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THE INTRINSIC SPREAD OF A CONFIGURATION IN $\mathbf{R}^d$

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Dedicated to Branko Grünbaum on the occasion of his sixtieth birthday

1. INTRODUCTION

How "spread out" is a finite set of points in general position in real affine d-space? On the line, a natural measure would be the ratio between the greatest distance and the smallest; this is invariant under affine transformations, so it depends only on the "affine shape" of the configuration. If we use the same definition in higher dimensions, the property of being invariant under affine transformations is lost: we can stretch in one direction but not in another, and the ratio will change. The same thing happens if we use the maximum ratio of distances from points to hyperplanes spanned by other points. Thus it seems most natural to use the following definition, which *is* affinely invariant and also generalizes the "correct" definition in the one-dimensional case:

Definition. If S is a configuration of points in general position in $\mathbf{R}^d$, its spread $\sigma(S)$ is defined by

$$(1)\quad \sigma(S) = \max_{P^{(0)},\dots,P^{(d)}\in S} \operatorname{vol}\langle P^{(0)},\dots,P^{(d)}\rangle \Big/ \min_{P^{(0)},\dots,P^{(d)}\in S} \operatorname{vol}\langle P^{(0)},\dots,P^{(d)}\rangle,$$

where $\langle P^{(0)},\dots,P^{(d)}\rangle$ denotes the simplex spanned by the points $P^{(0)},\dots,P^{(d)}$.

If we have a configuration $S = \{P^{(1)},\dots,P^{(n)}\}$ on a line, and we want to find a configuration S' in the same order ($S' \sim S$) with the smallest spread,

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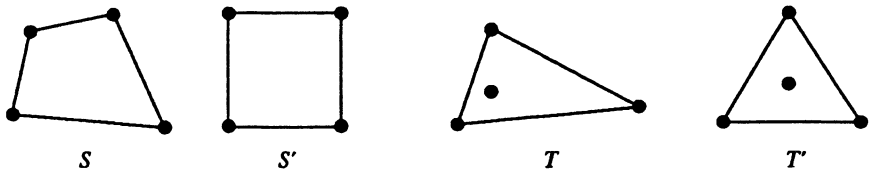

FIGURE 1

we obviously space the points equally, keeping their order the same. Thus the best we can achieve for n points on a line is a spread of n. We call this the *intrinsic spread* $\tilde{\sigma}$ of S.

This concept makes sense in higher dimensions as well. Recall [8] that two labeled configurations in $\mathbf{R}^d$, S and S', have the same *order type* ($S' \sim S$) if corresponding $(d+1)$-tuples are similarly oriented, i.e., if their associated oriented matroids are isomorphic [2, 3, 7]. It is then natural to ask for the infimum, over all $S' \sim S$, of the spread $\sigma(S')$; we call this the *intrinsic spread* $\tilde{\sigma}$ of S. It is not hard to see, by compactness, that this infimum is actually a minimum; see §4 below. On the line, the intrinsic spread of a configuration S of n points obviously depends only on n; in dimension $d \geq 2$ it is more interesting—a measure of how "evenly" it is possible to realize S. (For the planar configurations shown in Figure 1, for example, we have $\tilde{\sigma}(S) = \sigma(S') = 1$, $\tilde{\sigma}(T) = \sigma(T') = 3$.) Notice also that the problem of finding bounds for the minimum, over *all* planar configurations S of n points, of the spread $\sigma(S)$ is closely related to the well-known Heilbronn problem (see [19]); for example it follows immediately from results of Komlós, Pintz, and Szemerédi [16] that for the plane,

$$\min_{|S|=n} \sigma(S) < \frac{cn^2}{\log n}.$$

The main purpose of this paper is to prove that in the worst case the intrinsic spread of a configuration of n points in general position in $\mathbf{R}^d$, for fixed $d \geq 2$, is doubly exponential in n. We do this by relating it to yet another measure of how spread out a configuration is—the size of the smallest integer lattice on which a configuration of the same order type can be embedded.

It has been known for some time (see [13]) that configurations exist with no rational, hence no integral, realizations. If S is *simple*, however, i.e. if its points are in general position, then it clearly admits an integral realization. It is also known that there are comparatively few simple order types in the plane [8, 9]. A few years ago these observations led Bernard Chazelle to pose the problem of how large a grid was needed to accommodate all simple planar n-point configurations up to order type [4]. An answer to Chazelle's question is relevant to the computational problem of accurately representing configurations of points and arrangements of lines [6] in an environment of finite precision arithmetic; see also [5, 11, 14, 18, 20], in which the problem of finding robust

geometric algorithms in such an environment is addressed. In this paper we solve Chazelle's problem by proving

