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HOMOTOPY HYPERBOLIC 3-MANIFOLDS ARE VIRTUALLY HYPERBOLIC

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The main result of this paper is the only if part of

Theorem 0.1. *A closed irreducible* 3*-manifold* N *is homotopy equivalent to a hyperbolic* 3*-manifold if and only if* N *is finitely covered by a hyperbolic* 3*-manifold.*

Remark 0.2. The if direction is a well known, quick consequence of Mostow's Rigidity theorem. Here is the sketch. Let $p: M \to N$ be a finite regular covering map. Any covering translation of H^3 corresponding to an element of $\pi_1(N)$ is a lift of a covering transformation f of p, which by Mostow rigidity is homotopic to a unique isometry of M. It follows that $\pi_1(N) \cong \Gamma \subset \mathrm{Isom}(H^3)$ and H^3/Γ is a hyperbolic 3-manifold M'. Since M' and N are $K(\pi, 1)$'s, they are homotopy equivalent.

The proof of the only if direction is likewise a quick application of well-known results. Here is the sketch. If N is homotopy equivalent to M, then using the residual finiteness of $\pi_1(M)$ we can pass to a regular covering space M_1 of M which has a closed geodesic γ with an enormously thick embedded regular neighborhood U. Now lift the homotopy equivalence to $f_1: M_1 \to N_1$ where N_1 is the corresponding covering of N. Using the fact that the thurston norm equals the singular norm [to replace a singular torus by an embedded one in the same homology class in $f_1(U) - f_1(N(\gamma))$] and the observation that the homotopy equivalence keeps far away points of M_1 far away, it follows that in $f_1(U)$ we can find a curve with a thick collar W. The homotopy inverse g_1 is homotopic to a map which is a homeomorphism on W and on $N - W$ restricts to a π_1-injective degree-1 map. By Waldhausen g_1 is homotopic to a homeomorphism.

More details are provided in §1. Theorem 0.1 is used in §2 to reduce the general problem of homotopy equivalence implying homeomorphism for hyperbolic 3-manifolds to Conjecture 2.1. Other results related to the proof of Theorem 0.1 are stated in §2.

1. PROOF OF THEOREM 1.1

Notation 1.1. If $f: M \to N$ is a homotopy equivalence, let $g: N \to M$ be the homotopy inverse and $F: M \times I \to M$ be the homotopy of $g \circ f$ to id_M. Let

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$C > 2\,\mathrm{Sup}\{\mathrm{diam}\,\widetilde{F}(m \times I) \mid m \in M\}$, where $\widetilde{F}$ is a lift of F to the universal covering of M. $l(\gamma)$ denotes length, and $B(n, x) = \{z \in Z \mid d(x, z) \leq n\}$ where the space Z is clear from context. $N(X)$ denotes (thin) regular neighborhood, and $|E|$ denotes number of components of E.

Lemma 1.2. *If $f\colon M \to N$ is a homotopy equivalence, then $d(x, y) \geq C$ implies that $f(x) \cap f(y) = \varnothing$.* □

Lemma 1.3. *If M is a closed hyperbolic manifold, $n > 0$, then there exists a regular finite sheeted covering M_1 of M with injectivity radius $\geq n$.*

Proof. Let $p\colon (H^3, z) \to (M, x)$ the universal covering map. Let $d = \mathrm{diam}(M)$, and assume that $n > d$. Let $V = \{t \in p^{-1}(x) \mid d(z, t) < 4n\}$. Since $\pi_1(M)$ is residually finite [Ma], there exist regular coverings $q\colon (H^3, z) \to (M_1, y)$, $\pi\colon (M_1, y) \to (M, x)$ such that $p = \pi \circ q$ and $V \cap q^{-1}(y) = z$. To see this let $\{a_1, \dots, a_k\} = \{a \in \pi_1(M, x) \mid$ which lift to paths with the first end point z and the other in $V - z\}$. M_1 is a covering corresponding to a finite index normal subgroup which does not contain $\{a_1, \dots, a_k\}$. $q \mid B(2n, z)$ is an embedding, else there exists $w \in B(4n, z)$ such that $q(w) = q(z)$. Since M_1 is regular, $q \mid B(2n, z')$ is an embedding for each $z' \in p^{-1}(x)$. Finally for all $s \in H^3$, there exists $z' \in p^{-1}(x)$ such that $B(n, s) \subset B(2n, z')$. Thus $q \mid B(n, s)$ is an embedding. □

