

MANIFOLDS OF POSITIVE RICCI CURVATURE WITH ALMOST MAXIMAL VOLUME

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1°. In this note we consider complete Riemannian manifolds with Ricci curvature bounded from below. The well-known theorems of Myers and Bishop imply that a manifold M^n with $\text{Ric} \geq n - 1$ satisfies $\text{diam}(M^n) \leq \text{diam}(S^n(1))$, $\text{Vol}(M^n) \leq \text{Vol}(S^n(1))$. It follows from [Ch] that equality in either of these estimates can be achieved only if M^n is isometric to $S^n(1)$. The natural conjecture is that a manifold M^n with almost maximal diameter or volume must be a topological equivalent to S^n . With respect to diameter this is true only if M^n satisfies some additional assumptions; see [An, O, GP, E]. With respect to volume however no extra restriction is necessary.

Theorem 1. *For any integer $n \geq 2$ there exists $\delta_n > 0$ with the following property. Let M^n be a complete Riemannian manifold with $\text{Ric} \geq n - 1$. Suppose that $\text{Vol}(M^n) \geq (1 - \delta_n) \text{Vol}(S^n(1))$. Then M^n is homeomorphic to S^n .*

In fact, we prove only that $\pi_i(M^n) = 0$ for all $i < n$ and refer to the work of Hamilton [H] for $n = 3$ and to the solution of generalized Poincaré conjecture (Smale [S], Freedman [F]) for $n \neq 3$.

Vanishing of homotopy groups is a simple consequence of the Main Lemma below. Its further simple corollaries are a noncompact version of Theorem 1 and a corresponding finiteness theorem (cf. [P, Corollary B]).

Let $B^H(R)$ denote a ball of radius R in the simply connected space form of constant curvature H .

Theorem 2. *Let M^n be a complete Riemannian manifold with $\text{Ric} \geq 0$; $p \in M$. Suppose that $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^0(R))$ for all $R > 0$. Then M^n is contractible.*

Theorem 3. *For any n, H, \mathcal{D}, R the set $\mathcal{M}_{\delta_n}(n, H, \mathcal{D}, R)$ of all complete Riemannian manifolds M^n with $\text{diam}(M^n) \leq \mathcal{D}$, $\text{Ric} \geq (n - 1)H$, and $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^H(R))$ for all $p \in M^n$, contains only finitely many homotopy types.*

2°. Henceforward we fix $n \geq 2$ and ignore the dependence on n in our notations. We denote by M an arbitrary compact n -dimensional Riemannian manifold with $\text{Ric} \geq n - 1$; all parameters below are supposed to be independent of M .

Received by the editors January 7, 1993.

1991 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C23.

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0894-0347/94 \$1.00 + \$.25 per page

Main Lemma. For any $c_2 > c_1 > 1$ and integer $k \geq 0$ there exists $\delta = \delta_k(c_1, c_2) > 0$ with the following property. Let $p \in M$, $0 < R < \pi c_2^{-1}$. Suppose that $\text{Vol}(B_q(\rho)) \geq (1 - \delta) \text{Vol}(B^1(\rho))$ for every ball $B_q(\rho) \subset B_p(c_2 R)$. Then

(A) Any continuous map $f: S^k = \partial D^{k+1} \rightarrow B_p(R)$ can be continuously extended to a map $g: D^{k+1} \rightarrow B_p(c_1 R)$.

(B) Any continuous map $f: S^k \rightarrow M \setminus B_p(R)$ can be continuously deformed to a map into $M \setminus B_p(c_1 R)$.

Remark. The Main Lemma can obviously be modified for n -manifolds with $\text{Ric} \geq 0$ or $\text{Ric} \geq -(n-1)$, in the latter case δ may depend on R as $R \rightarrow \infty$.

We give below a detailed proof of (A) and outline a similar proof of (B) leaving the details to the reader.

3°. At first we state explicitly all the properties of manifolds with Ricci curvature bounded from below, which are relevant to the proof.

Let \overline{ab} denote a shortest geodesic with endpoints a, b .

