

## IMMANANT INEQUALITIES AND 0-WEIGHT SPACES

BERTRAM KOSTANT

### 1. INTRODUCTION

Let  $n \in \mathbb{N}$ , and let  $A = (a_{ij})$  be an  $n \times n$  complex matrix. Let  $S_n$  be the group of permutations of  $\mathcal{N} = \{1, \dots, n\}$ , and let  $\mathcal{P}_n$  be the set of all partitions of  $n$ . As one knows, to each  $\lambda \in \mathcal{P}_n$  we may associate an irreducible complex representation  $\nu_\lambda : S_n \rightarrow \text{Aut } Y_\lambda$  such that, using standard notation,  $\chi^\lambda$  is the character of  $\nu_\lambda$ . Generalizing the determinant and permanent of  $A$  one defines the immanants of  $A$  by putting, for any  $\lambda \in \mathcal{P}_n$ ,

$$\text{Imm}_\lambda(A) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) a_{1\sigma_1} \cdots a_{n\sigma_n}.$$

According to [GJ], although utilized much earlier by Schur, the term Immanant was introduced by Littlewood. In fact, in [Li], Littlewood uses immanants to define Schur functions. Indeed given any  $g \in Gl(m, \mathbb{C})$  Littlewood (see (6.2;7) in [Li]) constructs a matrix  $Z(g) \in M(n, \mathbb{C})$ , using power sums of the eigenvalues of  $g$ , such that, for any  $\lambda \in \mathcal{P}_n$ ,

$$(1) \quad \text{Imm}_\lambda(Z(g)) = \frac{1}{n!} \text{tr } \pi_\lambda(g),$$

where  $\nu_\lambda \times \pi_\lambda$  occurs as an irreducible component of the reduction of  $\otimes^n \mathbb{C}^m$  under the natural action of  $S_n \times Gl(m, \mathbb{C})$ .

A recent paper by Haiman [H] dealt with immanant inequality results and conjectures. Cited in particular were a result of Schur and conjectures and results of Stembridge. For any  $\lambda \in \mathcal{P}_n$ , let  $f_\lambda = \dim \nu_\lambda$ . Consider the validity of the statement: For any  $\lambda \in \mathcal{P}_n$

$$(2) \quad \text{Imm}_\lambda(A) \geq f_\lambda$$

for a matrix  $A \in Sl(n, \mathbb{C})$  (to which one is readily reduced). Two of the results stated in [H] for  $Sl(n, \mathbb{C})$  are as follows (see [Sc] and [St]):

**Theorem 1** (Schur). *One has (2) if  $A$  is positive definite.*

A matrix  $A$  is called totally positive if all square minors are non-negative.

---

Received by the editors November 16, 1993.

1991 *Mathematics Subject Classification*. Primary 22E46, 15A15.

Research supported in part by NSF grant DMS-9307460.

**Theorem 2** (Stembridge). *One has (2) if  $A$  is totally positive.*

In this paper spotlighting an interpretation of immanants based on 0-weight spaces, we give as an application a generalization, for all representations of  $SL(n, \mathbb{C})$ , of Theorems 1 and 2. In the case of Schur's theorem the generalization is straightforward. For the case of Stembridge's theorem we rely on a result of A. Whitney on the structure of totally positive matrices and a very deep result of George Lusztig on the coefficient non-negativity for the action of certain semigroups with respect to the canonical basis (see Theorem 22.1.7 in [Lu]).

