

INVARIANTS ON PROJECTIVE SPACE

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1. INTRODUCTION

This article is concerned with the construction of invariant nonlinear differential operators for projective space, $\mathbf{P}^n := \mathbf{P}(\mathbf{R}^{n+1})$. There is no loss in working at first on the *projective n -sphere* S^n , that is the space of rays in \mathbf{R}^{n+1} . The consequences for \mathbf{P}^n will be explained at the end of the article. Working on S^n avoids notational difficulties arising from the nonorientability of \mathbf{P}^n when n is even.

We shall use the general notion of a homogeneous bundle as in [2]. Suppose E and F are homogeneous vector bundles on X , a homogeneous space for some Lie group G . The polynomials of degree at most d on a vector space V themselves form a vector space, namely $\bigoplus_{h=0}^d \odot^h V^*$ (where \odot is the symmetric tensor product). Therefore, the G -invariant differential operators $E \rightarrow F$ of order k and which are polynomial in the jets of E are precisely the G -invariant homomorphisms

$$\bigoplus_{h=0}^d \odot^h J^k E \rightarrow F$$

for some d . Here $J^k E$ is the k th-jet bundle of E . Without loss of generality we can restrict attention to homomorphisms

$$\odot^h J^k E \rightarrow F$$

in which case the differential operator will be said to be of *degree h* .

Here $G = \mathrm{SL}(n+1)$ and our objective is to construct all such invariant differential operators on S^n when E is a line bundle and F is irreducible. We are almost successful in this endeavour. Statements of the results follow some preliminary definitions and notation (see Theorems 2.0.1, 2.0.2, and 2.0.3). These results have a purely algebraic interpretation as an analogue of Weyl's classical invariant theory [8] for certain P -modules. Similar problems are posed by Fefferman in [4, pp. 143 and 148] for other parabolics. These problems are given a geometric interpretation in [3] and the projective case is also noted there. It is this geometric interpretation which is our starting point in this article. Since

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this work an analogue of the approach used here together with some classical representation theory have been used by Bailey, Eastwood, and Graham [1] to solve Fefferman's problems completely.

As is common in differential geometry tensors are denoted by indexed quantities. These indices may be assumed to be abstract (see [6]) unless otherwise clear by the context. We will use parentheses (\dots) to indicate the completely symmetric part over the enclosed indices whilst brackets $[\dots]$ will indicate the completely antisymmetric part over the enclosed indices. When an index is repeated it indicates a contraction (for example, between a vector and its dual). Finally note that we will not distinguish vector bundles and their associated sheaves of germs of smooth sections and indeed will use the same symbol for each; which is intended should be clear from the context.

2. PRELIMINARIES

Let $W = \mathbf{R}^{n+1}$ be the defining representation space of $G = \mathrm{SL}(n+1, \mathbf{R})$ with volume form $\tilde{\epsilon}$. Denote by $e \in W$ the vector with coordinates

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and let P be the subgroup of G stabilising the ray through e . Then P is a parabolic subgroup consisting of elements of G of the form

$$\begin{pmatrix} \lambda & r_j \\ 0 & m'_j \end{pmatrix}, \quad \text{where } \lambda > 0,$$

and $G/P = S^n$, the projective n -sphere.

The homogeneous line bundles on S^n are induced from the representations

$$\begin{pmatrix} \lambda & r \\ 0 & m \end{pmatrix} \mapsto \lambda^{-w}, \quad w \in \mathbf{R},$$

and are denoted by $\mathcal{O}(w)$. (This notation is borrowed from complex projective space [7].) Let Z^I be standard coordinates on W . Sections of $\mathcal{O}(w)$ are represented by homogeneous functions $f(Z^I)$ of degree w , that is $f(\lambda Z^I) = \lambda^w f(Z^I)$ for $\lambda > 0$; such a function will be said to have *weight* w . The tangent bundle \mathcal{O}^i is induced from the representation

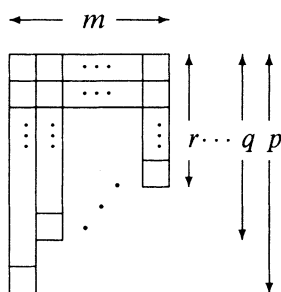
$$\begin{pmatrix} \lambda & r \\ 0 & m \end{pmatrix} \mapsto \lambda^{-1} m$$

and the dual representation induces \mathcal{O}_i , the cotangent bundle.

