", "\mathscr{S}_{\mathrm{Her}}"'] & & \big(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\big)^{W_0} \\
& \overline{\mathbb{F}}_q[e^{(0,1)}, e^{\pm(1,1)}]. \arrow[ur, "\sim", "e^{(0,1)}\mapsto z_1,\ e^{(1,1)}\mapsto z_2"'] &
\end{tikzcd}$$

Specializing 6.2.7 and using $R(\mathbb{T}^\vee) = \mathbb{Z}[\mathbb{T}]$, cf. 6.1.1, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_q} : (\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q})^{W_0} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V^{(1)}_{\widehat{\mathbf{T}},0}]^{W_0} = \big(\overline{\mathbb{F}}_q[\mathbb{T}][X,Y,z_2^{\pm 1}]/(XY)\big)^{W_0}.$$

7.2.2. **The mod p Bernstein isomorphism.** Specializing 6.1.2, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V^{(1)}_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[\mathbb{T}][X,Y,z_2^{\pm 1}]/(XY).$$

Moreover, similarly as in 6.1.2 but here using 7.1.2 and [PS20, 2.5.1], we get the *characteristic isomorphism*

$$c_{V^{(1)}_{\widehat{\mathbf{G}},0}} : R(V^{(1)}_{\widehat{\mathbf{T}},0}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0}).$$

Whence by 7.1.1 (and recalling 7.1.4) an isomorphism

$$c^{\mathrm{rel}}_{V^{(1)}_{\widehat{\mathbf{G}},0},\overline{\mathbb{F}}_q} \circ \mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd,\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}.$$

Also, specializing 6.1.3, $\mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q}$ splits as a product over $\mathbb{T}^\vee$ of $\overline{\mathbb{F}}_q$-algebras isomorphisms $\mathscr{B}^{\lambda}_{\overline{\mathbb{F}}_q}$, each of them being of the form

$$\mathscr{B}_{1,\overline{\mathbb{F}}_q} : \mathcal{A}_{1,\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY).$$

7.2.3. **The mod p central elements embedding.** Specializing 6.2.6, we get an embedding of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{Z}_{1,\overline{\mathbb{F}}_q} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}) \subset \mathcal{A}_{1,\overline{\mathbb{F}}_q} \subset \mathcal{H}_{1,\overline{\mathbb{F}}_q}$$

making the diagram

$$\begin{tikzcd}
\mathcal{A}_{1,\overline{\mathbb{F}}_q} \arrow[r, "\mathscr{B}_{1,\overline{\mathbb{F}}_q}", "\sim"'] & \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY) \\
\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \arrow[u, hook, "\mathscr{Z}_{1,\overline{\mathbb{F}}_q}"] \arrow[r, "\mathscr{S}_{\overline{\mathbb{F}}_q}", "\sim"'] & \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}]^{W_0} = \big(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\big)^{W_0} \arrow[u, hook]
\end{tikzcd}$$

commutative. Then $\mathscr{Z}_{1,\overline{\mathbb{F}}_q}$ coincides with the central elements construction of Ollivier [O14, Th. 4.3] for the case of $\mathbf{GL_2}$. This follows from the explicit formulas for the values of $\mathscr{Z}_1(\mathbf{q})$ on $T_{(1,0)}$ and $T_{(1,1)}$, cf. 6.2.4.

7.3. **The mod p parametrization.**

Definition 7.3.1. The category of quasi-coherent modules on the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ will be called the *category of mod p Satake parameters*, and denoted by $\mathrm{SP}_{\widehat{\mathbf{G}},0}$:

$$\mathrm{SP}_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\left(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0\right).$$

For $\gamma \in \mathbb{T}^\vee/W_0$, we also define $\mathrm{SP}^\gamma_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\left(V^\gamma_{\widehat{\mathbf{T}},0}/W_0\right)$, where $V^\gamma_{\widehat{\mathbf{T}},0} = \coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}},0}$.

7.3.1. Similarly to the generic case 6.3, the mod p pro-p-Iwahori Satake isomorphism induces an equivalence of categories

$$S : \mathrm{Mod}(Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})) \xrightarrow{\sim} \mathrm{SP}_{\widehat{\mathbf{G}},0}$$

that will be referred to as the *functor of mod p Satake parameters*, and which decomposes as a product over the finite set $\mathbb{T}^\vee/W_0$:

$$\begin{aligned} S = \prod_\gamma S^\gamma : \prod_\gamma \mathrm{Mod}(Z(\mathcal{H}^\gamma_{\overline{\mathbb{F}}_q})) &\xrightarrow{\sim} \prod_\gamma \mathrm{SP}^\gamma_{\widehat{\mathbf{G}},0} \\ &\simeq \prod_{\gamma\ \mathrm{reg}} \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}) \prod_{\gamma\ \text{non-reg}} \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}/W_0). \end{aligned}$$

For $\gamma = \{1\}$ and using 7.2.3 we get an equivalence

$$S^{\{1\}} : \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} = \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}/W_0).$$

Note that under this equivalence, the characters $\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \to \overline{\mathbb{F}}_q$ correspond to the skyscraper sheaves on $V_{\widehat{\mathbf{T}},0}/W_0$, and hence to its k-points. Choosing the Steinberg coordinates (z_1, z_2) on the k-scheme $V_{\widehat{\mathbf{T}},0}/W_0$, they may also be regarded as the k-points of $\mathrm{Spec}(k[\mathbb{X}^\bullet(\widehat{\mathbf{T}})_-])$, which are precisely the mod p Satake parameters defined by Herzig in [H11].

