HOMOTOPY HYPERBOLIC 3-MANIFOLDS
ARE VIRTUALLY HYPERBOLIC

DAVID GABAI

The main result of this paper is the only if part of

**Theorem 0.1.** A closed irreducible 3-manifold \( N \) is homotopy equivalent to a hyperbolic 3-manifold if and only if \( N \) is finitely covered by a hyperbolic 3-manifold.

**Remark 0.2.** The if direction is a well known, quick consequence of Mostow's Rigidity theorem. Here is the sketch. Let \( p: M \rightarrow N \) be a finite regular covering map. Any covering translation of \( H^3 \) corresponding to an element of \( \pi_1(N) \) is a lift of a covering transformation \( f \) of \( p \), which by Mostow rigidity is homotopic to a unique isometry of \( M \). It follows that \( \pi_1(N) \cong \Gamma \subset \text{Isom}(H^3) \) and \( H^3/\Gamma \) is a hyperbolic 3-manifold \( M' \). Since \( M' \) and \( N \) are \( K(\pi, 1) 's, they are homotopy equivalent.

The proof of the only if direction is likewise a quick application of well-known results. Here is the sketch. If \( N \) is homotopy equivalent to \( M \), then using the residual finiteness of \( \pi_1(M) \) we can pass to a regular covering space \( M_1 \) of \( M \) which has a closed geodesic \( \gamma \) with an enormously thick embedded regular neighborhood \( U \). Now lift the homotopy equivalence to \( f_1: M_1 \rightarrow N_1 \) where \( N_1 \) is the corresponding covering of \( N \). Using the fact that the thurston norm equals the singular norm [to replace a singular torus by an embedded one in the same homology class in \( f_1(U) - f_1(N(\gamma)) \) and the observation that the homotopy equivalence keeps far away points of \( M_1 \) far away, it follows that in \( f_1(U) \) we can find a curve with a thick collar \( W \). The homotopy inverse \( g_1 \) is homotopic to a map which is a homeomorphism on \( W \) and on \( N - W \) restricts to a \( \pi_1 \)-injective degree-1 map. By Waldhausen \( g_1 \) is homotopic to a homeomorphism.

More details are provided in §1. Theorem 0.1 is used in §2 to reduce the general problem of homotopy equivalence implying homeomorphism for hyperbolic 3-manifolds to Conjecture 2.1. Other results related to the proof of Theorem 0.1 are stated in §2.

1. PROOF OF THEOREM 1.1

**Notation 1.1.** If \( f: M \rightarrow N \) is a homotopy equivalence, let \( g: N \rightarrow M \) be the homotopy inverse and \( F: M \times I \rightarrow M \) be the homotopy of \( g \circ f \) to \( \text{id}_M \).
Lemma 1.2. If \( f: M \to N \) is a homotopy equivalence, then \( d(x, y) \geq C \) implies that \( f(x) \cap f(y) = \emptyset \). □

Lemma 1.3. If \( M \) is a closed hyperbolic manifold, \( n > 0 \), then there exists a regular finite sheeted covering \( M_1 \) of \( M \) with injectivity radius \( \geq n \).

Proof. Let \( p: (H^3, z) \to (M, x) \) the universal covering map. Let \( d = \text{diam}(M) \), and assume that \( n > d \). Let \( V = \{ t \in p^{-1}(x) \mid d(z, t) < 4n \} \). Since \( \pi_1(M) \) is residually finite [Ma], there exist regular coverings \( q: (H^3, z) \to (M_1, y) \), \( \pi: (M_1, y) \to (M, x) \) such that \( p = \pi \circ q \) and \( V \cap q^{-1}(y) = \emptyset \). To see this let \( \{ a_1, \ldots, a_k \} = \{ a \in \pi_1(M, x) \mid \text{which lift to paths with the first end point} \ z \text{and the other in} \ V - z \} \). \( M_1 \) is a covering corresponding to a finite index normal subgroup which does not contain \( \{ a_1, \ldots, a_k \} \). \( q|B(2n, z) \) is an embedding, else there exists \( w \in B(4n, z) \) such that \( q(w) = q(z) \). Since \( M_1 \) is regular, \( q|B(2n, z') \) is an embedding for each \( z' \in p^{-1}(x) \). Finally for all \( s \in H^3 \), there exists \( z' \in p^{-1}(x) \) such that \( B(n, s) \subseteq B(2n, z') \). Thus \( q|B(n, s) \) is an embedding. □

C > 2 \( \text{Sup}\{ \text{diam} \tilde{F}(m \times I) \mid m \in M \} \), where \( \tilde{F} \) is a lift of \( F \) to the universal covering of \( M \). \( l(\gamma) \) denotes length, and \( B(n, x) = \{ z \in Z \mid d(x, z) \leq n \} \) where the space \( Z \) is clear from context. \( N(X) \) denotes (thin) regular neighborhood, and \( |E| \) denotes number of components of \( E \).

