

DISCRETE GROUPS AND NON-RIEMANNIAN HOMOGENEOUS SPACES

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I. INTRODUCTION

A basic question in geometry is to understand compact locally homogeneous manifolds, i.e., those compact manifolds that can be locally modelled on a homogeneous space $J \backslash H$ of a finite-dimensional Lie group H . This means that there is an atlas on a manifold M consisting of local diffeomorphisms with open sets in $J \backslash H$ where the transition functions between these open sets are given by translations by elements of H . A basic example is the case of $M = J \backslash H / \Gamma$ where $\Gamma \subset H$ is a discrete subgroup acting freely and properly discontinuously on J / H , with a compact quotient. (These examples can be abstractly characterized as the “complete” case. See [G].) We then call Γ a cocompact lattice on $J \backslash H$. When J is compact, $J \backslash H$ has an H -invariant Riemannian metric, so that M itself becomes a Riemannian manifold in a natural way. Thus, the class of locally homogeneous Riemannian manifolds includes (but is not exhausted by) the locally symmetric spaces. For $H = \text{Aff}(\mathbb{R}^n)$ and $J = \text{GL}(n, \mathbb{R})$, the spaces $J \backslash H / \Gamma$ are the complete affinely flat manifolds, a non-Riemannian example which has been extensively studied.

When H is simple and J is not compact, the situation is far from being well understood. Some special series of such homogeneous spaces admitting a cocompact lattice have been constructed by Kulkarni [Ku] and by T. Kobayashi [K]. For example, Kulkarni shows the existence of such a discrete group Γ for the homogeneous spaces $\text{SO}(1, 2n) \backslash \text{SO}(2, 2n)$ and Kobayashi does the same for $\text{U}(1, n) \backslash \text{SO}(2, 2n)$. On the other hand, the constructions they give for these (and other similar) series are quite special, and there has been the general suspicion that most homogeneous spaces $J \backslash H$ with H simple and J not compact do not admit a cocompact lattice. (We recall that, when J is compact, i.e. the Riemannian case, such Γ always exist by a result of Borel [B].) One general situation in which it is known that no such Γ exists is the case in which $J \backslash H$ is noncompact and of reductive type and $\mathbb{R}\text{-rank}(H) = \mathbb{R}\text{-rank}(J)$. This situation was studied by Calabi-Markus [CM], Wolf [W], and Kobayashi [K], and the nonexistence of Γ in this situation is a special case of a more general result known as the Calabi-Markus phenomenon. Kobayashi and Ono [KO] use the Hirzebruch proportionality principle to show that some other special series do

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not admit cocompact lattices. Recently Benoist and Labourie [BL] have shown that if H is simple and the center $Z(J)$ contains a nontrivial \mathbb{R} -split one-parameter subgroup then no such Γ exists. Their result in fact applies more generally to show that there is no compact M locally modelled on $J \backslash H$.

These results are quite special of course and, in particular, give no insight into the question whether such Γ exist for the basic test case of $\mathrm{SL}(p, \mathbb{R}) \backslash \mathrm{SL}(n, \mathbb{R})$ where $p < n$ and the embedding is the standard one. The goal of this paper is to prove a (still special) result that clarifies the situation for the above examples in case $p < n/2$ and for many other natural examples as well. Here is the main result.

Theorem 1.1. *Let H be a real algebraic group and $J \subset H$ an algebraic subgroup. Suppose:*

- (i) H and J are both unimodular;
- (ii) There is a discrete group $\Gamma \subset H$ that is a cocompact lattice for $J \backslash H$; and
- (iii) The centralizer $Z_H(J)$ of J in H contains a subgroup G with the following properties:
 - (a) G is not contained in a proper normal subgroup of H ,
 - (b) G is a semisimple Lie group each of whose simple factors has \mathbb{R} -rank at least 2,
 - (c) every local homomorphism of G to J is trivial.

Then J is compact.

