

LIMITS OF WEIGHT SPACES, LUSZTIG'S q -ANALOGS,
 AND FIBERINGS OF ADJOINT ORBITS

RANEE KATHRYN BRYLINSKI

1. INTRODUCTION

Let G be a connected complex semisimple algebraic group, and T a maximal torus inside a Borel subgroup B , with \mathfrak{g} , \mathfrak{t} , and \mathfrak{b} their Lie algebras. Let V be a representation in the category \mathcal{O} for \mathfrak{g} . The \mathfrak{t} -decomposition $V = \bigoplus_{\mu \in \mathfrak{t}^*} V^\mu$ of V into a direct sum of finite-dimensional weight spaces is central in the representation theory of \mathfrak{g} .

In this paper, we introduce on weight spaces a new structure, the principal filtration $J_e(V^\mu)$, where e is a principal nilpotent in \mathfrak{g} chosen to be compatible with \mathfrak{t} ; for example, e can be the sum of the simple root vectors relative to $(\mathfrak{t}, \mathfrak{b})$. This filtration is constructed in a very simple way by taking $J_e^p(V^\mu)$ to be the space of vectors annihilated by the $(p + 1)$ th power of e , for $p \geq 0$. Our approach is motivated by Kostant's fundamental work [K1, K2] on actions of the principal TDS (three-dimensional subalgebra) and coordinate rings of regular adjoint orbits.

In Theorem 3.4, we give a new description, in terms of the dimension jumps of the principal filtration of the weight space, of Lusztig's [L] q -analog $m_\lambda^\mu(q)$ of dominant μ -weight multiplicity in a finite-dimensional irreducible \mathfrak{g} -representation V_λ . The polynomial $m_\lambda^\mu(q)$ was defined algebraically as an alternating sum over the Weyl group, through a q -analog of Kostant's weight multiplicity formula. We prove that $m_\lambda^\mu(q)$ is equal to the *jump polynomial*

$$r_\lambda^\mu(q) = \sum_{p \geq 0} \dim(J_e^p(V_\lambda^\mu) / J_e^{p-1}(V_\lambda^\mu)) q^p$$

of the principal filtration of V_λ^μ . This means that we compute Lusztig's q -analog directly from the data of two smaller actions on the \mathfrak{g} -representation: the actions of a Cartan subalgebra and of a (compatibly chosen) TDS. This result carries a broad hypothesis: \mathfrak{g} must have all components of classical type, or μ must be regular. (We expect Theorem 3.4 is true without these extra conditions.)

Received by the editors October 6, 1988 and, in revised form, January 18, 1989.
 1980 *Mathematics Subject Classification* (1985 Revision). Primary 20G05, 17B10, 22E46, 14M17.

Research supported in part by a NATO Postdoctoral Fellowship.

The theory in §2 gives another description of this result: the coefficients of Lusztig's polynomials are the dimensions of the eigenspaces of a certain regular semisimple element h of \mathfrak{g} (determined modulo \mathfrak{g}^e by e) on a special subspace $\lim_e V^\mu$ of $V^{\mathfrak{g}^e}$. For, associated to the principal filtration of V^μ is the *principal limit* $\lim_e V^\mu$, the sum $\sum_{p \geq 0} e^p \cdot J_e^p(V^\mu)$ inside V . The sum is direct, and $\lim_e V^\mu$ is annihilated, not just by the action of e , but also by the action of the full centralizer \mathfrak{g}^e of e . In addition, $\lim_e V^\mu$ carries an intrinsic grading, by the eigenspaces of any (appropriately scaled) regular semisimple element h lying in a TDS containing e ; for example, h can be the vector in \mathfrak{t} where all simple roots, relative to $(\mathfrak{t}, \mathfrak{b})$, take value 1. The dimension jump $\dim(J_e^p(V_\lambda^\mu)/J_e^{p-1}(V_\lambda^\mu))$ is equal to the dimension of the $(p + \mu(h))$ -eigenspace of h in $\lim_e V^\mu$.

Lusztig's polynomials are deep invariants of representations, having far reaching applications. They equal certain Kazhdan-Lusztig polynomials for the affine Weyl group which compute certain local intersection cohomology groups for Schubert varieties in infinite dimensional flag varieties [L, Kt, K-L]. They compute a certain group scheme cohomology with twisted coefficients in characteristic p (see [A-J], and, in the nontwisted case, [F-P]). Other sorts of results are discussed in the author's earlier papers [G1] and [G2].

The connection between principal filtrations and Lusztig's polynomials comes from the geometric formulations of each, in terms of twisted functions on a regular semisimple adjoint orbit Q and twisted functions on the cotangent bundle $\underline{T}_{G/B}^*$ of the flag variety. The first geometric formulation is found in Theorem 5.8, based on the notion of "fiber degree" discussed below. The second is found in Lemma 6.1, based on ideas of Hesselink in [Hs1]. The proof of Theorem 3.4 then requires a comparison of these two types of functions.

The main work here is accomplished by studying the geometry of Q fibered over G/B by a G -equivariant projection π . We observe that the fiber of π over an arbitrary point $x \in G/B$ is an affine space, equipped with a natural linear translation action of the cotangent space to G/B at x . This picture is a sort of local version of the "associated cone" construction of Borho and Kraft [B-K]. The pair $(Q, \underline{T}_{G/B}^*)$ forms an *affine bundle* over G/B (Proposition 5.3). In §4, we present an abstract theory of affine bundles.

We then take a homogeneous line bundle $\underline{L}^{-\mu}$ on G/B , and consider twisted functions on Q and $\underline{T}_{G/B}^*$, i.e., functions with values in the respective pullbacks of the line bundle. The translation action of the cotangent bundle leads to a natural notion of *fiber degree* of twisted functions on Q , and also to a degeneration mapping sending twisted functions of fiber degree p on Q to homogeneous degree p twisted functions on the cotangent bundle (Theorem 5.5). With this, and some appropriate vanishing of higher cohomology of sheaves on G/B (Hypothesis 6.2), we make the necessary comparison in Theorem 6.4. The extra assumptions in Theorem 3.4. insure that the needed vanishing is known; we use the results of Andersen and Jantzen and of Griffiths (see Theorem 6.3).

Finally, let us explain the relation with generalized exponents. For the case of zero-weight multiplicity, the polynomials $m_\lambda^0(q)$ were first constructed, independently, by Hesselink [Hs1] and Peterson [Pt]. They showed that, for each finite-dimensional irreducible representation V_λ of the adjoint group, and each $p \geq 0$, the coefficient of q^p in $m_\lambda^0(q)$ is equal to the multiplicity of V_λ in the space $R^p(N)$ of degree p homogeneous regular functions on the cone N of nilpotents in \mathfrak{g} . The latter is the multiplicity $k_\lambda(p)$ of p in Kostant's set of *generalized exponents* of V_λ . On the other hand, Kostant showed [K2] that $k_\lambda(p)$ is also equal to the multiplicity of V_λ in $R^{\leq p}(Q)/R^{\leq p-1}(Q)$, where $R^{\leq p}(Q)$ is the space of regular functions on Q obtained by restriction of polynomial functions on \mathfrak{g} of degree at most p . Out of our study of Q fibered over G/B , comes the construction (Definition 5.10) of μ -twisted *generalized exponents* of any irreducible G -representation V , for any weight μ . In this way, we generalize both Kostant's theory (cf. Lemma 5.4) and the Hesselink-Peterson result (cf. Corollary 5.11).

The author is grateful for the support of a NATO Postdoctoral Research Fellowship and the hospitality of University of Paris VI and the I.H.E.S. during 1985-86, in which time the first stage of this work was carried out. She also thanks R. Rentschler for his comments on an earlier version of this paper, preprinted under her former name, R. K. Gupta.

