LATTICES OF MINIMAL COVOLUME IN $SL_2$: A NONARCHIMEDEAN ANALOGUE OF SIEGEL’S THEOREM $\mu \geq \pi/21$

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A classical result of Siegel [Si] asserts that if $G = SL_2(\mathbb{R})$ and $\mu$ is its Haar measure (appropriately normalized) then $\min_{\Gamma} \mu(\Gamma \backslash G) = \pi/21$, where $\Gamma$ runs over all the discrete subgroups of $G$. The minimum is obtained for the triangle group $(2, 3, 7)$ which is a cocompact lattice in $G$ (see also [Gr]). For $G = SL_2(\mathbb{C})$, Meyerhoff [Me] has shown recently that among the nonuniform lattices in $G$ (that is, discrete subgroups of finite covolume but not cocompact) the minimum volume is obtained with $\Gamma = SL_2(\mathbb{O}_3)$ where $\mathbb{O}_3$ is the ring of integers of $\mathbb{Q}(\sqrt{-3})$, and it is approximately $0.0863$. (But, there are cocompact lattices with smaller covolumes and the minimum is not known, see [EGM, Th]).

In this paper, we address the analogue question for locally compact nonarchimedean fields. Our main result is: Let $F$ be a locally compact field of characteristic $p > 0$ with residue field of order $q = p^\alpha$. (Hence $F$ is isomorphic to $F_q((1/t))$, the field of Laurent formal power series in $1/t$).

**Theorem 1.** Let $G = SL_2(F)$ and $\mu$ be its Haar measure normalized so that $\mu(K) = 1$ for all maximal compact subgroups of $G$. Then

$$\min\{\mu(\Gamma \backslash G) | \Gamma \text{ a lattice in } G\} = (q - 1)^{-2}(q + 1)^{-1}.$$  

This minimum is obtained for $\Gamma_0 = SL_2(F_q[t])$, the characteristic $p$ modular group (as it is called by Weil [We]).

Theorem 1 is somewhat surprising in that the minimum is obtained for a nonuniform lattice. It should be interesting to understand what the typical situation is: the Siegel Theorem or our Theorem 1. Namely, for a semisimple Lie group over a local field, is a lattice of minimal covolume a cocompact or nonuniform lattice?
For cocompact lattices we have the stronger lower bounds:

**Theorem 2.**

\[
\min\{\mu(\Gamma \backslash G) | \Gamma \text{ a cocompact lattice of } G\} = \begin{cases} 
\frac{1}{24} & \text{if } q = 5, \\
\frac{1}{48} & \text{if } q = 7, \\
\frac{1}{120} & \text{if } q = 11, 19, 29, \\
\frac{1}{q+1} & \text{if } q = 2^a, \\
\frac{1}{2(q+1)} & \text{otherwise}.
\end{cases}
\]

Recall that for a local nonarchimedean field $F$ of characteristic 0, every lattice is cocompact (cf. [S2]). For these we have:

**Theorem 3.** Let $p$ be a prime, $\mathbb{Q}_p$ the field of $p$-adic numbers, $G = SL_2(\mathbb{Q}_p)$, and $\mu$ normalized to give maximal compact subgroups measure 1. Then

\[
\min\{\mu(\Gamma \backslash G) | \Gamma \text{ a lattice in } G\} = \begin{cases} 
\frac{1}{12} & \text{for } p = 2, \\
\frac{1}{24} & \text{for } p = 3, 5, \\
\frac{1}{48} & \text{for } p = 7, \\
\frac{1}{120} & \text{for } p = 11, 19, 29, \\
\frac{1}{2(p+1)} & \text{for all other } p.
\end{cases}
\]

For all cases of Theorems 2 and 3 we explicitly construct lattices with minimal covolume in §3. Section 2 is devoted to the proof of Theorem 1, and we end in §4 with some open problems.

## 1. Preliminaries and notation

In this section, we review some basic properties of $G = SL_2(F)$ and the tree associated with it. We also set some notation that will be used freely in the next sections.

Let $F$ be a locally compact nonarchimedean field, with valuation ring $D$ and a finite residue field $\mathbb{F}_q$. Hence if $\text{char}(F) = 0$ then $F$ is a finite extension of $\mathbb{Q}_p$ where $q = p^\alpha$, and if $\text{char}(F) = p > 0$ then $F$ is (isomorphic to) $\mathbb{F}_q((1/t))$, the field of Laurent formal power series over $\mathbb{F}_q$. Let $\pi$ be a uniformizer of $F$ (so we take $\pi = 1/t$ if $\text{char}(F) > 0$). Set $G = SL_2(F)$, $K_1 = SL_2(D)$, and $K_2 = \{ (\frac{a}{\pi^c}, \frac{b}{\pi^d}) | (\frac{a}{c}, \frac{d}{c}) \in K_1 \}$. Both $K_1$ and $K_2$ are maximal open compact subgroups of $G$ and every compact subgroup of $G$ is conjugate to a subgroup of either $K_1$ or $K_2$. Let $I = K_1 \cap K_2$ and be of index $q + 1$ in both. Any Haar measure of $G$, therefore, gives the same volume to $K_1$ and $K_2$. We will normalize the Haar measure to be 1 on $K_1$ and $K_2$.

Let $T$ be the Bruhat–Tits tree associated with $G$ (cf. [S2, Chapter II]). The vertices $V(T)$ of $T$ are the left cosets $G/K_1 \cup G/K_2$ and the edges are $G/I$ where $gI$ connects $gK_1$ to $gK_2$. We denote $v_1 = 1K_1$ and $v_2 = 1K_2$, so $d(v_1, v_2) = 1$ where $d$ is the distance function on $T$. The cosets of $K_1$ (resp. $K_2$) will be called the blue (resp. red) vertices. A more concrete realization of $T$ is given in [S2, Chapter II] via $D$-lattices in $F^2$. It is proved there that $T$ is
a \( (q + 1) \)-regular tree on which \( G \) acts through its action on left cosets. There are two orbits of vertices: the blue ones and the red ones, while the action on the edges is transitive.

A ray in \( T \) (with origin in a vertex \( x \)) is an infinite path in \( T \) (beginning in \( x \)) and without backtracking. A line is an infinite path—infinitesimal to both directions and without backtracking. Two rays are said to be equivalent if their intersection is infinite. An equivalent class of rays is called an end of \( T \). The set of ends \( \partial T \) is called the boundary of \( T \). From the description of \( T \) in [S2] one can easily see that \( \partial T \) is identified with the projective line \( \mathbb{P}^1(F) \).

Given \( x \in V(T) \) and \( \varepsilon \in \partial T \), there is a unique ray from \( x \) toward \( \varepsilon \), that is, a unique ray beginning in \( x \) and in the equivalent class of \( \varepsilon \).

