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SHARP EFFECTIVE NULLSTELLENSATZ

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1. INTRODUCTION

The aim of this article is to consider the following question:

Let $f_1, \dots, f_k$ be polynomials in n variables and assume that they have no common zero in $\mathbb{C}^n$. Then Hilbert's Nullstellensatz guarantees the existence of polynomials $g_1, \dots, g_k$ such that

$$\sum f_i g_i = 1.$$

The usual proofs of this result, however, give no information about the g_i's; for instance they give no bound on their degrees. This question was first considered by G. Hermann [H]. She used elimination theory to get a bound on the degree of the g_i's which was doubly exponential in the number of variables. Her results were later improved in [MW] and in [Th]. All these produce bounds that are doubly exponential in the number of variables.

A major breakthrough was achieved by Brownawell [B1] who proved the following result:

1.1. **Theorem.** *Let $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ be such that the f_i's have no common zero. Assume that* $\deg f_i \le D$. *Then one can find polynomials $g_1, \dots, g_k \in \mathbb{C}[x_1, \dots, x_n]$ such that*

$$\sum f_i g_i = 1$$

and

$$\deg g_i \le n \cdot \min(k, n) \cdot D^{\min(k,n)} + \min(k, n) \cdot D.$$

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Brownawell uses results from transcendental number theory to get the following estimate:

1.2. **Theorem.** *Given* $f_1, \dots, f_k$ *as above then*

$$\max_i |f_i(x)| \geq \text{constant} \cdot \|x\|^{1-(n-1)D^{\min(n,k)}}$$

where $\|x\| = \sqrt{|x_1^2| + \cdots + |x_n^2|}$.

Having this estimate one can use some results of Skoda [Sk] to get the bound on the degree of the g_i's. We refer to the article of Brownawell for more details and for further references. In *all* characteristics Shiffman [Sh] recently improved the bound given previously; his bound is however still doubly exponential in the number of variables.

Here a completely algebraic method of estimating the degree of the g_i's will be presented. The proof is mostly elementary ideal theory in homogeneous polynomial rings. The only nonstandard method we use is the definition and elementary properties of local cohomology groups. For this [G] can serve as a good reference. The proof will work in all characteristics and in most cases gives the best possible result. Roughly speaking we eliminate the factor $n \cdot \min(k, n)$ from 1.1.

The above problem will turn out to be a special case of a more general result comparing ideals and their radicals. To formulate this result it is most convenient to start with two definitions.

1.3. **Definition.** *Given a field K and natural numbers n and* $d_1, \dots, d_k$ *let*

$$N(n, d_1, \dots, d_k) = \min \Big\{ s \mid \textit{for any polynomials } f_1, \dots, f_k \in K[x_1, \dots, x_n] \textit{ such that } \deg f_i = d_i \textit{ and such that they have no common roots in the algebraic closure of } K, \textit{ there are polynomials } g_1, \dots, g_k \in K[x_1, \dots, x_n] \textit{ such that } \sum f_i g_i = 1 \textit{ and } \max_i \{\deg(f_i g_i)\} \leq s \Big\}.$$

(*Note that K is suppressed in the notation since it will turn out to be unimportant.*)

1.4. **Definition.** *Given a field K and natural numbers n and* $d_1, \dots, d_k$ *let*

$$N'(n, d_1, \dots, d_k) = \min \Big\{ s \mid \textit{for any homogeneous polynomials } \bar{f}_1, \dots, \bar{f}_k \in K[x_0, \dots, x_n] \textit{ such that } \deg \bar{f}_i = d_i \textit{ we have } \Big(\sqrt{(\bar{f}_1, \dots, \bar{f}_k)}\Big)^s \subseteq (\bar{f}_1, \dots, \bar{f}_k) \Big\}.$$

Now we can formulate our main result as follows:

1.5. **Theorem.** *Given a field K and natural numbers n and $d_1 \geq \cdots \geq d_k$ assume that all the d_i are different from 2. Then*

$$N'(n, d_1, \ldots, d_k) = \begin{cases} d_1 \cdot \cdots \cdot d_k & \text{if } k \leq n; \\ d_1 \cdot \cdots \cdot d_{n-1} \cdot d_k & \text{if } k > n > 1; \\ d_1 + d_k - 1 & \text{if } k > n = 1. \end{cases}$$

1.6. *Remarks.* (i) By definition, a polynomial h belongs to $(f_1, \ldots, f_k)$ iff we can write $h = \sum g_i f_i$ for some polynomials g_i. If we fix the degree of the g_i and we consider the coefficients of the g_i as unknowns we get a system of linear equations in these unknowns. Thus solvability in a field extension implies solvability in the base field. This shows that it is sufficient to prove 1.5 for K algebraically closed.