2. THE PRINCIPAL FILTRATION OF A WEIGHT SPACE

Throughout the paper, we retain the notations introduced in §1. The pair $(\mathfrak{t}, \mathfrak{b})$ determines the cone \mathcal{P}^{++} of dominant integral weights in the lattice \mathcal{P} of integral weights. Let ρ be the half-sum of the positive roots; its "dual" is the vector $h_\rho \in \mathfrak{t}$ on which all simple roots take value equal to 1.

G^{ad} will denote the adjoint group. \mathcal{U} is the universal enveloping algebra of \mathfrak{g} . The *Weyl group* is $W = N(T)/T$. A *representation* of an algebraic group will always mean a rational representation on a complex vector space. A *representation* of \mathfrak{g} will mean one in the Bernstein-Gelfand-Gelfand [B-G-G] "category \mathcal{O} ," or the differential of a locally finite G -representation.

Call $x \in \mathfrak{g}$ *regular* iff the adjoint orbit $\text{ad}_G x$ has maximal dimension, or equivalently, iff the Lie centralizer \mathfrak{g}^x has dimension equal to the rank ℓ of \mathfrak{g} . The regular nilpotents, also called *principal nilpotents*, form a dense adjoint orbit N° in the cone N of all nilpotents. Each principal nilpotent e lies in a unique Borel subalgebra $\mathfrak{b}^{(e)}$ of \mathfrak{g} .

Definition 2.1. Suppose V is a \mathfrak{g} -representation and e is a principal nilpotent in \mathfrak{g} . For any subspace U of V , the e -filtration of U is the finite filtration $J_e(U): 0 \subseteq J_e^0(U) \subseteq J_e^1(U) \subseteq \dots$, where $J_e^p(U) = \{u \in U \mid e^{p+1} \cdot u = 0\}$, $p \geq 0$. The e -limit of U is the subspace $\lim_e U := \sum_{p \geq 0} e^p \cdot J_e^p(U)$ of V .

The associated graded space to U is then $\text{gr}_e U := \bigoplus_{p \geq 0} J_e^p(U) / J_e^{p-1}(U)$; take $J_e^{-1}(U) := 0$ always. Clearly, $\lim_e U$ lies in V^e , and the actions of powers of e give a natural projection $\text{gr}_e U \rightarrow \lim_e U$.

To study the e -filtration and e -limit of a weight space of V , we will take e in a “good” position relative to \mathfrak{t} .

Definition 2.2. A principal nilpotent e in \mathfrak{g} is \mathfrak{t} -compatible iff $[h, e] = e$ for some $h \in \mathfrak{t}$. Call such a pair (e, h) a special pair (relative to \mathfrak{t}).

Special pairs exist, by TDS theory. For, let (e_+, h_0, e_-) be a principal S -triple, i.e., a standard basis of a principal three-dimensional subalgebra of \mathfrak{g} , so a nonzero triple in \mathfrak{g} with $e_+ \in N^\circ$ satisfying $[h_0, e_+] = e_+$, $[h_0, e_-] = -e_-$, and $[e_+, e_-] = 2h_0$. Such exist by the Jacobson-Morozov Theorem. By conjugating the triple as needed, we may assume $h_0 \in \mathfrak{t}$. Then (e_+, h_0) is a special pair. In fact, every special pair (e, h) arises in this way. For, let (e, h_0, e_-) be a principal S -triple through e . Then $h \in h_0 + \mathfrak{g}^e = \text{ad}_{G^e} h_0$, and hence h is regular semisimple, and some G^e -conjugate of (e, h_0, e_-) is equal to (e, h, e'_-) , for some e'_- . In a similar way we prove, using TDS theory as developed in [K1]:

Lemma 2.3. (1) All \mathfrak{t} -compatible principal nilpotents lie in one of the $|W|$ Borel subalgebras containing \mathfrak{t} . $N(T)$ operates simply transitively on the set of special pairs.

(2) A pair $(e, h) \in \mathfrak{b} \times \mathfrak{t}$ is special iff $h = h_\rho$ and $e = c_1 X_{\alpha_1} + \dots + c_\ell X_{\alpha_\ell}$, where all $c_i \neq 0$, and $X_{\alpha_1}, \dots, X_{\alpha_\ell}$ are the simple root vectors relative to $(\mathfrak{t}, \mathfrak{b})$.

For each weight μ , let W^μ be the Weyl group stabilizer of μ . The lemma easily gives

Proposition-Definition 2.4. Let μ be an integral weight of a \mathfrak{g} -representation V . Then

(1) As e varies over the set of \mathfrak{t} -compatible principal nilpotents inside \mathfrak{b} , the e -filtrations $J_e(V^\mu)$ coincide. We call this the principal filtration (relative to $(\mathfrak{t}, \mathfrak{b})$) of V^μ .

(2) As e varies over the set of \mathfrak{t} -compatible principal nilpotents inside Borel subalgebras containing \mathfrak{t} relative to which μ is dominant, the e -filtrations $J_e(V^\mu)$ are all W^μ -translates. In particular, as e varies over the set of all \mathfrak{t} -compatible principal nilpotents, the e -filtrations of V^0 are all W -translates.

The principal filtration of a weight space in a highest weight representation V_λ is a measure of distance from the highest weight space. In particular, computing the 0th filtration piece, we get $J_e^0(V_\lambda^\lambda) = V_\lambda^\lambda$, while $J_e^0(V_\lambda^\mu) = 0$ for $\mu \neq \lambda$.

Lemma 2.5. Let (e, h) be a special pair. Suppose h acts as the constant c on a subspace A of V . Then the natural projection $\text{gr}_e A \rightarrow \lim_e A$ is an isomorphism, and $\lim_e A = \bigoplus_{p \geq 0} e^p \cdot J_e^p(A)$.

Proof. In \mathcal{U} we find $he^p = pe^p + e^p h$, so it follows that h acts as the constant $c + p$ on $e^p \cdot A$.

Surprisingly, we have

Proposition 2.6. *Let V be a \mathfrak{g} -representation, and let e be any \mathfrak{t} -compatible principal nilpotent in \mathfrak{g} . Then for every weight μ of V , $\lim_e V^\mu$ lies inside $V^{\mathfrak{g}^e}$.*

Proof. The adjoint action of \mathfrak{g} on itself extends to a left action ad of \mathcal{U} on itself. Write $x * y$ for $\text{ad}_x y$, for any $x, y \in \mathcal{U}$.

Since $\lim_e \mathfrak{t} = \mathfrak{g}^e$ (combine the obvious inclusion with the dimension equality), we see that \mathfrak{g}^e has a basis consisting of elements of the form $z = e^k * s = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} e^i s e^{k-i}$, where $s \in \mathfrak{t}$ and $k \geq 1$ satisfy $e^{k+1} * s = 0$.

On the other hand, $\lim_e V^\mu$ has a basis consisting of vectors of the form $e^p \cdot v$, where $v \in V^\mu$ and $p \geq 0$ satisfy $e^{p+1} \cdot v = 0$. So to show that \mathfrak{g}^e annihilates $\lim_e V^\mu$, it is enough to show that for s, k, z, v , and p chosen in these ways, z annihilates $e^p \cdot v$. The above equations give us $(ze^p) \cdot v = ((e^k * s)e^p) \cdot v = (e^k s e^p) \cdot v$. To prove now that $(e^k s e^p) \cdot v$ vanishes, we consider a sequence of vectors in V : $t^i := (e^i s e^{k+p-i}) \cdot v$, $i = 0, \dots, k + p$.

