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SOME EXAMPLES OF SIMPLE GENERIC FI-MODULES IN POSITIVE CHARACTERISTIC

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Abstract. We give, in any characteristic $p > 0$, examples of simple generic FI-modules whose underlying representations are reducible in all sufficiently high degrees.

1. Introduction

In this paper, an *FI-module* (more precisely, an *FI-module over K*) is a functor from the category FI of finite sets and injections into K-modules for a field K. FI-modules were introduced by Church, Ellenberg, and Farb in [2] with numerous applications in topology, algebra, and number theory in mind, and have been since studied extensively (see e.g. [1, 3–5, 9–11, 17–23]). Stable phenomena of the representation theory of symmetric groups are encoded by the category of generic FI-modules, defined in a way to disregard elements which go to 0 in the representations of Σ_n for $n \gg 0$. This is analogous to the construction of the category of quasi-coherent sheaves from the category of graded modules over the projective coordinate ring of a projective scheme [25]. This analogy was in fact used by Sam and Snowden [24] to gain a good understanding of the category of generic FI-modules in characteristic 0. In particular, they identified all the simple objects of that category.

The case of characteristic $p > 0$ is more complicated. Nevertheless, simple generic FI-modules in positive characteristic were characterized by Nagpal [21], Theorem 1.11. R.Nagpal asked if the Σ_n-representation terms of a simple generic FI-module in positive characteristic are necessarily irreducible for infinitely many n. The main result of the present paper is to construct counterexamples for all primes p.

To discuss our result more precisely, we need some notation. Let $[n] = \{1, \dots, n\}$. For an FI-module X, we will sometimes write $X(N)$ or X_N instead of $X([N])$. For a given N, we identify a $K\Sigma_N$-module with the FI-module over K equal to it in degree N and 0 in other degrees. An FI-module X is called *torsion* if each of the elements of every $X(n)$ goes to $0 \in X(m)$ for some $m \gg 0$. Torsion FI-modules (over K) form a Serre subcategory of the category of FI-modules (over K), and taking the Serre quotient by them gives the category of *generic FI-modules (over K)* (see [8] for the details of this construction). Nagpal [21], Theorem 1.11 (see also Theorem 2) proved that in every characteristic, isomorphism classes of simple generic FI-modules are in bijective correspondence with p-regular Young diagrams.

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We denote the simple generic FI-module in positive characteristic corresponding to a p-regular Young diagram λ by $\mathscr{D}_\lambda$. Theorem 1 answers a question by Nagpal:

Theorem 1. *Suppose K is a field of characteristic $p > 0$.*

(1) *If $p = 2$, then for every $N \gg 0$, the Σ_N-representation*

$$\mathscr{D}_{(3,1)}(N)$$

is reducible.

(2) *If $p > 2$, then for every $N \gg 0$, the Σ_N-representation*

$$\mathscr{D}_{(p,2)}(N)$$

is reducible.

We will review the structure of simple generic FI-modules in Section 2. This is needed in our main argument. The proof of Theorem 1 requires different approaches depending on whether $p = 2$ or $p > 2$. The case of $p = 2$ is treated in Section 3, and the case of $p > 2$ is treated in Section 4.

2. Preliminaries and Nagpal's Theorem

We begin with some notation. A *Young diagram* is a k-tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \dots \geq \lambda_k$ are positive integers (this can be visualized as a diagram of boxes with k rows and λ_i boxes in the i-th row). For a Young diagram λ, let $|\lambda|$ denote the number of its boxes (i.e. $|\lambda| = \lambda_1 + \dots + \lambda_k$). Let S_λ denote the Specht module corresponding to a Young diagram λ. As a general reference for Specht modules, we recommend [12]. We denote by M_λ the *Spechtral FI-module* consisting of the Specht modules of the Young diagrams obtained by adding a row to the top of λ at each degree $\geq |\lambda| + \lambda_1$ ([2] Definition 2.2.6: they work in characteristic 0, but the construction works over $\mathbb{Z}$, see [16]).

A Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ is called p-regular if at most $p-1$ of the numbers $\lambda_1, \dots, \lambda_k$ are equal to any given number i. Recall that the set of Young diagrams with ℓ boxes has a natural ordering called *dominance* given by saying, for two partitions $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_m)$ of ℓ, $\mu \trianglerighteq \nu$ when

$$\mu_1 + \dots + \mu_k \geq \nu_1 + \dots + \nu_k$$

for all $k \geq 1$. In this note, we will also call a Young diagram μ *strictly dominant* over ν (write $\mu \triangleright \nu$) if we have $\mu \trianglerighteq \nu$ and $\mu \neq \nu$.

For every p-regular Young diagram λ, S_λ has a unique quotient D_λ which is a simple $K\Sigma_{|\lambda|}$-module. These form a complete set of representatives of isomorphism classes of simple $K\Sigma_{|\lambda|}$-modules. Moreover, for a p-regular Young diagram λ, all the other composition factors of S_λ are D_μ with $\mu \triangleright \lambda$ ([12], Section 12).