(κ) There exists a positive function κ , $\kappa(t) \rightarrow 0$ as $t \rightarrow 0$, such that

$$|ab| + |ac| - |bc| \leq \kappa \left(|a, \overline{bc}| / \min\{|ab|, |ac|\} \right) \cdot |a, \overline{bc}| \quad \text{for all } a, \overline{bc} \subset M.$$

This is a weakened version of the Abresch-Gromoll inequality [AG].

(γ) For any $c_2 > c_1 > 1$, $\epsilon > 0$ there exists $\gamma = \gamma(c_1, c_2, \epsilon)$ with the following property. Let $p \in M$, $0 < R < \pi c_2^{-1}$. Suppose that $\text{Vol}(B_p(c_2 R)) \geq (1 - \gamma) \text{Vol}(B^1(c_2 R))$. Then for every $a \in B_p(R)$ there exists $b \in M \setminus B_p(c_1 R)$ such that $|a, \overline{pb}| \leq \epsilon R$.

This is a simple corollary of (the proof of) the Bishop-Gromov volume comparison inequality.

Warning. In the proof of the Main Lemma, we do not use the existence of the injectivity radius and avoid explicit induction on R .

4°. **Outline of the proof of (A).** Assertion (A) is proved by induction on k . The case $k = 0$ is obvious. Assume that (A) holds in dimensions less than k . Fix $c_2 > c_1 > 1$ and let $d_0 > 0$ and $\delta > 0$ be small enough. Now given M, p, R , satisfying the conditions of (A), and a continuous map $f: S^k \rightarrow B_p(R)$, we can construct another continuous map $\tilde{f}: S^k \rightarrow B_p((1-d_0)R)$ such that the uniform distance between f and \tilde{f} is small in comparison with R . This is the crucial step, it uses both properties (κ), (γ) and the inductive assumption.

The map \tilde{f} is not known yet to be homotopic to f , and there is no obvious way to construct such a homotopy at once. To go around this difficulty, we take a fine triangulation of S^k and construct a “small” homotopy between f and \tilde{f} on the $(k-1)$ -skeleton of this triangulation. In fact, the homotopy is constructed consecutively on i -skeleta, $i = 0, 1, \dots, k-1$, using the inductive assumption.

The result of previous steps can be interpreted as an extension of f from $S^k = \partial D^{k+1}$ to the k -skeleton of a finite cell decomposition of D^{k+1} . Recall

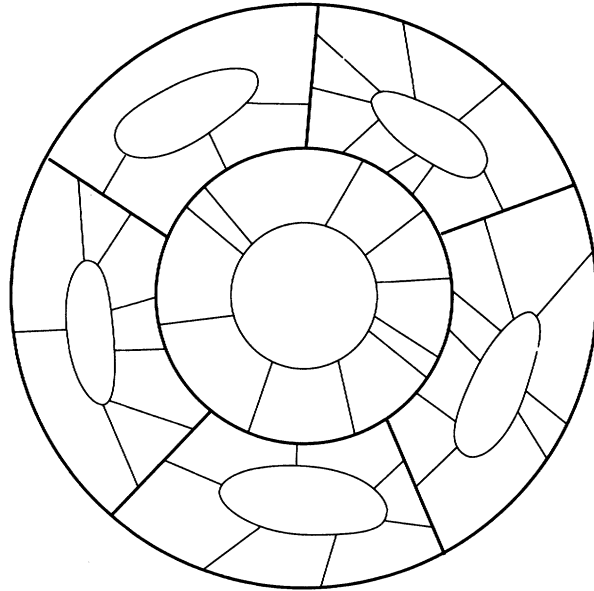


FIGURE 1

that the boundary of the “central” cell is mapped into $B_p((1 - d_0)R)$, and the size of the images of the boundaries of all other cells is small in comparison with R . Now we repeat the previous steps for each cell separately and obtain an extension of f to the k -skeleton of a finer cell decomposition (Figure 1), etc. The limit of the infinite repetition of this procedure is the required extension g .

Apparently the argument above cannot be convincing until the choice of “small” parameters is specified. We give a formal exposition below.