Definition 7.3.2. The category of quasi-coherent modules on the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}$ will be called the *category of mod p Bernstein parameters*, and denoted by $\mathrm{BP}_{\widehat{\mathbf{G}},0}$:

$$\mathrm{BP}_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\left(V^{(1)}_{\widehat{\mathbf{T}},0}\right).$$

7.3.2. Similarly to the generic case 6.3, the inclusion $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} \supset \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q}$ together with the mod p Bernstein isomorphism defines a *functor of mod p Bernstein parameters*

$$B : \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \longrightarrow \mathrm{BP}_{\widehat{\mathbf{G}},0}\,.$$

Moreover the category $\mathrm{BP}_{\widehat{\mathbf{G}},0}$ decomposes as a product over the finite group $\mathbb{T}^\vee$:

$$\mathrm{BP}_{\widehat{\mathbf{G}},0} = \prod_\lambda \mathrm{BP}^\lambda_{\widehat{\mathbf{G}},0} = \prod_\lambda \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}).$$

Notation 7.3.3. Let $\pi : V^{(1)}_{\widehat{\mathbf{T}},0} \to V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ be the canonical projection.

Definition 7.3.4. The *mod p parametrization functor* is the functor

$$P := S \circ \mathrm{Res}^{\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}}_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})} = \pi_* \circ B :$$

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \arrow[d] \\ \mathrm{SP}_{\widehat{\mathbf{G}},0}\,. \end{tikzcd}$$

7.3.3. The functor P decomposes as a product over the finite set $\mathbb{T}^\vee/W_0$:

$$P = \textstyle\prod_\gamma P^\gamma : \prod_\gamma \mathrm{Mod}(\mathcal{H}^\gamma_{\overline{\mathbb{F}}_q}) \xrightarrow{\ \sim\ } \prod_\gamma \mathrm{SP}^\gamma_{\widehat{\mathbf{G}},0}\,.$$

In the case of the trivial orbit $\gamma := \{1\}$, $P^{\{1\}}$ factors as

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}) \arrow[d, "\mathrm{Res}^{\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}}_{\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}}"'] \arrow[dr, "P^{\{1\}}"] & \\ \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}) \arrow[r, "S^{\{1\}}", "\sim"'] & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0}\,. \end{tikzcd}$$

7.4. The mod p spherical module.

Definition 7.4.1. We call

$$\mathscr{A}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{End}_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})}(\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q})$$

the *mod p spherical representation*, and the corresponding left $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}$ the *mod p spherical module.*

Proposition 7.4.2.

(1) *The mod p spherical representation is faithful.*

(2) *The mod p Bernstein parameter of the spherical module is the structure sheaf:*

$$B(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}) = \mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}},0}}.$$

(3) *The mod p Satake parameter of the spherical module is the $R_{\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{T}},0})^{\mathrm{rel},W_0}$-module of the relevant induced $V^{(1)}_{\widehat{\mathbf{G}},0}$-equivariant $K_{\overline{\mathbb{F}}_q}$-theory of the flag variety of $V^{(1)}_{\widehat{\mathbf{G}},0}$:*

$$c^{\mathrm{rel}}_{V^{(1)}_{\widehat{\mathbf{G}},0},\overline{\mathbb{F}}_q} : S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd,\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}.$$

Proof. Part (1) follows from 3.3.3 and 4.3.4, part (2) from the property (i) in 3.3.1 and 4.3.1, and part (3) from the characteristic isomorphism in 7.2.2. $\square$

Corollary 7.4.3. *The diagram*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \arrow[r, "B"] & \mathrm{BP}_{\widehat{\mathbf{G}},0} \\
\mathrm{Mod}(Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})) \arrow[u, "\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}\otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})}\bullet"] \arrow[r, "S", "\sim"'] & \mathrm{SP}_{\widehat{\mathbf{G}},0} \arrow[u, "\pi^*"']
\end{tikzcd}$$

is commutative.