One defines two functors

$$\Psi' : FI\text{-Mod} \to FI\text{-Mod}$$
$$\Phi' : FI\text{-Mod} \to FI\text{-Mod}$$

by

$$\Psi'(M_\bullet) : [N] \mapsto \mathrm{Hom}_{FI\text{-Mod}}(K\mathrm{Map}_{FI}([\bullet],[N])^\vee, M_\bullet) \tag{1}$$

$$\Phi'(M_\bullet) : [N] \mapsto K\mathrm{Map}_{FI}([N],[\bullet])^\vee \otimes_{FI\text{-Mod}} M_\bullet \tag{2}$$

for an FI-module $M_\bullet$. By definition, Φ' is left adjoint to Ψ'. It is also easy to see that applying Φ' to a torsion FI-module gives 0 (by surjectivity of morphisms in the first factor of the right hand side of (2)) and that applying Φ' to any FI-module gives a torsion FI-module. This shows that for every FI-modules $M_\bullet$, denoting by $M_{\geq N}$ the sub-FI-module in degree $\geq N$ (and 0 below), the projection induces a surjection

$$\Phi'(M_{\geq N}) \to \Phi'(M). \tag{3}$$

However, considering the additional relations in $\Phi'(M)$ involving $x \in M_n$ for $n < N$, one sees that they are also present in the source of (3). Thus, (3) is in fact an isomorphism.

Let $FI\text{-Mod}^{\text{gen}}$ denote the category of generic finitely generated FI-modules over K and let $FI\text{-Mod}^{\text{tor}}$ denote the full subcategory of FI-modules over K on finitely generated torsion FI-modules over K. Then Φ', Ψ' induce a pair of functors

$$\Phi : FI\text{-Mod}^{\text{gen}} \to FI\text{-Mod}^{\text{tor}}$$

$$\Psi : FI\text{-Mod}^{\text{tor}} \to FI\text{-Mod}^{\text{gen}}$$

where Φ is left adjoint to Ψ. (See [21], Section 1.)

In characteristic 0, by Schur-Weyl correspondence, the functors Ψ, Φ coincide with the functors of the same names in [24], where they are proved to be inverse equivalences of categories. This is false in characteristic $p > 0$.

Nagpal's Theorem can be restated as follows:

Theorem 2 ([21], Theorem 1.11)**.** *Let K be a field of characteristic p. For every p-regular Young diagram λ, there exists a canonical non-zero morphism of FI-modules over K*

$$\iota_\lambda : M_\lambda \to \Psi(D_\lambda)$$

such that $\mathscr{D}_\lambda = Im(\iota_\lambda)$ is a simple object in the category FI-Mod^{gen} of generic finitely generated FI-modules over K. Additionally, every simple generic finitely generated FI-module over K is isomorphic to $\mathscr{D}_\lambda = Im(\iota_\lambda)$ for a unique p-regular Young diagram λ.

In this paper, we denote the induction from a subgroup H to a group G by Ind_G^H with the philosophy that the superscript indicates a contravariant variable. The opposite convention also occurs in the literature. Note that one can identify

$$KMap_{FI}([m],[m]) \cong K\Sigma_n/\Sigma_{n-m}.$$

Note that a morphism of FI-modules is determined by a sequence of Σ_n-equivariant maps commuting with the structure maps corresponding to the standard inclusions $[n] \subset [n+1]$.

For our purposes, we will need to review the construction of the map ι_λ. First, one notes that for an FI-module X, $\Phi(X)(m)$ can be described as the colimit of a diagram of the form

$$\begin{array}{ccc} (X(n))_{\Sigma_{n-m}} & & (X(n-k))_{\Sigma_{n-k-m}} \\ \nwarrow{\scriptstyle \phi_+} & & \nearrow{\scriptstyle \phi_-} \\ & (Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k} X(n-k))_{\Sigma_{n-m}}. & \end{array} \tag{4}$$

Precisely speaking, the objects of the category I indexing the diagram (4) consist of a "top row" and a "bottom row." The objects in the top row are indexed by $n = m, m+1, m+2, \dots$. The objects in the bottom row are indexed by pairs of integers $(n, n-k)$ where $m \leq n-k \leq n$. The morphisms are those drawn in (4). The morphisms ϕ_+, ϕ_- are described as follows: ϕ_+ is given by taking Σ_{n-m}-cofixed points (also nown as coninvariants) of the natural

$$Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k} X(n-k) \to X(n).$$

The map ϕ_- is defined to be the composition

$$\begin{array}{c} (Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k}(X(n-k)))_{\Sigma_{n-m}} \\ \downarrow \\ (Ind_{\Sigma_n}^{\Sigma_{n-k}\times\Sigma_k}(X(n-k)))_{\Sigma_k\times\Sigma_{n-k-m}} \\ \downarrow \\ (X(n-k))_{\Sigma_{n-k-m}} \end{array}$$

where the top map is taking corestriction (i.e. summing over coset representatives of $\Sigma_{n-m}/\Sigma_k \times \Sigma_{n-m-k}$), and the lower map is the counit of adjunction of the induction as a right adjoint to cofixed points, followed by Σ_{n-k-m}-cofixed points.