5°. **Proof of (A).**

5.1. Consider the following general situation. Let $f: S^k \rightarrow B_p(R) \subset M$ be a continuous map, and let sequences of finite cell subdivisions K_j of D^{k+1} and continuous maps $f_j: \text{skel}_k K_j \rightarrow M$ satisfy

- (a) K_{j+1} is a cell subdivision of K_j and $f_{j+1} \equiv f_j$ on $\text{skel}_k(K_j)$.
- (b) For each $(k + 1)$ -cell $\sigma \in K_j$ there exist $p_\sigma \in B_p(c_1 R)$ and $R_\sigma > 0$ such that $f_j(\partial\sigma) \subset B_{p_\sigma}(R_\sigma)$ and

$$B_{p_{\sigma'}}(c_1 R_{\sigma'}) \subset B_{p_\sigma}(c_1 R_\sigma) , \quad R_{\sigma'} \leq (1 - d_0)R_\sigma$$

(for a positive constant d_0), in case $\sigma \in K_j$, $\sigma' \in K_{j+1}$, $\sigma' \subset \sigma$.

- (c) $\text{skel}_k(K_0) = S^k = \partial D^{k+1}$, $f_0 \equiv f$, $R_{\sigma_0} = R$ for $\sigma_0 = D^{k+1} \setminus S^k \in K_0$.

Then there exists a continuous map $g: D^{k+1} \rightarrow B_p(c_1 R)$, such that $g \equiv f_j$ on $\text{skel}_k(K_j)$ for all j .

Indeed, let $g(x) = \lim_{j \rightarrow \infty} p_{\sigma_j}$ for some sequence of $(k + 1)$ -cells $\sigma_j \in K_j$,

such that $\sigma_{j+1} \subset \sigma_j$ and $x \in \text{clos}(\sigma_j)$ for all j . Obviously p_{σ_j} form a Cauchy sequence for any such $\{\sigma_j\}$, and moreover, $|g(x)p_{\sigma_j}| \leq (1 - d_0)^j c_1 R$. The sequence σ_j is defined unambiguously if $x \notin \cup_j \text{skel}_k K_j$, and it is clear that $g(x) = f_j(x)$ if $x \in \text{skel}_k K_j$; therefore g is correctly defined and continuous.

5.2. Specify the choice of d_0 and δ in the following way. Let $d_0 > 0$ be so small that for suitably chosen positive numbers d_1, \dots, d_k ,

$$d_{i+1}/d_i > 100, \quad d_0/d_i > 100k \cdot \kappa(100d_i/d_{i+1}), \\ 100d_k < 10^{-k}(1 + d_0/2k)^{-k}(1 - c_1^{-1})$$

hold, and let

$$\delta = \delta_k(c_1, c_2) \\ = \min\{\gamma(c_1, c_2, d_0), \delta_i(1 + d_0/2k, c_2), i = 0, 1, \dots, k - 1\}.$$

5.3. Assume that the conditions of (A) are satisfied. Then the extensions f_j from 5.1 can be constructed inductively using the following key assertion (see 6° for the proof).

(C) Given $\rho > 0, q \in M$, such that $\text{Vol}(B_q(c_2\rho)) \geq (1 - \delta) \text{Vol}(B^1(c_2\rho))$, a continuous map $\phi: S^k \rightarrow B_q(\rho)$ and a triangulation T of S^k such that $\text{diam}(\phi(\Delta)) \leq d_0\rho$ for all $\Delta \in T$, there exists a continuous map $\tilde{\phi}: S^k \rightarrow B_q((1 - d_0)\rho)$ such that

$$\text{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta)) \leq 10^{-k-1}(1 + d_0/2k)^{-k}(1 - c_1^{-1})\rho$$

for all $\Delta \in T$.

Indeed, represent a $(k + 1)$ -cell $\sigma \in K_j$ as $S^k \times (0, 1] \cup \{0\}$, choose a fine triangulation T of S^k and apply (C) to $f_j: S^k \times \{1\} \rightarrow B_{p_\sigma}(R_\sigma)$. (The volume condition is satisfied since it follows from 5.1 (b) that $B_{p_\sigma}(c_2R_\sigma) \subset B_p(c_2R)$.) Define K_{j+1} by $\sigma \cap \text{skel}_k(K_{j+1}) = S^k \times \{1/2\} \cup S^k \times \{1\} \cup \text{skel}_{k-1}(T) \times [1/2, 1]$, and let $f_{j+1} \equiv f_j$ on $S^k \times \{1\}$ and $f_{j+1} \equiv \tilde{f}_j$ on $S^k \times \{1/2\}$. Now f_{j+1} can be extended consecutively to $\text{skel}_i(T) \times [1/2, 1]$, $i = 0, 1, \dots, k - 1$, in such a way that