Theorem A. *Let $f(n, d)$ be the smallest integer N such that every configuration S of n points in general position in $\mathbf{R}^d$ can be realized, up to order type, on the grid $G(N, d) = \{(i_1, \dots, i_d): -N \le i_j \le N\}$, and let $g(n, d)$ be the maximum of the intrinsic spread $\tilde{\sigma}(S)$ over all simple configurations S of n points in $\mathbf{R}^d$. Then there exist constants $c_1 = c_1(d)$, $c_2 = c_2(d)$ such that*

$$2^{2^{c_1 n}} \le f(n, d), g(n, d) \le 2^{2^{c_2 n}}.$$

In §2 we establish the lower bound by first constructing a "rigid" configuration that is very spread out in the intuitive sense, then modify it via a recent construction of [15] to a configuration of points in general position which achieves at least the same spread in every realization. §3 contains the proof of the doubly-exponential upper bound, which uses results of Grigor′ev and Vorobjov [12] on the size of solutions of polynomial inequalities. In §4, we relate the bounds on the grid size to bounds for the original problem of determining the intrinsic spread. Finally, in §5, we discuss the intrinsic spread $\tilde{\sigma}(\mathscr{P})$ of a convex polytope $\mathscr{P}$, which is defined as the minimum spread of the vertex sets of all polytopes combinatorially equivalent to $\mathscr{P}$. We prove by means of Gale diagrams that for d-polytopes with up to $d+3$ vertices the intrinsic spread is a linear function of d, while for those with at least $d+4$ vertices it is doubly-exponential in d.

Theorem B. *For every simplicial d-polytope $\mathscr{P}$ with $\le d+3$ vertices we have $\tilde{\sigma}(\mathscr{P}) \le d+2$. On the other hand, there is a constant $c > 0$ such that for every $d \ge 2$ a simplicial d-polytope $\mathscr{P}$ with $d+4$ vertices exists with $\tilde{\sigma}(\mathscr{P}) \ge 2^{2^{c\sqrt{d}}}$.*

We conclude with some remarks on a related problem.

We note that the lower bound in §2 has recently found application to the problem of determining the complexity of finding a minimum-crossing-number rectilinear planar layout of a graph; see [1], in which Bienstock uses this result to show that no polynomial-time algorithm exists for producing a rectilinear drawing of a graph which achieves its minimum crossing number.

2. The lower bound

Throughout §§2–4, d represents a fixed integer ≥ 2.

By the *norm* $\nu(S)$ of a configuration $S = \{P^{(1)}, \dots, P^{(n)}\}$ in $\mathbf{R}^d$ we mean the integer

$$\nu(S) = \min \max_{i,j} |x_j^{(i)}|,$$

the minimum being taken over all configurations

$$S' = \{(x_1^{(1)}, \dots, x_d^{(1)}), \dots, (x_1^{(n)}, \dots, x_d^{(n)})\}, \qquad x_j^{(i)} \in \mathbf{Z},$$

having the same order type as S. (If S has no integral realization $\nu(S)$ is undefined.)

Theorem 1. *There is a constant $c > 0$ such that for each n a configuration S of n points in general position in $\mathbf{R}^d$ exists with $\nu(S) \geq 2^{2^{cn}}$.*

Proof. Let $\iota\colon \mathbf{R}^d \to \mathbf{R}P^d$ be the inclusion $(x_1, \dots, x_d) \mapsto (1, x_1, \dots, x_d)$ of real affine d-space into projective d-space. If $S = \{P^{(1)}, \dots, P^{(n)}; P^{(i)} = (x_1^{(i)}, \dots, x_d^{(i)})\}$ is a configuration in $\mathbf{R}^d$, we define $\chi(S)$ as the maximum, over all ordered $(d+3)$-tuples of indices from 1 to n for which the denominator does not vanish, of the absolute cross-ratio

$$(i_1, \dots, i_{d-1}; j, k, l, m) = \left| \frac{[i_1 \cdots i_{d-1} jk][i_1 \cdots i_{d-1} lm]}{[i_1 \cdots i_{d-1} jl][i_1 \cdots i_{d-1} km]} \right|,$$

where, e.g., the bracket $[i_1 \cdots i_{d-1} jk]$ represents the determinant

$$\begin{vmatrix} 1 & x_1^{(i_1)} & \cdots & x_d^{(i_1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(i_{d-1})} & \cdots & x_d^{(i_{d-1})} \\ 1 & x_1^{(j)} & \cdots & x_d^{(j)} \\ 1 & x_1^{(k)} & \cdots & x_d^{(k)} \end{vmatrix},$$

and we let $\tilde{\chi}(S) = \inf_{S' \sim S} \chi(S')$. (Notice that $(i_1, \dots, i_{d-1}; j, k, l, m)$ is independent of the choice of projective coordinates of any of its points, and is nothing more than the absolute value of the cross-ratio of the four points P'_j, P'_k, P'_l, P'_m in which an arbitrary line in general position cuts the hyperplanes $\operatorname{aff}(P^{(i_1)}, \dots, P^{(i_{d-1})}, P^{(j)}), \dots, \operatorname{aff}(P^{(i_1)}, \dots, P^{(i_{d-1})}, P^{(m)})$, resp.)