(ii) The assumption that all the d_i are different from 2 is purely technical; I expect that it is not necessary. My proof works if at most three of the d_i are equal to 2, but in general the proof gives a bigger upper bound.

1.7. **Corollary.** *Given $f_1, \ldots, f_k$ and $h \in K[x_1, \ldots, x_n]$, assume that h vanishes on all common zeroes of $f_1, \ldots, f_k$ (in the algebraic closure of K). Let $d_i = \deg f_i$ and assume that none of the d_i is 2. Then one can find $g_1, \ldots, g_k \in K[x_1, \ldots, x_n]$ and a natural number s satisfying*

$$\sum g_i f_i = h^s$$

such that

$$s \leq N'(n, d_1, \ldots, d_k) \quad \text{and} \quad \deg g_i f_i \leq (1 + \deg h) \cdot N'(n, d_1, \ldots, d_k).$$

1.8. *Remark.* It is very interesting to compare the above result with a construction of Mayr and Meyer [MM]. They show that in the above situation it is possible that $h = \sum g_i f_i$ has a solution but the degree of the g_i grows doubly exponentially in the number of variables. It is quite surprising that if we allow taking a power of h then the solution will have lower degree.

1.9. **Corollary.** *Let $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$ such that they have no common zero in the algebraic closure of K. Let $d_i = \deg f_i$ and assume that none of the d_i is 2. Then one can find $g_1, \ldots, g_k \in K[x_1, \ldots, x_n]$ such that $\deg g_i f_i \leq N'(n, d_1, \ldots, d_k)$ satisfying*

$$\sum g_i f_i = 1.$$

In particular, $N(n, d_1, \ldots, d_k) = N'(n, d_1, \ldots, d_k)$.

If $k = n$, then there are no solutions with $\deg g_i f_i < N(n, d_1, \ldots, d_k)$ if and only if the closures of the hypersurfaces $f_i = 0$ in $\mathbb{P}^n$ intersect in one point only, the Zariski tangent space of the intersection is one dimensional and it is not contained in the hyperplane at infinity.

This corollary in turn at once gives an improved version of Theorem 1.2. It is, however, not too hard to give a direct proof that removes the restriction on the degrees. This gives the following:

1.10. **Proposition.** *Let $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ such that the f_i have only finitely many common zeroes. Let $d_i = \deg f_i$ and assume that $n \geq 2$. Then*

$$\max_i \{|f_i(x)| \cdot \|x\|^{-d_i}\} \geq \text{ constant } \cdot \|x\|^{-N(n, d_1, \dots, d_k)}$$

if $\|x\|$ is sufficiently large. Moreover the above exponent is best possible.

1.11. *Remarks.* (i) It is easy to see that an estimate of the above type holds if it holds after restriction to any algebraic curve in $\mathbb{C}^n$ (this is the valuative criterion of integral dependence, cf. [Te, 1.3.4]). The latter makes perfect sense even in characteristic p, and 1.10 is also true over any field if we adopt this definition.

(ii) Brownawell informed me that he recently proved a result similar to 1.10.

(iii) The results of Brownawell [B1] not only bound the degree of the polynomials f_i but they can be used to make the constant in 1.10 effective provided we know the height of the coefficients of the g_i ; in this direction see [B2].

1.12. *Remark.* In the context of 1.4 it is clear that the computation of $N'(n, d_1, \dots, d_k)$ tells us something about the primary decomposition of the ideal $(\bar{f}_1, \dots, \bar{f}_k)$. The primary decomposition is not unique, thus it is better to view it in the following way:

The module $M = K[x_0, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_k)$ can be filtered by submodules $M_i =$ (sections whose support has dimension at most i). The quotient M_i/M_{i-1} is the "well defined part" of the i-dimensional primary components. 1.5 can be interpreted as a statement that these quotients are not too big "in every direction." Unfortunately I do not know any technically precise and meaningful interpretation of this last claim.

The ideal result would be a bound on the length of these qoutients. This is however too much to hope for. Here is a simple example.