The irreducible homogeneous vector bundles, up to isomorphism, are given by

$$(p, q, \dots, r) \mathcal{O}_i(w) := (p, q, \dots, r) \mathcal{O}_i \otimes \mathcal{O}(w)$$

where $\underbrace{(p, q, \dots, r)}_m$ is a Young symmetry



meaning that sections of $(p, q, \dots, r) \mathcal{O}_i$ are covariant tensors

$$w_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q \dots \ell_1 \ell_2 \dots \ell_r} = w_{[i_1 i_2 \dots i_p][j_1 j_2 \dots j_q] \dots [\ell_1 \ell_2 \dots \ell_r]}$$

with further *underlying* symmetries as explained in [6]. We shall write v for the valence of a given tensor w , i.e., $v = p + q + \dots + r$. This classifies the irreducible homogeneous bundles save for the redundancy

$$(1) \quad (p, q, \dots, r) \mathcal{O}_i(w) = (n, p, q, \dots, r) \mathcal{O}_i(w + n + 1).$$

Weyl's classical invariant theory [8] constructs the invariants of a collection of tensors for any of the classical groups as linear combinations of an enumerable set of basic invariants. (It is not a classification since it does not distinguish the nonzero combinations.)

In a similar way one would like to construct the G -invariant differential operators

$$\mathcal{O}(w) \rightarrow E \quad \text{for } E \text{ an irreducible homogeneous bundle}$$

which are polynomial in the jets of $\mathcal{O}(w)$. From now on we shall refer to such operators as *invariants*. From our remarks above it suffices to construct all invariants

$$\mathcal{O}(w) \rightarrow Y \mathcal{O}_i(w')$$

for the various Young symmetries Y . This article shows how this can be done for $w \notin \{0, 1, 2, \dots\}$ whilst for $w \in \{0, 1, 2, \dots\}$ it is shown how to construct those invariants which are independent of the w -jet.

To explain the results precisely we need some more terminology and notation. Denote the homogeneous bundle $W \otimes \mathcal{O}(w)$ by $\mathcal{O}^I(w)$. Then the standard coordinates Z^I on W define a tautological section of $\mathcal{O}^I(1)$. The dual bundle to \mathcal{O}^I is written \mathcal{O}_I . There is a family of invariant differential operators

defined by

$$D_{\underbrace{IJ\dots K}_v} := \frac{\partial^v}{\partial Z^I \partial Z^J \dots \partial Z^K}.$$

Then for example

$$D_{IJ\dots K} : \mathcal{O}(w) \rightarrow \mathcal{O}_{IJ\dots K}(w - v).$$

The Euler sequence

$$0 \rightarrow \mathcal{O}_i(w) \rightarrow \mathcal{O}_I(w - 1) \xrightarrow{Z^I} \mathcal{O}(w) \rightarrow 0$$

implies that, for each Young symmetry Y , we have

$$(2) \quad Y\mathcal{O}_i(w) \hookrightarrow Y\mathcal{O}_I(w - v),$$

where the identification is with *saturated* tensors in $Y\mathcal{O}_I(w - v)$, that is tensors such that contraction of Z^I into any index results in annihilation. It is easy to construct saturated tensors. Suppose $Y = (\underbrace{p, q, \dots, r}_m)$. Let \tilde{Y} be the Young symmetry given by $(p + 1, q + 1, \dots, r + 1)$, and suppose

$$T_{I_0 I_1 \dots I_p J_0 J_1 \dots J_q \dots L_0 L_1 \dots L_r} \in \tilde{Y}\mathcal{O}_I(w' - v - m),$$

then

$$(3) \quad Z^{I_0} Z^{J_0} \dots Z^{L_0} T_{I_0 I_1 \dots I_p J_0 J_1 \dots J_q \dots L_0 L_1 \dots L_r} \in Y\mathcal{O}_I(w' - v)$$

and is saturated. Suppose now that T is constructed as follows. For $f \in \mathcal{O}(w)$ form a juxtaposition consisting of various $D_{IJ\dots K}f$ (and possibly f too) in such a way that the result has weight $w' - v - m$, the same valence as \tilde{Y} and degree h (so if $f \rightarrow \lambda f$ ($\lambda \in \mathbf{R}$) then $I(f) \rightarrow \lambda^h I(f)$). Now choose an ordering of the indices and apply the Young symmetry \tilde{Y} to form T . Then $w' = hw$ and (3) is an invariant

$$I : \mathcal{O}(w) \rightarrow Y\mathcal{O}_i(hw)$$

of degree h . We shall call invariants I which arise this way, and linear combinations thereof, *Weyl invariants*.

Weyl invariants are important since one can list a basic set of Weyl invariants from which all others are linear combinations.