Definition 7.4.4. The *mod p spherical functor* is the functor

$$\mathrm{Sph} := (\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})} \bullet) \circ S^{-1} :$$

$$\mathrm{SP}_{\widehat{\mathbf{G}},0} \longrightarrow \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}).$$

Corollary 7.4.5. *The diagram*

$$\begin{tikzcd}
 & & \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \arrow[d, "P"] \\
\mathrm{SP}_{\widehat{\mathbf{G}},0} \arrow[urr, "\mathrm{Sph}"] \arrow[r, "\pi^*"'] & \mathrm{BP}_{\widehat{\mathbf{G}},0} \arrow[r, "\pi_*"'] & \mathrm{SP}_{\widehat{\mathbf{G}},0}
\end{tikzcd}$$

is commutative.

7.4.1. The spherical functor Sph decomposes as a product of functors Sph^γ for $\gamma \in \mathbb{T}^\vee/W_0$, and accordingly the previous diagram decomposes over $\mathbb{T}^\vee/W_0$. In particular for $\gamma = \{1\}$ we have the commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Mod}(\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}) \arrow[d, "P^{\{1\}}"] \arrow[dr, "\mathrm{Res}^{\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}}_{\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}}"] & \\
\mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} \arrow[urr, "\mathrm{Sph}^{\{1\}}"] \arrow[r, "\pi^*"'] & \mathrm{BP}^{\{1\}}_{\widehat{\mathbf{G}},0} \arrow[r, "\pi_*"'] & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} & \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}). \arrow[l, "\sim", "S^{\{1\}}"']
\end{tikzcd}$$

7.4.2. Now, identifying the k-points of the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ with the skyscraper sheaves on it, the spherical functor Sph induces a map

$$\mathrm{Sph} : \big(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}.$$

Considering the decomposition of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ into its connected components, cf. 6.2.7,

$$V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)} V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0 \simeq \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}},0} \quad \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0,$$

the spherical map decomposes as a disjoint union of maps

$$\mathrm{Sph}^{\gamma} : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \simeq V_{\widehat{\mathbf{T}},0}(k) \longrightarrow \{\text{left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\text{-modules}\} \quad \text{for } \gamma \text{ regular},$$

$$\mathrm{Sph}^{\gamma} : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \simeq (V_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\text{-modules}\} \quad \text{for } \gamma \text{ non-regular}.$$

7.4.3. In the regular case, we make the standard choice of coordinates

$$V_{\widehat{\mathbf{T}},0}(k) = \left(\{(x,0) \mid x\in k\} \coprod_{(0,0)} \{(0,y) \mid y \in k\}\right) \times \{z_2 \in k^\times\}$$

and we identify $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}$ with $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$ using 3.1.3. A point $v \in V_{\widehat{\mathbf{T}},0}(k)$ corresponds by 3.1.4 to a character

$$\theta_v : Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q}) \simeq \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY) \longrightarrow \overline{\mathbb{F}}_q,$$

and then $\mathrm{Sph}^{\gamma}(v)$ identifies with the central reduction

$$\mathcal{A}_{2,\theta_v} := \mathcal{A}_{2,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q$$

of the mod p regular spherical representation $\mathscr{A}_{2,\overline{\mathbb{F}}_q}$ specializing 3.3.1. The latter being an isomorphism by 3.3.3, so is

$$\mathscr{A}_{2,\theta_v} : \mathcal{H}_{2,\theta_v} \xrightarrow{\ \sim\ } \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{2,\theta_v}).$$

Consequently $\mathcal{H}_{2,\theta_v}$ is a matrix algebra and $\mathcal{A}_{2,\theta_v}$ is the unique simple finite dimensional left $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module with central character θ_v, up to isomorphism. It is the *standard module with character* θ_v, with *standard basis* $\{\varepsilon_1, \varepsilon_2\}$ (in particular its $\overline{\mathbb{F}}_q$-dimension is 2). Conversely, any simple finite dimensional $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module has a central character, by Schur's lemma.

Following [V04], a central character θ is called *supersingular* if $\theta(X+Y)=0$, and the standard module with character θ is called *supersingular* if θ is. Since $XY=0$, one has $\theta(X+Y)=0$ if and only if $\theta(X)=\theta(Y)=0$.

Theorem 7.4.6. *Let $\gamma \in \mathbb{T}^\vee/W_0$ regular. Then the spherical map induces a bijection*

$$\mathrm{Sph}^{\gamma} : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \xrightarrow{\ \sim\ } \{\textit{simple finite dimensional left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim .$$

The singular locus of the parametrizing k-scheme $V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$ is given by $(0,0)\times \mathbb{G}_m \subset V_{\widehat{\mathbf{T}},0}$ in the standard coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence Sph^{γ}.