We begin by constructing a rational configuration $S^{(r)}$ of $3r + d + 6$ points in $\mathbf{R}P^d$ for each $r = 0, 1, 2, \dots$. Starting with the points

$$(1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1) \in \mathbf{R}P^2, \tag{2}$$

and letting $\overline{P, Q}$ denote the line segment with endpoints P and Q, we first construct the points

$$\begin{aligned} (1, 0, 0) &= \overline{(1, 1, 0), (0, 1, 0)} \cap \overline{(1, 0, 1)(0, 0, 1)}, \\ (1, 1, 1) &= \overline{(1, 1, 0), (0, 0, 1)} \cap \overline{(1, 0, 1)(0, 1, 0)}, \\ (0, 1, -1) &= \overline{(1, 1, 0), (1, 0, 1)} \cap \overline{(0, 1, 0)(0, 0, 1)}, \\ (1, 2, 0) &= \overline{(1, 1, 1), (0, 1, -1)} \cap \overline{(1, 1, 0)(0, 1, 0)}. \end{aligned}$$

Let $S^{(0)}$ be the resulting configuration of eight points.

Now notice that from $(1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1), (0, 1, -1), (1, a, 0)$ we can construct $(1, a^2, 0)$ by the sequence

$$\begin{aligned} (1, 0, a) &= \overline{(1, a, 0), (0, 1, -1)} \cap \overline{(1, 0, 1)(0, 0, 1)}, \\ (0, a, -1) &= \overline{(1, 0, 1), (1, a, 0)} \cap \overline{(0, 1, 0)(0, 0, 1)}, \\ (1, a^2, 0) &= \overline{(1, 0, a), (0, a, -1)} \cap \overline{(1, 1, 0)(0, 1, 0)}, \end{aligned}$$

as in Figure 2.

We now embed $\mathbf{R}P^2$ in $\mathbf{R}P^d$, by mapping each point (x_0, x_1, x_2) to the point $(x_0, \dots, x_d)$ with $x_3 = \cdots = x_d = 0$. Thus each step of the form "intersect $\overline{PQ}$ and $\overline{RS}$" in the construction above, when viewed as taking place in $\mathbf{R}P^d$, consists of intersecting the d hyperplanes

$$\begin{cases} \operatorname{aff}(P, Q, (0,0,0,1,0,\dots,0), \dots, (0,0,0,0,\dots,0,1)), \\ \operatorname{aff}(R, S, (0,0,0,1,0,\dots,0), \dots, (0,0,0,0,\dots,0,1)), \\ x_3 = 0, \dots, x_d = 0, \end{cases}$$

and to the projective basis (2) above we must add the $d-2$ points $(0, 0, 0, 1, 0, \dots, 0), \dots, (0, 0, 0, 0, \dots, 0, 1)$. Hence we can inductively construct a configuration $S^{(r)} = \{P^{(1)}, \dots, P^{(3r+d+6)}\}$ by this process, which will number among its points

$$P^{(i_1)}(0,0,1,0,\dots,0), P^{(i_2)}(0,0,0,1,0,\dots,0), \dots,$$
$$P^{(i_{d-1})}(0,0,0,0,\dots,0,1), P^{(j)}(1,0,0,0,\dots,0),$$
$$P^{(k)}(1, 2^{2^r}, 0, 0, \dots, 0), P^{(l)}(1,1,0,0,\dots,0), P^{(m)}(0,1,0,0,\dots,0),$$

whose absolute cross-ratio (as defined above) is

$$(i_1, \dots, i_{d-1}; j, k, l, m) = 2^{2^r},$$

as a quick calculation shows.

Apply a projective transformation π to $\mathbf{R}P^d$ which moves all the points in $S^{(r)}$ to finite distance. Since this leaves all the cross-ratios invariant, the resulting configuration

$$T^{(r)} = \{\iota^{-1}(\pi(P^{(1)})), \dots, \iota^{-1}(\pi(P^{(3r+d+6)}))\} \subset \mathbf{R}^d$$