Theorem 2.0.1. *Suppose $w \notin \{0, 1, 2, \dots\}$. All invariants of $\mathcal{O}(w)$ are Weyl invariants.*

Theorem 2.0.2. *Suppose $w \in \{1, 2, 3, \dots\}$. All invariants of $\mathcal{O}(w)$ which, at any point $x \in S^n$, are independent of $J_x^w(\mathcal{O}(w))$ are Weyl invariants except for those which depend only on $J_x^{w+1}(\mathcal{O}(w))$ and are of symmetry type (p, q, \dots, r) where $p = h$, the degree of the invariant. Invariants of the latter type are not Weyl invariants. On $\mathcal{O}(0)$ all invariants are Weyl invariants except for those nonzero invariants which are the exterior derivative of a monomial in the function.*

There are Weyl invariants which are not covered by these theorems. The following theorem goes some way toward remedying this situation.

Theorem 2.0.3. *For any weight $w \neq 0$, let $I(f)$ be an invariant of $f \in \mathcal{O}(w)$. Then for some $a \in \mathbf{N}$, $J(f) := f^a I(f)$ is a Weyl invariant.*

3. THE PROOFS

Before proving the theorems above we pause to consider what is already known about Weyl invariants using classical invariant theory and to prove a preliminary result (Theorem 3.1.3) using this. The proofs for the *generic weight case*, that is $w \notin \{0, 1, 2, \dots\}$, and the remaining exceptional weights then follow using elementary arguments.

3.1. Standard and primitive expressions. Denote by S_+^n the hemisphere of S^n where $Z^0 > 0$. S_+^n can be identified with R^n by

$$\begin{pmatrix} Z^0 \\ Z^i \end{pmatrix} \mapsto Z^i/Z^0 = y^i \in R^n.$$

S_+^n is an orbit of the subgroup H of G consisting of those elements of the form

$$\begin{pmatrix} \lambda & 0 \\ x^i & m_j^i \end{pmatrix} \quad \text{where } \lambda > 0.$$

In terms of the coordinates y^i the action is given by

$$y^i \mapsto \lambda^{-1}(x^i + m_j^i y^j).$$

Notice that if we restrict to $\lambda = 1$ then $\det m = 1$ and this gives the the proper (i.e., volume form preserving) affine transformations of R^n . The isotropy group of the origin is the subgroup of H with $x^i = 0$, i.e., the subgroup of G which is $H \cap P$:

$$H \cap P = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & m_j^i \end{pmatrix} \text{ s.t. } \lambda > 0 \text{ and } \lambda \det m = 1 \right\}.$$

Homogeneous bundles on S^n restrict to homogeneous bundles on $S_+^n = H/H \cap P$ and irreducible homogeneous bundles on S^n restrict to irreducible homogeneous bundles on S_+^n . Since $H \cap P$ is reductive all homogeneous bundles on S_+^n decompose into irreducibles. In particular, the Euler sequence splits as a sequence of bundles over S_+^n :

$$0 \rightarrow \mathcal{O}_i(w) \rightarrow \mathcal{O}_I(w-1) \rightarrow \mathcal{O}(w) \rightarrow 0.$$

In terms of *component functions*, with respect to the standard coordinates, the injection $\mathcal{O}_i(w) \hookrightarrow \mathcal{O}_I(w-1)$ is given by

$$f_i \mapsto (-(Z^0)^{-2} Z^i f_i, (Z^0)^{-1} f_i) \quad \text{where } i = 1, 2, \dots, n$$

whilst the splitting homomorphism is given by

$$g_I = (g_0, g_i) \mapsto Z^0 g_i.$$

Similarly, the various tensor powers of the Euler sequence split. In particular

$$\mathcal{O}_{(ij\dots k)}(w) \xrightarrow{\sim} \mathcal{O}_{(IJ\dots K)}(w - v)$$

where, as usual, v is the valence of the tensors. Hence we obtain the family of H -invariant differential operators

$$\nabla_{ij\dots k} : \mathcal{O}(w) \rightarrow \mathcal{O}_{(ij\dots k)}(w)$$

as the composition of these splitting homomorphisms with the operators

$$D_{IJ\dots K} : \mathcal{O}(w) \rightarrow \mathcal{O}_{(IJ\dots K)}(w - v).$$

In terms of the coordinates Z^I and $y^i = Z^i/Z^0$ ($i = 1, 2, \dots, n$) we have

$$\begin{aligned} \nabla_{ij\dots k} f &= (Z^0)^v \frac{\partial^v}{\partial Z^i \partial Z^j \dots \partial Z^k} f \\ &= \frac{\partial^v}{\partial y^i \partial y^j \dots \partial y^k} f. \end{aligned}$$

Thus we can write $\nabla_{ij\dots k} f = \nabla_i \nabla_j \dots \nabla_k f$ where ∇_i means the standard flat affine connection on $R^n (\cong S_+^n)$.