7.4.4. In the non-regular case, we make the Steinberg choice of coordinates

$$(V_{\widehat{\mathbf{T}},0}/W_0)(k) = \{z_1 \in k\} \times \{z_2 \in k^\times\}$$

and we identify $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_q}$ with $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$ using 4.1.3. A point $v \in (V_{\widehat{\mathbf{T}},0}/W_0)(k)$ corresponds to a character

$$\theta_v : Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}) \simeq \overline{\mathbb{F}}_q[z_1, z_2^{\pm 1}] \longrightarrow \overline{\mathbb{F}}_q,$$

and then $\mathrm{Sph}^\gamma(v)$ identifies with the central reduction

$$\mathcal{A}_{1,\theta_v} := \mathcal{A}_{1,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q$$

of the mod p non-regular spherical representation $\mathscr{A}_{1,\overline{\mathbb{F}}_q}$ specializing 4.3.1.

Now recall from [V04, 1.4] the classification of the simple finite dimensional $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-modules: they are the characters and the simple standard modules. The characters

$$\mathcal{H}_{1,\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[S, U^{\pm 1}] \longrightarrow \overline{\mathbb{F}}_q^\times$$

are parametrized by the set $\{0, -1\} \times \overline{\mathbb{F}}_q^\times$ via evaluation on the elements S and U. On the other hand, given $v = (z_1, z_2) \in k \times k^\times = \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q^\times$, a *standard module with character* θ_v over $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$ is defined to be a module of type

$$M_2(z_1, z_2) := \overline{\mathbb{F}}_q m \oplus \overline{\mathbb{F}}_q Um, \quad Sm = -m, \quad SUm = z_1 m, \quad U^2 m = z_2 m$$

(in particular its $\overline{\mathbb{F}}_q$-dimension is 2). The center $Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q})$ acts on $M_2(z_1, z_2)$ by the character θ_v. In particular such a module is uniquely determined by its central character. It is simple if and only if $z_2 \neq z_1^2$. It is called *supersingular* if $z_1 = 0$.

Lemma 7.4.7. *Set*

$$\mathscr{A}_{1,\theta_v} := \mathscr{A}_{1,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q : \mathcal{H}_{1,\theta_v} \longrightarrow \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{1,\theta_v}).$$

- *Assume $z_2 \neq z_1^2$. Then $\mathscr{A}_{1,\theta_v}$ is an isomorphism, and the $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-module $\mathcal{A}_{1,\theta_v}$ is isomorphic to the simple standard module $M_2(z_1, z_2)$.*
- *Assume $z_2 = z_1^2$. Then $\mathscr{A}_{1,\theta_v}$ has a 1-dimensional kernel, and the $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-module $\mathcal{A}_{1,\theta_v}$ is a non-split extension of the character $(0, z_1)$ by the character $(-1, -z_1)$.*

Proof. The proof of Proposition 4.3.2 shows that $\mathcal{H}_{1,\theta_v}$ has an $\overline{\mathbb{F}}_q$-basis given by the elements $1, S, U, SU$, and that their images

$$1,\ \mathscr{A}_{1,\theta_v}(S),\ \mathscr{A}_{1,\theta_v}(U),\ \mathscr{A}_{1,\theta_v}(S)\mathscr{A}_{1,\theta_v}(U)$$

by $\mathscr{A}_{1,\theta_v}$ are linearly independent over $\overline{\mathbb{F}}_q$ if and only if $z_1^2 - z_2 \neq 0$.

If $z_2 \neq z_1^2$, then $\mathscr{A}_{1,\theta_v}$ is injective, and hence bijective since $\dim_{\overline{\mathbb{F}}_q} \mathcal{A}_{1,\theta_v} = 2$ from 4.2.1. Moreover $S \cdot Y = -Y$ and $U \cdot Y = (z_1^2 - z_2) - z_1 Y$ and so $SUY = S((z_1^2 - z_2) - z_1 Y) = S(-z_1 Y) = z_1 Y$, so that

$$\mathcal{A}_{1,\theta_v} = \overline{\mathbb{F}}_q Y \oplus \overline{\mathbb{F}}_q U \cdot Y = M_2(z_1, z_2).$$

If $z_2 = z_1^2$, then the proof of Proposition 4.3.2 shows that $\mathscr{A}_{1,\theta_v}$ has a 1-dimensional kernel which is the $\overline{\mathbb{F}}_q$-line generated by $-z_1(1 + S) + U + SU$. Moreover $\overline{\mathbb{F}}_q Y \subset \mathcal{A}_{1,\theta_v}$ realizes the character $(-1, -z_1)$ of $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$, and $\mathcal{A}_{1,\theta_v}/\overline{\mathbb{F}}_q Y \simeq \overline{\mathbb{F}}_q 1$ realizes the character $(0, z_1)$. Finally the 0-eigenspace of S in $\mathcal{A}_{1,\theta_v}$ is $\overline{\mathbb{F}}_q 1$, which is not U-stable, so that the character $(0, z_1)$ does not lift in $\mathcal{A}_{1,\theta_v}$. $\square$

Remark 7.4.8. Geometrically, the function $z_2 - z_1^2$ on $V_{\widehat{\mathbf{T}},0}/W_0$ defines a family of parabolas

$$\begin{tikzcd} V_{\widehat{\mathbf{T}},0}/W_0, \arrow[d, "z_2 - z_1^2"] \\ \mathbb{A}^1 \end{tikzcd}$$

whose parameter is 4Δ, where Δ is the discriminant of the parabola. Then the locus of $V_{\widehat{\mathbf{T}},0}/W_0$ where $z_2 = z_1^2$ corresponds to the parabola at 0, having vanishing discriminant (at least if $p \neq 2$).