Now fix an end \( \varepsilon \) on the boundary. For two vertices \( x, y \in V(T) \) let \( \phi(x, y) \) be rays extending from them toward \( \varepsilon \). Take \( v \in \psi(x, y) \). We say that \( x \) and \( y \) are in the same horosphere (with respect to \( \varepsilon \)) if \( d(x, v) = d(y, v) \) (this is independent of the choice of \( v \)). An equivalent class of rays is called an end of \( T \).

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Given \( x \in V(T) \) and \( e \in \mathbb{P}^1(F) \), there is a unique ray from \( x \) to \( e \), that is, a unique ray beginning in \( x \) and in the equivalent class of \( e \).

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Now fix an end \( \varepsilon \) on the boundary. For two vertices \( x, y \in V(T) \) let \( r_x \) and \( r_y \) be rays extending from them toward \( \varepsilon \). Take \( v \in r_x \cap r_y \). We say that \( x \) and \( y \) are in the same horosphere (with respect to \( \varepsilon \)) if \( d(x, v) = d(y, v) \) (this is independent of the choice of \( v \)). We denote by \( S(\varepsilon, x) \) the horosphere of \( x \) around \( \varepsilon \). All vertices in \( S(\varepsilon, x) \) have the same color as \( x \).

We say that the horosphere \( S(\varepsilon, x) \) is closure to \( e \) than \( S(\varepsilon, y) \) if \( d(y, v) = d(x, v) \) for \( v \in \psi(x, y) \). Now choose some horosphere to be the zero horosphere denoted \( S^0(\varepsilon) \). Let \( x \in S^0(\varepsilon) \), \( y \in V(T) \), and \( v \in r_x \cap r_y \). Then the horosphere \( S(\varepsilon, y) \) will be labeled \( S^n(\varepsilon) \) if \( d(x, v) = d(y, v) + n \). It is easy to see that the labeling is well defined and unique up to translation. The horosphere \( S^n(\varepsilon) \) is closure to \( e \) than \( S^m(\varepsilon) \) iff \( n > m \).

We say that the horoball \( B(\varepsilon, x) \) is the union of all the horospheres that are closer to \( e \) than \( S(\varepsilon, x) \), that is, \( B(\varepsilon, x) = \bigcup_{k \geq 0} S(\varepsilon, x^k) \).

The group \( G \) acts on \( \partial T \) and after the identification with \( \mathbb{P}^1(F) \) the action is by Möbius transformations, that is, \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \varepsilon = \left( \frac{a\varepsilon + b}{c\varepsilon + d} \right) \). When \( G \) acts on a set \( X \), \( x \in X \), and \( H \) is a subgroup of \( G \), we denote \( H_x = \text{Stab}_H(x) = \{ h \in H | hx = x \} \). So when \( \varepsilon \in \partial T \), \( G_\varepsilon \) is a Borel subgroup, for example, \( \varepsilon = \infty \) and \( G_\infty = \{ (a b) \} \). The map \( \phi \) commutes with the action of \( G_\varepsilon \). Also, \( G_\varepsilon \) acts on the horospheres. We denote

\[
A_\varepsilon = \{ g \in G_\varepsilon | g(S^n(\varepsilon)) = S^n(\varepsilon) \text{ for every } n \in \mathbb{Z} \}
\]

\[
= \{ g \in G_\varepsilon | g(S^n(\varepsilon)) = S^n(\varepsilon) \text{ for some } n \in \mathbb{Z} \}.
\]

\( A_\varepsilon \) contains the unipotent radical \( U_\varepsilon \) of \( G_\varepsilon \). Also, \( G_\varepsilon = T_\varepsilon U_\varepsilon \) when \( T_\varepsilon \) is a torus, for example, when \( \varepsilon = \infty \), \( T_\varepsilon = \{ (a 0) | a \in F^* \} \), \( U_\varepsilon = \{ (0 b) | b \in F \} \), and \( A_\varepsilon = T_\varepsilon U_\varepsilon \) when \( T_\varepsilon = \{ (a 0) | a \in D^* \} \).

\( T_\varepsilon \) is a compact group. The torsion subgroup of \( T_\varepsilon \) is of order \( q - 1 \). The
group $U_\varepsilon$ acts simply transitive on $\partial T - \{\varepsilon\}$, so every $u \in U_\varepsilon$ has a unique fixed point $(\varepsilon)$ on the boundary. Moreover, $\partial T - \{\varepsilon\} = \lim \tau S^n(\varepsilon)$ and hence $U_\varepsilon$ acts transitively on every horosphere $S^n(\varepsilon)$.

If $\text{char}(F) = 0$ then $U_\varepsilon$ is a torsionfree group, but if $\text{char}(F) = p$, then every element of $U_\varepsilon$ is of order $p$. Moreover, every element of $G$ of order $p$ (or $p$-power) is in $U_\varepsilon$ for some $\varepsilon$.

2. Proof of Theorem 1

The proof of the theorem will be split into a sequence of propositions and lemmas, some of them of independent interest. We keep the notation of §1, and assume here that $\text{char}(F) > 0$.

If $x \in V(T)$ is a blue (resp. red) vertex then $G_x$ is conjugate to $K_1$ (resp. $K_2$). Also $K_1 \simeq K_2 \simeq SL_2(D)$ so for every $x \in V(T)$ there is an epimorphism $G_x \to SL_2(q)$ whose kernel $N(x)$ is well defined. $N(x)$ is a pro-$p$ group.

Definition 2.1. A nontrivial unipotent element $u \in G_x$ is called a dominant unipotent at $x$ if it is in $N(x)$.

Lemma 2.2. A unipotent element $u \in G_x$ is dominant if and only if $u(y) = y$ for every $y \in V(T)$ satisfying $d(y, x) = 1$.

Proof. The set of $q + 1$ vertices adjacent to $x$ can be identified with $P^1(q)$, the projective line over $F_q$. The action of $G_x$ on it factors through the action of $SL_2(q)$ on $P^1(q)$. $PSL_2(q)$ acts faithfully, so the set of elements of $G_x$ acting trivially on the neighbors of $x$ is a group $N^1$ containing $N(x)$ as a subgroup of index 2 if $(2, q) = 1$ and is equal to $N(x)$ if $q = 2^\alpha$. These observations suffice to prove the lemma.

Proposition 2.3. Let $H$ be a finite subgroup of $G$. Then

(i) $H \subseteq G_x$ for some (not necessarily unique) $x \in V(T)$.
(ii) The order of $H$ is $p^\beta m$ when $(m, p) = 1$ and $m|q^2 - 1$.
(iii) If $|H| = p^\beta m$ and $p^\beta > p^\alpha = q$ then $H$ contains a dominant unipotent with respect to $x$. In particular, if $|H| > q(q^2 - 1)$ then $H$ contains a dominant unipotent.
(iv) If $H \subseteq G_x$ and contains a dominant unipotent, then there exists a unique end $\varepsilon \in P^1(F)$ that is fixed by $H$, that is, $H \subseteq G_\varepsilon$.

