THE INVERSE EIGENVALUE PROBLEM
FOR REAL SYMMETRIC TOEPLITZ MATRICES

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INTRODUCTION

A Toeplitz matrix is one for which the entries are constant on diagonals: $c_{i,j} = c_{i-j}$. Thus Toeplitz matrices are discrete analogues of convolution operators. They are connected to analysis by the trigonometric moment problem and its many ramifications, and to applications by discrete time-invariant linear systems and stationary stochastic processes, which they represent. They have therefore been extensively studied, and far-ranging theory has grown from the problem of inverting them, and from questions about the quadratic forms which they define. Nevertheless, relatively little is known about their spectral properties. In particular, the fundamental question of whether or not a real symmetric Toeplitz matrix can have arbitrary real eigenvalues, which was posed by P. Delsarte and Y. Genin and solved by them for $n \leq 4$ [DG], has been open. The problem is challenging because of its analytic intractability and because the few available examples exhibit multiple solutions that form no apparent pattern [F]. Here we use a nonconstructive method to give an affirmative answer, within a class of Toeplitz matrices having a certain additional regularity. These matrices therefore constitute another canonical form for Hermitian matrices under unitary transformation.

To describe the result more precisely, let a vector $(v_1, \ldots, v_k)$ be called even if $v_{1+i} = v_{k-i}$ and odd if $v_{1+i} = -v_{k-i}$, $0 \leq i \leq k-1$. Delsarte and Genin showed that each eigenvector of a real symmetric Toeplitz matrix can be taken to be either even or odd, and that the numbers of even and odd eigenvectors differ by at most one at most, not only for the matrix as a whole, but also for each subspace of eigenvectors corresponding to a multiple eigenvalue. They pointed out that, in consequence, for the inverse eigenvalue problem to have a continuous solution, the association of eigenvectors to eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ must assign eigenvectors of opposite parity to adjacent eigenvalues. We extend this crucial observation by focusing on matrices, termed regular, for which all the principal submatrices also have this property. Following S. Friedland [F], we...
then consider the topological degree of the map which takes such a matrix to its eigenvalues. Using little more than a prior characterization [SL] of matrices with multiple eigenvalues, we succeed in showing that the degree does not vanish; it follows that the eigenvalues of matrices in this class already attain all possible \( n \)-tuples of real numbers. It is remarkable that a conclusion so inaccessible analytically nevertheless can be derived from such simple information.

**Notation and Preliminaries**

A symmetric Toeplitz matrix is determined by its top row, \((t_1, \ldots, t_n)\); accordingly, we denote such a matrix by \(M_n(t_1, \ldots, t_n)\). Limiting consideration to real matrices in which \(t_1 = 0\) and \(t_2 = 1\), we associate each \(M_n(0, 1, t_3, \ldots, t_n)\) with the point \((t_3, \ldots, t_n)\) of \(R^{n-2}\). Such a matrix has \(n\) real eigenvalues, which we label so that \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\); here

\[
\sum \lambda_i = \text{trace}(M_n) = 0,
\]

hence \(\lambda_1 < 0\), else all the eigenvalues vanish, and \(\lambda_n = - (\lambda_2 + \cdots + \lambda_{n-1})\). We can rescale these so that \(\lambda_1 = -1\), and so associate with \(M_n\) the normalized eigenvalues \(y_i = \lambda_i/|\lambda_1| = \lambda_i/(-\lambda_1)\), satisfying

\[
-1 \leq y_2 \leq \cdots \leq y_{n-1} \leq y_n = (-1 + y_2 + \cdots + y_{n-1}) ,
\]

which we view as a point \((y_2, \ldots, y_{n-1}) \in R^{n-2}\) lying in the simplex \(L_n\) defined by the \(n-1\) linear inequalities

\[
-1 \leq y_2 ,
\]

\[
y_i \leq y_{i+1}, \quad 2 \leq i \leq n-2 ,
\]

\[
y_2 + y_3 + \cdots + y_{n-2} + 2y_{n-1} \leq 1
\]

that correspond to (1). Let \(\Lambda = \Lambda(M_n)\) denote the function which maps a point \((t_3, \ldots, t_n)\) onto these normalized eigenvalues \((\lambda_2/|\lambda_1|, \ldots, \lambda_{n-1}/|\lambda_1|)\) of \(M_n(0, 1, t_3, \ldots, t_n)\). \(\Lambda\) is then a map of \(R^{n-2}\) into \(L_n \subset R^{n-2}\).

We want to show that every set of \(n\) real numbers \((y_1, \ldots, y_n)\), which we can label in increasing order, is the set of eigenvalues of some real symmetric \(n \times n\) Toeplitz matrix. Since the eigenvalues of \(M_n + \alpha I\) are a uniform translation by \(\alpha\) of those of \(M_n\), we may apply such a shift to the \(y_i\) to ensure \(\sum y_i = 0\), and again scale by \(|y_1| \neq 0\) to produce a point in \(L_n\). Our object therefore is to prove that the map \(\Lambda\) covers all of \(L_n\). We will show that this is so already when \(\Lambda\) is restricted to the subset of regular matrices \(M_n\) described earlier. The argument will be based on the topological degree of \(\Lambda\).

More specifically, on the strength of certain basic facts we will show that the set \(\mathcal{F}_n\) of normalized \(n \times n\) regular matrices is bounded, with boundary made up entirely of matrices having multiple eigenvalues. The complement of the image of this boundary under the eigenvalue map \(\Lambda\) therefore includes the
interior of $L_n$ in a connected component. To determine the topological degree of $\Lambda$, as a map of $\mathcal{F}_n$, at some point inside $L_n$ we begin with the boundary point of $L_n$ where the lowest $n-1$ eigenvalues coincide; this corresponds to a unique matrix. Following the action of $\Lambda$ in a neighborhood of that matrix, we show that it is locally one-to-one along a succession of boundary facets of $\mathcal{F}_n$ of increasing dimension, made up of matrices for which the lowest $k$ eigenvalues coincide, $k = n-2, \ldots, 1$. Ultimately this enables us to show that $\Lambda$ is one-to-one also at some point interior to $L_n$; the degree of $\Lambda$ is then not zero in $L_n$. Since the degree vanishes at a point outside $\Lambda(\mathcal{F}_n)$, it follows that $\Lambda(\mathcal{F}_n)$ covers $L_n$.

We begin with some basic properties of Toeplitz matrices. Throughout, $M_n$ and $M_k$ will denote, respectively, $M_n(t_1, t_2, \ldots, t_n)$ without the preceding normalizations, and $M_k(t_1, \ldots, t_k)$, its principal submatrix, $k \leq n$. We will always number the eigenvalues in nondecreasing order. Propositions 1 and 3(d) are known; we include brief proofs for completeness. Proposition 2 represents an elaboration of results in [SL]. In addition, we will frequently invoke the fact that the eigenvalues of $M_k$ interlace those of $M_{k+1}$. The motion of eigenvalues was studied also in [T1, T2].

**Definition.** For a vector $v = (v_1, \ldots, v_k)$, let $v_{\text{rev}} = (v_k, \ldots, v_1)$ denote the vector obtained by writing the coordinates of $v$ in reverse order; $v$ is even or odd as $v = \pm v_{\text{rev}}$. We will refer to the property of being even or odd as the parity of a vector. When there is no confusion, we will also use terms defined for eigenvalues to apply to the corresponding eigenvectors, and conversely, as for example by speaking of the smallest eigenvector or of an even eigenvalue. We will call an eigenvalue simple or single if its multiplicity is 1, and multiple if it exceeds 1.

**Proposition 1.** If $v$ is an eigenvector of $M_n$, so is $v_{\text{rev}}$. If $\lambda$ is a simple eigenvalue of $M_n$, then the corresponding eigenvector is either even or odd.

**Proof.** It is easy to see that, since $M_n$ is Toeplitz, $(M_n v_{\text{rev}}) = (M_n v)_{\text{rev}}$; hence if $M_n v = \lambda v$ then also $M_n v_{\text{rev}} = \lambda v_{\text{rev}}$. If the eigenspace is one-dimensional, $v_{\text{rev}}$ must be a constant multiple of $v$, and the constant can be only $\pm 1$; hence $v$ is even or odd.

**Definition.** For a vector $v = (v_1, \ldots, v_k)$, let $v_{\text{ext}} \equiv (v_1, \ldots, v_k, 0)$ denote $v$ extended by a last component of zero.

**Proposition 2.** Let $n_k(\lambda) > 0$ denote the multiplicity of $\lambda$ as an eigenvalue of $M_k$, and $E_k(\lambda)$ the corresponding $n_k(\lambda)$-dimensional subspace of eigenvectors. Then

(a) $n_{k+1}(\lambda) - n_k(\lambda) = \pm 1$;

(b) the following conditions are equivalent:
   (i) for every $v \in E_k(\lambda)$, $v_{\text{ext}} \in E_{k+1}(\lambda)$,
   (ii) $n_{k+1}(\lambda) = n_k(\lambda) + 1$. 