Definition 7.4.9. We will say that a pair of characters of $\mathcal{H}_{1,\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[S, U^{\pm 1}] \to \overline{\mathbb{F}}_q^\times$ is *spherical* if there exists $z_1 \in \overline{\mathbb{F}}_q^\times$ such that, after evaluating on (S, U), it is equal to

$$\{(0, z_1), (-1, -z_1)\}.$$

7.4.5. Note that the set of characters $\mathcal{H}_{1,\overline{\mathbb{F}}_q} \to \overline{\mathbb{F}}_q^\times$ is the disjoint union of the spherical pairs, by the very definition.

Theorem 7.4.10. *Let $\gamma \in \mathbb{T}^\vee/W_0$ non-regular. Consider the decomposition*

$$V^\gamma_{\widehat{\mathbf{T}},0}/W_0 = D(2)_\gamma \cup D(1)_\gamma,$$

where $D(1)_\gamma$ is the closed subscheme defined by the parabola $z_2 = z_1^2$ in the Steinberg coordinates z_1, z_2 and $D(2)_\gamma$ is the open complement. Then the spherical map induces bijections

$$\mathrm{Sph}^\gamma(2) : D(2)_\gamma(k) \xrightarrow{\sim} \{\textit{simple 2-dimensional left } \mathcal{H}^\gamma_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim,$$

$$\mathrm{Sph}^\gamma(1) : D(1)_\gamma(k) \xrightarrow{\sim} \{\textit{spherical pairs of characters of } \mathcal{H}^\gamma_{\overline{\mathbb{F}}_q}\}/\sim.$$

The branch locus of the covering

$$V_{\widehat{\mathbf{T}},0} \longrightarrow V_{\widehat{\mathbf{T}},0}/W_0 \simeq V^\gamma_{\widehat{\mathbf{T}},0}/W_0$$

is contained in $D(2)_\gamma$, with equation $z_1 = 0$ in Steinberg coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence $\mathrm{Sph}^\gamma(2)$.

Remark 7.4.11. The matrices of S, U and $S_0 = USU^{-1}$ in the $\overline{\mathbb{F}}_q$-basis $\{1, Y\}$ of the supersingular module $\mathcal{A}_{1,\theta_v} \cong M_2(0, z_2)$ are

$$S = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -z_2 \\ -1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The two characters of the *finite subalgebra* $\overline{\mathbb{F}}_q[S]$ corresponding to $S \mapsto 0$ and $S \mapsto -1$ are realized by 1 and Y. From the matrix of S_0, we see in fact that the whole *affine subalgebra* $\overline{\mathbb{F}}_q[S_0, S]$ acts on 1 and Y via the two *supersingular affine characters*, which by definition are the characters different from the trivial character $(S_0, S) \mapsto (0, 0)$ and the sign character $(S_0, S) \mapsto (-1, -1)$.

7.4.6. Finally, let v be any k-point of the parametrizing space $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. As a particular case of 7.4.5, the Bernstein parameter of the spherical module $\mathrm{Sph}(v)$ is the structure sheaf of the fibre of the quotient map π at v, and its Satake parameter is the underlying k-vector space:

$$B(\mathrm{Sph}(v)) = \mathcal{O}_{\pi^{-1}(v)} \quad \text{and} \quad S(\mathrm{Sph}(v)) = \pi_* \mathcal{O}_{\pi^{-1}(v)}.$$

7.5. **Central characters.** In this final subsection, we show that the dual parametrization 7.4.10 behaves naturally with respect to central characters.

7.5.1. Let $\omega : \mathbb{F}_q^\times \to k^\times$ be induced by the inclusion $\mathbb{F}_q \subset k$. Then $(\mathbb{F}_q^\times)^\vee = \langle \omega \rangle$ is a cyclic group of order $q-1$. An element ω^r defines a non-regular character of $\mathbb{T}$:

$$\omega^r(t_1, t_2) := \omega^r(t_1)\omega^r(t_2)$$

for all $(t_1, t_2) \in \mathbb{T} = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Composing with multiplication in $\mathbb{T}^\vee$, we get an action of $(\mathbb{F}_q^\times)^\vee$ on $\mathbb{T}^\vee$, which factors on the quotient set $\mathbb{T}^\vee/W_0$:

$$\mathbb{T}^\vee/W_0 \times (\mathbb{F}_q^\times)^\vee \longrightarrow \mathbb{T}^\vee/W_0,\ (\gamma, \omega^r) \mapsto \gamma\omega^r.$$

If $\gamma \in \mathbb{T}^\vee/W_0$ is regular (non-regular), then $\gamma\omega^r$ is regular (non-regular).