Proof. (i) is clear. (ii) and (iii) follow from the fact that $N(x)$ is a pro-$p$ group and $|SL_2(q)| = q(q^2 - 1)$. To prove (iv): $H \cap N(x)$ is a nontrivial finite $p$-group so all its elements are unipotent. Each $u \in H \cap N(x)$ fixes some unique end, but as $Z(H \cap N(x)) \neq \{1\}$ (since every finite $p$-group has a nontrivial center) all have the same fixed end $\varepsilon$. Now, $H$ normalizes $H \cap N(x)$ and hence fixes $\varepsilon$ as well.
Remark. Proposition 2.3(iv) is not valid if $H$ contains a unipotent but not a dominant unipotent. This can be seen by looking at $H = \text{SL}_2(q) \subseteq \text{SL}_2(D) = K_1 = G_{v_1}$. $H$ has many unipotent elements with different fixed ends as well as some elements which fix no end.

**Lemma 2.4.** Let $x \in V(T)$ and $u \in G_x$ a unipotent element fixing the end $e$. Then:

(i) For every $y \in B(e, x)$, $u(y) = y$.
(ii) If $u$ is dominant w.r.t. $x$ then it is dominant w.r.t. every $y \in B(e, x)$.

**Proof.** (i) Let $S(e, x)$ be the horosphere of $x$. Every $y \in S(e, x)$ can be written as $y = vx$ for some $v \in U_{e}$. As $U_{e}$ is abelian, we have $u(y) = u(vx) = v(uv) = vx = y$. Now if $z \in B(e, x)$ then $z = \varphi^{k}(y)$ for some $k \in \mathbb{N}$ and $y \in S(e, x)$. Hence $u(z) = u(\varphi^{k}(y)) = \varphi^{k}(u(y)) = \varphi^{k}(y) = z$.

(ii) If $u$ is a dominant unipotent and $S(e, x) = S^{l}(e)$ for some $l$, then from Lemma 2.2 and Lemma 2.4(i) it follows that $u$ fixes every vertex of $S^{l-1}(e)$. Hence $u$ is dominant for every $y \in S^{k}(e)$ for every $k \geq l$.

**Proposition 2.5.** Let $\Gamma$ be a discrete subgroup of $G$ and $u \in \Gamma, x$ a dominant unipotent w.r.t. $x$ fixing an end $e$. Then for $y, z \in B(e, x)$, if $y \in B(e, x)$ satisfies $\gamma(y) = z$ then $\gamma \in \Gamma_x$.

**Proof.** Look at $d(\gamma^{-1}uy, y) = d(uy, y) = d(uz, z) = 0$, whence $\gamma^{-1}uy$ is in $\Gamma_y$. $\Gamma_y$ is a finite group (since $\Gamma_y = \Gamma \cap G_y$, $\Gamma$ is discrete and $G_y$ is compact) and contains the dominant unipotent $u$. $u$ is a dominant unipotent at $y$ as well by 2.4(ii). By 2.3(iv), $\gamma^{-1}uy$ also fixes $e$ which forces $\gamma \in \Gamma_x$.

Before continuing, we will make some careful remarks about calculating $\mu(\Gamma \setminus G)$ for $\Gamma$ discrete.

In general, if $K$ is an open compact subgroup of $G$, and $S$ is a system of double coset representatives for $\Gamma \setminus G/K$, and for each $s \in S$ let $g(s)$ be the order of the finite group $\Gamma_s = \Gamma \cap sKs^{-1}$, then $\mu(\Gamma \setminus G) = \mu(K) \sum_{s \in S} 1/g(s)$ (cf. [S2, II 1.5, p. 84]). Now, in our case there are two natural possibilities for $K$, either $K_1$ or $K_2$. The double cosets $\Gamma \setminus G/K_1$ (resp. $\Gamma \setminus G/K_2$) are exactly the blue (resp. red) orbits of $\Gamma$ on $T$. So let $Y = \Gamma \setminus T$ and $E$ be a fundamental domain for $\Gamma$, that is, $E$ is a subtree of $T$ whose vertices are mapped bijectively to those of $Y$. Denote by $V_b(E)$ (resp. $V_r(E)$), the blue (resp. red) vertices of $E$. Then

\begin{equation}
\mu(\Gamma \setminus G) = \sum_{x \in V_b(E)} \frac{1}{|\Gamma_x|} = \sum_{x \in V_r(E)} \frac{1}{|\Gamma_x|}.
\end{equation}

**Example 2.7.** Let $\Gamma = \text{SL}_2(\mathbb{F}_q[t]) \subseteq G = \text{SL}_2(\mathbb{F}_q((1/t)))$. Then $\Gamma$ is a nonuniform lattice in $G$ with a fundamental domain, a ray $x_0, x_1, x_2, \ldots$, where $\{x_{2n} | n \in \mathbb{N}\}$ are blue and $\{x_{2n+1} | n \in \mathbb{N}\}$ are red. Also, $|\Gamma_{x_0}| = q(q^2 - 1)$ while $|\Gamma_{x_n}| = (q - 1)q^{n+1}$ for $n \geq 1$. Hence $\mu(\Gamma \setminus G) = 1/((q - 1)(q^2 - 1))$ (see [S2,
II.1.6, Exercise 6, p. 89] but notice the difference between our normalization and the normalization there).

Most of the rest of this section is devoted to proving that for any other lattice of $G$, $\mu(\Gamma \backslash G) \geq \mu(\overline{\Gamma} \backslash G)$. Cocompact lattices can easily be eliminated using the following easy lemma (in §3 we will give much better results on cocompact lattices).

**Lemma 2.8.** Let $\Gamma$ be a cocompact lattice in $G$. Then $\mu(\Gamma \backslash G) \geq 1/(q^2 - 1) \geq \mu(\overline{\Gamma} \backslash G)$.

**Proof.** The fundamental domain for $\Gamma$ contains at least one blue vertex $x$ (and one red vertex). $\Gamma_x$ does not contain elements of order $p$ since cocompact lattices do not contain unipotent elements (cf. [GGPS, p. 10]) hence $(|\Gamma_x|, p) = 1$, and by Lemma 2.3(ii) it is of order at most $q^2 - 1$. Hence $\mu(\Gamma \backslash G) \geq 1/(q^2 - 1)$.

From now on we can assume that $\Gamma$ is a nonunimodular lattice, that is, of finite covolume but not cocompact. Therefore if $E$ is a fundamental domain for $\Gamma$ on $T$, then $E$ is infinite but

$$\sum_{x \in E} \frac{1}{|\Gamma_x|} = 2\mu(\Gamma \backslash G) < \infty.$$  

**Definition 2.9.** An end $e \in \partial T$ is called a **cusp** of $\Gamma$ if for some $x \in V(T)$, $\Gamma_x$ contains a dominant unipotent fixing $e$.

Since $\Gamma$ is nonuniform, there is $x \in V(T)$ with $|\Gamma_x| > q(q^2 - 1)$. From Lemma 2.3(iv) we deduce that $\Gamma$ has a cusp, call it $e$.