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(iii) \( E_{k+1}(\lambda) \) contains a vector with nonzero first (equivalently, last) component;

(c) property (b)(iii) implies the same for \( E_k(\lambda) \).

Proof. The equivalence asserted in (b)(iii) derives from the reversibility of eigenvectors.

The fact that

\[
(3) \quad n_k(\lambda) - 1 \leq n_{k+1}(\lambda) \leq n_k(\lambda) + 1
\]

is a consequence of the interlacing of eigenvalues of \( M_k \) and \( M_{k+1} \), but it also has the following suggestive direct proof; for brevity, we use \( E_k \) and \( n_k \) to denote \( E_k(\lambda) \) and \( n_k(\lambda) \). Choose a basis \( \{w(i)\}, 1 \leq i \leq n_{k+1} \), of \( E_{k+1} \).

If the last component of one of these vectors, say of \( w^{(1)} \), does not vanish (case (b)(iii)), let us replace \( w(i) \) by \( w(i) - \alpha_i w^{(1)} \), \( i \geq 2 \), with \( \alpha_i \) chosen so that the last component of this linear combination vanishes; these \( (n_{k+1} - 1) \) vectors \( (n_k + 1) \) if all the last components of the basis vanish) remain linearly independent since \( \{w(i)\} \) are. As their last component vanishes, truncation to size \( k \) preserves linear independence and produces eigenvectors of \( M_k \). Thus \( n_k \geq n_{k+1} - 1 \), and equality, which is (b)(ii), implies (b)(iii). Similarly, let \( \{v(i)\}, 1 \leq i \leq n_k \), be a basis for \( E_k \), and extend each \( v^{(i)} \) to \( v^{(i)}_{\text{ext}} \). Then for the first \( k \) components, \( M_{k+1} v^{(i)}_{\text{ext}} = \lambda v^{(i)}_{\text{ext}} \), so \( v^{(i)}_{\text{ext}} \in E_{k+1} \), if and only if the last component of \( M_{k+1} v^{(i)}_{\text{ext}} \) vanishes. Either this happens for every \( v^{(i)}_{\text{ext}} \) or there is \( v^{(1)} \) for which it does not. In the former case, \( E_{k+1} \) contains all \( \{v^{(i)}_{\text{ext}}\} \) and so we have (b)(i). In the latter, again on replacing \( v^{(i)}_{\text{ext}} \) by \( v^{(i)}_{\text{ext}} - \beta_i v^{(1)}_{\text{ext}} \), \( i \geq 2 \), with \( \beta_i \) chosen so that the last component of \( M_{k+1} (v^{(i)}_{\text{ext}} - \beta_i v^{(1)}_{\text{ext}}) \) vanishes, we obtain \( n_k - 1 \) linearly independent vectors \( (n_k + 1) \) for case (b)(i) in \( E_{k+1} \). Thus for either situation \( n_{k+1} \geq n_k - 1 \), completing the proof of (3). We have also shown that (b)(ii) implies (b)(iii).

We return to (b)(i). Select \( v^{(1)} \) to have the earliest nonzero component among \( v \in E_k \); say it is the \( j \)th. Since \( v^{(i)}_{\text{rev}} \in E_k \) for each \( v \in E_k \), all the \( v^{(i)}_{\text{rev}} \) and \( v^{(i)}_{\text{rev}} \) compete for the role of \( v^{(1)} \), hence must all have their first and last \( j - 1 \) components vanish. The extensions \( v^{(i)}_{\text{ext}}, 1 \leq i \leq n_k \), remain linearly independent and, by the assumption of (b)(i), all are eigenvectors of \( M_{k+1} \); their last \( j \) components vanish. As \( v^{(i)}_{\text{ext}} \) is in \( E_{k+1} \), so is \( (v^{(i)}_{\text{ext}})^{\text{rev}} \), which places the nonvanishing \( j \)th component of \( v^{(i)}_{\text{ext}} \) in position \( (k + 2 - j) \), where the \( v^{(i)}_{\text{ext}} \) all vanish. Thus \( (v^{(i)}_{\text{ext}})^{\text{rev}} \in E_{k+1} \) is linearly independent of the \( v^{(i)}_{\text{ext}}, 1 \leq i \leq n_k \), and so \( n_{k+1} \geq n_k + 1 \). By (3), this shows that \( n_{k+1} = n_k + 1 \). Thus (b)(i) implies (b)(ii), which implies (b)(iii). Moreover, \( E_{k+1} \) is spanned by \( v^{(i)}_{\text{ext}} \) and \( v^{(i)}_{\text{ext}}^{\text{rev}} \), and as the former all have vanishing last component, (b)(iii) implies that \( (v^{(i)}_{\text{ext}})^{\text{rev}} \) does not. Hence \( v^{(1)} \) has nonzero first component,
as therefore does $v^{(1)} \in E_k$. Consequently (b)(i) also implies the conclusion of (c).

Next we show that (b)(iii) implies (b)(i). For suppose, by (b)(iii), that $w \in E_{k+1}$ has nonzero last component $w_{k+1}$. Let $v$ be any eigenvector in $E_k$, and denote by $\mu_{k+1}$ the last component of $M_{k+1}v_{\text{ext}}$. Since $M_{k+1}$ is symmetric,

$$(M_{k+1}w, v_{\text{ext}}) = (w, M_{k+1}v_{\text{ext}}).$$

By definition of $w$,

$$(M_{k+1}w, v_{\text{ext}}) = \lambda(w, v_{\text{ext}}),$$

and as the first $k$ components of $M_{k+1}v_{\text{ext}}$ coincide with those of $M_kv$, we find that

$$(w, M_{k+1}v_{\text{ext}}) = \lambda(w, v_{\text{ext}}) + w_{k+1}\mu_{k+1}.$$  

Thus $\mu_{k+1} = 0$, whence $v_{\text{ext}} \in E_{k+1}$, establishing (b)(i). This cycle of implications proves (b) and (c).

Finally, we consider (a). If (b)(ii), which is a subcase of (a), is not satisfied, neither is (b)(iii). Thus the last component of all the vectors of $E_{k+1}$ must vanish. Thereupon, the vectors of any basis of $E_{k+1}$, when truncated to size $k$, remain linearly independent, lie in $E_k$, and are extendible by $(\cdot)_{\text{ext}}$ to $E_{k+1}$. Since (b)(i) is also false, they cannot include all of $E_k$; hence $n_{k+1} \leq n_k - 1$. By (3), $n_{k+1} = n_k - 1$, completing the proof of (a).

**Corollary 1 [I].** In any block of consecutive values of $k$ for which $n_k(\lambda) > 0$, the sequence of numbers $n_k(\lambda)$ can have at most one relative maximum.

**Proof.** It follows from Proposition 2(b) that, if $n_{k+1}(\lambda) \leq n_k(\lambda)$, all the vectors of $E_{k+1}$ have vanishing first and last components. By Proposition 2(c), the same must be true for $E_{k+2}$, and so this condition propagates. Thus once the sequence $\{n_k(\lambda)\}$ starts to decrease, it continues to do so.

**Proposition 3.** The following statements describe the behavior of multiple eigenvectors:

(a) Suppose $n_{k-1}(\lambda) = 0$ and $n_k(\lambda) = 1$. Then for the eigenvector $v = (v_1, \ldots, v_k)$ of $M_k$ corresponding to $\lambda$ the component $v_1 \neq 0$, and $n_{k+1}(\lambda) = 2$ if and only if

$$t_{k+1} = -v_1^{-1}(t_kv_2 + \cdots + t_kv_k).$$

With this value of $t_{k+1}$, the eigenspace $E_{k+1}(\lambda)$ is spanned by $(v_1, \ldots, v_k)$ and $(0, v_1, \ldots, v_k)$.

(b) If $n_j(\lambda)$ increases from $n_k(\lambda) = 1$ to $n_{k+m}(\lambda) = m + 1$, $m \geq 2$, each matrix entry $t_{k+q}$, $1 \leq q \leq m$, is determined from the preceding $\{t_i\}$ by the convolution

$$(4) t_{k+q} = -v_1^{-1}(t_{k+q-1}v_2 + t_{k+q-2}v_3 + \cdots + t_{q+1}v_k),$$

defined by the components of the eigenvector $v$ of $M_k$. The corresponding eigenspace $E_{k+q}$ is spanned by the vectors of length $k + q$ formed by preceding the block $(v_1, \ldots, v_k)$ by $i$ zeros, and following it by $q - i$ zeros, $0 \leq i \leq q$.
In the convolution (4), when $1+1 \sim q \sim m$, the value of $t_{k+q}$ is unchanged when $v$ is replaced by any eigenvector $w = (w_1, \ldots, w_{k+1}) \in E_{k+1}(\lambda)$ having nonzero first component, viz.

$$t_{k+q} = -w_1^{-1}(t_{k+q-1}w_2 + \cdots + t_{q-1}w_{k+1}).$$

(c) If $t_{k+m+1}$ differs from the value given by (4), then $n_{k+m+1}(\lambda) = m$, and $n_{k+m+j}(\lambda)$ decreases monotonically to $n_{k+2m+1}(\lambda) = 0$. The eigenspace $E_{k+m+j}$ is spanned by the vectors of length $k + m + j$ formed by preceding the block $(v_1, \ldots, v_k)$ by $i$ zeros, and following it by $m + j - i$ zeros, $j \leq i \leq m$.