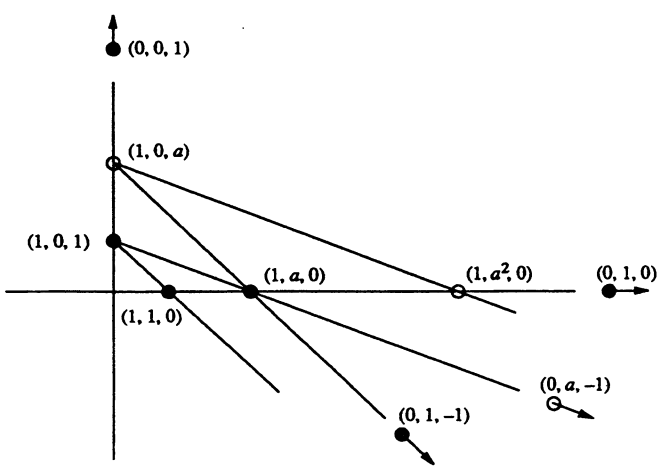

FIGURE 2

has the property

$$\chi(T^{(r)}) \geq 2^{2^r},$$

and since $S^{(r)}$ is "constructible" in the sense that each point of $S^{(r)}$ was uniquely determined by the choice of the first $d+2$ by construction, so that $S^{(r)}$ is unique up to a projective transformation, it follows that

$$\tilde{\chi}(T^{(r)}) \geq 2^{2^r}$$

as well. Let $Q^{(i)} = \iota^{-1}(\pi(P^{(i)}))$ for each i.

Now $T^{(r)}$ is very far from being in general position. To remedy this we use the "scattering" method used in [15], which was in turn suggested by earlier work of Las Vergnas [17]. To each constructible configuration U of n points in $\mathbf{R}^d$ there exists a configuration V of at most $d+2+2^d(n-d-2)$ points *in general position* in $\mathbf{R}^d$ and a continuous surjection ρ of the space of configurations $V' \sim V$ to the space of configurations $U' \sim U$; see [15]. V can be obtained from U by fixing $d+2$ points and successively replacing each of the remaining points that lies on the intersection of d "connecting hyperplanes" by 2^d points closely surrounding it, as shown in Figure 3 for $d = 2$; each point must be chosen sufficiently close, in comparison with its predecessor, to the original intersection point. In our case, applied to the configuration $T^{(r)}$, this means that since each of its points is constructed, starting from $d+2$ points in general position, by successively adjoining intersections of d connecting hyperplanes, we may replace each of the points (other than the first $d+2$) by 2^d points closely surrounding it to arrive at a configuration $\tilde{T}^{(r)}$ in general position; $\tilde{T}^{(r)}$ will have the property that for any configuration $\tilde{T} \sim \tilde{T}^{(r)}$ there is a configuration $T \sim T^{(r)}$ such that $\tilde{T}$ is obtained by the same process from T. Since $|T^{(r)}| = 3r+d+6$, we have

$$|\tilde{T}^{(r)}| = d+2+2^d(3r+4).$$

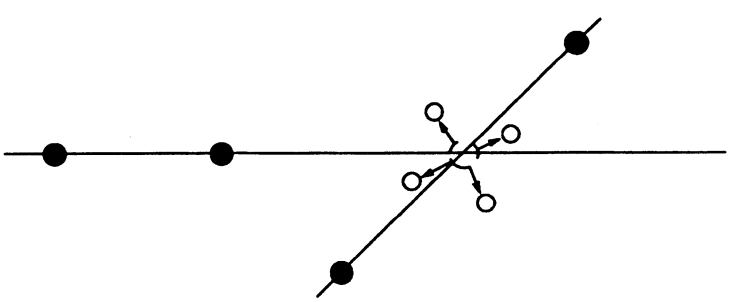

FIGURE 3

We claim next that $\widetilde{T}^{(r)}$ still contains $d+3$ points whose absolute cross-ratio is at least 2^{2^r}. This follows by applying Lemma 1 below to the points $Q^{(i_1)}, \dots, Q^{(i_{d-1})}, Q^{(j)}, Q^{(k)}, Q^{(l)}, Q^{(m)}$ constructed earlier. But more than this is true: for any configuration $\widetilde{T} \sim \widetilde{T}^{(r)}$ we still have $d+3$ points with absolute cross-ratio at least 2^{2^r}, by the same argument applied to the configuration T above. This shows that $\tilde{\chi}(\widetilde{T}^{(r)}) \geq 2^{2^r}$.

The last step is to relate $\tilde{\chi}$ to ν. Choose an integral configuration $T \sim \widetilde{T}^{(r)}$ of minimal norm, i.e., such that $\nu(\widetilde{T}^{(r)})$ is realized by T, and let $R^{(i_1)}, \dots, R^{(i_{d-1})}, R^{(j)}, R^{(k)}, R^{(l)}, R^{(m)} \in T$ be such that $(i_1, \dots, i_{d-1}; j, k, l, m) \geq 2^{2^r}$. Then

$$2^{2^r} \leq \left| \frac{[i_1 \cdots i_{d-1} jk][i_1 \cdots i_{d-1} lm]}{[i_1 \cdots i_{d-1} jl][i_1 \cdots i_{d-1} km]} \right| \leq \left| [i_1 \cdots i_{d-1} jk][i_1 \cdots i_{d-1} lm] \right|,$$

since the denominator is an integer $\neq 0$; hence

$$2^{2^r} \leq (d!(\nu(\widetilde{T}^{(r)}))^{d-1})^2,$$

from which it follows that $\nu(\widetilde{S}_r) \geq 2^{2^{(1-\varepsilon)r}}$. If we replace r by $(n-2^{d+2}-d-2)/(3 \cdot 2^d)$, the result follows. □

