MANIFOLDS OF POSITIVE RICCI CURVATURE
WITH ALMOST MAXIMAL VOLUME

G. PERELMAN

1°. In this note we consider complete Riemannian manifolds with Ricci curvature bounded from below. The well-known theorems of Myers and Bishop imply that a manifold $M^n$ with $\text{Ric} \geq n - 1$ satisfies $\text{diam}(M^n) \leq \text{diam}(S^n(1))$, $\text{Vol}(M^n) \leq \text{Vol}(S^n(1))$. It follows from [Ch] that equality in either of these estimates can be achieved only if $M^n$ is isometric to $S^n(1)$. The natural conjecture is that a manifold $M^n$ with almost maximal diameter or volume must be a topological equivalent to $S^n$. With respect to diameter this is true only if $M^n$ satisfies some additional assumptions; see [An, O, GP, E]. With respect to volume however no extra restriction is necessary.

Theorem 1. For any integer $n \geq 2$ there exists $\delta_n > 0$ with the following property. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq n - 1$. Suppose that $\text{Vol}(M^n) \geq (1 - \delta_n) \text{Vol}(S^n(1))$. Then $M^n$ is homeomorphic to $S^n$.

In fact, we prove only that $\pi_i(M^n) = 0$ for all $i < n$ and refer to the work of Hamilton [H] for $n = 3$ and to the solution of generalized Poincaré conjecture (Smale [S], Freedman [F]) for $n \neq 3$.

Vanishing of homotopy groups is a simple consequence of the Main Lemma below. Its further simple corollaries are a noncompact version of Theorem 1 and a corresponding finiteness theorem (cf. [P, Corollary B]).

Let $B^H(R)$ denote a ball of radius $R$ in the simply connected space form of constant curvature $H$.

Theorem 2. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$; $p \in M$. Suppose that $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^0(R))$ for all $R > 0$. Then $M^n$ is contractible.

Theorem 3. For any $n$, $H$, $D$, $R$ the set $\mathcal{M}_n(D, H, R)$ of all complete Riemannian manifolds $M^n$ with $\text{diam}(M^n) \leq D$, $\text{Ric} \geq (n - 1)H$, and $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^H(R))$ for all $p \in M^n$, contains only finitely many homotopy types.

2°. Henceforward we fix $n \geq 2$ and ignore the dependence on $n$ in our notations. We denote by $M$ an arbitrary compact $n$-dimensional Riemannian manifold with $\text{Ric} \geq n - 1$; all parameters below are supposed to be independent of $M$.

Received by the editors January 7, 1993.
1991 Mathematics Subject Classification. Primary 53C20; Secondary 53C23.

© 1994 American Mathematical Society
0894-0347/94 $1.00 + .25$ per page

299
Main Lemma. For any \( c_2 > c_1 > 1 \) and integer \( k \geq 0 \) there exists \( \delta = \delta_k(c_1, c_2) > 0 \) with the following property. Let \( p \in M, \ 0 < R < \pi c_2^{-1} \). Suppose that \( \text{Vol}(B_q(p)) \geq (1 - \delta) \text{Vol}(B^1(p)) \) for every ball \( B_q(p) \subset B_p(c_2 R) \). Then

(A) Any continuous map \( f: S^k = \partial D^{k+1} \to B_p(R) \) can be continuously extended to a map \( g: D^{k+1} \to B_p(c_1 R) \).

(B) Any continuous map \( f: S^k \to M \setminus B_p(R) \) can be continuously deformed to a map into \( M \setminus B_p(c_1 R) \).

Remark. The Main Lemma can obviously be modified for \( n \)-manifolds with Ric \( \geq 0 \) or Ric \( \geq -(n - 1) \), in the latter case \( \delta \) may depend on \( R \) as \( R \to \infty \).

We give below a detailed proof of (A) and outline a similar proof of (B) leaving the details to the reader.

3°. At first we state explicitly all the properties of manifolds with Ricci curvature bounded from below, which are relevant to the proof.

Let \( \overline{ab} \) denote a shortest geodesic with endpoints \( a, b \).

(K) There exists a positive function \( \kappa, \kappa(t) \to 0 \) as \( t \to 0 \), such that
\[
|ab| + |ac| - |bc| \leq \kappa \left( |a, \overline{bc}|/ \min \{|ab|, |ac|\} \right) \cdot |a, \overline{bc}| \quad \text{for all} \ a, \overline{bc} \subset M.
\]

This is a weakened version of the Abresch-Gromoll inequality [AG].