7.5.2. Restricting characters of $\mathbb{T}$ to the subgroup $\mathbb{F}_q^\times \simeq \{\mathrm{diag}(a, a) : a \in \mathbb{F}_q^\times\}$ induces a homomorphism $\mathbb{T}^\vee \to (\mathbb{F}_q^\times)^\vee$ which factors into a restriction map

$$\mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee,\ \gamma \mapsto \gamma|_{\mathbb{F}_q^\times}.$$

The relation to the $(\mathbb{F}_q^\times)^\vee$-action on the source $\mathbb{T}^\vee/W_0$ is given by the formula

$$(\gamma\omega^r)|_{\mathbb{F}_q^\times} = \gamma|_{\mathbb{F}_q^\times}\ \omega^{2r}.$$

We describe the fibres of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$.

Let $(\cdot)|^{-1}_{\mathbb{F}_q^\times}(\omega^{2r})$ be the fibre at a square element ω^{2r}. By the above formula, the action of ω^{-r} on $\mathbb{T}^\vee/W_0$ induces a bijection with the fibre $(\cdot)|^{-1}_{\mathbb{F}_q^\times}(1)$. The fibre

$$(\cdot)|^{-1}_{\mathbb{F}_q^\times}(1) = \{1 \otimes 1\} \coprod \{\omega \otimes \omega^{-1}, \omega^2 \otimes \omega^{-2}, \ldots, \omega^{\frac{q-3}{2}} \otimes \omega^{-\frac{q-3}{2}}\} \coprod \{\omega^{\frac{q-1}{2}} \otimes \omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. The $\frac{q-3}{2}$ elements in the middle of this list, i.e. the W_0-orbits represented by the characters $\omega^r \otimes \omega^{-r}$ for $r = 1, \ldots, \frac{q-3}{2}$, are all regular W_0-orbits. The two orbits at the two ends of the list are non-regular orbits (note that $\frac{q-1}{2} \equiv -\frac{q-1}{2} \mod (q-1)$). Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

On the other hand, let $(\cdot)|^{-1}_{\mathbb{F}_q^\times}(\omega^{2r-1})$ be the fibre at a non-square element ω^{2r-1}. The action of ω^{-r} induces a bijection with the fibre $(\cdot)|^{-1}_{\mathbb{F}_q^\times}(\omega^{-1})$. The fibre

$$(\cdot)|^{-1}_{\mathbb{F}_q^\times}(\omega^{-1}) = \{1 \otimes \omega^{-1}, \omega \otimes \omega^{-2}, \ldots, \omega^{\frac{q-1}{2}-1} \otimes \omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q-1}{2}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular W_0-orbits. Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a non-square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

Note that $\frac{q-1}{2}(\frac{q+1}{2} + \frac{q-1}{2}) = \frac{q^2-q}{2}$ is the cardinality of the set $\mathbb{T}^\vee/W_0$.

7.5.3. Recall the commutative k-semigroup scheme

$$V^{(1)}_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

together with its W_0-action, cf. 6.2.7: the natural action of W_0 on the factors $\mathbb{T}^\vee$ and $\mathrm{SingDiag}_{2\times 2}$ and the trivial one on $\mathbb{G}_m$. There is a commuting action of the k-group scheme

$$\mathcal{Z}^\vee := (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$$

on $V^{(1)}_{\widehat{\mathbf{T}},0}$: the (constant finite diagonalizable) group $(\mathbb{F}_q^\times)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in 7.5.1; an element $z_0 \in \mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\mathrm{diag}(z_0, z_0)$ on $\mathrm{SingDiag}_{2\times 2}$ and by multiplication with the square z_0^2 on $\mathbb{G}_m$. Therefore the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ inherits a $\mathcal{Z}^\vee$-action. Now, according to 7.4.2, one has the decomposition

$$V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 = \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}},0} \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0.$$

Then the $(\mathbb{F}_q^\times)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$, i.e. on the set of connected components of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$; as observed above, it preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ preserves each connected component.