Lemma 1. *Let $i_1, \dots, i_{d-1}, j, k, l, m$ be points in $\mathbf{R}^d$ with j, k, l, m collinear and $i_1, \dots, i_{d-1}, j, k$ in general position. Suppose each point $x = i_1, \dots, i_{d-1}, j, k, l, m$ is contained in the convex hull of points $x^{(1)}, \dots, x^{(s)}$, and suppose further that all $s(d+3)$ of the points $i_1^{(1)}, \dots, m^{(s)}$ are in general position. Then for some choice of "surrounding point" $t(x)$, $1 \leq t(x) \leq s$, for each index, we have*

$$(i_1^{t(i_1)}, \dots, i_{d-1}^{t(i_{d-1})}; j^{t(j)}, k^{t(k)}, l^{t(l)}, m^{t(m)}) \geq (i_1, \dots, i_{d-1}; j, k, l, m).$$

Proof. Since, to begin with, the points j, k, l, m are collinear, $(i_1, \dots, i_{d-1}; j, k, l, m)$ is nothing more than the ordinary (absolute) cross-ratio of (j, k, l, m), so for $t(i_1), \dots, t(i_{d-1})$ we may choose *any* of the numbers $1, \dots, k$. To choose $t(j)$, we observe that the cross-ratio of four collinear points is a monotone function of each point separately, when that point is restricted to the interval cut out by the remaining three. Hence for some choice of $t(j)$ the cross-ratio will go up.

The same fact allows us to choose $t(k)$, $t(l)$, and $t(m)$ successively. □

3. The upper bound

Theorem 2. *There is a constant $c = c(d) > 0$ such that for every configuration S of n points in general position in $\mathbf{R}^d$, $\nu(S) \leq 2^{2^{cn}}$.*

Proof. For the upper bound, we use results of Grigor′ev and Vorobjov on the solution of simultaneous inequalities to bound the size of a realization of S, then use this bound in turn to bound the size of the lattice needed to embed S.

Recall the following result [12, Lemma 10]:

Lemma 2 (Grigor'ev–Vorobjov). *Suppose that the polynomials* $h_1, \dots, h_k \in \mathbf{Z}[X_1, \dots, X_n]$ *satisfy the bounds* $\deg h_i < d$ *and* $l(h_i) \le M$, *where* $l(h)$ *is the maximum bit length of the coefficients of* h. *If* W *is any connected component of the semi-algebraic set defined by the system* $h_1 \ge 0, \dots, h_k \ge 0$, *then* W *meets the ball of radius* $R = \exp((M + \log k)(d^{qn}))$ *centered at the origin for some natural number* q.

We begin by replacing the system of $\binom{n}{d+1}$ inequalities

$$\{[i_1 \cdots i_{d+1}] > 0\} \tag{3}$$

(where we have chosen one positively oriented permutation of the indices $i_1, \dots, i_{d+1}$ from each set of distinct $(d+1)$-tuples) by the system

$$\{[i_1 \cdots i_{d+1}] \ge 1\}, \tag{4}$$

for the same $\binom{n}{d+1}$ ordered $(d+1)$-tuples of indices. Since (3) has a solution (namely S), so does (4), by dilatation; and conversely any solution of (4) will give us a configuration $S' \sim S$. Furthermore, (4) involves dn variables and satisfies the bounds $\deg[i_1 \cdots i_{d+1}] < d+1$, $l[i_1 \cdots i_{d+1}] \le 1$ in the notation of [12]. Hence by Lemma 2 there is a solution $S' = \{P_1 = (x_1^{(1)}, \dots, x_d^{(1)}), \dots, P_n' = (x_1^{(n)}, \dots, x_d^{(n)})\}$ of (4) satisfying

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{d}(x_j^{(i)})^2\right)^{1/2} \le \exp((1 + (d+1)\log n)(d+1)^{c'n}) \le 2^{2^{c''n}}$$

for appropriate constants $c', c'' > 0$.