In four homogeneous variables let q be the equation of a smooth quadric Q. Let $C \subset Q$ be the union of k disjoint lines. Let h_1, h_2 be degree k homogeneous polynomials such that $h_1 = h_2 = 0$ defines a curve $C \cup D$ where D is also smooth, $C \cup D$ has only nodes and D intersects Q only along C. If $\bar{*}$ denotes the cone over $*$ then we have an exact sequence

$$0 \to \mathscr{O}_{\bar{D}}(-\overline{C \cap D}) \to K[x_0, \dots, x_3]/(h_1, h_2) \to K[x_0, \dots, x_3]/(q, h_1, h_2) \to 0.$$

If m denotes the maximal ideal of the origin then we have

$$H^0_m(K[x_0, \dots, x_3]/(q, h_1, h_2)) = H^1_m(\mathscr{O}_{\bar{D}}(-\overline{C \cap D})) = H^1_m(\mathscr{O}_{\bar{D}}).$$

Since C and D are linked the latter group is dual to $H^1_m(\mathscr{O}_{\bar{C}})$ which can be computed easily. This gives that the length of $H^0_m(K[x_0, \dots, x_3]/(q, h_1, h_2))$ is $\binom{k+1}{3}$. For large k this is larger than the Bézout number $2k^2$.

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2. Preliminary remarks

2.1. *Proof of* 1.7. Introduce a new variable x_0 and let $\bar{f}_i$ be the homogenization of f_i and $\bar{h}$ the homogenization of h. Since h vanishes on all the common zeroes of the f_i, we see that $\bar{h}$ vanishes on all the common zeroes of $\bar{f}_i$ that lie outside the hyperplane at infinity. Therefore $x_0\bar{h}$ vanishes on all common zeroes of $\bar{f}_i$ and hence $x_0\bar{h}$ is contained in the radical of $(\bar{f}_1, \dots, \bar{f}_k)$. Therefore, by 1.5, $(x_0\bar{h})^s$ is contained in $(\bar{f}_1, \dots, \bar{f}_k)$ for some $s \le N'(n, d_1, \dots, d_k)$. Thus there are homogeneous polynomials $\bar{g}_i$ such that $\sum \bar{f}_i\bar{g}_i = (x_0\bar{h})^s$. We can assume that

$$\deg \bar{f}_i\bar{g}_i = s \cdot \deg x_0\bar{h} \le N'(n, d_1, \dots, d_k) \cdot (1 + \deg h).$$

Now we can dehomogenize the above relation to obtain the required result.

2.2. *Case* $n = 1$. In this case everything is very easy and is left to the reader. In the sequel we always assume that in fact $n \ge 2$.

2.3. **Example.** We give an example to show that $N(n, d_1, \dots, d_k)$ and $N'(n, d_1, \dots, d_k)$ are at least as big as 1.5 claims. The example is a slight modification of the one given by Masser, Philippon, and Brownawell [B1].

Given any n and $d_1, \dots, d_n$ consider the polynomials

$$x_1^{d_1}, x_1x_n^{d_2-1} - x_2^{d_2}, \dots, x_{n-2}x_n^{d_{n-1}-1} - x_{n-1}^{d_{n-1}}, x_{n-1}x_n^{d_n-1} - x_0^{d_n}.$$

We denote them by $\bar{t}_i$. Clearly $\deg \bar{t}_i = d_i$. Their only common zero is along the line $x_0 = \cdots = x_{n-1} = 0$. If we set $x_n = 1$, then we get

$$K[x_0, \dots, x_n]/(\bar{t}_1, \dots, \bar{t}_n, x_n - 1) \cong K[x_0]/(x_0)^{\prod d_i}.$$

Therefore we see that x_0 is in the radical of $(\bar{t}_1, \dots, \bar{t}_n)$, but $(x_0)^{\prod d_i - 1}$ is not contained in $(\bar{t}_1, \dots, \bar{t}_n)$.

Now if we have K, n, and d_i and $k \ge n$ then arrange the d_i in such a way that d_n is the smallest and take $\bar{f}_i = \bar{t}_i$ for $i \le n$ and $\bar{f}_i =$ appropriate multiple of $\bar{t}_n$ for $i > n$. This gives the necessary lower bound in 1.5. If $k < n$ then we can consider the above example with k variables and consider these as polynomials in n variables. This shows the required lower bound for 1.5.

We can dehomogenize x_0 to get the required examples for 1.9 and 1.10.