Lemma 1.4. If \( M_1 \) is a closed hyperbolic manifold with injectivity radius \( n \), then there exists a geodesic \( \gamma \) in \( M_1 \) with tube radius \( > n/2 \).

Proof. Let \( \gamma = \text{Sup}\{ \text{radii of embedded hyperbolic tubes about} \ \gamma \} = \frac{1}{2} \min \{ d(\gamma, \delta) \mid \delta \) is a distinct covering translate of \( \gamma \) in \( H^3 \} \).

Lemma 1.5. If \( M \) is a closed oriented hyperbolic 3-manifold and \( f: M \to N \) is a homotopy equivalence such that \( N \) is irreducible and \( M \) has a geodesic \( \gamma \) with tube radius \( > 4C \), then \( f \) is homotopic to a homeomorphism.

Proof. For \( 0 < i \leq 4 \) let \( S_i \) be the torus in \( M \) at distance \( iC \) from \( \gamma \), let \( V_i \) be the solid torus in \( M \) bounded by \( S_i \), and let \( K = f(S_2) \) and \( J = N(K) \cup (\text{components of} \ N - K \text{ disjoint from} \ f(S_1 \cup S_3)) \). Let \( V_0 \) also denote \( \gamma \).

Claim 1. (0) \( f^{-1}(J) \subseteq V_3 - V_1 \) and \( g(J) \subseteq \bar{V}_4 - V_0 \).

(i) \( |\partial J| = 2 \), one component of which bounds a region disjoint from \( J \) containing \( f(S_1) \) and the other component bounds a region disjoint from \( J \) containing \( f(S_3) \).

(ii) \( J \) is irreducible.

(iii) \( [K] \) generates \( H_2(J) = Z \).
Proof of Claim 1. (0) $K \cap (f(V_1) \cup f(M - \mathring{V}_3)) = \emptyset$ by Lemma 1.2. If $R$ is a component of $\partial N(K)$, then $g(R) \subset V_3$ and, hence, is homologically trivial, so $R$ bounds in $N$ since $f$ is a homotopy equivalence. Each of $f(M - \mathring{V}_3), f(V_1)$ lies in a unique component of $N - K$ and, hence, in a unique component of $N - J$, so $f^{-1}(J) \subset \mathring{V}_3 - V_1, g(J) \subset \mathring{V}_4 - V_0$ now follow from Lemma 1.2.

(i) If $x \in f(\gamma), y \in f(M - \mathring{V}_4)$, and $\alpha \subset N - K$ is a path from $x$ to $y$, then $\deg f = 1$ and choice of $C$ implies that (after possibly a tiny homotopy of $f$) some component $\beta$ of $f^{-1}(\alpha)$ is a path from some element of $f^{-1}(x) \in V_1$ to some element of $f^{-1}(y) \in M - V_3$ disjoint from $S_2$.

(ii) If there exists an essential 2-sphere $P$ in $J$, the irreducibility of $N$ would imply $P$ bounded a ball containing $f(V_1)$ or $f(M - \mathring{V}_3)$. This would contradict the $\pi_1$-injectivity of $f$.

(iii) $g \circ f \mid S_2$ is homotopic to $\text{id}$ in $V_3 - V_1 \subset V_4 - V_0$, and $[S_2]$ generates $H_2(V_4 - V_0)$; therefore, $[f(S_2)] = [K]$ is primitive in $H^2(J)$. Since each closed curve in $J$ can be homotoped out of $J$, $J$ contains no nonseparating surface, so by (i) $H_2(J) = Z$.

Claim 2. $J$ contains a homologically nontrivial torus $T$ which bounds in $N$ a solid torus $W$ containing $f(\gamma)$. Finally $g : T \to M - V_0$ and $in : T \to N - \mathring{W}$ are $\pi_1$-injective.