Dually, $\Psi(X)(N)$ is the limit of the diagram

$$(5) \qquad Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}(X(\ell-k)) \xrightarrow{\psi^+} Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}((X(\ell))^{\Sigma_k}) \xleftarrow{\psi^-} Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(X(\ell))$$

where the indexing category is I^{Op} where I is the indexing category of the diagram (4). The map ψ^+ is given by applying $Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}$ to the natural map

$$X(\ell-k) \to (X(\ell))^{\Sigma_k}.$$

The map ψ^- is defined as the composition

$$\begin{array}{c} Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(X(\ell)) \\ \downarrow \\ Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}}(Ind_{\Sigma_\ell}^{\Sigma_{\ell-k}\times\Sigma_k}(X(\ell)^{\Sigma_k})) \\ \downarrow \\ Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_{N-\ell+k}}(X(\ell)^{\Sigma_k}) \end{array}$$

where the top map is given by induction applied to the unit of adjunction of fixed points and induction, and the lower map, noting that

$$Ind_{\Sigma_N}^{\Sigma_\ell\times\Sigma_{N-\ell}} \circ Ind_{\Sigma_\ell}^{\Sigma_{\ell-k}\times\Sigma_k} = Ind_{\Sigma_N}^{\Sigma_{\ell-k}\times\Sigma_k\times\Sigma_{N-\ell}},$$

is given by corestriction (i.e. summing over coset representatives of

$$(\Sigma_\ell \times \Sigma_{N-\ell})/(\Sigma_{\ell-k} \times \Sigma_k \times \Sigma_{N-\ell})).$$

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a p-regular Young diagram and let $N \geq |\lambda| + \lambda_1$. Define

$$\lambda_N^+ = (N - |\lambda|, \lambda_1, \dots, \lambda_k).$$

We will sometimes omit N when it is implicit.

Lemma 3. *Suppose λ is a p-regular Young diagram, $N > |\lambda| + \lambda_1$.*

(A) *$\Psi(D_\lambda)(N)$ has a unique composition factor isomorphic to $D_{\lambda_N^+}$.*

(B) *Let X be a finitely generated FI-module. Suppose there exists a generic surjection $M_\lambda \twoheadrightarrow X$. Then there exists a canonical (up to scaling) surjection*

$$\Phi(X) \twoheadrightarrow D_\lambda. \tag{6}$$

Additionally, the map

$$X \to \Psi(D_\lambda) \tag{7}$$

adjoint to (6) sends the composition factor $D_{\lambda_N^+}$ to itself by an isomorphism. More precisely, there exist filtrations on $X(N)$ and $\Phi(D_\lambda)(N)$ compatible with the map, giving the stated isomorphism on the associated graded pieces.

Proof. By [14], Theorem 3, the induction to $N \gg 0$ of D_λ contains $D_{\lambda_N^+}$ as a unique composition factor, and all other composition factors are of the form D_μ for $\mu \rhd \lambda_N^+$. Additionally, $D_{\lambda_N^+}$ is not a composition factor in the induction of any Σ_n-module with $n < |\lambda|$. By the above description of the functor Ψ, this implies (A).

Also by [14], Theorem 3, for every N, the cofixed point $K\Sigma_{|\lambda|}$-module

$$(D_{\lambda_N^+})_{\Sigma_{N-|\lambda|}} \tag{8}$$

is D_λ and the cofixed point module of $D_{\lambda_N^+}$ under Σ_{N-i} with $i < |\lambda|$ is 0 (since D_λ occurs at the "top branching level" of $L(\lambda_N^+)$). Thus, by the description of the functor Φ as the colimit (4), D_λ is by definition a quotient of the module of generators of $\Phi(X)$. Additionally, the assumption guarantees that these generators are not killed by the relations (again by [14], Theorem 3, since, if $\mu_N^+ \rhd \lambda_N^+$, then $\mu \rhd \lambda$ or $|\mu| < |\lambda|$). This implies the first statement of (B).

For the last statement, we also observe that by [14], Theorem 3, we cannot have $\lambda_N^+ = \mu_N^+$ for $|\mu| < |\lambda|$ and thus, by the description of Ψ as the limit (5), $D_{\lambda_N^+}$ is a composition factor of $\Psi(D_\lambda)(N)$ (since there is no condition excluding this factor). Additionally, all other composition factors of $\Psi(D_\lambda)(N)$ are D_μ for $\mu \rhd \lambda_N^+$. Moreover, our construction of (6) from (8) implies that the adjoint (7) defines an isomorphism on the constituent factors $D_{\lambda_N^+}$. □

Now, by Lemma 3, for a p-regular Young diagram λ, we have a natural (non-zero) surjection

$$\beta_\lambda : \Phi(M_\lambda) \to D_\lambda.$$

Then since Φ and Ψ are adjoint, we obtain a non-zero map

$$\iota_\lambda : M_\lambda \to \Psi(D_\lambda).$$

For the remainder of the proof of Theorem 2, we refer the reader to [21].