2.4. **Example.** Assume now that $k = n$ which is somehow the main case. Assume furthermore that the hypersurfaces $F_i = (\bar{f}_i = 0)$ in $\mathbb{P}^n$ intersect in only finitely many points. By Bézout's Theorem the intersection is then a zero dimensional scheme of length $\prod d_i$ and its homogeneous ideal is generated by the $\bar{f}_i$. If a polynomial h vanishes on all the points of this intersection then

$h^{\prod d_i}$ is contained in its ideal. Moreover if $h^{\prod d_i - 1}$ is not contained in the ideal then the intersection has only one closed point, its Zariski tangent space is one dimensional and it is not contained in the tangent space of $h = 0$ at that point. This in particular shows the last part of 1.9 if we can prove that in case of equality the intersection is zero dimensional.

It is not easy to come up with many such examples. One might take for instance an elliptic plane curve C and on it a flex O and a point P such that $m(P - O) = 0$ in the Picard group. Then there is a degree m curve D such that $C \cap D = P$. If the order of $P - O$ is exactly m and $3 \nmid m$ then any such D is irreducible. Variants of this idea give a few more cases, but 2.3 is the only large series of examples that I know of. Since it is related to constructing curves with torsion points on them, this is not too surprising.

2.5. *Remark.* There are some easy cases when the proof of the theorems can be reduced to a simpler case. If for instance one of the f_i is reducible, then we can replace the f_i with one factor at a time and get a solution of the original problem this way. If one of the f_i is linear then we can eliminate one variable. Thus 1.5 reduces to the case when $d_i \geq 3$ for every i. We will assume this in the proof.

Next we prove an easy lemma that will be needed in the proof.

2.6. **Lemma.** *Let $Z \subset \mathbb{P}^n$ be a zero dimensional subscheme, and let h be a degree d homogeneous polynomial nowhere zero on Z. Let $\mathcal{I}$ be the homogeneous ideal of Z. Then*

$$(x_0, \dots, x_n)^{k+d} \subset (\mathcal{I}, h)$$

if one of the following conditions is satisfied:

(i) $k \geq$ length Z;

(ii) *There are hypersurfaces $F_1, \dots, F_n$ of degrees $d_1, \dots, d_n$ such that their intersection T is zero dimensional, Z is the union of some connected components of T and $k \geq \sum(d_i - 1)$.*

Proof. Let

$$M_Z(k) = \operatorname{im}[H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathcal{O}_Z(k))],$$

and consider the multiplication map

$$m_k : M_Z(k) \xrightarrow{h} M_Z(k+d).$$

Clearly

$$(x_0, \dots, x_n)^{k+d} \subset (\mathcal{I}, h)$$

iff m_j is surjective for $j \geq k$. The multiplication map is injective since h is nowhere zero on Z; thus m_k is surjective if $M_Z(k) = H^0(\mathcal{O}_Z(k))$. In case (i) let S be a connected component of Z and let m be the ideal of the closed point of S. Then

$$H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}/m^{k+1}(k)) \longrightarrow H^0(\mathcal{O}_S(k))$$

is surjective for $k \geq$ length S since $m^{\text{length } S}\mathscr{O}_S = 0$. For the same reason one can find a hypersurface of degree length S such that it is zero on S (scheme theoretically), and not zero at the other points of Z. Multiplying such polynomials together we get the required surjectivity.

In the second case we know by Macaulay (see e.g. [GH, Chapter 5]) that

$$H^0(\mathscr{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathscr{O}_T(k))$$

is surjective in the required range and hence so is the similar map for Z. This proves the lemma.

3. Proof of Theorem 1.5

3.1. Now consider our polynomials f_i. We can rearrange them so that d_1 is the smallest and then $d_2 \geq d_3 \geq \cdots \geq d_k$. We can now also replace f_i for $i \geq 2$ by a general linear combination of $f_i, \ldots, f_k$. This rearrangement will be used for the rest of the proof. Let $F_i = (f_i = 0)$ and let $R = F_1 \cap \cdots \cap F_k$ with reduced scheme structure and le $\mathscr{R}$ be the homogeneous ideal of R. Let U be the open set $\mathbb{P}^n \setminus R$. Let

$$Z_i = \text{ the closure of } (U \cap F_1 \cap \cdots \cap F_i).$$

By the assumptions $Z_i \cap U$ is a complete intersection of codimension i. We can write

$$(f_1, \ldots, f_i) = \mathscr{I}_{Z_i} \cap \mathscr{J}_i$$

where $\mathscr{I}_{Z_i}$ is the homogeneous ideal of Z_i and $\mathscr{J}_i$ is a homogeneous ideal whose cosupport is in R. Of course in general $\mathscr{J}_i$ is not unique. Let

$$a_i = \min\{s | \mathscr{R}^s \text{ annihilates } \mathscr{I}_{Z_i}/(f_1, \ldots, f_i)\}.$$

By assumption $\mathscr{I}_{Z_k}$ is the whole ring and thus a_k tells us which power of h is in the ideal of the f_i. Therefore our task is to estimaste the a_i from above. This will be done as follows.