We aim to show $t^k = 0$. Let us use the “symbolic method”: consider “ t^i ” as the i th power of an indeterminate t , modulo some space L of linear relations in $\mathbb{C}[t]$. We have $t^{k+p} = (e^{k+p} s) \cdot v = 0$ (as $s \cdot v$ is a multiple of v and $k \geq 1$), so the case $p = 0$ is settled, and we may assume $p \geq 1$. Since $e^{p+1} \cdot v = 0$, we have $1 = t = \dots = t^{k-1} = 0$ in $\mathbb{C}[t]/L$. But also $e^{k+1} * s = 0$, so we get, for $j = 1, \dots, p$,

$$0 = (e^{k+j} * s) \cdot e^{p-j} \cdot v = \sum_{i=0}^{k+j} (-1)^{k+j-i} \binom{k+j}{i} (e^i s e^{k+p-i}) \cdot v = (t-1)^{k+j}.$$

As t^k is a linear combination of the polynomials $1, \dots, t^{k-1}, (t-1)^{k+1}, \dots, (t-1)^{k+p}, t^{k+p}$, we conclude that t^k also vanishes.

Kostant [K2] defined the *generalized exponents* of a representation V of G^{ad} to be the eigenvalues $m_i(V)$, $i = 1, \dots, |V^{\mathfrak{g}^e}|$ (counted with multiplicity) of h_0 on the space $V^{\mathfrak{g}^e}$, for e and h_0 the first two members of a principal S -triple. The independence of these eigenvalues on the particular choices of e and h_0 followed from his geometrical interpretation (see §1).

As a consequence of Proposition 2.6, the generalized exponents can be computed from the principal filtration of the zero-weight space.

Corollary 2.7. *Suppose V is a representation of G^{ad} , and e is as in Proposition 2.6. Then $\lim_e V^0 = V^{\mathfrak{g}^e}$, and for each integer $p \geq 0$, the multiplicity of p as a generalized exponent of V is equal to the dimension jump $|J_e^p(V^0)| - |J_e^{p-1}(V^0)|$.*

Proof. The dimension equality [K2] $|V^0| = |V^{\mathfrak{g}^e}|$, together with the inclusion of Proposition 2.6, implies that $V^{\mathfrak{g}^e}$ is the e -limit of V^0 . Combining with Lemma 2.5, we get this description of generalized exponents.

TDS theory, and our notion of “special pair,” distinguishes a particular adjoint orbit.

Definition 2.8. Q_{TDS} is the regular semisimple adjoint orbit through h_ρ .

So Q_{TDS} consists of semisimple elements occurring in principal S -triples.

3. JUMP POLYNOMIALS AND LUSZTIG’S q -ANALOGS

We turn now to measuring the sizes of the principal filtration components for a weight space. Let e be a \mathfrak{t} -compatible principal nilpotent in \mathfrak{b} . For each weight μ of a \mathfrak{g} -representation V , define the *jump polynomial* of the principal filtration $J_e(V^\mu)$ to be

$$(3.1) \quad r_V^\mu(q) := \sum_{p \geq 0} (|J_e^p(V^\mu)| - |J_e^{p-1}(V^\mu)|)q^p.$$

Write $r_\lambda^\mu(q)$ when $V = V_\lambda$.

Alternatively, we may describe the jump polynomial in terms of a natural grading on the e -limit $\lim_e V^\mu$. V^e is the space of highest weight vectors in V under the action of the principal \mathfrak{sl}_2 subalgebra containing e and h_ρ . It follows that the eigenvalue p of an h_ρ -eigenvector $v \in V^e$ is $p = (d - 1)/2$, where d is the dimension of the irreducible \mathfrak{sl}_2 -subrepresentation spanned by v . Thus V^e and all its h_ρ -stable subspaces U , including $V^{\mathfrak{g}^e}$ and $\lim_e V^\mu$, are graded over the nonnegative half-integers by their eigenspaces for h_ρ . We define the *Hilbert series relative to e* of such a (finite-dimensional) space U to be $HS_e(U) := \sum_{p \geq 0} [U]^{h_\rho=p/2} q^{p/2}$. Note that, if we vary our choice of \mathfrak{t} , then h_ρ varies only modulo \mathfrak{g}^e , so that the h_ρ -stable subspaces of V^e and their Hilbert series are determined purely by e .

Let $(,)$ be the usual form on \mathfrak{t}^* , the dual of the restriction of the Cartan-Killing form on \mathfrak{g} to \mathfrak{t} .

Lemma 3.2. For each weight space V^μ , $r_V^\mu(q) = q^{-(\mu, \rho)} HS_e(\lim_e V^\mu)$.

Proof. Immediate from Lemmas 2.3(2) and 2.5.

Clearly, these polynomials $r_V^\mu(q)$ and $HS_e(\lim_e V^\mu)$ have nonnegative integral coefficients and take value $|V^\mu|$ at $q = 1$. They are “ q -analogs” of weight multiplicity.

Lusztig [L, (9.4)] introduced a fundamental q -analog

$$(3.3) \quad m_\lambda^\mu(q) := \sum_{w \in W} \text{sgn}(w) \wp_q(w(\lambda + \rho) - \mu - \rho),$$

of μ -weight multiplicity in V_λ , $\lambda \in \mathcal{P}^{++}$, for μ dominant. Here \wp_q is the q -analog of Kostant's partition function given by

$$\prod_{\phi > 0} (1 - qe^\phi)^{-1} = \sum_{\pi \in \mathcal{P}} \wp_q(\pi) e^\pi;$$

hence (3.3) is a q -analogue of Kostant's weight multiplicity formula and $m_\lambda^\mu(1) = |V_\lambda^\mu|$ (regardless of the dominance of μ). Though it is not at all apparent from the definition, these polynomials have nonnegative coefficients when μ is dominant, as explained in §1. This nonnegativity fails for nondominant μ .

Our main result relates Lusztig's polynomials to the principal filtrations.

Theorem 3.4. *Let μ be a dominant weight in an irreducible finite-dimensional representation V_λ of \mathfrak{g} . Assume that all components of \mathfrak{g} are of classical type, or that μ is regular. Then, the jump polynomial $r_\lambda^\mu(q)$ of the principal filtration (relative to (t, b)) of V_λ^μ is equal to Lusztig's polynomial $m_\lambda^\mu(q)$.*

The proof is given in §6 (see Theorem 6.4).

Remark 3.5. We expect that the theorem is valid without the extra assumptions of the second sentence. Indeed, we conjecture that the equality $m_V^\mu(q) = r_V^\mu(q)$ holds more generally, where $m_V^\mu(q)$ is defined in the natural way for any \mathfrak{g} -representation V in the category \mathcal{O} , by expressing the character of V as a linear combination of characters of Verma modules.

4. AFFINE BUNDLES

In this section, we work over a connected algebraic base variety X (over \mathbb{C}) with structure sheaf \mathcal{O}_X . All sheaves will be sheaves of \mathcal{O}_X -modules, and all linear maps of these sheaves will be \mathcal{O}_X -linear. We write $R(V)$ for the coordinate ring of regular functions on an algebraic variety V .

An affine space of dimension n is a pair (A, M) of an algebraic variety A and a vector space M , both n -dimensional, where M acts freely on A . We often regard just A as the affine space, and M , together with its linear action, as the linear structure of A ; then we write the action as addition. An affine linear function f on A is a regular function f satisfying $f(cv + a_0) = cf(v + a_0) + (1 - c)f(a_0)$, for all $c \in \mathbb{C}$, $v \in M$, $a_0 \in A$.

The familiar model is that M is a vector subspace of some vector space L , and A is a translate of M inside L . Indeed, any given affine space (A, M) can be realized in this way, by taking L to be $A^1 := \mathbb{C}a_0 \oplus M$, for an arbitrary $a_0 \in A$. Then A and M sit as hyperplanes in A^1 , M is a vector space acting on A by addition in A^1 , and we have the canonical projection, collapsing A to 1, of A^1 to \mathbb{C} from M . With respect to these properties, A^1 is uniquely determined up to linear isomorphism; we call A^1 the associated vector space to (A, M) . Intrinsically, A^1 is obtained as the dual to the space of affine linear functions on A , together with the natural embeddings of A and M .