That an invariant is polynomial in the jets of f at some point $x \in S_+^n$ is equivalent to it being polynomial in $f(x)$ and the various ∇_i derivatives of f at x , $\nabla_{ij\dots k} f(x)$. Moreover, an invariant restricted to S_+^n is certainly H -invariant. Now recall that H contains the n -dimensional proper affine group (as the $\lambda = 1$ case) so any projective invariant must be invariant under these affine motions. Thus, by Weyl's invariant theory for $SL(n + 1, \mathbf{R})$ [8] we conclude that each invariant can be written as a linear combination of terms each of which is a juxtaposition of f , various ∇_i derivatives of f and the volume form $\epsilon \in \wedge^n W$. We shall always remove occurrences of the volume form using (1).

Definition 3.1.1. Standard Expression: Such an expression for an invariant in terms of just f and the various $\nabla_{ij\dots k} f$ will take values in $Y \mathcal{O}_i(hw)$, where as usual h is the degree of the invariant and where Y is some Young symmetry. Without loss of generality we may assume that this expression enjoys the Young symmetry Y even as a formal expression. We call this a *standard* expression for the invariant.

One can use the invariance of the operators $D_{IJ\dots K}$ and (2) to write down examples of invariants. For example if $f \in \mathcal{O}(1)$ then $D_{(IJ} f D_{KL)} f$ is saturated so it is an invariant and

$$(4) \quad D_{(IJ} f D_{KL)} f = \nabla_{(ij} f \nabla_{k\ell)} f.$$

The right-hand side here is a standard expression for the invariant. We need a name for the left-hand side expression.

Definition 3.1.2. Primitive Expression: It follows from (2) that any saturated tensor in $Y \mathcal{O}_I(w')$ which is constructed as a linear combination of terms, each of which is a juxtaposition of $f \in \mathcal{O}(w)$, and various $D_{IJ\dots K}f$ is an invariant. An expression for it as a saturated tensor constructed from f and the $D_{IJ\dots K}f$ will be said to be a *primitive* expression.

Note that it follows from the definition of $\nabla_{ij\dots k}$ that a standard expression is obtained from a primitive expression simply by formally replacing each $D_{IJ\dots K}f$ by $\nabla_{ij\dots k}f$. Remarkably, this works the other way:

Theorem 3.1.3. *For all $w \in \mathbf{R}$ there is a primitive expression for each invariant of $f \in \mathcal{O}(w)$. It may be obtained from a standard expression by formally replacing each $\nabla_{ij\dots k}f$ by $D_{IJ\dots K}f$.*

Proof. Let I be a nonzero invariant of $f \in \mathcal{O}(w)$ with degree h and order k (as a differential operator). Then, at $e \in S^n$, I is given by a P -module homomorphism

$$(5) \quad \tilde{\phi} : \odot^h J_e^k \mathcal{O}(w) \rightarrow Y \mathcal{O}_i(hw)_e$$

for some Young symmetry Y ; here $Y \mathcal{O}_i(hw)_e$ is the fibre over $e \in S^n$ of $Y \mathcal{O}_i(hw)$ while $J_e^k \mathcal{O}(w)$ is the fibre over e of the bundle of k -jets of $\mathcal{O}(w)$. Instead we can regard the modules in (5) as \mathfrak{p} -modules where \mathfrak{p} is the Lie algebra of P . Then $\tilde{\phi}$ is equivalent to a \mathfrak{p} -epimorphism which we denote by the same symbol. We need the following result. Let \mathfrak{g} be the Lie algebra of G (so \mathfrak{p} is a parabolic subalgebra of \mathfrak{g}), F an irreducible \mathfrak{g} -module and $\mathcal{U}(\mathfrak{p})$ the universal enveloping algebra of \mathfrak{p} . Then the *highest weight space* of F is in the $\mathcal{U}(\mathfrak{p})$ orbit of any other weight space of F . (This is because all the *raising operators* of \mathfrak{g} are in \mathfrak{p} . Explanation of these terms and the associated theory can be found in [5].) It follows that any nonempty \mathfrak{p} -submodule of F must contain the highest weight space of F . Thus any two nonzero \mathfrak{p} -submodules of F intersect in a nonzero submodule.