3.2. **Inductive step.** Consider the following decomposition:

$$(\mathscr{I}_{Z_i}, f_{i+1}) = \mathscr{I}_{Z_{i+1}} \cap \mathscr{K}_{i+1} \cap \mathscr{E}_{i+1}$$

where $\mathscr{K}_{i+1}$ is the intersection of the primary components of codimension $i+1$ whose cosupport is in R and $\mathscr{E}_{i+1}$ is the intersection of the primary components whose cosupport has codimension at least $i+2$. Note that the cosupport of $\mathscr{E}_{i+1}$ is in R. Note also that $\mathscr{K}_{i+1}$ is uniquely defined.

Let

$$k_{i+1} = \min\{s | \mathscr{R}^s \text{ is contained in } \mathscr{K}_{i+1}\}$$

and let

$$e_{i+1} = \min\{s | \mathscr{R}^s \text{ annihilates } \mathscr{I}_{Z_{i+1}} \cap \mathscr{K}_{i+1}/(\mathscr{I}_{Z_i}, f_{i+1})\}.$$

Therefore $\mathscr{R}^{k_{i+1}+e_{i+1}}$ annihilates

$$\mathscr{I}_{Z_{i+1}}/(\mathscr{I}_{Z_i}, f_{i+1}).$$

Also $\mathscr{R}^{a_i}$ annihilates

$$(\mathscr{I}_{Z_i}, f_{i+1})/(f_1, \dots, f_i, f_{i+1}).$$

Putting all these together we get that $\mathscr{R}^{k_{i+1}+e_{i+1}+a_i}$ annihilates

$$\mathscr{I}_{Z_{i+1}}/(f_1, \dots, f_i, f_{i+1}).$$

Therefore $a_{i+1} \le k_{i+1} + e_{i+1} + a_i$.

k_{i+1} is at most as big as the degree of the scheme defined by $\mathscr{K}_{i+1}$ and so it is easy to deal with . The difficult part is to understand e_{i+1}, which comes from the embedded primes. This will be accomplished in the following way.

3.3. **Definition.** Given a scheme S and an ideal $\mathscr{R}$ we define

$$\operatorname{nil}(\mathscr{R}, S) = \{\min t \mid \forall Z \subset R \text{ and } \forall i < \operatorname{codim}_Z S \text{ we have } \mathscr{R}^t \cdot H^i_Z(\mathscr{O}_S) = 0\}.$$

3.4. **Lemma.** *Let X be a pure dimensional affine scheme, let f be a nonzero divisor, and let $\mathscr{R}$ be an ideal. Let $R = \operatorname{Spec} \mathscr{O}_X/\mathscr{R}$. Let*

$$(f) = \mathscr{I} \cap \mathscr{K} \cap \mathscr{E},$$

where $\mathscr{I}$ is the intersection of isolated primary ideals whose cosupport is not in R, $\mathscr{K}$ is the intersection of isolated primary ideals whose cosupport is in R, and $\mathscr{E}$ is the intersection of embedded primary ideals. Furthermore let X' be the scheme defined by $\mathscr{I}$.

Assume that $\mathscr{R}^k$ is contained in $\mathscr{K}$ and that $\operatorname{cosupp} \mathscr{E}$ *is contained in R (e.g. this holds if X is Cohen-Macaulay outside R).*

Then

$$\operatorname{nil}(\mathscr{R}, X') \le 3\operatorname{nil}(\mathscr{R}, X) + k.$$

Proof. The exact sequences

$$0 \to \mathscr{O}_X \xrightarrow{f} \mathscr{O}_X \to \mathscr{O}_X/(f) \to 0$$

and

$$0 \to \mathscr{I}/(f) \to \mathscr{O}_X/(f) \to \mathscr{O}_{X'} \to 0$$

give rise to the following cohomology sequences

$$H^i_Z(\mathscr{O}_X) \to H^i_Z(\mathscr{O}_X/(f)) \to H^{i+1}_Z(\mathscr{O}_X)$$

and

$$H^i_Z(\mathscr{O}_X/(f)) \to H^i_Z(\mathscr{O}_{X'}) \to H^{i+1}_Z(\mathscr{I}/(f)).$$