If γ is a closed geodesic in a hyperbolic 3-manifold, then the *tube radius* of $\gamma = \mathrm{Sup}\{$radii of embedded hyperbolic tubes about $\gamma\} = \frac{1}{2}\min\{d(\gamma, \delta) \mid \delta$ is a distinct covering translate of γ in $H^3\}$.

Lemma 1.4. *If M_1 is a closed hyperbolic manifold with injectivity radius n, then there exists a geodesic γ in M_1 with tube radius $> n/2$.*

Proof. Let γ be a shortest geodesic in M_1. Let γ_1, γ_2 be distinct lifts of γ in H^3. If $d(\gamma_1, \gamma_2) \leq n = \frac{1}{2}l(\gamma)$, then there exist $x_i \in \gamma_i$ which are covering translates of each other such that $d(x_1, x_2) < l(\gamma)$, which implies the existence of a geodesic shorter than γ. □

Lemma 1.5. *If M is a closed oriented hyperbolic 3-manifold and $f\colon M \to N$ is a homotopy equivalence such that N is irreducible and M has a geodesic γ with tube radius $> 4C$, then f is homotopic to a homeomorphism.*

Proof. For $0 < i \leq 4$ let S_i be the torus in M at distance iC from γ, let V_i be the solid torus in M bounded by S_i, and let $K = f(S_2)$ and $J = N(K) \cup ($components of $N - K$ disjoint from $f(S_1 \cup S_3))$. Let V_0 also denote γ.

Claim 1. (0) $f^{-1}(J) \subset V_3 - \overset{\circ}{V}_1$ and $g(J) \subset \overset{\circ}{V}_4 - V_0$.

(i) $|\partial J| = 2$, one component of which bounds a region disjoint from J containing $f(S_1)$ and the other component bounds a region disjoint from J containing $f(S_3)$.

(ii) J is irreducible.

(iii) $[K]$ generates $H_2(J) = Z$.

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Proof of Claim 1. (0) $K \cap (f(V_1) \cup f(M - \mathring{V}_3)) = \varnothing$ by Lemma 1.2. If R is a component of $\partial N(K)$, then $g(R) \subset V_3$ and, hence, is homologically trivial, so R bounds in N since f is a homotopy equivalence. Each of $f(M - V_3)$, $f(V_1)$ lies in a unique component of $N - K$ and, hence, in a unique component of $N - J$, so $f^{-1}(J) \subset \mathring{V}_3 - V_1$, $g(J) \subset \mathring{V}_4 - V_0$ now follow from Lemma 1.2.

(i) If $x \in f(\gamma)$, $y \in f(M - V_4)$, and $\alpha \subset N - K$ is a path from x to y, then $\deg f = 1$ and choice of C implies that (after possibly a tiny homotopy of f) some component β of $f^{-1}(\alpha)$ is a path from some element of $f^{-1}(x) \in V_1$ to some element of $f^{-1}(y) \in M - V_3$ disjoint from S_2.

(ii) If there exists an essential 2-sphere P in J, the irreducibility of N would imply P bounded a ball containing $f(V_1)$ or $f(M - V_3)$. This would contradict the π_1-injectivity of f.

(iii) $g \circ f \mid S_2$ is homotopic to id in $V_3 - V_1 \subset V_4 - V_0$, and $[S_2]$ generates $H_2(V_4 - V_0)$; therefore, $[f(S_2)] = [K]$ is primitive in $H_2(J)$. Since each closed curve in J can be homotoped out of J, J contains no nonseparating surface, so by (i) $H_2(J) = Z$. □

Claim 2. J contains a homologically nontrivial torus T which bounds in N a solid torus W containing $f(\gamma)$. Finally $g : T \to M - V_0$ and $in : T \to N - \mathring{W}$ are π_1-injective.