(\( \gamma \)) For any \( c_2 > c_1 > 1, \ \epsilon > 0 \) there exists \( \gamma = \gamma(c_1, c_2, \epsilon) \) with the following property. Let \( p \in M, \ 0 < R < \pi c_2^{-1} \). Suppose that \( \text{Vol}(B_p(c_2 R)) \geq (1 - \gamma) \text{Vol}(B^1(c_2 R)) \). Then for every \( a \in B_p(R) \) there exists \( b \in M \setminus B_p(c_1 R) \) such that \( |a, \overline{pb}| \leq \epsilon R \).

This is a simple corollary of (the proof of) the Bishop-Gromov volume comparison inequality.

Warning. In the proof of the Main Lemma, we do not use the existence of the injectivity radius and avoid explicit induction on \( R \).

4°. Outline of the proof of (A). Assertion (A) is proved by induction on \( k \). The case \( k = 0 \) is obvious. Assume that (A) holds in dimensions less than \( k \). Fix \( c_2 > c_1 > 1 \) and let \( d_0 > 0 \) and \( \delta > 0 \) be small enough. Now given \( M, p, R \), satisfying the conditions of (A), and a continuous map \( f: S^k \to B_p(R) \), we can construct another continuous map \( \tilde{f}: S^k \to B_p((1 - d_0) R) \) such that the uniform distance between \( f \) and \( \tilde{f} \) is small in comparison with \( R \). This is the crucial step, it uses both properties (K), (\( \gamma \)) and the inducational assumption.

The map \( \tilde{f} \) is not known yet to be homotopic to \( f \), and there is no obvious way to construct such a homotopy at once. To go around this difficulty, we take a fine triangulation of \( S^k \) and construct a "small" homotopy between \( f \) and \( \tilde{f} \) on the \((k - 1)\)-skeleton of this triangulation. In fact, the homotopy is constructed consecutively on \( i \)-skeleta, \( i = 0, 1, \ldots, k - 1 \), using the inducational assumption.

The result of previous steps can be interpreted as an extension of \( f \) from \( S^k = \partial D^{k+1} \) to the \( k \)-skeleton of a finite cell decomposition of \( D^{k+1} \). Recall
that the boundary of the “central” cell is mapped into $B_p((1 - d_0)R)$, and the size of the images of the boundaries of all other cells is small in comparison with $R$. Now we repeat the previous steps for each cell separately and obtain an extension of $f$ to the $k$-skeleton of a finer cell decomposition (Figure 1), etc. The limit of the infinite repetition of this procedure is the required extension $g$.

Apparently the argument above cannot be convincing until the choice of "small" parameters is specified. We give a formal exposition below.

5°. **Proof of (A).**

5.1. Consider the following general situation. Let $f: S^k \to B_p(R) \subset M$ be a continuous map, and let sequences of finite cell subdivisions $K_j$ of $D^{k+1}$ and continuous maps $f_j: \text{skel}_k K_j \to M$ satisfy

(a) $K_{j+1}$ is a cell subdivision of $K_j$ and $f_{j+1} \equiv f_j$ on $\text{skel}_k(K_j)$.

(b) For each $(k + 1)$-cell $\sigma \in K_j$ there exist $p_\sigma \in B_p(c_1R)$ and $R_\sigma > 0$ such that $f_j(\partial \sigma) \subset B_{p_\sigma}(c_1R_\sigma)$ and

$$B_{p_\sigma}(c_1R_\sigma') \subset B_{p_\sigma}(c_1R_{\sigma'}) , \quad R_{\sigma'} \leq (1 - d_0)R_\sigma$$

(for a positive constant $d_0$), in case $\sigma \in K_j$, $\sigma' \in K_{j+1}$, $\sigma' \subset \sigma$.

(c) $\text{skel}_k(K_0) = S^k = \partial D^{k+1}$, $f_0 \equiv f$, $R_{\sigma_0} = R$ for $\sigma_0 = D^{k+1} \setminus S^k \in K_0$.

Then there exists a continuous map $g: D^{k+1} \to B_p(c_1R)$, such that $g \equiv f_j$ on $\text{skel}_k(K_j)$ for all $j$.