Proof of Claim 2. Since the thurston norm on $H_2(J)$ equals the singular norm on $H_2(J)$ [G Corollary 6.18] and (iii) there exists an embedded nonbounding torus $T$ in $J$ such that $[T] = [K] \in H_2(J)$. Since $g \mid T$ is not $\pi_1$-injective as a map into $V_4$, it follows that $T$ is compressible in $N$. A compressible torus in an irreducible 3-manifold bounds either a solid torus or lives in a ball. $\pi_1$-injectivity of $f$ precludes the latter, and $Z \neq \pi_1(M - \mathring{V}_3)$ implies that the solid torus $W$ contains $f(\gamma)$. The $\pi_1$-injectivity of $g \mid T$ follows from the facts that $g \mid T$ is $\pi_1$-injective as a map into $V_4 - V_0$ (since each singular sphere in $V_4 - V_0$ is homologically trivial and $[g(T)] = [S_2]$) and $S_4$ is incompressible in $M - \mathring{V}_4$. Finally if $T$ is compressed in $N - \mathring{W}$, then an application of the loop theorem would imply that either some power of $f(\gamma)$ is homotopically trivial in $N$ or $N = S^2 \times S^1$.

Claim 3. Let $Q = N - \mathring{W}$. $g$ is homotopic to a map $h : N \to M$ such that $h \mid T$ is a homeomorphism onto $S_2, h \mid W$ is degree-1 onto $V_2, h \mid Q$ is degree-1 onto $M - \mathring{V}_2$, and $h \mid W$ is $\pi_1$-injective into $V_2$.

Proof of Claim 3. By Claim 1 the map on $T$ obtained by first applying $g$ and then projecting to $S_2$ (in $V_4 - V_0$) is a degree-1 map, so by [K] or [BE] it is homotopic to a homeomorphism. Therefore, to obtain $h$, first homotop $g$ to $g'$ via a homotopy supported in a tiny neighborhood of $T$ so that $g' \mid T$ is a homeomorphism, $g'(W) \subset V_4$, and $g'(Q) \subset M - V_0$. Applying the natural retractions of $V_4$ to $V_2$ and $M - V_0$ to $M - \mathring{V}_2$, to stuff the guts spilling out, we obtain $h$. The degree-1 conclusions follow from the fact that $g$ is degree-1. $h \mid W$ is obviously $\pi_1$-injective.
Claim 4. \( Q \) is irreducible and \( \pi_1 \)-injects into \( f(M - \hat{V}_1) \).

Proof of Claim 4. The irreducibility of \( Q \) follows from the irreducibility of \( N \) and the fact that \( f(\gamma) \) is homotopically nontrivial. If \( D \) is a singular disc in \( f(M - \hat{V}_1) \) such that \( \partial D \subset T \), then \( g(D) \subset M - V_0 \) and \( g \mid T : T \to M - V_0 \) is \( \pi_1 \)-injective implies that \( \partial D \) is homotopically trivial in \( T \). Therefore, \( T \) \( \pi_1 \)-injects into \( f(M - \hat{V}_1) \) and, since \( T \) is incompressible in \( Q \), Claim 4 follows.  

Claim 5. \( h \mid Q \) is \( \pi_1 \)-injective into \( M - V_2 \).

Proof of Claim 5. Let \( \delta \) be a closed curve in \( Q \). Let \( \alpha \) (resp. \( \beta \)) be the curve \( h(\delta) \) (resp. \( g(\delta) \)). By construction \( \alpha \subset M - V_2 \) and \( \beta \subset M - V_0 \). Furthermore \( \alpha \) is homotopic to \( \beta \) in \( M - V_0 \). \( \deg f = 1 \) implies that \( f^{-1}(\delta) \) contains a curve \( \varepsilon \in M - V_1 \) such that \( f \mid \varepsilon \) maps with nonzero degree to \( \delta \). \( \varepsilon \) is homotopic to a nonzero multiple of \( \beta \) and, hence, a nonzero multiple of \( \alpha \) in \( M - V_0 \). Therefore, if \( h(\delta) \) is homotopically trivial in \( M - V_2 \), then \( \varepsilon \) is homotopically trivial in \( M - V_1 \), so \( \delta \) is homotopically trivial in \( f(M - V_1) \) [\( \pi_1(f(M - \hat{V}_1)) \) being torsion free] and so \( \delta \) is homotopically trivial in \( Q \) by Claim 4.  

Claim 6. \( h \) is homotopic to a homeomorphism.

Proof of Claim 6. By Waldhausen [He] \( h : (Q, \partial Q) \to (M - \hat{V}_2, \partial V_2) \) (resp. \( h : (W, \partial W) \to (V_2, \partial V_2) \)) is homotopic to a homeomorphism via a homotopy fixed on the boundary.  