3. Proof of Theorem 1 at $p = 2$

First, note that we have a short exact sequence

$$0 \to S_{(4)} \to S_{(3,1)} \to D_{(3,1)} \to 0. \tag{9}$$

Thus,

$$dim(D_{(3,1)}) = dim(S_{(3,1)}) - dim(S_{(4)}) = 3 - 1 = 2,$$

which is also the dimension of $S_{(2,2)}$. Since, at $p = 2$, we have

$$(3,1) = (2,2)^r$$

where λ^r denotes the Young diagram obtained from shifting the boxes of λ as high as possible along each ladder (see [13, 15]), $D_{(3,1)}$ is a composition factor of $S_{(2,2)}$ (by [13], Theorem A). Thus,

$$D_{(3,1)} = S_{(2,2)}.$$

By Lemma 3, we have a natural surjection

$$\Phi(M_{(3,1)}) \twoheadrightarrow D_{(3,1)} = S_{(2,2)}.$$

Now we claim the following

Proposition 4. *There is a short exact sequence*

$$0 \to M_{(2,2)} \to \Psi(D_{(3,1)}) \to M_{(2)} \to 0.$$

First, note that by the Pieri rule, the restriction of the $K\Sigma_4$-module $D_{(3,1)} = S_{(2,2)}$ to Σ_3 is the Specht module $S_{(2,1)}$ (since the only removable box in $(2,2)$ is the bottom right corner). We thus obtain that the induction of $S_{(2,1)}$ has composition factors

$$D_{(3,1)},\ D_{(4)},\ D_{(3,1)},\ D_{(4)},\ D_{(3,1)}, \tag{10}$$

listed from top to bottom (i.e., with the piece that can be considered as a quotient listed first, and the piece that can be considered a submodule listed last).

Lemma 5. *The unit of adjunction*

$$S_{(2,2)} \to Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,2)}|_{\Sigma_3})$$

maps $S_{(2,2)}$ isomorphically to the bottom $D_{(3,1)}$ piece (10) *(coming from $S_{(2,1,1)}$).*

Proof. We can identify the non-zero elements of $S_{(2,2)}$ with 4-cycle subgraphs of the complete graph on vertices $[4] = \{1,2,3,4\}$. On the other hand, $S_{(2,1)}$ can be identified with the submodule of $K^{[3]}$ consisting of vectors whose coordinates have sum 0. Thus, $Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,1)})$ is a submodule of $Ind_{\Sigma_4}^{\Sigma_3}(K^{[3]})$, which is identified with $\mathrm{Map}_{FI}([3],[4])$ (where by our convention, the image of 1 is the new coordinate and the image of 2 comes from the coordinate in [3]). We encode an injective map $[2] \to [4]$ by a 4-tuple where we write i for the image of $i = 1, 2$, and 0's in the remaining places. Under these conventions, our unit of adjunction maps

$$S_{(2,1)} \ni \{1,2\} + \{2,3\} + \{3,4\} + \{4,1\} \longmapsto$$

$$\begin{gathered}(2,0,0,1) + (0,0,1,2) + (1,0,0,2) \\ +(0,1,2,0) + (0,0,2,1) + (0,2,1,0) \\ +(1,2,0,0) + (2,1,0,0).\end{gathered} \tag{11}$$

On the other hand, in this notation, the generators of the Specht module $S_{(2,1,1)} \subseteq \mathrm{Map}_{FI}([2],[4])$ can be identified with, choosing $i \in [4]$, the sum q_i of the six 4-tuples which are non-zero on i. We then see that (11) lies in this submodule, and namely, is equal to $q_1 + q_3$.

The images under the unit of adjunction of other elements of $S_{(2,2)}$ then also lie in the submodule

$$S_{(2,1,1)} \subseteq Ind_{\Sigma_4}^{\Sigma_3}(S_{(2,1)}).$$

□

Proof of Proposition 4. Now for induction from $S_{(2,2)}$ to a degree $N \gg 0$, the Pieri rule gives pieces (from top to bottom)

$$S_{(N-2,2)},\ S_{(N-3,2,1)},\ S_{(N-4,2,2)}.$$

The middle summand is eliminated by the above observation using the description of the functor Ψ in the beginning of Section 2 as the limit of the Diagram (5). Thus, we get generically

$$0 \to M_{(2,2)} \to \Psi(D_{(3,1)}) \to M_{(2)} \to 0.$$

□

Now any map of FI-modules

$$M_{(3,1)} \to M_{(2)}$$

is 0, since the map is necessarily 0 in degree 7 (since the composition factors of $S_{(3,3,1)}$ are $D_{(7)}$ and $D_{(4,2,1)}$, while $S_{(5,2)}$ is irreducible). Hence, the map $\iota_{(3,1)}$ factors through

$$\begin{array}{ccccc} & & & & M_{(3,1)} \\ & & & \overset{\kappa}{\swarrow} & \downarrow{\scriptstyle \iota_{(3,1)}} \\ 0 & \longrightarrow & M_{(2,2)} & \longrightarrow & \Psi(S_{(2,2)}) \end{array}$$

for some map

$$\kappa : M_{(3,1)} \to M_{(2,2)}.$$

At an FI-degree N, denote the cokernel

$$C = Coker(\kappa).$$

We claim the following

Lemma 6. *In degrees* $\gg 0$*, generically,*

$$C = M_{\emptyset}.$$

To prove this lemma, we will need calculations of $\Psi(S_{(4)})$ and $\Psi(S_{(3,1)})$, which we make in the following propositions:

Proposition 7. *Generically, there is a short exact sequence*

$$0 \to M_{(4)} \to \Psi(S_{(4)}) \to M_{\emptyset} \to 0.$$

Proof. First, the restriction of the Specht module $S_{(4)}$ to Σ_3 is exactly the Specht module $S_{(3)}$, whose induction to Σ_4 has pieces (listed from top to bottom) $S_{(4)},\ S_{(3,1)}$. The unit of adjunction (between restriction and induction) sends $S_{(4)}$ monomorphically to the lowest piece.