7.5.4. The two canonical projections from $V^{(1)}_{\widehat{\mathbf{T}},0}$ to $\mathbb{T}^\vee$ and $\mathbb{G}_m$ respectively induce two projection morphisms

$$\begin{tikzcd} & V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \arrow[dl, "\mathrm{pr}_{\mathbb{T}^\vee/W_0}"'] \arrow[dr, "\mathrm{pr}_{\mathbb{G}_m}"] & \\ \mathbb{T}^\vee/W_0 & & \mathbb{G}_m. \end{tikzcd}$$

Then we may compose the map $\mathrm{pr}_{\mathbb{T}^\vee/W_0}$ with the restriction map $(\cdot)|_{\mathbb{F}_q^\times} : \mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee$, set

$$\theta := \big((\cdot)|_{\mathbb{F}_q^\times} \circ \mathrm{pr}_{\mathbb{T}^\vee/W_0}\big) \times \mathrm{pr}_{\mathbb{G}_m}$$

and view $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ as fibred over the space $\mathcal{Z}^\vee$:

$$\begin{tikzcd} V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \arrow[d, "\theta"] \\ \mathcal{Z}^\vee. \end{tikzcd}$$

The relation to the $\mathcal{Z}^\vee$-action on the source $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is given by the formula

$$\theta(x.(\omega^r, z_0)) = \theta(x)(\omega^{2r}, z_0^2) = \theta(x)(\omega^r, z_0)^2$$

for $x \in V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ and $(\omega^r, z_0) \in \mathcal{Z}^\vee$. This formula follows from the formula in 7.5.2 and the definition of the $\mathbb{G}_m$-action in 7.5.3.

Definition 7.5.1. Let $\zeta \in \mathcal{Z}^\vee$. The *space of mod p Satake parameters with central character ζ* is the k-scheme

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

7.5.5. Let $\zeta = (\zeta|_{\mathbb{F}_q^\times}, z_2) \in \mathcal{Z}^\vee(k) = (\mathbb{F}_q^\times)^\vee \times k^\times$. Denote by $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{z_2}$ the fibre of $\mathrm{pr}_{\mathbb{G}_m}$ at $z_2 \in k^\times$. Then by 7.4.2 we have

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta = \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2} \quad \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2}/W_0.$$

Recall that the choice of standard coordinates x, y identifies

$$V_{\widehat{\mathbf{T}},0,z_2} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1$$

with two affine lines over k, intersecting at the origin, cf. 7.4.3. On the other hand, the choice of the Steinberg coordinate z_1 identifies

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1$$

with a single affine line over k, cf. 7.4.4.

Lemma 7.5.2. *Let $\zeta, \eta \in \mathcal{Z}^\vee$. The action of η on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ induces an isomorphism of k-schemes $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \simeq (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}$.*

Proof. Follows from the last formula in 7.5.4. □

7.5.6. Recall from 7.4.2 the spherical map

$$\mathrm{Sph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}/\sim .$$

The $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules in the image of this map are of length 1 or 2, cf. 7.4.6 and 7.4.10. We write $\mathrm{Sph}(v)^{\mathrm{ss}}$ for the semisimplification of the module $\mathrm{Sph}(v)$, for $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k)$.

Let $(\omega^r, z_0) \in \mathcal{Z}^\vee(k)$. Recall that the standard or irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules may be 'twisted by the character (ω^r, z_0)': in the regular case, the actions of X, Y, U^2 get multiplied by z_0, z_0, z_0^2 respectively and the component γ gets multiplied by ω^r, cf. [V04, 2.4]; in the non-regular case, the action of U gets multiplied by z_0, the action of S remains unchanged and the component γ gets multiplied by ω^r, cf. [V04, 1.6]. This gives an action of the group of k-points of $\mathcal{Z}^\vee$ on the standard or irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules. It extends to an action on semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules.

Proposition 7.5.3. *The map $\mathrm{Sph}(-)^{\mathrm{ss}}$ is $\mathcal{Z}^\vee(k)$-equivariant.*

Proof. Let $(\omega^r, z_0) \in \mathcal{Z}^\vee(k)$. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that γ is regular, choose an ordering $\gamma = (\chi, \chi^s)$ on the set γ and standard coordinates. Then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(v)$ is a simple two-dimensional standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. 7.4.6, i.e. of the form $M(x, y, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{Sph}(v.(\omega^r, z_0)) \simeq M(z_0x, z_0y, z_0^2z_2, \chi.\omega^r) \simeq \mathrm{Sph}(v).(\omega^r, z_0).$$

Suppose that $\gamma = \{\chi\}$ is non-regular and choose Steinberg coordinates. (a) If $v \in D(2)_\gamma(k)$, then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. 7.4.10, i.e. of the form $M(z_1, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{Sph}(v.(\omega^r, z_0)) \simeq M(z_0 z_1, z_0^2 z_2, \chi.\omega^r) \simeq \mathrm{Sph}(v).(\omega^r, z_0).$$

(b) If $v \in D(1)_\gamma(k)$, then the semisimplified module $\mathrm{Sph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters in the spherical pair $\mathrm{Sph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$. Similarly $\mathrm{Sph}(v.(\omega^r, z_0))^{\mathrm{ss}}$ is the direct sum of the characters $\{(0, z_0 z_1), (-1, -z_0 z_1)\}$ in the component $\gamma.\omega^r$, and hence is isomorphic to $\mathrm{Sph}(v)^{\mathrm{ss}}.(\omega^r, z_0)$. $\square$

7.5.7. We now explain the compatibility with central characters for G-representations. In order to do this, let us consider W_0 to be a subgroup of G, by sending s to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by identifying the group Λ with a subgroup of T via $(1, 0) \mapsto \mathrm{diag}(\varpi^{-1}, 1)$ and $(0, 1) \mapsto \mathrm{diag}(1, \varpi^{-1})$. We obtain for example (recall that $u = (1, 0)s \in W_0$)

$$u = \begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix}, \; u^{-1} = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \; us = \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \; su = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1})$.[6] Since

$$\begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} d & \varpi^{-1} c \\ \varpi b & a \end{pmatrix},$$

the element $u \in G$ normalizes the group $I^{(1)}$.

