2-TORSION INSTANTON INVARIANTS

RONALD FINTUSHEL AND RONALD J. STERN

Dedicated to the memory of Andreas Floer

1. INTRODUCTION

In [D2] S. Donaldson introduced new and useful invariants for smooth, closed, simply connected 4-manifolds by evaluating certain elements of the rational cohomology of the space of connections in an $SU(2)$-bundle over a 4-manifold $M$ on a homology class represented by the moduli space of self-dual connections. In this paper we utilize torsion in the cohomology of the space of connections to define a collection of mod 2 polynomial invariants for smooth, simply connected, closed spin 4-manifolds and show that this collection of polynomials is stable under connect sums with $S^2 \times S^2$ (Theorems 1.1 and 1.3). This contrasts with the vanishing theorems for Donaldson's integral polynomials for connected sums [D2]. It will then follow that if one could find two homotopy equivalent simply connected, smooth, closed, and spin 4-manifolds with Donaldson polynomials having different parity, then these manifolds are not diffeomorphic and remain nondiffeomorphic after connect summing with one or two copies of $S^2 \times S^2$. At present, no such example is known. However, we will show that the Donaldson polynomial invariants have limited utility in this vein. In fact, using the relation between the usual Donaldson invariants and the mod 2 polynomial invariants (Theorem 1.1) and through a detailed understanding of how moduli spaces decompose for manifolds which are connected sums, we are able obtain severe restrictions on when the Donaldson polynomials reduced mod 2 can be nonzero. (See Theorem 1.6.)

In order to describe these mod 2 polynomial invariants, recall that Donaldson's (integral) polynomial invariant $q_{\ell,M}$ is defined for a closed oriented simply connected 4-manifold $M$ with $b_+^2$ odd $> 1$ (and for $\ell$ a large enough positive integer) and has degree $d = 4\ell - \frac{1}{2}(1 + b_+^2)$). Consider the Banach manifold $\mathcal{B}_{M,\ell}$ of equivalence classes of irreducible connections of charge $\ell$. Donaldson's invariant is defined on homology classes $z_1, \ldots, z_d \in H_2(M; \mathbb{Z})$ by evaluating the cup product of the $d$ cohomology classes $\mu(z_i) \in H^2(\mathcal{B}_{M,\ell}; \mathbb{Z})$ ($\mu$ is defined in [D1]) against the fundamental class of the $2d$-dimensional

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moduli space $\mathcal{M}_{M, \ell}$ of anti-self-dual $SU(2)$ connections of charge $\ell$. See §2 for a more complete description. In case $b_M^+ > 1$ is even, we have $\dim \mathcal{M}_{M, k} = 8k - 3(1 + b_M^+) = 2d + 1$; so a similar polynomial invariant can be defined if there is a nontrivial 1-dimensional cohomology class in $\mathcal{B}^*_M$. If $k$ is even and $M$ is spin there is a nontrivial class $u_1 \in H^1(\mathcal{B}^*_M; \mathbb{Z}_2)$ (cf. [D1] and §2). Thus in this case there is (for large enough even $k$) a polynomial invariant $q_{k, u_1, M}$ of degree $d$ in $H_2(M; \mathbb{Z}_2)$ and defined with values in $\mathbb{Z}_2$. (See [D3] for a general discussion of such invariants.) As in the case of $q_{\ell, M}$, the mod 2 invariant $q_{k, u_1, M}$ is an invariant of the smooth structure of $M$.

Now suppose that $M$ is a closed oriented simply connected spin 4-manifold with $b^+_M$ odd $> 1$ and has a Donaldson polynomial invariant $q_{\ell, M}$ of degree $d$ and with $\ell$ odd. Then $M#S^2 \times S^2$ has $b^+$ even and the moduli space $\mathcal{M}_{M#S^2 \times S^2, \ell+1}$ has dimension $2d + 5$. Since $\ell + 1$ is even, we have the mod 2 invariant $q_{\ell+1, u_1, M#S^2 \times S^2}$ of degree $d + 2$ in $H_2(M#S^2 \times S^2; \mathbb{Z}_2)$. Our main theorem is

**Theorem 1.1.** Suppose that $M$ is a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{\ell, M}$ of degree $d$, where $\ell$ is odd. Then $q_{\ell+1, u_1, M#S^2 \times S^2}$ is defined and for any classes $z_1, \ldots, z_d \in H_2(M; \mathbb{Z})$ and for $x = [S^2 \times 0]$ and $y = [0 \times S^2]$ in $H_2(S^2 \times S^2; \mathbb{Z})$ we have

$$q_{\ell, M}(z_1, \ldots, z_d) \equiv q_{\ell+1, u_1, M#S^2 \times S^2}(z_1, \ldots, z_d, x, y) \mod 2.$$ 

In order to explain the relevance of this theorem to the problem at hand, let us make the following definition. Suppose that $M_1$ and $M_2$ are homotopy equivalent simply connected 4-manifolds. We shall say that their degree $d$ Donaldson invariants $q_{\ell, M_1}$ and $q_{\ell, M_2}$ have the same parity if for each isomorphism of intersection forms $f : H_2(M_1; \mathbb{Z}) \rightarrow H_2(M_2; \mathbb{Z})$ and for all $z_1, \ldots, z_d \in H_2(M_1; \mathbb{Z})$ we have

$$q_{\ell, M_1}(z_1, \ldots, z_d) \equiv q_{\ell, M_2}(f(z_1), \ldots, f(z_d)) \mod 2.$$ 

**Theorem 1.2.** Let $M_1$ and $M_2$ be homotopy equivalent closed simply connected spin 4-manifolds. If $M_1#S^2 \times S^2$ is diffeomorphic to $M_2#S^2 \times S^2$ then $q_{\ell, M_1}$ and $q_{\ell, M_2}$ have the same parity for all odd $\ell$.

This theorem is actually a corollary of Theorem 1.1 and Wall’s work on the diffeomorphisms of 4-manifolds [W1]. For if there is an odd $\ell$ and an isomorphism $f$ of intersection forms as above, then consider the isomorphism $f \oplus 1 : H_2(M_1#S^2 \times S^2; \mathbb{Z}) \rightarrow H_2(M_2#S^2 \times S^2; \mathbb{Z})$. Let $\Phi : M_1#S^2 \times S^2 \rightarrow M_2#S^2 \times S^2$ be a diffeomorphism; so $\Phi^{-1} \circ (f \oplus 1) = g$ is an automorphism of $H_2(M_1#S^2 \times S^2; \mathbb{Z})$. Then by [W1, Theorem 2], $g = G$, where $G$ is a self-diffeomorphism of $M_1#S^2 \times S^2$. Then $f \oplus 1$ is induced from the diffeomorphism $\Phi \circ G$. Theorem 1.2 now follows from Theorem 1.1 and the naturality of Donaldson’s invariant.
If $b^+_N$ is odd $> 1$, so $\dim \mathcal{M}_{N, k}$ is even, we may write $\dim \mathcal{M}_{N, k} = 2d + 2$. For $k$ odd and $N$ spin, $\pi_1(\mathcal{M}_{N, k}) = 0$ and there is a nontrivial class $u_2 \in H^2(\mathcal{M}_{N, k}; \mathbb{Z}_2)$ arising from the 2-torsion in $H^2(\mathcal{M}_{N, k}; \mathbb{Z})$. (Again, see [D1].) Thus we may form the degree $d$ polynomial invariant $q_{k, u_2, N}$ with values in $\mathbb{Z}_2$. We get

**Theorem 1.3.** Let $N$ be a closed simply connected spin 4-manifold with the degree $d$ Donaldson invariant $q_{k, u_2, N}$ with values in $\mathbb{Z}_2$. (Thus $b^+_N$ is even and $k$ is even.) Then the degree $d + 2$ invariant $q_{k+1, u_2, N \# S^2 \times S^2}$ is defined and for any $z_1, \ldots, z_d \in H_2(N; \mathbb{Z}_2)$

$$q_{k, u_2, N}(z_1, \ldots, z_d) = q_{k+1, u_2, N \# S^2 \times S^2}(z_1, \ldots, z_d, x, y) \mod 2,$$

where $x = [S^2 \times 0]$ and $y = [0 \times S^2]$.

A basic theorem in the theory of smooth 4-manifolds, due to C. T. C. Wall [W1,2], states that given two homotopy equivalent simply connected smooth closed 4-manifolds $M_1$ and $M_2$, there is an integer $k$ such that $M_1 \# k(S^2 \times S^2)$ is diffeomorphic to $M_2 \# k(S^2 \times S^2)$, i.e. $M_1$ and $M_2$ are stably diffeomorphic. A natural problem is to determine the minimal such integer $k$, denoted $sd(M_1, M_2)$. It now follows from Theorems 1.1, 1.2 and 1.3 that if $M_1$ and $M_2$ are homotopy equivalent closed simply connected spin 4-manifolds with Donaldson invariants $q_{\ell, M_1}$ and $q_{\ell, M_2}$ ($\ell$ odd) which do not have the same parity then actually $sd(M_1, M_2) \geq 3$. At present no such examples are known. A possible reason is given by Theorem 1.6 where it is shown that many Donaldson invariants are even.

For many closed simply connected 4-manifolds it is known that “homotopy equivalent implies diffeomorphic after a single connected sum with $S^2 \times S^2$.” For example, work of Mandelbaum [M] and Gompf [G] shows that this is true for simply connected elliptic surfaces. Hence

**Theorem 1.4.** Let $M_1$ and $M_2$ be homotopy equivalent closed simply connected spin elliptic surfaces. Then any Donaldson polynomial invariants $q_{\ell, M_1}$ and $q_{\ell, M_2}$, for $\ell$ odd, have the same parity.

Theorem 1.4 also follows from the explicit computations for spin elliptic surfaces given by Friedman and Morgan [FM].

Let $\text{Sym}^d_R(H_2(M; \mathbb{Z}))$ be the set of $d$-linear symmetric functions on $H_2(M; \mathbb{Z})$ with values in a ring $R$. The symmetric product $q_1 q_2 \in \text{Sym}^{d_1 + d_2}_R(H_2(M; \mathbb{Z}))$ of the symmetric functions $q_1 \in \text{Sym}^{d_1}_R(H_2(M; \mathbb{Z}))$ and $q_2 \in \text{Sym}^{d_2}_R(H_2(M; \mathbb{Z}))$ is defined by the rule

$$q_1q_2(x_1, \ldots, x_{p+q}) = \frac{1}{d_1! d_2!} \sum_{\sigma \in S_{d_1+d_2}} q_1(x_{\sigma(1)}, \ldots, x_{\sigma(d_1)})q_2(x_{\sigma(d_1+1)}, \ldots, x_{\sigma(d_1+d_2)}).$$
The degree $d$ Donaldson invariant $q_{\ell,M}$ is an element of $\text{Sym}_d^d(H_2(M;\mathbb{Z}))$. Similarly the intersection form $Q_M$ of $M$ is an element of $\text{Sym}_2^d(H_2(M;\mathbb{Z}))$. Define

$$Q_M^{(p)} = \frac{1}{p!}Q_M^p.$$ 

Reducing mod 2 we consider the algebra

$$\text{Sym}^d_2(H_2(M;\mathbb{Z})) = \bigoplus_d \text{Sym}^d_2(H_2(M;\mathbb{Z})).$$

In §8 we prove a vanishing theorem for $q_{\ell+1,u,M#S^2 \times S^2}$ (Theorem 8.1) reminiscent of Donaldson’s connected sum theorem [D2] which together with our calculations of $q_{\ell+1,u,X} \in \text{Sym}^{d+2}_2(H_2(X;\mathbb{Z}))$ and its invariance under the orthogonal transformations of $H_2(X;\mathbb{Z})$ induced from diffeomorphisms of $X = M#S^2 \times S^2$ will prove

**Theorem 1.5.** Let $M$ be a closed simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$ with $\ell$ odd. Then $q_{\ell,M} \equiv \epsilon_{\ell,M}Q_M^{(p)} \mod 2$ for some integer $p$ and $\epsilon_{\ell,M} \in \mathbb{Z}_2$.

Combining this result with the thesis of Y. Ruan [R], we obtain strong restrictions on the possibility of Donaldson’s invariants $q_{\ell,M}$ taking odd values.

**Theorem 1.6.** Let $M$ be a closed simply connected spin 4-manifold with a Donaldson invariant $q_{\ell,M}$ of degree $d$ with $\ell$ odd. If $b^+_M \neq 3 \mod 8$, or if $d > \text{rank}(H_2(M;\mathbb{Z}))$, then $q_{\ell,M} \equiv 0 \mod 2$.

The proof of the main Theorem 1.1 is accomplished via a degeneration of metrics argument of the sort utilized by Donaldson in [D2]. Two routes are available for carrying out this argument. The first is to split $M#S^2 \times S^2$ along an obvious $S^3$. Then the argument is a modification of the arguments of [D2], and we shall outline such an approach in a moment. We have chosen instead to view $M#S^2 \times S^2$ as the result of surgery along a circle in $M$. Then the interface between $M \setminus \{\text{tubular neighborhood of the circle}\}$ and the handle $S^2 \times D^2$ is the 3-manifold $S^2 \times S^1$. We study the result of stretching a tubular neighborhood of this $S^2 \times S^1$ to infinite length. The tool for comparing the resulting moduli spaces of anti-self-dual connections with the original moduli space is the thesis of Tom Mrówka [Mr]. Techniques from Mrówka’s thesis are becoming increasingly important in gauge theory (cf. [GM] and [MMR]), and we felt that it would be interesting to carry out our argument from Mrówka’s point of view.

For the experts, we outline the alternative approach using Donaldson’s work from [D1] and [D2]. Let $S^3$ be a 3-sphere in $M#S^2 \times S^2$ whose complement is the disjoint union of $M \setminus B^4$ and $S^2 \times S^2 \setminus B^4$. Suppose that one has a 1-parameter family of metrics $\{g_t\}$ on $M#S^2 \times S^2$ such that in $(M#S^2 \times S^2, g_t)$ our $S^3$ has diameter less than $d(t)$ and $d(t) \to 0$ as $t \to \infty$. Thus in a reasonable sense, the sequence of Riemannian manifolds $(M#S^2 \times S^2, g_t)$...
converges to the one-point union \((M, g_M) \vee (S^2 \times S^2, g_{S^2 \times S^2})\), and we choose the \(g_i\) so that both \(g_M\) and \(g_{S^2 \times S^2}\) are “generic”. For each \(i = 1, \ldots, d\), let \(V_i\) be a codimension 2 subvariety of the appropriate space of connections, such that \(V_i\) is a cocycle representative of \(\mu(z_i)\). (See §2.) Similarly choose \(V_x\) and \(V_y\). Then the invariant \(q_{t+1, M} \# S^2 \times S^2, \mu(z_1, \ldots, z_d, x, y)\) is found by evaluating \(u_1[V_1 \cap \cdots \cap V_d \cap V_x \cap V_y \cap \mathcal{M}_{S^2 \times S^2, t+1}^{g_{S^2 \times S^2}}]\), and \(q_{t, M}(z_1, \ldots, z_d)\) is the algebraic intersection number \(\# V_1 \cap \cdots \cap V_d \cap \mathcal{M}_M, t\).

Suppose that \(\{A_n\}\) is a sequence of connections such that for each \(n\) we have \(A_n \in V_1 \cap \cdots \cap V_d \cap V_x \cap V_y \cap \mathcal{M}_{S^2 \times S^2, t+1}^{g_{S^2 \times S^2}}\) where \(t_n \to \infty\). Then \(\{A_n\}\) converges to a pair of connections \(A \in \mathcal{M}_{M, k}^{g_M}\) and \(B \in \mathcal{M}_{S^2 \times S^2, j}^{g_{S^2 \times S^2}}\) together with possible instanton bubbles, and \(k + j + \#\text{(bubbles)} \leq \ell + 1\). Counting arguments as in [D2] and §5 show that the only possibility is that \(A \in V_1 \cap \cdots \cap V_d \cap \mathcal{M}_{M, t}^{g_M}\), \(B \in \mathcal{M}_{S^2 \times S^2}^{g_{S^2 \times S^2}}\) (so \(B\) is the trivial connection \(\Theta_{S^2 \times S^2}\)). Furthermore, a single bubble must occur at an intersection point of generic surfaces \(S_x\) and \(S_y\) representing \(x\) and \(y\). One then needs to argue that the calculation of \(V_1 \cap \cdots \cap V_d \cap V_x \cap V_y \cap \mathcal{M}_{S^2 \times S^2, t+1}^{g_{S^2 \times S^2}}\) for large \(n\) will follow from the solution of the gluing problem: To an \(A \in V_1 \cap \cdots \cap V_d \cap \mathcal{M}_{M, t}^{g_M}\) glue the connection \(\Theta_{S^2 \times S^2} \# I\) on \(S^2 \times S^2\) obtained from starting with \(\Theta_{S^2 \times S^2}\) and grafting in a charge 1 instanton \(I\) at a point of \(S_x \cap S_y\).

The gluing problem at hand is quite similar to the one considered by Donaldson in his proof of Theorem B of [D1]. In that case one needs to graft a pair of charge 1 instantons to the trivial connection on a \(b^+ = 1\) manifold. Here there is only one instanton on the \(b^+ = 1\) manifold \(S^2 \times S^2\), but there is a second gluing parameter coming from gluing the connection \(\Theta_{S^2 \times S^2} \# I\) on \(S^2 \times S^2\) to the connection \(A\) on \(M\). Finally, one needs to modify the argument of [D1, §V] to see that for each \(A \in V_1 \cap \cdots \cap V_d \cap \mathcal{M}_{M, t}^{g_M}\) and each intersection point of \(S_x\) and \(S_y\), one obtains a circle of connections of the form \(A \# (\Theta_{S^2 \times S^2} \# I)\) on which \(u_1\) evaluates nontrivially. Since \(x \cdot y = 1\) is odd, Theorem 1.1 will follow. This is discussed further in the proof of Theorem 8.1 below.

Here is an outline of the paper. In §2 we review Donaldson’s invariant and describe \(q_{k, u_1, N}\) in more detail. In §3 we present the necessary results of Mrówka [Mr] and of Taubes [T3] concerning gauge theory on manifolds with cylindrical ends. Mrówka’s thesis [Mr] is discussed in §4. In §5 we begin serious consideration of Theorem 1.1, whose proof is then reduced to a single calculation. This calculation is then carried out in §6. In §7 we study the invariant \(q_{k, u_2, M}\) and prove Theorem 1.3. Finally, in §8 we combine our gauge-theoretic calculations with some algebra to prove Theorems 1.5 and 1.6.

2. SOME TORSION INSTANTON INVARIANTS

Let \(M\) be a closed oriented simply connected 4-manifold and \(P\) a principal \(SU(2)\)-bundle over \(M\). The bundle \(P\) is classified topologically by its second Chern class, \(c_2(P) = k\). Let \(\mathcal{A} = \mathcal{A}(P)\) be the \(L^2\)-Sobolev space of connections on \(P\). It is acted on by the Hilbert Lie group \(\mathcal{G} = \mathcal{G}(P)\) of
$L_4^2$-gauge transformations. The quotient is $\mathcal{B}_{M,k} = \mathcal{B}(P)$, the space of equivalence classes of connections. Let $\mathcal{B}_{M,k}$ denote the irreducible connections in $\mathcal{B}_{M,k}$. (We do not distinguish in notation between a connection $A \in \mathcal{A}$ and its equivalence class $A \in \mathcal{B}_{M,k}$.) The moduli space of equivalence classes of anti-self-dual connections on $P$ is denoted by $\mathcal{M}_{M,k}$ or by $\mathcal{M}_{M,k}(g)$ when making explicit the Riemannian metric $g$ on $M$. For $k > 0$ and a generic choice of $g$, this moduli space is a manifold, which, if nonempty, has dimension $8k - 3(1 + b_2^+)$. (See [FU] for details.)

If $b_2^+ > 1$ is odd and if $k > \frac{3}{4}(1 + b_2^+)$ then the Donaldson polynomial invariant $q_{k,M}$ is defined as follows. The dimension of $\mathcal{M}_{M,k}$ is $8k - 3(1 + b_2^+) = 2d$, and $q_{k,M} \in \text{Sym}^d_\mathbb{Z}(H^2(M; \mathbb{Z}))$. For a generic surface $\Sigma$ in $M$, the restriction of an irreducible anti-self-dual connection over $\Sigma$ is again irreducible; let $r_\Sigma : \mathcal{M}_{M,k} \to \mathcal{B}_{\Sigma,k}^*$ denote the restriction map. Donaldson defines a complex line bundle $\mathcal{L}_{\Sigma}$ over $\mathcal{B}_{\Sigma,k}^* \cup \{\theta\}$ (where $\theta$ denotes the trivial connection on $\Sigma$) together with a section so that when pulled back by $r_\Sigma$ it gives a section of $r_\Sigma^*(\mathcal{L}_{\Sigma})$ whose zero set $V_{\Sigma}$ is a codimension 2 submanifold of $\mathcal{B}_{M,k} \cup \{\theta\}$ which meets all of the moduli spaces $\mathcal{M}_{M,l}, l \leq k$, transversely [D1]. We shall call $V_{\Sigma}$ “the divisor associated to $\Sigma$”.

Given homology classes $z_1, \ldots, z_d \in H_2(M; \mathbb{Z})$, represent them by generic surfaces $\Sigma_1, \ldots, \Sigma_d$ in general position. The intersection $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_{M,k}$ will then be discrete, and the condition $k > \frac{3}{4}(1 + b_2^+)$ will imply that it is compact. (The $V_{\Sigma_i}$ are also chosen to have transverse multiple intersections.) Donaldson’s polynomial invariant is defined to be

$$q_{k,M}(z_1, \ldots, z_d) = \#(V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap \mathcal{M}_{M,k})$$

where “#” denotes a count with signs. Donaldson [D2] proves that $q_{k,M}$ depends only on the smooth structure of $M$. More formally, $q_{k,M}$ can be viewed as follows. There is a homomorphism $\mu : H_2(M; \mathbb{Z}) \to H^2(\mathcal{B}_{M,k}^*; \mathbb{Z})$ defined in [D1], and under the hypotheses on $k$, the above intersection is compact. Then $q_{k,M}(z_1, \ldots, z_d) = \mu(z_1) \cup \cdots \cup \mu(z_d)[\mathcal{M}_{M,k}]$. In fact, the divisor $V_{\Sigma_i}$ is a cocycle representative for $\mu(z_i)$. The next proposition is well known.