Now the induction of $S_{(4)}$ to $N \geq 8$ has pieces (listed from top to bottom)

$$S_{(N)},\ S_{(N-1,1)},\ S_{(N-2,2)},\ S_{(N-3,3)},\ S_{(N-4,4)}.$$

The above observation, along with our description of the functor Ψ, eliminates all but the first and last piece. Thus, using the FI-module structure of the induction, we get generically

$$0 \to M_{(4)} \to \Psi(S_{(4)}) \to M_{\emptyset} \to 0.$$

□

Proposition 8. *We have*

$$\Psi(S_{(3,1)}) = M_{(3,1)}.$$

Proof. First, note that the restriction of the Specht module $S_{(3,1)}$ to Σ_3 has pieces $S_{(3)}$, $S_{(2,1)}$. The induction back to Σ_4 of the first piece is $S_{(3,1)}$, to which the bottom piece $D_{(4)}$ of $S_{(3,1)}$ injects by the unit of adjunction. The piece $S_{(2,1)}$ inducts to $S_{(3,1)}$ and $S_{(2,1,1)}$, to which the top piece $S_{(2,2)}$ of $S_{(3,1)}$ injects.

Now the induction of $S_{(3,1)}$ to $N \geq 8$ has pieces

$$S_{(N-1,1)},\ S_{(N-2,2)},\ S_{(N-2,1,1)},\ S_{(N-3,3)},\ S_{(N-3,2,1)},\ S_{(N-4,3,1)}.$$

The first, second, and fourth pieces are eliminated by the first part of the unit of adjunction (to the induction of $S_{(3)}$) and the third and fifth pieces are eliminated by the second part of the unit of adjunction (to the induction of $S_{(2,1,1)}$), similarly as in the proofs of Proposition 4 and Proposition 7. Thus,

$$\Psi(S_{(3,1)}) = M_{(3,1)}.$$

□

Proof of Lemma 6. Recall again the exact sequence

$$0 \to S_{(4)} \to S_{(3,1)} \to S_{(2,2)} \to 0.$$

Since Ψ is right adjoint to Φ, it is left exact, so we obtain

$$0 \longrightarrow \Psi(S_{(4)}) \longrightarrow \Psi(S_{(3,1)}) \xrightarrow{\rho} \Psi(S_{(2,2)}).$$

Then ρ factors through κ (since by above, $\Psi(S_{(3,1)}) = M_{(3,1)}$).

Thus, at every FI-degree $N \gg 0$, the dimension of $C(N)$ equals

$$\begin{aligned}
&dim(M_{(2,2)}(N)) - dim(M_{(3,1)}(N)) + dim(\Psi(S_{(4)})(N)) = \\
&= dim(M_{(2,2)}(N)) - dim(M_{(3,1)}(N)) + dim(M_{\emptyset}(N)) + dim(M_{(4)}(N)) = \\
&= dim(M_{\emptyset}(N)) = dim(S_{(N)}) = 1
\end{aligned}$$

(since, by the hook length formula,

$$dim(S_{(k,3,1)}) = \frac{(k+4)(k+3)(k+1)(k-2)}{8}$$
$$dim(S_{(k,4)}) = \frac{(k+4)(k+3)(k+2)(k-3)}{24}$$

and

$$dim(S_{(k,3,1)}) - dim(S_{(k,4)}) = \frac{(k+4)(k+3)k(k-1)}{12}) = dim(S_{(k,2,2)}).$$

Hence, $C(N)$ is a $K\Sigma_N$-module with dimension 1. Thus, for every N, $C(N) = S_{(N)}$, proving that, as FI-modules,

$$C = M_{\emptyset}.$$

□

Finally, to prove Theorem 1, we let $R_\lambda = K\Sigma_{\text{row}}^{\lambda}$ where $\Sigma_{\text{row}}^{\lambda}$ is the subgroup of $\Sigma_{|\lambda|}$ of permutations preserving the rows of a Young diagram λ.