In particular, all the points of S' lie in the ball of radius $r = 2^{2^{c''n}}$, and all the simplices formed by $d+1$ of the points are at least $1/d!$ in volume. Lemmas 3 and 4 below then show that the distance between each pair of flats of subcomplementary dimension spanned by the points of S', $\operatorname{aff}(P_{i_1}, \dots, P_{i_k})$ and $\operatorname{aff}(P_{i_{k+1}}, \dots, P_{i_{d+1}})$, must be at least $1/(k-1)!(d-k)!(2r)^{d-1}$, and that each point may therefore be moved a distance

$$\varepsilon = \min_{1 \le k \le d} \frac{1}{2(k-1)!(d-k)!(2r)^{d-1}} = \frac{1}{2(d-1)!(2r)^{d-1}}$$

in any direction, with no reversal of orientation of any $(d+1)$-tuple taking place.

Applying a dilatation by a factor of $\sqrt{d}/2\varepsilon$, we arrive at a configuration $S'' \sim S$ contained within the disk of radius $2^{2^{cn}}$ for appropriate c, which now has the property that every point may be moved a distance $\sqrt{d}/2$ with no crossover. But this is precisely enough to guarantee that each point may be moved to a lattice point, and we are done. □

Lemma 3. *If points* $P_1, \dots, P_{d+1}$ *lie in a ball of radius* r *and* $\operatorname{vol}\langle P_1, \dots, P_{d+1}\rangle = v$, *then the perpendicular distance* $h_{F,G}$ *between the flats* $F = \operatorname{aff}(P_1, \dots, P_k)$ *and* $G = \operatorname{aff}(P_{k+1}, \dots, P_{d+1})$ *is at least* $d\binom{d-1}{k-1}v/(2r)^{d-1}$.

Proof. This follows immediately from the inequality

$$d!v \leq (k-1)!\,\mathrm{vol}_{k-1}\langle P_1, \dots, P_k\rangle (d-k)!\,\mathrm{vol}_{d-k}\langle P_{k+1}, \dots, P_{d+1}\rangle h_{F,G}$$

(where vol_m is m-dimensional euclidean volume), since $\mathrm{vol}_m\langle P_{i_1}, \dots, P_{i_{m+1}}\rangle$ is clearly bounded above by $(2r)^{m-1}$. □

Lemma 4. *If the distance $h_{F,G}$ between the flats spanned by any two complementary faces of the simplex $\langle P_1, \dots, P_{d+1}\rangle$ is at least $a > 0$, then for every choice of point P_i' within the open ball $B(P_i, a/2)$, the simplex $\langle P_1', \dots, P_{d+1}'\rangle$ remains nondegenerate (hence the points $P_1', \dots, P_{d+1}'$ have the same orientation as $P_1, \dots, P_{d+1}$).*

Proof. Let ε be the common radius of the smallest set of equi-radial balls $B_1, \dots, B_{d+1}$ centered at $P_1, \dots, P_{d+1}$ that contain points $P_1', \dots, P_{d+1}'$ (resp.) not in general position. The hyperplane H that is tangent to $B_1, \dots, B_{d+1}$ at $P_1', \dots, P_{d+1}'$ (resp.) must separate some of these balls from the rest, say $B_1, \dots, B_k$ from $B_{k+1}, \dots, B_{d+1}$. Then each of the flats $\mathrm{aff}(P_1', \dots, P_k')$, $\mathrm{aff}(P_{k+1}', \dots, P_{d+1}')$ is parallel to H and at distance ε from it, from which the conclusion follows. □

4. The intrinsic spread

We show first that the intrinsic spread

$$\tilde{\sigma}(S) = \inf_{S' \sim S} \sigma(S')$$

is actually achieved by some configuration.

Proposition 1. *$\tilde{\sigma}(S) = \sigma(S_0)$ for some $S_0 \sim S$.*

Proof. Notice that every configuration has the same order type as one in which the sum of the squares of the distances between points is 1, and that the set of such configurations is compact. Hence the only problem we may encounter, after selecting a convergent subsequence, is that a sequence of configurations S_i' of the same order type, for which $\sigma(S_i')$ decreases to a limit $\tilde{\sigma}$, may converge to a configuration which is no longer in general position. If this were the case, then both the numerator and the denominator in the defining expression (1) for $\sigma(S_i')$ would have to approach 0. It is therefore sufficient to find a compact set Σ_n representing all of the affine equivalence classes of n-point configurations which has the property that $\max_{P^{(0)}, \dots, P^{(d)} \in S'} \mathrm{vol}\langle P^{(0)}, \dots, P^{(d)}\rangle$ is bounded away from 0 as S' runs through all the members of Σ_n which belong to a single (simple) order type.