Choose a standard form of the invariant. This is a formula in terms of f and the various $\nabla_{ij\dots k}f$. Let J be the expression obtained by formally replacing each $\nabla_{ij\dots k}f$ with $D_{IJ\dots K}f$. Then J is an invariant differential operator taking values in $Y \mathcal{O}_I(hw - v)$ where v is the valence of Y . Thus J corresponds to a \mathfrak{p} -homomorphism

$$\Phi : \odot^h J_e^k \mathcal{O}(w) \rightarrow Y \mathcal{O}_I(hw - v)_e.$$

Let ι be the natural \mathfrak{p} -monomorphism (c.f. (2))

$$\iota : Y \mathcal{O}_i(hw)_e \hookrightarrow Y \mathcal{O}_I(hw - v)_e.$$

Denote by ϕ the composition of this with $\tilde{\phi}$. Let ℓ be the Lie algebra of $H \cap P$. As an ℓ -module ι has a left inverse π and by the definition of $\nabla_{ij\dots k}$

we have

$$\tilde{\phi} = \pi_o \Phi.$$

It follows that the range of the \mathfrak{p} -homomorphism

$$(\Phi - \phi) : \odot^h J_e^k \mathcal{O}(w) \rightarrow Y \mathcal{O}_I(hw - v)_e$$

is a \mathfrak{p} -submodule of the irreducible \mathfrak{g} -module $Y \mathcal{O}_I(hw - v)_e$ whose intersection with $Y \mathcal{O}_i(hw)_e$ is zero. Therefore this range is zero as required. We have worked so far at e . However a similar argument applies at any other point of S_+^n . Hence the result follows. \square

3.2. The generic weight case. Showing all invariants of $\mathcal{O}(w)$ are Weyl invariants turns out to be straightforward when $w \notin \{0, 1, 2, \dots\}$. This is due to the following lemma.

Lemma 3.2.1. *For $w \notin \{0, 1, 2, \dots, k - 1\}$ the map*

$$(6) \quad \psi : J^k \mathcal{O}(w) \rightarrow \underbrace{\mathcal{O}_{(IJ \dots N)}}_k(w - k)$$

given at $x \in S^n$ by

$$f \mapsto D_{IJ \dots N} f(x),$$

for $f \in J_x^k \mathcal{O}(w)$, is an isomorphism.

Proof. If $f \in \mathcal{O}(w)$ then

$$Z^K D_{\underbrace{KL \dots N}_{\ell+1}} f = (w - \ell) D_{L \dots N} f.$$

Thus if $w \notin \{0, 1, \dots, k - 1\}$ and $f \in J_x^k \mathcal{O}(w)$ is in the kernel of ψ_x then (by the definition of $\nabla_{ij \dots k}$ in terms of $D_{IJ \dots K}$) it follows that

$$f(x) = 0 \text{ and } \nabla_{\underbrace{kl \dots n}_{\ell}} f(x) = 0 \quad 1 \leq \ell \leq k$$

and so $f = 0$. Thus ψ_x is a monomorphism and the result follows from dimensional considerations. \square

To prove Theorem 2.0.1 we shall begin with an invariant I of $\mathcal{O}(w)$ and describe explicitly how to construct a tensor J from which I arises as in (3). We use the isomorphism (6) of the lemma and the related result that at $x \in S^n$ the maps

$$f \mapsto D_{\underbrace{LM \dots N}_{\ell}} f(x),$$

for $f \in \mathcal{O}(w)$, correspond to contracting a complementary number of Z^I 's into the appropriate tensor in $\mathcal{O}_{IJ\dots N}(w-k)_x$

$$D_{IJ\dots KLM\dots N}f \mapsto \frac{1}{(w-k+1)(w-k+2)\dots(w-\ell)} \underbrace{Z^I Z^J \dots Z^K}_{k-\ell} D_{IJ\dots KLM\dots N}f = D_{LM\dots N}f.$$

We will also need the next proposition which is an elementary fact concerning Young symmetries. A tensor $T_{IJ\dots K}$ is said to X^I -saturated if contraction of X^I into any index results in annihilation.

Proposition 3.2.2. *Suppose for arbitrary X^I*

$$\underbrace{X^{I_0} X^{J_0} \dots X^{L_0} X^{M_0} X^{N_0} \dots X^{P_0}}_{m'} J_{I_0 J_0 \dots L_0 M_0 N_0 \dots P_0} I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r \neq 0$$

is X^I -saturated and has the Young symmetry given by the m -tuple (p, q, \dots, r) , $r > 0$. Then $m' \geq m$ and if we let

$$K_{I_0 I_1 \dots I_p J_0 J_1 \dots J_q \dots L_0 L_1 \dots L_r M_0 N_0 \dots P_0} := J_{(I_0 J_0 \dots L_0 M_0 N_0 \dots P_0) I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}$$

then

$$K_{[I_0 I_1 \dots I_p][J_0 J_1 \dots J_q]\dots[L_0 L_1 \dots L_r]M_0 N_0 \dots P_0}$$

has the Young symmetry

$$\underbrace{(p+1, q+1, \dots, r+1, 1, 1, \dots, 1)}_{m'}$$

and symmetrising over $I_0 J_0 \dots L_0 M_0 N_0 \dots P_0$ of the last tensor returns K up to scale.