**Lemma 2.10.** Assume $x \in V(T)$ and $\Gamma$ is a nonunimodular lattice in $G$ containing a dominant unipotent w.r.t. $x$ fixing an end $e$ (so $e$ is a cusp for $\Gamma$). Then

(i) $|\Gamma_e \backslash S(e, x)| < \infty$.

(ii) For some $k \in \mathbb{N}$, $|\Gamma_e \backslash S(e, \phi^k(x))| = 1$.

**Proof.** (i) First we note that from Proposition 2.5 it follows that $\Gamma_e \backslash S(e, x)$ is embedded in $Y = \Gamma \backslash X$. We will prove that if $z \in S(e, x)$ then $\Gamma_z$ is bounded. So, if there are infinitely many $\Gamma_z$ orbits in $S(e, x)$ we get in $Y$ infinitely many vertices with bounded stabilizers. This would contradict the finiteness of the volume of $Y$. To prove that $\Gamma_z$ is bounded: First recall from Lemma 2.4(ii) and Proposition 2.3(iv) that $\Gamma_z \subseteq G_e$. Secondly, $\Gamma_z$ is a finite subgroup of $G_e = T_x U_e$. The only torsion in $T_x$ is of order $q - 1$ so $[\Gamma_z : \Gamma_z \cap U_x] \leq q - 1$. Moreover, there exists $v \in U_e$ such that $z = vx$ and hence $\Gamma_z = \{ y \in \Gamma | yvx = vx \} = \{ \gamma \in \Gamma | v^{-1} \gamma vx = x \}$. Hence, $v \Gamma_z v^{-1} \subseteq G_x$; but because $\Gamma_z \cap U_e$ commutes with $v$, $\Gamma_z \cap U_e \subseteq \Gamma_x$. This proves that $|\Gamma_z| \leq (q - 1)|\Gamma_x|$ and (i) is proven.

To prove (ii): Let $y_1, \ldots, y_r$ be representatives for $\Gamma_e \backslash S(e, x)$. Then for some $k \in \mathbb{N}$, $\phi^k(y_1) = \phi^k(y_2) = \cdots = \phi^k(y_r)$. Since $\phi$ commutes with $\Gamma_e$, it follows that $|\Gamma_e \backslash S(e, \phi^k(x))| = 1$.  


Now let \( \epsilon \) be a cusp of \( \Gamma \) and \( x \in V(T) \) such that \( \Gamma_x \) contains a dominant unipotent fixing \( \epsilon \). Assume \( x \in S^n(\epsilon) \). We claim that for some \( m \), for no \( y \in S^{n-m}(\epsilon) \), \( \Gamma_y \) contains a dominant unipotent fixing \( \epsilon \). Otherwise, if \( y \) contains such an element we can assume \( \varphi^m(y) = x \). Hence \( \Gamma_y \subseteq \Gamma_x \). From Proposition 2.5 it follows that the ray from \( y \) toward \( \epsilon \) injects into \( Y = \Gamma \setminus T \). This implies \( m/|\Gamma_x| < \mu(\Gamma \setminus G) \). Hence \( m \) must be bounded.

Now let
\[
l = \min\{k \in \mathbb{Z} \mid \exists y \in S^k(\epsilon) \text{ such that } \Gamma_y \text{ contains a dominant unipotent fixing } \epsilon\}.
\]

The ball \( B^l(\epsilon) = \bigcup_{k \geq l} S^k(\epsilon) \) will be called the \( \epsilon \)-tritory w.r.t. \( \Gamma \). By Proposition 2.5, \( \Gamma \setminus B^l(\epsilon) \) injects into \( Y = \Gamma \setminus T \). For future use, we remark here that for two different cusps of \( \Gamma \) the corresponding tritories are disjoint.

We are now going to analyze the contribution of \( \Gamma \setminus B^l(\epsilon) \) to the volume of \( Y \).

Lemma 2.11. If \( x \in S^l(\epsilon) \), then \( |\Gamma_x| \leq (q-1)q^2 \).

Proof. First note that \( \Gamma_x \subseteq G_\epsilon \), hence it fixes \( \varphi(x) \). Let \( J = \varphi^{-2}(x) \), then \( |J| = q^2 \) and the abelian \( p \)-sylow subgroup \( P \) of \( \Gamma_x \) acts on \( J \). If \( |P| > q^2 \) then some nontrivial element \( u \in P \) fixes some \( y_0 \in J \) and by 2.4(i) it fixes every \( y \in J \). Thus \( u \) is a dominant unipotent in \( \Gamma_x \) for every \( z \in \varphi^{-1}(x) = \varphi(J) \). This contradicts the minimality of \( l \). Now \( \Gamma_x/P \) is a finite subgroup of the torus \( T \approx \mathbb{F}^* \), whence its order is at most \( q - 1 \). This prove the lemma.

Now fix \( x \in S^l(\epsilon) \) and let \( r = \{x = x_1, x_2, x_3, \ldots\} \) be the ray from \( x \) toward \( \epsilon \). By Proposition 2.5, \( r \) injects into \( Y \). We have just proved that \( \Gamma_{x_1} \leq (q-1)q^2 \). We will use it to continue by induction to prove that \( \Gamma_{x_n} \leq (q-1)q^{n+1} \). Indeed, assume this is true for \( n-1 \). Now, \( \Gamma_{x_{n-1}} \subseteq \Gamma_{x_n} \) (since \( \Gamma_{x_{n-1}} \subseteq \Gamma_\epsilon \)) and the index is at most \( q \) (since \( \Gamma_{x_n} \) fixes \( x_{n+1} \) and acts on the set of \( q \) elements \( \varphi^{-1}(x_n) \)). Hence \( \Gamma_{x_n} \leq (q-1)q^{n+1} \).

The ray \( r \) injects into \( Y \), as previously mentioned. Assume that \( x_1 \) is a blue (resp. red) vertex and compute the volume of \( Y \) (or equivalently of a fundamental domain containing this ray) along the blue (resp. red) vertices. This volume is at least \( \sum_{n \text{ odd}} 1/((q-1)(q^{n+1})) = 1/((q-1)(q^2-1)) \). Thus \( \mu(\Gamma \setminus G) \geq 1/((q-1)(q^2-1)) = \mu(\Gamma \setminus G) \) and the proof of Theorem 1 is complete.

Some additional corollaries can be deduced from the proof.

Corollary 2.12. If \( \Gamma \) is a nonuniform lattice in \( G \) with \( c \) equivalent classes of cusps then
\[\mu(\Gamma \setminus G) \geq \frac{c}{2} \frac{1}{(q-1)^2 q}.\]
Proof. We remarked in the proof that the tritories of the cusps are disjoint. We have to remark, however, that not every (equivalent class of a) cusp contributes the amount of \(1/((q^2 - 1)(q - 1))\): some of them are “blue cusps” and some are “red cusps,” that is, for some cusps \(e\), \(S^l(e)\) (where \(l\) is like in the proof) is a blue horosphere and for others it might be red. We can instead sum over all vertices blue and red: so, \(2\mu(\Gamma \setminus G) \geq c \cdot \sum_{n=1}^{\infty} 1/((q - 1)q^{n+1})\). This finishes the proof of the corollary.

