

POLYNOMIAL Diffeomorphisms of \mathbf{C}^2 . II: STABLE MANIFOLDS AND RECURRENCE

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1. INTRODUCTION

Friedland and Milnor [FM] have shown that from a dynamical point of view the polynomial diffeomorphisms of \mathbf{C}^2 fall naturally into two classes. The first class consists of diffeomorphisms with simple dynamics. The diffeomorphisms in this class have periodic points of at most finitely many periods and topological entropy zero. The second class contains the well-known Hénon map

$$f(x, y) = (y, y^2 - ax + c).$$

The diffeomorphisms in this class have complicated dynamics: in particular, they have periodic points of infinitely many periods and positive topological entropy (see [FM, S]).

We can distinguish between these classes by considering the growth of the degrees of iterates of the diffeomorphism. Define $\deg(f)$ to be the maximum of the degrees of the coordinate functions. This quantity is not a conjugacy invariant of f , hence not a dynamical invariant. We can construct a conjugacy invariant, which we call the *dynamical degree*, as follows:

$$d = d(f) = \lim_{n \rightarrow \infty} (\deg f^n)^{1/n} = \lim_{n \rightarrow \infty} (\deg f \circ \dots \circ f)^{1/n}.$$

The maps with simple dynamics are those with $d = 1$. The maps with complicated dynamics are those with $d > 1$; in the remainder of the paper we will restrict our attention to this second class.

In studying polynomial diffeomorphisms of \mathbf{C}^2 it is often useful to make analogies with the theory of polynomial maps of \mathbf{C} . If g is a polynomial map of \mathbf{C} then

$$K = \{q \in \mathbf{C} : \{g^n(q) : n = 1, 2, 3, \dots\} \text{ is bounded}\}$$

is the “filled Julia set,” and the boundary of K is the Julia set, which is denoted by J . Analogous terminology for polynomial diffeomorphisms has been introduced in [HO], where the authors define

$$K^\pm = \{q \in \mathbf{C}^2 : \{f^{\pm n}(q) : n = 1, 2, 3, \dots\} \text{ is bounded}\},$$

Received by the editors August 14, 1990.

1991 *Mathematics Subject Classification*. Primary 58F99, 32H99.

The first author was partially supported by NSF grant DMS-8602020. The second author was partially supported by NSF grant DMS-8803228.

and furthermore $J^\pm = \partial K^\pm$, $K = K^+ \cap K^-$, and $J = J^+ \cap J^-$. A point p is said to be *periodic* if $g^m(p) = p$ for some $m \geq 1$. A periodic point $p \in \mathbf{C}$ is *expanding* if $|Dg^m(p)| > 1$. A fundamental result of Fatou and Julia dating from approximately 1920 is the dynamical characterization of the Julia set as the closure of the set of expanding periodic points. In this paper we will prove an analogous dynamical characterization of the sets J^\pm .

The stable and unstable sets of a point p are defined as

$$W^s(p) = \left\{ q : \lim_{n \rightarrow \infty} d(f^n(p), f^n(q)) = 0 \right\},$$

$$W^u(p) = \left\{ q : \lim_{n \rightarrow -\infty} d(f^n(p), f^n(q)) = 0 \right\}.$$

Let $p \in \mathbf{C}^2$ be a periodic point with eigenvalues λ_1 and λ_2 of $Df^m(p)$ labeled so that $|\lambda_1| \leq |\lambda_2|$. We say p is a *saddle* if $|\lambda_1| < 1 < |\lambda_2|$. If p is a saddle then it follows from the Stable Manifold Theorem that the stable and unstable sets of p are immersed complex manifolds biholomorphically equivalent to \mathbf{C} .

Our dynamical characterization of J^\pm is given by the following theorem.

Theorem 1. *Let f be a polynomial diffeomorphism of \mathbf{C}^2 satisfying $d(f) > 1$. Let p be a saddle point of f . Then J^+ is the closure of the stable manifold $W^s(p)$ and J^- is the closure of $W^u(p)$.*

Theorem 1 was conjectured by J. Hubbard. For the special case of hyperbolic diffeomorphisms this result was proved in [BS1].

A periodic point p is a *sink* if both eigenvalues of $Df^m(p)$ are less than one in absolute value. If p is a sink, then the set $W^s(p)$ is an open set containing p , which is called the *basin of attraction* of p . The basin of attraction of a sink is biholomorphically equivalent to \mathbf{C}^2 (cf. [RR]), and if $d(f) > 1$, then it cannot be all of \mathbf{C}^2 . Proper subsets of \mathbf{C}^2 that are biholomorphically equivalent to \mathbf{C}^2 are known as Fatou-Bieberbach domains. Some results on the geometry of Fatou-Bieberbach domains constructed from polynomial diffeomorphisms are contained in [BS2]. Here we prove

Theorem 2. *The boundary of any basin of attraction is J^+ .*

It is possible to construct polynomial diffeomorphisms with arbitrarily many basins of attraction. According to Theorem 2 these basins must share a common boundary. Such examples are reminiscent of the ‘‘lakes of Wada’’ construction of three regions in the sphere with the same boundary. In such examples the geometry of the regions is forced to be quite intricate.

Corollary 1. *If f has more than one basin component then J^+ is not an embedded topological manifold at any point.*

Proof. Let q be a point of J^+ . If J^+ were an embedded manifold at q we could find a neighborhood U of q so that $U - J^+$ has two components. Since J^+ is the boundary of K^+ , the set $U - J^+$ must meet the complement of K^+ . By Theorem 2 the set $U - J^+$ also meets all basin components. Thus there can be at most one basin component.

In §5, we discuss recurrent domains, which we define to be connected components of $\text{int } K^+$ with some form of recurrence; basins of attraction are a special case. We show that all recurrent domains arise either as basins of attraction of sinks, Siegel disks, or Herman rings. In the case of a Siegel disk or Herman ring, we show that f is actually conjugate to the restriction of a linear map. We also show that the analog of Theorem 2 holds for these domains. In fact, the setting of recurrent domains seems to be most natural for the technique of the proof of Theorem 2.

Our approach will be to obtain information on J^+ by studying the current μ^+ , whose support is exactly J^+ . In §§2–4 we show that μ^+ is the limit of currents of the form $(d^{-n})[f^{-n}M]$, where M is a certain complex manifold, and $[f^{-n}M]$ denotes the current of integration over the set $f^{-n}M$. It had been shown earlier in [BS1,2] that μ^+ could be obtained this way when M was an algebraic subvariety of \mathbf{C}^2 . Here we obtain the more useful result (Theorem 3) that it suffices to take M to be an analytic disk satisfying rather minimal hypotheses. Perhaps the main ingredient to the proof of Theorem 3 is a uniqueness result (Proposition 1) that characterizes μ^+ as the unique positive closed current supported on J^+ , which has the correct invariance under f , i.e., $\frac{1}{d}f^*\mu^+ = \mu^+$. This is proved in §4.

We determine the nonwandering and the chain recurrent sets for polynomial diffeomorphisms. These are sets of points in \mathbf{C}^2 that have certain types of recurrent behavior. The case of diffeomorphisms that do not preserve volume appears in §6. The volume preserving case appears in the appendix. In the appendix we also discuss Siegel disks and Herman rings in the volume preserving case.

2. CONVERGENCE TO μ^+

Polynomial diffeomorphisms of \mathbf{C}^2 necessarily have polynomial inverses and are often called polynomial automorphisms. According to [FM] an automorphism f with $d(f) > 1$ is conjugate to an automorphism of the form $f = f_1 \circ \cdots \circ f_m$, where each f_j is a *generalized Hénon mapping*, which has the form

$$f_j(x, y) = (y, p_j(y) - a_jx),$$

where $p_j(y)$ is a polynomial of degree at least 2 and a_j is a nonzero complex number. Without loss of generality we will assume that our maps are not simply conjugate to maps of the form $f = f_1 \circ \cdots \circ f_m$ but are actually equal to maps of this form. For such maps $d(f) = \deg(f) = \prod \deg(f_j)$. We note that the Jacobian determinant $\delta(f)$ is constant and $\delta = \prod a_j$. In this paper we typically assume $|\delta| \leq 1$. The case $|\delta| > 1$ can be treated by replacing f by f^{-1} . If

*Added in proof: Related results have been obtained by J.-E. Fornæss and N. Sibony in their preprint *Complex Hénon mappings and Fatou-Bieberbach domains*. In particular, they give a general uniqueness theorem for positive, closed currents T satisfying $d^{-1}f^*T = T$, subject to a condition on the support of T .

$|\delta| < 1$ then it is easy to show (see [FM]) that K^- has no interior so $K^- = J^-$.

There is a filtration for f as discussed in [BS1]. We recall some of its properties here. We can write \mathbf{C}^2 as a disjoint union $\mathbf{C}^2 = V \cup V^+ \cup V^-$. The set V^- has the property that $f(V^-) \subset V^-$ so once a point enters V^- it remains in V^- . Furthermore $f^n(p)$ diverges to infinity as $n \rightarrow \infty$ if and only if $f^n(p) \in V^-$ for some positive n . The set V^+ has the property that $f^{-1}(V^+) \subset V^+$ and $f^n(p)$ diverges to infinity as $n \rightarrow -\infty$ if and only if $f^n(p) \in V^+$ for some negative n . The set V is compact. Since K is a closed subset of V , the set K is compact.

We will at times find it convenient to pass to higher powers of the mapping f . It follows from the observations above that the sets K^\pm are unchanged if we replace f by f^n for $n \geq 1$.