Corollary 1.2. *Let H be an almost simple real algebraic group. Suppose $J \subset H$ is a unimodular subgroup such that $Z_H(J)$ contains a simple Lie group of \mathbb{R} -rank at least 2 that does not have a local embedding in J . If $J \backslash H$ admits a cocompact lattice, then J is compact.*

Corollary 1.3. *If $2 \leq p < n/2$, $n > 4$, then $\mathrm{SL}(p, \mathbb{R}) \backslash \mathrm{SL}(n, \mathbb{R})$ does not admit a cocompact lattice. The same is true for $J \backslash \mathrm{SL}(n, \mathbb{R})$ where J is any unimodular noncompact subgroup of $\mathrm{SL}(p, \mathbb{R})$.*

The remainder of the paper is devoted largely to the proof of Theorem 1.1. The proof involves in a crucial way results and arguments of algebraic ergodic theory, using in particular superrigidity for actions on principal bundles [Z1] and Ratner's theorem showing the validity of Raghunathan's measure conjecture [R]. Some parts of the argument are closely related to the arguments of [Z5].

We conclude this introductory section with some remarks and questions.

Questions. (1) With $J \backslash H$ as in Theorem 1.1 and J noncompact, do there exist compact manifolds locally modelled on $J \backslash H$? While some of the ideas in the proof of Theorem 1.1 can be applied in this more general situation, more work is required to resolve this general case.

(2) Consider for simplicity the case of H simple and $J \subset H$ unimodular. It is natural to ask if Corollary 1.2 remains true under weaker hypotheses on $Z_H(J)$. The proof of Theorem 1.1 and the results of [CZ] show that the conclusion of Corollary 1.2 still remains true if we only assume $Z_H(J)$ contains a noncompact simple group with Kazhdan's property, e.g., $\mathrm{Sp}(1, n)$. On the other hand, there are no known examples of $J \backslash H$ admitting a compact lattice where H is simple, J is noncompact and unimodular, and $Z_H(J)$ contains

$GL(2, \mathbb{R})$. It would, of course, be of interest to know if the conclusion of Corollary 1.2 remains true in this case as well.

(3) Are there cocompact lattices for $SL(n-2, \mathbb{R}) \backslash SL(n, \mathbb{R})$ or $SL(n-1, \mathbb{R}) \backslash SL(n, \mathbb{R})$? Even the case $SL(2, \mathbb{R}) \backslash SL(3, \mathbb{R})$ remains open.

The author's interest in the general issues this paper deals with was stimulated by the paper of Benoist-Labourie and many conversations with Labourie. We would like to thank Y. Benoist, P. Foulon, and F. Labourie for some very useful conversations that helped clarify some important issues involved in the main result. Special thanks are also due to S. G. Dani with whom I spent many productive hours discussing the arguments of the proof, and to F. Labourie for pointing out a gap in the original version of this paper. Much of the work on this paper was completed at the Institut des Hautes Études Scientifiques which we thank for its hospitality.

2. PRELIMINARIES

In this section we provide one simple background result we shall need and briefly recall some standard notions we shall use freely.

Proposition 2.1. *Let H be a unimodular Lie group, $J \subset H$ a closed unimodular subgroup, and $\Gamma \subset H$ a discrete subgroup that is a cocompact lattice for $J \backslash H$. Then there is a volume form on the compact manifold $M = J \backslash H / \Gamma$ that is invariant under the natural left action of $Z_H(J)$ on M .*

Proof. Since both H and J are unimodular, there is an H -invariant volume form ω on $J \backslash H$ that is unique up to a constant multiple. If $g \in Z_H(J)$, then g acts on $J \backslash H$ on the left commuting with the right action of H . Since $g^* \omega$ is then also H -invariant, it must be a scalar multiple of ω , say $g^* \omega = c(g) \omega$. Since ω is Γ -invariant, it factors to a volume form ω_0 , and on M we then clearly also have $g^* \omega_0 = c(g) \omega_0$. Since M is compact, ω_0 defines a finite measure μ on M and $g_* \mu = c(g) \mu$. Since the total measure is preserved, $c(g) = 1$, and ω_0 is $Z_H(J)$ -invariant.

We shall deal extensively with actions of a Lie group by automorphisms of a principal bundle. We recall some standard notation regarding this situation.