Proof of Theorem 2.0.1. Let $I_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q \dots \ell_1 \ell_2 \dots \ell_r}$ be an invariant of degree h in $f \in \mathcal{O}(w)$ and with the Young symmetry given by the m -tuple (p, q, \dots, r) . By Theorem 3.1.3 there exists a primitive expression for this invariant,

$$I_{I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}$$

Recall that this is saturated, also has the irreducible symmetry (p, q, \dots, r) and is constructed entirely from f and various $D_{IJ\dots K}f$. Let us assume that the order of I is k (i.e., the invariant depends only on the k -jets of f).

If now in the primitive expression for I we formally replace each $D_{LM\dots N}f$ with

$$\frac{1}{(w-k+1)(w-k+2)\dots(w-\ell)} \underbrace{Z^I Z^J \dots Z^K}_{k-\ell} D_{IJ\dots KLM\dots N}f$$

then we get

$$(7) \quad I_{I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} = \underbrace{Z^{I_0} Z^{J_0} \dots Z^{P_0}}_{m'} J_{I_0 J_0 \dots P_0} I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r$$

Here the tensor

$$J_{I_0 J_0 \dots L_r} \in \odot^h(\mathcal{O}_{\underbrace{LM \dots N}_k}(w - k))$$

is a linear combination of terms each of which is an h -fold juxtaposition of

$$T_{IJ \dots K} := D_{\underbrace{IJ \dots K}_k} f$$

with various permutations of the indices. Write $J_{I_0 J_0 \dots L_r}(T)$ to indicate this. Furthermore (7) is Z^I -saturated and has an irreducible symmetry in the class (p, q, \dots, r) . Now by the lemma the $D_{IJ \dots K} f(x)$ (for various $f \in \mathcal{O}(w)$) span $\mathcal{O}_{IJ \dots K}(w - k)_x$ (which as a vector space is isomorphic to $\odot^k W_I$). It follows that the expression (7) remains saturated if we replace $T_{IJ \dots K}$ with any other tensor in $\mathcal{O}_{IJ \dots K}(w - k)$. Therefore

$$X^{I_0} X^{J_0} \dots X^{P_0} J_{I_0 J_0 \dots P_0 I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}(T)$$

must be X^I -saturated for arbitrary $X^I \in W^I$. Thus $m' \geq m$ and if we form K from J as in the proposition then

$$\begin{aligned} & Z^{M_0} Z^{N_0} \dots Z^{P_0} K_{[I_0 I_1 \dots I_p][J_0 J_1 \dots J_q] \dots [L_0 L_1 \dots L_r] M_0 N_0 \dots P_0} \\ & \in (p + 1, q + 1, \dots, r + 1) \mathcal{O}_I(hw - v), \end{aligned}$$

where $v = hk - m' + m$, and

$$Z^{I_0} Z^{J_0} \dots Z^{L_0} Z^{M_0} Z^{N_0} \dots Z^{P_0} K_{[I_0 I_1 \dots I_p][J_0 J_1 \dots J_q] \dots [L_0 L_1 \dots L_r] M_0 N_0 \dots P_0}$$

is I up to scale. \square

Example 3.2.3. Consider the invariant

$$(8) \quad wf \nabla_{k\ell} f - (w - 1) \nabla_k f \nabla_\ell f.$$

The primitive expression for this is

$$wf D_{IJ} f - (w - 1) D_I f D_J f.$$

If in this we replace f by $\frac{1}{w(w-1)} Z^K Z^L D_{KL} f$ and each $D_I f$ by $\frac{1}{w-1} Z^K D_{KI} f$ we obtain

$$\frac{2}{(w-1)} Z^K Z^L D_{K[L} f D_{I]J} f.$$

3.3. The exceptional weight case. Let us begin by treating a simple example. Recall the invariant (8) once more. Let I_{IJ} be the primitive expression for this and consider

$$J_{KLJ} := D_{[K]I_{LJ}}.$$

Note that

$$Z^K J_{KLJ} = \frac{u+1}{2} I_{LJ} \text{ and } Z^J J_{KLJ} = I_{[KL]} = 0,$$

where u is the weight of I_{IJ} . Using these properties it is easy to show that

$$K_{KLIJ} := D_{[I]J_{|KL|J}}$$

(where $|KL|$ means the skew symmetrisation ignores the indices KL) satisfies

$$Z^I Z^K K_{KLIJ} = \frac{(u+1)u}{4} I_{LJ}.$$

Thus, since K_{KLIJ} has the Young symmetry



I is shown to be a Weyl invariant unless $u = -1$ or $u = 0$ (i.e., $w = 1/2$ or $w = 1$).

