HOMOTOPY HYPERBOLIC 3-MANIFOLDS
ARE VIRTUALLY HYPERBOLIC

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The main result of this paper is the only if part of

Theorem 0.1. A closed irreducible 3-manifold \( N \) is homotopy equivalent to a
hyperbolic 3-manifold if and only if \( N \) is finitely covered by a hyperbolic 3-
manifold.

Remark 0.2. The if direction is a well known, quick consequence of Mostow's
Rigidity theorem. Here is the sketch. Let \( p: M \rightarrow N \) be a finite regular covering
map. Any covering translation of \( H^3 \) corresponding to an element of \( \pi_1(N) \)
is a lift of a covering transformation \( f \) of \( p \), which by Mostow rigidity is
homotopic to a unique isometry of \( M \). It follows that \( \pi_1(N) \cong \Gamma \subset \text{Isom}(H^3) \)
and \( H^3/\Gamma \) is a hyperbolic 3-manifold \( M' \). Since \( M' \) and \( N \) are \( K(\pi,1) \)'s,
they are homotopy equivalent.

The proof of the only if direction is likewise a quick application of well-
known results. Here is the sketch. If \( N \) is homotopy equivalent to \( M \), then
using the residual finiteness of \( \pi_1(M) \) we can pass to a regular covering space
\( M_1 \) of \( M \) which has a closed geodesic \( \gamma \) with an enormously thick embedded
regular neighborhood \( U \). Now lift the homotopy equivalence to \( f_1: M_1 \rightarrow N_1 \)
where \( N_1 \) is the corresponding covering of \( N \). Using the fact that the thurston
norm equals the singular norm [to replace a singular torus by an embedded one
in the same homology class in \( f_1(U) - f_1(N(\gamma)) \)] and the observation that the
homotopy equivalence keeps far away points of \( M_1 \) far away, it follows that
in \( f_1(U) \) we can find a curve with a thick collar \( W \). The homotopy inverse
\( g_1 \) is homotopic to a map which is a homeomorphism on \( W \) and on \( N - W \)
restricts to a \( \pi_1 \)-injective degree-1 map. By Waldhausen \( g_1 \) is homotopic to a
homeomorphism.

More details are provided in §1. Theorem 0.1 is used in §2 to reduce the
general problem of homotopy equivalence implying homeomorphism for hy-
perbolic 3-manifolds to Conjecture 2.1. Other results related to the proof of
Theorem 0.1 are stated in §2.

1. PROOF OF THEOREM 1.1

Notation 1.1. If \( f: M \rightarrow N \) is a homotopy equivalence, let \( g: N \rightarrow M \) be the
homotopy inverse and \( F: M \times I \rightarrow M \) be the homotopy of \( g \circ f \) to \( \text{id}_M \). Let
$C > 2 \text{ Sup}\{\text{diam} \tilde{F}(m \times I) \mid m \in M\}$, where $\tilde{F}$ is a lift of $F$ to the universal covering of $M$. $l(\gamma)$ denotes length, and $B(n, x) = \{z \in Z \mid d(x, z) \leq n\}$ where the space $Z$ is clear from context. $N(X)$ denotes (thin) regular neighborhood, and $|E|$ denotes number of components of $E$.

**Lemma 1.2.** If $f : M \to N$ is a homotopy equivalence, then $d(x, y) \geq C$ implies that $f(x) \cap f(y) = \emptyset$. □

**Lemma 1.3.** If $M$ is a closed hyperbolic manifold, $n > 0$, then there exists a regular finite sheeted covering $M_1$ of $M$ with injectivity radius $\geq n$.

**Proof.** Let $p : (H^3, z) \to (M, x)$ the universal covering map. Let $d = \text{diam}(M)$, and assume that $n > d$. Let $V = \{t \in p^{-1}(x) \mid d(z, t) < 4n\}$. Since $\pi_1(M)$ is residually finite [Ma], there exist regular coverings $q : (H^3, z) \to (M_1, y)$, $\pi : (M_1, y) \to (M, x)$ such that $p = \pi \circ q$ and $V \cap q^{-1}(y) = z$. To see this let $\{a_1, \ldots, a_k\} = \{a \in \pi_1(M, x) \mid$ which lift to paths with the first end point $z$ and the other in $V - z\}$. $M_1$ is a covering corresponding to a finite index normal subgroup which does not contain $\{a_1, \ldots, a_k\}$. $q | B(2n, z)$ is an embedding, else there exists $w \in B(4n, z)$ such that $q(w) = q(z)$. Since $M_1$ is regular, $q | B(2n, z')$ is an embedding for each $z' \in p^{-1}(x)$. Finally for all $s \in H^3$, there exists $z' \in p^{-1}(x)$ such that $B(n, s) \subset B(2n, z')$. Thus $q | B(n, s)$ is an embedding. □

If $\gamma$ is a closed geodesic in a hyperbolic 3-manifold, then the *tube radius* of $\gamma = \text{Sup}\{\text{radii of embedded hyperbolic tubes about } \gamma\} = \frac{1}{2} \min\{d(\gamma, \delta) \mid \delta$ is a distinct covering translate of $\gamma$ in $H^3\}$.