Let M be a manifold and $P \rightarrow M$ a principal H -bundle where H is a Lie group, and suppose G acts on P by principal bundle automorphisms. If V is any left H -space, we can form the associated bundle $E_V = (P \times V) / H \rightarrow P / H = M$ with V as fiber, and G acts by bundle automorphisms of $E_V \rightarrow M$. Any measurable section s of $P \rightarrow M$ (which always exists) defines a measurable trivialization $P \cong M \times H \rightarrow M$. The G action on $M \times H$ corresponding to the action on P under this trivialization will be given by $g \cdot (m, h) = (gm, \alpha(g, m)h)$ where $\alpha : G \times M \rightarrow H$ is a measurable cocycle. We recall that two cocycles $\alpha, \beta : G \times M \rightarrow H$ are equivalent if there is a measurable $\varphi : M \rightarrow H$ such that $\varphi(gm)^{-1} \alpha(g, m) \varphi(m) = \beta(g, m)$. Different trivializations of P yield cocycles equivalent to α , and, conversely, any cocycle equivalent to α comes from a trivialization. There are a large number of results concerning the structure of measurable cocycles or, equivalently, the measurable structure of actions on principal bundles. In particular, the superrigidity theorem for

cocycles [Z1–Z3] is such a result, and there are several such results related to amenability and Kazhdan's property [Z7]. The notion of algebraic hull of a cocycle (or of an action on a principal bundle) is a basic invariant. We refer the reader to [Z1, Z4, Z9] for background information on all these and related issues.

3. PROOF OF MAIN THEOREM

Let H be a connected simple Lie group with finite center and $J \subset H$ an algebraic unimodular subgroup. We can assume J is connected. Let $\Gamma \subset H$ be a discrete group, which we assume acts freely and properly discontinuously on $J \backslash H$ such that $M = J \backslash H / \Gamma$ is compact. We assume $Z_H(J)$ contains a simple Lie group of real rank at least 2. We write the Lie algebra $\mathfrak{z}_h(j) = \mathfrak{l} \rtimes \mathfrak{r}$ where \mathfrak{r} is the radical and \mathfrak{l} is a semisimple Levi factor. We let $G \subset Z_H(J)$ be the connected semisimple group corresponding to the sum of all simple factors of \mathfrak{l} of real rank at least 2. Then G is a semisimple Lie group each of whose simple factors has \mathbb{R} rank of at least 2, and if $\sigma : G \rightarrow Z_H(J)$ is any faithful representation then $\sigma(G)$ and G are conjugate in $Z_H(J)$.

We view $J \backslash H \rightarrow M$ as a principal Γ -bundle on which G acts by Γ -bundle automorphisms. We have a natural embedding $\Gamma \rightarrow H$, and we let $Q \rightarrow M$ be the associated H -bundle, i.e., $Q = (J \backslash H \times H) / \Gamma$. The group G then acts on Q by H -bundle automorphisms, and we wish to apply superrigidity for actions on principal bundles. To this end, we let (E, ν) be the space of ergodic components of the G -action on M . Thus, (E, ν) is a measure space and there is a G -invariant measurable map $p : M \rightarrow E$ such that, letting μ be the measure on M , we have $\mu = \int_{e \in E}^{\oplus} \mu_e d\nu(e)$, where μ_e is a measure on $M_e = p^{-1}(e)$. Furthermore, a.e. μ_e is a G -invariant ergodic probability measure. For each e , we let $Q_e \rightarrow M_e$ the (measurable) principal H -bundle obtained by restricting Q to the fibers over M_e . Since the action of G on M_e is ergodic, there is a well-defined algebraic hull for the action of G . We recall that there this is the unique conjugacy class of algebraic subgroups of H such that there is a measurable G -invariant reduction of the bundle to an element of this conjugacy class but not to any smaller algebraic subgroup. We shall use basic properties of algebraic hulls freely and refer the reader to [Z4] for discussion of this notion.

We shall also find the following notation (from [Z9]) convenient.

Definition 3.1. If $P \rightarrow M$ is a principal Γ -bundle with Γ discrete, $\Lambda \subset \Gamma$ is a subgroup, and G acts by automorphisms, we say that the G -action is Λ -reducible if there is a measurable G -invariant reduction of P to Λ .

We then have the following easy fact. (Cf. [Z9].)