The next corollary is a special case of a result that is true for every nonuniform lattice in any rank one semisimple group over \(F\) (see [R, Lu]). We have already proved almost all that is necessary for our \(G\); we introduce this corollary from which we will deduce another result on covolumes of lattices.

**Corollary 2.13.** Let \(\Gamma\) be a nonuniform lattice in \(G\). Then \(Y = \Gamma \setminus T\) is a graph built up from a finite set of vertices plus finitely many infinite rays.

**Proof.** We show in the proof (Lemma 2.10(ii)), that for every cusp \(e\) the image of some horoball around \(e\) in \(Y\) is just a ray. From the previous corollary, it follows that there are only finitely many equivalent classes of cusps. It remains only to show that if \(E\) is a fundamental domain for \(\Gamma\), then any ray in \(E\) is going toward a cusp of \(\Gamma\). Let \(r = \{x_1, x_2, \ldots\}\) be such a ray in \(E\) going toward \(e\) in \(E\). Since \(\sum_{i=1}^{\infty} 1/|\Gamma_{x_i}| < \infty\), it follows that, for almost all \(i\), \(|\Gamma_{x_i}| > (q + 1)!\) So by slightly changing the ray, we can assume this for every \(i\).

This implies: (i) \(\Gamma_{x_i}\) contains a dominant unipotent and so \(\Gamma_{x_i}\) fixes an end \(e_i\); (ii) \(\Gamma_{x_i}\) and \(\Gamma_{x_{i+1}}\) have a nontrivial intersection whence \(e_i = e_{i+1}\) and by induction they all fix the same end \(e'\).

We claim that \(e = e'\). If not, the ray \(r'\) from \(x_1\) to \(e'\) satisfies \(r \cap r' = \{x_1, \ldots, x_N\}\). Hence for \(i \geq N\), as \(\Gamma_{x_i}\) fixes \(e'\), \(\Gamma_{x_i}\) is a subgroup of \(\Gamma_{x_N}\). This is impossible since the orders of \(\Gamma_{x_i}\) are unbounded. This proves that the only way to go to infinity in \(E\) is toward cusps of \(\Gamma\) and the corollary is proved.

**Corollary 2.14.** For every lattice \(\Gamma\) in \(G\), \(\mu(\Gamma \setminus G)\) is a rational number.

**Proof.** This is clear for cocompact lattices. For nonuniform lattices we claim that the contribution of a ray going toward a cusp must be a sum of geometric progression. Indeed, if \(x_1, x_2, \ldots\) is such a ray then for sufficiently large \(i\), \(\Gamma_{x_i} \subseteq \Gamma_{x_{i-1}}\) and the only neighbors of \(x_i\) in \(Y\) are \(x_{i-1}\) and \(x_{i+1}\). By [S2, I.5.3] we must have: \(|\Gamma_{x_i} : \Gamma_{x_{i-1}}| + 1 = q + 1\). So \(|\Gamma_{x_i} : \Gamma_{x_{i-1}}| = q\). Hence the contribution of every cusp is rational and the rest follows from (2.13).

It might be interesting to say more on the set of numbers which appears as covolumes of lattices in \(G\). In the case of \(G = SL_2(C)\) a very interesting situation occurs (see [Th, Chapter 6]). On the other hand, H. Bass showed that for \(G = Aut(X_k)\) \(X_k\) the \(k\)-regular tree \((k \geq 3)\), every positive real number is the covolume of some lattice in \(G\).
3. Cocompact lattices

Again let $F$ be a local nonarchimedean field of characteristic $l \geq 0$, with a residue field of order $q = p^\alpha$ (where $p = l$ if $l > 0$). We begin with a Lemma that will enable us later to construct cocompact lattices in $G = SL_2(F)$. We continue the previous notation. In particular, $T$ is the tree associated with $G$.

**Lemma 3.1.** Let $x_1$ and $x_2$ be two adjacent vertices of $T$ and $A_i$ ($i = 1, 2$) a finite subgroup of $G$ such that

(a) $A_i$ acts transitively on the set of $q + 1$ neighbors of $x_i$,
(b) $\text{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$.

Then $\Gamma = \langle A_1, A_2 \rangle$ the group generated by $A_1$ and $A_2$ is a cocompact lattice in $G$ with a fundamental domain on $T$ consisting of two vertices $x_1$ and $x_2$. Moreover, $\Gamma$ is isomorphic to the free product with amalgamation $\Gamma \simeq A_1 *_{A_1 \cap A_2} A_2$ and $\Gamma_{x_1} = A_i$.

**Proof.** Let $\Delta$ be the abstract free product with amalgamation $\Delta = A_1 *_{A_1 \cap A_2} A_2$ and $\varphi$ the obvious homomorphism from $\Delta$ to $\Gamma$. Let $a_1, \ldots, a_r$ (resp. $b_1, \ldots, b_r$) be representative of the nontrivial cosets of $A_1 \cap A_2$ in $A_1$ (resp. $A_2$). In fact, $r = s = q$ but we do not need it now. A word $w \in \Delta$ has a normal form $w = a_{i_1} b_{j_1} \cdots a_{i_t} b_{j_t} c$ where $c \in A_1 \cap A_2$ (and with the possibility that $a_{i_t} = 1$ or $b_{j_t} = 1$).

Induction on $t$ shows that if $e$ is in the edge between $x_1$ and $x_2$ then $d(we, e) \geq t$ (when distance between edges is defined in the obvious way). In particular $w(e) \neq e$ unless $w \in A_1 \cap A_2$. This proves that $\Gamma$ is isomorphic to $\Delta$ and that $\Gamma_{x_1} = A_i$. Hence $\Gamma$ is also discrete. Moreover, from general nonsense, it follows that the set $E = \{ e' | e' \text{ an edge of } T \text{ and } d(e', e) \leq d(\gamma e', e), \forall s \in \Gamma \}$ forms the set of edges of a connected subtree which contains a representative from every orbit of edges of $T$. But $E$ contains $e$ and none of the edges adjacent to $e$ (by condition (a)). Hence $E = \{ e \}$ and so $\Gamma$ acts transitively on the edges of $T$. This finishes the proof of the lemma.

**Definition 3.2.** Let $P = \{ q | q$ is a prime power in $N \}$. Define the function $\lambda: P \to \mathbb{Q}$ by

(i) $\lambda(q) = 1/(2(q + 1))$ if $(q, 2) = 1$ and $q \neq 5, 7, 11, 19, 29$,
(ii) $\lambda(q) = 1/(q + 1)$ if $q = 2^\alpha$,
(iii) $\lambda(q) = 1/120$ for $q = 11, 19, 29$,
(iv) $\lambda(q) = 1/48$ for $q = 7$,
(v) $\lambda(q) = 1/24$ for $q = 5$.