The notation underlying the statement (2) deals with arbitrary  $n, m \in \mathbb{N}$ . Now assume that  $m = n$ . Furthermore regard  $\pi_\lambda$  as a representation of  $Sl(n, \mathbb{C})$ . Thus, as one knows,  $\otimes^n \mathbb{C}^n$  is a multiplicity free  $(S_n \times Sl(n, \mathbb{C}))$ -module and its complete reduction into irreducible components can be written as

$$(3) \quad \otimes^n \mathbb{C}^n = \sum_{\lambda \in \mathcal{P}_n} Y_\lambda \otimes V_\lambda,$$

where  $\pi_\lambda : Sl(n, \mathbb{C}) \rightarrow \text{Aut } V_\lambda$  is an irreducible representation of  $Sl(n, \mathbb{C})$ . Let  $H$  be the Cartan subgroup of all diagonal matrices in  $Sl(n, \mathbb{C})$  so that  $V_\lambda^H$  is the 0-weight space for the representation  $\pi_\lambda$ . Let  $P_\lambda : V_\lambda \rightarrow V_\lambda^H$  be that projection operator on the 0-weight space which commutes the action of  $\pi_\lambda(H)$ .

We first establish Theorem 3 below—a 0-weight interpretation of immanants, for matrices  $A \in Sl(n, \mathbb{C})$ . If we extend the action of  $\pi_\lambda$  to be a multiplication preserving map of  $M(n, \mathbb{C})$ , then our proof of Theorem 3 in fact establishes the same characterization of immanants for any  $n \times n$  matrix. The first statement of Theorem 3 is known. The second statement, although easy to prove (e.g., it is deducible from §II.2 in [B]), appears to be new.

**Theorem 3.** *Let  $\lambda \in \mathcal{P}_n$ . Then*

$$(4) \quad \dim V_\lambda^H = f_\lambda.$$

*Now let  $A \in Sl(n, \mathbb{C})$ . Then*

$$(5) \quad \text{Imm}_\lambda(A) = \text{tr } P_\lambda \pi_\lambda(A) P_\lambda.$$

*Remark.* If we extend  $\pi_\lambda$  to all of  $M(n, \mathbb{C})$ , then, as established in [Li], (1) is valid for any  $g \in M(n, \mathbb{C})$  and hence the extended Theorem 3 yields the following somewhat startling trace equality for  $\pi_\lambda$ :

$$(6) \quad \text{tr } P_\lambda \pi_\lambda(Z(g)) P_\lambda = \frac{1}{n!} \text{tr } \pi_\lambda(g).$$

The representations  $\{\pi_\lambda\}$ ,  $\lambda \in \mathcal{P}_n$ , appearing in Theorem 3 are of course only a finite subset of the set of all the irreducible representations of  $Sl(n, \mathbb{C})$ . Actually this finite set can be characterized by the condition, satisfied for example by the adjoint representation, that twice a root is not a weight.

By scaling, Theorem 1 follows immediately from the more restricted statement where  $A$  is assumed to lie in  $Sl(n, \mathbb{C})$ . The same is true of Theorem 2 by virtue of the non-singular approximation result—Theorem 1 in [W]—for totally positive matrices. In view of Theorem 3 the following result is then a generalization of Theorems 1 and 2.

**Theorem 4.** *Let  $\pi : Sl(n, \mathbb{C}) \rightarrow \text{Aut } V$  be any finite (holomorphic) dimensional representation of  $Sl(n, \mathbb{C})$ , and let  $P : V \rightarrow V^H$  be the  $\pi(H)$ -projection on the 0-weight space  $V^H$ . Then*

$$(7) \quad \text{tr } P\pi(A)P \geq \dim V^H$$

whenever  $A \in Sl(n, \mathbb{C})$  is either positive definite or totally positive.

We wish to thank R. Stanley and A. Zelevinsky for informative conversations.

### 2. THE 0-WEIGHT SPACE IN $\otimes^n \mathbb{C}^n$

We retain the notation of the Introduction. The natural representation of  $S_n \times Sl(n, \mathbb{C})$  on  $\otimes^n \mathbb{C}^n$  will be denoted by  $\rho$ . Explicitly if  $v_i \in \mathbb{C}^n, i = 1, \dots, n, A \in Sl(n, \mathbb{C})$ , and  $\tau \in S_n$ , then

$$\begin{aligned} \rho(A)(v_1 \otimes \dots \otimes v_n) &= Av_1 \otimes \dots \otimes Av_n, \\ \rho(\tau)(v_1 \otimes \dots \otimes v_n) &= v_{\tau^{-1}1} \otimes \dots \otimes v_{\tau^{-1}n}. \end{aligned}$$