**Proposition 2.1.** Let $M$ be a closed simply connected 4-manifold. Then $\pi_1(\mathcal{B}_{M,k}^*) = 0$ unless $M$ is spin and $k$ is even, in which case $\pi_1(\mathcal{B}_{M,k}^*) \cong \mathbb{Z}_2$.

**Proof.** This proof can be ferreted out of [FU] as follows. Let $\mathcal{F} = \mathcal{F}/\pm 1$. Then $\mathcal{F}$ is a principal $\mathcal{G}$-bundle over $\mathcal{B}_{M,k}^*$; so $\pi_1(\mathcal{B}_{M,k}^*) \cong \pi_0(\mathcal{F})$. If $\mathcal{F}$ is the based gauge group (of gauge transformations restricting to the identity over a basepoint) then $\mathcal{F}$ fibers over $SU(2)$ with fiber $\mathcal{F}$; so $\pi_0(\mathcal{F}) \cong \pi_0(\mathcal{F}^0)$. It is easy to see as in [FU, §5] that $\pi_0(\mathcal{F}^0) \cong [M, S^3]$, and by the Steenrod Classification Theorem this is $\mathbb{Z}_2$ if $M$ is spin and 0 if $M$ is not spin. We have

$$\mathbb{Z}_2 \xrightarrow{i} \pi_0(\mathcal{F}) \xrightarrow{\pi_0(\mathcal{F}) \to \mathbb{Z}_2} 0;$$
so \( \pi_0(\mathcal{G}) = 0 \) if \( M \) is not spin. If \( M \) is spin then [FU] shows that \( j_* \) is onto in the case that \( k = 1 \) by identifying its image with a generator \( u \in \pi_1(S^3) \cong \pi_0(\mathcal{G}') \cong \pi_0(\mathcal{G}) \). For general \( k \) the argument shows that its image is \( g^*(u) \), where \( g : S^4 \to S^4 \) is a degree \( k \) map; hence \( j_* \) is onto if and only if \( k \) is odd, completing the proof. \( \square \)

Now consider a simply connected spin 4-manifold \( M \) with \( b^+_M \) even. The moduli space of anti-self-dual connections on the \( SU(2) \)-bundle over \( M \) with \( c^2 = k \) has formal dimension \( 8k - 3(1 + b^+_M) = 2d + 1 \). Let homology classes \( z_1, \ldots, z_d \in H_2(M; \mathbb{Z}) \) be represented by generic surfaces \( \Sigma_1, \ldots, \Sigma_d \) as above. If \( k > \frac{3}{4}(1 + b^+_M) + \frac{1}{4} \) then the intersection \( \Sigma_1 \cap \cdots \cap \Sigma_d \cap \mathcal{M}_{M,k} \) will be a compact 1-manifold in \( \mathcal{B}_{M,k}^* \) (for a generic metric on \( M \)). Suppose \( k \) is even. This intersection represents a class in \( H^1(\mathcal{B}_{M,k}^*; \mathbb{Z}) \cong \mathbb{Z} \). Just as for \( q_{k,M} \) this class can be shown to depend only on \( z_1, \ldots, z_d \) and the smooth structure of \( M \). This defines an invariant which Donaldson calls \( q_{k,u,M} \in \text{Sym}^d_{\mathbb{Z}}(H_2(M; \mathbb{Z})) \). Donaldson's definition of this invariant in [D3] is basically the same. It goes as follows. Coupling the Dirac operator \( D \) on \( M \) to connections on \( P \) gives a family of operators and hence a virtual bundle \( \text{Ind} D_A \) over \( \mathcal{A} \). The operator \( D_A \) can be regarded as a real operator, and so there is a real line bundle \( \text{det} \text{Ind} D_A \) over \( \mathcal{A} \). When \( k \) is even this bundle descends to a real line bundle \( \eta \) over \( \mathcal{B}_{M,k}^* \) (see [D1]). The class \( u_1 \in H^1(\mathcal{B}_{M,k}^*; \mathbb{Z}) \) is defined to be \( u_1 = w_1(\eta) \). Then for \( k \) even and in the range above, \( q_{k,u,M}(z_1, \ldots, z_d) = \mu(z_1) \cup \cdots \cup \mu(z_d) \cup u_1[\mathcal{M}_{M,k}] \).

Similarly for \( M \) spin, \( b^+_M \) odd, and \( k \) odd one obtains a class \( u_2 \in H^2(\mathcal{B}_{M,k}^*; \mathbb{Z}) \) as in [D1]. Then for \( k > \frac{3}{4}(1 + b^+_M) + \frac{1}{2} \) one obtains an invariant \( q_{k,u,M} \in \text{Sym}^d_{\mathbb{Z}}(H_2(M; \mathbb{Z})) \) where \( \dim \mathcal{M}_{M,k} = 8k - 3(1 + b^+_M) = 2d + 2 \) defined by \( q_{k,u,M}(z_1, \ldots, z_d) = \mu(z_1) \cup \cdots \cup \mu(z_d) \cup u_2[\mathcal{M}_{M,k}] \).

3. Gauge theory on manifolds with cylindrical ends

Given a simply connected oriented 4-manifold \( X \) with nonempty boundary \( \partial X \), we let \( X_+ \) denote the extended manifold \( X \cup (\partial X \times [-1, \infty)) \). We shall only consider Riemannian metrics on \( X_+ \) which differ on \( \partial X \times [1, \infty) \) from a product metric by an exponentially decaying term. We call such metrics "asymptotically cylindrical". Gauge theory on these manifolds has been studied by several authors, notably Mrówka [Mr], Taubes [T2,3], Morgan, Mrówka, and Ruberman [MMR], and Floer [F]. The appropriate theory in our context is based on connections which decay faster than \( e^{-\delta t} \) for a fixed small constant \( \delta \).

Let \( \mathcal{A}(P) \) denote the space of smooth connections on \( P \) with finite energy, i.e. \( \int_{X^+} \text{Tr}(F_A \wedge F_A) < \infty \) with the coarsest topology compatible with smooth convergence on compact sets and the continuity of \( A \mapsto \int_{X^+} \text{Tr}(F_A \wedge F_A) \). The
gauge group $G$ consists of gauge transformations on $P$ with the topology of smooth convergence on compact sets. In $\mathcal{M}(P)$ consider the finite energy anti-self-dual connections

$$m_{X,k} = \left\{ A \in \mathcal{A} \mid \frac{1}{8\pi^2} \int_{X^s} \text{Tr}(F_A \wedge F_A) = k < \infty \text{ and } *F_A = -F_A \right\}.$$ 

Its quotient by $G$ is the moduli space $\mathcal{M}_{X,k} \subset B$. The "charge" $k$ need not be an integer.

In auspicious circumstances the moduli space $\mathcal{M}_{X,k}$ has nice local properties and one can proceed without worries. This is explained in [T3, Theorem 1.8] which we shall now paraphrase. We need to work with based moduli spaces; so we fix a basepoint $x_0 \in \partial X \times \{0\}$ and let $\mathcal{G}^0$ be the subgroup of gauge transformations in $G$ which restrict to the identity on the fiber over $x_0$. Set $\mathcal{B}^0 = \mathcal{A}^0 / \mathcal{G}^0$ and $\mathcal{M}_{X,k}^0 = m_{X,k} / \mathcal{G}^0$. Consider an $\alpha \in \mathcal{R}(\partial X)$, the representation space $\text{Hom}(\pi_1(\partial X), SU(2))$. The action of $SU(2)$ by conjugation induces an effective action of $SO(3)$ on $\mathcal{R}(\partial X)$ whose quotient is the character variety $\chi(\partial X)$. Let $\Gamma_\alpha$ be the isotropy group of $\alpha$ and let $U_\alpha$ be a slice in $\mathcal{R}(\partial X)$ to this action at $\alpha$; so $U_\alpha \times_{\Gamma_\alpha} SO(3)$ models a neighborhood of the $SO(3)$ orbit of $\alpha$ in $\mathcal{R}(\partial X)$.

**Theorem 3.1** (Taubes [T3; Theorem 1.8], Mrówka [Mr]). There is a locally constant continuous function $R$ on $\mathcal{M}_{X,k}^0$ with values in the set of connected components of $\mathcal{R}(\partial X)$ with the following significance. If $K$ is a connected component of $\mathcal{R}(\partial X)$ such that for each $\alpha \in K$ the dimension of $U_\alpha$ equals the dimension of the twisted cohomology $H^1(\partial X; \text{ad } \alpha)$ and $k_0 \in \mathbb{Z}^+$, then for each $k \leq k_0$:

(i) Each $A \in \mathcal{M}_{X,k}^0 \cap R^{-1}(K)$ has a representative in $m_{X,k}$ such that if $A_t = A|_{\partial X \times \{t\}}$ then $\lim_{t \to \infty} A_t$ exists, is a flat connection, and the assignment $A \to r^0(A) = \lim_{t \to \infty} A_t$ defines an $SO(3)$-equivariant continuous map

$$r^0 : \mathcal{M}_{X,k}^0 \cap R^{-1}(K) \to \mathcal{R}(\partial X)$$

which descends to

$$r : \mathcal{M}_{X,k} \cap R^{-1}(K) / SO(3) \to \chi(\partial X).$$

(ii) For a dense set of asymptotically cylindrical metrics on $X$, the moduli space defined by $\mathcal{M}_{X,k}^0(\alpha) = \mathcal{M}_{X,k} \cap r^{-1}(\alpha)$ is a manifold (except at reducible connections in case $X$ has a negative definite intersection form). If nonempty, $\mathcal{M}_{X,k}^0(\alpha)$ has dimension $8k - 3(\text{E}(X) + \text{sign}(X)) - \frac{1}{2} h - \frac{1}{2} p_\alpha$, where $E(X)$ and $\text{sign}(X)$ denote the Euler characteristic and signature (with compact supports) of $X$, $p_\alpha$ is the $p$-invariant of $[APS2]$ and $h_\alpha$ is the sum of the 0th and 1st betti numbers of $H^*(\partial X; \text{ad } \alpha)$.

(iii) If $W$ is an open subset of $\mathcal{R}(\partial X)$ which contains only smooth points, then for a dense set of asymptotically cylindrical metrics on $X$, $\mathcal{M}_{X,k}^0 \cap (r^0)^{-1}(W)$ is a manifold (except at reducible connections).
(iv) For a dense set of asymptotically cylindrical metrics, \( r^0 : M^0_{X,k} \cap R^{-1}(K) \to R(\partial X) \) is an \( SO(3) \)-equivariant generic map. In particular, it can be made transverse to any \( SO(3) \) invariant complex.

This follows since in this situation there is a \( \delta > 0 \) such that, for \( A \in M^0_{X,k} \cap R^{-1}(K) \), \( e^{\delta t} \text{Tr}(F_A \wedge F_A) \) is integrable. As examples, note that if \( \partial X \) is a Brieskorn homology sphere then \( \chi(\partial X) \) is discrete, and if \( \alpha \in \chi(\partial X) \) is nontrivial, then \( H^1(\partial X ; \text{ad } \alpha) = 0 \) (cf. [FS1]). For the trivial representation, \( H^1(\partial X ; \text{ad } \alpha) = H^1(\partial X ; \mathbb{R}^3) = 0 \) as well. Thus the hypothesis of Theorem 3.1 holds for all of \( R(\partial X) \). Of course, this already appears in the work of Floer [F].

In this paper we are especially interested in the case \( \partial X \cong S^2 \times S^1 \). We have

\[
R(S^2 \times S^1) = \text{Hom}(\mathbb{Z}, SU(2)) \cong S^3,
\]

and

\[
\chi(S^2 \times S^1) \cong SU(2)/SO(3) \cong [-1, 1].
\]

Let \( \alpha \) be a representation corresponding to a point in the open interval \((-1, 1)\). Then \( \Gamma_\alpha \cong S^1 \), \( U_\alpha \) is 1-dimensional, and

\[
U_\alpha \times_{\Gamma_\alpha} SO(3) \cong (\text{interval}) \times_{S^1} SO(3) \cong (\text{interval}) \times S^2.
\]

Now \( H^1(S^2 \times S^1 ; \text{ad } \alpha) \) is the group cohomology \( H^1(\mathbb{Z} ; \text{ad } \alpha) = H^1(S^1 ; \text{ad } \alpha) \). But the twisted Euler characteristic of \( S^1 \) is the untwisted Euler characteristic with coefficients in \( \mathbb{R}^3 \), namely 0; so \( \text{dim } H^1(S^2 \times S^1 ; \text{ad } \alpha) = \text{dim } H^0(S^1 ; \text{ad } \alpha) = \text{dim } \Gamma_\alpha = 1 = \text{dim } U_\alpha \). For \( \tau = \pm 1 \in \chi(S^2 \times S^1) \) we have \( \Gamma_\tau \cong SO(3) \) and so \( U_\tau \) is 3-dimensional. But the same argument as above shows

\[
\text{dim } H^1(S^2 \times S^1 ; \text{ad } \alpha) = \text{dim } H^0(S^2 \times S^1 ; \text{ad } \alpha) = \text{dim } \Gamma_\tau = 3 = \text{dim } U_\tau.
\]

Thus \( K = R(S^2 \times S^1) \) satisfies the hypothesis of (3.1).

4. Mrówka’s thesis

Our technique for proving Theorems 1.1 and 1.3 requires an understanding of what happens when a metric on \( M \) degenerates along a codimension 1 submanifold. In particular, suppose we have a fixed metric \( g \in M \) and a codimension 1 submanifold \( Y \) of \( M \). Suppose that \( Y \) splits \( M \) into submanifolds \( X_1 \) and \( X_2 \); so \( M = X_1 \cup_Y X_2 \). We assume that \( g \) is close to a metric on \( M \) which is a product in a neighborhood “tube” \( Y \times [-1, 1] \). We then wish to study the effect of changing \( g \) in a family \( \{ g_t \} \) where \( g_t \) is within \( \epsilon_t \) of the metric on \( M \) which agrees with \( g \) off \( Y \times [-1, 1] \) but has stretched the tube to \( Y \times [-t, t] \), and \( \lim_{t \to \infty} \epsilon_t = 0 \). Suppose further, for simplicity, that all components of \( R(\partial Y) \) satisfy the hypothesis of Theorem 3.1. For \( t \) large, we need to understand how \( M_{M,k}(g_t) \) relates to \( M_{X_1,k_1}(g_{t,1}) \) and \( M_{X_2,k_2}(g_{t,2}) \).

In one direction, there is Uhlenbeck’s Compactness Theorem [U]. In this situation it yields the following result.
Theorem 4.1 (Uhlenbeck (cf. [Mr])). Consider an increasing unbounded sequence of integers \( \{t_i\} \). Suppose \( A_i \) is a sequence with \( A_i \in \mathcal{M}_{k,t_i}(g_i) \). Then there are finite sets of points \( \{x_i\}_{i=1,\ldots,m} \) in \( X_1 \) and \( \{y_j\}_{j=1,\ldots,n} \) in \( X_2 \) such that after passing to subsequences, \( \{A_i\} \) converges uniformly in \( C^\infty \) on compact subsets of \( X_1 \setminus \{x_1, \ldots, x_m\} \) and \( X_2 \setminus \{y_1, \ldots, y_n\} \) to anti-self-dual connections.

This phenomenon is called “weak convergence”.

As for the other direction, when \( Y \) is a homology 3-sphere, the relationship can be studied via the Floer homology of \( Y \); for example see [F, FS 1,2, A]. In this case, the Donaldson invariant of \( M \) can be computed via a pairing of relative Donaldson invariants on \( X_1^+ \) and \( X_2^+ \) which take their values in the Floer homology of \( Y \).

The general case is studied in the thesis of T. Mrówka [Mr] and by Taubes [T3]. (See the forthcoming work of Morgan, Mrówka, and Ruberman [MMR] for further details.) We next proceed to give a synopsis of some of the results of Mrówka’s thesis.

Let \( U^0 \) be an open subset of the smooth points of \( \mathcal{R}(Y) \) and for \( j = 1,2 \) let \( \mathcal{N}_j^0 \) be a precompact open subset of \( (r_j)^{-1}(U^0) \) where \( r_j : \mathcal{M}_{X_j^+,k_j}^0 \rightarrow \mathcal{R}(Y) \) is given by Theorem 3.1. Let \( \mathcal{N}^0 \) be the fibered product (with \( SO(3) \)-action coming from the diagram):

\[
\begin{array}{ccc}
\mathcal{N}^0 & \xleftarrow{r_1^0} & \mathcal{N}_1^0 \\
\rightarrow & & \downarrow \\
U^0 & \xrightarrow{r_2^0} & \mathcal{N}_2^0 \\
\end{array}
\]

Then Mrówka shows that for generic (asymptotically cylindrical) metrics on \( X_1^+ \) and \( X_2^+ \), the restriction maps \( r_1^0 \) and \( r_2^0 \) are transverse; so \( \mathcal{N}^0 \) is a manifold.

For \( i = 1, 2 \), let \( X_{i,t} = X_i \cup Y \times (-t,t) \subset M(g_i) \). Then as in [D1,2] we say that an \( A \in \mathcal{B}_{k_i,k_2}^+ \) is “\((\eta,t)\)-close” to \((A_1,A_2) \in \mathcal{N}_1 \times \mathcal{N}_2\) if for \( i = 1, 2 \) we have

\[
\| A_{X_{i,t}} - A_{i,X_{i,t}} \|_{L^4(X_{i,t})} < \eta,
\]

where the subscripts denote restriction (and \( \mathcal{N}_i = \mathcal{N}^0_i / SO(3) \)).

Mrówka’s Theorem 4.3 (Part I) [Mr]. If \( k_1 \) and \( k_2 \) are both positive then there are real numbers \( \eta_0 \) and \( t_0 \) such that for all \( t \geq t_0 \) and \( 0 < \eta \leq \eta_0 \) there is an \( SO(3) \)-equivariant map \( \gamma_t^0 : \mathcal{N}^0 \rightarrow \mathcal{M}_{k_1+k_2}^0(g_i) \) satisfying:

(a) The image of \( \gamma_t^0 \) is open in \( \mathcal{M}_{k_1+k_2}^0(g_i) \) and contains all points \((A, \xi)\) such that \( A \) is \((\eta,t)\)-close to a point of \( \mathcal{N}_1 \times \mathcal{N}_2 \).

(b) \( \gamma_t^0 \) is a homeomorphism onto its image.
(c) For fixed \( ((A_1, \xi_1), (A_2, \xi_2)) \in \mathcal{N}^0 \), the sequence \( \gamma_i^0((A_1, \xi_1), (A_2, \xi_2)) \) converges weakly to
\[
[(A_1, \xi_1), (A_2, \xi_2)] \in \mathcal{M}_1^0 \times \mathcal{M}_2^0.
\]

Here elements of \( \mathcal{B}^0 \) are given by gauge equivalence classes of pairs \( (A, \xi) \) where \( A \in \mathfrak{g} \) and \( \xi \in \mathcal{P}_{x_0} \), the fiber over the basepoint.

The theorem above should be compared with [D1, Theorem 4.53] and [D2, Proposition 4.6]. The map \( \gamma_i^0 \) is obtained from an \( SO(3) \)-equivariant map \( \beta_i^0 : \mathcal{N}^0 \to \mathcal{B}^{*0}_{M, k_1 + k_2} \) where \( \beta_i^0((A_1, \xi_1), (A_2, \xi_2)) \) is equal to \( A_1 \) on \( X_1 \) and \( A_2 \) on \( X_2 \) and patches the two connections together on the tubes \( \partial X_i \times [0, 2t] \) and has framing \( \xi_i \xi_i^{-1} \). The map \( \gamma_i^0 \) is obtained from a deformation of \( \beta_i^0 \).

In particular, the image \( \beta_i(\mathcal{N}) \) is homologous to \( \gamma_i(\mathcal{N}) \) in \( \mathcal{B}^{*,0}_{M, k_1 + k_2} \).