7.5.8. Let $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the category of smooth G-representations over k. Taking $I^{(1)}$-invariants yields a functor $\pi \mapsto \pi^{I^{(1)}}$ from $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ to the category $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$. If $F = \mathbb{Q}_p$, it induces a bijection between the irreducible G-representations and the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules, under which supersingular representations correspond to supersingular Hecke modules [V04].

For future reference, let us recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \mathrm{Ind}_B^G(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$.

In the regular case, one chooses the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set γ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}}).$$

In the non-regular case, one has

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(\chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}}).$$

These standard modules are irreducible if and only if $\chi \neq \chi^s$ [V04, 4.2/4.3].[7]

Let $F = \mathbb{Q}_p$. If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, \dots, p-1$ and a character $\eta : \mathbb{Q}_p^\times \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^s\}$ represented by the character

[6] Note that our element u equals the element u^{-1} in [Be11], [Br07] and [V04].

[7] Our formulas differ from [V04, 4.2/4.3] by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04, Appendix A.5].

$\chi := (\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$, cf. [Br07, 5.1/5.3]. If π is the trivial representation $\mathbb{1}$ or the Steinberg representation St, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character $(0,1)$ or $(-1,-1)$ respectively.

7.5.9. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)}uI^{(1)} = uI^{(1)}$. It follows that the convolution action of the Hecke operator U (resp. U^2) on $\pi^{I^{(1)}}$ is therefore induced by the action of u (resp. u^2 on π). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup I and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbf{T}(o_F)$-action on π.

7.5.10. We identify $F^\times$ with the center $Z(G)$ via $a \mapsto \mathrm{diag}(a,a)$. A (smooth) character

$$\zeta : Z(G) = F^\times \longrightarrow k^\times$$

is determined by its value $\zeta(\varpi^{-1}) \in k^\times$ and its restriction $\zeta|_{o_F^\times}$. Since the latter is trivial on the subgroup $1+\varpi o_F$, we may view it as a character of $\mathbb{F}_q^\times$; we will write $\zeta|_{\mathbb{F}_q^\times}$ for this restriction in the following. Thus the group of characters of $Z(G)$ gets identified with the group of k-points of the group scheme $\mathscr{Z}^\vee = (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$:

$$Z(G)^\vee \xrightarrow{\sim} \mathscr{Z}^\vee(k),\ \zeta \mapsto (\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1})).$$

Proposition 7.5.4. *Suppose that $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ has a central character $\zeta : Z(G) \to k^\times$. Then the Satake parameter $S(\pi^{I^{(1)}})$ of $\pi^{I^{(1)}} \in \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$ has central character ζ, i.e. it is supported on the closed subscheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0.$$

Proof. If M is any $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module, then

$$M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \varepsilon_\gamma M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \oplus_{\lambda \in \gamma} \varepsilon_\lambda M,$$

and $\mathbb{T} \subset \overline{\mathbb{F}}_q[\mathbb{T}] \subset \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\varepsilon_\lambda M$ through the character $\lambda : \mathbb{T} \to \mathbb{F}_q^\times$. Now if $M = \pi^{I^{(1)}}$, then the $\mathbb{T}$-action on M is the factorization of the $\mathbf{T}(o_F)$-action on π, cf. 7.5.9. In particular, the restriction of the $\mathbb{T}$-action along the diagonal inclusion $\mathbb{F}_q^\times \subset \mathbb{T}$ is the factorization of the action of the central subgroup $o_F^\times \subset Z(G)$ on π, which is given by $\zeta|_{o_F^\times}$ by assumption. Hence

$$\varepsilon_\gamma M \neq 0 \quad \Longrightarrow \quad \forall \lambda \in \gamma,\ \lambda|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times} \text{ i.e. } \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}.$$

Moreover, the element $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1}) \in Z(G)$ acts on π by multiplication by $\zeta(\varpi^{-1})$ by assumption. Therefore, by 7.5.9, the Hecke operator $z_2 := U^2 \in \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\pi^{I^{(1)}}$ by multiplication by $\zeta(\varpi^{-1})$. Thus we have obtained that $S(\pi^{I^{(1)}})$ is supported on

$$\coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})} \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})}/W_0$$
$$= (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))}.$$

□

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