Therefore $\mathscr{R}^{2\operatorname{nil}(\mathscr{R}, X)}$ annihilates $H^i_Z(\mathscr{O}_X/(f))$ and we have to estimate which power of $\mathscr{R}$ will annihilate $H^{i+1}_Z(\mathscr{I}/(f))$. For this it is sufficient to know which power of $\mathscr{R}$ annihilates $\mathscr{I}/(f)$. To see this consider the sequence

$$0 \to \mathscr{I} \cap \mathscr{K}/(f) \to \mathscr{I}/(f) \to \mathscr{I}/\mathscr{I} \cap \mathscr{K} \to 0.$$

By assumption $\mathscr{R}^k$ annihilates $\mathscr{I}/\mathscr{I} \cap \mathscr{K}$. Also if Z is the support of $\mathscr{I} \cap \mathscr{K}/(f)$ then

$$\mathscr{I} \cap \mathscr{K}/(f) = H^0_Z(\mathscr{I} \cap \mathscr{K}/(f)) = H^0_Z(\mathscr{O}_X/(f)) \hookrightarrow H^1_Z(\mathscr{O}_X);$$

thus $\mathscr{R}^{\mathrm{nil}(\mathscr{R},X)}$ annihilates $\mathscr{I} \cap \mathscr{K}/(f)$. Putting these together we get the conclusion of the lemma.

3.5. **Counting losses.** For notational simplicity we assume that $k \geq n$. The other case is the same. Now consider our sequence of hypersurfaces F_i for $i = 1, \dots, n$. We have defined our schemes Z_i and we have a well-defined sequence of integers $k_1, \dots, k_n$. Since Z_1 is a hypersurface we have $\mathrm{nil}(\mathscr{R}, Z_1) = 0$. Thus we get recursive upper bounds for $\mathrm{nil}(\mathscr{R}, Z_i)$. To get estimates for the numbers a_j we look at the map

$$\mathscr{I}_j \cap \mathscr{K}_j/(f_j) = H^0_Z(\mathscr{I}_j \cap \mathscr{K}_j/(f_j)) \hookrightarrow H^1_Z(\mathscr{O}_{Z_{j-1}}).$$

Thus we get that $e_j \leq \mathrm{nil}(\mathscr{R}, Z_{j-1})$. This in turn yields the following estimate

$$\begin{aligned} a_n &\leq \sum_{i=1}^{n} (k_i + \mathrm{nil}(\mathscr{R}, Z_{i-1})) \\ &\leq \sum_{i=1}^{n} \left(k_i + \sum_{j=1}^{i-1} k_j \cdot 3^{i-1-j} \right) \\ &= \sum_{i=1}^{n} k_i \cdot \left(1 + \sum_{j=i+1}^{n} 3^{j-i-1} \right) \\ &= \sum_{i=1}^{n} k_i \cdot \frac{3^{n-i} + 1}{2}. \end{aligned}$$

3.6. **Counting gains.** Z_n is a zero dimensional subscheme and we would like to compute its degree. This can be done using the formula

$$\deg Z_{i+1} = d_{i+1} \cdot \deg Z_i - \deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_{i+1}.$$

Now h is contained in the radical of $\mathscr{K}_{i+1}$ and this ideal is unmixed. Therefore

$$\mathscr{R}^{\deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_{i+1}} \in \mathscr{K}_{i+1};$$

in particular, $k_{i+1} \leq \deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_{i+1}$. Thus we get

$$\deg Z_{i+1} \leq d_{i+1} \cdot \deg Z_i - k_{i+1}.$$

Using this repeatedly, we obtain the following estimate

$$\deg Z_n \leq \prod_{i=1}^{n} d_i - \sum_{i=1}^{n} k_i \cdot \prod_{j=i+1}^{n} d_j.$$

Since we assumed that $d_j \geq 3$, this gives that

$$\deg Z_n \leq \prod_{i=1}^{n} d_i - \sum_{i=1}^{n} k_i \cdot 3^{n-i-1} d_n.$$

The following is a routine computation:

3.7. **Lemma.**

$$k_i \cdot \frac{3^{n-i}+1}{2} \le k_i \cdot 3^{n-i-1} \cdot d_n ,$$

and

$$d_n + k_i \cdot \frac{3^{n-i}+1}{2} \le k_i \cdot 3^{n-i-1} \cdot d_n ,$$

unless $i = n-1$ *and* $k_i = 1$ *or* 2.