The function

$$G^\pm(x, y) = \lim_{n \rightarrow \infty} (d^{-n}) \log^+ |f^{\pm n}(x, y)|$$

gives the rate of escape of the orbit of (x, y) to infinity in forward/backward time. G^\pm is continuous on \mathbf{C}^2 , G^\pm is pluriharmonic on $\{G^\pm > 0\}$, and $K^\pm = \{G^\pm = 0\}$. This serves as the Green function for K^\pm with logarithmic singularity at infinity, and it satisfies the functional equation

$$(1) \quad \frac{1}{d} G^\pm(f^{\pm 1}) = G^\pm$$

We will make use of the operator $dd^c = 2i\partial\bar{\partial}$ where

$$\partial\bar{\partial} = \sum_{1 \leq j, k \leq 2} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

We define the current $\mu^\pm := dd^c G^\pm$, and the support of μ^\pm is exactly J^\pm . We may view equation (1) in terms of the pull-back of G^\pm under f^\pm . Thus it is natural to consider the equation

$$\frac{1}{d} f^{\pm 1*} \mu^\pm = \mu^\pm$$

for the pull-backs of μ^\pm , where we define $f^* \mu^\pm := dd^c(f^* G^\pm)$. These facts are contained in [BS1].

Let M denote a locally closed complex submanifold of \mathbf{C}^2 , i.e., for every $q \in M$ there is an open ball \mathcal{B} in \mathbf{C}^2 about q such that $M \cap \mathcal{B}$ is a closed submanifold of \mathcal{B} . The currents μ^\pm induce measures on M by

$$\mu^\pm|_M := (dd^c)_M(G^\pm|_M),$$

where $(dd^c)_M$ denotes the operator dd^c acting intrinsically on M and $G^\pm|_M$ is the restriction of G^\pm to M .

We will consider locally closed submanifolds $M \subset \mathbf{C}^2$ of the following types:

- (i) $M \subset J^+$ or
- (ii) $M \subset X$, where X is algebraic.

Theorem 1 will be a consequence of a characterization of the currents μ^\pm .

Theorem 3. *Let M be a locally closed complex submanifold satisfying (i) or (ii) above, let χ denote a test function on M , and set $c := \int_M \chi \mu^-|_M$. Then*

$$\lim_{n \rightarrow \infty} (d^{-n})f^{*n}(\chi[M]) = \frac{c}{4\pi^2} \mu^+,$$

where $f^{*n}[M]$ denotes the current of integration over $f^{-n}M$ and the convergence holds in the sense of currents.

This result was obtained already in [BS1,2] in the analogous case where M is an algebraic variety and $\chi = 1$.

We will use Theorem 3 to prove Theorem 1.

Proof of Theorem 1. As in Proposition 5.1 of [BS1], it follows that $W^s(p) \subset J^+$. Let M_0 be an open subset of $W^s(p)$ that contains the saddle point p and is locally closed in \mathbb{C}^2 . Since $|Df(p)v| = |\lambda_1 v|$ if v is a tangent vector to $W^s(p)$ at p , it follows that $f^{-n}|_{W^s(p)}$ is not a normal family in a neighborhood of p in $W^s(p)$. Thus p lies in the closure of $\{q \in M_0 : G^-(q) > 0\}$. By the maximum principle, $G^-|_{M_0}$ cannot be harmonic in a neighborhood of q , and thus q is in the support of the Laplacian $(dd^c)_{M_0} G^-|_{M_0}$. It follows that

$$(2) \quad \mu^-|_{M_0}(M_0) = (dd^c)_{M_0} G^-|_{M_0} > 0.$$

By Theorem 3 then, we know that the currents of integration $(d^{-n})[f^{-n}M_0]$ converge to a current whose support is all of J^+ . On the other hand, it is evident that the support of the limit current is contained in the closure of $W^s(p) = \bigcup_n f^{-n}M_0$, which completes the proof.

The hypothesis that f has a saddle point is satisfied for all polynomial diffeomorphisms with $d(f) > 1$. In fact,

- *If $d(f) > 1$ then f has infinitely many saddle points.*

This follows from the formula of Katok [K] relating saddle points and entropy in two dimensions (see also the discussion in [FM, §4]). Theorem 1 together with the observation above gives a new proof of the connectivity of J^\pm , which was proved in [BS1].

3. CLOSED CURRENTS

Let M denote a manifold satisfying the hypotheses of Theorem 3, and let χ denote a test function with $\text{spt } \chi \cap M$ compact. Without loss of generality, we may assume $0 \leq \chi \leq 1$. Let $[M]$ denote the current of integration over M , and let $\mathcal{S}(\chi[M])$ denote the set of all positive $(1, 1)$ -currents that are obtained as limits (in the sense of currents) of subsequences of the set $\{(d^{-n})f^{*n}(\chi[M]) : n = 1, 2, 3, \dots\}$. The proof of Theorem 3 then, is equivalent to showing that $\mathcal{S}(\chi[M])$ consists of a single element $c\mu^+$. It is immediate that the set $\mathcal{S}(\chi[M])$ satisfies

$$\frac{1}{d} f^* \mathcal{S}(\chi[M]) = \mathcal{S}(\chi[M]).$$

In case (i) $M \subset J^+$, and it is evident that the currents of $\mathcal{S}(\chi[M])$ are supported on J^+ . In case (ii) M is contained in an algebraic variety X , so $\chi[M] \leq [X]$. By [BS1,2] then, the elements of $\mathcal{S}(\chi[M])$ are all dominated by a constant times μ^+ . So in this case, too, the elements of $\mathcal{S}(\chi[M])$ are supported on J^+ .

This section will be devoted to showing that $\mathcal{S}(\chi[M])$ consists of closed currents. Applying Corollary 6.2 of [BS1], we have

$$(3) \quad \lim_{n \rightarrow \infty} d^{-n} A_n = \frac{1}{2\pi} \text{Area}(D) \mu^-|_M(M_0) > 0,$$

where $D \subset \mathbb{C}$ and $M_0 \subset\subset M$ are open sets, $\pi_x(x, y) = x$ is the projection to the x -axis, and A_n denotes the area (with multiplicity) of the projection of $\pi_x^{-1}(D) \cap f^{-n}(M_0)$ to the x -axis. A similar estimate involving the projection to the y -axis shows that the currents $(d^{-n})[f^{-n}M_0]$ have uniformly bounded mass on compact subsets of \mathbb{C}^2 , and so any subsequence of this sequence has a further subsequence that converges to a nonzero element of $\mathcal{S}(\chi[M])$. In particular, $\mathcal{S}(\chi[M])$ is nonempty.

We may also consider $M_a = \{\chi > a\} \cap M$, and so we have

$$c := \int \chi \mu^-|_M = \int_0^1 da \mu^-|_M(M_a).$$

Applying this to (3) as in [BS1], we obtain

$$(3') \quad \lim_{n \rightarrow \infty} d^{-n} (\pi_* f^{*n}(\chi[M]))(D) = \frac{c}{2\pi} \text{Area}(D).$$

Let us recall that the mass norm of a $(1, 1)$ -current T is given by

$$\mathbf{M}[T] = \sup_{|\varphi| \leq 1} |T(\varphi)|,$$

where φ is a test form, $|\varphi(x)|$ denotes the euclidean length of $\varphi(x)$, and $|\varphi| = \sup_x |\varphi(x)|$. If T denotes the current of integration over M then $\mathbf{M}[T]$ will denote the two-dimensional measure of the manifold M . Since M is a one-dimensional complex manifold, $f^{-n}|_M$ is a conformal mapping of M to $f^{-n}(M)$. Thus the (two-dimensional) jacobian determinant of $f^{-n}|_M$ is given by $|Df^{-n}|_M|^2$. We may estimate the mass norm of $\chi[M]$ by

$$\begin{aligned} \mathbf{M}[f^{n*}(\chi[M])] &= \mathbf{M}[\chi \circ f^n[f^{-n}M]] \\ &= \int_{f^{-n}M} \chi \circ f^n = \int_M \chi |Df^{-n}|_M|^2. \end{aligned}$$

We will also be interested in the amount of mass that lies inside a fixed set of the form $\pi_x^{-1}(D)$ for some bounded open set $D \subset \mathbb{C}$. The amount of mass in this set is

$$\mathbf{M}[f^{n*}(\chi M) \llcorner \pi_x^{-1}(D)] = \int_{M_n} \chi |Df^n|_M|^2,$$

where we use the notation

$$M_n := M \cap \text{spt } \chi \cap f^{-n} \pi_x^{-1} D.$$

Lemma 1. $\lim_{n \rightarrow \infty} (d^{-n})\partial[f^{*n}(\chi[M])] = 0$ holds in the sense of currents. Thus the elements of $\mathcal{S}(\chi[M])$ are closed.