Remarks. If \( \gamma \) has a larger tube radius, e.g. \( 12C \), then Claims 4–5 can be replaced by the observation that the homotopy equivalence splits along \( S_6 \) and \( T_6 \) to ones on \( V_6 \) and \( W \) and \( M - \hat{N}(V_6) \) and \( Q \). Hint: there is an embedded torus \( T_i \) near \( f(S_i) \) for \( i=2, 6, 10 \) which bounds a solid torus; furthermore, \( T_2, T_{10} \) bound a product homeomorphic to Torus \( \times I \). In this setting we now have enough room to homotope \( f \) so that \( f(S_6) = T_6 \). I thank Mike Freedman for suggesting this simplification.

Proof of Theorem 1.1. By Lemmas 1.3, 1.4, \( M \) has a finite covering space \( M_1 \) with a geodesic of tube radius \( > 4C \). Let \( N_1 \) be the associated covering space of \( N \). By [MSY] or [D] \( N_1 \) is irreducible. Now apply Lemma 1.5.  

2. Related results and a conjecture

Conjecture 2.1. Let \( G \) and \( H \) be isomorphic finitely generated groups such that \( G \subset \text{PSL}(2, \mathbb{C}) \subset \text{Homeo}(B^3) \) and \( H \subset \text{Homeo}(B^3) \). Suppose further:

(a) \( G \) and \( H \) act freely on \( \hat{B}^3 \) with closed 3-manifold quotients;
(b) \( H \mid S^2 = G \mid S^2 \); and
(c) there exist subgroups \( H' \), \( G' \) of finite index in \( H \) and \( G \) such that \( H' \mid B^3 = G' \mid B^3 \).
Then $H$ is conjugate to $G$ in Homeo $B^3$.

**Remark 2.2.** Mostow rigidity and Conjecture 2.1 imply the conjecture “If $f: M \to N$ is a homotopy equivalence where $M$ is hyperbolic and $N$ is irreducible, then $f$ is homotopic to a homeomorphism.” For if $N$ is an irreducible homotopy hyperbolic 3-manifold, then by Theorem 0.1 it has a regular finite sheeted covering space $N_1$ which is a hyperbolic 3-manifold. The well-known argument of Remark 0.2 shows that there exists a hyperbolic 3-manifold $M'$ homotopy equivalent to $N$ such that $M$ is finitely covered by $N_1$ and if $G$ (resp. $H$) is the group of covering transformations of $H^3$ corresponding to $M'$ (resp. $N$), and extended to act on $B^3$, then $G$ and $H$ satisfy (a), (b), and (c), where $H'$ and $G'$ are the groups associated to $N_1$. The conclusion of Conjecture 2.1 implies that $N$ is homeomorphic to $M'$, and another application of Mostow rigidity shows that $M = M'$ and that the homotopy equivalence $f$ is homotopic to a homeomorphism.

**Theorem 2.3.** Let $f: M \to N$ be a homotopy equivalence between closed irreducible 3-manifolds with residually finite fundamental group. Suppose further that there exists an element $\gamma \in \pi_1(M)$ which generates a maximal abelian subgroup $\langle \gamma \rangle$ whose associated covering space $M_\gamma = D^2 \times S^1$; then $M$ and $N$ have homeomorphic finite sheeted coverings.

**Proof.** Fix any Riemannian metric on $M$. Let $V_i$, $i = 0, 1, \ldots, 4$, be parallel solid tori in $M$ containing $\gamma$ as a core with $\partial V_i = S_i$ at least $C$ distance apart. It is well known (to algebraists, see [L]) that maximal abelian subgroups are separable (so given $a_1, \ldots, a_n \in \pi_1(M) - \langle \gamma \rangle$ there exists a subgroup of finite index containing $\langle \gamma \rangle$ but missing $a_1, \ldots, a_n$). An argument related to the one of Lemma 1.3 shows that there exists a finite covering $M_1$ of $M$ such that $M_1$ is covered by $M_\gamma$ and the projection of $V_1$ to $M_1$ is an embedding.

We abuse notation by continuing to call the image in $M_1$ of $V_1$ by the same name. Let $N_1$ be the associated finite covering of $N$. Again by [MSY] or [D] $N_1$ is irreducible. The argument of Lemma 1.5 now shows that $M_1$ and $N_1$ are homeomorphic.

Combining Waldhausen [W] with the idea of the proof of Lemma 1.5 we obtain.

**Theorem 2.4.** If $f: M \to M$ is a homeomorphism homotopic to the identity and $M$ is a hyperbolic 3-manifold, then there exists a finite covering space of $M$ such that a lift of $f$ is isotopic to the identity. $\square$

**Remark 2.5.** Actually $M$ need only satisfy the hypothesis of Theorem 2.3.

**References**


Department of Mathematics, California Institute of Technology, Pasadena, California 91125-0001

E-mail address: Gabai@juliet.caltech.edu