Indeed, let $g(x) = \lim_{j \to \infty} p_{\sigma_j}$ for some sequence of $(k + 1)$-cells $\sigma_j \in K_j$,
such that $\sigma_{j+1} \subset \sigma_j$ and $x \in \text{clos}(\sigma_j)$ for all $j$. Obviously $p_{\sigma_j}$ form a Cauchy sequence for any such $\{\sigma_j\}$, and moreover, $|g(x)p_{\sigma_j}| \leq (1 - d_0)^j c_1 R$. The sequence $\sigma_j$ is defined unambiguously if $x \notin \cup_j \text{skel}_k K_j$, and it is clear that $g(x) = f_j(x)$ if $x \in \text{skel}_k K_j$; therefore $g$ is correctly defined and continuous.

5.2. Specify the choice of $d_0$ and $\delta$ in the following way. Let $d_0 > 0$ be so small that for suitably chosen positive numbers $d_1, \ldots, d_k$,

$$d_{i+1}/d_i > 100, \quad d_0/d_i > 100k \cdot \kappa(100d_i/d_{i+1}),$$

and let

$$100d_k < 10^{-k}(1 + d_0/2k)^{-k}(1 - c_1^{-1})$$

hold, and let

$$\delta = \delta_k(c_1, c_2) = \min\{\gamma(c_1, c_2, d_0), \delta_i(1 + d_0/2k, c_2), i = 0, 1, \ldots, k - 1\}.$$

5.3. Assume that the conditions of (A) are satisfied. Then the extensions $f_j$ from 5.1 can be constructed inductively using the following key assertion (see 6.0 for the proof).

(C) Given $\rho > 0$, $q \in M$, such that $\text{Vol}(B_q(c_2 \rho)) \geq (1 - \delta) \text{Vol}(B^1(c_2 \rho))$, a continuous map $\phi: S^k \to B_q(\rho)$ and a triangulation $T$ of $S^k$ such that $\text{diam}(\phi(\Delta)) \leq d_0 \rho$ for all $\Delta \in T$, there exists a continuous map $\tilde{\phi}: S^k \to B_q((1 - d_0) \rho)$ such that

$$\text{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta)) \leq 10^{-k-1}(1 + d_0/2k)^{-k}(1 - c_1^{-1})\rho$$

for all $\Delta \in T$.

Indeed, represent a $(k + 1)$-cell $\sigma \in K_j$ as $S^k \times (0, 1]\cup\{0\}$, choose a fine triangulation $T$ of $S^k$ and apply (C) to $f_j: S^k \times \{1\} \to B_{p_{\sigma}}(R_\sigma)$. (The volume condition is satisfied since it follows from 5.1 (b) that $B_{p_{\sigma}}(c_2 R) \subset B_p(c_2 R)$.)

Define $K_{j+1}$ by $\sigma \cap \text{skel}_k(K_{j+1}) = S^k \times \{1/2\} \cup S^k \times \{1\} \cup \text{skel}_{k-1}(T) \times [1/2, 1]$, and let $f_{j+1} \equiv f_j$ on $S^k \times \{1\}$ and $f_{j+1} \equiv \tilde{f}_j$ on $S^k \times \{1/2\}$. Now $f_{j+1}$ can be extended consecutively to $\text{skel}_i(T) \times [1/2, 1]$, $i = 0, 1, \ldots, k - 1$, in such a way that

$$\text{diam } f_{j+1}(\Delta \times [1/2, 1]) \leq 10^{i-k}(1 + d_0/2k)^{i+1-k}(1 - c_1^{-1})R_\sigma$$

for all $\Delta \in \text{skel}_i(T)$. (Each extension to $\Delta \times [1/2, 1]$ from its boundary, for $\Delta \in \text{skel}_i T$, is ensured by the inducational assumption in dimension $i$ and the inequality $\delta \leq \delta_i(1 + d_0/2k, c_2)$.) It is easy to check that $f_{j+1}: \text{skel}_k K_{j+1} \to M$ satisfies the conditions of 5.1, since the boundary of the “central” cell $S^k \times (0, 1/2] \cup \{0\}$ is mapped into $B_{p_{\sigma}}((1 - d_0) R_\sigma)$, and the images of the boundaries of all other cells have diameters less than $(1/2)(1 - c_1^{-1}) R_\sigma$.
6°. **Proof of (C).** We construct \( \tilde{\phi} \) consecutively on \( \text{skel}_i(T) \), \( i = 0, \ldots, k \), to satisfy \( \tilde{\phi}|_{\Delta} = \phi|_{\Delta} \) if \( \phi(\Delta) \subset B_q(\rho(1 - d_0)) \),

\[
(1) \quad \tilde{\phi}(\Delta) \subset B_q((1 - d_0(2 - i/k))\rho),
\]

\[
(2) \quad \text{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta)) \leq 10d_i\rho
\]

for all \( \Delta \in \text{skel}_i(T) \).