Proof. It will suffice to show that the total mass of $(d^{-n})\partial f_*^n(\chi[M])$ on the set $\pi_x^{-1}D$ tends to zero, i.e.,

$$(4) \quad \lim_{n \rightarrow \infty} (d^{-n})\mathbf{M}[(\partial f^{*n}(\chi[M])) \llcorner \pi_x^{-1}D] = 0.$$

As above, we may compute the mass norm as

$$(5) \quad \begin{aligned} \mathbf{M}[(\partial f^{*n}(\chi[M])) \llcorner \pi_x^{-1}D] &= \sup_{|\varphi| \leq 1} \int_{f^{-n}M \cap \text{spt } \chi \cap \pi_x^{-1}D} \varphi \wedge d(f^{*n}\chi) \\ &= \sup_{|\varphi| \leq 1} \int_{M_n} \varphi \wedge d\chi |D(f^{-n}|_M)|. \end{aligned}$$

We estimate the mass in the current in (5) to obtain

$$(6) \quad \begin{aligned} \mathbf{M}[(\partial f^{*n}(\chi[M])) \llcorner \pi_x^{-1}D] &\leq |\varphi \wedge d\chi| \int_{M_n} |Df^{-n}|_M| \\ &\leq C \left(\int_{M_n} |Df^{-n}|_M|^2 \right)^{1/2} \leq Cd^{n/2}, \end{aligned}$$

where the second inequality follows from the Schwarz inequality, and the last follows from (3) and (4). Thus we have (6), which completes the proof.

4. A UNIQUENESS THEOREM

For a closed $(1, 1)$ -current ν we use the notation

$$\tilde{\nu}^{(x, \bar{y})} = \nu \llcorner \frac{i}{2} d\bar{x} \wedge dy, \quad \tilde{\nu}^{(y, \bar{y})} = \nu \llcorner \frac{i}{2} dx \wedge d\bar{x}, \text{ etc.}$$

Since ν is closed, we have the relations

$$(7) \quad \partial_{\bar{x}} \tilde{\nu}^{(x, \bar{y})} = \partial_{\bar{y}} \tilde{\nu}^{(x, \bar{x})}, \quad \partial_{\bar{x}} \tilde{\nu}^{(y, \bar{y})} = \partial_{\bar{y}} \tilde{\nu}^{(y, \bar{x})}, \text{ etc.}$$

Let $\psi(x, y)$ be a nonnegative real-valued function with support in the unit ball such that $\int \psi = 1$ and $\psi(x, y) = \psi(e^{i\theta}x, e^{i\tau}y)$ for all real θ, τ . Setting

$$\psi_\epsilon(x, y) = \epsilon^{-2n} \psi(x/\epsilon, y/\epsilon),$$

we see that $\{\psi_\epsilon\}$ is a usual family of smoothing kernels. For $\epsilon > 0$ and $\nu \in \mathcal{S}(\chi[M])$, we set $\nu_\epsilon = \psi_\epsilon * \nu$. Thus for any bounded set $D \subset \mathbb{C}$ we see that $\text{spt } \nu_\epsilon \cap \pi_x^{-1}(D)$ is bounded. If ν_ϵ is smooth then $\tilde{\nu}_\epsilon^{(y, \bar{y})}$ is represented by a smooth function times Lebesgue measure. Since the support is bounded in the y -direction, we may define the function

$$(8) \quad U_{\nu_\epsilon}(x, y) := \frac{2}{\pi} \int \log|y - \xi| \tilde{\nu}_\epsilon^{(y, \bar{y})}(x, \xi) \mathcal{L}^2(d\xi),$$

where $\mathcal{L}^2(d\xi)$ denotes Lebesgue two-dimensional measure in the variable ξ .

Lemma 2. $dd^c U_{\nu_\epsilon} = \nu_\epsilon.$

Proof. We can write the operator dd^c as

$$dd^c = 2i(\partial_x \partial_{\bar{x}} dx \wedge d\bar{x} + \partial_x \partial_{\bar{y}} dx \wedge d\bar{y} + \partial_y \partial_{\bar{x}} dy \wedge d\bar{x} + \partial_y \partial_{\bar{y}} dy \wedge d\bar{y}).$$

It is clear that $\partial_y \partial_{\bar{y}} U_{\nu_\epsilon} = \tilde{\nu}^{(y, \bar{y})}$ since $(2/\pi) \log|y|$ is the fundamental solution for $\partial_y \partial_{\bar{y}}$ (which is $1/4$ times the usual Laplacian in the y -variable). Next we compute the $\partial_y \partial_{\bar{x}}$ and $\partial_{\bar{x}} \partial_x$ derivatives. By (7) and (8) we have

$$\begin{aligned} \partial_x U_{\nu_\epsilon} &= \frac{2}{\pi} \int \log|y - \xi| \partial_x \tilde{\nu}_{x, \epsilon}^{(y, \bar{y})}(d\xi) \\ &= \frac{2}{\pi} \int \log|y - \xi| \partial_y \tilde{\nu}_{x, \epsilon}^{(x, \bar{y})}(d\xi) = \frac{1}{\pi} \int (y - \xi)^{-1} \tilde{\nu}_{x, \epsilon}^{(x, \bar{y})}(d\xi), \end{aligned}$$

where the last equality is an integration by parts, which is valid since the measure $\tilde{\nu}_{x, \epsilon}^{(x, \bar{y})}$ has compact support.

Applying $\partial_{\bar{x}}$ to this equation, we obtain

$$\begin{aligned} \partial_{\bar{x}} \partial_x U_{\nu_\epsilon} &= \frac{1}{\pi} \int (y - \xi)^{-1} \partial_{\bar{x}} \tilde{\nu}_{x, \epsilon}^{(x, \bar{y})}(d\xi) \\ &= \frac{1}{\pi} \int (y - \xi)^{-1} \partial_{\bar{y}} \tilde{\nu}_{x, \epsilon}^{(x, \bar{x})}(d\xi). \end{aligned}$$

Since $\tilde{\nu}_{x, \epsilon}^{(x, \bar{x})}$ is compactly supported, we may integrate by parts in the y -variable and use the fact that y^{-1} is the fundamental solution for $\partial_{\bar{y}}$ and thus find that $\partial_x \partial_{\bar{x}} U_{\nu_\epsilon} = \tilde{\nu}_\epsilon^{(x, \bar{x})}$.

A similar argument takes care of the $\partial_y \partial_{\bar{x}}$ derivative, which completes the proof.

Now let us estimate the size of U_{ν_ϵ} . We note that U_{ν_ϵ} is pluriharmonic on sets of the form $\{|y| > R, |x| < \rho\}$, and $U_{\nu_\epsilon}(x, y)$ grows like $\gamma(x) \log|y| + O(1)$ for fixed x as $y \rightarrow \infty$. Then it follows that the $O(1)$ term is a bounded pluriharmonic function on this set. Thus $\gamma(x) \log|y|$ is pluriharmonic, and so $\gamma(x) = \gamma$ is constant. It follows that the total mass of $\tilde{\nu}^{(y, \bar{y})}$ on any vertical slice $\{x = x_0\}$ is equal to $2\pi\gamma$. Further, if the support of the slice is contained in $\{|y| < R\}$, then for $|y| > R$ we have

$$(9) \quad \gamma \log(|y| - R) \leq U_{\nu_\epsilon}(x, y) \leq \gamma \log(|y| + R).$$

By the concavity of the logarithm function, this yields

$$(10) \quad \gamma \log|y| - \frac{\gamma R}{|y| - R} \leq U_{\nu_\epsilon}(x, y) \leq \gamma \log|y| + \frac{\gamma R}{|y|}$$

for $|y| > R$.

Lemma 3. *The functions U_{ν_ϵ} are monotone increasing in ϵ . If we set $U_\nu := \lim_{\epsilon \rightarrow 0} U_{\nu_\epsilon}$, then U_ν is plurisubharmonic and $dd^c U_\nu = \nu$.*

Proof. Let us consider a second smoothing $\nu_{\epsilon, \delta} := (\nu_\epsilon) * \psi_\delta$. It is evident that $U_{\nu_{\epsilon, \delta}} = \psi_\delta * U_{\nu_\epsilon} = \psi_\epsilon * U_{\nu_\delta}$. If U is a psh function then $U * \psi_\epsilon$ is monotone

increasing in ϵ and $U * \psi_\epsilon$ decreases to U as ϵ decreases to 0. Thus after we let δ decrease to zero, we see that U_{ν_ϵ} is increasing in ϵ .

Now, if we let ϵ decrease to 0, we see that $\lim_{\epsilon \rightarrow 0} U_{\nu_\epsilon}$ decreases to an upper semicontinuous function, which is psh if it is not identically $-\infty$. However, the estimates (9) and (10) hold for ν_ϵ independently of ϵ , and thus the limit is psh. Since the sequence is monotone, it follows that U_{ν_ϵ} converges to U_ν locally in L^1 . Thus $\nu = \lim_{\epsilon \rightarrow 0} \nu_\epsilon = dd^c U_\nu$.

Lemma 4. For $\nu \in \mathcal{S}(\chi[M])$ the function U_ν is the unique psh function of logarithmic growth such that $dd^c U_\nu = \nu$.

Proof. If V is any other function with logarithmic growth satisfying $dd^c V = \nu$, then $V - U_\nu$ is a pluriharmonic function with logarithmic growth. This function is then constant along all vertical and horizontal slices and thus constant.

Lemma 5. If $\lambda \in \mathcal{S}(\chi[M])$, then $\nu = (d^{-n})f^{n*}\lambda \in \mathcal{S}(\chi[M])$ and $U_\lambda \circ f^n = (d^n)U_\nu$.

Proof. We calculate

$$dd^c(U_\lambda \circ f^n) = dd^c(f^{*n}U_\lambda) = f^{n*}dd^cU_\lambda = f^{n*}\lambda = (d^n)\nu,$$

and the lemma follows.

Lemma 6. For $(x, y) \notin K^+$ and $\nu \in \mathcal{S}(M_0)$, $\gamma G^+(x, y) = U_\nu(x, y)$.