Let  $e_i \in \mathbb{C}^n, i = 1, \dots, n$ , be the standard basis so that any pure monomial tensor product of the  $e_i$  in  $\otimes^n \mathbb{C}^n$  is a weight vector with respect to  $\rho(H)$ . It is immediate then that for  $\rho|Sl(n, \mathbb{C})$  the 0-weight space  $(\otimes^n \mathbb{C}^n)^H$  is  $n!$  dimensional and is given by

$$(\otimes^n \mathbb{C}^n)^H = \sum_{\tau \in S_n} \mathbb{C} e_{\tau^{-1}1} \otimes \dots \otimes e_{\tau^{-1}n}.$$

Now if  $A = \{a_{ij}\} \in Sl(n, \mathbb{C})$ , then

$$A e_{\tau^{-1}j} = \sum_{i=1}^n a_{i\tau^{-1}j} e_i.$$

Thus

$$\rho(A)(e_{\tau^{-1}1} \otimes \dots \otimes e_{\tau^{-1}n}) = \sum_{(i_1, \dots, i_n) \in \mathcal{N}^n} a_{i_1\tau^{-1}1} \dots a_{i_n\tau^{-1}n} e_{i_1} \otimes \dots \otimes e_{i_n}.$$

Now let  $Q : (\otimes^n \mathbb{C}^n) \rightarrow (\otimes^n \mathbb{C}^n)^H$  be the projection operator which commutes with the action of  $H$ . Then clearly

$$Q\rho(A)(e_{\tau^{-1}1} \otimes \dots \otimes e_{\tau^{-1}n}) = \sum_{\sigma \in S_n} a_{\sigma^{-1}1\tau^{-1}1} \dots a_{\sigma^{-1}n\tau^{-1}n} e_{\sigma^{-1}1} \otimes \dots \otimes e_{\sigma^{-1}n};$$

that is, if  $z = e_1 \otimes \dots \otimes e_n$  and  $\alpha(\sigma, \tau) = a_{\sigma^{-1}1\tau^{-1}1} \dots a_{\sigma^{-1}n\tau^{-1}n}$ , then

$$(8) \quad Q\rho(A)\rho(\tau)z = \sum_{\sigma \in S_n} \alpha(\sigma, \tau)\rho(\sigma)z.$$

On the other hand if  $W_n$  denotes the Weyl group of  $Sl(n, \mathbb{C})$  with respect to  $H$ , then  $\rho$  induces a representation  $\gamma$  of  $W_n$  on the 0-weight space  $(\otimes^n \mathbb{C}^n)^H$ . Since elements of  $W_n$  may be represented, modulo  $H$ , by permutation matrices with signs—so as to have determinant 1—we may clearly identify  $W_n$  with  $S_n$  in such a fashion that for  $\sigma, \tau \in S_n$  one has

$$(9) \quad \gamma(\sigma)\rho(\tau)z = sg(\sigma)\rho(\tau)\rho(\sigma^{-1})z.$$

The group algebra  $\mathbb{C}[S_n]$  is an  $(S_n \times S_n)$ -module with respect to left and right multiplication so that if  $g \in \mathbb{C}[S_n]$  and  $(\tau, \sigma) \in S_n \times S_n$ , then  $(\tau, \sigma) \cdot g = \tau g \sigma^{-1}$ . On the other hand  $(\otimes^n \mathbb{C}^n)^H$  is an  $(S_n \times S_n)$ -module, where  $(\tau, \sigma)$  operates by  $sg(\sigma)\rho(\tau)\gamma(\sigma)$ . It is clear then that there is a unique  $S_n \times S_n$  isomorphism

$$(10) \quad \eta : (\otimes^n \mathbb{C}^n)^H \rightarrow \mathbb{C}[S_n]$$

such that  $\gamma(z) = \epsilon$ , where  $\epsilon$  is the identity element of  $S_n$ . Now recalling the complete reduction (3) of  $\rho$ , let  $\gamma_\lambda : W_n \rightarrow \text{Aut } V_\lambda^H$  be the 0-weight space representation of the Weyl group corresponding to  $\pi_\lambda$ . By (10) and the Peter-Weyl theorem one has an identification