3.8. *Case* $k \le n$. In this case for a_k we get the following formula (note that Z_k is empty)

$$\begin{aligned} a_k &\le \sum_{i=1}^{k} k_i \cdot \frac{3^{n-i}+1}{2} \le \sum_{i=1}^{k} k_i \cdot 3^{n-i-1} \cdot d_n \\ &\le \prod_{i=1}^{k} d_i - \deg Z_k = \prod_{i=1}^{k} d_i . \end{aligned}$$

Thus we have the required bound.

3.9. *Case* $k_1 = \cdots = k_{n-1} = 0$. In this case the hypersurfaces F_i intersect in a zero dimensional subscheme of $\mathbb{P}^n$. Thus we can use 2.6 to conclude that the multiplication map

$$M_k \xrightarrow{f_{n+1}} M_{k+d_{n+1}}$$

is surjective for $k \ge \sum_{i=1}^{n}(d_i - 1)$. In particular, if we look at the quotient

$$K[x_0, \dots, x_n]/(f_1, \dots, f_{n+1}),$$

then for $t \ge d_{n+1} + \sum_{i=1}^{n}(d_i - 1)$ the degree d graded piece has support in H. By Bézout the cosupport of $\mathscr{K}_n$ has length at most $\prod_{i=1}^{n} d_i$, thus $\mathscr{R}^{\prod_{i=1}^{n} d_i}$ is contained in $\mathscr{K}_n$. Since $\prod_{i=1}^{n} d_i \ge d_{n+1} + \sum_{i=1}^{n}(d_i - 1)$ we see that

$$\mathscr{R}^{\prod_{i=1}^{n} d_i} \in (f_1, \dots, f_{n+1}).$$

This is what we wanted to prove.

3.10. *Case* $k_i > 0$ *for some* $i \le (n-1)$. By definition $\mathscr{R}^{a_n}$ annihilates $\mathscr{I}_n/(f_1, \dots, f_n)$ and therefore it also annihilates

$$(\mathscr{I}_n, f_{n+1})/(f_1, \dots, f_n, f_{n+1}).$$

On the other hand by 2.6 we know that $\mathscr{R}^{\deg Z_n + d_{n+1}}$ is contained in $(\mathscr{I}_n, f_{n+1})$. Therefore

$$\mathscr{R}^{a_n + \deg Z_n + d_{n+1}} \in (f_1, \dots, f_{n+1}).$$

On the other hand, by 3.7 we have

$$\begin{aligned} a_n + \deg Z_n + d_{n+1} &\le \sum_{i=1}^{n} k_i \cdot \frac{3^{n-i}+1}{2} + \prod_{i=1}^{n} d_i - \sum_{i=1}^{n} k_i \cdot 3^{n-i-1} d_n + d_{n+1} \\ &\le \prod_{i=1}^{n} d_i , \end{aligned}$$

unless $k_1 = \cdots = k_{n-2} = 0$, $k_{n-1} = 1$ or 2.

Thus except for these special cases the required inequality is again proved.

3.11. *Remaining cases.* In the remaining cases the hypersurfaces $F_1, \dots, F_n$ intersect in a curve C of degree c and in finitely many other points. Thus $\deg Z_n = \prod_{i=1}^n d_i - cd_n - \deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_n$, and so, as in 3.10, we need the inequality

$$\deg Z_n + 2k_{n-1} + k_n + d_{n+1} \leq \prod_{i=1}^n d_i .$$

This is clearly satisfied if $c \geq 3$. Hence the remaining cases are when C has degree 1 or 2. In this case we are also done if by accident

$$\deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_n \geq k_n + 2 .$$

This indeed will be the case by excess intersection theory. Since C has degree at most two, it is a local complete intersection curve and so we can use 9.1.1 in [F] to conclude that

$$F_1 \cap \cdots \cap F_n = C \cup \left(\prod_{i=1}^n d_i - \left(\sum_{i=1}^n d_i - n - 1 \right) c - 2\chi(\mathscr{O}_C) \right) \text{ points},$$

(some possibly embedded). From this we get that $\operatorname{Spec} \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_n$ has at least

$$\left(\sum_{i=1}^n d_i - n - 1 \right) c + 2\chi(\mathscr{O}_C) - cd_n$$

points as a subscheme of $\mathscr{O}_C$. Therefore

$$\deg \mathscr{O}_{\mathbb{P}^n}/\mathscr{K}_n - k_n \geq \left(\sum_{i=1}^n d_i - n - 1 \right) c + 2\chi(\mathscr{O}_C) - cd_n - c \geq 2 ,$$

provided $n \geq 3$. We are left with the case when $n = 2$. Here the common curve of intersection becomes a common irreducible factor for the f_i and so we are in the reducible case which can be treated by reducing it to a simpler case. This completes the proof of 1.5.