Proof. We will use the notation $(x_n, y_n) = f^n(x, y)$ and $\nu_n = (d^{-n})f^{*n}\nu$. Thus by Lemma 5,

$$U_\nu(x, y) = U_{\nu_n}(x_n, y_n).$$

Now we apply (10) and use the fact that it holds independently of the choice of $\nu_n \in \mathcal{S}(M_0)$ to obtain

$$\gamma d^{-n} \log|y_n| - \frac{\gamma R_n}{|y_n| - R_n} \leq U_\nu(x, y) \leq \gamma d^{-n} \log|y_n| + \frac{\gamma R_n}{|y_n|}.$$

Now there is a constant C such that $|y_n| \geq C + |x_n|^{d_1}/C$ for $n = 1, 2, 3 \dots$ (cf. [BS1]). Furthermore, C may be chosen such that

$$J^+ \subset \{(x, y) : |x| + C \geq |y|^{d_1}/C\}.$$

Thus for n large we may take $R_n \leq C(|x_n|^{1/d_m} + 1)$, and choosing C still larger if necessary, we have

$$R_n \leq C(|y_n|^{(d_1 d_m)^{-1}} + 1).$$

This gives

$$|U_\nu(x, y) - \gamma(d^{-n}) \log|y_n|| \leq C|y_n|^{(d_1 d_m)^{-1}-1},$$

and so we have $U_\nu(x, y) = \gamma G^+(x, y)$.

Proposition 1. *Let \mathcal{S} denote a set of positive, closed currents with the properties*

- (i) $\text{spt } \nu \subset J^+$ for all $\nu \in \mathcal{S}$, and
- (ii) $\frac{1}{d} f^* \mathcal{S} = \mathcal{S}$.

Then for each $\nu \in \mathcal{S}$ there is a real constant c such that $\nu = c\mu^+$.

Proof. Let us choose $\nu \in \mathcal{S}$. We will show that $U_\nu = 0$ on ∂K^+ . By Lemma 6, we know that $U_\nu \geq 0$ on ∂K^+ . Let us fix $(x_0, y_0) \in \partial K^+$. By Lemma 5, we have

$$(11) \quad U_\nu(x_0, y_0) = (d^{-n})U_{\lambda_n}g^n(x_0, y_0),$$

where $\lambda_n = (d^{-n})g^{n*}\nu$. Since (x_0, y_0) belongs to K^+ , the forward orbit $\{g^n(x_0, y_0) : n > 0\}$ stays bounded. Now (10) is independent of $\nu \in \mathcal{S}(M_0)$, so the right-hand side of (11) is (d^{-n}) times a bounded quantity, and so we conclude that $U_\nu(x_0, y_0) = 0$.

Finally, since the support of ν is contained in J^+ , U_ν is pluriharmonic on $\mathbb{C}^2 - J^+$. Applying the maximum principle to U_ν on a vertical slice and using the fact that $U_\nu = 0$ on J^+ , we see that $U_\nu = 0$ on K^+ . Thus $U_\nu = \gamma G^+$, and it follows that $\nu = \gamma\mu^+$.

Proof of Theorem 3. Let M satisfy the hypotheses of Theorem 3, and without loss of generality we may assume that $\chi \geq 0$. By Lemma 1, the set $\mathcal{S}(M)$ consists of positive, closed currents with support in J^+ . Then it follows from Proposition 1 that $\lim_{n \rightarrow \infty} (d^{-n})f^{*n}(\chi[M]) = \gamma\mu^+$ for some real constant γ . Now we observe that

$$\mu^-|_M = \mu^- \wedge [M] = (d^{-n})f^{-n*}\mu^- \wedge [M] = \mu^- \wedge (d^{-n})[f^{-n}M].$$

Letting $n \rightarrow \infty$, we see that this converges to $\gamma\mu^- \wedge \mu^+$ (see [BS1, Lemma 6.8]). However the total mass of $\chi\mu^-|_M$ is c and the total mass of $\mu^+ \wedge \mu^-$ is $4\pi^2$, so we see that $\gamma = c/4\pi^2$.

5. RECURRENT DOMAINS

In this section we restrict ourselves to the case where the complex Jacobian of f is δ with $|\delta| < 1$, i.e., f strictly contracts volume. Our idea is to look at a component Ω of $\text{int } K^+$ that exhibits some form of recurrence, and show that in this case Ω contains either a sink or a rotational domain and that Ω is precisely the set of points attracted to this sink or rotational domain.

Definition. Let Ω be a connected component of $\text{int } K^+$. We say that Ω is *recurrent* if there is some point $p \in \Omega$ and some compact set $C \subset \Omega$ so that $f^n(p) \in C$ for infinitely many positive values of n .

It is straightforward to show that

- *If Ω is recurrent, then Ω is actually a periodic domain.*

A recurrent domain is thus a periodic domain such that not all points tend to $\bigcup_j f^j \partial\Omega$.

We recall the following standard definition. The ω -limit set of a point p , written $\omega(p)$, is the set of q for which there is a sequence $n_i \rightarrow \infty$ such that $f^{n_i}(p) \rightarrow q$. The set Ω is recurrent if and only if Ω has a nonempty intersection with some set $\omega(p)$.

We also have:

- If Ω is a recurrent component of $\text{int } K^+$ then $\Omega \cap J^- \neq \emptyset$.

This is easily seen. Let p be a point so that $\omega(p)$ intersects Ω . The set $\omega(p)$ is invariant and contained in the set V (see discussion in §2). It follows that $\omega(p)$ consists of orbits bounded in backward time, thus $\omega(p) \subset K^- = J^-$.

By a Siegel disk we mean a set of the form $\mathcal{D} = \varphi(\Delta)$, where $\varphi : \Delta \rightarrow \mathbb{C}^2$ is an injective holomorphic mapping of the disk $\Delta \subset \mathbb{C}$ with the property that

$$(12) \quad f(\varphi(\zeta)) = \varphi(\alpha\zeta)$$

holds for $\alpha = e^{i\pi a}$ for some irrational a , and for all $\zeta \in \Delta$. It follows from the fact that the zeros of $D\varphi$ are isolated and from (12) that φ is nonsingular at all points other than zero. In Proposition 4 we will prove that φ is nonsingular at zero.

A Herman ring is a set of the form $\mathcal{H} = \varphi(A)$, where $\varphi : A \rightarrow \mathbb{C}^2$ is an injective holomorphic mapping of the annulus $A = \{\zeta \in \mathbb{C} : r_1 < |\zeta| < r_2\}$ such that (12) holds. The previous argument shows that \mathcal{H} is a nonsingular complex submanifold of \mathbb{C}^2 .

We will refer to Siegel disks and Herman rings collectively as rotation domains. We use the symbol \mathcal{R} for rotation domains. We note that $\mathcal{R} \subset K$.

Lemma 7. *If M is a bounded, connected, one-dimensional complex submanifold of $\text{int } K^+$ with $f(M) = M$, then M is recurrent if and only if it is a rotation domain.*

Proof. It is clear that a rotation domain is recurrent. Conversely, if M is recurrent, there exists a point $p \in \mathbb{C}^2$ such that $q = \lim_{j \rightarrow \infty} f^{n_j}(p) \in M$. Passing to a subsequence if necessary, we may assume that $m_j := n_{j+1} - n_j$ tends to infinity as $j \rightarrow \infty$. Since $\{f^n\}$ is a normal family in a neighborhood of q , we have $|Df^n| \leq \text{const}$ in a neighborhood of q . Thus

$$\begin{aligned} |f^{n_{j+1}}(p) - f^{m_j}(q)| &= |f^{m_j} f^{n_j}(p) - f^{m_j}(q)| \\ &\leq \|Df^{m_j}\| |f^{n_j}(p) - q| \leq \text{const} |f^{n_j}(p) - q| \end{aligned}$$

if j is sufficiently large. Thus $\lim_{j \rightarrow \infty} f^{m_j}(q) = q$.

Since M is bounded, $\{f^n|_M\}$ is a normal family in the sense that each sequence $\{f^{n_j}|_M\}$ is either compactly divergent or has a normally convergent subsequence with a limit $h : M \rightarrow M$. The set \mathcal{G} of all holomorphic mappings $h : M \rightarrow \overline{M}$ that arise as limits to subsequences of $\{f^{m_j}|_M\}$ is a subset of $\text{Aut}(M)$. This is seen because if $h \in \mathcal{G}$, then $h(q) \in M$, and thus $h : M \rightarrow M$; so by a theorem of H. Cartan on the limits of automorphisms $h \in \text{Aut}(M)$.

By the same argument, \mathcal{G} is a compact subgroup of $\text{Aut}(M)$. To see this, suppose that $\lim_{j \rightarrow \infty} f^{m_j}|_M = h$. Then pass to a subsequence so that $m_{j+1} -$

$2m_j \rightarrow \infty$ and $\tilde{h} = \lim_{j \rightarrow \infty} f^{m_{j+1}-2m_j}$ exists. Thus it is evident that $\tilde{h}h = \text{id}$, so $\tilde{h} = h^{-1}$.

Now it was observed in [FM] that the fixed point set of f^p is finite, so $f^p|_M$ cannot be the identity on M for any $p > 0$. It follows that \mathcal{G} (and thus $\text{Aut}(M)$) is an infinite group, and so the Riemann surface M must be equivalent either to the disk or to an annulus. In either case, since $f|_M$ generates an infinite group, it must be conjugate to an irrational rotation on the disk/annulus.