Lemma 3.2. *Suppose $P \rightarrow M$ is a principal Γ -bundle and that G acts ergodically on M . Suppose $\Gamma \rightarrow H$ is a discrete faithful representation into an algebraic group H . Let Q be the associated H -bundle. If the algebraic hull for the G action on P is compact, then $P \rightarrow M$ is Λ -reducible for some finite $\Lambda \subset \Gamma$.*

(The proof of Lemma 3.2 also follows from the proof of Proposition 3.6 below.)

Proposition 3.3. *For almost all $e \in E$, the algebraic hull of the action of G on $Q_e \rightarrow M_e$ is noncompact.*

Proof. If this were not the case, then there would be a set of $e \in E$ of positive measure for which the algebraic hull is compact. By Lemma 3.2, on a set $e \in E$ of positive measure, the G action on $(J \setminus H)_e \rightarrow M_e$ is Λ_e -reducible for some finite $\Lambda_e \subset \Gamma$. It follows that each such $(J \setminus H)_e$ supports a finite G -invariant measure projecting to M_e , and hence (by integrating) $J \setminus H$ supports a finite G -invariant measure projecting to the restriction of μ to a subset of positive μ -measure in M . Since the fibers of $J \setminus H \rightarrow M$ are discrete, it follows that $J \setminus H$ admits a finite G -invariant measure that is the restriction of a smooth measure to a G -invariant set. Since the action of G on $J \setminus H$ is an action on a variety, the Borel density theorem implies that finite invariant measures are supported on fixed points. Therefore, we would have a set of fixed points of positive Lebesgue measure. Since the action is algebraic, this implies that G acts trivially on $J \setminus H$, which is clearly impossible because G is not contained in a proper normal subgroup of H .

Remarks. (a) Exactly the same argument can be applied to the action of any simple normal subgroup of G .

(b) In general, by passing to a finite ergodic extension, the algebraic hull will become a subgroup of finite index. Thus, Proposition 3.3 remains true if $Q_e \rightarrow M_e$ is replaced by the pullback of Q_e to a finite ergodic extension of M_e .

We now fix an ergodic component e that satisfies the conclusions of Proposition 3.3 and apply superrigidity for cocycles (i.e., for actions on principal bundles) in the form of [Z1, Z3]. We obtain:

Theorem 3.4. *Fix e as above. Then there is a finite ergodic extension $M'_e \rightarrow M_e$ of M_e , a locally faithful representation $\sigma : \tilde{G} \rightarrow H$, a compact subgroup $K \subset H$ with $K \subset Z_H(\sigma(\tilde{G}))$ and a measurable function $\psi : M'_e \rightarrow H$ such that*

$$\psi(gm)^{-1} \alpha'(g, m) \psi(m) = \sigma(g) c(g, m)$$

for $g \in \tilde{G}$, $m \in M'_e$ where $\alpha : G \times M_e \rightarrow \Gamma$ is the cocycle corresponding to $(J \setminus H)_e \rightarrow M_e$, α' is its lift to $\tilde{G} \times M'_e$, and $c(g, m) \in K$.

We remark that σ is nontrivial by Proposition 3.3 (and Remark (b)) and, in fact, is locally faithful by Remark (a).

By passing to a further ergodic extension with finite invariant measure, we can obtain the same conclusion but with c trivial. Namely, let $K_0 \subset K$ be the algebraic hull of c . Then c is equivalent to a cocycle c_0 taking values in K_0 , and we can form the G -space $X_e = M'_e \times_{c_0} K_0$. Since K is compact, this will be ergodic [Z6] (with respect to the finite G -invariant product measure). The pullback of c_0 to a cocycle $\tilde{G} \times X_e \rightarrow K_0$ will be trivial. Therefore, we deduce

Corollary 3.5. *There is a measurable function $\varphi : X_e \rightarrow H$ such that*

$$\varphi(gx)^{-1} \alpha_x(g, m) \varphi(gx) = \sigma(g)$$

where α_X is the lift of α to a cocycle $\tilde{G} \times X \rightarrow \Gamma$.

