HOMOTOPY HYPERBOLIC 3-MANIFOLDS ARE VIRTUALLY HYPERBOLIC

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The main result of this paper is the only if part of

Theorem 0.1. A closed irreducible 3-manifold $N$ is homotopy equivalent to a hyperbolic 3-manifold if and only if $N$ is finitely covered by a hyperbolic 3-manifold.

Remark 0.2. The if direction is a well known, quick consequence of Mostow's Rigidity theorem. Here is the sketch. Let $p:M \rightarrow N$ be a finite regular covering map. Any covering translation of $H^3$ corresponding to an element of $\pi_1(N)$ is a lift of a covering transformation $f$ of $p$, which by Mostow rigidity is homotopic to a unique isometry of $M$. It follows that $\pi_1(N) \cong \Gamma \subset \text{Isom}(H^3)$ and $H^3/\Gamma$ is a hyperbolic 3-manifold $M'$. Since $M'$ and $N$ are $K(\pi, 1)$'s, they are homotopy equivalent.

The proof of the only if direction is likewise a quick application of well-known results. Here is the sketch. If $N$ is homotopy equivalent to $M$, then using the residual finiteness of $\pi_1(M)$ we can pass to a regular covering space $M_1$ of $M$ which has a closed geodesic $\gamma$ with an enormously thick embedded regular neighborhood $U$. Now lift the homotopy equivalence to $f_1:M_1 \rightarrow N_1$ where $N_1$ is the corresponding covering of $N$. Using the fact that the thurston norm equals the singular norm [to replace a singular torus by an embedded one in the same homology class in $f_1(U) - f_1(N(\gamma))$] and the observation that the homotopy equivalence keeps far away points of $M_1$ far away, it follows that in $f_1(U)$ we can find a curve with a thick collar $W$. The homotopy inverse $g_1$ is homotopic to a map which is a homeomorphism on $W$ and on $N-W$ restricts to a $\pi_1$-injective degree-1 map. By Waldhausen $g_1$ is homotopic to a homeomorphism.

More details are provided in §1. Theorem 0.1 is used in §2 to reduce the general problem of homotopy equivalence implying homeomorphism for hyperbolic 3-manifolds to Conjecture 2.1. Other results related to the proof of Theorem 0.1 are stated in §2.

1. PROOF OF THEOREM 1.1

Notation 1.1. If $f:M \rightarrow N$ is a homotopy equivalence, let $g:N \rightarrow M$ be the homotopy inverse and $F:M \times I \rightarrow M$ be the homotopy of $g \circ f$ to $\text{id}_M$. Let

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\[ C > 2 \text{Sup}\{\text{diam} \tilde{F}(m \times I) \mid m \in M\}, \] where \( \tilde{F} \) is a lift of \( F \) to the universal covering of \( M \). \( l(\gamma) \) denotes length, and \( B(n, x) = \{z \in Z \mid d(x, z) \leq n\} \) where the space \( Z \) is clear from context. \( N(X) \) denotes (thin) regular neighborhood, and \( |E| \) denotes number of components of \( E \).

**Lemma 1.2.** If \( f: M \to N \) is a homotopy equivalence, then \( d(x, y) \geq C \) implies that \( f(x) \cap f(y) = \emptyset \). \( \square \)

**Lemma 1.3.** If \( M \) is a closed hyperbolic manifold, \( n > 0 \), then there exists a regular finite sheeted covering \( M_1 \) of \( M \) with injectivity radius \( \geq n \).

**Proof.** Let \( p: (H^3, z) \to (M, x) \) the universal covering map. Let \( d = \text{diam}(M) \), and assume that \( n > d \). Let \( V = \{t \in p^{-1}(x) \mid d(z, t) < 4n\} \). Since \( \pi_1(M) \) is residually finite [Ma], there exist regular coverings \( q: (H^3, z) \to (M_1, y) \), \( \pi: (M_1, y) \to (M, x) \) such that \( p = \pi \circ q \) and \( V \cap q^{-1}(y) = z \). To see this let \( \{a_1, \ldots, a_k\} = \{a \in \pi_1(M, x) \mid \text{which lift to paths with the first end point} \ z \) and the other in \( V - z\} \). \( M_1 \) is a covering corresponding to a finite index normal subgroup which does not contain \( \{a_1, \ldots, a_k\} \). \( q|B(2n, z) \) is an embedding, else there exists \( w \in B(4n, z) \) such that \( q(w) = q(z) \). Since \( M_1 \) is regular, \( q|B(2n, z') \) is an embedding for each \( z' \in p^{-1}(x) \). Finally for all \( s \in H^3 \), there exists \( z' \in p^{-1}(x) \) such that \( B(n, s) \subset B(2n, z') \). Thus \( q|B(n, s) \) is an embedding. \( \square \)

If \( \gamma \) is a closed geodesic in a hyperbolic 3-manifold, then the **tube radius** of \( \gamma = \text{Sup}\{\text{radii of embedded hyperbolic tubes about} \ \gamma\} = \frac{1}{2} \min\{d(\gamma, \delta) \mid \delta \) is a distinct covering translate of \( \gamma \) in \( H^3\} \).