(d) [DG] $E_{k+q}(\lambda)$ can be decomposed into the sum of subspaces of even and odd vectors, the dimensions of which differ by 1 at most. Moreover, when $n_{k+q}(\lambda)$ is odd, the subspace of larger dimension corresponds to vectors having the same parity as $v$, the single eigenvector in $E_k(\lambda)$.

(e) If $\lambda$ is the largest eigenvalue of $M_k$ then its multiplicity as an eigenvalue of $M_{k-1}$ is $n_{k-1}(\lambda) = n_k(\lambda) - 1$, and if $n_k(\lambda) \geq 2$ then $\lambda$ is also the largest eigenvalue of $M_{k-1}$. The analogous assertion holds true for the smallest eigenvalue.

Proof. If $n_{k-1}(\lambda) = 0$ and $n_k(\lambda) = 1$, then $\lambda$ is an eigenvalue of $M_k(t_1, \ldots, t_k)$ with eigenvector $v = (v_1, \ldots, v_k)$, but $\lambda$ is not an eigenvalue of $M_{k-1}(t_1, \ldots, t_{k-1})$. Consequently $v_1 \neq 0$, else $(v_2, \ldots, v_k)$ is an eigenvector of $M_{k-1}$ with eigenvalue $\lambda$, contrary to hypothesis. By Proposition 2(b)(i), $n_{k+1}(\lambda) = 2$ if and only if $v_{\text{ext}}$ remains an eigenvector of $M_{k+1}$, that is, providing the last component of $M_{k+1}(v_{\text{ext}})$ vanishes; this means that

$$t_{k+1} = -v_1^{-1}(t_k v_2 + \cdots + t_2 v_k).$$

With $t_{k+1}$ so determined, the eigenspace $E_{k+1}(\lambda)$ is spanned by $v_{\text{ext}} = (v_1, \ldots, v_k, 0)$ and $(v_{\text{ext}})^{\text{rev}} = (0, v_k, \ldots, v_1)$; since $v$ is even or odd, the second of these is equivalent to $(0, v_1, \ldots, v_k)$. This establishes (a).

To maintain the increase of $n_k(\lambda)$, we must ensure that each of these basis vectors, when extended by $(\cdot)^{\text{ext}}$, remains an eigenvector of $M_{k+2}$, but by Proposition 2(b)(iii) it is sufficient to verify this for any vector of $E_{k+1}(\lambda)$ with nonzero first component, in particular for $(v_1, \ldots, v_k, 0)$. Once more, this is equivalent to

$$t_{k+2} = -v_1^{-1}(t_{k+1} v_2 + \cdots + t_3 v_k).$$

Thereupon, $E_{k+1}$ is spanned by $(v_1, \ldots, v_k, 0, 0)$, $(0, v_1, \ldots, v_k, 0)$, and their reversed versions, equivalently by $(v_1, \ldots, v_k, 0, 0)$, $(0, v_1, \ldots, v_k, 0)$, and $(0, 0, v_1, \ldots, v_k)$, of which only the first has nonzero initial component. Continuing in this way, we see that if $n_j(\lambda)$ increases from 1 to $m+1$ as $j$ increases from $j = k$ to $j = k + m$, then each $t_{k+q}$, for $1 \leq q \leq m$, is determined from the preceding $t_i$ by the same convolution

$$t_{k+q} = -v_1^{-1}(t_{k+q-1} v_2 + t_{k+q-2} v_3 + \cdots + t_{q+1} v_k).$$
defined by the components of the eigenvector $v$ of $M_k$. This expresses the requirement that every eigenvector of $E_{k+q-1}(\lambda)$ extend to one of $E_{k+q}(\lambda)$; as we have seen from Proposition 2(b)(iii), any eigenvector $w$ of $E_{k+q-1}(\lambda)$ having a nonzero initial component can replace $v$ for this purpose. The corresponding eigenspace $E_{k+q}$ is spanned as described in (b).

Thereafter, if $t_{k+m+1}$ is not given by the convolution (4), the eigenvalue multiplicity $n_{k+m+1}(\lambda) = n_{k+m}(\lambda) - 1 = m$, and by Corollary 1 it continues to decrease for succeeding indices, regardless of the value of the corresponding $t_{k+m+j}$, $2 \leq j \leq m$; each of the eigenspaces $E_{k+m+j}$, $1 \leq j \leq m$, of dimension $m+1-j$, is formed from the preceding one by removing the basis vector having the fewest leading zeros and extending the remaining vectors by an appended component of zero, so is spanned as described in (c).

We can generate the even and odd subspaces of $E_{k+q}(\lambda)$ by forming pairs $\{v^{(i)} \pm v^{(i)}_{\text{rev}}\}$, with $\{v^{(i)}\}$ the basis vectors for $E_{k+q}(\lambda)$ described in (a), (b), or (c), consisting of $v$ preceded and followed by suitable blocks of zeros; $\{v^{(i)}_{\text{rev}}\}$ are also in this collection. For each $i$, the span of these even and odd vectors includes $v^{(i)}$ and $v^{(i)}_{\text{rev}}$. If the numbers of leading and trailing zero components for $v^{(i)}$ differ, $v^{(i)}$ and $v^{(i)}_{\text{rev}}$ are linearly independent, so account for distinct members of the basis. Thus there are $[n_{k+q}(\lambda)/2]$ such pairs ($[\ ]$ denoting the integer part) and the dimension of each of the subspaces constructed from them can be no greater than this number. However, the span of these subspaces includes all of the $v^{(i)}$ used, hence has dimension no smaller than $2[n_{k+q}(\lambda)/2]$. Consequently each subspace must have dimension $[n_{k+q}(\lambda)/2]$. When $n_{k+q}(\lambda)$ is odd, there remains the vector $v^{(i)}$, consisting of $v$ preceded and followed by the same number of zeros, for which $v^{(i)} \pm v^{(i)}_{\text{rev}}$ generate only a single nonzero vector, having the same parity as $v$. Because the span now includes the additional $v^{(i)}$, this vector must increase the dimension of its subspace. This establishes (d).

Finally, let $\lambda$ be the largest eigenvalue of $M_k$ and have multiplicity exceeding 1. By Proposition 2(a), $\lambda$ is also an eigenvalue of $M_{k-1}$, and, from the interlacing of eigenvalues of these matrices, $\lambda$ is the largest eigenvalue of $M_{k-1}$. The interlacing shows also that $n_{k-1}(\lambda)$ cannot exceed $n_k(\lambda)$ even when $n_k(\lambda) = 1$. The same considerations apply to the smallest eigenvalue of $M_k$. This completes the proof of Proposition 3.

**Regular Toeplitz Matrices**

From Proposition 3(d) [DG, p. 204] the authors drew the important conclusion that for the existence of a continuous map from ordered eigenvalues $(\lambda_1, \ldots, \lambda_n)$ to Toeplitz matrices it is necessary that the eigenvectors corresponding to successive eigenvalues alternate in parity. For otherwise, if both $\lambda_j$ and $\lambda_{j+1}$, say, correspond to even eigenvectors, a smooth deformation of the
eigenvalue sequence which makes \( \lambda_j \) and \( \lambda_{j+1} \) coincide could not be accompanied by a similarly smooth deformation of the matrices, since one of the eigenvectors corresponding to the multiple eigenvalue is necessarily odd. We amplify this crucial observation by restricting attention to matrices \( M_n(t_1, \ldots, t_n) \) in which not only \( M_n \), but also each principal submatrix \( M_k(t_1, \ldots, t_k) \), \( 1 \leq k \leq n \), has the property that its eigenvalues are distinct and alternate being even and odd, with the largest one even. Let \( \mathcal{F}_n \) denote the set of regular matrices \( M_n(0, 1, t_3, \ldots, t_n) \).