There is also a version of Mrówka’s theorem which holds when one of the moduli spaces has zero charge. Assume that \( k_1 > 0 \) and \( k_2 = 0 \). Then \( \mathcal{M}_{X_1, +, k_1}^0 \) can be identified with the representation space \( \mathcal{B}(X_2) \) and we can consider \( \mathcal{N}_2^0 \) as an open subset of the smooth points of \( \mathcal{B}(X_2) \). Thus, as the \( \alpha \in \mathcal{N}_2^0 \) vary, the \( H^2(X_2; \text{ad}_A) \) fit together to form an \( SO(3) \)-equivariant bundle \( \Xi_2^0 \to \mathcal{N}_2^0 \). Pull back \( \Xi_2^0 \) to an \( SO(3) \)-equivariant bundle \( \Xi^0 \) over \( \mathcal{N}^0 \). Mrówka’s result in this case is

**Mrówka’s Theorem 4.4 (Part II) [Mr].** There are real numbers \( \eta_0 \) and \( t_0 \) such that for all \( t \geq t_0 \) and \( 0 < \eta \leq \eta_0 \) there is an \( SO(3) \)-equivariant map \( \gamma_i^0 : \mathcal{N}^0 \to \mathcal{B}^{*0}_{M, k_1} \) and an \( SO(3) \)-equivariant section \( s^0 : \mathcal{N}^0 \to \Xi^0 \) such that

- (a) \( \gamma_i^0(\mathcal{N}^0) \cap \mathcal{M}_{M, k_1}^0(\eta_t) \) is open in \( \mathcal{M}_{M, k_1}^0(\eta_t) \) and contains all points \( (A, \xi) \) such that \( A \) is \( (\eta, t) \)-close to a point of \( \mathcal{N}_i \times \mathcal{N}_2 \).
- (b) \( \gamma_i^0((s^0)^{-1}(0)) = \gamma_i^0(\mathcal{N}^0) \cap \mathcal{M}_{M, k_1}^0(\eta_t) \), and \( \gamma_i^0((s^0)^{-1}(0)) \) is a homeomorphism onto its image.
- (c) For fixed \( \mathcal{N}_1((A_1, \xi_1), (A_2, \xi_2)) \in \mathcal{N}^0 \), the sequence \( \gamma_i^0((A_1, \xi_1), (A_2, \xi_2)) \) converges weakly to
\[
[(A_1, \xi_1), (A_2, \xi_2)] \in \mathcal{M}_1^0 \times \mathcal{M}_2^0.
\]

We next need to look more closely at the divisor \( V_{\Sigma} \) corresponding to an oriented surface \( \Sigma \subset X \). For any such surface which is generic in the sense that restriction induces \( r_{\Sigma} : \mathcal{M}_{X, k}^* \to \mathcal{B}_{\Sigma}^* \) for all \( k \) (i.e. an irreducible anti-self-dual connection over \( X \) restricts to an irreducible connection over \( \Sigma \)) the divisor \( V_{\Sigma} \) is defined to be the zero set of a generic section of \( r_{\Sigma}^*(\mathcal{L}_{\Sigma}) \) where \( \mathcal{L}_{\Sigma} = \det \text{Ind} D_{\Sigma, A} \) and \( D_{\Sigma, A} \) is the Dirac operator on \( \Sigma \) coupled to connections. The operator \( D_{\Sigma, A} \) has numerical index 0; thus generic fibers of the index virtual bundle \( \text{Ind} D_{\Sigma, A} \) are 0-dimensional, however at certain “jumping points” \( A \in \mathcal{B}_* \), \( \ker D_{\Sigma, A} \cong \text{coker} D_{\Sigma, A} \neq 0 \). These points constitute the divisor of the determinant line bundle \( \det \text{Ind} D_{\Sigma, A} \). It is shown in [DK] that if \( \mathcal{V}_{\Sigma} \) is defined by this divisor, then \( \mathcal{V}_{\Sigma} \) is also a cocycle representative for
Thus for our purposes, it suffices to identify \( V_\Sigma \) with \( \mathbb{V}_\Sigma \). Then an anti-self-dual connection \( A \) over \( X \) lies in \( V_\Sigma \) if and only if its restriction \( r_\Sigma(A) \) over \( \Sigma \) is a jumping point for the index virtual bundle of the Dirac operator on \( \Sigma \) coupled to connections.

**Proposition 4.5.** In the situation of Theorem 4.3 suppose that \( \Sigma \subset X_1 \). Then \( \beta_!(A_1, A_2) \in V_\Sigma \) if and only if \( A_1 \in V_\Sigma \).

**Proof.** Let \( \beta \) denote restriction of a connection to \( \Sigma \); so \( A_1 = \beta_!(A_1, A_2) \). Then \( \beta_!(A_1, A_2) \in V_\Sigma \) if and only if \( \ker D_{\Sigma,A} \neq 0 \), i.e. if and only if \( A_1 \in V_\Sigma \). \( \square \)

It is more interesting to ask about a surface \( \Sigma \) in \( X \) such that \( \Sigma = \Sigma_1 \cup \Sigma_2 \) with \( \Sigma_i \subset X_i \) and such that the homology class of \( \Sigma \) is represented neither in \( X_1 \) nor \( X_2 \). Assume that \( \Sigma_1 \cap \Sigma_2 \) is a circle, and let \( \Sigma_1^{+} = \Sigma_1 \cup (\partial \Sigma_1 \times [0, \infty)) \subset X_{1^{+}} \). The surface \( \Sigma_1^{+} \) has a Dirac operator \( D_{\Sigma_1} \) defined on \( L^2 \)-sections.

**Theorem 4.6** (Morgan, Mrówka, and Ruberman). Let \( ((A_1, \xi_1), (A_2, \xi_2)) \in \mathcal{M}^0 \). Suppose that \(-1 \notin U^0 \). Then there is a \( t_0 \) such that, for all \( t \geq t_0 \), \( \beta_!(A_1, \xi_1), (A_2, \xi_2) \in V_\Sigma \) if and only if \( (A_1, \xi_1) \in V_{\Sigma_1}, \) or \( (A_2, \xi_2) \in V_{\Sigma_2} \).

### 5. Connected sums with \( S^2 \times S^2 \)

Let \( M \) be a smooth closed simply connected spin 4-manifold with \( b_M^+ \geq 3 \) and odd, and let \( \ell \) be an odd integer, \( \ell > \frac{3}{4}(1 + b_M^+) \). The moduli space of anti-self-dual connections on the \( SU(2) \)-bundle \( P \) over \( M \) with \( c_2(P) = \ell \) gives rise to a Donaldson polynomial \( q_{\ell,M} \in \text{Sym}^d_Z(H_2(M; \mathbb{Z})) \) where \( d = 4\ell - \frac{3}{4}(1 + b_M^+) \). (Notice that the conditions placed on \( b_M^+ \) and \( \ell \) imply that \( d > \frac{3}{4}(1 + b_M^+) \geq 6 \).) Let \( X = M \# S^2 \times S^2 \) and consider the \( SU(2) \) bundle over \( X \) with \( c_2 = \ell + 1 \). The formal dimension of the moduli space \( \mathcal{M}_{X,\ell+1} \) is \( 8(\ell + 1) - 3(1 + (b_M^+ + 1)) = 2d + 5 \). Since \( X \) is spin and \( \ell + 1 \) is even and \( > \frac{3}{4}(b_M^+ + 1) + \frac{3}{4} \), the \( \mathbb{Z}_2 \)-polynomial invariant \( q_{\ell+1,\ell+1} \) of degree \( d + 2 \) is defined. Let \( z_1, \ldots, z_d \in H_2(M, \mathbb{Z}) \) be represented by generic oriented surfaces \( \Sigma_1, \ldots, \Sigma_d \) in \( M \) as in \( \S 2 \) and let \( x, y \) be the classes represented by \( S^2 \times 0 \) and \( 0 \times S^2 \). Here we are viewing \( H_2(M, \mathbb{Z}) \) as a subgroup of \( H_2(X, \mathbb{Z}) \). We mean to evaluate \( q_{\ell+1,\ell+1}(z_1, \ldots, z_d, x, y) \).

It is useful to view the process of taking connected sums with \( S^2 \times S^2 \) as the result of surgery along a homotopically trivial circle. Let \( C \) denote a circle in \( M \), and let \( M_0 \) be the simply connected manifold \( M \setminus N(C) \), where \( N(C) \) is an open tubular neighborhood of \( C \). Let \( C' \) be a circle in \( S^4 \), and \( K = S^2 \setminus N(C') \simeq S^2 \times D^2 \); then \( X = M_0 \cup_{S^2 \times S^2} K \). Let \( g_{M_0} \) and \( g_K \) be asymptotically cylindrical generic metrics on \( M_{0,+} \) and \( K_+ \) and let \( g = g_1 \) be a generic metric on \( X \) such that the family of metrics \( \{g_t \mid t \geq 1\} \) on \( X \), obtained as in \( \S 4 \), converges to \( g_{M_0} \parallel g_K \).
Since the character variety $\mathcal{X}(S^2 \times S^1) = [-1, 1]$ is connected, all Chern-Simons invariants of flat $SU(2)$-connections over $S^2 \times S^1$ are trivial. Thus a connection $A$ with finite action on the restricted bundle over $M_0$ has integral charge $(-1/8\pi^2) \int_{M_0} \text{Tr}(F_A \wedge F_A)$.

Recall from Theorem 3.1 the restriction map $r_{M_0} : \mathcal{M}_{M_0} \to \mathcal{X}(S^2 \times S^1)$. Let $\mathcal{M}_{M_0,m}(\alpha)$ denote the moduli space of anti-self-dual connections $A$ over $M_0$ with charge equal to $m$ and with $r_{M_0}(A) = \alpha$.

**Proposition 5.1.** The formal dimension of $\mathcal{M}_{M_0,m}(\alpha)$ is

$$\dim \mathcal{M}_{M_0,m}(\alpha) = \begin{cases} 2d + 8(m - \ell) - 1, & \alpha \neq \pm 1, \\ 2d + 8(m - \ell), & \alpha = \pm 1. \end{cases}$$

**Proof.** The Atiyah, Patodi, Singer Theorem gives

$$\dim \mathcal{M}_{M_0,m}(\alpha) = 8m - 3(1 + b_M^+) - \frac{1}{2} h_\alpha - \frac{1}{2} \rho_\alpha$$

where $\rho_\alpha$ is the $\rho$-invariant of [APS2] and $h_\alpha$ is the sum of the 0th and 1st betti numbers of $H^*(S^2 \times S^1, \text{ad} \alpha)$. To compute $\rho_\alpha$, note that the representation $\alpha$ of $\pi_1(S^2 \times S^1)$ extends as a representation of $\pi_1(D^3 \times S^1)$. Thus $\rho_\alpha = 3 \text{ sign}(D^3 \times S^1) - \text{sign}_{\text{ad} \alpha}(D^3 \times S^1) = 0$.

The calculation at the end of §3 shows that $h_\alpha = 2$ if $\alpha \neq \pm 1$ and $h_\alpha = 6$ if $\alpha = \pm 1$. Now $8\ell - 3(1 + b_M^+) = 2d$; so we have $\dim \mathcal{M}_{M_0,m}(\alpha) = 2d + 8(m - \ell) - \frac{1}{2} h_\alpha$, as desired. \(\square\)

**Corollary 5.2.** If $2d + 8(m - \ell) \geq 1$ the moduli space $\mathcal{M}_{M_0,m}$ (with respect to the generic metric $g^{M_0}$) is a manifold whose formal dimension is $2d + 8(m - \ell)$.

**Proof.** Since $\mathcal{R}(S^2 \times S^1) \cong S^3$ is a smooth manifold, Theorem 3.1(iii) implies that $\mathcal{M}^0_{M_0,m}$ is a manifold. Since this moduli space contains no reducible connections, the group $SO(3)$ acts freely on $\mathcal{M}^0_{M_0,m}$. Also $SO(3)$ acts on $\mathcal{R}(S^2 \times S^1)$ with $S^2$ as principal orbit type. Thus the quotient $\mathcal{M}^0_{M_0,m}$ is also a manifold, and according to (3.1) $r_{M_0}^0$ is a generic $SO(3)$-equivariant map. Thus for a generic metric, $r_{M_0} : \mathcal{M}_{M_0,m} \setminus r_{M_0}^{-1}(\pm 1) \to (-1, 1)$ is transverse, and so $\dim \mathcal{M}_{M_0,m} = 1 + \dim \mathcal{M}_{M_0,m}(\alpha)$ where $\alpha \neq \pm 1$. \(\square\)

Similarly we have

**Proposition 5.3.** The formal dimension of $\mathcal{M}_{K,n}(\alpha)$ is

$$\dim \mathcal{M}_{K,n}(\alpha) = \begin{cases} 8n - 4, & \alpha \neq \pm 1, \\ 8n - 6, & \alpha = \pm 1. \end{cases}$$

**Corollary 5.4.** If $n \geq 1$ the moduli space $\mathcal{M}_{K,n}$ is a manifold whose formal dimension is $8n - 3$. With the metric $g_{M_0}$ on $M_0$, the moduli spaces $\mathcal{M}_{M_0,m}(\alpha)$ will either be empty or manifolds of dimension given in Proposition 5.1. Now let $V_1, \ldots, V_d$
be the divisors associated with the surfaces $\Sigma_i \subset M_0$, $[\Sigma_i] = z_i$. Let $J_{M_0}$ denote the zero-dimensional intersection $J_{M_0} = V_1 \cap \cdots \cap V_d \cap M_{0, \ell}$.

**Proposition 5.5.** If $A \in J_{M_0}$ then $r_{M_0}(A) \neq \pm 1$.

**Proof.** The subcomplex $r_{M_0}^{-1}(\pm 1)$ of $M_{0, \ell}$ has dimension $2d - 3$ and is met transversely by the codimension $2d$ submanifold $V_1 \cap \cdots \cap V_d$. □

**Proposition 5.6.** $J_{M_0}$ is compact.

**Proof.** If not, there is a sequence of connections $\{A_n\}$ in $J_{M_0}$ converging weakly to an $A_\infty \in M_{0, m}$, $m < \ell$, together with instantons at points $x_1, \ldots, x_r \in M_0$, and perhaps instantons on tubes $S^2 \times S^1 \times \mathbb{R}$. (See e.g. [FS2].) Since the Chern-Simons invariant of any flat connection on $S^2 \times S^1$ is $0 \in \mathbb{R}/\mathbb{Z}$, the moduli spaces containing the instantons on tubes account for an integral total charge $T \geq 0$. If $A_\infty \notin M_{0, \ell}$ then $r + T > 0$. The surfaces $\Sigma_i$ are in general position; so the points $x_1, \ldots, x_r$ lie on at most $2r$ of the surfaces. Suppose that $0 < m < \ell$. Then $A_\infty$ lies in at least $d - 2r$ of the codimension 2 varieties $V_i$. So by Corollary 5.2 and transversality, $2d + 8(m - \ell) \geq 2(d - 2r)$. But also counting charge we get

$$\ell \geq m + r + T \geq \ell + \frac{r}{2} + T.$$  

This gives a contradiction unless $r + T = 0$.

In case $A_\infty \in M_{0, 0}$, then $A_\infty = \theta$, the trivial connection. But the formal dimension of $M_{0, 0}$ is negative by Proposition 5.1. Thus transversality implies that $\theta \notin V_i$ for any $i$. This means that each $\Sigma_i$ contains some $x_j$, and so $2r \geq d = 4\ell - \frac{3}{2}(1 + b_M^+)$. Again counting charge, we have $\ell \geq r + T \geq 2\ell - \frac{3}{2}(1 + b_M^+) + T$. This contradicts our basic assumption that $\ell > \frac{3}{4}(1 + b_M^+)$. Thus $J_{M_0}$ is compact. □

**Proposition 5.7.** The intersection $J_{M_0}$ consists of a finite number of points; modulo 2 this number is $q_{\ell, M}(z_1, \ldots, z_d)$.

**Proof.** First we apply Mrówka’s Theorem 4.4 to $M = M_0 \cup N(C)$. Since $\mathcal{R}(S^2 \times S^1) \cong S^3$, we can take $U^0 = \mathcal{R}(S^2 \times S^1) \setminus \{\pm 1\}$. The tubular neighborhood $N(C) \cong \mathbb{R}^3 \times S^1$; so $\mathcal{R}(N(C))$ can be identified with $\mathcal{R}(S^2 \times S^1)$, and the fibered product $\mathcal{N}^0$ can be identified with $M_{0, \ell}(C_f)$. The obstruction bundle $\Xi_{M_0}$ has fiber $H^2(\mathbb{R}^3 \times S^1; \text{ad} \rho)$ for $\rho \in \mathcal{R}(\mathbb{R}^3 \times S^1)$. This cohomology group vanishes; so Theorem 4.4 implies that for large $t$ we get maps $\mathcal{N}^0 \to \mathcal{M}_{M, \ell}(g_t)$ which are homeomorphisms onto their (open) image.

The intersection $J_{M_0} \subset r_{M_0}^{-1}(-1, 1)$ by Propositions 5.5 and 5.6. Hence we may consider $J_{M_0} \subset \mathcal{N}$. We claim that for large $t$ the homeomorphism $\gamma_t$ identifies $J_{M_0}$ with $J_{M}(t) = M_{M, \ell}(g_t) \cap V_1 \cap \cdots \cap V_d$. (Cf. [D2; Proof of Theorem 4.8].) First, if $A \in J_{M_0}$, then by Theorem 4.4(c) $\{\gamma_t(A)\}$ converges to
[A, r_{M_0}(A)] \in \mathcal{M}_{M_0, t} \times \mathfrak{R}(\mathbb{R}^3 \times \mathbb{S}^1); \text{ so as } t \to \infty, \gamma_t(A) \in \mathcal{M}_{M, t}(g_t) \text{ converges to a point of } \mathcal{V}_t \cap \cdots \cap \mathcal{V}_d. \text{ But } \mathcal{V}_t \cap \cdots \cap \mathcal{V}_d \text{ intersects } \mathcal{M}_{M, t}(g_t) \text{ transversely in a dimension } 0 \text{ submanifold. So for large enough } t \text{ by Theorem 4.4(a) there is a unique point of } \mathcal{J}_M(t) \text{ close to } \gamma_t(A). \text{ We claim that for large enough } t \text{ these points comprise all of } \mathcal{J}_M(t). \text{ If not, then there is a sequence } A_n \in \mathcal{J}_M(t_n), t_n \to \infty, \text{ which fails to converge strongly to some connection } [A, r_{M_0}(A)]. \text{ In other words, } \{A_n\} \text{ converges to } [A_{M_0}, A_c] \in \mathcal{M}_{M_0, k_1} \times \mathcal{M}_{N(C), k_2} \text{ together with instantons at points } x_1, \ldots, x_r \in M_0 \text{ and } y_1, \ldots, y_s \in N(C), \text{ and perhaps also loses some integral charge } T \text{ on the tube } S^2 \times S^1 \times \mathbb{R}. \text{ (See the proof of Proposition 5.6.) Since we are assuming that } \{A_n\} \text{ fails to converge strongly to some } [A, r_{M_0}(A)], \text{ we must have } k_2 + r + s + T > 0. \text{ But the standard counting argument as used in Proposition 5.6 implies that } k_2 + r + s + T = 0. \text{ This means that for large } t \text{ we can identify } \mathcal{J}_M \text{ with } \mathcal{J}_M(t). \text{ The Donaldson invariant } q_{\ell, M}(z_1, \ldots, z_d) \text{ is calculated by computing the signed number (finite) of points in } \mathcal{J}_M(t); \text{ so our proposition follows.} \square

Notice that we have not yet used reduction mod 2 in a substantial way. Given a homology orientation [D2], \( q_{\ell, M}(z_1, \ldots, z_d) \) is the count of signed points in \( \mathcal{J}_M(t) \); so our proposition follows.

Let \( \Sigma_x \) and \( \Sigma_y \) be generic surfaces representing the homology classes \( x = [S^2 \times 0] \) and \( y = [0 \times S^2] \) in \( X = M \# S^2 \times S^2 \), and with divisors \( V_x \) and \( V_y \). We are viewing \( X = M_0 \cup (S^2 \times D^2) \), and we may clearly assume \( \Sigma_x \subset S^2 \times D^2 \). We need to focus our attention on \( \Sigma_y \). The surface \( \Sigma_y \) restricts to \( M_0 \) and \( K \) to give bounded surfaces \( \Sigma_y, M_0 \) and \( \Sigma_y, K \). We also use this same notation to denote the corresponding surfaces with cylindrical ends in \( M_{0+} \) and \( K_+ \).

For \( A \in \mathcal{D}_{M_0, k_1} \) and \( B \in \mathcal{D}_{K, k_2} \), let \( A' \) and \( B' \) denote the restrictions to \( \Sigma_y, M_0 \) and \( \Sigma_y, K \). Since \( \mathcal{J}_{M_0} \) consists of a finite number of connections, and since the divisor \( V_{\Sigma_x, M_0} \) can be chosen transverse to \( \mathcal{J}_{M_0} \), we have \( \mathcal{J}_{M_0} \cap V_{\Sigma_x, M_0} = \emptyset \). Then it follows from Proposition 4.6 that if \((A, \xi), (B, \eta) \in \mathcal{N}^0\), then for large enough \( t \), the image \( \beta_t^1((A, \xi), (B, \eta)) \in V_{\Sigma_y} \) if and only if \((B, \eta) \in V_{\Sigma_y, K} \).

For \((A, \xi) \in \mathcal{M}^0_{M_0, t} \) and \((B, \eta) \in \mathcal{M}^0_{K, t} \) with \( r_{M_0}(A, \xi) = \alpha = r_K(B, \eta) \), we have in the pullback diagram (4.2) restricted to the fibers of \((A, \xi)\) and \((B, \eta)\):

\[
SO(3) \cdot A \quad SO(3) \cdot B \\
\downarrow \quad \downarrow \\
SU(2)/\Gamma_\alpha \cong S^2
\]

where \( SO(3) \cdot A = \{(A, \xi) | \xi \in P_{M_0, \Sigma_y} \} \) and similarly for \( SO(3) \cdot B \).

Now let \( \mathcal{J}_K \) denote the intersection \( \mathcal{M}_{K, 1} \cap V_x \cap V_y, K \). By Corollary 5.4 this is a 1-manifold. Let \( \mathcal{J}^0_{M_0} \ast \mathcal{J}^0_K \) denote the \( SO(3) \)-equivariant fibered product.
over $\mathcal{R}(S^2 \times S^1)$.

We see from (5.8) that if $(A, \xi) \in \mathcal{J}^0_{M_0}$ and $\alpha = r^0_{M_0}(A, \xi)$ then the corresponding fiber of $\mathcal{J}^0_{M_0} \ast \mathcal{J}^0_K$ is an $SO(3)$-equivariant $\Gamma_\alpha \cong S^1$-bundle over the 3-manifold

$$\mathcal{J}^0_K \cap (r^0_K)^{-1}(SU(2)/\Gamma_\alpha) = \mathcal{J}^0_K(\alpha).$$

Taking the quotient by $SO(3)$ we get the 1-manifold $(\mathcal{J}^0_{M_0} \ast \mathcal{J}^0_K)/SO(3) = \mathcal{J}_{M_0} \ast \mathcal{J}_K$ which is an $S^1$-fibration over $\mathcal{J}_K \cap r^{-1}_K r^0_{M_0}(\mathcal{J}_{M_0})$. Note that $\mathcal{J}_{M_0} \ast \mathcal{J}_K$ is not a fibered product. It is easy to see, using Uhlenbeck’s compactness theorem [U], that each $\mathcal{J}_K(\alpha)$ is a compact 0-manifold; thus $\mathcal{J}_{M_0} \ast \mathcal{J}_K$ consists of a finite number of the isotropy circles $\Gamma_\alpha$, $\alpha \in r^{-1}_M(\mathcal{J}_{M_0})$.