**Lemma 1.4.** If $M_1$ is a closed hyperbolic manifold with injectivity radius $n$, then there exists a geodesic $\gamma$ in $M_1$ with tube radius $> n/2$.

**Proof.** Let $\gamma$ be a shortest geodesic in $M_1$. Let $\gamma_1, \gamma_2$ be distinct lifts of $\gamma$ in $H^3$. If $d(\gamma_1, \gamma_2) \leq n = 1/2l(\gamma)$, then there exist $x_i \in \gamma_i$ which are covering translates of each other such that $d(x_1, x_2) < l(\gamma)$, which implies the existence of a geodesic shorter than $\gamma$. □

**Lemma 1.5.** If $M$ is a closed oriented hyperbolic 3-manifold and $f : M \to N$ is a homotopy equivalence such that $N$ is irreducible and $M$ has a geodesic $\gamma$ with tube radius $> 4C$, then $f$ is homotopic to a homeomorphism.

**Proof.** For $0 < i \leq 4$ let $S_i$ be the torus in $M$ at distance $iC$ from $\gamma$, let $V_i$ be the solid torus in $M$ bounded by $S_i$, and let $K = f(S_2)$ and $J = N(K) \cup (\text{components of } N - K \text{ disjoint from } f(S_1 \cup S_3))$. Let $V_0$ also denote $\gamma$.

**Claim 1.** (0) $f^{-1}(J) \subset V_3 ^c - V_1 ^c$ and $g(J) \subset V_4 ^c - V_0 ^c$.

(i) $|\partial J| = 2$, one component of which bounds a region disjoint from $J$ containing $f(S_1)$ and the other component bounds a region disjoint from $J$ containing $f(S_3)$.

(ii) $J$ is irreducible.

(iii) $[K]$ generates $H_2(J) = Z$. 

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Proof of Claim 1. (0) \(K \cap (f(V_1) \cup f(M - V_3)) = \emptyset\) by Lemma 1.2. If \(R\) is a component of \(\partial N(K)\), then \(g(R) \subset V_3\) and, hence, is homologically trivial, so \(R\) bounds in \(N\) since \(f\) is a homotopy equivalence. Each of \(f(M - V_3)\), \(f(V_1)\) lies in a unique component of \(N - K\) and, hence, in a unique component of \(N - J\), so \(f^{-1}(J) \subset \hat{V}_3 - V_1\), \(g(J) \subset \hat{V}_4 - V_0\) now follow from Lemma 1.2.

(i) If \(x \in f(\gamma)\), \(y \in f(M - V_4)\), and \(\alpha \subset N - K\) is a path from \(x\) to \(y\), then \(\deg f = 1\) and choice of \(C\) implies that (after possibly a tiny homotopy of \(f\)) some component \(\beta\) of \(f^{-1}(\alpha)\) is a path from some element of \(f^{-1}(x) \subset V_1\) to some element of \(f^{-1}(y) \subset M - V_3\) disjoint from \(S_2\).

(ii) If there exists an essential 2-sphere \(P\) in \(J\), the irreducibility of \(N\) would imply \(P\) bounded a ball containing \(f(V_1)\) or \(f(M - V_3)\). This would contradict the \(\pi_1\)-injectivity of \(f\).

(iii) \(g \circ f\)|\(S_2\) is homotopic to id in \(V_3 - V_1 \subset V_4 - V_0\), and \([S_2]\) generates \(H_2(V_4 - V_0)\); therefore, \([f(S_2)] = [K]\) is primitive in \(H_2(J)\). Since each closed curve in \(J\) can be homotoped out of \(J\), \(J\) contains no nonseparating surface, so by (i) \(H_2(J) = Z\).

Claim 2. \(J\) contains a homologically nontrivial torus \(T\) which bounds in \(N\) a solid torus \(W\) containing \(f(\gamma)\). Finally \(g : T \to M - V_0\) and \(in : T \to N - \hat{W}\) are \(\pi_1\)-injective.