We shall prove Theorems 2.0.2 and 2.0.3 by generalising this procedure. Let

$$I_{I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} \in (p, q, \dots, r) \mathcal{O}_I(u)$$

be a primitive form of an invariant. Define

$$J_{I_0 I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} := D_{[I_0] I_{I_1 I_2 \dots I_p} J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}.$$

Then

$$Z^{I_0} J_{I_0 I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} = \frac{u+p}{p+1} I_{I_1 I_2 \dots L_r},$$

since $I_{I_1 I_2 \dots L_r}$ is saturated and

$$Z^I D_I I_{JK \dots T} = u I_{JK \dots T}.$$

Thus contraction of Z^I into any of the first $p + 1$ indices of J yields $\pm \frac{u+p}{p+1} I$. On the other hand contraction of Z^I into any other index of J gives zero. For the next stage we wish to ignore the first $(p + 1)$ indices of $J_{I_0 I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}$. Let us write $J_{*J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}$ to suggest this (rather than use $|\dots|$ as in the above example). Now form

$$K_{*J_0 J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} := D_{[J_0] J_{*J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}}$$

where the suppressed indices are not involved in the skewing. By an almost identical calculation to that above, one can verify that contracting Z^I into any one of the indices $J_0, J_1, J_2, \dots, J_q$ of K gives $\pm \frac{u+q-1}{q+1} J$ whereas K is killed by contraction of Z^I into any other (nonsuppressed) index. Now we can also suppress the indices $J_0, J_1, J_2, \dots, J_q$ of K and repeat this process once again. Continuing in this manner until all the sets of skew indices of I (i.e., the sets

corresponding to the columns of its Young diagram) have been dealt with, let us denote by ΔI the resulting tensor.

By construction ΔI has Young symmetry of type

$$\underbrace{(p + 1, q + 1, \dots, r + 1)}_m$$

and

$$\begin{aligned} & Z^{I_0} Z^{J_0} \dots Z^{L_0} \Delta I_{I_0 I_1 \dots I_p J_0 J_1 \dots J_q \dots L_0 L_1 \dots L_r} \\ &= \frac{(u+p)(u+q-1) \dots (u+r-m+1)}{(p+1)(q+1) \dots (r+1)} I_{I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r} \end{aligned}$$

It follows immediately that if

$$(9) \quad (u + p)(u + q - 1) \dots (u + r - m + 1) \neq 0$$

then I is a Weyl invariant.

Proof of Theorem 2.0.2. Let f be homogeneous of weight $w \neq 0$ and suppose

$$I_{i_1 i_2 \dots i_p j_1 j_2 \dots j_q \dots \ell_1 \ell_2 \dots \ell_r}(f) \in (p, q, \dots, r) \mathcal{O}_i(hw)$$

is an invariant of the type considered in the theorem. Let $I_{I_1 I_2 \dots I_p J_1 J_2 \dots J_q \dots L_1 L_2 \dots L_r}$ be a primitive expression for this. Then since

$$D_{\underbrace{IJ \dots K}_{w+1}} f$$

has weight -1 it follows that the weight u of $I_{I_1 I_2 \dots L_r}$ satisfies with equality if and only if, at each point x , I depends only on $J_x^{w+1}(\mathcal{O}(w))$. Here, as usual, h denotes the degree of $I_{I_0 I_1 \dots L_r}$. Now if I is nonzero then $p \leq h$ and thus

$$u \leq -p < -(q - 1) < \dots < -(r - m + 1).$$

We consider the cases $u < -p$ and $u = -p$:

- If $u < -p$ then it follows at once from (9) that I is a Weyl invariant.
- If $u = -p$ then without loss of generality we may assume that, at each point x , I depends only on $J_x^{w+1}(\mathcal{O}(w))$. If also I is a Weyl invariant then $I = 0$ since it must arise (in the manner of (3)) from a tensor $T_{IJ \dots K}$ which is of degree $h = p$ in f yet possesses the Young symmetry $(p + 1, q + 1, \dots, r + 1)$. $T_{IJ \dots K} = 0$ is the only such tensor.