**Lemma 1.4.** If \( M_1 \) is a closed hyperbolic manifold with injectivity radius \( n \), then there exists a geodesic \( \gamma \) in \( M_1 \) with tube radius \( \geq n/2 \).

**Proof.** Let \( \gamma \) be a shortest geodesic in \( M_1 \). Let \( \gamma_1, \gamma_2 \) be distinct lifts of \( \gamma \) in \( H^3 \). If \( d(\gamma_1, \gamma_2) \leq n = \frac{1}{2} l(\gamma) \), then there exist \( x_1, x_2 \) which are covering translates of each other such that \( d(x_1, x_2) < l(\gamma) \), which implies the existence of a geodesic shorter than \( \gamma \). \( \square \)

**Lemma 1.5.** If \( M \) is a closed oriented hyperbolic 3-manifold and \( f: M \to N \) is a homotopy equivalence such that \( N \) is irreducible and \( M \) has a geodesic \( \gamma \) with tube radius \( > 4C \), then \( f \) is homotopic to a homeomorphism.

**Proof.** For \( 0 < i \leq 4 \) let \( S_i \) be the torus in \( M \) at distance \( iC \) from \( \gamma \), let \( V_i \) be the solid torus in \( M \) bounded by \( S_i \), and let \( K = f(S_2) \) and \( J = N(K) \cup (\text{components of} \ N - K \text{ disjoint from} \ f(S_1 \cup S_3)) \). Let \( V_0 \) also denote \( \gamma \).

**Claim 1.** (0) \( f^{-1}(J) \subset V_3 - \overset{\circ}{V_1} \) and \( g(J) \subset \overset{\circ}{V_4} - V_0 \).

(i) \( |\partial J| = 2 \), one component of which bounds a region disjoint from \( J \) containing \( f(S_1) \) and the other component bounds a region disjoint from \( J \) containing \( f(S_3) \).

(ii) \( J \) is irreducible.

(iii) \( |K| \) generates \( H_2(J) = Z \).
Proof of Claim 1. (0) \( K \cap (f(V_1) \cup f(M - \mathbf{\hat{V}}_3)) = \emptyset \) by Lemma 1.2. If \( R \) is a component of \( \partial N(K) \), then \( g(R) \subset V_3 \) and, hence, is homologically trivial, so \( R \) bounds in \( N \) since \( f \) is a homotopy equivalence. Each of \( f(M - V_3) \), \( f(V_1) \) lies in a unique component of \( N - K \) and, hence, in a unique component of \( N - J \), so \( f^{-1}(J) \subset \mathbf{\hat{V}}_3 - V_1 \). \( g(J) \subset \mathbf{\hat{V}}_4 - V_0 \). Now follow from Lemma 1.2.

(i) If \( x \in f(\gamma) \), \( y \in f(M - V_4) \), and \( \alpha \subset N - K \) is a path from \( x \) to \( y \), then \( \deg f = 1 \) and choice of \( C \) implies that (after possibly a tiny homotopy of \( f \)) some component \( \beta \) of \( f^{-1}(\alpha) \) is a path from some element of \( f^{-1}(x) \in V_1 \) to some element of \( f^{-1}(y) \in M - V_3 \) disjoint from \( S_2 \).

(ii) If there exists an essential 2-sphere \( P \) in \( J \), the irreducibility of \( N \) would imply \( P \) bounded a ball containing \( f(V_1) \) or \( f(M - V_3) \). This would contradict the \( \pi_1 \)-injectivity of \( f \).

(iii) \( g \circ f \mid S_2 \) is homotopic to id in \( V_3 - V_1 \subset V_4 - V_0 \), and \( [S_2] \) generates \( H_2(V_4 - V_0) \); therefore, \( [f(S_2)] = [K] \) is primitive in \( H_2(J) \). Since each closed curve in \( J \) can be homotoped out of \( J \), \( J \) contains no nonseparating surface, so by (i) \( H_2(J) = Z \). \( \square \)

Claim 2. \( J \) contains a homologically nontrivial torus \( T \) which bounds in \( N \) a solid torus \( W \) containing \( f(\gamma) \). Finally \( g : T \to M - V_0 \) and \( \text{id} : T \to N - \mathbf{\hat{W}} \) are \( \pi_1 \)-injective.