That such a set Σ_n exists follows from the observation that every configuration of n points in $\mathbf{R}^d$ may be realized, up to affine equivalence, by projecting the n vertices of a fixed $(n-1)$-simplex $\Delta \subset \mathbf{R}^{n-1}$ orthogonally onto a d-space

which is free to rotate through the origin. The following simple proof is due to Peter Ungar [22]. Since, as is easily seen, two configurations

$$\{P^{(i)}: P^{(i)} = (x_1^{(i)}, \dots, x_d^{(i)}),\ 1 \le i \le n\}$$

and

$$\{Q^{(i)}: Q^{(i)} = (y_1^{(i)}, \dots, y_d^{(i)}),\ 1 \le i \le n\}$$

in $\mathbf{R}^d$ are affinely equivalent if and only if the vectors

$$(1, \dots, 1), (x_1^{(1)}, \dots, x_1^{(n)}), \dots, (x_d^{(1)}, \dots, x_d^{(n)})$$

and

$$(1, \dots, 1), (y_1^{(1)}, \dots, y_1^{(n)}), \dots, (y_d^{(1)}, \dots, y_d^{(n)})$$

span the same $(d+1)$-space, we may extend the vector $(1, \dots, 1)/\sqrt{n}$ to an orthonormal basis of this $(d+1)$-space, and then extend this further to an orthonormal basis

$$(1, \dots, 1), (x_1^{(1)}, \dots, x_1^{(n)}), \dots, (x_{n-1}^{(1)}, \dots, x_{n-1}^{(n)})$$

of $\mathbf{R}^n$. Since the transpose of an orthogonal matrix is orthogonal, one sees immediately that the points

$$v_i = (x_1^{(i)}, \dots, x_{n-1}^{(i)}), \qquad 1 \le i \le n,$$

form the vertices of a regular simplex, as desired.

If we inscribe a sphere of radius r, say, in Δ, it follows that every projected configuration must contain a ball of d-volume

$$V(d, r) = 2\pi^{d/2} r^d / d\Gamma(\tfrac{d}{2})$$

in its convex hull. Since the latter has no more than n vertices, triangulating it yields at most $\binom{n}{d+1}$ simplices (actually far fewer), so that any projected configuration must contain $d+1$ points spanning a simplex of volume at least $V(d, r)/\binom{n}{d+1}$. Hence, in particular, this volume cannot be too close to 0. □

Next, let us note that the bounds on $f(n, d)$ and $g(n, d)$ announced in §1 follow from Theorems 1 and 2:

Proof of Theorem A. Theorems 1 and 2 immediately give the desired lower and upper bounds, respectively, for $f(n, d)$.

The lower bound for $g(n, d)$ follows from the argument in Theorem 1 for the corresponding bound for $f(n, d)$, by noting that

$$(i_1, \dots, i_{d-1}; j, k, l, m) = \left| \frac{[i_1 \cdots i_{d-1} jk][i_1 \cdots i_{d-1} lm]}{[i_1 \cdots i_{d-1} jl][i_1 \cdots i_{d-1} km]} \right| \ge M$$

for points $P_{i_1}, \dots, P_{i_{d-1}}, P_j, P_k, P_l, P_m$ of a configuration S implies that

$$\sigma(S) \ge M^{1/2}.$$

Finally, the upper bound for $g(n, d)$ follows from Theorem 2 since for any d-dimensional configuration S embedded in the grid $G(N, d)$ we must have $\sigma(S) \leq d!(2N)^d$. □

5. The intrinsic spread of simplicial polytopes with few vertices

In this section we apply our results to point configurations in convex position. Given a simplicial d-polytope $\mathscr{P} \subset \mathbf{R}^d$ with n vertices, we define its *intrinsic spread* $\tilde{\sigma}(\mathscr{P})$ as the infimum of the numbers $\sigma(S)$, where S ranges over all configurations of n points in general position in $\mathbf{R}^d$ whose convex hull $\mathscr{P}' := \operatorname{conv}(S)$ is combinatorially equivalent to $\mathscr{P}$. It follows from Proposition 1 that this infimum is actually a minimum. Here we shall investigate the intrinsic spread of simplicial d-polytopes with $d+3$ vertices and $d+4$ vertices respectively, as a function of d. While the spread is linear in d for up to $d+3$ vertices, we show that simplicial polytopes with $d+4$ vertices can be doubly-exponentially "spread out". These results are obtained using Gale diagram techniques, and are closely related to the known negative results concerning the polytopal realizability of $(d-1)$-spheres with $d+4$ vertices.

We recall that an *affine Gale diagram* [21] of a d-polytope $\mathscr{P}$ with n vertices is a point configuration $S = \{P_1, P_2, \ldots, P_n\} \subset \mathbf{R}^{n-d-2}$, together with a partition $S = S_+ \cup S_-$, such that the linear subspace $\mathscr{D}(\mathscr{P})$ of affine dependencies on the vertices of $\mathscr{P}$ is the orthogonal complement in $\mathbf{R}^n$ of the linear subspace $\mathscr{D}(S_+, S_-)$ of *signed affine dependencies* of S. More precisely, the latter space is defined as

$$\mathscr{D}(S_+, S_-) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n : \sum_{P_i \in S_+} \lambda_i = \sum_{P_j \in S_-} \lambda_j \text{ and } \sum_{P_i \in S_+} \lambda_i P_i = \sum_{P_j \in S_-} \lambda_j P_j \right\}.$$