D. Slepian has proved that \( A_n \equiv M_n(1, \rho, \ldots, \rho^{n-1}) \) has even eigenvectors with components
\[
v_j = \cos(n + 1 - 2j)\theta, \quad 1 \leq j \leq n,
\]
for \( \theta \) a solution of \( \tan n\theta = [(1 - \rho)/(1 + \rho)]\cot \theta \), \( 0 < \theta < \pi/2 \), and odd eigenvectors with components
\[
v_j = \sin(n + 1 - 2j)\theta, \quad 1 \leq j \leq n,
\]
for \( \theta \) a solution of \( \tan n\theta = -[(1 + \rho)/(1 - \rho)]\tan \theta \), \( 0 < \theta < \pi/2 \). The corresponding eigenvalues are
\[
\lambda = 1 - \rho^2)/(1 - \rho \cos 2\theta + \rho^2).\]
This shows explicitly that \( A_n \) is regular. We can also verify this without calculation by considering \( T_n \), the inverse of \(-A_n\), which is tridiagonal and extendible to a Jacobi matrix. Viewing this matrix as a three-term recursion which defines a sequence \( \{P_n(x)\} \) of orthogonal polynomials [A, p. 5], the eigenvalues \( \lambda_i \) of \( T_n \) are the zeros of \( P_n(x) \), with eigenvectors \( 1, P_i(\lambda_i), \ldots, P_{n-1}(\lambda_i) \); by Proposition 1, these are even or odd according to the sign of \( P_{n-1}(\lambda_i) \). Since the zeros of orthogonal polynomials are distinct and interlace, the eigenvectors of \( T_n \), equivalently those of \( A_n \), alternate parity in the way required for \( A_n \) to be regular. Thus the set of regular matrices is not empty.

We henceforth return to the normalization \( t_1 = 0, t_2 = 1 \). Since single eigenvectors and eigenvalues deform smoothly under variations of the matrix, the set of regular matrices corresponds to an open set of points \((t_3, \ldots, t_n) \in \mathbb{R}^{n-2}\).

To simplify notation, we will identify points \((t_3, \ldots, t_n) \) with the corresponding matrices \( M_n(0, 1, t_3, \ldots, t_n) \) and refer to them interchangeably. We now describe the geometry of \( \mathcal{F}_n \).

**Lemma 1.** (a) For \( k > 2 \), \( \mathcal{F}_k \) corresponds to a set in \( \mathbb{R}^{k-2} \) coordinatized by \((t_3, \ldots, t_k)\) for which \((t_3, \ldots, t_{k-1}) \in \mathcal{F}_{k-1} \) and whose boundary consists entirely of the closure of (portions of) the \((k-1)\) surfaces \( B_i \), defined as functions over \( \mathcal{F}_{k-1} \) by
\[
B_i : \quad t_k = -v_1^{(i)}t_{k-1}v_2^{(i)} + t_{k-2}v_3^{(i)} + \cdots + t_3v_{k-2}^{(i)} + v_{k-1}^{(i)},
\]
with \( v_i^{(i)} = (v_1^{(i)}, v_2^{(i)}, \ldots, v_{k-1}^{(i)}) \) the \( i \)th eigenvector of the matrix \( M_{k-1}(0, 1, t_3, \ldots, t_{k-1}) \), \( 1 \leq i \leq k-1 \). The points of \( B_i \) represent those matrices for which \( \lambda_i = \lambda_{i+1} \).

(b) The set \( \mathcal{F}_k \) lies above \( B_i \) if \( v_i^{(i)} \) is even, and below \( B_i \) if \( v_i^{(i)} \) is odd.
If $M_k$ corresponds to a boundary point of $\mathcal{T}_k$, then, for each of its eigenvalues, $n_k(\lambda) = n_{k-1}(\lambda) + 1$.

(d) The closure $\overline{\mathcal{T}_k}$ is compact.

Proof. We proceed by induction. $M_2(0, 1)$ has eigenvectors $(1, 1)$ and $(1, -1)$ corresponding to eigenvalues $\pm 1$. $M_3(0, 1, t_3)$ has odd eigenvector $(1, 0, -1)$ with eigenvalue $-t_3$; its even eigenvalues are $(t_3 \pm \sqrt{t_3^2 + 8})/2$. Thus $M_3(0, 1, t_3)$ is regular provided

$$t_3 - \sqrt{t_3^2 + 8} < -2t_3 < t_3 + \sqrt{t_3^2 + 8},$$

that is, for $-1 < t_3 < 0$. The statement concerning the boundary behavior is immediate. This verifies the lemma for $\mathcal{T}_3$.

If $(t_3, \ldots, t_k) \in \mathcal{T}_k$, then $(t_3, \ldots, t_{k-1}) \in \mathcal{T}_{k-1}$ by definition. $\mathcal{T}_k$ can therefore be pictured as a subset of the cylinder in $\mathbb{R}^{k-2}$ obtained by restricting $(t_3, \ldots, t_{k-1})$ to the base $\mathcal{T}_{k-1}$, and letting $t_k$ be unconstrained. We now consider the $(k-3)$-dimensional boundary of the $(k-2)$-dimensional set $\mathcal{T}_k$. One part of this is accounted for by the matrices $M_k(0, 1, t_3, \ldots, t_k)$ having a double eigenvalue, but other possible boundaries are in the cylindrical wall, that is, consist of matrices for which $M_{k-1}(0, 1, t_3, \ldots, t_{k-1})$ is regular provided

$$t_3 - \sqrt{t_3^2 + 8} < -2t_3 < t_3 + \sqrt{t_3^2 + 8},$$

that is, for $-1 < t_3 < 1$. The statement concerning the boundary behavior is immediate. This verifies the lemma for $\mathcal{T}_3$.

If $(t_3, \ldots, t_k) \in \mathcal{T}_k$, then $(t_3, \ldots, t_k) \in \mathcal{T}_{k-1}$ by definition. $\mathcal{T}_k$ can therefore be pictured as a subset of the cylinder in $\mathbb{R}^{k-2}$ obtained by restricting $(t_3, \ldots, t_{k-1})$ to the base $\mathcal{T}_{k-1}$, and letting $t_k$ be unconstrained. We now consider the $(k-3)$-dimensional boundary of the $(k-2)$-dimensional set $\mathcal{T}_k$. One part of this is accounted for by the matrices $M_k(0, 1, t_3, \ldots, t_k)$ having a double eigenvalue, but other possible boundaries are in the cylindrical wall, that is, consist of matrices for which $M_{k-1}(0, 1, t_3, \ldots, t_{k-1})$ lies in the $(k-4)$-dimensional boundary of $\mathcal{T}_{k-1}$, with arbitrary $t_k$. We will eliminate the latter possibility.

Starting with $M_k \in \mathcal{T}_k$, we examine first the effect on a matrix $M_k \in \mathcal{T}_k$ of varying $t_k$, while keeping $M_{k-1} \in \mathcal{T}_{k-1}$ fixed. Suppose $w = (w_1, \ldots, w_k)$ is an eigenvector of $M_k$ with eigenvalue $\lambda$. If $0 = w_1 = \pm w_k$, then $(w_2, \ldots, w_{k-1}, 0)$ and $(0, w_2, \ldots, w_{k-1})$ form linearly independent eigenvectors of $M_{k-1}$, a contradiction since $M_{k-1} \in \mathcal{T}_{k-1}$ has only simple eigenvalues. Thus $w_1 \neq 0$. On replacing $t_k$ by $t_k \pm \epsilon$, with $\epsilon > 0$, we obtain

$$M_k^\epsilon = M_k(0, 1, t_3, \ldots, t_k) \pm \epsilon M_k(0, \ldots, 0, 1), \quad \epsilon > 0.$$

We know that $M_k^\epsilon$ has an eigenvector $w(\epsilon)$ near $w$, of the same parity, and corresponding neighboring eigenvalue $\lambda(\epsilon)$. More precisely, by the theory of analytic (in $\epsilon$) perturbations of symmetric matrices [K, p. 120; RN, p. 376], there exists a vector $u$ and a constant $\mu$ such that

$$w(\epsilon) = w + \epsilon u + O(\epsilon^2), \quad \lambda(\epsilon) = \lambda + \epsilon \mu + O(\epsilon^2).$$

By substituting (6) into (5), and collecting terms of order $\epsilon$, we find

$$\mu = \left. \frac{d\lambda(\epsilon)}{d\epsilon} \right|_{\epsilon=0^+} = (\pm M_k(0, \ldots, 0, 1)w, w)/\|w\|^2.$$

Since

$$(M_k(0, \ldots, 0, 1)w, w) = \pm 2w_1^2/\|w\|^2,$$

according as $w$ is even or odd, we see that, as $t_k$ increases (that is, when $+\$ is chosen in (5) and (7)), the even eigenvalues of $M_k$ increase monotonically and
the odd ones decrease monotonically. $M_k$ reaches the boundary of $\mathcal{T}_k$ when, in this process, two eigenvalues coincide; since $M_{k-1} \in \mathcal{T}_{k-1}$, the eigenvectors of $M_{k-1}$ are simple, their first components do not vanish, and the explicit formula of (a) comes from Proposition 3(a). The value of restricting $M_{k-1}$ to $\mathcal{T}_{k-1}$ lies in the fact that the eigenvectors $v^{(i)}$ of $M_{k-1}$ are distinct, smooth, unambiguously determined functions of $(t_3, \ldots, t_{k-1})$, so also this formula, having the form $t_k = f(t_3, \ldots, t_{k-1})$, defines a smooth surface $B_i$ over $\mathcal{T}_{k-1}$. This establishes (a) when $M_{k-1} \in \mathcal{T}_{k-1}$.