Now let $\lambda : X_e \rightarrow \Gamma \backslash H$ be the composition of φ with the projection of H to $\Gamma \backslash H$. Since α_X takes values in Γ , we deduce $\lambda(gx) = \lambda(x) \cdot \sigma(g)^{-1}$, i.e., that λ is a \tilde{G} map. The measure $\lambda_*(\mu_e)$ is then a finite ergodic $\sigma(\tilde{G})$ -invariant measure on $\Gamma \backslash H$. As in [Z5] we can apply Ratner's theorem [R] to this situation.

Proposition 3.6. *There is a connected Lie group $L \subset H$ such that:*

- (i) $\Lambda = L \cap \Gamma$ is a lattice in L , and
- (ii) the G action on the principal Γ -bundle $P_X \rightarrow X$, the pullback of $(J \backslash H)_e \rightarrow M_e$ to X , is Λ -reducible.

Proof. By Ratner's Theorem [R], there is a connected Lie group L , $\sigma(G) \subset L \subset H$, such that $\lambda_*(\mu_e)$ lies on an L -orbit in $\Gamma \backslash H$ of the form $h\Gamma h^{-1} \cap L \backslash L$, where $h \in H$ and $h\Gamma h^{-1} \cap L$ is a lattice in L . In other words, there is some $h \in H$ such that $\varphi(X_e) \subset \Gamma hL$ (a.e.). Thus, we can write $\varphi(x) = \omega(x)h\theta(x)$ where $\omega : X_e \rightarrow \Gamma$ and $\theta : X_e \rightarrow L$ are measurable. We then have

$$\theta(gx)^{-1}h^{-1}\omega(gx)^{-1}\alpha_X(g, x)\omega(x)h\theta(x) = \sigma(g).$$

The cocycle $\beta(g, x) = \omega(gx)^{-1}\alpha_X(g, x)\omega(x)$ is equivalent (as a cocycle into Γ) to α_X and satisfies

$$\beta(g, x) = h\theta(gx)\sigma(g)\theta(x)^{-1}h^{-1} \in hLh^{-1}.$$

Therefore, $\beta(G \times X) \subset \Gamma \cap hLh^{-1}$. Hence, the G -action on P_X is Λ -reducible for $\Lambda = \Gamma \cap hLh^{-1}$ and Λ is a lattice in hLh^{-1} .

Lemma 3.7. *Let L be as in Proposition 3.6. Then L acts properly on $J \backslash H$.*

Proof. Γ acts properly by assumption, and hence so does Λ . Since Λ is a lattice in L , L will also act properly. For convenience, we recall the proof of this latter assumption.

We first recall there is a compact symmetric set $B \subset L$ such that $L = B\Lambda B$. Namely, choose a compact symmetric $B \subset L$ whose projection $[B]$ to L/Λ has measure at least $3/4$. Then, for any $h \in L$, $h([B]) \cap [B]$ has positive measure; i.e., $hB \cap B\Lambda$ is nontrivial and so $h \in B\Lambda B$. Now take any $A \subset J \backslash H$ that is compact. If $h \in L$ and $Ah \cap A \neq \emptyset$, write $h = b\lambda c$ where $\lambda \in \Lambda$, and $b, c \in B$. Then $(AB)\lambda \cap Ac^{-1} \neq \emptyset$, i.e., $(AB)\lambda \cap AB \neq \emptyset$.

Letting $\Lambda_0 \subset \Lambda$ be $\{\lambda \in \Lambda \mid AB\lambda \cap AB \neq \emptyset\}$, we have Λ_0 is finite since AB is compact, and $h \in B\Lambda_0 B$ which is compact.

Proposition 3.8. *There is a discrete subgroup $F \subset \Gamma$ with the following properties:*

- (i) F is finitely generated.
- (ii) The action of G on $P_X \rightarrow X$ is F -reducible.
- (iii) If we let N be the Zariski closure of F , then N acts properly on $J \backslash H$.

Proof. The group $[L, L] \subset H$ is algebraic. Let $\beta : G \times X \rightarrow \Lambda$ be a cocycle corresponding to the G -invariant Λ -reduction of P_X , and $\pi : L \rightarrow L/[L, L]$.