We will show in Proposition 6 that f may be linearized in a neighborhood of a Herman ring; here we perform the first step in that direction. Since \mathcal{H} is nonsingular, $\varphi := (\varphi_1, \varphi_2)$ satisfies $(\varphi'_1(\zeta), \varphi'_2(\zeta)) \neq (0, 0)$ for all $\zeta \in A = \{r_1 < |\zeta| < r_2\}$. Thus there exist holomorphic functions $\psi_1, \psi_2 \in \mathcal{O}(A)$ such that

$$(13) \quad \psi_1(\zeta)(-\varphi'_2(\zeta)) + \psi_2(\zeta)\varphi'_1(\zeta) = 1 \quad \text{for all } \zeta \in A.$$

We consider the mapping

$$(14) \quad s(\zeta, \eta) = (\varphi_1(\zeta) + \eta\psi_1(\zeta), \varphi_2(\zeta) + \eta\psi_2(\zeta)),$$

which is defined for $(\zeta, \eta) \in A \times \mathbb{C}$. By (13) it is clear that s is a local diffeomorphism in a neighborhood of $A \times \{0\}$ and

$$Ds(\zeta, 0) = \begin{pmatrix} \varphi'_1 & \psi_1 \\ \varphi'_2 & \psi_2 \end{pmatrix}.$$

Thus the determinant of Ds is 1 on $A \times \{0\}$, and so there is a neighborhood \mathcal{N} of $A \times \{0\}$ such that the mapping $F := s^{-1}fs$ is defined and the following hold:

$$\begin{aligned} F(\zeta, 0) &= (\alpha\zeta, 0), \\ F : \mathcal{N} &\rightarrow F(\mathcal{N}) \text{ is a diffeomorphism,} \\ \det DF(\zeta, 0) &= \delta. \end{aligned}$$

Since $F(A \times \{0\}) = A \times \{0\}$, there is a function $F_{1,1}(\zeta) \in \mathcal{O}(A)$ such that

$$DF(\zeta, 0) = \begin{pmatrix} \alpha & F_{1,1}(\zeta) \\ 0 & \delta/\alpha \end{pmatrix}$$

holds on A . Thus

$$F(\zeta, \eta) = (\alpha\zeta + \eta F_{1,1}(\zeta), \delta\eta/\alpha) + O(\eta^2)$$

holds for $\zeta \in A$. Without loss of generality we may assume that $F_{1,1}$ is bounded on A . Since $|\delta| < 1$, we may apply the Contraction Mapping Principle on the space $H^\infty(A)$ to obtain a unique solution $h_{1,1}$ to the functional equation

$$(15) \quad h_{1,1}(\zeta) - \frac{\delta}{\alpha^2}h_{1,1}(\alpha\zeta) = \frac{-1}{\alpha}F_{1,1}(\zeta).$$

Thus if we define

$$h^{(1)}(\zeta, \eta) = (\zeta + \eta h_{1,1}(\zeta), \eta),$$

we have

$$F \circ h^{(1)} = h^{(1)} \circ L + O(\eta^2),$$

where $L(\zeta, \eta) = (\alpha\zeta, \delta\eta/\alpha)$.

Thus we may assume that

$$(16) \quad F(\zeta, \eta) = (\alpha\zeta, \delta\eta/\alpha) + O(\eta^2).$$

If A_0 is an annulus with $A_0 \subset \bar{A}_0 \subset A$, then there exist numbers ϵ and M such that if $\zeta \in A_0$ and $|\eta| < \epsilon$ then $(\zeta', \eta') = F(\zeta, \eta)$ satisfies

$$|\zeta' - \alpha\zeta| \leq M|\eta|^2 \quad \text{and} \quad |\eta'| \leq |\delta\eta| + M|\eta|^2.$$

Let $(\zeta_n, \eta_n) = F^n(\zeta_0, \eta_0)$ denote the forward orbit of (ζ_0, η_0) . It is evident that for $\zeta_0 \in A_0$, $|\eta_0|$ may be chosen sufficiently small that $|\eta_n|$ will tend to zero sufficiently rapidly that $\zeta_n \in A_0$ for all $n = 1, 2, 3, \dots$.

Proposition 2. *Every rotation domain is contained in $\text{int } K^+$.*

Proof. If \mathcal{H} is a Herman ring then the proposition follows from the argument given above. If \mathcal{D} is a Siegel disk then let us write $\mathcal{D} = \varphi(\Delta)$ and $\mathcal{H} = \varphi(\Delta - \{0\})$. Since \mathcal{H} is a Herman ring, we have $\mathcal{H} \subset \text{int } K^+$. Consider $\mathcal{D}' = \varphi\{|\zeta| < 1/2\}$, and note that $\text{dist}(\varphi\mathcal{D}', \varphi \text{int } K^+) = \delta > 0$. Thus if we make a euclidean translation $\mathcal{D}' + v$ of \mathcal{D}' in a direction v with $|v| < \delta$, then $\partial(\mathcal{D}' + v) \subset \text{int } K^+$. For each v there is a constant C such that $|f^n| \leq C$ for all points of $\partial\mathcal{D}' + v$ and all $n \geq 1$. Now we may apply the maximum principle to $f^n|_{\mathcal{D}' + v}$ and conclude that $\{f^n : n = 1, 2, 3, \dots\}$ remains bounded at all points of $\mathcal{D}' + v$ for $|v| < \delta$. So we see that \mathcal{D}' , and thus \mathcal{D} , is contained in $\text{int } K^+$.

Proposition 3. *Let Ω be a connected component of $\text{int } K^+$ that is recurrent and has period m . Then one of the following occurs:*

- (i) *There is an attracting fixed point $p \in \Omega$ for f^m , and Ω is the basin of attraction of p under f^m .*
- (ii) *There is a retraction $\rho : \Omega \rightarrow \Omega$ onto a smooth subvariety $\mathcal{D} = \rho(\Omega)$ that is invariant under f^m . Further, \mathcal{D} is either a Siegel disk or a Herman ring.*

Proof. By replacing f by f^m , we may assume that $f(\Omega) = \Omega$. Since Ω is recurrent, there exists $p \in \Omega$ and a sequence $n_j \rightarrow \infty$ such that $f^{n_j}(p)$ converges to some point in Ω for some fixed $p \in \Omega$. The iterates $\{f^{n_j}|_{\Omega}\}$ form a normal family of automorphisms of Ω . Let $g : \Omega \rightarrow \bar{\Omega}$ denote the limit of a subsequence of these iterates. It is immediate that $fg = gf$. Since f decreases volume, the map g must be degenerate, i.e., it must have rank either 0 or 1. The set $\Omega \cap f(\Omega)$ is recurrent and thus a rotation domain by Lemma 7. By Proposition 2 then, we conclude that $f(\Omega) \subset \Omega$.

By [B] g has the following structure: there is a retraction $\rho : \Omega \rightarrow \Omega$ such that $M := \rho(\Omega)$ is a smooth subvariety, and there is an automorphism $\psi \in$

$\text{Aut}(M)$ such that $g = \psi \circ \rho$. Then it follows that $g\Omega = \rho\Omega = M$ and $fM = M$.

In case g has rank 0, g is a constant mapping and M is a fixed point for f . Since $Df^n(p)$ converges to zero, it follows that the eigenvalues of $Df(p)$ must be less than 1 in modulus, and thus p is a sink. Let U be the basin of attraction of p . Let $q \in \Omega$. Since U is a neighborhood of p there is some n so that $f^n(q) \in U$. But this implies that $q \in U$. So Ω is the basin of attraction of p .

In case g has rank 1, the proposition follows from Lemma 7, which completes the proof.

A Herman ring $\mathcal{H} = \varphi(A)$ with $A = \{0 = r_1 < |\zeta| < r_2\}$ is bounded since $\mathcal{H} \subset K$, and thus extends to a Siegel disk $\mathcal{D} = \hat{\varphi}(|\zeta| < r)$. The following proposition shows that there is no ‘‘punctured Siegel disk,’’ or equivalently, every maximal Herman ring has $0 < r_1 < r_2 < \infty$. This answers a question raised by Milnor in [Bi]. Siegel disks, however, may be singular in the volume preserving case (see the Appendix).

Proposition 4. *A Siegel disk \mathcal{D} is a nonsingular submanifold of \mathbb{C}^2 .*

Proof. By Proposition 3, $\mathcal{D} \subset \text{int } K^+$, and there is a retraction $\rho : \Omega \rightarrow \Omega$. It follows that $M := \rho(\Omega) \supset \mathcal{D}$, and the nonsingularity follows from that of M .

In the sequel we will assume that rotation domains are maximal with respect to inclusion. Thus we may write a Herman ring as $\mathcal{H} = \varphi(A)$ with $A = \{r^{-1} < |\zeta| < r\}$. There is a dichotomy between the cases of rings and disks given by polynomial convexity. (Recall that a compact set X is *polynomially convex* if $X = \{z \in \mathbb{C}^2 : |p(z)| \leq |p|_X \text{ for all polynomials } p\}$.) Let γ be a real-analytic closed curve with $f\gamma = \gamma$.

• *If γ is polynomially convex, then γ is contained in a Herman ring. Otherwise, γ is in a Siegel disk.*

(Since γ is real analytic, it has a maximal complexification, M , which satisfies $fM = M$. Since f must act as a rotation on M , it follows from Proposition 2 and Lemma 7 that M contains either a ring or a disk. If γ lies in a Siegel disk, it is clearly not polynomially convex. Conversely, if the polynomial hull $\hat{\gamma}$ is nontrivial, i.e., $\hat{\gamma} - \gamma \neq 0$, then $\hat{\gamma} - \gamma$ is an f -invariant subvariety of $\mathbb{C}^2 - \gamma$. Since $\hat{\gamma} - \gamma$ must coincide with M near γ , it is clear that $\hat{\gamma} - \gamma$ is a Siegel disk.)