If $M_k \in B_i$ with $M_{k-1} \in \mathcal{T}_{k-1}$, it has a double eigenvalue $\lambda_i = \lambda_{i+1}$, with a corresponding even and odd eigenvector, having nonzero first components, and the perturbed matrix $M_k(\epsilon)$ has a neighboring even and odd eigenvalue $\lambda_e(\epsilon)$ and $\lambda_o(\epsilon)$. On comparing the parity of eigenvalues of $M_k$ and $M_{k-1}$, we see that for $M_k(\epsilon)$ to lie in $\mathcal{T}_k$ we require $\lambda_o(\epsilon) < \lambda_e(\epsilon)$, hence an increase in $t_k$ when $v^{(i)}$ is even, and the reverse inequality, hence a decrease in $t_k$ when $v^{(i)}$ is odd. Once $t_k$ crosses the boundary surface $B_i$, the alternation of eigenvalues required for $M_k$ cannot be restored without another exchange of even and odd eigenvalues between $\lambda_i$ and $\lambda_{i+1}$, but this coincidence occurs only on $B_i$. Thus $\mathcal{T}_k$ lies entirely on one side of each $B_i$, in the manner described by (b).

We now consider a sequence of matrices $M_{k,n} \in \mathcal{T}_k$ for which $M_{k-1}$ approaches a matrix $M'_{k-1}$ in the boundary of $\mathcal{T}_{k-1}$. By choosing a subsequence of $M_{k,n}$ if necessary, we may suppose that each eigenvector of $M_{k-1}$, when normalized to unit length, also converges. By the induction hypothesis (a), $M'_{k-1}$ has a multiple eigenvalue $\lambda$, and $n_{k-1}(\lambda) > n_{k-2}(\lambda)$ by (c). It follows from Proposition 2(b)(iii) that the eigenspace $E^*_{k-1}(\lambda)$ has a vector with nonzero first component, and from the construction proving Proposition 3(d) that both the even and the odd subspaces of $E^*_{k-1}(\lambda)$ do also. The eigenvectors of $M^{(n)}_{k-1}$ corresponding to the eigenvalues which approach $\lambda$ are mutually orthogonal, include even and odd vectors, and converge to some orthonormal basis for $E^*_{k-1}(\lambda)$. At least one of these even eigenvectors must have its first component converge to a nonzero value, since the even subspace of $E^*_{k-1}(\lambda)$ contains such a vector; say it is the $i$th eigenvector of $M^{(n)}_{k-1}$, and denote its limit by $v_e$. It follows from (a) that the corresponding lower bounding surface $B_i$ of $\mathcal{T}_k$ converges along the sequence $M^{(n)}_{k-1}$, and its height at $M^*_{k-1}$ is given by (a), with $v^{(i)} = v_e \in E^*_{k-1}(\lambda)$, having nonzero first component. By the identical argument, some $j$th, odd, eigenvector of $M^{(n)}_{k-1}$ converges to $v_o \in E^*_{k-1}(\lambda)$, having nonzero first component, and the corresponding upper bounding surface $B_j$ of $\mathcal{T}_k$ has height at $M^*_{k-1}$ given by (a) with $v^{(j)} = v_o$. However, by Proposition 3(b), these two values coincide; hence $t_k$ is determined as the intersection of $B_i$ and $B_j$ at $M^*_{k-1}$. As $t_k$ depends only on $E^*_{k-1}(\lambda)$, the choice of sequence $M^{(n)}_{k-1} \to M^*_{k-1}$ plays no role, and so the surfaces $B_i$ and $B_j$ are continuous at $M^*_{k-1}$, with limit $t_k$ there. This shows that they account for the
boundary of $\mathcal{F}_k$ also over $\mathcal{F}_{k-1}$, and completes the proof of (a).

Let $\lambda$ be an eigenvalue of $M_k \in \mathcal{F}_k$. If $n_{k-1}(\lambda) = 0$, then (c) is automatically true. Otherwise, by Proposition 2(a), $n_{k-1}(\lambda) \geq 2$; hence $M_{k-1}$ lies on the boundary of $\mathcal{F}_{k-1}$, and the argument just given shows that $t_k$ is determined to satisfy (a). By Proposition 3(b), this choice of $t_k$ produces an increase in the multiplicity, and so establishes (c).

Finally, since $\mathcal{F}_{k-1}$ is compact by the induction hypothesis, and $t_k$ is bounded there, $\mathcal{F}_k$ remains compact. This concludes the proof of Lemma 1.

To illustrate for $n = 4$, $\mathcal{I}_3$ is the interval $|t_3| < 1$, and $\mathcal{I}_4$ is bounded by the curves

\[
B_1: t_4 = -1 + \left( t_3^2 + t_3 \sqrt{t_3^2 + 8} \right) / 2, \quad 0 \leq t_3 < 1.
\]
\[
B_2: t_4 = 1 ,
\]
\[
B_3: t_4 = -1 + \left( t_3^2 - t_3 \sqrt{t_3^2 + 8} \right) / 2, \quad -1 < t_3 \leq 0.
\]

We remark that this example is atypical, in that, for $k \geq 5$, the projection of $\mathcal{F}_k$ onto $t_k = 0$ is strictly smaller than $\mathcal{F}_{k-1}$; that is, not all matrices of $\mathcal{F}_{k-1}$ are extendible to $\mathcal{F}_k$. In particular, if $M_{k-1} \in \mathcal{F}_{k-1}$ has two different multiple eigenvalues, the heights $t_k$ determined in the preceding construction by the corresponding eigenspaces of $M_{k-1}$ are not likely to coincide, and if they differ there can be no matrix in $\mathcal{I}_k$ above $M_{k-1}$.

The eigenvalue map

Following S. Friedland [F], we next focus on the topological degree $[S, B]$ of the normalized eigenvalue map $\Lambda$. This map takes $\mathcal{F}_k \subset R^{k-2}$ into $L_k \subset R^{k-2}$, and, by Lemma 1(a), the boundary $\partial \mathcal{F}_k$ into the hyperplane boundaries of the simplex $L_k$. The topological degree, defined for a point disjoint from $\Lambda(\partial \mathcal{F}_k)$, the image of the boundary, measures the number of times the point is covered by the map $\Lambda$ acting on $\mathcal{F}_k$, counted with orientation. Like the winding number in two dimensions, it is known to remain constant in each connected component of the complement of $\Lambda(\partial \mathcal{F}_k)$, hence in particular in $L_k$. At a point not covered by $\Lambda(\mathcal{F}_k)$, the degree is zero. Therefore to prove that all of $L_k$ is covered, it is sufficient to exhibit a single point in $L_k$ where the degree is not zero. We do not do this explicitly, not knowing enough about $\Lambda$ to invert it anywhere in $L_k$. Instead, we proceed indirectly, beginning with a special boundary point of $L_k$.

Let $M_k(0, 1, t_3, \ldots , t_k)$ have eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = -1$, $\lambda_k = k - 1$; thus $n_k(-1) = k - 1$. By Proposition 3(e), (b) we then find that $n_i(-1) = i - 1$, for $2 \leq i \leq k$; hence every $t_i$, $3 \leq i \leq k$, is given by (4), with $v$ the eigenvector $(1, -1)$ of $M_2(0, 1)$ corresponding to $\lambda = -1$. We conclude that $M_k$ is unique, with $t_3 = \cdots = t_k = 1$; it lies in the boundary.

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of $\mathcal{T}_k$, being the limit of $M_k(0, 1, \rho, \ldots, \rho^{k-2})$ as $\rho \to 1$. This is not yet useful for our purpose, because it is only a boundary point of $\mathcal{T}_k$, and one at which $\Lambda$ is not differentiable. However, starting with this matrix, whose first $k - 1$ eigenvalues coincide, we will perturb it, freeing one eigenvalue at a time. We will be able to parametrize this sequence of boundary facets of $\mathcal{T}_k$ in a way which will allow us to follow $\Lambda$ from the boundary of $\mathcal{T}_k$ to its interior, and to show that in this progression $\Lambda$ is locally one-to-one. (For example, for $n = 4$, we find how $\Lambda$ extends from the point $t_3 = t_4 = 1$, first along a segment of $B_1$ which is mapped onto a segment of $\lambda_2 = -1$ near $(-1, -1)$, and thence to the interior of $\mathcal{T}_4$.) Finally, the uniqueness of the starting point will show that $\Lambda$, as a map of $\mathcal{T}_k$, must also be globally one-to-one at some nearby point inside $L_k$, and at such a point the degree does not vanish.

**Definition.** A matrix $M_k(0, 1, t_3, \ldots, t_k)$ is boundary-regular if, for each $j \leq k$, the eigenvalues of $M_j(0, 1, t_3, \ldots, t_j)$ are single except for a possibly multiple lowest eigenvalue, and eigenvectors can be chosen in that eigenspace so that all the eigenvectors of $M_k$ alternate in parity in the manner required of $\mathcal{T}_k$.