**Theorem 3.3.** Let $F$ be a local nonarchimedean field with a residue field of order $q = p^\alpha$. Then,

(i) $G = SL_2(F)$ has a cocompact lattice of covolume $\lambda(q)$.
(ii) If $\text{char}(F) = p = l > 0$ then every cocompact lattice in $G$ has covolume greater-equal $\lambda(q)$. 
For the proof of the theorem, we need the following result (see [La, Theorem 2.3, p. 185], [F2, Theorem 1, p. 201] and [S4, p. 281]).

**Proposition 3.4.** Let \( F \) be a field of characteristic \( 1 \) and \( S \) a finite subgroup of \( SL_2(F) \) of order prime to \( 1 \). Let \( H \) be the image of \( S \) in \( PSL_2(F) \). Then

(a) One of the following cases occurs:

(i) \( H \) is cyclic and \( S \) is contained in a Cartan subgroup;

(ii) \( H \) is dihedral and \( S \) is contained in the normalizer of a Cartan subgroup but not in the Cartan subgroup itself; or

(iii) \( H \) is isomorphic to \( A_4, S_4 \), or \( A_5 \).

(b)

(i) If \( \text{char}(F) > 2 \) then \( PSL_2(F) \) contains a subgroup isomorphic to \( A_4 \).

(ii) If \( F = F_q \) and \( \text{char}(F) \neq 2, 3 \) then \( PSL_2(F) \) contains a subgroup isomorphic to \( S_4 \) if and only if \( q \equiv \pm 1 \) (mod 8).

(iii) If \( F = F_q \) and \( \text{char}(F) = 2, 3, 5 \) then \( PSL_2(F) \) contains a subgroup isomorphic to \( A_5 \) if and only if \( q \equiv \pm 1 \) (mod 5).

The reader is referred to the above references for the definition of split and nonsplit Cartan subgroups.

**Lemma 3.5.** For every \( q \in \mathbb{P} \), \( SL_2(q) \) has a subgroup \( M_q \) of order \( \lambda(q)^{-1} \) that acts transitively on the projective line \( \mathbb{P}^1(q) \).

**Proof.** If \( q \neq 2^a, 5, 7, 11, 19, 29 \) take \( M_q \) to be the normalizer of a nonsplit Cartan subgroup. This is a group of order \( 2(q+1) \) whose intersection with "the" Borel subgroup \( B_q \) of \( SL_2(q) \) is \( \pm I \). Since \( B_q \) is the stabilizer of \( \infty \in \mathbb{P}^1(q) \), this shows that the action of \( M_q \) is transitive. Similarly, for \( q = 2^a \) we take as \( M_q \) a nonsplit Cartan subgroup. It is of order \( q+1 \) and has trivial intersection with \( B_q \), and hence acts transitively on \( \mathbb{P}^1(q) \).

Now if \( q = 11, 19, 29 \), then \( PSL_2(q) \) contains \( A_5 \) by (3.4). Take \( M_q \) to be the preimage of \( A_5 \) in \( SL_2(q) \). So \( |M_q| = 120 \) and \( |B_q| = (q-1)q \). Hence \((|M_q|, |B_q|) = 10, 6, 4 \) if \( q = 11, 19, 29 \) respectively. So \( |M_q \cap B_q| \leq 10, 6, 4 \) respectively. But this bound is exactly equal to \( 120/(q+1) \) in all these three cases which proves that the action of \( A_5 \) on \( \mathbb{P}^1(q) \) is transitive in these cases.

Similarly, for \( q = 7 \), take \( M_q \) to a preimage of \( S_4 \) (which exists by (3.4)), so \( |M_q| = 48 \). As \((48, (q-1)q) = 6 = 48/(q+1) \), \( M_q \) acts transitively on \( \mathbb{P}^1(q) \). Finally, for \( q = 5 \), let \( M_q \) be a preimage of \( A_4 \). So \( |M_q| = 24, (24, (q-1)q) = 4 = 24/(q+1) \) and the lemma is proven.

Incidentally, we note that in all cases, we took \( M_q \) to contain \( \pm I = \text{Ker}(SL_2(q) \to PSL_2(q)) \). Of course if \( q = 2^a \) then \( -I = I \).

**Proof of Theorem 3.3.** Let \( K_1 = G_{v_1} \) and \( K_2 = G_{v_2} \) as in §1. Then \( K_1 \simeq K_2 \simeq SL_2(D) \) and the projection \( \phi: K_1 \to SL_2(q) \) has a kernel that is a pro-p
group. If \( M \) is a subgroup of \( SL_2(q) \) of order prime to \( p \), then by the Schur–Zassenhaus Theorem [Su, p. 235] applied to the pro-finite group \( \varphi^{-1}(M) \), \( M \) can be lifted to \( K_i \). (If \( \text{char}(F) > 0 \), \( SL_2(q) \) is anyway a subgroup of \( K_i \).)

In particular, take the \( M_q \) presented in (3.5). Because \( (\lambda(q)^{-1}, q) = 1 \), \( M_q \) can be considered as a subgroup of \( K_2 \). Moreover, it has an intersection with \( K_2 \) which is of order equal to \( M_q \cap B_q \). This is because the set of neighbors of \( x_1 \) can be identified with \( \mathbb{P}^1(q) \) and so the stabilizer of \( x_2 \) in \( M_q \) is like the stabilizer of a point of \( \mathbb{P}^1(q) \). A case by case checking shows that a group \( M_q' \) can be chosen in \( K_2 \) of order \( \lambda(q) - 1 \), acting transitively on the neighbors of \( x_2 \) and such that \( M_q' \cap K_1 = M_q \cap K_2 = M_q' \cap M_q \cdot \) (Only the last property needs some verification which is left to the reader.)

We can now take \( \Gamma = \langle M_q', M_q' \rangle \). Lemma 3.1 shows that \( \Gamma \) is a cocompact lattice of covolume \( |M_q'|^{-1} = \lambda(q) \) and part (i) of the theorem is proven.

To prove part (ii) we recall again that a cocompact lattice does not contain unipotent elements ([GGPS, p. 10]) and that an element of order \( p \) in \( G \) is unipotent. Hence if \( E \) is a fundamental domain of a cocompact lattice \( \Gamma \) and \( x \) is a blue vertex in \( E \) then \( \text{Vol}(\Gamma \backslash G) \geq 1/\Gamma_x \) and \( \Gamma_x \) is a subgroup of \( K_1 \simeq SL_2(D) \) of order prime to \( p \), whence mapped injectively into \( SL_2(q) \).

By (3.4a) the maximal subgroups of \( SL_2(q) \) of order prime to \( p \) are of orders \( 2(q+1) \) (normalizers of nonsplit Cartan subgroups), \( 2(q-1) \) (normalizers of split Cartan subgroups), \( 120 \), \( 48 \) and \( 24 \) (pre-images of \( A_5 \), \( S_4 \), and \( A_4 \) respectively). This is for odd \( p \), for \( p = 2 \) the possibilities are \( q + 1 \) or \( q - 1 \).