As A has no origin, we have no notion of a homogeneous function on A . However, we do have two related notions. Choose any $a_0 \in A$ as pseudo-origin. Given $f \in R(A)$, manufacture $f^\# \in R(M)$, by $f^\#(v) := f(v + a_0)$, all $v \in M$. Then the *affine degree* $\text{deg}_{\text{aff}}(f)$, or just $\text{deg}(f)$, is the degree of $f^\#$ as a polynomial function on M . The *symbol* $f^+ \in R^{\text{deg}(f)}(M)$ of f is the top homogeneous component of $f^\#$. Easily, the affine degree and symbol are independent of the choice of a_0 . We can also determine them by computing a limit, for, if $p = \text{deg}(f)$, then

$$(4.1) \quad f^+(v) = \lim_{t \rightarrow \infty} f(a_0 + tv)/t^p,$$

for all $v \in M$. Moreover, $\text{deg}(f)$ is the smallest value of p such that the limit exists.

Thus $R(A)$ is filtered by affine degree of functions. This filtration is also the one naturally inherited from $R(A^1)$; the space $R^p(A^1)$ of homogeneous functions of degree p on A^1 identifies via restriction with the space $R^{\leq p}(A)$ of functions of affine degree at most p on A , for all $p \geq 0$.

We view the symbol map ($f \mapsto f^+$) as a degeneration map $\Delta: R(A) \rightarrow R(M)$. Although Δ is nonlinear, it induces a graded algebra isomorphism $\text{gr}\Delta: \text{gr} R(A) \xrightarrow{\sim} R(M)$.

Before globalizing these notions, let us make an observation about affine degree. The degree of $f \in R(A)$ is the maximum of the degrees of the restrictions of f to the affine lines in A , and the degree of f on such a line $a + Cv$ depends on both the direction vector v and the position vector a . However, the following useful fact is verified immediately:

Lemma 4.2. *Let (A, M) be an affine space and f a regular function on A . Then the affine degree of f on parallel lines in A is the same, provided that f takes maximal degree on any one of these lines.*

An *affine bundle of rank n* is a pair $(\underline{A}, \underline{M})$ of a locally trivial algebraic fiber bundle \underline{A} and vector bundle \underline{M} over X , both with n -dimensional fibers, where \underline{M} acts freely over X on \underline{A} . Thus the pair $(\underline{A}_x, \underline{M}_x)$ of fibers over any point $x \in X$ forms an affine space, with associated vector space A_x^1 . Using the gluing data for \underline{A} , we construct the *associated vector bundle* \underline{A}^1 over X having fibers $\underline{A}_x^1 = A_x^1$, together with some canonical bundle maps over X : the inclusion $\underline{A} \rightarrow \underline{A}^1$, and the exact sequence $0 \rightarrow \underline{M} \rightarrow \underline{A}^1 \rightarrow \underline{C}_X \rightarrow 0$.

Example 4.3. Let X be the Grassmannian variety $G_{k,n}$ of k -dimensional vector subspaces in \mathbb{C}^n , with universal subbundle \underline{S} and universal quotient bundle \underline{Q} . Let \underline{M} be the cotangent bundle of X , then

$$\underline{M} = \underline{\text{Hom}}_X(\underline{Q}, \underline{S}) = \{(L, f) \in X \times \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \mid \text{Im}(f) \subseteq L \subseteq \text{Ker}(f)\}.$$

The dual Grassmannian to X is $\widehat{X} = G_{n-k,n}$. Let \underline{A} be the open subvariety of $X \times \widehat{X}$ consisting of pairs (L, P) of subspaces meeting only at the origin. Both \underline{M} and \underline{A} map to X by projection to the first factor.

Define an action $\varphi: \underline{M} \times_X \underline{A} \rightarrow \underline{A}$ as follows. Given $(L, f) \in \underline{M}$ and $(L, P) \in \underline{A}$, form the composition $f': P \rightarrow \mathbf{C}^n \rightarrow \mathbf{C}^n/L \rightarrow L$, where the last map is the one induced by f . Then set $N = \{p + f'(p) \mid p \in P\}$ (so $N \cap L = \{0\}$), and define $\varphi((L, f), (L, P)) = (L, N)$. The action is free, since the natural map $P \rightarrow \mathbf{C}^n/L$ is an isomorphism. Thus $(\underline{A}, \underline{M})$ becomes an affine bundle over X .

Let α , β , and α^1 be the projection maps of \underline{A} , \underline{M} , and \underline{A}^1 to X , and let \mathcal{M} and \mathcal{A}^1 be the sheaves of germs of sections over X of \underline{M} and \underline{A}^1 .

For each open set U of X , define the *fiber degree* $\deg_{\text{fib}}(f)$, or just $\deg(f)$, of a regular function f on $\alpha^{-1}(U)$ to be the maximum of the affine degrees of the restrictions of f to the fibers of α over U . Then $(\alpha_* \mathcal{O}_{\underline{A}})(U)$ is filtered by the spaces $\Gamma^{\leq p}(U, \alpha_* \mathcal{O}_{\underline{A}})$ of sections of fiber degree at most p . These sets give the data of a sub- \mathcal{O}_X -module $(\alpha_* \mathcal{O}_{\underline{A}})^{\leq p}$ of $\alpha_* \mathcal{O}_{\underline{A}}$, for each $p \geq 0$.

Note that the canonical gradings of $\beta_* \mathcal{O}_{\underline{M}} = S(\mathcal{M}^*)$ and $\alpha_*^1 \mathcal{O}_{\underline{A}^1} = S(\mathcal{A}^{1*})$ are fiber degree gradings, in exactly the same sense. Our fiber degree filtration of $\alpha_* \mathcal{O}_{\underline{A}}$ is the one inherited from $\alpha_*^1 \mathcal{O}_{\underline{A}^1}$; i.e., $S^p(\mathcal{A}^{1*})$ identifies with $(\alpha_* \mathcal{O}_{\underline{A}})^{\leq p}$.

The symbol maps $\Delta_x: R(\underline{A}_x) \rightarrow R(\underline{M}_x)$ on fibers patch together over X to give a (nonlinear) *symbol map* $\Delta: \alpha_* \mathcal{O}_{\underline{A}} \rightarrow \beta_* \mathcal{O}_{\underline{M}}$ of sheaves which induces a graded isomorphism $\text{gr} \Delta: \text{gr} \alpha_* \mathcal{O}_{\underline{A}} \xrightarrow{\sim} \beta_* \mathcal{O}_{\underline{M}}$ of \mathcal{O}_X -algebras.

These constructions go through to the case of twisted functions. Let \underline{F} be a line bundle on X , with $\underline{L} = \alpha^* \underline{F}$ its pullback to \underline{A} . Let \mathcal{F} and \mathcal{L} be the corresponding sheaves of sections on X and \underline{A} .

For, suppose s is a regular section of \underline{L} over an open set $\alpha^{-1}(U)$ of \underline{A} , and consider the restriction $s|_{\underline{A}_x}$ of s to a fiber \underline{A}_x of α over U . By projecting down from \underline{L} to \underline{F} the values of $s|_{\underline{A}_x}$, we obtain a regular function f (with values in \underline{F}_x) on the affine space \underline{A}_x . Call the affine degree of f the *affine degree of $s|_{\underline{A}_x}$* . As we did earlier, we then define the *fiber degree* $\deg_{\text{fib}}(s)$, or just $\deg(s)$, of the section s to be the maximum of the affine degrees of its restrictions to fibers of α over X . Again, for each $p \geq 0$, the space $\Gamma^{\leq p}(U, \alpha_* \mathcal{L})$ of sections of fiber degree at most p gives the data of a sub- \mathcal{O}_X -module $(\alpha_* \mathcal{L})^{\leq p}$ of $\alpha_* \mathcal{L}$.