The definition of a boundary-regular matrix differs from that of a regular one only in allowing a multiple lowest eigenvalue.

**Lemma 2.** Let $M_k$ be a boundary-regular matrix with lowest eigenvalue of multiplicity $m$. If $m = 1$, $M_k$ is regular. If $m \geq 2$, then $M_k$ is the endpoint of an arc $M_k^{(e)}$, $\epsilon \to 0+$, of boundary-regular matrices with lowest eigenvalue of multiplicity $m - 1$.

**Proof.** Suppose $M_k$ is boundary-regular with single lowest eigenvalue $\lambda$. Then by Proposition 3(e), the lowest eigenvalue $\mu$ of $M_{k-1}$ must be larger than $\lambda$. Since $M_k$ is boundary-regular, $n_k(\mu) \leq 1$; hence $n_{k-1}(\mu) \leq 2$ by Proposition 2(a). However, if $n_{k-1}(\mu) = 2$, then $n_k(\mu) = 1$, and $n_{k-2}(\mu) = 1$ by Proposition 3(e). If $(v_1, \ldots, v_{k-2})$ is the eigenvector of $M_{k-2}$ corresponding to $\mu$, then by Proposition 3(a), (c) that of $M_k$ is $(0, v_1, \ldots, v_{k-2}, 0)$, hence has the same parity. However, the alternation of parity required of the eigenvalues of $M_k$ and $M_{k-2}$, which are single, shows that the lowest eigenvalue of $M_{k-2}$ and the second-lowest of $M_k$ must have opposite parities. This contradiction shows that $n_{k-1}(\mu) = 1$, so that $M_{k-1}$ also has a single lowest eigenvalue. By iterating this argument, the same is true of each $M_j$, $j \leq k$, hence $M_k \in \mathcal{T}_k$.

Suppose $n_k(\lambda) = m \geq 2$. By Proposition 3(e), $M_{k-m+1}$ has $\lambda$ as a single lowest eigenvalue with corresponding eigenvector $v$, and by Proposition 3(b) each $t_j$, $k - m + 2 \leq j \leq k$, is determined by (4) from $v$. As we have seen, $M_{k-m+1} \in \mathcal{T}_{k-m+1}$; hence $M_{k-m+2}$ lies in the boundary surface of $\mathcal{T}_{k-m+2}$ defined by $v$ in Lemma 1(a). It follows from Lemma 1(b) that the perturbation

$$t_{k-m+2}^{(e)} = t_{k-m+2} \pm \epsilon$$
\( \epsilon > 0 \), with the sign positive or negative as \( v \) is even or odd, respectively, will move \( M_{k-m+2} \) to \( M_{k-m+2}^{(\epsilon)} \in \mathcal{T}_{k-m+2} \), thereby separating the double lowest eigenvalue \( \lambda \) of \( M_{k-m+2} \) into \( \lambda^- (\epsilon) < \lambda^+ (\epsilon) \); by definition of \( \mathcal{T}_{k-m+2} \) and the interlacing of eigenvalues with those of \( M_{k-m+1} \), the parity of \( \lambda^- (\epsilon) \) is opposite to that of \( v \). We now define \( t_j (\epsilon), k - m + 3 \leq j \leq k \), by (4), using the eigenvector \( w(\epsilon) \) of \( M_{k-m+2}^{(\epsilon)} \) corresponding to \( \lambda^- (\epsilon) \). This produces a matrix \( M_k^{(\epsilon)} \) with lowest eigenvalue \( \lambda^- (\epsilon) \) of multiplicity \( m - 1 \), and an eigenvalue \( \lambda_m (\epsilon) \) for which, by the interlacing property,

\[
\lambda_m (\epsilon) \geq \lambda^+ (\epsilon) .
\]

We now show that \( M_k^{(\epsilon)} \) approaches \( M_k \) as \( \epsilon \to 0^+ \), and that for all \( \epsilon \) sufficiently small, \( M_k^{(\epsilon)} \) is boundary-regular.

The lowest eigenvector \( w(\epsilon) \) of \( M_{k-m+2}^{(\epsilon)} \) has for its limit as \( \epsilon \to 0^+ \) that one of the two lowest eigenvectors of \( M_{k-m+2} \) which has the parity of \( w(\epsilon) \), specifically, the one of opposite parity to \( v \); by Proposition 3(b), the entries \( t_j \) of \( M_k \), for \( k - m + 3 \leq j \leq k \), are also definable by (4) using this eigenvector. Thus \( M_k^{(\epsilon)} \) approaches \( M_k \) as \( \epsilon \to 0^+ \). Now for \( \epsilon \) sufficiently small, eigenvectors of \( M_j \) are perturbed to corresponding eigenvectors of \( M_j^{(\epsilon)} \) without a change of parity or increase of multiplicity; hence the single eigenvalues of \( M_j \) yield neighboring single eigenvalues of the same parity for \( M_j^{(\epsilon)} \), while the single eigenvalue split from the block of lowest eigenvalues of \( M_j \), \( k - m + 2 \leq j \leq k \), is the next-to-smallest in \( M_j^{(\epsilon)} \). We now determine the parity of this eigenvalue; in order to maintain the alternation required for boundary-regularity of \( M_k^{(\epsilon)} \), it must be that of \( v \) for each \( j \).

The multiple block of eigenvectors of \( M_j \) is generated, as in Proposition 3(b), by \( v \), while that of \( M_j^{(\epsilon)} \) comes from \( w(\epsilon) \), of opposite parity. By Proposition 3(d), when the block of \( M_j \) has odd multiplicity, its subspace of vectors with the parity of \( v \) has the larger dimension, whereas in the block remaining in \( M_j^{(\epsilon)} \) even and odd vectors have equal dimension; consequently, the eigenvector removed from the block by \( M_j^{(\epsilon)} \) has the parity of \( v \). Similarly, if the block of \( M_j \) has even multiplicity, then that remaining in \( M_j^{(\epsilon)} \) has an excess of vectors with the parity of \( w(\epsilon) \), opposite to \( v \); hence again the eigenvector removed has the parity of \( v \). This shows that \( M_k^{(\epsilon)} \) is boundary-regular, completing the proof of Lemma 2.

**Corollary 2.** A boundary-regular matrix \( M_k \) lies in \( \mathcal{F}_k \).

**Proof.** By Lemma 2, we can approximate \( M_k \) arbitrarily closely by boundary-regular matrices for which the smallest eigenvalue has lower multiplicity. On iterating this, we can reduce the multiplicity to 1, obtaining approximants in \( \mathcal{F}_k \). Thus \( M_k \in \mathcal{F}_k \).
Definition. Let $l_j$ be the $j$-dimensional boundary face

$$-1 = \lambda_1 = \lambda_2 = \cdots = \lambda_{k-j-1}$$

of $L_k$. With $k$ fixed, denote by $S_j$, $0 \leq j \leq k - 3$, the subset of $F_k$ sent by $A$ to points of $l_j$. Let $A_j$ represent the restriction of $A$ to $S_j$.

Identifying $S_{k-2}$ with $F_k$ for consistency, we see that $S_i \subset S_{i+1}$, and that $S_j$, for $j < k - 2$, lies in the boundary of $F_k$. In terms of matrices, $S_j$ corresponds to those for which the lowest $k - j - 1$ eigenvalues coincide; hence

$$S_j = F_k \bigcap_{i=1}^{k-j-2} B_i.$$

Lemma 3. For each $j$, $1 \leq j \leq k - 2$, $S_j$ contains a subset $O_j$ of boundary-regular matrices, which is open in $S_j$ and includes a neighborhood of $O_{j-1}$ in $S_j$, such that $A_j$, construed as a map of $S_j$ into $l_j$, is locally one-to-one in $O_j$, with nonvanishing Jacobian.

Proof. $S_{j-1}$ serves as a boundary of $S_j$. We will proceed inductively, by extending $A_{j-1}$ locally from $S_{j-1}$ into $S_j$. Accordingly, we start with $O_0 = S_0$ which, as we have seen, is the single point $(1, 1, \ldots, 1)$ corresponding to the boundary-regular $M_k(0, 1, \ldots, 1)$, and we consider $S_1$ in the neighborhood of its boundary point $O_0$. Since for a matrix in $S_1$ the multiplicity of the smallest eigenvalue is $(k - 2)$, it follows as before from Proposition 3(e), (b) that the matrix consists of the extension (4) of some matrix of $F_3$ near $M_3(0, 1, 1)$, using for $v$ the smallest eigenvector. This is exactly the construction of Lemma 2, applied to the boundary-regular $M_k(0, 1, \ldots, 1)$. We conclude that, sufficiently near $S_0$, the set $S_1$ is an arc, parametrized by the boundary-regular $M_k(\epsilon)$, $0 < \epsilon < \epsilon_0$, formed by (4) from the smallest eigenvector $w(\epsilon)$ of

$$M_3(\epsilon) = M_3(0, 1, 1) - \epsilon M_3(0, 0, 1),$$

which is even since $M_3(\epsilon) \in F_3$; the negative sign in the perturbation is dictated by Lemma 1(b), and implies that $S_1$ departs from $S_0$ in one direction only.