Proof of Claim 2. Since the thurston norm on \(H_2(J)\) equals the singular norm on \(H_2(J)\) [G Corollary 6.18] and (iii) there exists an embedded nonbounding torus \(T\) in \(J\) such that \([T]\) = \([K]\) \in \(H_2(J)\). Since \(g\)|\(T\) is not \(\pi_1\)-injective as a map into \(V_4\), it follows that \(T\) is compressible in \(N\). A compressible torus in an irreducible 3-manifold bounds either a solid torus or lives in a ball. \(\pi_1\)-injectivity of \(f\) precludes the latter, and \(Z \neq \pi_1(M - V_3)\) implies that the solid torus \(W\) contains \(f(\gamma)\). The \(\pi_1\)-injectivity of \(g\)|\(T\) follows from the facts that \(g\)|\(T\) is \(\pi_1\)-injective as a map into \(V_4 - V_0\) (since each singular sphere in \(V_4 - V_0\) is homologically trivial and \([g(T)] = [S_2]\)) and \(S_4\) is incompressible in \(M - \hat{V}_4\). Finally if \(T\) is compressed in \(N - \hat{W}\), then an application of the loop theorem would imply that either some power of \(f(\gamma)\) is homotopically trivial in \(N\) or \(N = S^2 \times S^1\).

Claim 3. Let \(Q = N - \hat{W}\). \(g\) is homotopic to a map \(h : N \to M\) such that \(h\)|\(T\) is a homeomorphism onto \(S_2\), \(h\)|\(W\) is degree-1 onto \(V_2\), \(h\)|\(Q\) is degree-1 onto \(M - V_2\), and \(h\)|\(W\) is \(\pi_1\)-injective into \(V_2\).

Proof of Claim 3. By Claim 1 the map on \(T\) obtained by first applying \(g\) and then projecting to \(S_2\) (in \(V_4 - V_0\)) is a degree-1 map, so by \([K]\) or \([BE]\) it is homotopic to a homeomorphism. Therefore, to obtain \(h\), first homotop \(g\) to \(g'\) via a homotopy supported in a tiny neighborhood of \(T\) so that \(g'|T\) is a homeomorphism, \(g'(W) \subset V_4\), and \(g'(Q) \subset M - V_0\). Applying the natural retractions of \(V_4\) to \(V_2\) and \(M - V_0\) to \(M - \hat{V}_2\), to stuff the guts spilling out, we obtain \(h\). The degree-1 conclusions follow from the fact that \(g\) is degree-1. \(h\)|\(W\) is obviously \(\pi_1\)-injective.
Claim 4. \( Q \) is irreducible and \( \pi_1 \)-injects into \( f(M - \hat{V}_1) \).

Proof of Claim 4. The irreducibility of \( Q \) follows from the irreducibility of \( N \) and the fact that \( f(\gamma) \) is homotopically nontrivial. If \( D \) is a singular disc in \( f(M - \hat{V}_1) \) such that \( \partial D \subset T \), then \( g(D) \subset M - V_0 \) and \( g \mid T : T \to M - V_0 \) is \( \pi_1 \)-injective implies that \( \partial D \) is homotopically trivial in \( T \). Therefore, \( T \pi_1 \)-injects into \( f(M - \hat{V}_1) \) and, since \( T \) is incompressible in \( Q \), Claim 4 follows.

Claim 5. \( h \mid Q \) is \( \pi_1 \)-injective into \( M - V_2 \).

Proof of Claim 5. Let \( \delta \) be a closed curve in \( Q \). Let \( \alpha \) (resp. \( \beta \)) be the curve \( h(\delta) \) (resp. \( g(\delta) \)). By construction \( \alpha \subset M - V_2 \) and \( \beta \subset M - V_0 \). Furthermore \( \alpha \) is homotopic to \( \beta \) in \( M - V_0 \). \( \deg f = 1 \) implies that \( f^{-1}(\delta) \) contains a curve \( \epsilon \in M - V_1 \) such that \( f \mid \epsilon \) maps with nonzero degree to \( \delta \). \( \epsilon \) is homotopic to a nonzero multiple of \( \beta \) and, hence, a nonzero multiple of \( \alpha \) in \( M - V_0 \). Therefore, if \( h(\delta) \) is homotopically trivial in \( M - V_2 \), then \( \epsilon \) is homotopically trivial in \( M - V_1 \), so \( \delta \) is homotopically trivial in \( f(M - V_1) \) \( \{ \pi_1(f(M - \hat{V}_1)) \) being torsion free\} and so \( \delta \) is homotopically trivial in \( Q \) by Claim 4.

Claim 6. \( h \) is homotopic to a homeomorphism.

Proof of Claim 6. By Waldhausen [He] \( h : (Q, \partial Q) \to (M - \hat{V}_2, \partial V_2) \) (resp. \( h : (W, \partial W) \to (V_2, \partial V_2) \)) is homotopic to a homeomorphism via a homotopy fixed on the boundary.