3.12. *The case of equality.* The preceeding argument shows that if $\mathscr{R}^{N(n, d_1, \dots, d_k)-1}$ is not contained in $(f_1, \dots, f_k)$ then we have either the case of 2.4 (which we want) or one of the cases treated in 3.11. In this latter case it is again clear that only some very special cases can give equality. These can be treated by ad hoc methods that are not worth writing down in detail.

4. Proof of Proposition 1.10

It is clear that 1.10 is local around the points at infinity. If we change coordinates to center around a point at infinity and pick x_1 as the equation of the

hyperplane at infinity then 1.10 is equivalent to the following:

4.1. **Proposition.** *Let $f_1, \dots, f_k$ be polynomials in $\mathbb{C}^n$ such that near the origin their common zero set is contained in $(x_1 = 0)$. We may assume that $k \le n$. Let $d_i = \deg f_i$. Then*

$$\max_i |f_i(x)| \ge \text{constant} \cdot |x_1|^{\prod d_i},$$

near the origin.

Proof. We consider the ideal $\mathcal{J}$ generated by the f_i and we blow it up. Taking the closure we get an algebraic variety $\Gamma \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$. Let p, respectively q, be the projections of Γ to $\mathbb{P}^n$, respectively $\mathbb{P}^{n-1}$. By definition of the blow-up, $p^*\mathcal{J} \subset \mathcal{O}_\Gamma$ is locally free above the origin and it is generated by the sections $p^* f_i$.

Let $E \subset \Gamma$ be any p-exceptional divisor such that the origin is contained in $p(E)$. Assume that we can prove the following:

$\triangle$: *Some $p^* f_i$ vanishes along E with multiplicity at most $\prod d_i$ (as a local section of $\mathcal{O}_\Gamma$).*

Since $p(E)$ is contained in $(x_1 = 0)$, $p^* x_1$ vanishes along E, thus $p^* x_1^{\prod d_i}$ vanishes along every exceptional divisor with a multiplicity at least as large as some generating section of $p^*\mathcal{J}$. Therefore we can cover Γ above the origin with open sets such that within each subset we have

$$p^* x_1^{\prod d_i} = p^* f_i \cdot (\text{a regular function}).$$

Since p^* does not change the value of a function, this implies 1.10.

Thus we have to prove $\triangle$. If $p(E)$ has positive dimension then a general affine $\mathbb{C}^{n-1}$ inside $\mathbb{C}^n$ intersects $p(E)$ and we can test $\triangle$ in one dimension less. Thus we are done by induction. Now we have to treat the main case when $p(E) =$ the origin.

Let $(y_1 : \cdots : y_n)$ be coordinates on $\mathbb{P}^{n-1}$. Let furthermore B, respectively A, be the cohomology classes of $q^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, respectively $p^*\mathcal{O}_{\mathbb{P}^n}(1)$, on $\mathbb{P}^{n-1} \times \mathbb{P}^n$. The multiplicity of vanishing of f_1 along E is bounded by

$$[(f_1 = 0)] \cdot [\Gamma] \cdot B^{n-1},$$

where [ ] denotes the cohomology class.

Γ is defined by equations $y_i f_j - y_j f_i = 0$ (and maybe some others). Thus $(f_1 = 0) \cap \Gamma$ is defined by equations $y_1 f_j = 0$ and $f_1 = 0$. Since $\mathbb{P}^{n-1} \times \mathbb{P}^n$ is homogeneous, effective cycles intersect nonnegatively. Therefore

$$[(f_1 = 0)] \cdot [\Gamma] \cdot B^{n-1} \le d_1 A \cdot \prod_{i=2}^{n} (B + d_i A) \cdot B^{n-1} = \prod_{i=1}^{n} d_i.$$

This proves $\triangle$ and thereby 1.10.

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