In analogy to this fiber degree filtration of $\alpha_* \mathcal{L}$, we have fiber degree gradings $\beta_* \beta^* \mathcal{F} = \bigoplus_{p \geq 0} (\beta_* \beta^* \mathcal{F})^p$ and $\alpha_*^1 \alpha^{1*} \mathcal{F} = \bigoplus_{p \geq 0} (\alpha_*^1 \alpha^{1*} \mathcal{F})^p$. Again, $(\alpha_*^1 \alpha^{1*} \mathcal{F})^p$ identifies with $(\alpha_* \mathcal{L})^{\leq p}$.

These constructions are functorial, in that we have natural identifications $(\alpha_* \mathcal{L})^{\leq p} = \mathcal{F} \otimes (\alpha_* \mathcal{O}_{\underline{A}})^{\leq p}$, $(\beta_* \beta^* \mathcal{F})^p = \mathcal{F} \otimes S^p(\mathcal{M}^*)$, etc., of \mathcal{O}_X -modules.

Twisting Δ , we obtain the *twisted symbol map* $\Delta^{\mathcal{F}}: \alpha_* \mathcal{L} \rightarrow \beta_* \beta^* \mathcal{F}$, and

Theorem 4.4. *Let $(\underline{A}, \underline{M})$, with projections α and β , be an affine bundle over X . Let \mathcal{F} a line bundle over X , with $\mathcal{L} = \alpha^* \mathcal{F}$. Then $\alpha_* \mathcal{L}$ and $\beta_* \beta^* \mathcal{F}$ have, respectively, a natural fiber degree filtration and fiber degree grading by*

sheaves of \mathcal{O}_X -modules. The twisted symbol map $\Delta^{\mathcal{F}}$ induces a graded isomorphism $\text{gr} \Delta^{\mathcal{F}} : \text{gr} \alpha_* \mathcal{L} \xrightarrow{\sim} \beta_* \beta^* \mathcal{F}$ of modules over the \mathcal{O}_X -algebra $\text{gr} \alpha_* \mathcal{O}_{\underline{A}}$ (use $\text{gr} \Delta$ to make $\text{gr} \alpha_* \mathcal{O}_{\underline{A}}$ act on $\beta_* \beta^* \mathcal{F}$).

All our constructions and results, including the finiteness of our fiber degree, are verified by working over a finite open cover of X where \underline{A} and \underline{M} , and hence also \underline{A}^1 , trivialize. There are no difficulties in doing this; we omit the details.

Let us emphasize the local nature of the symbol map. Let U be an open set of X containing a point x . Suppose s is a regular section of \underline{L} over $\alpha^{-1}(U)$, with s of fiber degree p . Then the value of the symbol $s^+ = \Delta^{\mathcal{F}}(s) \in \Gamma^p(\beta^{-1}(U), \beta^* \mathcal{F})$ on the fiber \underline{M}_x is given, for all $v \in \underline{M}_x$ and any $a \in \underline{A}_x$, by the formula

$$(4.5) \quad s^+(v) = \lim_{t \rightarrow \infty} s(a + tv)/t^p,$$

provided that we make the following fiber identifications. We identify, by projection, all fibers \underline{L}_{a+tv} to the single fiber \underline{F}_x , which in turn identifies with its pullback $(\beta^* \underline{F})_v$. Thus (4.5) becomes a limit of points in $(\beta^* \underline{F})_v$.

In §6, we will compare functions, as opposed to *germs* of functions. Set $\Gamma^{\leq p}(\underline{A}, \mathcal{L}) := \Gamma^{\leq p}(X, \alpha_* \mathcal{L})$, and $\Gamma^p(\underline{M}, \beta^* \mathcal{F}) := \Gamma^p(X, \beta_* \beta^* \mathcal{F})$, for all $p \geq 0$.

Corollary 4.6. *On global sections, the twisted symbol map induces a graded linear injection $\text{gr} \Delta^{\mathcal{F}} : \text{gr} \Gamma(\underline{A}, \mathcal{L}) \rightarrow \Gamma(\underline{M}, \beta^* \mathcal{F})$ of modules over $\text{gr} R(\underline{A}) = R(\underline{M})$; $\Delta^{\mathcal{F}}_X$ is an isomorphism if, for all $p \geq 0$, $H^1(X, \mathcal{F} \otimes S^p(\mathcal{M}^*)) = 0$.*

Proof. The short exact sequence $0 \rightarrow (\alpha_* \mathcal{L})^{\leq p-1} \rightarrow (\alpha_* \mathcal{L})^{\leq p} \rightarrow (\beta_* \beta^* \mathcal{F})^p \rightarrow 0$ induces a long exact sequence on cohomology, for each $p \geq 0$. Hence, we get the injection $\text{gr} \Delta^{\mathcal{F}}_X$. But also, the vanishing of the first sheaf cohomology group of $\beta_* \beta^* \mathcal{F} = \mathcal{F} \otimes S^p(\mathcal{M}^*)$ for all $p \geq 0$ forces the vanishing of $H^1(X, (\alpha_* \mathcal{L})^{\leq p})$ for all $p \geq 0$.

Suppose now that an algebraic group P acts on X , and that $(\underline{A}, \underline{M})$ is an affine P -bundle, i.e., an affine bundle equipped with actions of P on \underline{A} and \underline{M} over X , in such a way that the defining morphism $\underline{M} \times_X \underline{A} \rightarrow \underline{A}$ is P -equivariant. Then all constructions above are P -equivariant, and all maps are P -linear.

We will apply these results in the next section to homogeneous bundles over quotient varieties. Each homogeneous space P/H , H any algebraic subgroup of P , has a canonical structure of P -variety. If H is a connected,

solvable subgroup, then for any H -variety F , the quotient space $P \times^H F = (P \times F)/\{(ph, y) = (p, h \cdot y)\}$ has a natural structure of algebraic variety, and is, canonically, a P -homogeneous locally trivial algebraic fiber bundle over P/H . See [S]. We easily obtain

Fact 4.7. Suppose H is a connected, solvable subgroup of an algebraic group P , and (A, M) is an affine H -space with associated vector space A^1 . Then $(P \times^H A, P \times^H M)$ is canonically an affine bundle over P/H with associated vector bundle $P \times^H A^1$.

5. GEOMETRY OF REGULAR SEMISIMPLE ADJOINT ORBITS
OVER THE FLAG VARIETY

From now on, we work over the flag variety $X = G/B$, the variety of Borel subalgebras of \mathfrak{g} . We write $x_{\mathfrak{b}_1}$ when we consider the Borel \mathfrak{b}_1 as a point of X . We take $x_{\mathfrak{b}}$ as the base point of X .

In this section, we determine an affine bundle structure on each regular semisimple adjoint orbit Q fibered over X , with the cotangent bundle of X supplying the linear structure. We then study the fiber degree of twisted functions with values in a homogeneous line bundle \underline{L}^μ over Q , and the symbol map. We explain in Lemma 5.6 how one can use the G -action to identify the fibers of \underline{L}^μ lying over a fiber of Q over X , in order to compute the fiber degree and symbol. In Theorem 5.8, we explicitly relate the principal filtration of weight spaces to the fiber degree filtration of twisted functions on our distinguished orbit Q_{TDS} from §2. At the end of the section, we then define “twisted generalized exponents” of representations, and obtain the geometric description of the jump polynomial of the principal filtration of a weight space.

Let (Q, π) be a *regular semisimple adjoint orbit over X* ; i.e., Q is a regular semisimple adjoint orbit and $\pi: Q \rightarrow X$ is a G -equivariant projection. (Given Q , the Weyl group operates simply transitively on the set of such G -projections.) The fibers of π are easy to describe.

Observation 5.1. Let \mathfrak{b}_1 be a Borel subalgebra with nilpotent radical \mathfrak{m}_1 . Then the fiber of π over the point $x = x_{\mathfrak{b}_1}$ is a linear coset $h_1 + \mathfrak{m}_1$ in \mathfrak{b}_1 . Consequently, the pair $(\pi^{-1}(x), \mathfrak{m}_1)$ is an affine space, and each Cartan subalgebra of \mathfrak{g} lying inside \mathfrak{b}_1 meets $\pi^{-1}(x)$ in a unique point.