Remarks. If \( \gamma \) has a larger tube radius, e.g. 12\( C \), then Claims 4-5 can be replaced by the observation that the homotopy equivalence splits along \( S_6 \) and \( T_6 \) to ones on \( V_6 \) and \( W \) and \( M - \hat{N}(V_6) \) and \( Q \). Hint: there is an embedded torus \( T_i \) near \( f(S_i) \) for \( i=2, 6, 10 \) which bounds a solid torus; furthermore, \( T_2, T_{10} \) bound a product homeomorphic to Torus \( \times I \). In this setting we now have enough room to homotope \( f \) so that \( f(S_6) = T_6 \). I thank Mike Freedman for suggesting this simplification.

Proof of Theorem 1.1. By Lemmas 1.3, 1.4, \( M \) has a finite covering space \( M_1 \) with a geodesic of tube radius \( > 4C \). Let \( N_1 \) be the associated covering space of \( N \). By [MSY] or [D] \( N_1 \) is irreducible. Now apply Lemma 1.5.

2. RELATED RESULTS AND A CONJECTURE

Conjecture 2.1. Let \( G \) and \( H \) be isomorphic finitely generated groups such that \( G \subset \text{PSL}(2, C) \subset \text{Homeo}(B^3) \) and \( H \subset \text{Homeo}(B^3) \). Suppose further:

(a) \( G \) and \( H \) act freely on \( \hat{B}^3 \) with closed 3-manifold quotients;
(b) \( H \mid S^2 = G \mid S^2 \); and
(c) there exist subgroups \( H' \), \( G' \) of finite index in \( H \) and \( G \) such that \( H' \mid B^3 = G' \mid B^3 \).
Then $H$ is conjugate to $G$ in $\text{Homeo} B^3$.

Remark 2.2. Mostow rigidity and Conjecture 2.1 imply the conjecture "If $f: M \to N$ is a homotopy equivalence where $M$ is hyperbolic and $N$ is irreducible, then $f$ is homotopic to a homeomorphism." For if $N$ is an irreducible homotopy hyperbolic 3-manifold, then by Theorem 0.1 it has a regular finite sheeted covering space $N_I$ which is a hyperbolic 3-manifold. The well-known argument of Remark 0.2 shows that there exists a hyperbolic 3-manifold $M'$ homotopy equivalent to $N$ such that $M$ is finitely covered by $N_I$ and if $G$ (resp. $H$) is the group of covering transformations of $H^3$ corresponding to $M'$ (resp. $N$), and extended to act on $B^3$, then $G$ and $H$ satisfy (a), (b), and (c), where $H'$ and $G'$ are the groups associated to $N_I$. The conclusion of Conjecture 2.1 implies that $N$ is homeomorphic to $M'$, and another application of Mostow rigidity shows that $M = M'$ and that the homotopy equivalence $f$ is homotopic to a homeomorphism.

Theorem 2.3. Let $f: M \to N$ be a homotopy equivalence between closed irreducible 3-manifolds with residually finite fundamental group. Suppose further that there exists an element $\gamma \in \pi_1(M)$ which generates a maximal abelian subgroup $\langle \gamma \rangle$ whose associated covering space $M_\gamma = D^2 \times S^1$; then $M$ and $N$ have homeomorphic finite sheeted coverings.

Proof. Fix any Riemannian metric on $M$. Let $V_i$, $i = 0, 1, \ldots, 4$, be parallel solid tori in $M_\gamma$ containing $\gamma$ as a core with $\partial V_i = S_i$ at least $C$ distance apart. It is well known (to algebraists, see [L]) that maximal abelian subgroups are separable (so given $a_1, \ldots, a_n \in \pi_1(M) - \langle \gamma \rangle$ there exists a subgroup of finite index containing $\langle \gamma \rangle$ but missing $a_1, \ldots, a_n$). An argument related to the one of Lemma 1.3 shows that there exists a finite covering $M_1$ of $M$ such that $M_1$ is covered by $M_\gamma$ and the projection of $V_i$ to $M_1$ is an embedding. We abuse notation by continuing to call the image in $M_1$ of $V_i$ by the same name. Let $N_1$ be the associated finite covering of $N$. Again by [MSY] or [D] $N_1$ is irreducible. The argument of Lemma 1.5 now shows that $M_1$ and $N_1$ are homeomorphic.

Combining Waldhausen [W] with the idea of the proof of Lemma 1.5 we obtain.

Theorem 2.4. If $f: M \to M$ is a homeomorphism homotopic to the identity and $M$ is a hyperbolic 3-manifold, then there exists a finite covering space of $M$ such that a lift of $f$ is isotopic to the identity.

Remark 2.5. Actually $M$ need only satisfy the hypothesis of Theorem 2.3.

References


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