To begin with, define \( \tilde{\phi} \) on \( \text{skel}_0(T) \) by

\[
\tilde{\phi}(x) \in \phi(x)q, \quad |q\tilde{\phi}(x)| = \rho(1 - 2d_0) \quad \text{if} \quad |q\phi(x)| > \rho(1 - 2d_0).
\]

Assume that \( \tilde{\phi} \) is defined on \( \text{skel}_i(T) \) for some \( i < k \) and consider a \((i + 1)\)-simplex \( \Delta \), such that \( \phi(\Delta) \not\subset B_q(\rho(1 - 2d_0)) \). Applying \((\gamma)\) choose a point \( r_\Delta \in M \setminus B_q(c_1\rho) \), such that \( |q_\Delta r_\Delta|, \phi(\Delta)| \leq d_0\rho \) and let \( q_\Delta \in B_q(\rho(1 - d_{i+1})) \) be such that \( |q_\Delta| = \rho(1 - d_{i+1}) \); see Figure 2 on the next page. It follows from (2) and the choice of \( \{d_j\} \) that for any \( x \in \partial \Delta \)

\[
|q_\Delta(x), q_\Delta r_\Delta| < 20d_i\rho, \quad |\tilde{\phi}(x) q_\Delta| > d_{i+1}\rho/2, \quad |\tilde{\phi}(x) r_\Delta| > d_{i+1}\rho/2.
\]

Hence we can apply \((\kappa)\) to \( \tilde{\phi}(x), q_\Delta r_\Delta \) and obtain

\[
|\tilde{\phi}(x) r_\Delta| + |\tilde{\phi}(x) q_\Delta| - |q_\Delta r_\Delta| < 20\kappa(100d_i/d_{i+1})d_i\rho.
\]

Adding this to the triangle inequality

\[
|q_\Delta r_\Delta| + \rho(1 - d_{i+1}) = |q r_\Delta| \leq |\tilde{\phi}(x) r_\Delta| + |\tilde{\phi}(x) q|
\]

and taking (1) into account we get

\[
\tilde{\phi}(\partial \Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - i/k) + 20\kappa(100d_i/d_{i+1})d_i))
\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (2i + 1)/2k))),
\]

where the last inclusion follows from the choice of \( \{d_i\} \).

Since \( \dim \Delta = i + 1 \leq k \), the inductional assumption can be applied to extend \( \tilde{\phi} \) from \( \partial \Delta \) to \( \Delta \). It follows from the choice of \( \delta \) that the extension satisfies

\[
\tilde{\phi}(\Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (i + 1)/k))).
\]

It remains to observe that the last inclusion implies (1), (2) with \( i \) replaced by \( i + 1 \).

7°. The proof of (B) can be carried out along the same lines. An argument similar to the proof of (C) shows that a map \( f: S^k \to M \setminus B_p(\rho), \ R \leq \rho \leq c_1R \), can be transformed to a map \( \tilde{f} \) with image outside a markedly larger ball, in such a way that \( \text{diam}(f(\Delta) \cup \tilde{f}(\Delta)) \) is small for every simplex \( \Delta \) of a fine triangulation \( T \) of \( S^k \). A deformation from \( f \) to \( \tilde{f} \) can be constructed consecutively on \( \text{skel}_i(T) \), \( i = 0, \ldots, k \) making use of the assertion (A). After a bounded number of such deformations we obtain the required map with image outside \( B_p(c_1R) \).
I would like to thank Jeff Cheeger and the referee for suggestions that led to improvement of the exposition.

REFERENCES


St. Petersburg branch of Steklov Mathematical Institute, St. Petersburg, Russia

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012, USA

Current address: Department of Mathematics, University of California at Berkeley, Berkeley, California 94720

E-mail address: perelman@math.berkeley.edu