$$\text{diam } f_{j+1}(\Delta \times [1/2, 1]) \leq 10^{i-k}(1 + d_0/2k)^{i+1-k}(1 - c_1^{-1})R_\sigma$$

for all $\Delta \in \text{skel}_i(T)$. (Each extension to $\Delta \times [1/2, 1]$ from its boundary, for $\Delta \in \text{skel}_i T$, is ensured by the inductual assumption in dimension i and the inequality $\delta \leq \delta_i(1 + d_0/2k, c_2)$.) It is easy to check that $f_{j+1}: \text{skel}_k K_{j+1} \rightarrow M$ satisfies the conditions of 5.1, since the boundary of the ‘‘central’’ cell $S^k \times (0, 1/2] \cup \{0\}$ is mapped into $B_{p_\sigma}((1 - d_0)R_\sigma)$, and the images of the boundaries of all other cells have diameters less than $(1/2)(1 - c_1^{-1})R_\sigma$.

6°. **Proof of (C).** We construct $\tilde{\phi}$ consecutively on $\text{skel}_i(T)$, $i = 0, \dots, k$, to satisfy $\tilde{\phi}|_\Delta \equiv \phi|_\Delta$ if $\phi(\Delta) \subset B_q(\rho(1 - 2d_0))$,

$$(1) \quad \tilde{\phi}(\Delta) \subset B_q((1 - d_0(2 - i/k))\rho) ,$$

$$(2) \quad \text{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta)) \leq 10d_i\rho$$

for all $\Delta \in \text{skel}_i(T)$.

To begin with, define $\tilde{\phi}$ on $\text{skel}_0(T)$ by

$$\tilde{\phi}(x) \in \overline{\phi(x)q} , \quad |q\tilde{\phi}(x)| = \rho(1 - 2d_0) \quad \text{if } |q\phi(x)| > \rho(1 - 2d_0) .$$

Assume that $\tilde{\phi}$ is defined on $\text{skel}_i(T)$ for some $i < k$ and consider a $(i + 1)$ -simplex Δ , such that $\phi(\Delta) \not\subset B_q(\rho(1 - 2d_0))$. Applying (y) choose a point $r_\Delta \in M \setminus B_q(c_1\rho)$, such that $|\overline{qr_\Delta}, \phi(\Delta)| \leq d_0\rho$ and let $q_\Delta \in \overline{qr_\Delta}$ be such that $|qq_\Delta| = \rho(1 - d_{i+1})$; see Figure 2 on the next page. It follows from (2) and the choice of $\{d_i\}$ that for any $x \in \partial\Delta$

$$|\tilde{\phi}(x), \overline{q_\Delta r_\Delta}| < 20d_i\rho , \quad |\tilde{\phi}(x)q_\Delta| > d_{i+1}\rho/2 , \quad |\tilde{\phi}(x)r_\Delta| > d_{i+1}\rho/2 .$$

Hence we can apply (κ) to $\tilde{\phi}(x)$, $\overline{q_\Delta r_\Delta}$ and obtain

$$|\tilde{\phi}(x)r_\Delta| + |\tilde{\phi}(x)q_\Delta| - |q_\Delta r_\Delta| < 20\kappa(100d_i/d_{i+1})d_i\rho .$$

Adding this to the triangle inequality

$$|q_\Delta r_\Delta| + \rho(1 - d_{i+1}) = |qr_\Delta| \leq |\tilde{\phi}(x)r_\Delta| + |\tilde{\phi}(x)q|$$

and taking (1) into account we get

$$\begin{aligned} \tilde{\phi}(\partial\Delta) &\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - i/k) + 20\kappa(100d_i/d_{i+1})d_i)) \\ &\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (2i + 1)/2k))) , \end{aligned}$$

where the last inclusion follows from the choice of $\{d_i\}$.