Since $\alpha = e^{ina}$ with a irrational, a maximal ring satisfies

• *φ cannot be continued analytically beyond any point of ∂A .*

(For suppose that φ extends to a neighborhood of $re^{i\theta_0}$. Then choose n such that $0 < 2na + \theta_0 - \theta_1 < \epsilon \pmod{2}$. $\mathcal{H} = \varphi(A)$ is again a Herman ring for f^n , with α replaced by $\alpha^n = e^{nnia}$. But the functional equation (12) permits us to extend φ to a neighborhood of $re^{i\theta_1}$, which contradicts the maximality of H .)

If X is any set, we recall the standard notation

$$W^s(X) = \{q : d(f^n(q), f^n(X)) \rightarrow 0\}.$$

Proposition 5. *Let X denote either a sink or a rotation domain. Then the stable set $W^s(X)$ is a recurrent connected component of $\text{int } K^+$.*

Proof. Let $X = \{p\}$ be a sink. It is clear that $W^s(p)$ is a recurrent, connected open set. Let Ω denote the connected component of $\text{int } K^+$ containing $W^s(p)$. If $\Omega \neq W^s(p)$, then choose $q \in \Omega \cap \partial W^s(p)$ and let $B \subset \bar{B} \subset \Omega$ be a neighborhood containing q . The iterates $\{f^n|_B\}$ form a normal family, so a subsequence $\{f^{n_j}|_B\}$ converges to a holomorphic mapping $g : B \rightarrow \mathbb{C}^2$. Since $g(B \cap W^s(p)) = \{p\}$, it follows that $g(B) = \{p\}$. So $B \subset K^+$, and thus $\Omega = W^s(p)$.

If \mathcal{R} is a rotation domain, then by Proposition 3, $\mathcal{R} \subset \text{int } K^+$. Let Ω be the connected component of $\text{int } K^+$ containing \mathcal{R} . Then by Proposition 2, there is a retraction ρ and a subvariety $\mathcal{D} = \rho\Omega$, which is a rotation domain. It is evident that $\mathcal{D} \supset \mathcal{R}$, and so $\mathcal{D} = \mathcal{R}$ by the maximality of \mathcal{R} . By the previous paragraph, we have $\Omega = W^s(X)$, which completes the proof.

Now we will complete our linearization of f in a neighborhood of a Herman ring. We will show that there exists a local diffeomorphism $H(\zeta, \eta)$ that is defined and holomorphic in a neighborhood of $A \times \{0\}$ and such that

$$(17) \quad H \circ F = L \circ H.$$

By Proposition 5, there is a recurrent component Ω of $\text{int } K^+$ and a retraction ρ of Ω onto \mathcal{H} . Transplanting via the map s , we may assume that ρ is defined in a neighborhood of $A \times \{0\}$. We consider mappings of the form

$$(18) \quad H(\zeta, \eta) = (\rho(\zeta, \eta), \eta + \eta^2 h(\zeta, \eta)),$$

where h is analytic in a neighborhood of $A \times \{0\}$. We note that the first component of equation (17) is $\rho(F) = \alpha\rho(\zeta, \eta)$, which holds independently of h since ρ and F commute, i.e., $\rho F = F\rho$.

Since F has the form (16), we may write it as

$$F = \left(\alpha\zeta + \eta^2 f_1(\zeta, \eta), \frac{\delta\eta}{\alpha} + \eta^2 f_2(\zeta, \eta) \right).$$

In this notation, the second component of the equation (17) is

$$\eta^2 f_2 + \left(\frac{\delta\eta}{\alpha} + \eta^2 f_2 \right)^2 h(F) = \frac{\delta}{\alpha} \eta^2 h,$$

which is equivalent to

$$(19) \quad h - \frac{\delta}{\alpha} \left(1 + \frac{\alpha}{\delta} \eta f_2 \right)^2 h(F) = \frac{\delta}{\alpha} f_2.$$

Let \mathcal{N} denote a neighborhood of $A \times \{0\}$ where F and ρ are defined, and let $A_0 \subset \bar{A}_0 \subset A$ be a relatively compact subannulus. If we set $D = \rho^{-1}(A_0) \cap \{|\eta| <$

$c\}$, then for $c > 0$ sufficiently small, $D \subset \mathcal{N}$ and $F(D) \subset D$. Thus the composition operator

$$C_F : H^\infty(D) \rightarrow H^\infty(D)$$

given by $C_F(f) = h \circ F$ is defined, and $\|C_F\| = 1$. Now choose c small enough that

$$|\delta| \left(1 + \frac{c}{|\delta|} \|f_2\|_{L^\infty(D)} \right)^2 < 1.$$

Then by the Contraction Mapping Principle there exists a unique solution $h \in H^\infty(D)$ of (19).

The function H defined by (18) is a local diffeomorphism in a small enough neighborhood of $A \times \{0\}$ and so satisfies (17). The mapping $\tilde{h} := H \circ s^{-1}$ then satisfies

$$\tilde{h} \circ f = L \circ \tilde{h}$$

in a neighborhood of \mathcal{H} . Iterating the functional equation $\tilde{h} = L^{-1} \circ \tilde{h} \circ f$, we have $\tilde{h} = L^{-n} \circ \tilde{h} \circ f^n$, which we may use to extend the domain of definition of \tilde{h} to the basin of attraction $\Omega = W^s(\mathcal{H})$ of \mathcal{H} . Similarly, we may extend \tilde{h}^{-1} to the basin of attraction $A \times \mathbb{C}$ of $A \times \{0\}$ for L , so we obtain a biholomorphic mapping

$$\tilde{h} : \Omega \rightarrow A \times \mathbb{C}.$$

We may summarize the preceding discussion as follows.

Proposition 6. *If \mathcal{H} is a Herman ring, then there is a biholomorphism $\tilde{h} : W^s(\mathcal{H}) \rightarrow A \times \mathbb{C}$ with the properties:*

- (i) $\tilde{h}(\mathcal{H}) = A \times \{0\}$, and
- (ii) $f = \tilde{h}^{-1} \circ L \circ \tilde{h}$, where $L(\zeta, \eta) = (\alpha, \zeta, \beta\eta/\alpha)$.

We remark that the linearization of a Herman ring applies equally to a Siegel disk.

Corollary 2. *Let \mathcal{R} denote a rotational domain. Then the component of $\text{int } K^+$ containing \mathcal{R} is given by $\Omega = \bigcup_{x \in \mathcal{R}} W^s(x)$.*

Corollary 3. *If Ω is a recurrent component of $\text{int } K^+$, then Ω is biholomorphically equivalent to \mathbb{C}^2 , $A \times \mathbb{C}$, or $\Delta \times \mathbb{C}$. In the last two cases, $f|_\Omega$ is conjugate to the restriction of a linear map.*

The case of \mathbb{C}^2 corresponds to an attracting fixed point, and it is classical (cf. [RR]) that f may be conjugated on Ω to a mapping in “normal form.” If there are no resonances between the eigenvalues of Df at the fixed point, then the normal form is in fact linear.

We end this section with a discussion of boundaries of periodic domains.*

*Note added in proof: Mappings with fixed point of the form $f(x, y) = (x + x^2 + \dots, by + \dots)$, for $|b| < 1$, have been studied by T. Ueda in a recent preprint *Local structure of analytic transformations of two complex variables*. II. For such f he considers the domain Ω of points which converge locally uniformly to $(0, 0)$ as $n \rightarrow +\infty$. He shows that $\Omega \cap K^- \neq \emptyset$, and so it will follow that $\partial\Omega = J^+$ by Proposition 7.

Proposition 7. *Assume that $|\delta| < 1$. Let U be a periodic component of $\text{int } K^+$ that has a nonempty intersection with K^- . Then $\partial U = J^+$.*

Proof. By replacing f by a power we may assume that $f(U) = U$. An easy argument gives $\partial U \subset \partial K^+ = J^+$. We will show the opposite inclusion $\partial U \supset J^+$. Let $q \in U \cap K^-$. Since $|\delta| < 1$ the set K^- has empty interior. Thus we can find a disk D that: (1) is contained in a complex line, (2) contains q , (3) is contained in U , and (4) is not contained in K^- .

By the maximum principle, $G^-|_D$ cannot be harmonic in a neighborhood of q , and thus q is in the support of the Laplacian $(dd^c)_D G^-|_D$. It follows as before that

$$\mu^-|_D(D) = (dd^c|_D)G^-|_D > 0.$$

Since D is contained in an algebraic variety, Theorem 3 tells us that the currents of integration $(d^{-n})[f^{-n}D]$ converge to a current whose support is all of J^+ . On the other hand, it is evident that the support of the limit current is contained in the closure of $\bigcup_n f^{-n}D$. Thus J^+ is contained in the closure of U , which completes the proof.

Theorem 4. *If X is either a sink or a rotational domain then $\partial W^s(X) = J^+$.*

Proof. $W^s(X)$ is a connected, periodic component of $\text{int } K^+$ with $W^s(X) \cap K^- \neq \emptyset$. Thus the theorem follows from Proposition 7.

Proof of Theorem 2. It is clear that Theorem 2 is a special case of Theorem 4.

6. THE NONWANDERING SET AND THE CHAIN RECURRENT SET

In this section we determine the nonwandering and the chain recurrent sets for the map f when f strictly contracts volume (that is to say $|\delta| < 1$). We begin by proving some results that do not use the hypothesis that $|\delta| < 1$.