We are going to count the circles in $\mathcal{J}_{M_0} \ast \mathcal{J}_K$. Our already cumbersome notation will be kept simpler by assuming that for $A_1, A_2 \in \mathcal{J}_{M_0}$, we have $r^{-1}_{M_0}(A_1) \neq r^{-1}_{M_0}(A_2)$. Since $\mathcal{J}_{M_0}$ is a finite set, there is no loss in making this assumption. The general case will follow by keeping track of multiplicities.

**Proposition 5.9.** For large enough $t$, the image $\gamma_t(\mathcal{J}_{M_0} \ast \mathcal{J}_K)$ is homologous to $\mathcal{J}_X(t) = \mathcal{M}_{X, t+1}(g_t) \cap V_1 \cap \cdots \cap V_d \cap V_x \cap V_y$ in $\mathcal{R}^*_{X, t+1}$.

**Proof.** Let $\alpha \in r^{-1}_M(\mathcal{J}_{M_0})$ and let $U$ be a small interval in $\mathcal{X}(S^2 \times S^1) \setminus \{\pm 1\}$ such that $U \cap r^{-1}_{M_0}(\mathcal{J}_{M_0}) = \{\alpha\}$. Our running assumption is that $r^{-1}_M(\alpha) = \{A\}$ is a single connection. Let $\mathcal{N}_{M_0} = \mathcal{M}_{M_0, t} \cap r^{-1}_{M_0}(U)$ and $\mathcal{N}_K = \mathcal{M}_{K, 1} \cap r^{-1}_K(U)$ and form the $SO(3)$-equivariant fibered product:

For large enough $t$, Mrówka’s theorem 4.3 gives $SO(3)$-equivariant homeomorphisms $\gamma^0_t : \mathcal{N}^0 \to \mathcal{M}_{X, t+1}(g_t)$ onto open subsets. Now

$$((\mathcal{J}^0_{M_0} \ast \mathcal{J}^0_K) \cap \mathcal{N}_0)/SO(3) = (\mathcal{J}_{M_0} \ast \mathcal{J}_K) \cap \mathcal{N}$$

is an $S^1$-fibration over $\mathcal{J}_K \cap r^{-1}_K (r^{-1}_{M_0}(\mathcal{J}_{M_0}) \cap U) = \mathcal{J}_K(\alpha)$. If $B \in \mathcal{J}_K(\alpha)$, denote the fiber over $B$ by $\Gamma_\alpha \cdot B$. Thus $\mathcal{N} \cap (\mathcal{J}^0_{M_0} \ast \mathcal{J}^0_K) = \Gamma_\alpha \cdot \mathcal{J}_K(\alpha)$. 

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By Proposition 4.5 and Theorem 4.6 we have that for large enough \( t \), 
\[ \beta_{t}(A, \xi, (B, \eta)) \in \mathcal{F}_{x} \cap \ldots \cap \mathcal{F}_{x} \cap V_{y} \cap V_{y}, \text{if and only if } A \in \mathcal{F}_{x} \cap \ldots \cap \mathcal{F}_{x} \cap V_{y} \text{ and } B \in V_{x} \cap V_{y}, \text{for some } y \text{ is a point} \]
that is, if and only if \( ((A, \xi), (B, \eta)) \in \mathcal{F}_{M_{0}} \cap \mathcal{F}_{K} \). Taking
the quotient by the action of \( SO(3) \) we see that \( \beta_{t}(\mathcal{N}) \cap \mathcal{V}_{x} \cap \ldots \cap \mathcal{V}_{y} \cap V_{x} \cap V_{y} = \beta_{t}(\Gamma_{\alpha} \cdot \mathcal{F}_{K}(\alpha)) \) which is homologous to \( \gamma_{t}(\Gamma_{\alpha} \cdot \mathcal{F}_{K}(\alpha)) \). But again by the definition of \( \gamma_{t} \) and the fact that intersections with the divisors \( V_{x} \) are transverse, the intersection \( \beta_{t}(\mathcal{N}) \cap \mathcal{V}_{x} \cap \ldots \cap \mathcal{V}_{y} \cap V_{x} \cap V_{y} \) is homologous to \( \gamma_{t}(\mathcal{N}) \cap \mathcal{F}_{x}(t) \).

To complete the proof, we must show (as in Proposition 5.7) that for large enough \( t \) there are no other points of \( \mathcal{F}_{x}(t) \). If there are other points then there is a sequence \( A_{n} \in \mathcal{F}_{x}(t_{n}), t_{n} \to \infty \), which fails to converge strongly to a point \( (A, B, h) \in \mathcal{F}_{M_{0}} \times \mathcal{F}_{K} \times \Gamma_{\alpha} \) where \( r_{M_{0}}(A) = r_{K}(B) = \alpha \). Thus \( \{A_{n}\} \)
converges weakly to some \( (A', B') \in \mathcal{F}_{M_{0}} \times \mathcal{F}_{K} \) together with instantons at points \( x_{1}, \ldots, x_{r} \in M_{0} \) and \( y_{1}, \ldots, y_{s} \in K \) and with an integral charge \( T \geq 0 \) lost on the tube \( S^{2} \times S^{1} \times \mathbb{R} \). By assumption \( r + s + T > 0 \).

We now proceed with a counting argument as in Proposition 5.6, but there are a few more complications. First assume that \( k_{1} > 0 \) and \( k_{2} > 0 \). Then we get \( 2d+8(k_{1}+\ell) \geq 2d-2r \), i.e. \( k_{1} \geq \ell-\frac{1}{4}r \), and \( 8k_{2}-3 \geq (1-2s), \) i.e. \( k_{2} \geq \frac{1}{4} - \frac{1}{4}s \). By a count of charge we have: \( \ell + 1 \geq k_{1} + k_{2} + r + s + T \geq \ell + \frac{1}{2} + \frac{1}{2}r + \frac{3}{4}s + T \), which implies that \( s + T = 0 \) and \( r = 1 \). But if \( r = 1 \), then \( k_{1} \geq \ell - \frac{1}{2} \); so \( k_{1} \geq \ell \). Also \( s = 0 \) implies that \( k_{2} \geq \frac{1}{2} \); so \( k_{2} \geq 1 \). The charge count now gives \( \ell + 1 \geq k_{1} + k_{2} + r + s + T \geq \ell + 1 + 1 \), a contradiction.

In case \( k_{1} = 0 \) and \( k_{2} > 0 \), we have (as in Proposition 5.6) \( r \geq 2\ell - \frac{3}{4}(1+b_{M}^{+}) \)
and \( k_{2} \geq \frac{1}{2} - \frac{1}{4}s \). Now the basic charge count yields \( \ell + 1 \geq (\ell + \frac{1}{2} - \frac{1}{4}s) + (2\ell - \frac{3}{4}(1+b_{M}^{+})) + s + T \). However our basic assumption on \( \ell \) which guarantees the existence of the Donaldson invariant \( q_{\ell,M} \) is that \( \ell \geq \frac{3}{4}(1+b_{M}^{+}) \). Combining this with the above inequality for \( \ell \) we have \( 0 \leq \frac{1}{2} - \frac{1}{4}s + T \). Thus \( s = T = 0 \), and so \( k_{2} \geq \frac{1}{2} \); so in fact \( k_{2} \geq 1 \). Recalculate the basic charge count: \( \ell + 1 \geq 1 + r \geq 1 + 2\ell - \frac{3}{4}(1+b_{M}^{+}) \), which implies that \( \ell \leq \frac{3}{4}(1+b_{M}^{+}) \), a contradiction.

Next consider the case \( k_{1} > 0, k_{2} = 0 \). Then \( k_{1} \geq \ell - \frac{1}{2}r \) and \( s \geq 1 \). We get \( \ell + 1 \geq k_{1} + r + s + T \geq \ell + \frac{1}{2}r + 1 + T \); so \( r = T = 0 \). But then \( A' \in \mathcal{F}_{M_{0}} \), and \( B' = \Theta \), and since \( T = 0, r(A') = r(B') = 1 \), which contradicts Proposition 5.5.

Finally suppose that \( k_{1} = k_{2} = 0 \). Then \( r \geq 2\ell - \frac{3}{4}(1+b_{M}^{+}) \) and \( s \geq 1 \). The charge count gives \( \ell + 1 \geq 2\ell - \frac{3}{4}(1+b_{M}^{+}) + 1 + T \); so \( \ell \leq \frac{3}{4}(1+b_{M}^{+}) - T \), which contradicts the basic assumption on \( \ell \). \( \square \)

Recall our standing hypothesis that \( r_{M_{0}}(\alpha) = 1 \) when restricted to \( \mathcal{F}_{M_{0}} \).

**Proposition 5.10.** \( q_{\ell+1, u_{i}, \lambda}(z_{1}, \ldots, z_{d}, x, y) \equiv \sum \{ \# \mathcal{F}_{K}(\alpha) \mid \alpha \in r_{M_{0}}(\mathcal{F}_{M_{0}}) \} \mod 2 \).

**Proof.** Consider the isotropy circles \( \gamma_{t}(\Gamma_{\alpha} \cdot B) \). Referring to (5.8) we first study the family of connections \( SO(3) \cdot A \). Since \( M_{0} \) is spin, the Dirac
operator gives us a family of real operators over $SO(3) \cdot A$. The index bundle of this family can be pulled back over $SU(2)$ where it becomes trivial. Now $-1 \in SU(2)$ acts nontrivially on spinors, and the index bundle of the Dirac family $\text{Ind}_R(\mathcal{D}, \gamma_1(SO(3) \cdot A))$ is the quotient of this trivial bundle. (See [D1].) It follows that as a class in $KO(SO(3))$ the index bundle is $m_1 \cdot \eta_1$ where $m_1 = 2 \text{ind}(\mathcal{D}_{M_0, 0}) + \ell$ is the numerical index and $\eta_1$ is the class of the Hopf (real) line bundle. Similarly over $SO(3) \cdot B$ we obtain $m_2 \cdot \eta_2$, where $m_2 = 2 \text{ind}(\mathcal{D}_{K, 0}) + 1$.

If $A \in \mathcal{S}^0_{M_0}$ and $B \in \mathcal{S}^0_K$, the connected component of the fibered product $\mathcal{S}^0_{M_0} \ast \mathcal{S}^0_K$ obtained from (5.8) is $S^1 \times SO(3)$. Now fix $\alpha \in r^0_{M_0} (SO(3) \cdot A) = r^0_K (SO(3) \cdot B)$. Then, if $A' \in (r^0_{M_0})^{-1}(\alpha) \cap SO(3) \cdot A = \Gamma_\alpha \cdot A' \cong S^1$ and similarly $(r^0_K)^{-1}(\alpha) \cap SO(3) \cdot B = \Gamma_\alpha \cdot B' \cong S^1$. (Note that $\Gamma_\alpha \cdot B' \subset SO(3) \cdot B$ is not to be confused with $\Gamma_\alpha \cdot B \subset N$.) The pullback in $S^1 \times SO(3)$ of $\Gamma_\alpha \cdot A'$ and $\Gamma_\alpha \cdot B'$ is a copy of $S^1 \times S^1$. Under the quotient $M^0 \to N$ this $S^1 \times S^1$ family projects to $\Gamma_\alpha \cdot B \subset \mathcal{S}^0_{M_0} \ast \mathcal{S}^0_K \subset N$.

We compute $\text{Ind}_R(\mathcal{D}, \gamma_1 (S^1 \times S^1))$ by using the excision property for indices. Pull the connection $B'$ back over $\Gamma_\alpha \cdot A'$ to get an $S^1$ family of connections in $N^0$. Then pull back over $S^1 \times S^1$; so we may think of this as an $S^1 \times S^1$ family, $\mathcal{S}^0_{M_0}$ of connections (constant in one direction). The excision property implies that $\text{Ind}_R(\mathcal{D}, S^1 \times S^1) - \text{Ind}_R(\mathcal{D}, \mathcal{S}^0_{M_0})$ is the pullback of

$$\text{Ind}_R(\mathcal{D}, \Gamma_\alpha \cdot B') - \text{Ind}_R(\mathcal{D}, B') = m_2 \cdot \eta_2 - m_2 \cdot 1,$$

where $(\mathcal{D}, B')$ denotes the Dirac operator on $K_+$ twisted over a constant $S^1$-family of connections. Similarly,

$$\text{Ind}_R(\mathcal{D}, \mathcal{S}^0_{M_0}) = (\text{Ind}_R(\mathcal{D}, \Gamma_\alpha \cdot A') - \text{Ind}_R(\mathcal{D}, A')) + \text{Ind}_R(\mathcal{D}, (A', B'))$$

$$= m_1 \cdot \eta_1 - m_1 \cdot 1 + (m_1 + m_2) \cdot 1 = m_1 \cdot \eta_1 + m_2 \cdot 1.$$

Hence

$$\text{Ind}_R(\mathcal{D}, \gamma_1(S^1 \times S^1)) = (\text{Ind}_R(\mathcal{D}, \Gamma_\alpha \cdot B') - \text{Ind}_R(\mathcal{D}, B')) + \text{Ind}_R(\mathcal{D}, \mathcal{S}^0_{M_0})$$

$$= m_1 \cdot \eta_1 + m_2 \cdot \eta_2.$$

We get a transversal of $S^1 \times S^1 \to \Gamma_\alpha \cdot B \subset N$ by fixing the first factor. Over this transversal we have the index bundle $m_2 \cdot \eta_2 + m_1 \cdot 1$. Restricted to the family $\Gamma_\alpha \cdot B$ the determinant is $\det(m_2 \cdot \eta_2) = \eta_2$ since $m_2$ is odd. For large $t$ the restriction of $u_1$ to the family $\gamma_1(\Gamma_t \cdot B)$ is the first Stiefel-Whitney class of the restricted index bundle; i.e., over $\Gamma_\alpha \cdot B$, we have $u_1 = w_1(\eta_2) \neq 0$, since $\Gamma_\alpha$ is an essential circle in $SO(3)$; so the Hopf bundle is twisted over it. This means that the isotropy circles $\gamma_1(\Gamma_\alpha \cdot B)$ represent the nontrivial element of $\pi_1(\mathcal{B}_X, t+1)$. Thus $q_{t+1, u_1, X} (z_1, \ldots, z_d, x, y)$ is the mod 2 count of these circles in $\mathcal{S}^0_{M_0} \ast \mathcal{S}^0_K$, and this is given above. □
We need now to make one final appeal to Mrówka's thesis. We have $r^0_K : M_{K,1}^0 \to \mathcal{R}(S^2 \times S^1) \cong S^3$ which according to Theorem 3.1(iv) is an $\text{SO}(3)$-equivariant generic map. Taking the quotient by $\text{SO}(3)$, this means that $r^0_K : r^0_K(-1,1) \cap M_{K,1}^0 \to (-1,1)$ is transverse to subcomplexes. In particular, consider $\alpha_0 < \alpha_1 \in (-1,1)$. For any $\alpha \in (-1,1)$, the 4-manifold $M_{K,1}(\alpha)$ is compact since such an $\alpha$ cannot be the restriction of a flat connection on $K \cong S^2 \times D^2$. The 1-manifold $\mathcal{F}_K \cap r^0_K(\alpha_0, \alpha_1) = r^{-1}_K(\alpha_0, \alpha_1) \cap M_{K,1}^0 \cap V_x \cap V_y$ gives a cobordism of the compact 0-manifolds $\mathcal{F}_K(\alpha_0)$ and $\mathcal{F}_K(\alpha_1)$. Thus the mod 2 intersection number $\#\mathcal{F}_K(\alpha)$ is independent of $\alpha \neq \pm 1$.

Similarly, given a generic 1-parameter family of asymptotically cylindrical metrics $\{g_K,t \mid t \in [0,1]\}$ on $K$, the family $\{\mathcal{F}_K, g_t\}$ gives a homology between $\mathcal{F}_K, g_0(\alpha)$ and $\mathcal{F}_K, g_1(\alpha)$ when $\alpha \neq \pm 1$. Thus the mod 2 intersection number $N_K = \#\mathcal{F}_K(\alpha)$ is independent of $\alpha \neq \pm 1$ and choice of generic metric.

**Theorem 5.11.** $q_{t+1, u_1, x(z_1, \ldots, z_d, x,y)} \equiv N_K \cdot q_{t,M}(z_1, \ldots, z_d) \mod 2$.

**Proof.** Since we have avoided multiplicity questions by assuming that $r_{M_0} \mid \mathcal{F}_{M_0}$ is 1-1, we have by Proposition 5.7 that $q_{t, M} \equiv \#r_{M_0}(\mathcal{F}_{M_0}) \mod 2$. Also $\#\mathcal{F}_K(\alpha) = N_K$ for all $\alpha \neq \pm 1$; so the result follows from Proposition 5.10. If we remove the assumption on $r_{M_0} \mid \mathcal{F}_{M_0}$, we get the same result by keeping track of multiplicities. \qed

6. **Calculation of $N_K$**

In this section we shall calculate $N_K$ by studying a specific example. Let $Q$ denote the negative definite $E_8$ plumbing manifold. Its boundary $Y = \partial Q$ is the Poincaré homology 3-sphere $\Sigma(2,3,5)$ with its negative orientation. Let $W$ be the result of performing surgery on any circle in the interior of $Q$. Since $Q$ is simply connected, all such surgeries are trivial. Then $W$ is diffeomorphic to $Q \# S^2 \times S^2$ and its intersection form is $-E_8 \oplus H$, where $H$ is the hyperbolic form.

Consider the moduli space of charge 2, asymptotically trivial self-dual connections $\mathcal{M}_{W,2}(\partial)$. (Since $Y$ is a Brieskorn homology 3-sphere, it follows from §3 that there is a single small $\delta$ such that we can base all our moduli spaces on connections with exponential $\delta$-decay.) The moduli space $\mathcal{M}_{W,2}(\partial)$ is nontrivial [T1,2] and is a 10-dimensional manifold. Choose a pair of generic oriented surfaces $\Sigma_1, \Sigma_2$ in $Q$, representing classes $z_1, z_2$ with $z_1^2 = z_2^2 = -2$ and $z_1 \cdot z_2 = 1$, and let $\Sigma_x, \Sigma_y$ be as in §5. We let $\mathcal{N}^2(\partial) = \mathcal{M}_{W,2}(\partial) \cap V_1 \cap V_2 \cap V_x \cap V_y$. As in the proof of Donaldson's Theorem B (see [D1] and [FS2]), $\mathcal{N}^2(\partial)$ is a 2-manifold with “internal” ends arising from instantons which bubble off at pairs of points of intersection of $\Sigma_1, \Sigma_2, \Sigma_x,$ and $\Sigma_y$. In fact, Donaldson shows that there is exactly one such end of $\mathcal{N}^2(\partial)$ arising from each such intersection. Our choice of surfaces then gives us an odd number of these ends. Donaldson shows, furthermore, that each such end is a circle on which the class $u_1 \in H^1(\mathcal{M}_W^*, \mathbb{Z}_2)$ evaluates nontrivially. Thus, as in [FS2],
there are an odd number of “asymptotic ends” of $\mathcal{M}^2(\theta)$ on which $u_1$ evaluates nontrivially. Dimension counting shows that the only possibility is for such ends to arise from nontrivial splittings of the form

$$\mathcal{M}_W(\rho) \times \mathcal{M}_Y(\rho, \theta) \to \mathcal{M}_{W,2}(\theta)$$

where $\rho \in \mathcal{B}(Y)$ and $\mathcal{M}_Y(\rho, \theta)$ is a moduli space of anti-self-dual connections on $Y \times \mathbb{R}$ with asymptotic conditions $\rho$ at $-\infty$ and $\theta$ at $+\infty$. Furthermore $V_1 \cap V_2 \cap V_x \cap V_y$ intersects $\mathcal{M}_W(\rho)$ transversely and nontrivially; so $\dim \mathcal{M}_W(\rho) \geq 8$, and thus $0 \leq \dim \mathcal{M}_Y(\rho, \theta) \leq 2$. Furthermore, since the character variety $\chi(Y)$ consists of isolated points, the only 0-dimensional anti-self-dual moduli space $\mathcal{M}_Y(\rho, \theta)$ consists of the singleton $\theta$. The next proposition follows directly from computations in [FS1].

**Proposition 6.1.** Let $Y = \Sigma(2, 3, 5)$ with its negative orientation. Up to conjugacy there are two nontrivial representations $\xi, \omega: \pi_1(Y) \to SU(2)$. The mod 8 dimensions of the corresponding moduli spaces of anti-self-dual connections on $\pm Y \times \mathbb{R}$ are $\dim \mathcal{M}_Y(\xi, \theta) \equiv 1$, $\dim \mathcal{M}_Y(\omega, \theta) \equiv 5$, $\dim \mathcal{M}_{-Y}(\xi, \theta) \equiv 4$, $\dim \mathcal{M}_{-Y}(\omega, \theta) \equiv 0$. \(\Box\)

It follows from this proposition that each asymptotic end of $\mathcal{M}^2(\theta)$ comes from a splitting $\mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \theta)$ where $\dim \mathcal{M}_Y(\xi, \theta) = 1$ and $\dim \mathcal{M}_W(\xi) = 9$. Since $\mathcal{M}_Y(\xi, \theta)$ is 1-dimensional, translational invariance of the anti-self-duality equation in the temporal gauge [F] implies that $\mathcal{M}_Y(\xi, \theta)$ is a disjoint union of copies of $\mathbb{R}$.

Viewed in the language of Theorem 4.3, given a family $\{g_t\}$ of generic asymptotically cylindrical metrics on $W_+$ which stretch a given segment $Y \times [-1, 1]$ of the end of $W_+$ to infinite length, we get for $t \geq t_0$, embeddings

$$\gamma_t: \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \theta) \to \mathcal{M}_{W,2}(\theta; g_t),$$

and $\gamma_t(A, B) \to (A, B) \in \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \theta)$ as $t \to \infty$. Now $V_1 \cap V_2 \cap V_x \cap V_y \cap (\mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \theta)) = \{V_1 \cap V_2 \cap V_x \cap V_y \cap \mathcal{M}_W(\xi)\} \times \mathcal{M}_Y(\xi, \theta) = \mathcal{N}^1(\xi) \times \mathcal{M}_Y(\xi, \theta)$ for a 1-manifold $\mathcal{N}^1(\xi)$. The standard dimension counting argument shows that $\mathcal{N}^1(\xi)$ is compact.