In view of this, we take $h_0 := \mathfrak{t} \cap \pi^{-1}(x_{\mathfrak{b}})$ as the *base point of (Q, π) relative to $(\mathfrak{t}, \mathfrak{b})$* . Let \mathfrak{m} be the nilpotent radical of \mathfrak{b} .

To globalize Observation 5.1, we introduce the the cotangent bundle \underline{T}_X^* , with projection τ to X , for \underline{T}_X^* identifies canonically with $G \times^B \mathfrak{m}$, and hence with the variety of pairs (z_1, \mathfrak{b}_1) of a nilpotent vector z_1 inside a Borel

subalgebra \mathfrak{b}_1 ; write $\xi_{z_1, \mathfrak{b}_1}$ for the corresponding cotangent vector. Let \mathcal{F}_X be the tangent sheaf of X . We have the diagram

$$\begin{array}{ccc} Q & & \underline{T}_X^* \\ \pi \searrow & & \swarrow \tau \\ & X & \end{array}$$

The affine bundle theory will give us a map from functions on Q to functions on \underline{T}_X^* .

Construction 5.2. We define an action of \underline{T}_X^* on Q over X through addition in \mathfrak{g} in the following way. Given points $\xi_{z_1, \mathfrak{b}_1}$ and h_1 in the fibers of \underline{T}_X^* on Q over the point $x_{\mathfrak{b}_1}$ of X , set $\xi_{z_1, \mathfrak{b}_1} \cdot h_1 := h_1 + z_1$.

Proposition 5.3. *The pair (Q, \underline{T}_X^*) is an affine G -bundle over X with associated vector bundle $G \times^B (\mathfrak{C}h_0 + \mathfrak{m})$.*

Proof. Q is a G -homogeneous fiber bundle over X with typical fiber $h_0 + \mathfrak{m}$, since $Q = \text{ad}_G h_0 = G \times^B \text{ad}_B h_0 = G \times^B (h_0 + \mathfrak{m})$. But $(h_0 + \mathfrak{m}, \mathfrak{m})$ is an affine space, so that $(G \times^B (\mathfrak{C}h_0 + \mathfrak{m}), G \times^B \mathfrak{m})$ is an affine bundle by Fact 4.7. This action agrees with the one of Construction 5.2, so (Q, \underline{T}_X^*) identifies with this affine bundle.

As (Q, \underline{T}_X^*) is an affine bundle, the theory of the last section gives a fiber degree filtration on the coordinate ring of Q . This turns out to be the same as the usual one.

Lemma 5.4. *The fiber degree filtration $\{\Gamma^{\leq p}(Q, \mathcal{O}_Q)\}_{p \geq 0}$ of $R(Q)$ coincides with the filtration of $R(Q)$ inherited from $S(\mathfrak{g}^*)$.*

Proof. Let us fix a graded G -stable complement H in $S(\mathfrak{g}^*)$ to the ideal of the nullcone N ; for example H can be the space of “harmonic functions.” Then H is also a complement to the ideal of Q ; see [K2].

To show the two filtrations coincide, we need to show that, if $F \in H$ is a nonzero homogeneous function of degree p , then the restriction $f = F|_Q$ has fiber degree p . Clearly, f has fiber degree at most p . On the other hand, consider the restriction of f to an arbitrary fiber $\pi^{-1}(x_{\mathfrak{b}_1}) = h_1 + \mathfrak{m}_1$ of π . For any $e_1 \in \mathfrak{m}_1$, we have $f(h_1 + te_1) = F(h_1 + te_1) = F(h_1/t + e_1)t^p$. So f has affine degree p on the line $h_1 + Ce_1$ if $F(e_1) \neq 0$. So f has affine degree p on the fiber $\pi^{-1}(x_{\mathfrak{b}_1})$ if F does not vanish on the nilpotent radical of \mathfrak{b}_1 . This last condition certainly holds for some \mathfrak{b}_1 , so f has fiber degree p on Q .

We may write $R^{\leq p}(Q)$ for $\Gamma^{\leq p}(Q, \mathcal{O}_Q)$.

Fix an integral weight μ of \mathfrak{t} . Let \mathbf{C}^μ be the complex line equipped with the action of T or B through the character \exp^μ (extend trivially over the unipotent radical of B to get a character of B). Form the homogeneous line

bundle $\underline{F}^\mu := G \times^B \mathbf{C}^\mu$ over X . Let $\underline{L}^\mu = \pi^* \underline{F}^\mu$ be its pullback to Q ; all homogeneous line bundles over Q arise in this way. The base fibers are $\underline{F}_{x_b}^\mu$ and $\underline{L}_{h_0}^\mu$. The sheaves \mathcal{F}^μ and \mathcal{L}^μ of sections are the sheaves of “ μ -twisted functions” on X and Q .

The work of the last section gives

Theorem 5.5. *Let $\mathcal{L}^\mu = \pi^* \mathcal{F}^\mu$ be the sheaf of sections of a homogeneous line bundle on a regular semisimple adjoint orbit (Q, π) over X . Then the sheaf $\pi_* \mathcal{L}^\mu$ is filtered by the G -stable \mathcal{O}_X -submodules $(\pi_* \mathcal{L}^\mu)^{\leq p}$, the sheaves of sections of fiber degree at most p , for $p \geq 0$. There is a G -equivariant symbol map which induces a graded G -linear isomorphism $\text{gr} \Delta^\mu : \text{gr} \pi_* \mathcal{L}^\mu \xrightarrow{\sim} \mathcal{F}^\mu \otimes S(\mathcal{F}_X)$ of modules over the \mathcal{O}_X -algebra $\text{gr} \pi_* \mathcal{O}_Q \xrightarrow{\sim} S(\mathcal{F}_X)$.*

From (4.5), we have the local description of $\text{gr} \Delta^\mu$. In particular, if s is a local section of $\pi_* \mathcal{L}^\mu$ around $x = x_{b_1}$, then the affine degree of s restricted to any line $h_1 + \mathbf{C}e$ in the fiber $\pi^{-1}(x)$ is the degree of the \underline{F}_x^μ -valued polynomial $\pi_* s(h_1 + te)$ in t . On the other hand, we can compute the degree by making the fiber identifications in a different way, using the action of G . G acts on sections in the usual way: $(g \cdot s)(h) = g \cdot (s(\text{ad}_{g^{-1}} h))$.

Lemma 5.6. *Fix two distinct points h_1 and h'_1 in the fiber $\pi^{-1}(x)$, for $x = x_{b_1}$, and set $e := h'_1 - h_1$. Then there exists a unique unipotent element $u_t = u(h_1, h_1 + te)$ of G such that $\text{ad}_{u_t} h_1 = h_1 + te$. The affine degree of s restricted to the affine line $h_1 + \mathbf{C}e$ is equal to the degree of the $\underline{L}_{h_1}^\mu$ -valued polynomial $(u_t^{-1} \cdot s)(h_1)$ in t .*

Proof. Suppose U_1 is the unipotent radical of corresponding Borel subgroup B_1 . Since $\pi^{-1}(x)$ is a U_1 -orbit, the transporter in G from h_1 to $h_1 + te$ is a coset of the maximal torus G^{h_1} through an element $u \in U_1$. This determines $u_t = u$. Furthermore, by construction, U_1 acts trivially on \underline{F}_x^μ , so that the action of U_1 on the fibers of \underline{L}^μ over $\pi^{-1}(x)$ commutes with projection to \underline{F}^μ . So the degree of the \underline{F}_x^μ -valued polynomial $\pi_* s(h_1 + te)$ in t is equal to the degree of the $\underline{L}_{h_1}^\mu$ -valued polynomial $u_t^{-1} \cdot s(h_1 + te)$ in t .