To describe the resulting behavior of eigenvalues, we again appeal to the basic properties of perturbations. Let $v_e$ and $v_o$, with first component $v_1 \neq 0$, be the even and odd vectors in the lowest eigenspace of $M_3(0, 1, 1)$. The perturbed $w(\epsilon) = w_\epsilon(\epsilon)$ is differentiable in $\epsilon$, and we know its first component to be bounded away from 0. Consequently the entries $t_j(\epsilon)$ of $M_k(\epsilon)$, defined successively by (4) from $w(\epsilon)$, are likewise differentiable in $\epsilon$. Since both sides of (9) approach the same limit as $\epsilon \to 0+$, for the eigenvalue $\lambda_{k-1}(\epsilon)$ of $M_k(\epsilon)$ we find, by (9) and (7),

$$\frac{d\lambda_{k-1}(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0^+} \geq \frac{d\lambda^+(\epsilon)}{d\epsilon} \bigg|_{\epsilon=0^+} \frac{2v_1^2}{\|v_o\|^2}. $$
The normalized map $A$ takes $M_k^{(e)}$ into $(-1, \ldots, -1, \lambda_{k-1}(e)/(-\lambda^-(e)))$. Since $\lambda_{k-1}(e)$ likewise approaches $-1$ as $e \to 0^+$, we find

$$
\frac{d}{de} \left( \frac{\lambda_{k-1}(e)}{-\lambda^-(e)} \right) \bigg|_{e=0^+} \geq 2v_1^2 \frac{1}{\|v_0\|^2} + 2v_1^2 \frac{1}{\|v_o\|^2}.
$$

Being continuous, this expression remains positive in an open interval of the arc $S_1$, having $O_0$ as boundary point, on which $M_k^{(e)}$ remains boundary-regular; we take this to be $O_1$. In turn, the positivity of the derivative means that the map $\Lambda_1$ is one-to-one in the neighborhood of every point of the closed arc $\overline{O_1}$, that is, on $O_1$ and its endpoint $O_0$. This proves the lemma for $j = 1$.

Now suppose $O_j \subset S_j$ has been determined to satisfy the requirements of the lemma. We argue as earlier. Let $M_k(0, 1, t_3, \ldots, t_k) \in O_j$ be boundary-regular. One way of moving $M_k$ into $S_{j+1}$ is to apply the construction of Lemma 2. It splits the double lowest eigenvalue of $M_{j+3}$ into $\lambda^-(e) < \lambda^+(e)$ and yields a boundary-regular

$$
M_k^{(e)} = M_k(0, 1, t_3, \ldots, t_{j+2}, t_{j+3} \pm e, t_{j+4}(e), \ldots, t_k(e)),
$$

in which $t_i(e), j + 4 \leq i \leq k$, are determined as in (4) to produce a block of $k - j - 2$ coincident eigenvalues at $\lambda^-(e)$; for the next eigenvalue we have, as in (9), $\lambda_{k-j-1}(e) \geq \lambda^+(e)$. By the same argument as for $S_1$, $M_k^{(e)}$ is differentiable in $e$. The normalized eigenvalue map $\Lambda_{j+1}$ takes $M_k^{(e)} \in S_{j+1}$ onto $(-1, \ldots, -1, (\lambda_{k-j-1}(e)/-\lambda^-(e)), \ldots, (\lambda_k(e)/-\lambda^-(e)))$, and we find, just as earlier for $S_0$,

$$
\frac{d}{de} \left( \frac{\lambda_{k-j-1}(e)}{-\lambda^-(e)} \right) \bigg|_{e=0^+} \geq \frac{v_1^2}{|\lambda|} \left( \frac{2}{\|v_e\|^2} + \frac{2}{\|v_o\|^2} \right) > 0,
$$

with $\lambda \neq 0$ the lowest eigenvalue of $M_k$, and $v_e$ and $v_o$ the lowest eigenvectors of $M_{j+3}$. This shows that $\Lambda_j$ can be extended from $S_j$ to $S_{j+1}$, with $M_k^{(e)}$ boundary-regular.

Now consider an arbitrary $M_k^{(e)}(0, 1, t_3(e), \ldots, t_k(e)) \in S_{j+1}$ that approaches a boundary-regular $M_k(0, 1, t_3, \ldots, t_k) \in S_j$. As before, by Proposition 3(e), (b), its lowest eigenvalue $\lambda(e)$ is a single eigenvalue of $M_{j+3}$, $t_i(e)$ in the range $j + 4 \leq i \leq k$ is defined successively by (4), with $v = w(e)$ the lowest eigenvector of $M_{j+3}^{(e)}$, and the $(k - j - 2)$-dimensional eigenspace $E_k^{(e)}$ of $M_k^{(e)}$ corresponding to $\lambda(e)$ is generated from this $w(e)$ as in Proposition 2(b). As $e \to 0^+$, $w(e)$ approaches that of the two lowest eigenvectors of $M_{j+3}$ which has the same parity as $w(e)$; since $M_{j+3}^{(e)} \in S_{j+3}$, we know this to be even or odd, according as the integer $j$ is. Consequently this limit, $w(0)$, is determined independently of the perturbation, and the eigenspace $E_k^{(e)}$ generated by $w(e)$ has as its limit the eigenspace $E_k^{(0)}$ of $M_k$ analogously generated by $w(0)$. The eigenvector $v_{k-j-1}(e)$ of $M_k^{(e)}$ corresponding to the eigenvalue
\[ \lambda_{k-j-1}(e), \] which has been split from the block of common eigenvalues of \( M_k \), being orthogonal to \( E_k^{(e)} \), then approaches the orthogonal complement in the \((k - j - 1)\)-dimensional lowest eigenspace of \( M_k \) of the \((k - j - 2)\)-dimensional subspace \( E_k^{(0)} \). This is a vector \( w \) which depends only on \( M_k \). We conclude that for any local variation of \( M_k \in S_j \) into \( S_{j+1} \) the eigenvectors beyond the \((k - j - 2)\)th are perturbations of fixed vectors that do not depend on the variation. We point out that this is not generally true of all variations of \( S_j \) into \( \mathcal{S}_k \)—a fact which explains why \( A \) is not differentiable at low-dimensional boundary facets of \( L_k \).

The following argument for the differentiability of the perturbed eigenvalues was suggested by P. Deift. Let \( M_{j+3}^{(e)} = M_{j+3}(0, 1, t_3 \pm \epsilon_3, \ldots, t_j \pm \epsilon_j) \), the signs chosen so that \( M_{j+3}^{(e)} \in \mathcal{S}_{j+3} \) for \( \epsilon \to 0^+ \). By perturbation theory for single eigenvectors [K, p. 119], each single eigenvector of \( M_{j+3}^{(e)} \) is perturbed analytically in all the components \( \epsilon_3, \ldots, \epsilon_j \) of \( \epsilon \). The same is true of the two lowest (multiple) even and odd eigenvectors of \( M_{j+3}^{(e)} \), since the perturbation of each is definable by orthogonality to the remaining eigenvectors of \( M_{j+3}^{(e)} \) of the same parity. In turn, the analyticity of \( w(\epsilon) \) propagates to \( t_i(\epsilon), \ j + 4 \leq i \leq k \), by (4), and to the \( j \) multiple lowest eigenvectors of \( M_{j+3}^{(e)} \) constructed from \( w(\epsilon) \) as in the proof of Proposition 3(d). Since \( M_{j+3}^{(e)} \) is now an analytic perturbation of \( M_k \), so again is that of each single eigenvector of \( M_k \), and the remaining \((j + 1)\)st eigenvector of \( M_{j+3}^{(e)} \) which has been split from the multiple eigenspace of \( M_k \) is analytic as well, being definable by orthogonality to all the others. Thereupon the eigenvalues are also analytic in \( \epsilon_3, \ldots, \epsilon_j \), and by differentiating the formula \( \lambda_i(\epsilon) = (M_{k}^{(e)}v_i(\epsilon), v_i(\epsilon))/\|v_i(\epsilon)\|^2 \) we obtain

\[
\left. \frac{d\lambda_i(\epsilon)}{d\epsilon} \right|_{\epsilon=0^+} = \frac{(T_k v_i, v_i)}{\|v_i\|^2}, \quad k - j - 1 \leq i \leq k,
\]

where \( v_{k-j-1} = w \), the vector just defined, \( v_i \) is the \( i \)th eigenvector of \( M_k \), \( k - j \leq i \leq k \), and \( T_k = dM_k^{(e)}/d\epsilon|_{\epsilon=0^+} \); for \( i \leq k - j - 2 \), the eigenvalues are equal and