Let $\mathcal{M}_Y(\xi, \theta)$ be a transversal to the $\mathbb{R}$-action on $\mathcal{M}_Y(\xi, \theta)$. Set $T_i = \gamma_i(N^1(\xi) \times \mathcal{M}_Y(\xi, \theta))$. The argument above using the proof of [D1, Theorem B] shows that $u_1[T_i] \neq 0$. Recall from §2 that $u_1 = w_1(\det \text{Ind}_R \mathcal{D}_A)$. It follows that for any $A_1, A_2 \in \mathcal{M}_Y(\xi, \theta)$, $u_1[\gamma_1(N^1(\xi) \times A_1)] = u_1[\gamma_1(N^1(\xi) \times A_2)]$; so $\gamma_1^*(u_1) \in H^1(N^1(\xi) \times \mathcal{M}_Y(\xi, \theta); \mathbb{Z}_2) = \bigoplus H^1(N^1(\xi); \mathbb{Z}_2)$ is a diagonal class which restricts to the class $u \in H^1(N^1(\xi); \mathbb{Z}_2)$ in each component. The class $u$ is obviously independent of $t$. We have

$$u_1[T_i] = \gamma_1^*(u_1)[N^1(\xi) \times \mathcal{M}_Y(\xi, \theta)] = \#\mathcal{M}_Y(\xi, \theta) \cdot u[N^1(\xi)].$$

Hence $[N^1(\xi)] \neq 0$ in $H_1(\mathcal{M}_W(\xi); \mathbb{Z}_2)$ since $u$ evaluates nontrivially on it.

Suppose $N^1(\xi)$ was the boundary of $\mathbb{Z}_2$-chain $C$ of $\mathcal{B}_W^*(\xi)$. Consider the map $\beta_1: \mathcal{M}_W(\xi) \times \mathcal{M}_Y(\xi, \theta) \to \mathcal{B}_W^{*,2}(\theta)$ described in §4. Its image is
approximately anti-self-dual. The map $\beta_t$ is deformed into $\gamma_t$; so in particular $\beta_t(N^1(\xi) \times \widetilde{M}_Y(\xi, \theta))$ is homologous to $\gamma_t(N^1(\xi) \times \widetilde{M}_Y(\xi, \theta))$ in $\mathcal{B}^*_W(\theta)$. The map $\beta_t$ is constructed by a process involving truncation and matching overlaps. (See [Mr] and [F].) Thus $\beta_t$ extends over all of $C \times \widetilde{M}_Y(\xi, \theta)$. This would imply that $T_t = \gamma_t(N^1(\xi) \times \widetilde{M}_Y(\xi, \theta))$ is nullhomologous, i.e. that $u_1[T_t] = 0$, a contradiction. Thus $[N^1(\xi)]$ is nonzero in $H_1(\mathcal{B}^*_W(\xi); \mathbb{Z}_2)$.

**Lemma 6.2.** $\pi_1(\mathcal{B}^*_W(\xi)) = \mathbb{Z}_2$.

**Proof.** We just apply the proof of Proposition 2.1 to our asymptotically flat end situation. The gauge group $\mathcal{G}_g$ consists of maps $g : X_+ \to SU(2)$ in $L^1, \text{loc}$ such that $\| \nabla_\theta g \|_{\delta, 3} < \infty$. Note that this group does not depend on the asymptotic condition $\xi$. Let $\mathcal{F} = \mathcal{G}/\mathbb{Z}_2$, so that $\pi_1(\mathcal{B}^*_W(\xi)) \cong \pi_0(\mathcal{F})$. Now $\pi_0(\mathcal{F}) = \{(W_+, \theta), (S^3, 1)\}$, which, since $W_+$ is spin, is the group $\mathbb{Z}_2$. One can now show that when one deforms the constant map $W_+ \to S^3$ with value $-1$ so that it sends $\partial W_+$ to $1$, then the resulting map is homotopic rel boundary to the map to $1 \in S^3$. (Use the great circle connecting $1$ to $-1$.) The result follows. □

**Corollary 6.3.** The class $u \in H^1(N^1(\xi); \mathbb{Z}_2)$ is the restriction of the nontrivial element of $H^1(\mathcal{B}^*_W(\xi); \mathbb{Z}_2)$. □

Next consider the moduli space $\mathcal{M}_Q(\xi)$. Again it follows from Taubes [T2] that $\mathcal{M}_Q(\xi)$ is a nonempty 5-manifold.

**Proposition 6.4.** There is a 4-dimensional moduli space $\mathcal{M}_Q(\xi)$ which gives rise to a degree 2 relative Donaldson invariant, and $#(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2) \equiv 1 \mod 2$.

**Proof.** First notice that if we can find a 4-dimensional moduli space $\mathcal{M}_Q(\xi)$, then for any classes $a, b \in H_2(Q; \mathbb{Z})$ with corresponding divisors $V_a$ and $V_b$ in $\mathcal{B}^*_Q(\xi)$, the intersection $\mathcal{M}_Q(\xi) \cap V_a \cap V_b$ is compact—for there is not enough charge for an instanton bubble to occur (any moduli space left would have nonnegative dimension), and Proposition 6.1 implies that there is no splitting of the form

$$((\mathcal{M}_Q(\rho) \cap V_a \cap V_b) \times \mathcal{M}_Y(\rho, \xi)) \to \mathcal{M}_Q(\xi) \cap V'_a \cap V'_b$$

for any $\rho$. Thus we would have a degree 2 Donaldson invariant.

The intersection $I = \mathcal{M}_Q(\xi) \cap V_1 \cap V_2$ is a 1-manifold in $\mathcal{B}^*_Q(\xi)$ which can have asymptotic ends as described above, ends coming from instanton bubbles occurring at points of intersection of $\Sigma_1$ and $\Sigma_2$ (there are an odd number of these by the choice of $\Sigma_1$ and $\Sigma_2$), and reducible connections. It follows from [D1] that the number of ends of $\mathcal{M}_Q(\xi)$ which are reducible is

$$\frac{1}{2} \sum \{(z_1 \cdot e)(z_2 \cdot e) | e \in H_2(Q_+; \mathbb{Z}), e^2 = -1\} = 0$$

since $H^2(Q_+; \mathbb{Z}) = -E_8$ is an even form. Now an argument similar to the one given above shows that there are an odd number of asymptotic ends of $I$.
coming from splittings
\[(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2) \times \mathcal{M}_Y(\xi, \vartheta) \mod 2.\]

Furthermore, there are no other asymptotic ends of \(I\). Thus Proposition 6.1 implies that \(\mathcal{M}_Q(\xi)\) is 4-dimensional, and it follows that \(\#(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2)\) is odd. 

Notice that this degree 2 relative Donaldson invariant \(q_{Q, \xi}\) takes its values in the 1-dimensional Floer homology group generated by the class of \(\xi\). Thus we can identify the values of \(q_{Q, \xi}\) with the integers, and then \(q_{Q, \xi}(z_1, z_2) \equiv 1 \mod 2\) for the classes \(z_1, z_2\) chosen above.

**Digression 6.5.** The basic technique of the proof of Proposition 6.4 can be used to prove the following “folk theorem”.

Let \(X\) be a compact simply connected negative definite 4-manifold with homology sphere boundary \(Y\). Suppose that the intersection form of \(X\) is not diagonalizable over \(\mathbb{Z}\). Then the 1-dimensional Floer instanton homology group \(I_1(Y) \neq 0\).

The proof is basically given above. Counting ends provides a nonzero (in fact, odd) pairing of the Floer 1-cycle of \(Y\) given by the degree 2 relative Donaldson invariant with the Floer 1-cocycle (which is a Floer cycle of \(-Y\)) given by \(\sum(\#\mathcal{M}_Y^1(\rho, \vartheta))\rho\) where \(\mathcal{M}_Y^1(\rho, \vartheta)\) consists of the 1-dimensional components of \(\mathcal{M}_Y(\rho, \vartheta)\) and \(\mathcal{M}_Y^1(\rho, \vartheta) = \mathcal{M}_Y(\rho, \vartheta)/\mathbb{R}\). A simple algebraic exercise shows that for a non-diagonal negative definite intersection form there always is a pair of classes \(z_1, z_2\) such that \(z_1 \cdot z_2 = -\frac{1}{2} \sum (e \cdot z_1)(e \cdot z_2) | e^2 = -1 | \equiv 0 \mod 2\). Thus both the Floer homology class given by the Donaldson invariant and the Floer cohomology class of the cocycle are nontrivial.

**Theorem 6.6.** \(N_K \equiv 1 \mod 2\).

**Proof.** We apply the arguments of §5 to \(W_+ = Q_{+,0} \cup S^2 \times D^2 = Q_{+,0} \cup K\) where \(Q_{+,0} = Q_+ \setminus (S^1 \times \mathbb{R}^3)\), and \(S^1 \times \mathbb{R}^3\) is a tubular neighborhood of the circle on which surgery is performed to construct \(W\). The proof of Theorem 5.11 applies to \(V_1 \cap V_2 \cap V_x \cap V_y \cap \mathcal{M}_W(\xi) = N^1(\xi)\) without change because, by a dimension counting argument as above, no sequence of connections in \(\mathcal{M}_W(\xi) \cap V_1 \cap V_2 \cap V_x \cap V_y\) can lose charge on the tube \(Y \times \mathbb{R}^+\). Since \([N^1(\xi)] \neq 0\), we get \(1 \equiv \#(\mathcal{M}_Q(\xi) \cap V_1 \cap V_2) \cdot N^1_K \equiv N_K \mod 2\). 

This together with Theorem 5.11 completes the proof of Theorem 1.1.

7. The \(u_2\)-invariant

In this section we shall study the invariant \(q_{k, u_2, X}\) and prove Theorem 1.3. We shall rely heavily on the work already done in §5 and §6. Let \(N\) be a closed simply connected spin 4-manifold with \(b_2^+\) even and \(\geq 2\) and let \(k\) be an even integer, \(k > \frac{3}{4}(1 + b_2^+) + \frac{1}{4}\); so we have a \(\mathbb{Z}_2\)-polynomial invariant \(q_{k, u_1, N}\) of degree \(d = 4k - \frac{3}{2}(1 + b_2^+) - \frac{1}{2}\). Let \(X = N \# S^2 \times S^2\) which is spin and \(k + 1\)
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\[ \frac{1}{2}(1 + b_X^+) + \frac{1}{2} \] is odd; so the \( \mathbb{Z}_2 \)-polynomial invariant \( q_{k+1, u_x, x} \) of degree \( d + 2 \) is also defined. Let \( z_1, \ldots, z_d \in H_2(N; \mathbb{Z}) \subset H_2(X; \mathbb{Z}) \) be represented by generic oriented surfaces \( \Sigma_1, \ldots, \Sigma_d \) as earlier and let \( \chi = [S^2 \times 0] \) and \( y = [0 \times S^2] \) with representative surfaces \( \Sigma_x \) and \( \Sigma_y \) as before. We must evaluate \( q_{k+1, u_x, x}(z_1, \ldots, z_d, x, y) \).

Write \( X = \tilde{N}_0 \cup S^2 \times S^1 \) where \( \tilde{N}_0 \) is \( N \) with the tubular neighborhood of a circle removed and \( K \cong S^2 \times D^2 \). We adopt the notation and follow the arguments of \( \S 5 \).

**Proposition 7.1.** The formal dimension of \( \mathcal{M}_{\tilde{N}_0, m}(\alpha) \) is

\[
\dim \mathcal{M}_{\tilde{N}_0, m}(\alpha) = \begin{cases} 
2d + 8(m - k), & \alpha \neq \pm 1, \\
2d + 8(m - k) - 2, & \alpha = \pm 1.
\end{cases}
\]

**Corollary 7.2.** If \( 2d + 8(m - k) \geq 0 \) the moduli space \( \mathcal{M}_{\tilde{N}_0, m} \) (with respect to the generic metric \( g_{\tilde{N}_0} \)) is a manifold whose formal dimension is \( 2d + 1 + 8(m - k) \).

Next consider the 1-dimensional intersection \( \mathcal{I}_{\tilde{N}_0} = V_1 \cap \cdots \cap V_d \cap \mathcal{M}_{\tilde{N}_0, k} \).

**Proposition 7.3.** If \( A \in \mathcal{I}_{\tilde{N}_0} \) then \( r_{\tilde{N}_0}(A) \neq \pm 1 \).

**Proof.** The subcomplex \( r_{\tilde{N}_0}^{-1}(\pm 1) \) of \( \mathcal{M}_{\tilde{N}_0, k} \) has dimension \( 2d - 2 \) and is met transversely by the codimension \( 2d \) submanifold \( V_1 \cap \cdots \cap V_d \).

**Proposition 7.4.** The 1-dimensional intersection \( \mathcal{I}_{\tilde{N}_0} \) is a compact manifold.

**Proof.** This follows from Uhlenbeck weak compactness and from a counting argument as in Proposition 5.6.

The cohomology class \( u_1 \in H^1(\mathcal{R}_{N, k}; \mathbb{Z}_2) \) restricts to a class, which we shall still call \( u_1 \) in \( H^1(\mathcal{R}_{\tilde{N}_0, k}; \mathbb{Z}_2) \cong \mathbb{Z}_2 \) (cf. Lemma 6.2).

**Proposition 7.5.** The intersection \( \mathcal{I}_{\tilde{N}_0} \) consists of a finite number of circles; modulo 2 the number of these circles which are nontrivial in \( \pi_1(\mathcal{R}_{N, k}^*; k) \cong \mathbb{Z}_2 \) is \( q_{k, u_1, N}(z_1, \ldots, z_d) \).

**Proof.** As in the proof of Proposition 5.7, for large \( t \) we get maps: \( N \to \mathcal{M}_{N, k}(g_t) \) which are homeomorphisms onto their image, and \( \mathcal{I}_{\tilde{N}_0} \subset N \). If \( I \) is a component (a circle) of \( \mathcal{I}_{\tilde{N}_0} \) then as \( t \to \infty \), \( \{\gamma_t(I)\} \) converges to \( \{[A, r_{\tilde{N}_0}(A)]| A \in I\} \subset \mathcal{M}_{N, k}(g_t) \times \chi(\mathbb{R}^3 \times S^1) \). But \( I \) is contained in \( V_1 \cap \cdots \cap V_d \) which intersects \( \mathcal{M}_{N, k}(g_t) \) transversely, and so by Theorem 4.3(a) for large enough \( t \) there is a unique circle of \( \mathcal{M}_{N, k}(g_t) \cap V_1 \cap \cdots \cap V_d \) close to \( \gamma_t(I) \). Then as in Proposition 5.7 a counting argument shows that these circles comprise all of \( \mathcal{I}_N(t) = \mathcal{M}_{N, k}(g_t) \cap V_1 \cap \cdots \cap V_d \). The invariant \( q_{k, u_1, N}(z_1, \ldots, z_d) \) counts (modulo 2) the number of homotopically nontrivial components of \( \mathcal{I}_N(t) \), but \( \gamma_t \) induces an isomorphism on \( \pi_1 \); so the proposition follows.
Let $\mathcal{J}_X(t)$ denote $\mathcal{M}_{X,k}(g_t) \cap V_1 \cap \ldots \cap V_d \cap V_x \cap V_y$. The technique that proves Proposition 5.9 then gives

**Proposition 7.6.** For large enough $t$, the image $\gamma_t(\mathcal{J}_{N_0} \ast \mathcal{J}_K)$ is homologous to $\mathcal{J}_X(t)$ in $\mathcal{B}^*_{X,k+1}$. □

**Lemma 7.7.** The inclusion induced homomorphism $\pi_1(\mathcal{M}_{K,1}\{\pm 1\}) \to \pi_1(\mathcal{B}^*_{K,1})$ is trivial.

**Proof.** Let $N_K = \mathcal{M}_{K,1}\{\pm 1\}$. Viewing $S^4$ as $K \cup D^3 \times S^1$, and referring again to Proposition 5.7, we have the pullback diagram

$\begin{array}{ccc}
\mathcal{N} & \to & \mathcal{B}(R^3 \times S^1) \{\pm 1\} \\
\mathcal{N}_K & \to & \mathcal{B}(S^2 \times S^1) \{\pm 1\} \\
\end{array}$

and $\mathcal{N} \to \mathcal{N}_K$ is a homeomorphism. Recall the embedding $\beta_t : N_K \cong N \to B^*_{S^4,1}$, given by $\beta_t(A) = A \# r_K(A)$ over $K \cup (R^3 \times S^1) = S^4$. For large enough $t$ this map is homotopic to the embedding $\gamma_t : N_K \cong N \to M_{S^4,g_t}$. Let $r'_K : B^*_{S^4,1} \to B^*_{K,1}$ be the restriction. Then $r'_K \circ \beta_t$ is homotopic to the inclusion of $N_K$ in $B^*_{K,1}$, but $\pi_1(B^*_{S^4,1})$ is trivial. □

Before proceeding further we need to give a description of the cohomology class $u_2 \in H^2(B^*_{X,k+1}; Z_2)$ of [D1]. Since $X$ is spin there is a family of real elliptic operators $\{\mathcal{L} A | A \in \mathcal{S}_{X,k+1}\}$ given by coupling the Dirac operator on $X$ to connections. This family descends to $B^0_{X,k+1}$; so for any compact subset $T^0$ of $B^0_{X,k+1}$ we obtain $\text{Ind}_R(\mathcal{L}, \{A\}) \in KO(T^0)$. This class does not descend directly to $B^*_{X,k+1}$; however Donaldson shows that $\text{Ind}_R(\mathcal{L}, \{A\}) \otimes \det(\text{Ind}_R(\mathcal{L}, \{A\}))$ descends to $B^*_{X,k+1}$ (since $k+1$ is odd). The class $u_2$ is defined to be $u_2 = w_2((\text{Ind}_R(\mathcal{L}, \{A\}) \otimes \det(\text{Ind}_R(\mathcal{L}, \{A\})))$.

Next we need to set the stage for the proof of Theorem 1.3. First we trivialize the projection

$\mathcal{B}(S^2 \times S^1) \{\pm 1\} = SU(2) \{\pm 1\} = (-1, 1) \times S^2 \to \chi(S^2 \times S^1) \{\pm 1\} = (-1, 1)$.

Then we can trivialize the projections $\mathcal{J}_{N_0}^0 \to \mathcal{J}_{N_0}$ and $\mathcal{J}_K^0 \to \mathcal{J}_K$ so that $\mathcal{J}_{N_0}^0 = \mathcal{J}_{N_0} \times SO(3)$ and $\mathcal{J}_K^0 = (r_K, \pi)$, where $\pi : SO(3) \to S^2$ is the projection of the circle bundle, and similarly $\mathcal{J}_K^0 = \mathcal{J}_K \times SO(3)$ where $r_K^0 = (r_K, \pi)$. Now let $\mathcal{J}_{N_0} \cup \mathcal{J}_K$ denote the actual pullback of

$\begin{array}{c}
\mathcal{J}_{N_0} \\
\downarrow \\
\mathcal{J}_K \\
\to (-1, 1)
\end{array}$

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(Compare with $\mathcal{I}_{N_0} \ast \mathcal{I}_K$ which is the quotient of the pullback $\mathcal{I}_{N_0}^0 \ast \mathcal{I}_K^0$ by the action of $SO(3)$.) Working with based moduli spaces we have that the pullback

\[ \mathcal{I}_{N_0}^0 \ast \mathcal{I}_K^0 = \{ ((A, \xi), (B, \eta)) | r_{N_0}(A) = r_K(B), \pi(\xi) = \pi(\eta) \} \]

\[ = (\mathcal{I}_{N_0} \circ \mathcal{I}_K) \times (S^1 \times SO(3)) \]

where $S^1 \times SO(3)$ is the pullback of diagram (5.8).

Now $q_{k+1} \circ u_i \circ x(z_1, \ldots, z_d, x, y)$ is given by evaluating $u_2$ on the class of the 2-cycle $\mathcal{I}_k(i) \in H_2(\mathcal{B}_{x, k+1}; \mathbb{Z}_2)$. So by Proposition 7.6 we need to compute the value of $u_2$ on $\gamma_i(\mathcal{I}_{N_0} \ast \mathcal{I}_K)$. This will be done as in Proposition 5.10 by using the excision property for indices. Fixing a basepoint $0 \in S^2$, we have $(-1, 1) \times \{0\}$ contained in $SU(2) \setminus \{\pm 1\}$ as a transversal to the projection to $(-1, 1)$ and there is the pullback diagram

\[ (\mathcal{I}_{N_0} \times S^1) \ast (\mathcal{I}_K \times S^1) \]

Under the projection map from based equivalence classes to ordinary equivalence classes of connections, the pullback $(\mathcal{I}_{N_0} \times S^1) \ast (\mathcal{I}_K \times S^1) = (\mathcal{I}_{N_0} \circ \mathcal{I}_K) \times (S^1 \times S^1)$ maps onto $\mathcal{I}_{N_0} \ast \mathcal{I}_K$.