Proof of Claim 2. Since the thurston norm on \( H_2(J) \) equals the singular norm on \( H_2(J) \) \( [G \text{ Corollary 6.18}] \) and (iii) there exists an embedded nonbounding torus \( T \) in \( J \) such that \( [T] = [K] \subset H_2(J) \). Since \( g \mid T \) is not \( \pi_1 \)-injective as a map into \( V_4 \), it follows that \( T \) is compressible in \( N \). A compressible torus in an irreducible 3-manifold bounds either a solid torus or lives in a ball. \( \pi_1 \)-injectivity of \( f \) precludes the latter, and \( Z \neq \pi_1(M - V_3) \) implies that the solid torus \( W \) contains \( f(\gamma) \). The \( \pi_1 \)-injectivity of \( g \mid T \) follows from the facts that \( g \mid T \) is \( \pi_1 \)-injective as a map into \( V_4 - V_0 \) (since each singular sphere in \( V_4 - V_0 \) is homologically trivial and \( [g(T)] = [S_2] \) and \( S_4 \) is incompressible in \( M - \mathbf{\hat{V}}_4 \). Finally if \( T \) is compressed in \( N - \mathbf{\hat{W}} \), then an application of the loop theorem would imply that either some power of \( f(\gamma) \) is homotopically trivial in \( N \) or \( N = S^2 \times S^1 \). \( \square \)

Claim 3. Let \( Q = N - \mathbf{\hat{W}} \). \( g \) is homotopic to a map \( h : N \to M \) such that \( h \mid T \) is a homeomorphism onto \( S_2 \), \( h \mid W \) is degree-1 onto \( V_2 \), \( h \mid Q \) is degree-1 onto \( M - V_2 \), and \( h \mid W \) is \( \pi_1 \)-injective into \( V_2 \).

Proof of Claim 3. By Claim 1 the map on \( T \) obtained by first applying \( g \) and then projecting to \( S_2 \) (in \( V_4 - V_0 \)) is a degree-1 map, so by \([K]\) or \([BE]\) it is homotopic to a homeomorphism. Therefore, to obtain \( h \), first homotope \( g \) to \( g' \) via a homotopy supported in a tiny neighborhood of \( T \) so that \( g' \mid T \) is a homeomorphism, \( g'(W) \subset V_4 \), and \( g'(Q) \subset M - V_0 \). Applying the natural retractions of \( V_4 \) to \( V_2 \) and \( M - V_0 \) to \( M - \mathbf{\hat{V}}_2 \), to stuff the guts spilling out, we obtain \( h \). The degree-1 conclusions follow from the fact that \( g \) is degree-1. \( h \mid W \) is obviously \( \pi_1 \)-injective. \( \square \)
Claim 4. \( Q \) is irreducible and \( \pi_1 \)-injects into \( f(M - \mathring{V}_1) \).

Proof of Claim 4. The irreducibility of \( Q \) follows from the irreducibility of \( N \) and the fact that \( f(\gamma) \) is homotopically nontrivial. If \( D \) is a singular disc in \( f(M - \mathring{V}_1) \) such that \( \partial D \subset T \), then \( g(D) \subset M - V_0 \) and \( g | T : T \to M - V_0 \) is \( \pi_1 \)-injective implies that \( \partial D \) is homotopically trivial in \( T \). Therefore, \( T \pi_1 \)-injects into \( f(M - \mathring{V}_1) \) and, since \( T \) is incompressible in \( Q \), Claim 4 follows.

Claim 5. \( h | Q \) is \( \pi_1 \)-injective into \( M - V_2 \).

Proof of Claim 5. Let \( \delta \) be a closed curve in \( Q \). Let \( \alpha \) (resp. \( \beta \)) be the curve \( h(\delta) \) (resp. \( g(\delta) \)). By construction \( \alpha \subset M - V_2 \) and \( \beta \subset M - V_0 \). Furthermore \( \alpha \) is homotopic to \( \beta \) in \( M - V_0 \). \( \deg f = 1 \) implies that \( f^{-1}(\delta) \) contains a curve \( \epsilon \in M - V_1 \) such that \( f | \epsilon \) maps with nonzero degree to \( \delta \). \( \epsilon \) is homotopic to a nonzero multiple of \( \beta \) and, hence, a nonzero multiple of \( \alpha \) in \( M - V_0 \). Therefore, if \( h(\delta) \) is homotopically trivial in \( M - V_2 \), then \( \epsilon \) is homotopically trivial in \( M - V_1 \), so \( \delta \) is homotopically trivial in \( f(M - V_1) \) [\( \pi_1(f(M - \mathring{V}_1)) \) being torsion free] and so \( \delta \) is homotopically trivial in \( Q \) by Claim 4.

Claim 6. \( h \) is homotopic to a homeomorphism.

Proof of Claim 6. By Waldhausen [He] \( h : (Q, \partial Q) \to (M - \mathring{V}_2, \partial V_2) \) (resp. \( h : (W, \partial W) \to (V_2, \partial V_2) \)) is homotopic to a homeomorphism via a homotopy fixed on the boundary.