Then $\pi \circ \beta$ is a cocycle taking values in the countable abelian group $\pi(\Lambda)$. It follows from [Z1, Theorem 9.1.1] (using Kazhdan's property) that $\pi \circ \beta$ is equivalent to a cocycle taking values in a finite subgroup, and hence β is equivalent to a cocycle taking values in a subgroup $\Lambda_1 \subset \Lambda$ such that $\Lambda_1 \supset \Lambda \cap [L, L]$ as a subgroup of finite index. It follows that Λ_1 is contained in an algebraic group L_1 such that $L_1 \cap [L, L]$ is of finite index in L_1 . Since G is a Kazhdan group, any cocycle into a countable group is equivalent to one taking values in a finitely generated subgroup [Z8, Lemma 2.2]. Therefore, there is a finitely generated $F \subset \Lambda_1$ such that the G -action on $P_X \rightarrow X$ is F -reducible. Letting N be the Zariski closure of F , we have $N \subset L_1$, and hence $N/N \cap L$ is finite. Properness of the N action on $J \backslash H$ then follows from Lemma 3.7.

Now let $Q_X \rightarrow X$ be the principal H -bundle obtained by pulling back $Q_e \rightarrow (J \backslash H)_e$ to X . Alternatively, Q_X is the principal H -bundle associated to the Γ -bundle $P_X \rightarrow X$ and the embedding $\Gamma \rightarrow H$. Since the G -action on P_X is F -reducible, it is clear that the algebraic hull of the G -action on Q_X is contained in N . On the other hand, this algebraic hull is exactly $\overline{\sigma(\tilde{G})}$ (the Zariski closure) in which $\sigma(\tilde{G})$ is of finite index. (This assertion follows from the Borel density theorem. See, e.g., [Z4].) Thus a conjugate of $\sigma(\tilde{G})$ is contained in N . We now wish to show that, in fact, some conjugate of G itself is contained in N . (This would follow of course if we knew σ was conjugate to the original realization $\tilde{G} \rightarrow G \subset H$, but this is what we have not yet established.)

Lemma 3.9. *G is contained in a conjugate of N .*

Proof. Since there is a G -invariant reduction of $P_X \rightarrow X$ to F and $F \subset N$, the algebraic hull of the G -action on Q_X is clearly contained in N .

Consider now the right H -action on $J \backslash H$. For each $y \in J \backslash H$, let $\mathfrak{h}_y \subset \mathfrak{h}$ be the Lie algebra of the stabilizer of y . Then $y \mapsto \mathfrak{h}_y$ defines a Γ -map $\eta : J \backslash H \rightarrow \text{Gr}(\mathfrak{h})$ (in fact, an H -map) where $\text{Gr}(\mathfrak{h})$ is the Grassmann variety of \mathfrak{h} . Since G acts on the left of $J \backslash H$, it is straightforward to check that $\eta(gy) = \eta(y)$ for all $y \in J \backslash H$ and $g \in G$.

Write $J = R \times U$ a Levi decomposition, so U is the unipotent radical. Let \mathfrak{u} be the Lie algebra of U , on which J acts. We can write $\mathfrak{o} \subset \mathfrak{u}_1 \subset \mathfrak{u}_2 \subset \dots \subset \mathfrak{u}_r = \mathfrak{u}$, where $\mathfrak{u}_i \subset \mathfrak{u}$ is a J -invariant ideal and the J -action on $\mathfrak{u}_{i+1}/\mathfrak{u}_i$ is irreducible (and, in particular, is trivial on U). Since each \mathfrak{u}_i is J -invariant, we can define an H -equivariant map $w_i : J \backslash H \rightarrow \text{Gr}(\mathfrak{u})$ (the Grassmann variety of \mathfrak{u}) by $w_i(y) = \text{Ad}(h)^{-1}\mathfrak{u}_i$ where $y = Jh$. For any $g \in G$, let $l(g)$ be left translation on $J \backslash H$. Then $w_j \circ l(g)$ will also be H -equivariant and $[w_j \circ l(g)]([e]) = w_j(g) = \text{Ad}(g)^{-1}\mathfrak{u}_j = \mathfrak{u}_j$ (since $\text{Ad}(G)$ is trivial on \mathfrak{u}) $= w_j([e])$. Since $w_j \circ l(g)$ and w_j agree at a point and are H -invariant, they are equal.