Consider the family of real elliptic operators $(\mathcal{I}_K \times S^1) \ast (\mathcal{I}_k \times \{1\})$ which we may view as a family over $(\mathcal{I}_{N_0} \times S^1) \ast (\mathcal{I}_K \times S^1)$ which is constant in the direction of the final $S^1$. By the excision principle we have

\[ \text{Ind}_R(\mathcal{I}_K \times S^1) = \text{Ind}_R(\mathcal{I}_K \times \{1\}) - \text{Ind}_R(\mathcal{I}_K \times \{1\}) \]

where this last difference must be pulled back over $(\mathcal{I}_{N_0} \times S^1) \ast (\mathcal{I}_K \times S^1)$. Since $\mathcal{I}_K$ consists of arcs and (by Lemma 7.7) nullhomotopic circles, we have that $\text{Ind}_R(\mathcal{I}_K \times \{1\})$ is trivial, i.e. $\text{Ind}_R(\mathcal{I}_K \times \{1\}) = m_1 \cdot 1$, where $m_1 = 2 \text{ind} \mathcal{I}_K^0 + 1$ is the numerical index. Each component of $\mathcal{I}_K \times S^1$ is homotopic to $\{\text{point}\} \times S^1$ for a point in $\mathcal{B}_{K, 0}^*$. It thus suffices to compute the index of the Dirac operator twisted by the family $\{B' \# I_{y_0} | \rho \in S^1 \subset SO(3)\}$, where $B'$ is a fixed connection in $\mathcal{B}_{K, 0}^{0,*}$ which is flat near the point $y_0 \in K$, $I_{y_0}$ is an instanton at $y_0$, and the grafted connections have fixed positive scale. Then it follows as in [D1] that

\[ \text{Ind}_R(\mathcal{I}_K \times S^1) = \eta_1 + (m_1 - 1) \cdot 1, \]

where $\eta_1$ is the restriction of the Hopf real line bundle. Thus

\[ \text{Ind}_R(\mathcal{I}_K \times S^1) - \text{Ind}_R(\mathcal{I}_K \times \{1\}) = \eta_1 - 1. \]
In the same way
\[
\text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \times S^1) \ast (\mathcal{F}_K \times \{1\})) - \text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \times \{1\}) \ast (\mathcal{F}_K \times \{1\}))
= \text{Ind}_R(\mathcal{D}, \mathcal{F}_0 \times S^1) - \text{Ind}_R(\mathcal{D}, \mathcal{F}_0 \times \{1\}).
\]

For a fixed \( \mathcal{A}' \in \mathcal{B}_{N_0,k}^* \) consider the family
\[
\{ \mathcal{A}' \# \rho_0 \# \rho_i \# \rho_i | \rho_i \in S^1 \subset SO(3) \}.
\]
The image of this family in \( \mathcal{F}_{N_0,k}^* \) is a circle which generates \( \pi_1(\mathcal{B}_{N_0,k}^*) \cong H_1(\mathcal{B}_{N_0,k}^*, \mathbb{Z}) = \mathbb{Z}_2 \). (Cf. [D1].) We have
\[
\text{Ind}_R(\mathcal{D}, \{ \mathcal{A}' \# \rho_0 \# \rho_i \# \rho_i \rho_0 \text{ fixed} \}) = \eta_2 + \eta_3 + (m_2 - 2) \cdot 1.
\]
where \( m_2 = 2 \text{ind}_{\mathcal{F}_N} + k \). A connected component \( I \) of \( \mathcal{F}_{N_0} \) which is homotopically nontrivial is homotopic to the image of the above family in \( \mathcal{B}_{N_0,k}^* \), and for such a component
\[
\text{Ind}_R(\mathcal{D}, I \times S^1) = \eta_2 + \eta_3 + (m_2 - 2) \cdot 1.
\]

Similarly,
\[
\text{Ind}_R(\mathcal{D}, I \times \{1\}) = \text{Ind}_R(\mathcal{D}, \{ \mathcal{A}' \# \rho_0 \# \rho_i \# \rho_i \rho_0 \text{ fixed} \}) = \eta_3 + (m_2 - 1) \cdot 1.
\]
So
\[
\text{Ind}_R(\mathcal{D}, I \times S^1) - \text{Ind}_R(\mathcal{D}, I \times \{1\}) = \eta_2 - 1.
\]
For a connected component \( J \) of \( \mathcal{F}_{N_0} \) which is homotopically trivial, we have
\[
\text{Ind}_R(\mathcal{D}, J \times S^1) - \text{Ind}_R(\mathcal{D}, J \times \{1\}) = (\eta_2 + (m_2 - 1) \cdot 1) - m_2 \cdot 1 = \eta_2 - 1;
\]
so we get the same result in either case.

Finally, consider
\[
\text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \times \{1\}) \ast (\mathcal{F}_K \times \{1\})) = \text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \circ \mathcal{F}_K) \times \{(1, 1)\})
\]
where \( (1, 1) \in S^1 \times S^1 \). Now \( r_K : \mathcal{F}_K \rightarrow (-1, 1) \) is a finite-to-one map which by Theorem 6.7 has odd degree. Hence as a homology class in \( \mathcal{B}_{K,1}^* \text{ rel } (r_K^{-1}(\pm 1)) \), \( \mathcal{F}_K \) is homologous to an odd number of arcs each mapping homeomorphically onto \((-1, 1) \) via \( r_K \). (This uses Lemma 7.7.) It is easy to see that components of \( \mathcal{F}_K \) which are homologically trivial rel \( (r_K^{-1}(\pm 1)) \) cannot contribute nontrivially to \( \text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \circ \mathcal{F}_K) \times \{(1, 1)\}) \). I.e. if \( J' \) is such a component, then
\[
\text{Ind}_R(\mathcal{D}, (\mathcal{F}_0 \circ \mathcal{J}') \times \{(1, 1)\}) = m_3 \cdot 1
\]
where \( m_3 = 2 \hat{A}(X) + k + 1 \). There are an odd number of components which are essential rel \( r_K^{-1}(\pm 1) \). Working with homology, we may assume that such a component is a single arc, \( I' \), and such that \( r_K : I' \cap r_K^{-1}(-1, 1) \rightarrow (-1, 1) \) is a homeomorphism. Then \( \mathcal{F}_0 \circ I' \cong \mathcal{F}_0 \). If \( I \) is a component of \( \mathcal{F}_0 \) then
Ind_R(\(\varphi, (I \circ I') \times \{(1, 1)\}\)) is Ind_R(\(\varphi, I\)) pulled back over \(I \circ I' \times S^1 \times S^1\).

Thus if \(I\) is trivial in \(\pi_1(\mathcal{B}_{N_0}^*, k)\),

\[
\text{Ind}_R(\varphi, (I \circ I') \times \{(1, 1)\}) = m_3 \cdot 1,
\]

and if \(I\) is nontrivial then

\[
\text{Ind}_R(\varphi, (I \circ I') \times \{(1, 1)\}) = \eta_3 + (m_3 - 1) \cdot 1.
\]

Adding the above expressions we obtain

**Lemma 7.8.** (a) For each component \(I\) of \(\mathcal{J}_{N_0}\) which is nontrivial in \(\pi_1(\mathcal{B}_{N_0}^*, k)\) and component \(I'\) of \(\mathcal{J}_K\) which is essential rel \(r_{K}^{-1}(\pm 1)\),

\[
\text{Ind}_R(\varphi, (I \times S^1) \ast (I' \times S^1)) = \eta_1 + \eta_2 + \eta_3 + (m_3 - 3) \cdot 1.
\]

(b) For each component \(J\) of \(\mathcal{J}_{N_0}\) which is trivial in \(\pi_1(\mathcal{B}_{N_0}^*, k)\),

\[
\text{Ind}_R(\varphi, (J \times S^1) \ast (I' \times S^1)) = \eta_1 + \eta_2 + (m_3 - 2) \cdot 1.
\]

(c) For each component \(J'\) of \(\mathcal{J}_K\) which is inessential rel \(r_{K}^{-1}(\pm 1)\),

\[
\text{Ind}_R(\varphi, (J \times S^1) \ast (J' \times S^1)) = \eta_1 + \eta_2 + (m_3 - 2) \cdot 1.
\]

**Proof of Theorem 1.3.** A transversal of \((\mathcal{J}_{N_0} \times S^1) \ast (\mathcal{J}_K \times S^1)\) can be obtained by fixing the final \(S^1\), say. For components \(I\) of \(\mathcal{J}_{N_0}\) which is nontrivial in \(\pi_1(\mathcal{B}_{N_0}^*, k)\) and \(I'\) of \(\mathcal{J}_K\) which is essential rel \(r_{K}^{-1}(\pm 1)\), let \(T_I\) be the corresponding transversal of \((I \times S^1) \ast (I' \times S^1)\). Then

\[
\text{Ind}_R(\varphi, T_I) \otimes \text{det} \text{Ind}_R(\varphi, T_I) = (\eta_2 + \eta_3 + (m_3 - 2) \cdot 1)(\eta_2 \eta_3).
\]

Let the cohomology generators of the two \(S^1\)-factors be \(t_2\) and \(t_1\), respectively, and let \(t_3\) be the cohomology generator of \(I\). The total Stiefel-Whitney class of \(\text{Ind}_R(\varphi, T_I) \otimes \text{det} \text{Ind}_R(\varphi, T_I)\) is \(w = (t_2 + t_3 + (m_3 - 2))(t_2 t_3)\); so \(u_2([T_I]) = (m_3 - 2)(t_2 t_3)\). If instead we evaluate on a homotopically trivial component \(J\) of \(\mathcal{J}_{N_0}\), we obtain

\[
\text{Ind}_R(\varphi, T_J) \otimes \text{det} \text{Ind}_R(\varphi, T_J) = (\eta_2 + (m_3 - 1) \cdot 1) \eta_2.
\]

So \(w = (t_2 + (m_3 - 1)) t_2\), and \(u_2([T_I]) = 0\). Similarly \(u_2([T_I]) = 0\) for a transversal of \((\mathcal{J}_{N_0} \times S^1) \ast (J' \times S^1)\) for any component \(J'\) of \(\mathcal{J}_K\) which is essential rel \(r_{K}^{-1}(\pm 1)\). Since the number of components \(I'\) is odd, this means that on the one hand, \(u_2([\mathcal{J}_{N_0} \ast \mathcal{J}_K])\) is (modulo 2) the number of homotopically nontrivial components of \(\mathcal{J}_{N_0}\), and by Proposition 7.5 this is just \(q_{k+1, u_1, y}(z_1, \ldots, z_d)\). On the other hand, by definition, \(u_2([\mathcal{J}_{N_0} \ast \mathcal{J}_K]) = q_{k+1, u_2, x}(z_1, \ldots, z_d, x, y)\). □

**8. Invariant theory and the Donaldson polynomial mod 2**

For \(M\) a closed simply connected 4-manifold, let \(\text{Sym}_d^R(H_2(M; \mathbb{Z}))\) be the ring of \(d\)-linear symmetric functions on \(H_2(M; \mathbb{Z})\) with values in a ring \(R\).
The symmetric product \( \gamma_1 \gamma_2 \in \text{Sym}^{d_1+d_2}(H_2(M; \mathbb{Z}) \text{ of } \gamma_1 \in \text{Sym}^{d_2}(H_2(M; \mathbb{Z}) \) and \( \gamma_2 \in \text{Sym}^{d_1}(H_2(M; \mathbb{Z}) \) is defined by the rule
\[
\gamma_1 \gamma_2(x_1, \ldots, x_{d_1+d_2}) = \frac{1}{d_1! d_2!} \sum_{\sigma \in S_{d_1+d_2}} \gamma_1(x_{\sigma(1)}, \ldots, x_{\sigma(d_1)}) \gamma_2(x_{\sigma(d_1+1)}, \ldots, x_{\sigma(d_1+d_2)})
\]
where \( S_{d_1+d_2} \) denotes the symmetric group on \( d_1 + d_2 \) letters. The intersection form \( Q_M \) of \( M \) is an element of \( \text{Sym}^2(H_2(M; \mathbb{Z})) \), and the degree \( d \) Donaldson invariant \( q_{t,M} \) is an element of \( \text{Sym}^d(H_2(M; \mathbb{Z})) \). Define
\[
Q_m^{(p)} \in \text{Sym}^{2p}(H_2(M; \mathbb{Z}))
\]
by
\[
Q_m^{(p)} = \frac{1}{p!} Q_m^p.
\]

It is interesting to note that if the homology classes \( z_1, \ldots, z_{2p} \) are represented by surfaces \( \Sigma_1, \ldots, \Sigma_{2p} \) in general position, then \( Q_m^{(p)}(z_1, \ldots, z_{2p}) \) is the number of ways of placing \( p \) points on the intersections of pairs of these surfaces (counted with suitable signs) such that each surface contains a point.

The results of Wall [W1] mentioned in the introduction imply that the diffeomorphism group of \( X = M \# S^2 \times S^2 \) maps onto the orthogonal group \( O_X \) of automorphisms of \( H_2(X; \mathbb{Z}) \) which preserve \( Q_X \). In this case, classical invariant theory implies that if \( \gamma \) is a nonzero element of \( \text{Sym}^d(H_2(X; \mathbb{Z})) \) which is a diffeomorphism invariant (hence is left invariant by \( O_X \)), then \( d = 2p \) and \( \gamma \) is a multiple of \( Q_X^{(p)} \). Such a result is false for \( \text{Sym}^d(H_2(X; \mathbb{Z})) \); so we are not able to prove Theorem 1.5 by an appeal to algebra. Instead, our proof combines algebra with specific knowledge of the invariant \( q_{t+1,u_1,x} \) deduced via gauge theory. We begin with the gauge-theoretic arguments. The next proposition is a version of Donaldson’s connected sum theorem [D2] in the context of the invariant \( q_{t+1,u_1,x} \). Our argument follows the lines of a proof of Donaldson’s theorem given by John Morgan.

**Theorem 8.1.** Let \( M \) be a closed simply connected spin 4-manifold with a degree \( d \) Donaldson invariant \( q_{t,M} \) with \( t \) odd. Let \( X = M \# S^2 \times S^2 \) and let \( x \) and \( y \) denote the homology classes \( [S^2 \times 0] \) and \( [0 \times S^2] \). Let \( z_1, \ldots, z_d \in H_2(M; \mathbb{Z}) \) and \( w_j = x \) or \( y \), \( j = 1, \ldots, d + 2 - r \). Suppose \( r \neq 0 \) or \( d + 2 \). Then the mod 2 invariant \( q_{t+1,u_1,x}(z_1, \ldots, z_r, w_1, \ldots, w_{d+2-r}) = 0 \) unless \( r = d \) and \( \{w_{d+1}, w_{d+2}\} = \{x, y\} \).

**Proof.** We apply an argument derived from considering a sequence of metrics \( \{g_\nu\} \) on \( X \) whose limit is the one-point union \( (M \# S^2 \times S^2, g_M \lor g_{S^2 \times S^2}) \) where \( g_M \) and \( g_{S^2 \times S^2} \) are generic. Fix points \( p_0 \in M \) and \( q_0 \in S^2 \times S^2 \) and embedded geodesic balls \( B_M(p_0, \nu) \) and \( B_{S^2 \times S^2}(q_0, \nu) \) of radius \( \nu \). Then \( (M \# S^2 \times S^2, g_\nu) \) is obtained by identifying boundary collars in \( M_0(\nu) = M \setminus B_M(p_0, \nu) \) and \( S^2 \times S^2 \setminus B_{S^2 \times S^2}(q_0, \nu) \) in such a way that outside of a small
neighborhood of the neck where the identification is made, \( g_\nu \) agrees with \( g_M \) and \( g_{S^2 \times S^2} \). (See [D2, §IV].) If \( q_{l+1}, u_1, x(z_1, \ldots, z_r, w_1, \ldots, w_{d+2-r}) \neq 0 \), then for each \( \nu \) there is an \( A_\nu \in \mathcal{M}_{X, l+1}(g_\nu) \cap V_1 \cap \cdots \cap V_{d+2} \), where the \( V_i \)'s are the divisors corresponding to good surfaces representing the \( z_i \) and \( w_j \). As usual, Uhlenbeck's theorems on compactness and removability of singularities imply that there are connections \( A_M \in \mathcal{M}_{M, m}(g_M) \) and \( A_{S^2 \times S^2} \in \mathcal{M}_{S^2 \times S^2, n}(g_{S^2 \times S^2}) \) such that \( A_\nu \) converges to \( A_M \lor A_{S^2 \times S^2} \) together with instantons at \( \rho \) points in \( M \) and at \( \sigma \) points in \( S^2 \times S^2 \).

At first we suppose that \( 0 < r < d \). Suppose that \( m > 0 \) and \( n > 0 \). Since surfaces representing the \( z_j \) and \( w_k \) are chosen in general position, no instanton point lies on more than two of these surfaces. Thus \( A_M \) must lie on at least \( r - 2\rho \) of \( V_1, \ldots, V_r \); hence \( 2d - 8(\ell - m) \geq 2(r - 2\rho) \). Similarly, from \( A_{S^2 \times S^2} \) we deduce that \( 8n - 6 \geq 2(d + 2 - r - 2\sigma) \). Combining these inequalities with the charge count \( m + n + \rho + \sigma \leq \ell + 1 \) leads to a contradiction. If \( m = 0, n = 0 \), then \( 2d - 8(\ell - m) \geq 2(r - 2\rho) \) and \( 2\sigma \geq d + 2 - r \). So \( \ell + 1 \geq m + \rho + \sigma \geq \ell + 1 - \frac{\ell}{4} + \frac{d}{2} \). This contradicts the assumption \( r < d \). If \( m = 0, n = 0 \) then \( 2\rho \geq r \) and \( 2\sigma \geq d + 2 - r \); and then the charge count \( \ell + 1 \geq \rho + \sigma \) contradicts the basic inequality \( \ell > \frac{3}{4}(1 + b_M) \).

To complete the proof in the case when \( 0 < r < d \), we need to consider the situation where all the connections in the 1-dimensional intersection

\[
\mathcal{F}_X(\nu) = \mathcal{M}_{X, l+1}(g_\nu) \cap V_1 \cap \cdots \cap V_{d+2}
\]

limit weakly to the trivial connection \( \Theta_M \) on \( M \) and none limit weakly to the trivial connection on \( S^2 \times S^2 \). A counting argument does not suffice here. This is precisely the situation encountered in the proof of the connected sum theorem.

As in Donaldson's proof, for large \( \nu \), we need to consider an open subset \( U \) of \( \mathcal{B}_{X, l+1}^* \) consisting of connections whose restrictions to \( M_0(\nu) \) are close to \( \Theta_M \) off a finite number of small balls where the charge is concentrated. To define \( U \), fix \( \varepsilon > 0 \). Then \( A \in \mathcal{B}_{X, l+1}^* \) is in \( U \) if there are a finite number of disjoint balls \( B_i \) in \( M_0(\nu) \) with centers \( p_i \) and radii \( \lambda_i \) such that

\[
\begin{align*}
1 & \quad \int_{M_0(\nu) \setminus \cup B_i} |F_A|^2 < \varepsilon, \\
2 & \quad \sum_i \lambda_i^2 < \varepsilon, \\
3 & \quad |\int_{B_i} |F_A^\pm|^2 - 8\pi^2 m_i| < \varepsilon \quad \text{for some positive integers } m_i.
\end{align*}
\]

The upshot of the counting argument given above is that if we fix \( \varepsilon \), then for small enough \( \nu \), the intersection \( \mathcal{F}_X(\nu) \) is contained in \( U \). The subset of \( U \) consisting of connections \( A \) for which some \( m_i \) is greater than one is of codimension at least 4 in \( U \). Since \( \mathcal{F}_{X, \rho}(\nu) \) is 1-dimensional, we can modify the third condition defining \( U \):

\[
(3') \quad |\int_{B_i} |F_A^\pm|^2 - 8\pi^2 | < \varepsilon \quad \text{for each } i.
\]

Note that \( U \) breaks up into connected components such that all connections in any given component have approximately the same charge \( \sum m_i \). To show that \( \mathcal{F}_X(\nu) \) is homologically trivial in \( \mathcal{B}_{X, l+1}^* \), it suffices to work with the
piece $\mathcal{F}_{X,\rho}(\nu)$ lying in, say, $U_\rho$, the union of the components of $U$ with charge approximately constant and equal to $\sum m_i = \rho$. We are assuming that $r \geq 1$; so $\rho \geq \frac{r}{2} > 0$. Also, we may as well assume that $\mathcal{F}_{X,\rho}(\nu)$ is nonempty.

Consider now $U_{\rho,M} \subset \mathcal{B}_{M,\rho}^*$ which consists of connections in $\mathcal{B}_{M,\rho}^*$ satisfying the defining conditions for $U$. According to the theory of Taubes and Donaldson, for small enough $\varepsilon$, the anti-self-dual connections in $U_{\rho,M}$ can be described as follows (cf. [T1, D1]). Let $F_M^+$ denote the bundle oriented orthonormal frames of the space of self-dual 2-forms on $M$, and let $S^\rho(F_M^+ \times \mathbb{R}^+)$ denote the complement in the symmetric product of $\rho$ copies of $F_M^+ \times \mathbb{R}^+$ of the preimage of the “fat diagonal” $\Delta_M^\rho$ under the obvious projection. (Here $\Delta_M^\rho = \{\{x_i\} \in \text{Symm}^\rho(M) \mid x_i = x_j, \text{some } i \neq j \} \}$. We consider the diagonal action of $SO(3)$ on $S^\rho(F_M^+ \times \mathbb{R}^+)$ (where $SO(3)$ acts trivially on the $\mathbb{R}^+$ factor). Then there is an $SO(3)$-equivariant embedding

$$\gamma^0 : S^\rho(F_M^+ \times \mathbb{R}^+) \to \mathcal{B}_{M,\rho}^*$$

and an $SO(3)$-equivariant map

$$\psi^0 : S^\rho(F_M^+ \times \mathbb{R}^+) \to \mathbb{R}^{3b_M^*}$$

such that $\gamma^0(\{(f_i, \lambda_i)\} \mid \sum \lambda_i^2 < \varepsilon) = U_{\rho,M}^0$ and such that $\gamma^0((\psi^0)^{-1}(0)) = U_{\rho,M} \cap \mathcal{M}_{M,\rho}^0$. Taking the quotient by $SO(3)$, this descends to

$$\gamma : S^\rho(F_M^+ \times \mathbb{R}^+)/SO(3) \to \mathcal{B}_{M,\rho}^*$$

and to a section $\psi$ of the rank $3b_M^*$ vector bundle

$$\eta_M = \bigoplus_{b_M^*} [S^\rho(F_M^+ \times \mathbb{R}^+) \times_{SO(3)} \mathbb{R}^3]$$

such that $\gamma(\psi^{-1}(0)) = U_{\rho,M} \cap \mathcal{M}_{M,\rho}$. By letting the scales $\lambda_i$ assume the value 0, this set-up extends naturally to the Uhlenbeck compactification $\mathcal{M}_{M,\rho}$, where the section gives obstructions to the lower charge problem with points of concentration (instanton points). (See [D1,5.6] and [DK,§7.2.8].)