The second part of the theorem, dealing with the $w = 0$ case is proved similarly. Any invariant of $\mathcal{O}(0)$ has $u \leq -p$. The only cases of $u = -p$ are for the exceptional invariants of the type mentioned in the statement of the theorem. Following the same line of argument as above one quickly concludes that again invariants satisfying $u < -p$ are Weyl invariants whereas otherwise $u = -p = -1$ and the invariant is not a Weyl invariant. \square

Proof of Theorem 2.0.3. Let $I_{ij\dots k}$ be an invariant of $\mathcal{O}(w)$, $w \neq 0$. Then, for any $a \in \mathbf{N}$,

$$J := f^a I(f)$$

is also an invariant. It is clearly possible to choose a sufficiently large so that the degree u of J satisfies (9) and so J is a Weyl invariant. \square

Remark 3.3.1. An example of an invariant which is not a Weyl invariant is $\nabla_{ij}g$ for $g \in \mathcal{O}(1)$. Although not directly relevant to the present article, it is interesting to observe that this invariant and the Weyl invariant (8) are part of the same family in the sense that

$$wf\nabla_{ij}f - (w - 1)\nabla_i f \nabla_j f = w^2 f^{2-1/w} \nabla_{ij} f^{1/w}.$$

One can generate families of interesting nonlinear invariants in this way beginning with any linear invariant operator. More generally all invariants on densities of any given nonzero weight can be obtained from the list of all invariants on densities of any other given nonzero weight using this technique. As an application of this it is easy to construct an alternative proof of Theorem 2.0.3 using this idea and Theorem 2.0.2.

4. RESULTS FOR \mathbf{P}^n

The results above for S^n carry over to \mathbf{P}^n either directly or with some reinterpretation according to the parity of n . Let us consider each case in turn.

Suppose n is even. Then $G = \text{SL}(n + 1, \mathbf{R})$ acts simply and transitively on \mathbf{P}^n , and the subgroup

$$P = \left\{ \begin{pmatrix} \lambda & r_j \\ 0 & m_j^i \end{pmatrix} \right\}$$

stabilises the line in \mathbf{R}^{n+1} through e (where e is as in §2). Thus there are two families of homogeneous line bundles on \mathbf{P}^n :

$$\mathcal{E}(w) \text{ induced from the representations } \begin{pmatrix} \lambda & r \\ 0 & m \end{pmatrix} \mapsto |\lambda|^{-w},$$

and

$$\mathcal{F}(w) \text{ induced from the representations } \begin{pmatrix} \lambda & r \\ 0 & m \end{pmatrix} \mapsto \text{sign}(\lambda)|\lambda|^{-w}.$$

Similarly if we write \mathcal{E}_i to denote the tangent bundle to \mathbf{P}^n then corresponding to each even bundle $\mathcal{E}_{ij\dots k}(w) := \underbrace{\otimes^\ell \mathcal{E}_i}_{\ell} \otimes \mathcal{E}(w)$ is the associated odd bundle

$\mathcal{F}_{ij\dots k} := \mathcal{F}(0) \otimes \mathcal{E}_{ij\dots k}$. Now all the analysis above for S^n carries over to \mathbf{P}^n if we just keep track of which sort of bundle we have. For example an invariant acting on $\mathcal{E}(w)$ will take values in another even bundle if the valence of the operator is even, otherwise it will take values in an odd bundle. With this taken

into account the results from S^n apply also to \mathbf{P}^n in particular Theorems 3.1.3, 2.0.1, 2.0.2, and 2.0.3 hold.

If n is odd then $G = \mathrm{SL}(n+1, \mathbf{R})/\pm I$ (where I is the identity) acts simply and transitively on \mathbf{P}^n in the obvious way. The subgroup

$$P = \left\{ \begin{pmatrix} \lambda & r_j \\ 0 & m_j \end{pmatrix} \right\} / \pm I \cong \left\{ \begin{pmatrix} \lambda & r_j \\ 0 & m_j \end{pmatrix} \text{ such that } \lambda > 0 \right\}$$

stabilises the line through $e \in \mathbf{R}^{n+1}$. Since P is isomorphic to the isotropy group of S^n discussed above it follows easily that the G -invariant differential operators on \mathbf{P}^n all arise naturally from the $\mathrm{SL}(n+1)$ -invariant differential operators on the 2-1 covering space S^n and the Theorems 3.1.3, 2.0.1, 2.0.2, and 2.0.3 carry over unchanged in this case also.

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