Define $w : J \backslash H \rightarrow \text{Gr}(\mathfrak{h}) \times \prod_i \text{Gr}(\mathfrak{u}) = V$ by $w = (\eta, w_1, \dots, w_r)$. This is then an H -equivariant, G -invariant map. The map $y \mapsto (y, w(y))$ therefore factors to a map $s : J \backslash H/\Gamma \rightarrow (J \backslash H \times V)/\Gamma$. Since the image of $y \mapsto w(y)$ clearly lies in the orbit of $(j, \mathfrak{u}_1, \dots, \mathfrak{u}_r)$ under H , the map s can be viewed as a section of the bundle $(J \backslash H \times H/H_0)/\Gamma \rightarrow J \backslash H/\Gamma$ where H_0 is the stabilizer of $(j, \mathfrak{u}_1, \dots, \mathfrak{u}_r)$ in H . It follows from [Z4] that the algebraic hull of the

G -action on the principal H -bundle $(J \backslash H \times H) / \Gamma \rightarrow J \backslash H / \Gamma$ is contained in H_0 .

It follows that the same is true for the algebraic hull of the G -action on the pullback of this bundle to X , which is clearly isomorphic to Q_X . As we have already observed, this algebraic hull is contained in N . We clearly have that $H_0 = N_H(j) \cap \bigcap_i N_H(u_i)$. It follows that for some $h \in H$ and some conjugate G_1 of $\sigma(\tilde{G})$ that

$$G_1 \subset hNh^{-1} \cap N_H(j) \cap \bigcap_i N_H(u_i).$$

By Proposition 3.8, N , and hence any conjugate of N , acts properly on $J \backslash H$. Therefore, G_1 acts properly on the right of $J \backslash H$, so the intersection of any conjugate of G_1 with J is compact. Since G_1 is semisimple with no compact factors and G_1 normalizes j , we have $\mathfrak{g}_1 \cap \mathfrak{j} = (0)$.

Now consider the subgroup $G_1 J = G_1 R U$. Note that each u_j is an ideal, i.e. is invariant under all $G_1 J$. Since G_1 and R are reductive, G_1 normalizes $R U$, and $\mathfrak{g}_1 \cap \mathfrak{j} = (0)$, in the quotient group $G_1 R U / U$ we can conjugate $G_1 U / U$ so that it centralizes $R U / U$. (We are using here the fact that there are no nontrivial local homomorphisms from G to J .) Choosing a lift of this semisimple group in $G_1 R U$, we deduce the following: there is a conjugate G_2 of G_1 in $G_1 R U$ such that G_2 centralizes R . Furthermore, $G_2 R$ acts on each u_{i+1} / u_i , and the representation of R on this space is irreducible. It follows that G_2 acts by scalars on u_{i+1} / u_i and, hence, that G_2 acts on u by a solvable group. Since G_2 is simple, G_2 acts trivially on u . Thus G_2 commutes with both R and U , and hence $G_2 \subset Z_H(j)$. This shows that some conjugate of $\sigma(\tilde{G})$ is contained in $hNh^{-1} \cap Z_H(j)$. But by remarks at the beginning of this section, since any representation $\tilde{G} \rightarrow Z_H(j)$ has image equal to a conjugate of G in $Z_H(j)$, it follows that for some (possibly different) h , $hNh^{-1} \cap Z_H(j) \supset G$. This completes the proof.

Lemma 3.10. *There is a finitely generated discrete group $F \subset \Gamma$ such that:*

- (a) *There is a G -invariant subset of finite (smooth) measure in $J \backslash H / F$;*
- (b) *If N is the Zariski closure F , then N acts properly on $J \backslash H$; and*
- (c) *G is contained in N .*

Proof. Consider the diagram

$$\begin{array}{ccc} P_X & \longrightarrow & P_e = (J \backslash H)_e \\ \downarrow & & \downarrow \\ X & \longrightarrow & M_e \end{array}$$

This yields a diagram

$$\begin{array}{ccc} P_X / F & \longrightarrow & (J \backslash H)_e / F \\ \downarrow & & \downarrow \\ X & \longrightarrow & M_e \end{array}$$