Likewise, there is an “obstruction bundle” description of $U_\rho \cap \mathcal{M}_{X,\ell+1}(g_\nu)$ given by an $SO(3)$-equivariant map

$$\sigma^0 : S^\rho(F_M^+ \times \mathbb{R}^+) \times \mathcal{M}_{S^2 \times S^2,n}^0 \to \mathbb{R}^{3b_M^*}$$

(again, diagonal action) where $n = \ell + 1 - \rho$. Thus $U_\rho$ can be identified with

$$S^\rho(F_M^+ \times \mathbb{R}^+) \times \mathcal{M}_{S^2 \times S^2,n}^0/SO(3),$$

and $U_\rho \cap \mathcal{M}_{X,\ell+1}(g_\nu)$ can be identified with the zeros of a section $\sigma(\nu)$ of

$$\bigoplus_{b_M^*} [(S^\rho(F_M^+ \times \mathbb{R}^+) \times \mathcal{M}_{S^2 \times S^2,n}^0) \times SO(3), \mathbb{R}^3]$$

$$[S^\rho(F_M^+ \times \mathbb{R}^+) \times \mathcal{M}_{S^2 \times S^2,n}^0]/SO(3)$$
Again, this description extends to $\mathcal{M}_{t+1, X}(g_{u})$ where we let some $\lambda_{i}$'s equal 0.

The open set $U$ has two distinct ends. The first consists of connections which are concentrated; this corresponds to the situation where some of the $\lambda_{i} = 0$. The second end, which we shall denote $\text{Fr}(U)$, consists of those $[(f_{i}, \lambda_{i})] \in U$ such that $\sum \lambda_{i}^{2} = \epsilon$. Since $\mathcal{F}_{X}(\nu) \subset \text{Int} \ U$, the section $\sigma$ restricted over $\text{Fr}(U) \cap V_{1} \cap \cdots \cap V_{d+2}$ has no zeros.

It follows from the description in [T1] that as $\nu \to 0$, the section $\sigma(\nu)$ "decouples", that is, it limits to the sum of sections $\sigma_{M} = \psi$ of $\eta_{M}$ and $\sigma_{S^{2} \times S^{2}}$ of

$$\eta_{S^{2} \times S^{2}} = \bigoplus_{b_{M}}(\mathcal{M}_{S^{2} \times S^{2}, n} \times SO(3); R^{3})$$

Since $\mathcal{M}_{S^{2} \times S^{2}, n}$ consists of anti-self-dual connections, the limiting section $\sigma_{S^{2} \times S^{2}} = 0$. Thus for small enough neck radius $\nu$, the corresponding section $\sigma(\nu)$ is almost a sum $\sigma(\nu) = \sigma_{M}(\nu) + \sigma_{S^{2} \times S^{2}}(\nu)$, and $\sigma_{S^{2} \times S^{2}}(\nu) \to 0$ as $\nu \to 0$. Furthermore, for fixed $\nu$, $\sigma_{M}(\nu) \{ (f_{i}, \lambda_{i}) \} \to 0$ as $\sum \lambda_{i}^{2} \to 0$.

In particular, this means that for small $\nu$, $\sigma(\nu)$ is asymptotically $\sigma_{M}(\nu)$ as $\sum \lambda_{i}^{2} \to \epsilon$. We fix such a small $\nu$ and now drop it from the notation. Then we have our neighborhood $U_{\rho}$, and $\sigma \sim \sigma_{M}$ near $\text{Fr}(U_{\rho})$. The intersection $\mathcal{F}_{X, \rho}$ is cut out by the zero set of $\sigma$ restricted to $U_{\rho} \cap V_{1} \cap \cdots \cap V_{d+2} = W_{\rho}$. Thus on $\text{Fr}(W_{\rho})$ the section $\sigma \sim \sigma_{M}$ is nonvanishing. Since $W_{\rho}$ consists of connections which are almost anti-self-dual (more precisely, which satisfy conditions (1)–(3) above) the notion of Uhlenbeck compactification $W_{\rho}$ makes sense. Note also that the compactness of $\mathcal{F}_{X}$ implies that $\sigma$ has no zeros on the singular set of $W_{\rho}$.

The formal situation is this—we have a compact singular space $W_{\rho}$ with singular set (corresponding to the lower strata of the Uhlenbeck compactification) of codimension $\geq 3$. Over $W_{\rho}$ there is a vector bundle $\eta$ of rank $3b_{M}^{+}$ with a section $\sigma$, nonvanishing over the singular set, and over the boundary $\text{Fr}(W_{\rho})$, $\sigma$ is pulled back from the section $\sigma_{M}$ of $\eta_{M}|_{\text{Fr}(W_{\rho}, M)}$. This set-up gives us a relative Euler class $e \in H^{3b_{M}^{+}}(W_{\rho}, \text{Fr}(W_{\rho}); Z)$. Clearly, $\dim W_{\rho} = 3b_{M}^{+} + 1$, and if $S$ denotes the singular set of $W_{\rho}$ then Poincaré duality gives

$$H_{1}(W_{\rho}; Z) \cong H_{c}^{3b_{M}^{+}}(W_{\rho}, \text{Fr}(W_{\rho}) \cup S; Z) \cong H^{3b_{M}^{+}}(W_{\rho}, \text{Fr}(W_{\rho}); Z)$$

because of the codimension of $S$. The Poincaré dual of $e$ in $H_{1}(W_{\rho}; Z)$ is represented by $\mathcal{F}_{X, \rho}$. Thus it will suffice to show that $e = 0$.

Since $\mathcal{F}_{X, \rho}$ is nonempty, the usual counting argument shows that $\rho \geq \frac{\epsilon}{2}$ and $8n - 6 \geq 2(d + 2 - r)$. By counting parameters in the base spaces of the obstruction bundles $\eta_{M}$ and $\eta$, we see that $\dim W_{\rho, M} = 8\rho - 3 - 2r$ and $\dim W_{\rho} = 8\rho + 8n - 6 - 2(d + 2)$. Thus $3b_{M}^{+} + 1 = \dim W_{\rho} = \dim W_{\rho, M} + [8n - 3 - 2(d + 2 - r)] \geq \dim W_{\rho, M} + 3$. So $\sigma|_{\text{Fr}(W_{\rho})}$ is pulled back from the
section $\sigma_M|_{\text{Fr}(W_{\rho,M})}$, and the base $W_{\rho,M}$ has dimension at least 2 less than the rank $3b_M^+$ of $\eta_M$. Note that any nonvanishing section of $\eta_M|_{\text{Fr}(W_{\rho,M})}$ is homotopic through nonvanishing sections to $\sigma_M|_{\text{Fr}(W_{\rho,M})}$ since the obstructions to such homotopies lie in $H^i(\text{Fr}(W_{\rho,M}); \pi_i(S^{3b_M^+ - 1})) = 0$.

We now need to separate the argument into two cases. Suppose first that $3b_M^+ + 1 > \dim W_{\rho,M} + 3 = 8\rho - 2r$. In this case our plan is to construct nonvanishing sections $\tau_M$ of $\eta_M$ over $W_{\rho,M}$ and $\tau_{S^2 \times S^2}$ of $S^2 \times S^2$ over $Y = S^2 \times S^2 \cap V_{r+1} \cap \cdots \cap V_{d+2}$ and combine them to get a nonvanishing section $\tau$ of $\eta$ over $W_{\rho}$. This section will be nonvanishing as well over the singular set of $W_{\rho}$ and over $\text{Fr}(W_{\rho})$ it will be pulled back from $\tau_M$. Since $\sigma_M|_{\text{Fr}(W_{\rho,M})}$ is homotopic to $\eta_M|_{\text{Fr}(W_{\rho,M})}$ through nonvanishing sections of $\sigma_M|_{\text{Fr}(W_{\rho,M})}$, the Poincaré dual of the Euler class $e$ will be represented by the (empty) zero set of $\tau$; hence $e = 0$.

We first construct $\tau_M$. Let $k$ be the unique integer such that

$$8\rho - 2r - 3 \leq 3k < 8\rho - 2r \leq 3b_M^+,$$

and consider the rank $3k$ vector bundle

$$\bigoplus_{i=1}^{k} \left( W_{\rho,M}^0 \times_{SO(3)} R^3 \right)$$

$$\downarrow$$

$W_{\rho,M}$

This is a subbundle of $\eta_M|_{W_{\rho,M}}$. Obstructions to the existence of a nonvanishing section of this bundle lie in $H^i(W_{\rho,M}; \pi_{i-1}(S^{3k-1}))$. Since $H^{3k}(W_{\rho,M}; Z) = 0$, all obstructions vanish and there is such a section $\tau_M$. Note that the total space $\bigoplus_{i=1}^{k} (W_{\rho,M}^0 \times_{SO(3)} R^3)$ is the quotient by $SO(3)$ of $\bigoplus_{k} (W_{\rho,M}^0 \times R^3) \cong W_{\rho,M}^0 \times R^{3k}$; so we equivalently have a nonvanishing $SO(3)$-equivariant map $\tau_0^M: W_{\rho,M}^0 \rightarrow R^{3k}$.

Next, consider the situation on $Y = S^2 \times S^2 \cap V_{r+1} \cap \cdots \cap V_{d+2}$ which has dimension $\dim Y = 8n - 6 - 2(d + 2 - r) = 3b_M^+ + 1 - (8\rho - 2r) < 3b_M^+ + 1 - 3k$. Thus $\dim Y \leq 3(b_M^+ - k)$. We now wish to find a nonvanishing section $\tau_{S^2 \times S^2}$ of the rank $3(b_M^+ - k)$ vector bundle $\bigoplus_{i=k+1}^{b_M^+} (Y^0 \times_{SO(3)} R^3)$ over $Y$. (The point of the case we are considering is that $b_M^+ - k > 0$.) The only possible obstruction arises in case $\dim Y = 3(b_M^+ - k)$. In this case the obstruction lies in $H^{3(b_M^+ - k)}(Y; Z)$ and is $c^{b_M^+ - k}$ where $c \in H^3(Y; Z)$ is the Euler class of the vector bundle $Y^0 \times_{SO(3)} R^3$ associated to the basepoint fibration over $Y$.

Since this bundle has odd rank, its Euler class $c$ is 2-torsion; so $c^{b_M^+ - k}$ is 2-torsion as well. However, $H^{3(b_M^+ - k)}(Y; Z)$ is torsion-free, and so the obstruction vanishes. Again, since $\bigoplus_{i=k+1}^{b_M^+} (Y^0 \times_{SO(3)} R^3) = \bigoplus_{i=k}^{b_M^+} (Y^0 \times R^3)/SO(3)$, we have
a nonvanishing $SO(3)$-equivariant map $\tau^0_{S^2 \times S^2} : Y^0 \to \mathbb{R}^{3(b^+_M - k)}$.

To define $\tau : W^0_{\rho} \to \eta| W^0_{\rho} \cong (W^0_{\rho,M} \times Y^0) \times \mathbb{R}^{3b^+_M}$, consider a typical point $Q = \{(f_i, \lambda_i)\}; (A, \xi) \in W^0_{\rho,M} \subset S^0(F^+_M \times \mathbb{R}^+) \times \mathbb{R}^{3b^+_M}$.

Let $\lambda(Q) = \sum \lambda_i^2$, and define

$$\tau(Q) = \lambda(Q) \cdot \tau^0_{M}\{f_i, \lambda_i\} + (\varepsilon - \lambda(Q)) \cdot \tau^0_{S^2 \times S^2}(A, \xi).$$

(Here the image of $\tau^0_M$ lies in the first $k$ of the $\mathbb{R}^3$-summands of $\mathbb{R}^{3b^+_M}$ and the image of $\tau^0_{S^2 \times S^2}$ lies in the last $b^+_M - k$ of them.) The section $\tau^0$ is $SO(3)$-equivariant, nonvanishing, and nonvanishing when extended to the singular set of $W^0_{\rho}$ (where some (or all) of the $\lambda_i = 0$). As we described above, this completes the proof in the case where $3b^+_M + 1 > 8\rho - 2r$.

Next, we need to consider the case where $\dim W^0_{\rho,M} = 8\rho - 2r = 3b^+_M + 1$.

Then we have $\dim W^0_{\rho} = \dim W^0_{\rho,M} + \dim Y^0 - 3 = 3b^+_M + 1 = \dim W^0_{\rho,M}$, so $\dim Y = \dim Y^0 - 3 = 0$. Thus the cut-down moduli space $Y = M_{S^2 \times S^2} \cap V_{r+1} \cap \cdots \cap V_{d+2}$ consists of a finite number of points. This means that $\mathcal{F}_{\rho \lambda}$ breaks up into unions of connected components corresponding to these points. It thus suffices to assume that $Y$ is a single connection and show that the homology class of $\mathcal{F}_{\rho \lambda}$ vanishes. Then $Y^0$ is a copy of $SO(3)$; so $W^0_{\rho} = W^0_{\rho,M} \times SO(3)$, and the obstruction bundle is:

$$\eta = \bigoplus_{b^+_M}(W^0_{\rho,M} \times SO(3) \mathbb{R}^3) \cong \bigoplus_{b^+_M}(W^0_{\rho,M} \times \mathbb{R}^3) \cong W^0_{\rho,M} \times \mathbb{R}^{3b^+_M} \downarrow$$

$$W^0_{\rho} = (W^0_{\rho} \times SO(3))/SO(3) \cong W^0_{\rho,M}.$$

The obstruction bundle is trivial; however this is not enough to deduce that the homology class $[\mathcal{F}_{\rho \lambda}]$ is trivial because $[\mathcal{F}_{\rho \lambda}]$ is Poincaré dual to a relative Euler class in $H^{3b^+_M}(W^0_{\rho,M} \times \text{Fr}(W^0_{\rho,M}); \mathbb{Z})$.

Recall that the condition on a section which defines the relative class is that over $\text{Fr}(W^0_{\rho,M})$ it must be homotopic to the pullback of $\sigma_M$, a section of $\eta_M| \text{Fr}(W^0_{\rho,M})$. Since $\text{Fr}(W^0_{\rho,M})$ is a manifold of dimension $3(b^+_M - 1)$, the Euler class argument given in the construction of $\tau^0_{S^2 \times S^2}$ works here as well to show that there is a nonvanishing section $\tau_{\beta}$ of $\eta_M$ over $\text{Fr}(W^0_{\rho,M})$ such that $\tau_{\beta}$ takes its values in a subbundle of rank $3(b^+_M - 1)$. (This argument needs $3(b^+_M - 1) > 0$ to work, for if $\text{Fr}(W^0_{\rho,M})$ were 0-dimensional, $\tau_{\beta}$ would still need to take values in a bundle of positive rank. Of course we have the standard assumption that $b^+_M > 1$.)

Now return to this relative Euler class problem of extending a nonvanishing section of

$$W^0_{\rho,M} \times \mathbb{R}^{3b^+_M} \downarrow$$

$$W^0_{\rho,M}$$
from \( \text{Fr}(W_{\rho, M}) \) over all of \( W_{\rho, M} \). As in the previous case, this problem is independent of the section over \( \text{Fr}(W_{\rho, M}) \) as long as it is pulled back from a section of \( \eta_M \)—so we work with the pullback of \( \tau_\rho \). Up to homotopy through nonvanishing sections, the pullback is the same as a map

\[
\tau'_\rho : \text{Fr}(W^0_{\rho, M}) \to S^{3(b^+_{\rho, M} - 1)} \subset S^{3b^+_{\rho, M} - 1}.
\]

Since \( \tau'_\rho \) is not a surjection onto \( S^{3b^+_{\rho, M} - 1} \), it is nullhomotopic and therefore it extends to a map \( W^0_{\rho, M} \to S^{3b^+_{\rho, M} - 1} \). This means that the relative Euler class vanishes and our proof of Theorem 8.1 is completed for the case where \( 0 < r < d \).

Before continuing with the other cases, we wish to make a few comments on the argument above. First of all, \( SO(3) \)-equivariance forces us to work with whole \( \mathbb{R}^3 \)-summands of \( \mathbb{R}^{3b^+_{\rho, M}} \). It would not be sufficient, for example, to get a nonvanishing section of \( \eta_\rho \) over \( \text{Fr}(W_{\rho, M}) \) which takes its values in a subbundle of rank \( 3b^+_{\rho, M} - 2 \), because the pullback over \( \text{Fr}(W^0_{\rho, M}) \) would not take its values in a proper subbundle of \( \eta \).

Second, this last argument comes close to filling in the details in the alternate proof of Theorem 1.1 which was sketched in the introduction. In that case we have a 0-dimensional cut-down moduli space over \( M \), and on \( S^2 \times S^2 \) there is a single point instanton with center \((S^2 \times S^2) \cap (0 \times S^2)\). Thus \( \rho = 1 \), \( W^0_{\rho, S^2 \times S^2} \cong cSO(3) \), \( Y^0 \) is a finite union of copies of \( SO(3) \), and \( b^+_{S^2 \times S^2} = 1 \). For simplicity we work with a single copy of \( SO(3) \). Then we get \( W^0_{\rho} \cong W^0_{\rho, S^2 \times S^2} \cong cSO(3) \) and \( \eta \) is the trivial bundle over \( W^0_{\rho} \) with fiber \( \mathbb{R}^3 \). The relative Euler class lives in \( H^3(W^0_{\rho}; \text{Fr}(W^0_{\rho}); \mathbb{Z}) \cong H^3(cSO(3), SO(3); \mathbb{Z}) \cong \mathbb{Z}_2 \), and it is the obstruction to the nonvanishing extension of a map \( \text{Fr}(W^0_{\rho}) \cong SO(3) \to \mathbb{R}^3 \) pulled back from \( \tau_\rho : \{\text{point}\} \to \mathbb{R}^3 \). In fact, if \( \tau_\rho = v \in \mathbb{R}^3 \) then the pullback \( SO(3) \to \mathbb{R}^3 \) is given by \( g \to g(v) \), and this realizes the nontrivial cohomology class.

To finish the proof of Theorem 8.1, we still need to dispense with the rest of the cases. Suppose that \( q_{t+1, u, (z_1, \ldots, z_{d+1}, x)} \neq 0 \) (so \( r = d + 1 \)). Arguing as above in case \( m > 0, n > 0 \), we obtain inequalities \( 8m \geq 8\ell + 2(1 - 2\rho) \) and \( 8n - 6 \geq 2(1 - 2\sigma) \). These inequalities are again incompatible with the charge count. We similarly rule out the cases \( m = 0 \) and/or \( n = 0 \).

Finally, we need to compute \( q_{t+1, u, (z_1, \ldots, z_{d}, x), (x)} \). This calculation follows §§5 and 6, except that in Theorem 5.11 we obtain

\[
q_{t+1, u, (z_1, \ldots, z_{d}, x), (x)} \equiv N(x, x) \cdot q_{t, M}(z_1, \ldots, z_{d}) \mod 2
\]

where \( N(x, x) \) is the mod 2 intersection number

\[
N(x, x) \equiv \#(\mathcal{M}_{K,1} \cap V'_x \cap V_x \cap r^{-1}_K(\alpha)) \quad \text{for} \quad \alpha \in r_{M_0}'(\mathcal{F}_{M_0})
\]

(The divisor \( V'_x \) corresponds to a second surface representing \( x \).) Refer now to §6. The proof preceding Lemma 6.2, which uses an argument based on [D1,
Theorem B], shows that in this case \( N^1(\alpha) = 0 \in H_1(\mathcal{R}_\omega^*(\alpha); \mathbb{Z}_2) \) since there are an even number of "internal" ends of \( \mathcal{M}^2(\vartheta) \). Then the proof of Theorem 6.7 shows \( N(x, x) \equiv 0 \mod 2 \). Similarly we can show
\[
q_{t+1, u_1, x}(z_1, \ldots, z_d, y, y) = 0.
\]
(Alternatively we can prove that \( q_{t+1, u_1, x}(z_1, \ldots, z_d, y, y) = 0 \) by applying the transformation \( S_0 \) defined below to the equation
\[
q_{t+1, u_1, x}(z_1, \ldots, z_d, x, x) = 0
\]
and then using the invariance of \( q_{t+1, u_1, x} \) under diffeomorphisms.) □

We continue with the hypothesis that \( M \) is a closed simply connected spin 4-manifold with a Donaldson invariant \( q_t(M) \) of degree \( d \) with \( e \) odd. In particular \( b_M^+ \geq 3 \). Now \( H_2(M; \mathbb{Z}) \cong 2aE_b \oplus bH \) where \( a, b \in \mathbb{Z} \) and \( b \geq 0 \). It follows from [D1] that \( a = 0 \) if \( b \leq 2 \); so \( b \geq 3 \). Let \( X = M\#S^2 \times S^2 \) and let \( s_X \) denote the symplectic form on \( H_2(X; \mathbb{Z}_2) \) given by \( s_X(u, v) = Q_X(u, v) \) mod 2 where \( u \) and \( v \) are any lifts of \( u, v \) to \( H_2(X; \mathbb{Z}) \). (Here we are using the fact that \( H_1(X; \mathbb{Z}) = 0 \).) Let \( \Phi_X : H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}_2 \) be given by \( \Phi_X(u) = \frac{1}{2}Q_X(u, u) \mod 2 \), and let \( \varphi_X : H_2(X; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \) be the induced quadratic form. Note that \( \varphi_X(u + v) + \varphi_X(u) + \varphi_X(v) = s_X(u, v) \) and that the orthogonal group \( O_X \) leaves \( \Phi_X \) invariant. For \( u, v \in H_2(X; \mathbb{Z}) \) write \( u = v \mod 2 \) provided \( v = u + 2w \) for some \( w \in H_2(X; \mathbb{Z}) \).

**Lemma 8.2.** Let \( u, v \in H_2(X; \mathbb{Z}) \) be primitive vectors satisfying \( \Phi_X(u) = \Phi_X(v) \). Then there is an \( h \in O_X \) such that \( h(u) \equiv v \mod 2 \).