Combinatorially, the affine Gale diagram is just the dual oriented matroid after a reorientation of the subset S_-, but, in addition, it satisfies the following metric correspondence. The volume of a d-simplex spanned by $d+1$ vertices of $\mathscr{P}$ is equal, up to a global constant, to the volume of the $(n-d-2)$-simplex spanned by the $(n-d-1)$ complementary points in S. This follows from the fact that orthogonal pairs of subspaces in $\mathbf{R}^n$ have the same Plücker coordinates up to sign, and that the Plücker coordinates of $\mathscr{D}(\mathscr{P})$ and $\mathscr{D}(S_+, S_-)$ are the volumes in question. We have proved the following.

Lemma 5. *The intrinsic spread $\tilde{\sigma}(\mathscr{P})$ of a simplicial polytope $\mathscr{P}$ equals the minimum of the intrinsic spreads $\tilde{\sigma}(S')$, where S' ranges over all affine Gale diagrams of polytopes $\mathscr{P}'$ combinatorially equivalent to $\mathscr{P}$.*

Proof of Theorem B. If $\mathscr{P}$ has $d+1$ vertices, $\tilde{\sigma}(\mathscr{P}) = 1$. If $\mathscr{P}$ has $d+2$ vertices, $\tilde{\sigma}(\mathscr{P}) \leq \sigma(\mathscr{P}')$, where $\mathscr{P}'$ is the union of two regular simplices sharing a common facet; it is easy to see that $\sigma(\mathscr{P}') = d/2$. Suppose, then, that

$\mathscr{P}$ has $d+3$ vertices. Every affine Gale diagram of $\mathscr{P}$ is then a configuration $S = S_+ \cup S_-$ of $d+3$ points in affine 1-dimensional space. Lemma 5 implies $\tilde{\sigma}(\mathscr{P}) \leq \tilde{\sigma}(S) \leq d+2$ and hence proves the claim. (Using affine Gale diagrams, in fact, we can easily see that if $\mathscr{P}$ is the cyclic d-polytope with $d+3$ vertices, then $\tilde{\sigma}(\mathscr{P}) = d+2$ for d even and $\tilde{\sigma}(\mathscr{P}) = d+1$ for d odd.)

For the second part of Theorem B, we use the construction introduced in [21, §4] or its refinement described in [3, §6.2]. Starting with a configuration S_+ of n points in general position in the plane, we place a negatively signed point into every open triangle spanned by points in S_+. The resulting configuration $S = S_+ \cup S_-$ of $n + \binom{n}{3}$ points has the following properties.

(a) S is the affine Gale diagram of a simplicial polytope $\mathscr{P}$, and

(b) every affine Gale diagram S' of a polytope $\mathscr{P}'$ combinatorially equivalent to $\mathscr{P}$ contains a set S'_+ of n points whose order type equals the order type of S_+.

In fact, using the argument given in [3, p. 108], we see that it suffices to place $n(n-1)/2$ negative points into S_+, i.e., we may suppose $|S_-| \leq n(n-1)/2$ for the affine Gale diagram with properties (a) and (b). By property (a), $\mathscr{P}$ is a simplicial d-polytope with $d+4$ vertices, where $d := n + n(n-1)/2 - 4$. By property (b) and Lemma 5, we have $\tilde{\sigma}(\mathscr{P}) \geq \tilde{\sigma}(S_+)$. Now the desired result follows from Theorem 1 because we can choose S_+ to have spread $\tilde{\sigma}(S_+) \geq 2^{2^{c'n}} \geq 2^{2^{c\sqrt{d}}}$. □

We conclude with a related problem on convex polytopes. Rather than considering the spread of the *vertex set* of a polytope $\mathscr{P}$, one can consider instead the function $\sigma_k(\mathscr{P})$ which is the ratio between the k-volume of a maximal-volume k-face of $\mathscr{P}$ and that of a minimal-volume k-face. (Notice that $\sigma_k(\mathscr{P})$ is no longer an affine invariant of $\mathscr{P}$, for $k < d$.) The expression $\tilde{\sigma}_k(\mathscr{P})$ would then be defined as the minimum of $\sigma_k(\mathscr{P}')$ over all polytopes $\mathscr{P}'$ combinatorially equivalent to $\mathscr{P}$. It is easy to see that Theorem A implies a doubly-exponential upper bound for $\tilde{\sigma}_k(\mathscr{P})$ if we consider simplicial polytopes with n vertices, say, as well as for the size of a grid needed to embed such polytopes with their vertices at lattice points. This may be far from the truth, however, and we ask for reasonable bounds on the functions corresponding to $f(n, d)$ and $g(n, d)$ in Theorem A.

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