Hence in all cases if \( q \geq 59 \), \( |\Gamma_x| \leq \lambda(q)^{-1} \) and we are done.

If \( 30 < q < 59 \), then a lattice \( \Gamma \) can have a covolume \( \lambda(q) \) only if a fundamental domain \( E \) for \( \Gamma \) has only one blue vertex and one red vertex (that is, \( \Gamma \) is transitive on edges) and \( |\Gamma_x| = 120 \) for every \( x \in E \). But in such a case, if \( E = \{x_1, x_2\} \) is a fundamental domain and \( e \) the edge between \( x_1 \) and \( x_2 \) then \( |\Gamma_x : \Gamma_e| \) should be equal to \( q + 1 \) for \( i = 1, 2 \) (cf. [S2, 1.5.3]). This is impossible since \( \Gamma_e \) contains \( \pm I \) and \( A_5 \) does not have a subgroup of index \( q + 1 \) for \( 30 < q < 59 \).

Hence we can assume \( q < 29 \). If \( q \) is divisible by 2 or 3, \( \Gamma_x \) cannot be of the type described in (3.4a(iii)), and for \( q = 11, 19, 29 \) we might have \( |\Gamma_x| = 120 \) and indeed we constructed a lattice of covolume \( 1/120 \). This leaves us to consider \( q = 5, 7, 13, 17, 23, 25 \). If \( q = 25 \) then \( A_5 \) is impossible as \( \Gamma_x \) and so \( |\Gamma_x| \leq 2(q+1) \) and again we are done. For \( q = 23 \), \( PSL_2(q) \) does not contain \( A_5 \) by (3.4b(iii)) and so again \( |\Gamma_x| \leq 2(q+1) \). Similarly for \( q = 13 \), \( PSL_2(q) \) contains neither \( A_5 \) nor \( S_4 \) and hence \( |\Gamma_x| \leq 2(q+1) \). For \( q = 17 \) an additional argument is needed: Indeed \( PSL_2(q) \) contains \( S_4 \) but for having a lattice of covolume less than \( 1/(2(q+1)) \) we need that such \( \Gamma \) would act transitively on edges and \( S_4 \) should have a subgroup of index 18 which is impossible.

If \( q = 7 \), \( PSL_2(q) \) contains \( S_4 \) and indeed we have a lattice of covolume
1/48. Since any other subgroup of order prime to 7 is of order less than 48, this is the minimal possible covolume. Similarly, for \( q = 5 \) and \( A_4 \) and now the theorem is completely proven.

We turn our attention now to the situation when \( \text{char}(F) = 0 \). Here the list of finite subgroups of \( SL_2(F) \) is also very limited. Still, for general local field \( F \), \( SL_2(F) \) might contain a large finite cyclic \( p \)-group of size larger than \( 2(q+1) \) (see [F1, 3.5]). We do not know if typically \( 1/(2(q+1)) \) is the minimal covolume since these \( p \)-groups (which are not unipotent) can in principle appear in cocompact lattices. It is not even clear to us that the minimal covolume is obtained with lattices whose fundamental domain has only two vertices. (Note that this was the situation in all cases when \( \text{char}(F) > 0 \), but we deduce it only using case by case analysis. It will be interesting to understand it from a more conceptual point of view. The theory developed in [Ba, BK] might be useful for that purpose.)

Anyway for the most interesting fields of characteristic 0, that is \( F = Q_p \) the \( p \)-adic numbers, we can give a definite result:

**Theorem 3.6.** Let \( F = Q_p \) the field of \( p \)-adic numbers, and \( G = SL_2(F) \). Then

\[
\min\{\mu(\Gamma \backslash G) | \Gamma \text{ a lattice in } G\} = \begin{cases} 
\lambda(p) & \text{if } p \neq 2, 3, \\
\frac{1}{12} & \text{if } p = 3, \\
\frac{1}{24} & \text{if } p = 2.
\end{cases}
\]

*Proof.* Every lattice in \( G \) is cocompact by a result of Tamagawa (cf. [S2, p. 84]).

If \( p \neq 2, 3 \) we claim that \( SL_2(Q_p) \) does not contain elements of order \( p \). Indeed, if \( \alpha \) is such an element then its eigenvalues \( \beta \) and \( \beta^{-1} \) satisfy \( \beta^p = 1 \). But as \( \alpha \) is a \( 2 \times 2 \) matrix, the irreducible polynomial of \( \beta \) over \( Q_p \) is of degree \( \leq 2 \). Just like over the rationals, the Eisenstein criterion implies that this is impossible for \( p \neq 2, 3 \).

Thus, every finite subgroup \( H \) of \( SL_2(Q_p) \) is of order prime to \( p \) and it is also isomorphic to a subgroup of \( SL_2(p) \). So all the discussion in the proof of Theorem 3.3(ii) is valid also in this case and so volume \( (\Gamma \backslash G) \geq \lambda(p) \). By 3.3(i), lattices with covolume \( \lambda(p) \) do exist and the case \( p \neq 2, 3 \) is done.

The case \( p = 3 \): We claim that a finite subgroup \( S \) of \( SL_2(Q_p) \) is of order at most 24. Indeed such a subgroup can be assumed (by (2.3)(i)) to be in \( SL_2(Z_3) \). Let \( N = \text{Ker}(SL_2(Z_3) \to SL_2(F_3)) \). Then \( N \) is a pro-3 group. If \( \alpha \) is a matrix in \( N \) different from the identity and satisfying \( \alpha^3 = 1 \), then its eigenvalues are \( \{\lambda, \lambda^{-1}\} \) the set of two nontrivial cubic roots of the identity. Hence \( \text{tr}(\alpha) = -1 \) and \( \det(\alpha) = 1 \). Thus \( \alpha^2 + \alpha + 1 = 0 \). On the other hand \( \alpha \equiv 1 \mod 3 \) and so

\[
9(\alpha^2 + \alpha + 1)^2 = A^2 - 2A + 1 = A^2 - 2A + 1 - (A^2 + A + 1) = -3A.
\]

Hence \( 3|A \) which is a contradiction. This proves that \( N \) is torsionfree and so \( S \) must inject into \( SL_2(F_3) \). The order of the latter is 24 whence \( |S| \leq 24 \).
Now we claim that $SL_2(\mathbb{Q}_3)$ does have a subgroup of order 24. (Note that $SL_2(\mathbb{Q})$ does not have such a subgroup!) Here is a way to see it:

Let $H$ be the Hamiltonian quaternion algebra. Then $H$ splits over all odd primes and hence $H(\mathbb{Q}_p) \cong M_2(\mathbb{Q}_p)$. The isomorphism takes the group $H'(\mathbb{Q}_p)$ of elements of norm one in $H(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$. Now, $H'(\mathbb{Q})$ has a subgroup of order 24, the group $\{\pm 1, \pm i, \pm j, \pm k\} \cdot \{\frac{1}{2}(1+i+j+k)\}$ and hence $SL_2(\mathbb{Q}_p)$ does as well. (A more conceptual way to prove such a result is to use the theory of Schur indices and to apply it to the representations of $SL_2(\mathbb{F}_3)$ over $\mathbb{Q}_3$—see [F3] or [S3].)