\[
\left. \frac{d\lambda_i(\epsilon)}{d\epsilon} \right|_{\epsilon=0^+} = \frac{(T_k u, u)}{\|u\|^2},
\]

for any eigenvector \( u \) of \( M_k \) in the subspace \( E_k^{(0)} \), that is, any eigenvector corresponding to the lowest eigenvalue, but orthogonal to \( w \). We conclude that the extension \( \Lambda_{j+1} \) of \( \Lambda_j \) is smooth and locally linear in the perturbation; indeed, (12) and (13) exhibit the tangent space to \( \Lambda_{j+1} \). Consequently, the variation of \( M_k \) to \( M_k^{(e)} \) can be decomposed into one within \( S_j \), followed by one of the form (10). This means that (10) gives the general parametrization of \( S_{j+1} \) near \( S_j \); as with \( S_1 \), since \( \epsilon > 0 \), \( S_{j+1} \) lies locally on only one side of \( S_j \).
Finally, we show that if $\Lambda_j$ is locally one-to-one on $O_j$ at a point $M_k$, its extension $\Lambda_{j+1}$ remains so. Intuitively, this happens because $\Lambda_j$ maps $S_j$ into $l_j$ and, by (11), a direction transverse to $S_j$ into one transverse to $l_j$. More precisely, we recall that the independent variables of a matrix $M_k \in S_j$ are $t_3, \ldots, t_{j+2}$, since the remaining $t_i$ are determined by (4) to produce the required multiplicity of the lowest eigenvalue. If $\Lambda_j$ is locally one-to-one at $M_k$, then, by the local linearity, the $j \times j$ Jacobian matrix $J$ of $(\Lambda_{k-j}/-\lambda_1), \ldots, (\lambda_{k-1}/-\lambda_1)$ with respect to these independent variables $t_3, \ldots, t_{j+2}$ is not singular at $M_k$. In extending the map to $\Lambda_{j+1}$, we increase its domain and range by one dimension, corresponding to the variables $t_{j+3}$ and $\lambda_{k-j-1}/(-\lambda_1)$, respectively. Let $K$ be the enlarged $(j+1) \times (j+1)$ Jacobian matrix of $(\lambda_{k-j}/-\lambda_1), (\lambda_{k-j}/-\lambda_1), \ldots, (\lambda_{k-1}/-\lambda_1)$ with respect to $t_3, \ldots, t_{j+2}, t_{j+3}$.

When viewed as a subset of $t_3, \ldots, t_{j+2}, t_{j+3}$, the independent variables of $S_{j+1}$, the matrices of $S_j$ correspond to values $t_3, \ldots, t_{j+2}, f(t_3, \ldots, t_{j+2})$, with $f$ determined, as usual, by (4), using for $v$ the lowest eigenvector of $M_{j+2}$. Therefore the tangent space $Z$ to $S_j$ at $M_k$ is generated by the vectors

$$\left( \delta t_3, \ldots, \delta t_{j+2}, \sum_{i=3}^{j+2} \frac{\partial f}{\partial t_i} \delta t_i \right).$$

The last entry being linear in $\{\delta t_i\}$, $Z$ forms a subspace of dimension $j$ in the $(j+1)$-dimensional space of $(\delta t_3, \ldots, \delta t_{j+2}, \delta t_{j+3})$, and is the domain of $J$ in that larger space; that is, $K$ coincides with $J$ on $Z$. Because $J$ is nonsingular, the image $KZ$ is likewise $j$-dimensional, and hence there exists a unique vector $p \neq 0$ with

$$\langle KZ, p \rangle = 0;$$

indeed, $p$ is the vector in $l_{j+1}$ orthogonal to $l_j$, that is, $(1, 0, \ldots, 0)$ in the present coordinates. By definition of $K$, we see from (11) that

$$\langle K\delta t_{j+3}, p \rangle \neq 0.$$
Proof. By Lemma 3, the Jacobian of \( \Lambda \) does not vanish on the open subset \( O_{k-2} \) of \( \mathcal{F}_k \). We consider the points in \( L_k \) onto which \( \Lambda \) takes \( O_{k-2} \). If the assertion of the lemma is false, each of these is covered more than once by \( \Lambda(\mathcal{F}_k) \). Thus to each point \( \tau \in O_{k-2} \) there corresponds \( \tau' \in \mathcal{F}_k \) with \( \tau' \neq \tau \) but

\[
\Lambda(\tau') = \Lambda(\tau). 
\]

Let \( \tau \) approach a point \( \sigma \) of the boundary \( O_{k-3} \) of \( O_{k-2} \). On taking a subsequence if necessary, \( \tau' \) likewise converges to \( \sigma' \) and, by (15), \( \Lambda(\sigma') = \Lambda(\sigma) \).

But since \( \sigma \in O_{k-3} \subset S_{k-3} \), the image \( \Lambda(\sigma) \) lies in the boundary face \( l_{k-3} \); hence \( \sigma' \in S_{k-3} \). The points \( \sigma \) and \( \sigma' \) cannot coincide, since by Lemma 3 the map \( \Lambda \) is one-to-one in an \( O_{k-2} \)-neighborhood of \( \sigma \), so once this neighborhood contains an approximant \( \tau_k \) of \( \sigma \), it cannot contain the corresponding \( \tau_k' \). We thus find that \( O_{k-3} \) inherits from \( O_{k-2} \) the property that to each of its points there exists a different point in \( S_{k-3} \) with the same image under \( \Lambda \). Continuing the argument for \( O_i \) with decreasing \( i \), we derive the same conclusion for \( O_0 = S_0 \). But as \( S_0' \) is a single point, it cannot accommodate the distinct \( \sigma \) and \( \sigma' \). This contradiction proves Lemma 4.

We now have the ingredients for a proof of the main result.

**Theorem 1.** For every \( k \), \( \mathcal{F}_k \in R^{k-2} \) is bounded by portions of all the surfaces \( B_i \), \( 1 \leq i \leq k-1 \), and the image of \( \mathcal{F}_k \) under \( \Lambda \) covers all of \( L_k \).

Proof. The image of the boundary of \( \mathcal{F}_k \) lies exclusively in the bounding hyperplanes of \( L_k \). By Lemma 4, there exists a point \( \tau \) interior to \( \mathcal{F}_k \) at which the Jacobian of \( \Lambda \) does not vanish, and which is the only solution of \( \Lambda(x) = \Lambda(\tau) \), for \( x \in \mathcal{F}_k \). Set \( y = \Lambda(\tau) \), and let \( \mathcal{A}_y \) denote the connected component of the complement of \( \Lambda(\partial \mathcal{F}_k) \) in \( R^{k-2} \) which contains \( y \); clearly \( L_k \subset \mathcal{A}_y \).

For the smooth map \( \Lambda \) of \( \mathcal{F}_k \) into \( R^{k-2} \), the degree at \( y \notin \Lambda(\partial \mathcal{F}_k) \) is definable as \([S, B]\)

\[
\text{deg}(y, \Lambda, \mathcal{F}_k) = \sum_{x \in \mathcal{F}_k, \Lambda(x) = y} \text{sign} \left( \text{Jacobian of } \Lambda \text{ at } x \right). 
\]

This is an integer, known to be constant on all of \( \mathcal{A}_y \). Since \( x = \tau \) is unique, \( \text{deg}(y, \Lambda, \mathcal{F}_k) = \pm 1 \neq 0 \). This implies that \( \Lambda(\mathcal{F}_k) \) covers all of \( \mathcal{A}_y \), for if there were a point \( y' \in \mathcal{A}_y \) with no solution to the equation \( \Lambda(x) = y' \), \( x \in \mathcal{F}_k \), then by (16)

\[
\text{deg}(y', \Lambda, \mathcal{F}_k) = 0 \neq \text{deg}(y, \Lambda, \mathcal{F}_k),
\]

a contradiction. Moreover, if \( B_i \) is not in the boundary of \( \mathcal{F}_k \), then \( \Lambda(\partial \mathcal{F}_k) \) does not include the boundary hyperplane \( \lambda_i = \lambda_{i+1} \) of \( L_k \); hence \( \mathcal{A}_y \) also contains that boundary. Since \( \Lambda(\mathcal{F}_k) \) covers \( \mathcal{A}_y \), this implies in turn that
contains points in its interior for which $\lambda_i = \lambda_{i+1}$, a contradiction. This completes the proof of the theorem.

**REMARKS**

It is tempting to conjecture that the above correspondence between eigenvalues and regular Toeplitz matrices is one-to-one. Of course, it would also be very interesting to find an algorithm for carrying it out.

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**REFERENCES**


**ABSTRACT.** We show that every set of $n$ real numbers is the set of eigenvalues of an $n \times n$ real symmetric Toeplitz matrix; the matrix has a certain additional regularity. The argument—based on the topological degree—is nonconstructive.

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