Remarks. If \( \gamma \) has a larger tube radius, e.g. \( 12C \), then Claims 4–5 can be replaced by the observation that the homotopy equivalence splits along \( S_6 \) and \( T_6 \) to ones on \( V_6 \) and \( W \) and \( M - \mathring{N}(V_6) \) and \( Q \). Hint: there is an embedded torus \( T_i \) near \( f(S_i) \) for \( i = 2, 6, 10 \) which bounds a solid torus; furthermore, \( T_2, T_{10} \) bound a product homeomorphic to \( \text{Torus} \times I \). In this setting we now have enough room to homotop \( f \) so that \( f(S_6) = T_6 \). I thank Mike Freedman for suggesting this simplification.

Proof of Theorem 1.1. By Lemmas 1.3, 1.4, \( M \) has a finite covering space \( M_1 \) with a geodesic of tube radius \( > 4C \). Let \( N_1 \) be the associated covering space of \( N \). By [MSY] or [D] \( N_1 \) is irreducible. Now apply Lemma 1.5.

2. Related results and a conjecture

Conjecture 2.1. Let \( G \) and \( H \) be isomorphic finitely generated groups such that \( G \subset \text{PSL}(2, C) \subset \text{Homeo}(B^3) \) and \( H \subset \text{Homeo}(B^3) \). Suppose further:

(a) \( G \) and \( H \) act freely on \( \mathring{B}^3 \) with closed 3-manifold quotients;
(b) \( H | S^2 = G | S^2 \); and
(c) there exist subgroups \( H', G' \) of finite index in \( H \) and \( G \) such that \( H' | B^3 = G' | B^3 \).
Then $H$ is conjugate to $G$ in $\text{Homeo} B^3$.

Remark 2.2. Mostow rigidity and Conjecture 2.1 imply the conjecture “If $f : M \rightarrow N$ is a homotopy equivalence where $M$ is hyperbolic and $N$ is reducible, then $f$ is homotopic to a homeomorphism.” For if $N$ is an irreducible homotopy hyperbolic 3-manifold, then by Theorem 0.1 it has a regular finite sheeted covering space $N_1$ which is a hyperbolic 3-manifold. The well-known argument of Remark 0.2 shows that there exists a hyperbolic 3-manifold $M'$ homotopy equivalent to $N$ such that $M$ is finitely covered by $N_1$ and if $G$ (resp. $H$) is the group of covering transformations of $H^3$ corresponding to $M'$ (resp. $N$), and extended to act on $B^3$, then $G$ and $H$ satisfy (a), (b), and (c), where $H'$ and $G'$ are the groups associated to $N_1$. The conclusion of Conjecture 2.1 implies that $N$ is homeomorphic to $M'$, and another application of Mostow rigidity shows that $M = M'$ and that the homotopy equivalence $f$ is homotopic to a homeomorphism.

Theorem 2.3. Let $f : M \rightarrow N$ be a homotopy equivalence between closed irreducible 3-manifolds with residually finite fundamental group. Suppose further that there exists an element $\gamma \in \pi_1(M)$ which generates a maximal abelian subgroup $\langle \gamma \rangle$ whose associated covering space $M_\gamma = D^2 \times S^1$; then $M$ and $N$ have homeomorphic finite sheeted coverings.

Proof. Fix any Riemannian metric on $M$. Let $V_i, i = 0, 1, \ldots, 4$, be parallel solid tori in $M_\gamma$ containing $\gamma$ as a core with $\partial V_i = S_i$ at least $C$ distance apart. It is well known (to algebraists, see [L]) that maximal abelian subgroups are separable (so given $a_1, \ldots, a_n \in \pi_1(M) - \langle \gamma \rangle$ there exists a subgroup of finite index containing $\langle \gamma \rangle$ but missing $a_1, \ldots, a_n$). An argument related to the one of Lemma 1.3 shows that there exists a finite covering $M_1$ of $M$ such that $M_1$ is covered by $M_\gamma$ and the projection of $V_i$ to $M_1$ is an embedding. We abuse notation by continuing to call the image in $M_1$ of $V_i$ by the same name. Let $N_1$ be the associated finite covering of $N$. Again by [MSY] or [D] $N_1$ is irreducible. The argument of Lemma 1.5 now shows that $M_1$ and $N_1$ are homeomorphic.

Combining Waldhausen [W] with the idea of the proof of Lemma 1.5 we obtain.

Theorem 2.4. If $f : M \rightarrow M$ is a homeomorphism homotopic to the identity and $M$ is a hyperbolic 3-manifold, then there exists a finite covering space of $M$ such that a lift of $f$ is isotopic to the identity. □

Remark 2.5. Actually $M$ need only satisfy the hypothesis of Theorem 2.3.

References


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