Since $\dim \Delta = i + 1 \leq k$, the inductual assumption can be applied to extend $\tilde{\phi}$ from $\partial\Delta$ to Δ . It follows from the choice of δ that the extension satisfies

$$\tilde{\phi}(\Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (i + 1)/k))) .$$

It remains to observe that the last inclusion implies (1), (2) with i replaced by $i + 1$.

7°. The proof of (B) can be carried out along the same lines. An argument similar to the proof of (C) shows that a map $f: S^k \rightarrow M \setminus B_p(\rho)$, $R \leq \rho \leq c_1R$, can be transformed to a map \bar{f} with image outside a markedly larger ball, in such a way that $\text{diam}(f(\Delta) \cup \bar{f}(\Delta))$ is small for every simplex Δ of a fine triangulation T of S^k . A deformation from f to \bar{f} can be constructed consecutively on $\text{skel}_i(T)$, $i = 0, \dots, k$ making use of the assertion (A). After a bounded number of such deformations we obtain the required map with image outside $B_p(c_1R)$.

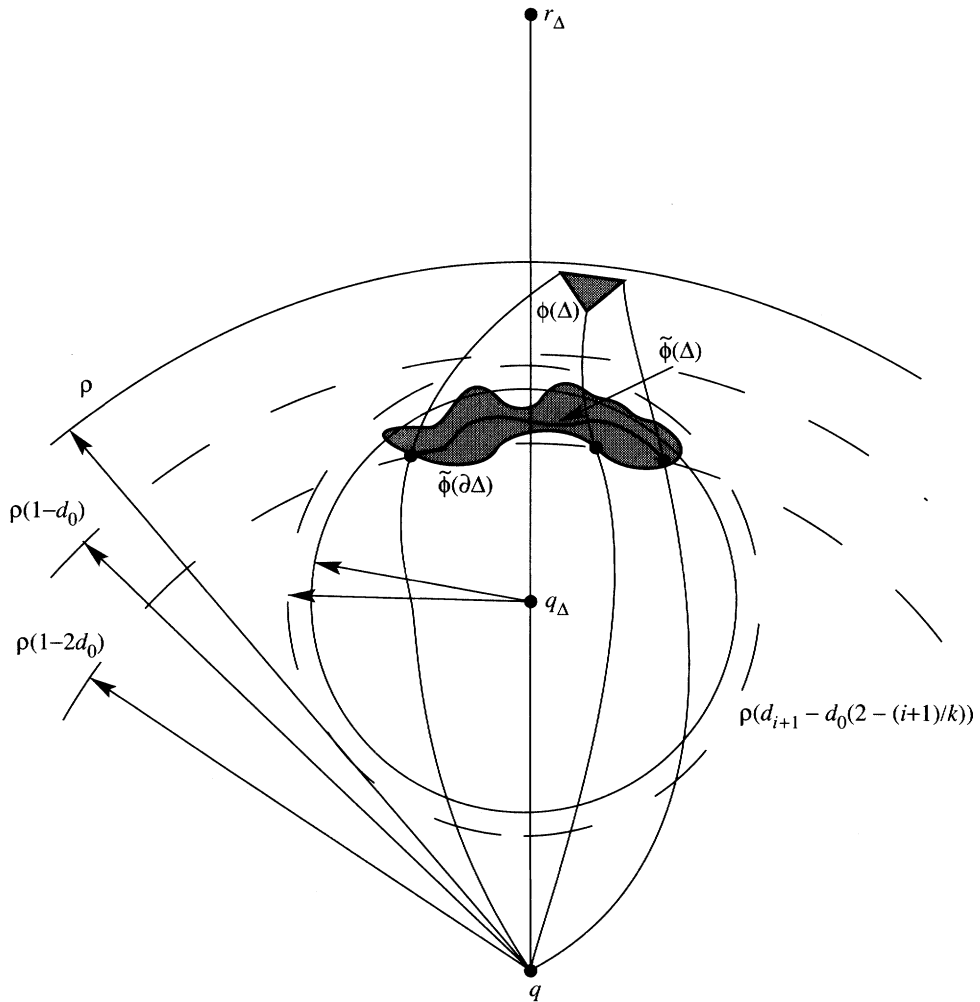


FIGURE 2

ACKNOWLEDGMENT

I would like to thank Jeff Cheeger and the referee for suggestions that led to improvement of the exposition.

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