Proof of Claim 2. Since the thurston norm on $H_2(J)$ equals the singular norm on $H_2(J)$ [G Corollary 6.18] and (iii) there exists an embedded nonbounding torus T in J such that $[T] = [K] \in H_2(J)$. Since $g \mid T$ is not π_1- injective as a map into V_4, it follows that T is compressible in N. A compressible torus in an irreducible 3-manifold bounds either a solid torus or lives in a ball. π_1-injectivity of f precludes the latter, and $Z \neq \pi_1(M - V_3)$ implies that the solid torus W contains $f(\gamma)$. The π_1-injectivity of $g \mid T$ follows from the facts that $g \mid T$ is π_1-injective as a map into $V_4 - V_0$ (since each singular sphere in $V_4 - V_0$ is homologically trivial and $[g(T)] = [S_2]$) and S_4 is incompressible in $M - \mathring{V}_4$. Finally if T is compressed in $N - \mathring{W}$, then an application of the loop theorem would imply that either some power of $f(\gamma)$ is homotopically trivial in N or $N = S^2 \times S^1$. □

Claim 3. Let $Q = N - \mathring{W}$. g is homotopic to a map $h : N \to M$ such that $h \mid T$ is a homeomorphism onto S_2, $h \mid W$ is degree-1 onto V_2, $h \mid Q$ is degree-1 onto $M - V_2$, and $h \mid W$ is π_1-injective into V_2.

Proof of Claim 3. By Claim 1 the map on T obtained by first applying g and then projecting to S_2 (in $V_4 - V_0$) is a degree-1 map, so by [K] or [BE] it is homotopic to a homeomorphism. Therefore, to obtain h, first homotop g to g' via a homotopy supported in a tiny neighborhood of T so that $g' \mid T$ is a homeomorphism, $g'(W) \subset V_4$, and $g'(Q) \subset M - V_0$. Applying the natural retractions of V_4 to V_2 and $M - V_0$ to $M - \mathring{V}_2$, to stuff the guts spilling out, we obtain h. The degree-1 conclusions follow from the fact that g is degree-1. $h \mid W$ is obviously π_1-injective. □

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Claim 4. Q is irreducible and π_1-injects into $f(M - \mathring{V}_1)$.

Proof of Claim 4. The irreducibility of Q follows from the irreducibility of N and the fact that $f(\gamma)$ is homotopically nontrivial. If D is a singular disc in $f(M - \mathring{V}_1)$ such that $\partial D \subset T$, then $g(D) \subset M - V_0$ and $g \mid T : T \to M - V_0$ is π_1-injective implies that ∂D is homotopically trivial in T. Therefore, T π_1-injects into $f(M - \mathring{V}_1)$ and, since T is incompressible in Q, Claim 4 follows. □

Claim 5. $h \mid Q$ is π_1-injective into $M - V_2$.

Proof of Claim 5. Let δ be a closed curve in Q. Let α (resp. β) be the curve $h(\delta)$ (resp. $g(\delta)$). By construction $\alpha \subset M - V_2$ and $\beta \subset M - V_0$. Furthermore α is homotopic to β in $M - V_0$. $\deg f = 1$ implies that $f^{-1}(\delta)$ contains a curve $\epsilon \in M - V_1$ such that $f \mid \epsilon$ maps with nonzero degree to δ. ϵ is homotopic to a nonzero multiple of β and, hence, a nonzero multiple of α in $M - V_0$. Therefore, if $h(\delta)$ is homotopically trivial in $M - V_2$, then ϵ is homotopically trivial in $M - V_1$, so δ is homotopically trivial in $f(M - V_1)$ [$\pi_1(f(M - \mathring{V}_1))$ being torsion free] and so δ is homotopically trivial in Q by Claim 4. □

Claim 6. h is homotopic to a homeomorphism.