Our next goal is to explain how to compute *algebraically* the geometric filtration of $\Gamma(Q, \mathcal{L}^\mu)$ by fiber degree. The computation will turn on the following fact:

Lemma 5.7. *Fix an affine line ℓ in Q , with principal nilpotent direction, lying inside some fiber of π ; so $\ell = h + \mathbf{C}e$, for some $h \in Q$ and principal nilpotent $e \in \pi(h)$. Then every global section s of \underline{L}^μ attains its fiber degree on some G -conjugate of ℓ .*

Proof. The section s attains its fiber degree on some fiber $A = \pi^{-1}(x_{b_1})$, and hence on some affine line ℓ_1 within A . Choose any point h_1 on ℓ_1 ; then $\ell_1 = h_1 + \mathbf{C}e_1$, for some nilpotent $e_1 \in \mathfrak{b}_1$ (cf. Observation 5.1). Since the

principal nilpotents are dense in the nilradical of \mathfrak{b}_1 , we may assume, with no loss, that e_1 is principal.

Now ℓ has a G -conjugate parallel to ℓ_1 , namely $\text{ad}_g \ell = \text{ad}_g h + Ce_1$, where we take $g \in G$ so that $\text{ad}_g e = e_1$. As a principal nilpotent uniquely determines the Borel subalgebra containing it, it must happen that $\text{ad}_g \mathfrak{b} = \mathfrak{b}_1$, so that $\text{ad}_g \ell$ also lies in A . We can now apply Lemma 4.2, to conclude that s attains its fiber degree on $\text{ad}_g \ell$.

Let V be a G -representation. The fiber degree of a G -linear map $\omega: V \rightarrow \Gamma(Q, \mathcal{L}^{-\mu})$ is the maximum of the fiber degrees of the sections $\omega_v, v \in V$. In this way, $\text{Hom}_G(V^*, \Gamma(Q, \mathcal{L}^{-\mu}))$ acquires a fiber degree filtration by the spaces $\text{Hom}_G(V^*, \Gamma^{\leq p}(Q, \mathcal{L}^{-\mu}))$, $p \geq 0$.

We fix a \mathbb{C} -linear identification of the base fiber $\underline{L}_{h_0}^\mu$ with \mathbb{C} , so that we can treat maps to the base fiber as ordinary functions. Then algebraic Frobenius reciprocity gives a natural linear isomorphism

$$ev_{h_0}: \text{Hom}_G(V^*, \Gamma(Q, \mathcal{L}^{-\mu})) \rightarrow V^\mu$$

by evaluation of sections at h_0 .

In Definition 2.8, we picked out a particular regular semisimple adjoint orbit, Q_{TDS} . Let $\sigma: Q_{\text{TDS}} \rightarrow X$ be the G -projection sending the base point h_ρ of Q_{TDS} to the base point x_b of X .

Our key result is

Theorem 5.8. *Keep the notation of Theorem 5.5, but assume also that $(Q, \pi) = (Q_{\text{TDS}}, \sigma)$, with basepoint $h_0 = h_\rho$. Let μ be a weight of a G -representation V . Then the evaluation isomorphism ev_{h_ρ} identifies the fiber degree filtration of $\text{Hom}_G(V^*, \Gamma(Q_{\text{TDS}}, \mathcal{L}^{-\mu}))$ with the principal filtration of V^μ .*

To establish the theorem, we prove a more general, but also more technical, result.

Proposition 5.9. *Keep the notation of Theorem 5.5, and let h_0 be the basepoint of (Q, π) relative to $(\mathfrak{t}, \mathfrak{b})$. Suppose e is any principal nilpotent in \mathfrak{b} , with μ and V as above. Set $u_t = u(h_0, h_0 + te)$ (notation from Lemma 5.6). Then, for any $\omega \in \text{Hom}_G(V^*, \Gamma(Q, \mathcal{L}^{-\mu}))$, the fiber degree of ω is equal to the degree of the V -valued polynomial $u_t \cdot ev_{h_0}(\omega)$ in t .*

Proof. Let d be the fiber degree of ω . The vector $v = ev_{h_0}(\omega) \in V$ identifies with the linear functional on V^* given by $v(\lambda) = \omega_\lambda(h_0), \lambda \in V^*$.

Fix the affine line $\ell_0 := h_0 + Ce$ in the base fiber $\pi^{-1}(x_b)$. For some $\lambda \in V^*$, the fiber degree of the section ω_λ is equal to d , and by Lemma 5.7, is attained on some G -conjugate $\text{ad}_g \ell_0$ of ℓ_0 . Hence, the section $\omega_{g^{-1}\lambda}$ attains fiber degree d on ℓ_0 itself. So d can be computed by evaluating sections just on ℓ_0 , as d is the maximum, as λ varies inside V^* , of the degrees of the $\underline{F}_{x_b}^{-\mu}$ -valued polynomials $\pi_* \omega_\lambda(h_0 + te)$ in t .

But, by Lemma 5.6, the degree of $\pi_*\omega_\lambda(h_0 + te)$ is equal to the degree of the $\underline{L}_{h_0}^{-\mu}$ -valued polynomial $(u_t^{-1} \cdot \omega_\lambda)(h_0)$ in t . As ω is G -linear, we find that $(u_t^{-1} \cdot \omega_\lambda)(h_0) = \omega_{u_t^{-1}\lambda}(h_0) = v(u_t^{-1} \cdot \lambda) = (u_t \cdot v)(\lambda)$. So d is equal to the maximum, as λ varies inside V^* , of the degrees of the polynomials $(u_t \cdot v)(\lambda)$. But this maximum is precisely the degree of the V -valued polynomial $u_t \cdot v$.

Proof of Theorem 5.8. Suppose $e \in \mathfrak{b}$ is a \mathfrak{t} -compatible principal nilpotent. Then $u_t = u(h_\rho, h_\rho + te)$ is equal to the exponential of $-te$. But for any vector v in a G -representation V , the degree of the V -valued polynomial $\exp(-te) \cdot v = (1 - te + t^2e^2 - \dots) \cdot v$ is clearly equal to the least value of p such that the $(p + 1)$ th power of e kills v .

Definition 5.10. The list of μ -twisted generalized exponents, measured on (Q, π) , of an irreducible G -representation V is the list of nonnegative integers where p occurs as many times as V^* occurs in $\Gamma^{\leq p}(Q, \mathcal{L}^{-\mu})/\Gamma^{\leq p-1}(Q, \mathcal{L}^{-\mu})$.

When $\mu = 0$, we recover Kostant's generalized exponents of adjoint group representations, regardless of our choice of (Q, π) , in view of Lemma 5.4. and the fact that $R^{\leq p}(Q)/R^{\leq p-1}(Q)$ is a self-dual representation (cf. §1).

Theorem 5.8 tells us that, for each integer $p \geq 0$, the multiplicity of p as a μ -twisted generalized exponent, measured on (Q_{TDS}, σ) , of V is equal to the dimension jump $|J_e^p(V^\mu)| - |J_e^{p-1}(V^\mu)|$. This exactly generalizes Corollary 2.7.

Let $\langle \cdot, \cdot \rangle = \dim \text{Hom}_G(\cdot, \cdot)$. Theorem 5.8 gives

Corollary 5.11. *Keep the notation of Theorem 5.8; let $V = V_\lambda$. Then*

$$r_\lambda^\mu(q) = \sum_{p \geq 0} \langle V_\lambda^*, \Gamma^{\leq p}(Q_{TDS}, \mathcal{L}^{-\mu})/\Gamma^{\leq p-1}(Q_{TDS}, \mathcal{L}^{-\mu}) \rangle q^p.$$

On the other hand,

Lemma 5.12. *Keep the notation of Theorem 5.5. Assume $H^1(X, \mathcal{F}^{-\mu} \otimes S^p(\mathcal{I}_X)) = 0$, for all $p \geq 0$. Then, for each $p \geq 0$, the G -representations $\Gamma^{\leq p}(Q, \mathcal{L}^{-\mu})/\Gamma^{\leq p-1}(Q, \mathcal{L}^{-\mu})$ and $\Gamma^p(\mathbb{I}_X^*, \tau^* \mathcal{F}^{-\mu}) = \Gamma(X, \mathcal{F}^{-\mu} \otimes S^p(\mathcal{I}_X))$ are isomorphic. In particular, the μ -twisted generalized exponents of any G -irreducible V_λ are independent of the choice of (Q, π) .*

Proof. The p th component of the symbol map $\Delta_X^{-\mu}$ gives the desired G -linear isomorphism, by Corollary 4.6. and Theorem 5.5.