The above discussion shows that the map $SL_2(\mathbb{Z}_3) \rightarrow SL_2(\mathbb{F}_3)$ splits and so $G_x = SL_2(\mathbb{Z}_3)$ has a subgroup of order 24 which acts transitively on the neighbors of $x_1$ and we can use, as before, Lemma 3.1 to construct a lattice of volume 1/24 in $SL_2(\mathbb{Q}_3)$.

The case $p = 2$: Denote $N = \text{Ker}(SL_2(\mathbb{Z}_2) \rightarrow SL_2(\mathbb{F}_2))$. This time $N$ has torsion $\{\pm 1\}$. We claim that this is the only torsion. Indeed if $A \in N$ is an element of order two then $A^2 = 1$ and so the eigenvalues are $\{\pm 1\}$. Since $\det(A) = 1$, $A$ must be either $I$ or $-I$. Moreover, if $A \in N$ is of order 4, and $\det A = 1$ we know that the eigenvalues of $A$ must be $\{\pm i\}$ where $i = \sqrt{-1}$ and so $\text{tr} A = 0$. Hence $A$ satisfies the characteristic polynomial $A^2 + 1 = 0$. Since $A \equiv I \pmod{2}$, we have $4| (A - I)^2 = A^2 - 2A + 1 = -2A$; hence $2|A$, a contradiction. The group $N$ is a pro-$2$ group, so every finite subgroup of it is a 2-group. The above argument shows that a finite subgroup of $N$ is of exponent 2 and hence it is a subgroup of $\{\pm I\}$. This together with the fact that $|PSL_2(\mathbb{F}_2)| = 6$, shows that a finite subgroup of $SL_2(\mathbb{Z}_2)$ is of order at most 12.

There is in $SL_2(\mathbb{Z}_2)$ a subgroup of order 12. (Again, this is not the case for $SL_2(\mathbb{Z})$ or $SL_2(\mathbb{Q})$!) A quick way to see it: $\mathbb{Z}_2$ contains a cubic root of unity $w$ (by Hensel lemma). The group generated by $-I$, $\begin{pmatrix} w & 0 \\ 0 & w^2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is of order 12. This group is mapped onto $SL_2(\mathbb{F}_2)$ and again one can use (3.1) to build a lattice of covolume 1/12, which is the minimum possible.

**Remark.** For every odd $p$, we have an arithmetic lattice in $SL_2(\mathbb{Q}_p)$ constructed using the Hamiltonian quaternion algebra (see [GVP, §IX.1]). This lattice has covolume 1/24 so for $p = 3, 5$ this is of minimal possible covolume.

We end with a corollary: Hurwitz Theorem asserts that if $S_g$ is a Riemann surface of genus $g$ then $|\text{Aut}(S)| \leq 84(g - 1)$. This is equivalent to: Let $\Gamma$ be a torsionfree cocompact subgroup of $PSL_2(\mathbb{R})$ and assume $\Gamma$ is isomorphic to $\pi_1(S_g)$ (note that every such $\Gamma$ satisfies this assumption for some $g \geq 2$) then $[\text{NG}(\Gamma): \Gamma] \leq 84(g - 1)$. Hurwitz theorem can be deduced from Siegel's theorem (cf. [Gr]), and similarly we have:

**Theorem 3.7.** Let $F$ be a local nonarchimedean field with residue field of order $q$ and $M_F = \min\{\text{Vol}(\Gamma \backslash SL_2(F)) | \Gamma \text{ a cocompact lattice in } SL_2(F)\}$.
Let $\Gamma$ be a torsion free cocompact lattice in $SL_2(F)$. Then:

(i) $\Gamma$ is a free group on say $g$ generators; and

(ii) $[N_G(\Gamma): \Gamma] \leq (g - 1)/(M_F(q - 1))$.

Note that we have computed $M_F$ for all fields of positive characteristic and for $F = \mathbb{Q}_p$. For $q \geq 60$ and odd we had $M_F = 1/(2(q + 1))$ and hence $[N_G(\Gamma): \Gamma] \leq 2(q + 1)/(q - 1)$.

Proof. (i) is due to Ihara (cf. [S2, p. 82]). To prove (ii) note that $\Gamma \backslash T$ is a regular $(q + 1)$-graph so if it has $n$ vertices it has $\frac{1}{2}(q + 1)n$ edges. The fundamental group of $\Gamma \backslash T$ is free on $g = \frac{1}{2}(q + 1)n - (n - 1) = \frac{1}{2}(q - 1)n + 1$ generators. Hence, $n = 2(g - 1)/(q - 1)$. Half of them are blue and half are red, so $\text{Vol}(\Gamma \backslash G) = (g - 1)/(q - 1)$. Thus $M_F \leq \text{Vol}(N_G(\Gamma) \backslash G) = \frac{\text{Vol}(\Gamma \backslash G)}{[N_G(\Gamma): \Gamma]} = (g - 1)(q - 1) \cdot [N_G(\Gamma): \Gamma]^{-1}$ and so $[N_G(\Gamma): \Gamma] \leq (g - 1)/(q - 1)$.

4. Questions

The most interesting question we leave open is the one mentioned in the introduction; namely, given a semisimple Lie group $G$, let $\Gamma$ be a lattice of minimal covolume in $G$. For which groups $G$, is $\Gamma$ cocompact and for which groups is it a nonuniform lattice? We do not know the answer for any other groups but those mentioned in this paper. Even the case $G = SO(n, 1)$ for $n \geq 4$ is open and seems to suggest an interesting geometric question. On the other hand for higher rank groups, in light of Margulis’s arithmeticity theorem and the recent formula of Prasad [Pr], our problem gives an interesting number theoretic problem.

Note that our Theorems 1 and 2 show that for $G = SL_2(F_2((1/t)))$ the minimal covolume of lattices is $\frac{1}{3}$ and this minimum is obtained by a cocompact lattice as well as by a nonuniform lattice. Are there other Lie groups with this property?

A second problem left open is computing the minimal covolume of lattices in $SL_2(F)$ for fields $F$ that are finite extensions of $\mathbb{Q}_p$.

The uniqueness of the lattices of minimal covolume was not studied. It is also of interest, in comparison to the beautiful result for $SL_2(\mathbb{C})$ (see [Th]) to understand the subset of $\mathbb{R}$ of those numbers that are covolumes of lattices in $SL_2(F_q((1/t)))$.

More results and questions on covolumes of lattices can be found in [Bo, BP, Me, EGM, MR] and references therein.

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