**Proof.** This is a consequence of a result of Wall [W3, Theorem 6] that since \( b_X^+ \geq 2 \) (in fact \( \geq 4 \) in our case) and \( X \) is spin, \( O_X \) is transitive on primitive vectors with a given norm \( Q_X(u, u) \). Suppose \( \Phi_X(u) = \Phi_X(v) \). Then \( Q_X(v, v) = Q_X(u, u) + 4r \) for some \( r \in \mathbb{Z} \). We wish to show that there is a primitive vector \( v \equiv 2w \) mod 2 with \( Q_X(v, v) \equiv Q_X(u, u) \) mod 2, for then Wall's theorem provides an \( h \in O_X \) with \( h(u) = v \equiv v \mod 2 \). Since \( H_2(X; \mathbb{Z}) \cong 2aE_b \oplus (b+1)H \), we can find \( e_1, f_1, e_2, f_2 \in H_2(X; \mathbb{Z}) \) such that \( Q_X(e_1, e_2) = Q_X(f_1, f_2) = 0 \), and \( Q_X(e_1, f_1) = \delta_{i, j} \). Let \( \alpha = \frac{1}{2}Q_X(v, v) \). Applying Wall's theorem we obtain a \( g \in O_X \) with \( g(v) = \alpha e_1 + f_1 \). If \( r = 2m \) define \( z \) by
\[
g(z) = e_2 - mf_2.
\]
If \( r = 2(m + \alpha) + 1 \), define \( z \) by
\[
g(z) = e_2 - mf_2 - (1 + \alpha)e_1 + f_1.
\]
Now set \( \vartheta = v - 2z \). Its norm \( Q_X(v, \vartheta) = Q_X(u, u) \). Furthermore, \( \vartheta \) is primitive since if \( r = 2m \) then \( Q_X(g(\vartheta), e_1) = 1 \), and in the other case \( Q_X(g(\vartheta), e_1 + f_2) = 1 \). Thus Lemma 8.2 follows. □

Let \( Sp_X \) denote the group of linear transformations of \( H_2(X; \mathbb{Z}_2) \) which preserve the symplectic form \( s_X \), and let \( O_{+, X} \) be the subgroup which preserves the quadratic form \( \varphi_X \). There is a homomorphism \( \Gamma : O_X \rightarrow O_{+, X} \) induced by reduction mod 2.
Lemma 8.3. The image $\Gamma(O_X)$ acts transitively on $\varphi_X^{-1}(0) \setminus \{0\}$ and on $\varphi_X^{-1}(1)$.

Proof. If $u, v \in H_2(X; \mathbb{Z}_2)$ are nonzero vectors such that $\varphi_X(u) = \varphi_X(v)$ we can find primitive vectors $\tilde{u}, \tilde{v} \in H_2(X; \mathbb{Z})$ which reduce mod 2 to $u$ and $v$. Apply Lemma 8.2 to $\tilde{u}$ and $\tilde{v}$. $\square$

Before continuing, let us recall that $\mathbb{Z}_2$- quadratic spaces which correspond to a symplectic form are classified by their dimension and Arf invariant. (See [S].) Now $H_2(X; \mathbb{Z}) = 2aE_8(\oplus (b+1)H)$; so its Arf invariant is 0 and $(H_2(X; \mathbb{Z}_2), \varphi_X) \cong (8|a| + b + 1)H$ as a $\mathbb{Z}_2$- quadratic space. (Note that $2E_8$ has Arf invariant 0, and so is isomorphic to $8H$ over $\mathbb{Z}_2$.)

Lemma 8.4. The homomorphism $f': O_X \to O_{+,X}$ is surjective.

Proof. The main theorem of [McL] states that the image of $O_X$ in $Sp_X$ is either all of $O_{+,X}$ or is a symmetric group on $2n + 1$ or $2n + 2$ letters, where $2n = \dim(H_2(X; \mathbb{Z}_2))$. To see that it is all of $O_{+,X}$, we need to describe the embeddings of $S_{2n+1}$ and $S_{2n+2}$ in $O_{+,X}$. (See [LPS, p. 34].) For any $m$ the symmetric group $S_m$ fixes the natural quadratic form $\langle (\xi) \rangle, (\eta) \rangle = \sum \xi_i \eta_i$ on $S(m) = \{(\xi) \in (\mathbb{Z}_2)^m | \sum \xi_i = 0\}$, and $S_m$ acts on $S(m)$ by permutation of the coordinates. Define $P_m : S(m) \to \mathbb{Z}_2$ by $P_m(\xi_1, \ldots, \xi_m) = \begin{cases} 1, & \text{if } \# \{i \mid \xi_i = 1\} \equiv 2 \text{ mod } 4, \\ 0, & \text{if } \# \{i \mid \xi_i = 1\} \equiv 0 \text{ mod } 4. \end{cases}$ Then $P_m$ is a quadratic form on $S(m)$ with associated quadratic form $\langle \cdot, \cdot \rangle$. When $n \equiv 0 \text{ mod } 4$, the Arf invariant of the $\mathbb{Z}_2$- quadratic space $(S(2n + 1), P_{2n+1})$ is 0; so $(S(2n + 1), P_{2n+1})$ and $(H_2(X; \mathbb{Z}_2), \varphi_X)$ are isomorphic. The symmetric group $S_{2n+1}$ only embeds in $O_{+,X}$ when $n \equiv 0 \text{ mod } 4$, and is then given by the action of $S_{2n+1}$ on $S(2n + 1)$.

For the case of $S_{2n+2}$, again start with the action of $S_{2n+2}$ on $S(2n + 2)$. The diagonal, $d = (1, \ldots, 1) \in S(2n + 2)$ is fixed by the action of $S_{2n+2}$. Thus $S(2n + 2)/\text{span}(d)$ is a $S_{2n+2}$-space. If $n \equiv 3 \text{ mod } 4$ the Arf invariant of $(S(2n + 2)/\text{span}(d), P_{2n+2})$ vanishes, and so $(S(2n + 2)/\text{span}(d), P_{2n+2})$ is isomorphic to $(H_2(X; \mathbb{Z}_2), \varphi_X)$. This gives the embedding of $S_{2n+2}$ in $O_{+,X}$. (If $n \equiv 3 \text{ mod } 4$ then $S_{2n+2}$ does not embed in $O_{+,X}$.)

Let $c = (1, 1, 0, \ldots, 0)$ and $c' = (1, 1, 1, 1, 1, 0, \ldots, 0) \in (\mathbb{Z}_2)^{2n+1}$. Then $(c, c') \in S(2n + 1)$, and $P_{2n+1}(c) = 1 = P_{2n+1}(c')$. There is no $\sigma \in S_{2n+1}$ such that $\sigma(c) = c'$. However by Lemma 8.3, $\Gamma(O_X)$ is transitive on $P_{2n+1}^{-1}$. Thus $\Gamma(O_X)$ is strictly larger than $S_{2n+1}$. A similar argument shows that the image of $O_X$ is also strictly larger than $S_{2n+2}$. $\square$

Let $2m = \dim H_2(M; \mathbb{Z}_2)$; so $H_2(X; \mathbb{Z}_2) = H_2(M \# S^2 \times S^2; \mathbb{Z}_2) \cong (m + 1)H$ as a $\mathbb{Z}_2$-quadratic space. Thus there is a basis $\{e_i, f_j | i = 0, \ldots, m\}$ of $H_2(X; \mathbb{Z}_2)$ such that $\varphi_X(e_i) = 0 = \varphi_X(f_j)$, and $s_X(e_i, e_j) = s_X(f_i, f_j) = 0$. 

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\(s_X(e_i, f_j) = \delta_{i,j} \). We choose this basis so that \(e_0\) is the mod 2 reduction of \(x \in H_2(S^2 \times S^2; \mathbb{Z})\) and \(f_0\) is the mod 2 reduction of \(y\). Let \(\{\alpha_i, \beta_i\}\) be a dual basis. Define the families \(R_{i,j}\) and \(S_i\) in \(O_{\pi, \mathbb{X}}\) by

\[
\begin{align*}
R_{i,j}(e_i) &= e_j, \\
R_{i,j}(e_j) &= e_i, \\
R_{i,j}(f_j) &= f_i, \\
R_{i,j}(f_i) &= f_j, \\
R_{i,j}(e_k) &= e_k, \\
R_{i,j}(f_k) &= f_k \quad \text{if } k \neq i, j
\end{align*}
\]

and

\[
\begin{align*}
S_i(e_i) &= f_i, \\
S_i(e_j) &= e_i, \\
S_i(f_i) &= f_j, \\
S_i(f_j) &= e_i, \\
S_i(e_k) &= e_k, \\
S_i(f_k) &= f_k \quad \text{if } k \neq i.
\end{align*}
\]

It follows from Lemma 8.4 that there are transformations \(\tilde{R}_{i,j}\) and \(\tilde{S}_i\) in \(O_{\mathbb{X}}\) such that \(\Gamma(\tilde{R}_{i,j}) = \tilde{R}_{i,j}\) and \(\Gamma(\tilde{S}_i) = \tilde{S}_i\). By [W2, Theorem 2] each transformation in \(O_{\mathbb{X}}\) is induced from a diffeomorphism of \(\mathbb{X}\). Hence the \(R_{i,j}\) and \(S_i\) are induced from diffeomorphisms.

We will be working with \(\text{Sym}^{\mathbb{Z}_2} (H_2(X; \mathbb{Z}_2))\). Since \(H_1(X; \mathbb{Z}) = 0\), the mod 2 Donaldson polynomial invariant \(q_{\ell+1, u_1, X}\) can be viewed as an element of \(\text{Sym}^{d+2} (H_2(X; \mathbb{Z}_2))\), and the dual basis vectors \(\alpha_i\) and \(\beta_j\) belong to \(\text{Sym}^1 (H_2(X; \mathbb{Z}_2))\). Define \(\alpha_i^k \in \text{Sym}^k_Z (H_2(X; \mathbb{Z}_2))\) given by \(\alpha_i^k (e_1, \ldots, e_l) = 1\), and \(\alpha_i^k = 0\) on all other sets of \(k\) basis vectors. Similarly define \(\beta_j^k\). The power \(\alpha_i^k\) is not the same as the symmetric product of \(k\) copies of \(\alpha_i\) (which is 0 if \(k \geq 2\)). As is pointed out by Ruan [R]

\[
\alpha_i^k \alpha_i = \begin{cases} 
\alpha_i^{k+1}, & \text{if } i \text{ is even}, \\
0, & \text{if } i \text{ is odd}.
\end{cases}
\]

Ruan further points out that \(\text{Sym}^{p\mathbb{Z}_2} (H_2(X; \mathbb{Z}_2))\) is generated by monomials

\[
\lambda_{R, S} = \alpha_{r_0}^0 \cdots \alpha_{r_m}^0 \beta_{s_0}^0 \cdots \beta_{s_m}^0
\]

where \(R = (r_0, \ldots, r_m)\) and \(S = (s_0, \ldots, s_m)\) with \(\sum(r_i + s_i) = p\).

**Theorem 8.5.** Let \(M\) be a closed simply connected spin 4-manifold with a Donaldson polynomial invariant \(q_{\ell, M}\) of degree \(d\) where \(\ell\) is odd. Let \(X = M#S^2 \times S^2\). Then

\[
q_{\ell+1, u_1, X} = \begin{cases} 
\epsilon_{\ell+1, X} Q_X^{(p+1)} + P_{\ell+1, X} \bmod 2, & \text{if } d = 2p, \\
P_{\ell+1, X} \bmod 2, & \text{if } d \text{ is odd},
\end{cases}
\]

where \(\epsilon_{\ell+1, X} \in \mathbb{Z}_2\), and \(P_{\ell+1, X}\) is a sum (mod 2) of monomials of the form \(\alpha_i^r \beta_i^s\) with \(r_i + s_i = d + 2\).

**Proof.** Write

\[
(*) \quad q_{\ell+1, u_1, X} = \sum \epsilon_{R, S} \lambda_{R, S} \in \text{Sym}^{d+2} (H_2(X; \mathbb{Z}_2))
\]

where \(\lambda_{R, S}\) are monomials as above and \(\epsilon_{R, S} = 0\) or 1. Suppose that a monomial \(\lambda_{R, S}\) in this expression has some \(r_j \geq 2\) and \(\epsilon_{R, S} = 1\). Then
there are vectors \( v_1, \ldots, v_{d+2-r_j} \) such that each \( v_k \) is a basis vector \( e_i \) or \( f_i \) but not \( e_j \), and \( \lambda_{R,S}(e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j}) = 1 \). Since all of the other monomials \( \lambda_{R',S'} \) vanish on \( (e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j}) \), we have

\[
q_{d+1, u_1, x}(e_j, \ldots, e_j, v_1, \ldots, v_{d+2-r_j}) = 1.
\]

Now \( q_{r+1, u_1, x} \) is invariant under diffeomorphisms of \( X \), and the transformations \( R_{i,j} \) are all induced from diffeomorphisms. Applying \( R_{0,j} \) we get

\[
1 = q_{r+1, u_1, x}(e_0, \ldots, e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+2-r_j})).
\]

Since \( e_0 = x \), unless all the \( R_{0,j}(v_i) = f_0 = y \), Theorem 8.1 implies that this equals 0. The exceptional case where \( R_{0,j}(v_i) = f_0 \), \( i = 1, \ldots, d + 2 - r_j \), occurs when \( v_i = f_j, i = 1, \ldots, d + 2 - r_j \), in other words when \( \lambda_{R,S} = \alpha_{i,j}^r \beta_{s,j}^s \) with \( r_i + s_i = d + 2 \). This means that in (*) no monomial \( \lambda_{R,S} \) for which any \( r_j \geq 2 \) can have a nonzero coefficient, except for those monomials of the type which occur in \( P_{r+1, x} \). In particular, if any \( \lambda_{R,S} \) with nonzero coefficient is not of the special type \( \alpha_{i,j}^r \beta_{s,j}^s \), then counting its exponents we have

\[
d + 2 = \sum (r_j + s_j) \leq 2m + 2 = \dim H_2(X; \mathbb{Z}_2).
\]

Thus without loss we may assume that \( d \leq 2m \).

Suppose that there is a monomial \( \lambda_{R,S} \) with nonzero coefficient in \( q_{r+1, u_1, x} \) for which \( r_j = 1 \) and \( s_j = 0 \). Then, as above, there are standard basis vectors \( v_1, \ldots, v_{d+1} \) in \( H_2(X; \mathbb{Z}_2) \), none of which are \( e_j \) or \( f_j \) and such that

\[
q_{r+1, u_1, x}(e_j, v_1, \ldots, v_{d+1}) = \lambda_{R,S}(e_j, v_1, \ldots, v_{d+1}) = 1.
\]

Again applying \( R_{0,j} \) we get

\[
1 = q_{r+1, u_1, x}(e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+1})).
\]

But \( e_0 = x \) and \( R_{0,j}(v_k) = y \) for \( k = 1, \ldots, d + 1 \); so Theorem 8.1 implies that

\[
q_{r+1, u_1, x}(e_0, R_{0,j}(v_1), \ldots, R_{0,j}(v_{d+1})) = 0,
\]

a contradiction. Thus for each monomial \( \lambda_{R,S} \) not of the form \( \alpha_{i,j}^r \beta_{s,j}^s \) appearing with nonzero coefficient in \( q_{r+1, u_1, x} \), if \( r_j \neq 0 \), then also \( s_j \neq 0 \). Such a \( \lambda_{R,S} \) has the form

\[
\alpha_{i_1} \beta_{i_1} \cdots \alpha_{i_k} \beta_{i_k}
\]

where \( 2k = d + 2 \). This already shows that \( d \) must be even if \( q_{r+1, u_1, x} \neq P_{r+1, x} \). Let \( d = 2p \); so \( k = p + 1 \).

Let \( \sigma = \tau_t \cdots \tau_1 \in S_{m+1} \) be a permutation, written as a product of transpositions. For any transposition \( t = (i, j) \) let \( R_t = R_{i,j} \). Let \( R = R_{t_p} \circ \cdots \circ R_{t_1} \).

If \( \alpha_{i_1} \beta_{i_1} \cdots \alpha_{i_k} \beta_{i_k} \) appears with nonzero coefficient in \( q_{r+1, u_1, x} \) then

\[
1 = q_{r+1, u_1, x}(e_{i_1}, f_{i_1}, \ldots, e_{i_k}, f_{i_k}) = q_{r+1, u_1, x}(R(e_{i_1}), R(f_{i_1}), \ldots, R(e_{i_k}), R(f_{i_k})) = q_{r+1, u_1, x}(e_{\sigma(i_1)}, f_{\sigma(i_1)}, \ldots, e_{\sigma(i_k)}, f_{\sigma(i_k)}).
\]
Thus $\alpha_{\sigma(i_1)}\beta_{\sigma(i_1)}\cdots\alpha_{\sigma(i_k)}\beta_{\sigma(i_k)}$ also appears with nonzero coefficient. Since the $\alpha_i$ and $\beta_i$ commute, if we sum these expressions over $\sigma \in S_{m+1}$ then each expression appears $k! = (p+1)!$ times. Thus if $q_{t+1,u_1,x} \neq P_{t+1,x}$, then

$$q_{t+1,u_1,x} = P_{t+1,x} + \frac{1}{k!} \sum_{\sigma \in S_{m+1}} \alpha_{\sigma(1)}\beta_{\sigma(1)}\cdots\alpha_{\sigma(k)}\beta_{\sigma(k)} \equiv P_{t+1,x} + Q^{(p+1)}_X \mod 2.$$ 

We have already shown that if $d > 2m$ then $q_{t+1,u_1,x} = P_{t+1,x}$, and furthermore, $Q^{(p+1)}_X \equiv 0 \mod 2$ in that case. □

**Proof of Theorem 1.5.** If $z_1, \ldots, z_d \in H_2(M; \mathbb{Z})$, then Theorems 1.1 and 8.5 imply

$$q_{t,M}(z_1, \ldots, z_d) \equiv q_{t+1,u_1,x}(z_1, \ldots, z_d, x, y) \equiv (\epsilon_{t+1,x} Q^{(p+1)}_X + P_{t+1,x})(z_1, \ldots, z_d, x, y) \equiv \epsilon Q^{(p)}_M(z_1, \ldots, z_d) \mod 2,$$

since $P_{t+1,x}(z_1, \ldots, z_d, x, y) \equiv 0$. □

Since the form $Q^{(p)}_M$ lies in $\text{Sym}^{2p}(H_2(M; \mathbb{Z}))$ and since $Q^{(p)}_M \equiv 0 \mod 2$ for $2p > \text{rank}(H_2(M; \mathbb{Z}))$, we have

**Theorem 8.6.** Suppose that $M$ is a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{t,M}$ of degree $d$, where $\ell$ is odd. If $b^{+}_M \equiv 1 \mod 4$, or if $d > \text{rank}(H_2(M; \mathbb{Z}))$ then $q_{t,M} \equiv 0 \mod 2$. □

We now need to refer to a recent theorem of Y. Ruan [R] which, when combined with Theorem 1.5, will prove Theorem 1.6.

(8.7) **Theorem** (Ruan). Let $M$ be a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{t,M}$ of degree $d$, where $\ell$ is odd. Then the symmetric product $Q_M q_{t,M} \equiv 0 \mod 2$.

If $b^{+}_M \equiv 1 \mod 4$, then Theorem 8.7 follows as well from our Theorem 8.6. Furthermore, note that

$$Q_M Q^{(p)}_M = (p+1)Q^{(p+1)}_M \mod 2.$$ 

Also note that when $b^{+}_M$ is odd and $M$ is spin, the mod 2 intersection form on $H_2(M; \mathbb{Z})$ is just the standard form on the direct sum of an odd number, say $k$, hyperbolic pairs. Thus mod 2, $Q^{(j)}$ will take on nonzero values for $j \leq k$ and will be identically 0 (mod 2) for $j > k + 1$.

**Theorem 8.8.** Let $M$ be a simply connected spin 4-manifold with a Donaldson invariant $q_{t,M}$, $\ell$ odd and with $b^{+}_M \equiv 7 \mod 8$. Then, $q_{t,M} \equiv 0 \mod 2$.

**Proof.** If $b^{+}_M \equiv 7 \mod 8$, then the Donaldson invariant $q_{t,M}$, $\ell$ odd has degree $4p$ for some $p$. If $Q^{(p)}_M \equiv 0 \mod 2$, then Theorem 1.5 implies $q_{t,M} \equiv 0 \mod 2$. If $Q^{(p)}_M \not\equiv 0 \mod 2$, then our comments above imply that also
$Q_M^{(2p+1)} \neq 0 \mod 2$. Theorem 1.5 implies $q_{\ell, M} \equiv \epsilon_{\ell, M} Q_M^{(2p)} \mod 2$. Then by Ruan’s result (Theorem 8.7)

$$0 \equiv Q_M q_{\ell, M} \equiv \epsilon_{\ell, M} Q_M Q_M^{(2p)} = (2p + 1) \epsilon_{\ell, M} Q_M^{(2p+1)} \mod 2$$

so that $\epsilon_{\ell, M} = 0$ and again $q_{\ell, M} \equiv 0 \mod 2$. □

Theorem 1.6 now follows from Theorems 8.6 and 8.8.

In summary, the mod 2 Donaldson polynomials vanish except possibly when either $\ell$ is even or when $\ell$ is odd, $b^+_M \equiv 3 \mod 8$, and the degree $d \leq \text{rank}(H_2(M; \mathbb{Z}))$. In [DK, p. 417] it is pointed out that if $M$ is the $K3$ surface (so $b^+_M = 3$) then $q_{5, M} = Q_5^{(7)}$. Known examples lead to the following conjecture.

**Conjecture 8.9.** Let $M$ be a simply connected spin 4-manifold with a Donaldson invariant $q_{\ell, M}$, $\ell$ odd. Then $q_{\ell, M} \neq 0 \mod 2$ if and only if $b^+_e = 3$ and $\ell = 5$.

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**References**


**DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824**

_E-mail address_: ronfint@math.msu.edu

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92717**

_E-mail address_: rstern@math.uci.edu

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