Proof of Theorem 1. Suppose $N \geq 8$ is odd. We consider the morphism

$$\theta_{T_1} : R_{(N-3,2,1)} \to R_{(N-4,2,2)} \tag{12}$$

of [12] given by the tableau T_1 with rows

3	3	2	1	...	1
2	1				
1					

We calculate that, using the notation of [12],

$$N_{1,1}(T_1) = N - 6,\ N_{2,1}(T_1) = 1,\ N_{3,1}(T_1) = 2,$$
$$N_{1,2}(T_1) = 1,\ N_{2,2}(T_1) = 1,\ N_{3,2}(T_1) = 0,$$
$$N_{1,3}(T_1) = 1,\ N_{2,3}(T_1) = 0,\ N_{3,3}(T_1) = 0,$$

and thus T_1 satisfies the condition of Theorem 24.6, (ii), [12] (since N is assumed to be odd). Hence, by Theorem 24.6, (ii), [12], the restriction of θ_{T_1} is a non-zero homomorphism

$$\theta_{T_1}|_{S_{(N-3,2,1)}} : S_{(N-3,2,1)} \to S_{(N-4,2,2)}.$$

Since T_1 is reverse semistandard, by the proof of Theorem 24.6,

$$Im(\theta_{T_1}|_{S_{(N-3,2,1)}}) \subseteq S_{(N-4,2,2)}$$

contains the composition factor $D_{(N-3,2,1)}$. Therefore, this composition factor must be present in $Im(\iota_{(3,1)})(N) \cong Im(\kappa)(N)$, which is therefore not simple, since it also contains the composition factor $D_{(N-4,3,1)}$.

Suppose $N \geq 8$ is even. We consider the morphism

$$\theta_{T_2} : R_{(N-2,1,1)} \to R_{(N-4,2,2)} \tag{13}$$

given by the tableau T_2 with rows

3	3	2	1	...	1
2					
1					

We calculate, using the notation of [12],

$$N_{1,1}(T_1) = N - 5,\ N_{2,1}(T_1) = 1,\ N_{3,1}(T_1) = 2,$$
$$N_{1,2}(T_1) = 0,\ N_{2,2}(T_1) = 2,\ N_{3,2}(T_1) = 0,$$
$$N_{1,3}(T_1) = 1,\ N_{2,3}(T_1) = 0,\ N_{3,3}(T_1) = 0,$$

and thus, again, T_2 satisfies the condition of Theorem 24.6, (ii), [12] (since N is assumed to be even). Hence, the restriction of θ_{T_2} is a non-zero homomorphism

$$\theta_{T_2}|_{S_{(N-2,1,1)}} : S_{(N-2,1,1)} \to S_{(N-4,2,2)}.$$

Now all composition factors of $S_{(N-2,1,1)}$ are of the form D_λ where $\lambda \rhd (N-2,1,1)$ (by Theorem 12.1 of [12]). Then $\theta_{T_2}|_{S_{(N-2,1,1)}}$ must be non-zero on at least one such D_λ, and therefore D_λ must be a composition factor of $Im(\theta_{T_2}|_{S_{(N-2,1,1)}}) \subseteq S_{(N-4,2,2)}$. Hence, this D_λ is also a composition factor of $Im(\iota_{(3,1)}) \cong Im(\kappa)$. By Theorem 24.4 of [12], $\lambda \neq (N)$. In addition, since $\lambda \rhd (N-2,1,1)$, we also have $\lambda \neq (N-4,3,1)$. Therefore, since $Im(\iota_{(3,1)})(N) \cong Im(\kappa)(N)$ also contains the composition factor $D_{(N-4,3,1)}$, it cannot be simple. $\square$

4. Proof of Theorem 1 at $p > 2$

Suppose $p > 2$. First, we have the following

Proposition 9. *There is a short exact sequence*

$$0 \to S_{(p+1,1)} \to S_{(p,2)} \to D_{(p,2)} \to 0.$$

Proof. If $h_\lambda(a,b)$ is the hook length of a box (a,b) in a Young diagram λ, we say that the box (a,b) is *bad* if $v_p(h_\lambda(a,b)) > 0$ and there are boxes $(x,b),(a,y)$ in λ such that $v_p(h_\lambda(a,b)) \neq v_p(h_\lambda(x,b))$ and $v_p(h_\lambda(a,b)) \neq v_p(h_\lambda(a,y))$.

First note that since $(p,2)$ contains a bad box, $S_{(p,2)}$ must be reducible (see [6,7]). It therefore contains a submodule of the form D_λ where $\lambda \rhd (p,2)$. The only options for λ are $(p+1,1)$ and $(p+2)$. By [12], Theorem 24.4, $D_{(p+2)} = S_{(p+2)}$ is not a submodule of $S_{(p,2)}$ since p is not -1 mod p. Thus, $D_{(p+1,1)} = S_{(p+1,1)}$ (the equality holds since $(p+1,1)$ has no bad boxes) is a submodule of $S_{(p,2)}$.

To prove the Proposition, by [12], Section 11, it suffices to show

$$S^{\perp}_{(p,2)} \cap S_{(p,2)} = S_{(p+1,1)}, \tag{14}$$

where $S^{\perp}_{(p,2)}$ is the orthogonal complement of $S_{(p,2)}$ in $R_{(p,2)}$ (the standard permutation module basis of $R_{(p,2)}$ is orthonormal). By the above discussion, we already know $S^{\perp}_{(p,2)} \cap S_{(p,2)} \supseteq S_{(p+1,1)}$ in (14).