By Proposition 3.8, there is a G -invariant section of $P_X/F \rightarrow X$ and, hence, a finite G -invariant measure θ_e in P_X/F that projects to the given measure on X . Pushing this measure forward to $(J\backslash H)_e/F$, we obtain a finite G -invariant measure on this space that projects to μ_e . This is all still within one ergodic component. The space of finitely generated subgroups of Γ is countable. Therefore, there is a set of ergodic components of positive ν -measure, say E_0 , on which the assignment $e \mapsto F$ will be constant. The measure $\theta = \int_{e \in E_0}^{\oplus} \theta_e$ is a finite G -invariant measure on $J\backslash H/F$ that projects to a set of positive measure in M (namely, the union of the ergodic components $e \in E_0$). Since the bundle $J\backslash H/F \rightarrow M = J\backslash H/\Gamma$ has discrete fibers, θ is the restriction of the smooth measure on $J\backslash H/F$ to a set of positive measure. The remaining assertions of Lemma 3.10 now follow from Proposition 3.8, Lemma 3.9, and a conjugation.

Lemma 3.11. G fixes every point of $J\backslash H/N$.

Proof. By Rosenlicht's theorem [Ro], there is a Zariski open (\mathbb{C} -connected) $J \times G$ -invariant subset U of H/N such that $J\backslash U$ is a variety on which G acts algebraically. Pushing the measure of Lemma 3.10(a) forward under the map $J\backslash H/F \rightarrow J\backslash H/N$, we deduce that there is a finite G -invariant measure on $J\backslash U$ that is the restriction of a smooth measure to a subset of positive measure. Since the G -action on $J\backslash U$ is algebraic, the Borel density theorem implies any finite invariant measure is supported on G -fixed points. Therefore, we deduce that there is a set of G -fixed points in $J\backslash U$ of positive smooth measure, and, since the action is algebraic, G pointwise fixes $J\backslash U$. This means there is a dense set of G -fixed points in $J\backslash H/N$. However, since the N action on $J\backslash H$ is proper, $J\backslash H/N$ is Hausdorff, and hence G fixes all points of $J\backslash H/N$.

Completion of Proof of Theorem. Choose a 1-parameter \mathbb{R} -split subgroup $a(t)$ in G . Since the root spaces for $a(t)$ in \mathfrak{h} corresponding to nonzero roots generate a normal subgroup of H , it suffices to see that all these root spaces are in \mathfrak{n} . If $\mathfrak{u} \subset \mathfrak{h}$ is such a 1-dimensional eigenspace and $\mathfrak{u} \subset \mathfrak{j} + \mathfrak{n}$, then we must have $\mathfrak{u} \subset \mathfrak{n}$ since G (and in particular $a(t)$) centralizes \mathfrak{j} . Therefore, we need only consider $\mathfrak{u} \not\subset \mathfrak{j} + \mathfrak{n}$. Let $U(s) \subset H$ be the corresponding 1-parameter subgroup. Then

$$\begin{aligned} a(t)U(s)N &= a(t)U(s)a(t)^{-1}N \\ &= U(e^{\lambda t}s)N \quad (\text{since } a(t) \in G \subset N) \end{aligned}$$

for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. This means that $U^+ = \{U(s)N | s > 0\}$ is contained in a single G -orbit in H/N . However, since G acts trivially on $J\backslash H/N$, it follows that U^+ is contained in a single J -orbit. By Lemma 3.10, N acts properly on $J\backslash H$, so $J\backslash H/N$ is Hausdorff and the J -orbits in H/N are closed. Therefore, U^+ is contained in the J -orbit of eN , i.e. in $JN \subset H$. However, JN is a closed submanifold whose tangent space at $e \in H$ is $\mathfrak{j} + \mathfrak{n}$, and, since $\mathfrak{u} \not\subset \mathfrak{j} + \mathfrak{n}$, we have $U(s) \notin JN$ for $s > 0$ but sufficiently small. This contradiction shows all such \mathfrak{u} are in \mathfrak{n} and, hence, that $\mathfrak{n} = \mathfrak{h}$, $N = H$, and J is compact (by Lemma 3.10).

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