6. COMPARISON OF q -ANALOGS AND PROOF OF THEOREM 3.4

In this section, we prove Theorem 3.4. Let V_λ be an irreducible G -representation, with character χ_λ , and take $\mu \in \mathcal{P}$.

In the last section, we obtained Corollary 5.11 and Lemma 5.12, the geometric formulae for the jump polynomial $r_\lambda^\mu(q)$. On the other hand, the geometric description of Lusztig's polynomials comes from Hesselink's work [Hs1]. The author is grateful to David Vogan for explaining this to her some years ago.

Form the graded Euler characteristic character

$$\chi_{\underline{T}_X}^{-\mu} := \chi(\underline{T}_X^*, \tau^* \mathcal{F}^{-\mu}) = \sum_{i,p \geq 0} (-1)^i \text{ch}(H^i(X, \mathcal{F}^{-\mu} \otimes S^p(\mathcal{F}_X)))q^p.$$

Lemma 6.1. *For all $\mu \in \mathcal{P}$, we have $\chi_{\underline{T}_X}^{-\mu} = \sum_{\lambda \in \mathcal{P}^{++}} m_{\lambda}^{\mu}(q)\chi_{\lambda}^*$.*

Proof. We proceed as in [Hs1]. One can filter $\mathfrak{g}/\mathfrak{b}$ by B -stable subspaces such that the consecutive quotients are (as \mathfrak{t} -modules) the negative root spaces. This induces on \mathcal{F}_X a filtration by G -homogeneous sheaves such that the consecutive quotients are the invertible sheaves $\mathcal{F}^{-\phi}$, as ϕ ranges over the set of positive roots, in some order.

Additivity of the Euler characteristic then gives

$$\chi_{\underline{T}_X}^{-\mu} = \sum_{\theta \in \mathcal{P}} \wp_q(\theta - \mu)\chi(X, \mathcal{F}^{-\theta}).$$

The Borel-Weil-Bott Theorem computes the cohomology of homogeneous line bundles on X . It implies that $\chi(X, \mathcal{F}^{-\theta}) = 0$ unless $\theta + \rho$ is regular, in which case $\chi(X, \mathcal{F}^{-\theta}) = \text{sgn}(w)\chi_{w(\theta+\rho)-\rho}^*$, where w is the unique Weyl group element such that $w(\theta + \rho) - \rho$ is dominant. Thus $\chi(X, \mathcal{F}^{-\theta}) = \pm\chi_{\lambda}^*$ iff $\theta = w^{-1}(\lambda + \rho) - \rho$, so that the coefficient of χ_{λ}^* in $\chi_{\underline{T}_X}^{-\mu}$ is equal to the expression on the right-hand side of (3.3).

Our comparison will require that the Euler characteristic reduces to the 0th cohomology.

Hypothesis 6.2. *For all $i > 0$ and $p \geq 0$, $H^i(X, \mathcal{F}^{-\mu} \otimes S^p(\mathcal{F}_X)) = 0$. This is a condition on both the group G and the integral weight μ .*

Theorem 6.3. *Hypothesis 6.2 is satisfied if*

- (1) [A-J] *the Lie algebra \mathfrak{g} is of classical type, and $\mu + \rho$ is dominant; or if*
- (2) [Gr] *the weight μ is regular and dominant.*

The author is grateful to H. H. Andersen and J. C. Jantzen for referring her to their results, and to D. A. Vogan for showing her Griffiths' result.

Proof. (1) Combine Proposition 5.4 and Remark 5.5 in [A-J]. (2) We apply Theorem G in [Gr] to the tangent sheaf \mathcal{F}_Y on any smooth, projective complex variety Y , and any line bundle \underline{F} on Y with sheaf of sections \mathcal{F} . Then we obtain the vanishing $H^i(Y, \mathcal{F} \otimes S^p(\mathcal{F}_Y)) = 0$ for all $i > 0$, $p \geq 0$, if \mathcal{F} is positive, or equivalently, ample (cf., Theorems A and C in [Gr]). To make this application, one needs to compute the tensor product of the canonical bundle \underline{K}_Y with the determinant of the tangent bundle \underline{T}_Y . But \underline{K}_Y is the determinant of the cotangent bundle, so $\underline{K}_Y \otimes \det(\underline{T}_Y)$ is trivial.

In our situation, the homogeneous line bundle $\underline{F}^{-\mu}$ on X is ample iff μ is regular and dominant.

Finally, we have

Theorem 6.4. *Let μ be a weight in an irreducible finite-dimensional G -representation V_λ . Assume Hypothesis 6.2 is satisfied. Then the jump polynomial $r_\lambda^\mu(q)$ of the principal filtration (relative to $(\mathfrak{t}, \mathfrak{b})$) of V_λ^μ is equal to Lusztig's polynomial $m_\lambda^\mu(q)$.*

Proof. In the presence of vanishing of the higher cohomology, Lemma 6.1 says that

$$\sum_{p \geq 0} \langle V_\lambda^*, \Gamma(X, \mathcal{F}^{-\mu} \otimes S^p(\mathcal{I}_X)) \rangle q^p = m_\lambda^\mu(q).$$

So the theorem follows at once by comparison with Corollary 5.11 and Lemma 5.12.

REFERENCES

- [A-J] H. H. Andersen and J. C. Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann. **269** (1984), 487–525.
- [B-G-G] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Structure of representations generated by highest weight vectors*, Funktsional Anal. i Prilozhen **5** (1971), 1–9, English transl. in Functional Anal. Appl. **5** (1971), 1–8..
- [B-K] W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen re-duktiver Gruppen*, Comment. Math. Helv. **54** (1979), 61–104.
- [F-P] E. M. Friedlander and B. J. Parshall, *On the cohomology of algebraic and related finite groups*, Invent. Math. **74** (1983), 85–117.
- [Gr] P. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis, Princeton Univ. Press, Princeton, NJ, 1970, pp. 185–251.
- [G1] R. K. Gupta, *Generalized exponents via Hall-Littlewood symmetric functions*, Bull. Amer. Math. Soc. (N.S.) **16** (1987), 287–291.
- [G2] —, *Characters and the q -analog of weight multiplicity*, J. London Math. Soc. (2) **36** (1987), 68–76.
- [Hs1] W. H. Hesselink, *Cohomology and the resolution of the nilpotent variety*, Math. Ann. **225** (1976), 249–252.
- [Hs2] —, *Characters of the nullcone*, Math. Ann. **252** (1980), 179–182.
- [K-L] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, Proc. Sympos. Pure Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.
- [Kt] S. Kato, *Spherical functions and a q -analog of Kostant's weight multiplicity formula*, Invent. Math. **66** (1982), 461–468.
- [K1] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex semi-simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.
- [K2] —, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.
- [L] G. Lusztig, *Singularities, character formulas, and a q -analog of weight multiplicities*, Analyse et Topologie sur les Espaces Singuliers (II–III), Astérisque **101-102** (1983), 208–227.
- [Pt] D. Peterson, *A formula for the generalized exponents of representations*, manuscript, M.I.T., 1978.
- [S] J. P. Serre, *Espaces fibres algébriques*, Séminaire C. Chevalley (Anneaux de Chow et applications), 1958, pp. 1-01–1-37.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, MCALLISTER BUILDING
312, UNIVERSITY PARK, PA 16802