To prove the other inclusion in (14), first, by the hook formula, we have

$$dim(S_{(p,2)}) = \frac{(p+2)!}{(p+1)p(p-2)!2} = \frac{(p+2)(p-1)}{2},$$

and we also have

$$dim(R_{(p,2)}) = \frac{(p+2)!}{p!2} = \frac{(p+2)(p+1)}{2}.$$

So

$$dim(R_{(p,2)}) - dim(S_{(p,2)}) = \frac{2(p+2)}{2} = p+2. \tag{15}$$

Let

$$V_n = K\Sigma_n/\Sigma_{n-1} = R_{(n-1,1)}.$$

Then we have a homomorphism

$$\psi_{1,1} : R_{(p,2)} \to V_{p+2}$$

and $S_{(p,2)} \subseteq ker(\psi_{1,1})$ (by [12], Corollary 17.18), where $\psi_{1,1}$ is defined as a sum of standard basis elements obtained by moving one box from the second row to the first row. In fact, in this case $\psi_{1,1}$ is surjective since its image contains sums of every pair of standard basis elements in V_{p+2} and $p > 2$.

Thus, since $dim(V_{p+2}) = p+2$, by (15), we have a short exact sequence

$$0 \longrightarrow S_{(p,2)} \longrightarrow R_{(p,2)} \xrightarrow{\psi_{1,1}} V_{p+2} \longrightarrow 0.$$

Hence, $S^{\perp}_{(p,2)} \cong V_{p+2}$, and in particular,

$$dim(S^{\perp}_{(p,2)} \cap S_{(p,2)}) \leq p+2.$$

To prove (14), since we already know the $\supseteq$-inclusion, it suffices to show

$$dim(S^{\perp}_{(p,2)} \cap S_{(p,2)}) \leq p+1 = dim(S_{(p+1,1)}).$$

To this end, it suffices to find an element in $S_{(p,2)}^{\perp} \setminus S_{(p,2)}$. Consider the map

$$R_{(p,2)} \to K,$$

given by sending a basis element to $1 \in K$ if it has a 2 in a given position and to $0 \in K$ else. This is equivalent to taking the dot product with the sum v of such basis elements, of which there are $p+1$. Thus, the dot product of the element v with itself is $p+1$ which is non-zero, and thus, v is not in $S_{(p,2)} = ker(\psi_{1,1})$. Thus, (14) is proven, concluding the proof of the Proposition. □

Again, since Ψ is a right adjoint, it is left exact, giving

$$0 \to \Psi(S_{(p+1,1)}) \to \Psi(S_{(p,2)}) \to \Psi(D_{(p,2)}). \tag{16}$$

We then claim the following

Proposition 10. *We have*

$$\Psi(S_{(p+1,1)}) = M_{(p+1,1)}.$$

Proof. Letting

$$V_n = K(\Sigma_n/\Sigma_{n-1}) \cong K^n,$$

we have

$$S_{(p+1,1)} = K\{(v_1, \dots, v_{p+2}) \in V_{p+2} | \sum_{i=1}^{p+2} v_i = 0\}.$$

Consider the unit of adjunction between induction and restriction

$$S_{(p+1,1)} \to Ind_{\Sigma_{p+2}}^{\Sigma_{p+1}} Res_{\Sigma_{p+1}}^{\Sigma_{p+2}} S_{(p+1,1)}. \tag{17}$$

Using the isomorphism

$$Ind_{\Sigma_{p+2}}^{\Sigma_{p+1}} Res_{\Sigma_{p+1}}^{\Sigma_{p+2}} S_{(p+1,1)} \cong K(\Sigma_{p+2}/\Sigma_{p+1}) \otimes_K S_{(p+1,1)}$$

the map (17) can be described as sending $(v_1, \dots, v_{p+2}) \in S_{(p+1,1)}$ to $(1, 1, \dots, 1) \otimes (v_1, \dots, v_{p+2})$.

Now the restriction of $S_{(p+1,1)}$ to Σ_{p+1} has pieces $S_{(p+1)}$, $S_{(p,1)}$, with $S_{(p+1)}$ above $S_{(p,1)}$. The image of (17) must be contained in the induction of $S_{(p,1)}$ since any $(1, \dots, 1) \otimes (v_1, \dots, v_{p+2})$ in the image of (17) can be expressed as the sum

$$\sum_{i=1}^{p+2} (0, \dots, 0, 1, 0, \dots, 0) \otimes (v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_{p+2})$$

(where in the ith summand, the 1 is in the ith place).

The only piece of the induction of $S_{(p+1,1)}$ to $N \gg 0$ that is not a piece in the induction of $S_{(p,1)}$ is $S_{(N-p-2,p+1,1)}$. Thus, by the description (5) of Ψ,

$$\Psi(S_{(p+1,1)}) = M_{(p+1,1)}.$$

(The FI-module structure again follows from the FI-module structure on the induction.) □

Proof of Theorem 1*:* Fix some $N \gg 0$. Denote by φ the first map of (16). By Proposition 10, the injection is of the form

$$\varphi : S_{(N-p-2,p+1,1)} \to \Psi(S_{(p,2)})(N).$$

We therefore obtain the short exact sequence

$$0 \to \varphi^{-1}(S_{(N-p-2,p,2)}) \to S_{(N-p-2,p,2)} \to (Im(\iota_{(p,2)}))(N) \to 0. \tag{18}$$

(For the sake of brevity, let us write $k = N - p - 2$.)