A point $p \in \mathbb{C}^2$ belongs to the *nonwandering set* if and only if, for every neighborhood U of the p , there is an n such that $g^n(U)$ intersects U . A point $p \in \mathbb{C}^2$ belongs to the *chain recurrent set* if for any $\epsilon > 0$ there exist points $x_1 = x, x_2, \dots, x_n = x$ (n depends on ϵ) such that $d(f(x_i), x_{i+1}) < \epsilon$ for $1 \leq i \leq n - 1$. These sets are closed and invariant. The nonwandering set is always contained in the chain recurrent set.

Lemma 8. *If $p_1 \in J^+$ and $p_2 \in J^-$ then for any neighborhoods U_i of p_i there is an n such that $f^n(U_1) \cap U_2 \neq \emptyset$. In particular, J is contained in the nonwandering set.*

Proof. We can construct a linear disk D in U_1 that contains point in J^+ and points in the complement of K^+ . As in the proof of Theorem 1, the sequence of currents defined by $(d^{-n})[f^n D]$ converges to a nonzero multiple of μ^- . The support of the current μ^- is equal to J^- so $\bigcup_n f^n(D)$ must meet U_2 .

Before we begin the calculation of the nonwandering and chain recurrent sets we resolve a question from [BS1].

Theorem 5. *The following are equivalent:*

- (i) *f has a hyperbolic splitting over the chain recurrent set;*
- (ii) *f has a hyperbolic splitting over the nonwandering set;*
- (iii) *f has a hyperbolic splitting over J .*

Proof. That assertion (iii) implies (i) was proved in [BS1]. Since the chain recurrent set contains the nonwandering set, (i) implies (ii). By Lemma 8 the nonwandering set contains J so (ii) implies (iii).

Each item in Theorem 5 could serve as a definition of hyperbolicity for polynomial diffeomorphisms. In [BS1] we adopted item (iii). According to the theorem this is equivalent to items (i) and (ii), which seem to be more natural definitions from a dynamical point of view.

For the remainder of this section we make the assumption that $|\delta| < 1$. Since the nonwandering set and chain recurrent set for f are the same as those for f^{-1} , the following theorems also apply when $|\delta| > 1$ with the appropriate substitution of the word “source” for the word “sink.”

Theorem 6. *When $|\delta| < 1$ the nonwandering set of f is the union of J , all rotational domains and all sink orbits.*

Proof. By Lemma 8, the nonwandering set contains J . On the other hand, the nonwandering set is contained in K . It suffices to show that the intersection of the nonwandering set with each component of $\text{int } K^+$ is as described in the statement of the theorem. It is easy to see that the intersection of the nonwandering set with a nonperiodic component of $\text{int } K^+$ is empty. Consider next the case of a recurrent component. The intersection of the nonwandering set with a domain that contains a rotational domain is the rotational domain (all other points are attracted to the rotational domain). The remaining possibility is that the component is periodic but not recurrent. Let Ω be such a component. Let $p \in \Omega$. Let U be a neighborhood of p such that the closure of U is compact in Ω . If there is a sequence n_i such that $f^{n_i}(U) \cap U \neq \emptyset$, then using a normal families argument, we can find a subsequence m_i so that f^{m_i} converges uniformly on compact subsets to a mapping $g : \Omega \rightarrow \overline{\Omega}$. If $g(\Omega) \cap (\Omega) \neq \emptyset$, then Ω is recurrent, so we conclude that $g(\Omega) \subset \partial\Omega$. But this contradicts the assertion that $f^{m_i}(U)$ intersects U for every i , and we conclude that the intersection of the nonwandering set with Ω is empty.

If p is a sink we will use the term *punctured basin* to refer to the set $W^s(p) - \{p\}$.

Theorem 7. *When $|\delta| < 1$ the chain recurrent set of f is equal to the set of bounded orbits (in forward/backward time) not in punctured basins.*

Before we give the proof we introduce some notation. We have defined $\omega(p)$ to be the set of q for which there is a sequence $n_i \rightarrow \infty$ such that $f^{n_i}(p) \rightarrow q$. We define $\alpha(p)$ to be the set of q for which there is a sequence $n_i \rightarrow -\infty$ such that $f^{n_i}(p) \rightarrow q$.

Proof. It is clear that the chain recurrent set is contained in K and that it contains no points in the punctured basin. It follows from Lemma 8 that J is contained in the chain recurrent set. It remains to consider points in the interior of K^+ that are in $K^- = J^-$ and not in basins of sinks. Choose such a point p . The α and ω limit sets of p are compact and invariant. If either is contained in $\text{int} K^+$ then the component of p contains a compact invariant set and thus is a recurrent component.

Assume first that p is not contained in a recurrent component. So $\alpha(p)$ meets J^+ . Since $\alpha(p)$ is contained in J^- we conclude that $\alpha(p)$ meets J . Since $\omega(p)$ is not contained in $\text{int} K^+$ it intersects J^+ . Since $\omega(p)$ consists of orbits bounded in backward time, $\omega(p)$ is contained in $K^- = J^-$. So we conclude that $\omega(p)$ meets J . Given $\epsilon > 0$ we can construct a periodic ϵ pseudo-orbit containing p as follows. We can find an $n < 0$ and an $m > 0$ so that $f^n(p)$ and $f^m(p)$ lie within ϵ of points q_1 and q_2 of J . Using Lemma 8 we can construct an ϵ chain connecting q_2 to q_1 . Concatenating these chains gives an ϵ chain from p to itself.

We now consider the case in which p is contained in a recurrent component. Proposition 3 shows that $\alpha(p)$ is disjoint from the orbit of the component. It follows that $\alpha(p) \subset J$. The set $\omega(p)$ is a point in a Siegel disk or Herman ring. Given $\epsilon > 0$ we can construct an ϵ -chain from $\omega(p)$ to some point in J . The rest of the ϵ -chain can be constructed as before. This completes the proof.

- The chain recurrent set of a polynomial diffeomorphism with $|\delta| < 1$ is equal to the nonwandering set if and only if the interior of K^+ consists exclusively of basins of sinks.

Chain recurrent sets have a natural decomposition into chain transitive components. For Axiom A diffeomorphisms this decomposition coincides with the Spectral Decomposition. The Spectral Decomposition for Axiom A polynomial diffeomorphisms was computed in [BS1]. Here we compute the chain transitive components for a general polynomial diffeomorphism.

An ϵ -chain from p to q is a sequence $x_1 = p, x_2, \dots, x_n = q$ such that $d(f(x_i), x_{i+1}) < \epsilon$ for $1 \leq i \leq n$. An invariant set is *chain transitive* if given any two points p and q in the set and any $\epsilon > 0$ we can construct an ϵ -chain from p to q . The decomposition of the chain recurrent set of a polynomial diffeomorphism into invariant chain transitive components is given by the following theorem. Let $R(f)$ denote the chain recurrent set of f . Let O_i for $i = 1, 2, \dots$ denote the sink orbits of f . Let $R' = R(f) - \bigcup_i O_i$.

Theorem 8. *For a polynomial diffeomorphism f the chain transitive components are the sink orbits, O_i , and the set R' .*

Proof. It is obvious that sink orbits are chain transitive since a periodic orbit is an ϵ -chain for any $\epsilon > 0$. The ϵ -chains required for proving that R' is chain transitive are constructed by concatenating ϵ -chains constructed in the proof of the previous theorem. Details are left to the reader.

APPENDIX. VOLUME PRESERVING MAPS

In this Appendix we will discuss the behavior of f on $\text{int } K$ in the case $|\delta| = 1$. As was observed in [BS1, Lemma 5.5], $\text{int } K^+ = \text{int } K^- = \text{int } K$. Since K is bounded, $\text{int } K$ has finite volume. Thus the (finitely many) components with volume equal to $c > 0$ are permuted among themselves by f . In particular, every component is periodic. In fact, it follows from Poincaré’s recurrence theorem that each component is recurrent.

Let us fix a component U of $\text{int } K$. Replacing f by f^n if necessary, we may assume that $fU = U$. Recalling that $\{f^j|_U\}$ is a normal family, we let

$$\mathcal{G}(U) = \left\{ h = \lim_{j \rightarrow \infty} f^{n_j} \text{ on } U \text{ for some subsequence } \{n_j\} \right\}.$$

Since f preserves volume, it follows that h , too, must preserve volume. Thus $h(U) \subset U$, and so by H. Cartan’s theorem $h \in \text{Aut}(U)$. In fact, passing to further subsequences, we see that

- $\mathcal{G}(U)$ is a compact abelian subgroup of $\text{Aut}(U)$.

Since $\text{Aut}(U)$ is a Lie group, so is \mathcal{G} . Since \mathcal{G} contains the infinite subset $\{f^n|_U\}$, it must have dimension at least one. The connected component of the identity \mathcal{G}_0 is then a torus \mathbf{T}^k for some $k \geq 1$. Since \mathcal{G}_0 acts effectively on U , the orbit of a generic point $p \in U$, $\mathcal{G}_0 \cdot p$, will be a smooth submanifold of U that is diffeomorphic to a k -dimensional torus. (See [BBD, Lemmas 1.2, 1.3].) Since the orbit must be totally real, we must have $k \leq 2$. If p is not generic, the orbit can be a torus of dimension less than k . In either case, the full orbit $\mathcal{G} \cdot p$ is a finite union of tori, and by the construction of \mathcal{G} , it is clear that the forward and backward orbits of p under f are $\mathcal{G} \cdot p$.