Proof of Claim 6. By Waldhausen [He] $h : (Q, \partial Q) \to (M - \mathring{V}_2, \partial V_2)$ (resp. $h : (W, \partial W) \to (V_2, \partial V_2)$) is homotopic to a homeomorphism via a homotopy fixed on the boundary. □

Remarks. If γ has a larger tube radius, e.g. $12C$, then Claims 4–5 can be replaced by the observation that the homotopy equivalence splits along S_6 and T_6 to ones on V_6 and W and $M - \mathring{N}(V_6)$ and Q. Hint: there is an embedded torus T_i near $f(S_i)$ for i=2, 6, 10 which bounds a solid torus; furthermore, T_2, T_{10} bound a product homeomorphic to Torus × I. In this setting we now have enough room to homotop f so that $f(S_6) = T_6$. I thank Mike Freedman for suggesting this simplification.

Proof of Theorem 1.1. By Lemmas 1.3, 1.4, M has a finite covering space M_1 with a geodesic of tube radius $> 4C$. Let N_1 be the associated covering space of N. By [MSY] or [D] N_1 is irreducible. Now apply Lemma 1.5. □

2. RELATED RESULTS AND A CONJECTURE

Conjecture 2.1. *Let G and H be isomorphic finitely generated groups such that* $G \subset \mathrm{PSL}(2, C) \subset \mathrm{Homeo}(B^3)$ *and* $H \subset \mathrm{Homeo}(B^3)$. *Suppose further:*

(a) *G and H act freely on $\mathring{B}^3$ with closed 3-manifold quotients;*
(b) *$H \mid S^2 = G \mid S^2$; and*
(c) *there exist subgroups H', G' of finite index in H and G such that $H' \mid B^3 = G' \mid B^3$.*

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Then H is conjugate to G in Homeo B^3.

Remark 2.2. Mostow rigidity and Conjecture 2.1 imply the conjecture "If $f: M \to N$ is a homotopy equivalence where M is hyperbolic and N is irreducible, then f is homotopic to a homeomorphism." For if N is an irreducible homotopy hyperbolic 3-manifold, then by Theorem 0.1 it has a regular finite sheeted covering space N_1 which is a hyperbolic 3-manifold. The well-known argument of Remark 0.2 shows that there exists a hyperbolic 3-manifold M' homotopy equivalent to N such that M is finitely covered by N_1 and if G (resp. H) is the group of covering transformations of H^3 corresponding to M' (resp. N), and extended to act on B^3, then G and H satisfy (a), (b), and (c), where H' and G' are the groups associated to N_1. The conclusion of Conjecture 2.1 implies that N is homeomorphic to M', and another application of Mostow rigidity shows that $M = M'$ and that the homotopy equivalence f is homotopic to a homeomorphism.

Theorem 2.3. *Let $f: M \to N$ be a homotopy equivalence between closed irreducible* 3*-manifolds with residually finite fundamental group. Suppose further that there exists an element $\gamma \in \pi_1(M)$ which generates a maximal abelian subgroup $\langle\gamma\rangle$ whose associated covering space $M_\gamma = D^2 \times S^1$; then M and N have homeomorphic finite sheeted coverings.*

Proof. Fix any Riemannian metric on M. Let V_i, $i = 0, 1, \dots, 4$, be parallel solid tori in M_γ containing γ as a core with $\partial V_i = S_i$ at least C distance apart. It is well known (to algebraists, see [L]) that maximal abelian subgroups are separable (so given $a_1, \dots, a_n \in \pi_1(M) - \langle\gamma\rangle$ there exists a subgroup of finite index containing $\langle\gamma\rangle$ but missing $a_1, \dots, a_n$). An argument related to the one of Lemma 1.3 shows that there exists a finite covering M_1 of M such that M_1 is covered by M_γ and the projection of V_4 to M_1 is an embedding. We abuse notation by continuing to call the image in M_1 of V_i by the same name. Let N_1 be the associated finite covering of N. Again by [MSY] or [D] N_1 is irreducible. The argument of Lemma 1.5 now shows that M_1 and N_1 are homeomorphic.

Combining Waldhausen [W] with the idea of the proof of Lemma 1.5 we obtain.

Theorem 2.4. *If $f: M \to M$ is a homeomorphism homotopic to the identity and M is a hyperbolic* 3*-manifold, then there exists a finite covering space of M such that a lift of f is isotopic to the identity.* □

Remark 2.5. Actually M need only satisfy the hypothesis of Theorem 2.3.

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