Now consider the map

$$\theta_T : R_{(\lfloor \frac{k}{p} \rfloor p+p, k-\lfloor \frac{k}{p} \rfloor p+1, 1)} \to R_{(k,p,2)} \tag{19}$$

(again using the notation and definitions given in [12]) where T is the reverse semistandard tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
3 & 3 & 2 & \dots & 2 & 2 & \dots & 2 & 1 & \dots & 1 \\
\hline
2 & 1 & 1 & \dots & 1 \\
\cline{1-5}
1 \\
\cline{1-1}
\end{array}$$

which has

$$N_{1,1}(T) = \left\lfloor \frac{k}{p} \right\rfloor p - 1,\ N_{2,1}(T) = p - 1,\ N_{3,1}(T) = 2$$
$$N_{1,2}(T) = k - \left\lfloor \frac{k}{p} \right\rfloor p,\ N_{2,2}(T) = 1,\ N_{3,2}(T) = 0$$
$$N_{1,3}(T) = 1,\ N_{2,3}(T) = 0,\ N_{3,3}(T) = 0.$$

This satisfies the conditions of Theorem 24.6, (ii), [12] and therefore (19) restricts to a non-zero map

$$\widehat{\theta_T} : S_{(\lfloor \frac{k}{p} \rfloor p+p, k-\lfloor \frac{k}{p} \rfloor p+1, 1)} \to S_{(k,p,2)}.$$

It therefore suffices to show $\widehat{\theta_T}$ does not lift to a map

$$S_{(\lfloor \frac{k}{p} \rfloor p+p, k-\lfloor \frac{k}{p} \rfloor p+1, 1)} \to \varphi^{-1}(S_{(k,p,2)}) \subseteq S_{(k,p+1,1)}, \tag{20}$$

for (18) (since then $(Im(\iota_{(p,2)}))(N)$ will have composition factors $D_{(k,p,2)}$ and D_λ for some λ dominant or equal to $(\left\lfloor \frac{k}{p} \right\rfloor p + p, k - \left\lfloor \frac{k}{p} \right\rfloor p + 1, 1)$ and therefore be reducible, having two different composition factors).

Suppose a lifting (20) exists. If p divides k, then $(k, p+1, 1)$ contains no bad boxes, so $S_{(k,p+1,1)}$ is irreducible, thus already forming a contradiction since then (20) is 0. So, suppose p does not divide k. By [12], Theorem 13.13, it suffices to show all linear combinations of $\widehat{\theta}_T$ for semistandard $(\left\lfloor \frac{k}{p} \right\rfloor p + p, k - \left\lfloor \frac{k}{p} \right\rfloor p + 1, 1)$-tableaux T of type $(k, p+1, 1)$ which have image contained in the Specht module $S_{(k,p+1,1)}$ are 0. The only semistandard $(\left\lfloor \frac{k}{p} \right\rfloor p + p, k - \left\lfloor \frac{k}{p} \right\rfloor p + 1, 1)$-tableau T of type $(k, p+1, 1)$ is

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & \dots & 1 & 1 & \dots & 1 & 2 & \dots & 2 \\
\hline
2 & 2 & \dots & 2 \\
\cline{1-4}
3 \\
\cline{1-1}
\end{array}\ . \tag{21}$$

We will prove that $Im(\widehat{\theta_T}) \nsubseteq S_{(k,p+1,1)}$ using [12], Corollary 17.18 by finding i, v with $\psi_{i-1,v}(Im(\widehat{\theta_T})) \neq 0$, where

$$\psi_{i-1,v} : R_\lambda \to R_{(\lambda_1,\dots,\lambda_{i-2},\lambda_{i-1}+\lambda_i-v,v,\lambda_{i+1},\dots)}$$

is obtained by moving $\lambda_i - v$ boxes from the ith row to the $(i-1)$th row.

Let us choose $i = 2$, $v = p$. Applying $\psi_{i-1,v}$ then involves summing over the different tableaux T' arising from taking un-signed row permutations and then taking the sum of signed column permutations of tableaux T'' arising from T' by replacing one 2 in (21) by a 1.

It then suffices to show that there exists a T'' with no two numbers the same in any column and this T'' arises a number of times that is not divisible by p. Consider the T'' given as the $(\left\lfloor \frac{k}{p} \right\rfloor p+p, k-\left\lfloor \frac{k}{p} \right\rfloor p+1, 1)$-tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
2 & 1 & \dots & 1 & 1 & \dots & 1 & 2 & \dots & 2 \\
\hline
1 & 2 & \dots & 2 \\
\cline{1-4}
3 \\
\cline{1-1}
\end{array} \quad . \tag{22}$$

This can arise in two fashions:

(1) T' arises by moving the first 2 in the first row to the first column and T'' then arises by replacing the first 2 in the second row with a 1. This yields one positive summand.

(2) T' arises by moving the first 2 in the first row to any of the first $k+1$ spots of the first row (including the possibility of letting it stay in the same spot), and T'' then arises by replacing this same 2 by a 1, and switching the 1 and 2 in the first column. This gives $k+1$ negative summands.

Thus, the coefficient of the summand T'' in the linear combination is $-k$. By our assumption, p does not divide k (and thus also does not divide $-k$), hence concluding the proof. ☐

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