Theorem 9. *When $|\delta| = 1$ the nonwandering and the chain recurrent sets both coincide with K .*

Proof. The chain recurrent set lies inside K (cf. [BS1, Corollary 2.7]). Since the chain recurrent set contains the nonwandering set, it suffices to show that the nonwandering set contains K . Now

$$K = K^+ \cap K^- = (J^+ \cup \text{int } K) \cap (J^- \cup \text{int } K) = J \cup \text{int } K.$$

We showed that J is contained in the nonwandering set, and it will suffice to show that $\text{int } K$ is contained in the nonwandering set. In fact, a stronger statement is true: each $p \in \text{int } K$ is the limit of its forward iterates. Let U be the component of $\text{int } K$ that contains p . Since the automorphism group of U is a compact group, there is a sequence n_i such that $f^{n_i}(p) \rightarrow p$.

To discuss some further properties of a component U of $\text{int } K$, we recall that we may write

$$K = \bigcap_{n=-\infty}^{\infty} \{|f^n| \leq C\}$$

for some $C < \infty$. Thus K is polynomially convex. Let us recall that a domain U is a *Runge domain* if any analytic function on U may be approximated uniformly on compact subsets of polynomials.

Proposition 8. *U is a Runge domain, and $H^n(U, \mathbf{C}) = 0$ for $n \geq 2$.*

Proof. To see that $H^2(U, \mathbf{C}) = 0$, we let X be a compact subset of U and show that $\widehat{X} \subset U$. By the Oka-Weil theorem, it suffices to show that if X is a compact subset of U then the polynomial hull \widehat{X} is contained in U . If v is any vector in \mathbf{C}^2 such that the euclidean translation $X + v \subset U \subset K$ then it follows that $\widehat{X} + v = \widehat{X + v} \subset \widehat{K} = K$. Since this holds for all v such that $|v| < \text{dist}(X, \partial U)$, we see that \widehat{X} is contained in the interior in K . By the Oka-Weil theorem, the function that is 0 on $U \cap \widehat{X}$ and 1 on $\widehat{X} - U$ may be uniformly approximated on \widehat{X} by polynomials, so it follows that $\widehat{X} \subset U$ and U is Runge.

A well-known property (cf. [Ho]) of Runge domains in \mathbf{C}^2 is that $H^2(U, \mathbf{C}) = 0$. Any Runge domain is a domain of holomorphy, and so it follows that $H^3(U, \mathbf{C}) = 0$.

Since $H^2(U, \mathbf{C}) = 0$ it follows that U cannot have the topology of a product of annuli. This answers a question of Milnor posed in [Bi].

There are two cases, according to the dimension of \mathcal{G}_0 :

(i) $\mathcal{G}_0 = \mathbf{T}^1$. We do not know very much about this case. However, let us suppose that f has a fixed point $(x_0, y_0) = (0, 0) \in U$. H. Cartan showed that in this case the \mathbf{T}^1 -action is conjugate to a (p, q) -action locally near the fixed point. That is, there is a change of coordinates in a neighborhood of the origin such that the action becomes

$$\theta \mapsto (e^{ip\theta}x, e^{iq\theta}y),$$

where $p, q \in \mathbf{Z}$ are relatively prime. The sets

$$V_c = \{y^p = cx^q\}$$

are invariant under \mathcal{G} , and by the remark above V_c determines an invariant complex variety of U . If $pq > 0$, then $\{V_c\}$ is a one-complex parameter family of Siegel disks. If, in addition, $p = \pm 1$ or $q = \pm 1$, then each V_c is nonsingular; otherwise, V_c is a ‘‘punctured Siegel disk.’’ In the latter case, there are only two nonsingular Siegel disks, which correspond to the x - and y -axes. If $p > 0 > q$ then the x - and y -axes again correspond to Siegel disks.

(ii) $\mathcal{G}_0 = \mathbf{T}^2$. Recall that a domain $G \subset \mathbf{C}^2$ is *Reinhardt* if it is invariant under the action $(\theta_1, \theta_2) \mapsto (e^{i\theta_1}x, e^{i\theta_2}y)$ for all $\theta_1, \theta_2 \in \mathbf{R}$ and all $(x, y) \in G$. It is shown in [BBD] that if U has a \mathbf{T}^2 action then U is equivariantly equivalent to a Reinhardt domain G . That is, there is a biholomorphic mapping $\Phi : U \rightarrow G$ such that

$$\Phi((\theta_1, \theta_2) \cdot (x, y)) = (e^{i\theta_1}, e^{i\theta_2})\Phi(x, y)$$

holds for all $(x, y) \in U$. In other words, the action of \mathcal{E}_0 is taken into the standard Reinhardt action. The action of f on U corresponds to $\tau_\alpha(x, y) = (\alpha_1 x, \alpha_2 y)$ for a pair of complex numbers $|\alpha_j| = 1$, $j = 1, 2$, and the numbers (α_1^k, α_2^k) , $k = 1, 2, 3, \dots$, are dense in the torus \mathbf{T}^2 .

We note that a connected pseudoconvex Reinhardt domain in \mathbf{C}^2 can have one of three topological types: $\Delta \times \Delta$, $\Delta \times A$, or $A \times A$, ($\Delta =$ disk and $A =$ annulus). The third type cannot occur here, since $H^2(U, \mathbf{C}) = 0$, as was observed above. In the first case, G intersects each coordinate axis in a disk. Thus the functions

$$\begin{aligned}\varphi_1(\zeta) &:= \Phi^{-1}(\zeta, 0) : \{|\zeta| < r_1\} \rightarrow U, \\ \varphi_2(\zeta) &:= \Phi^{-1}(\zeta, 0) : \{|\zeta| < r_2\} \rightarrow U,\end{aligned}$$

give a pair of Siegel disks. Since G does not contain any other one-dimensional complex manifolds that are invariant under τ_α , U does not contain any other complex manifolds that are invariant under f . The point $\Phi^{-1}(0, 0)$ is evidently the unique fixed point for f in U .

If G is topologically equivalent to $\Delta \times A$, then G intersects only one axis, the x -axis, say, in an annulus. This yields a Herman ring, which is the only f -invariant complex manifold in U ; and f has no fixed point in U .

Case (ii) can actually occur, at least in the subcase where f has a fixed point in U . To see this, we choose a mapping $f(x, y) = (\alpha_1 x, \alpha_2 y) + \dots$, where the dots indicate terms of degree at least 2 and $|\alpha_1| = |\alpha_2| = 1$. It follows from the preceding discussion that $(0, 0) \in \text{int} K$ if and only if f can be linearized in a neighborhood of $(0, 0)$. If we choose α_1 and α_2 to satisfy also a diophantine condition, then f may be linearized at $(0, 0)$ (cf. Zehnder [Z]). Case (i) can also occur, at least for a (p, q) -action with $pq > 0$. For if α satisfies a diophantine condition and if p, q are relatively prime, then so does the pair $\alpha_1 = \alpha^q$, $\alpha_2 = \alpha^p$. Again by [Z] f may be linearized at $(0, 0)$, and f will generate a (p, q) -action.

BIBLIOGRAPHY

- [BBD] D. Barret, E. Bedford, and J. Dadok, T^n -actions on holomorphically separable complex manifolds, *Math. Z.* **202** (1989), 65–82.
- [B] E. Bedford, *On the automorphism group of a Stein manifold*, *Math. Ann.* **266** (1983), 215–227.
- [BS1] E. Bedford and J. Smillie, *Polynomial diffeomorphisms of \mathbf{C}^2 : currents, equilibrium measure and hyperbolicity*, *Invent. Math.* **103** (1991), 69–99.
- [BS2] ———, *Fatou-Bieberbach domains arising from polynomial automorphisms*, *Indiana U. Math. J.* **40** (1991), 789–792.
- [Bi] B. Bielefeld, *Conformal dynamics problem list*, SUNY Stony Brook Institute for Mathematical Sciences, preprint, 1990.
- [FM] S. Friedland and J. Milnor, *Dynamical properties of plane polynomial automorphisms*, *Ergodic Theory Dynamical Systems* **9** (1989), 67–99.
- [H] M. Herman, *Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of \mathbf{C}^n near a fixed point*, VIIIth International

- Congress on Mathematical Physics (Marseille, 1986), World Scientific Publ. Singapore, 1987, pp. 138–184.
- [Ho] L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, 1973.
- [HO] J. H. Hubbard and R. Oberste-Vorth, *Hénon mappings in the complex domain*, in preparation.
- [K] A. Katok, *Nonuniform hyperbolicity and structure of smooth dynamical systems*, Proc. Internat. Congr. Math., Warsaw, 1983.
- [N] S. Newhouse, *The creation of non-trivial recurrence in the dynamics of diffeomorphisms*, Les Houches session XXXVI, Chaotic Behavior of Deterministic Systems (G. Iooss et al., eds.), North-Holland, 1983, pp. 381–442.
- [RR] J.-P. Rosay and W. Rudin, *Holomorphic maps from \mathbb{C}^n to \mathbb{C}^n* , Trans. Amer. Math. Soc. **310** (1988), 47–86.
- [S] J. Smillie, *The entropy of polynomial diffeomorphisms of \mathbb{C}^2* , Ergodic Theory Dynamical Systems (to appear).
- [Z] E. Zehnder, *A simple proof of a generalization of a Theorem by C. L. Siegel*, Geometry and Topology (J. Palis and M. do Carmo, eds.), Lecture Notes in Math., vol. 597, Springer-Verlag, New York, 1977, pp. 855–866.

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