$$(11) \quad V_\lambda^H = Z_\lambda^*,$$

where  $Z_\lambda^*$  is the dual space to  $Z_\lambda$ , and an equivalence (since  $\nu_\lambda$  is self-contragredient)

$$\gamma_\lambda \simeq \nu_\lambda \otimes sg.$$

With the identification (11) we may write

$$(12) \quad (\otimes^n \mathbb{C}^n)^H = \oplus_{\lambda \in \mathcal{P}_n} Z_\lambda \otimes Z_\lambda^*.$$

In particular recalling that  $f_\lambda = \dim Z_\lambda$  so that  $f_\lambda = \chi^\lambda(\epsilon)$  one has

$$(13) \quad \dim V_\lambda^H = f_\lambda,$$

establishing (4).

### 3. PROOF OF (5) IN THEOREM 3

One knows that for the extension of any representation  $\nu$  of  $S_n$  to the group algebra  $\mathbb{C}[S_n]$  the image under  $\nu$  of the element  $F_\lambda \in \mathbb{C}[S_n]$  given by

$$F_\lambda = \frac{f_\lambda}{n!} \sum_{\tau \in S_n} \chi^\lambda(\tau)\tau$$

is the projection operator on the  $\nu_\lambda$  primary component of  $\nu$ . But then, if  $\{v_i\}$ ,  $i = 1, \dots, f_\lambda$ , is a basis of  $Z_\lambda$  and  $\{u_i\}$  is the dual basis of  $Z_\lambda^* = V_\lambda^H$ , one has, by the Peter-Weyl theorem, that

$$(14) \quad \frac{f_\lambda}{n!} \sum_{\tau \in S_n} \chi^\lambda(\tau)\rho(\tau)z = \sum_{i=1}^{f_\lambda} v_i \otimes u_i.$$

Then upon applying  $Q\rho(A)$  to the left side of (14) it follows from (8) that

$$(15) \quad \sum_{i=1}^{f_\lambda} v_i \otimes P_\lambda \pi_\lambda(A) u_i = \frac{f_\lambda}{n!} \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \alpha(\sigma, \tau) \chi^\lambda(\tau) \rho(\sigma) z.$$

Now using the natural  $*$ -operation in the group algebra  $\mathbb{C}[S_n]$  one defines a Hilbert space structure on  $(\otimes^n \mathbb{C}^n)^H$  by putting, for  $x, y \in (\otimes^n \mathbb{C}^n)^H$ ,

$$\{x, y\} = \text{tr } L_{\eta(x)} L_{(\eta(y))^*}$$

recalling (10), where if  $F \in \mathbb{C}[S_n]$ , then  $L_F$  is the operator of left translation in  $\mathbb{C}[S_n]$  by  $F$ . One notes then that

$$\{\rho(\sigma)z, \rho(\tau)z\} = n! \delta_{\sigma, \tau},$$

and if the  $v_i$  are chosen to be an orthonormal basis of  $Z_\lambda$  with respect to an  $S_n$ -invariant Hilbert space structure in  $Z_\lambda$ , then

$$\{v_i \otimes u_k, v_j \otimes u_l\} = f_\lambda \delta_{ij} \delta_{kl}.$$

But now the inner product of the right side of (14) with the left side of (15) is the same as the inner product of the right side of (15) with  $z$ . Thus

$$(16) \quad f_\lambda \text{tr } P_\lambda \pi_\lambda(A) = f_\lambda \sum_{\tau \in S_n} \alpha(\epsilon, \tau) \chi^\lambda(\tau).$$

Division by  $f_\lambda$  yields (5). Q.E.D.

#### 4. PROOF OF THEOREM 4 WHEN $A$ IS POSITIVE DEFINITE

Let  $\langle u, v \rangle$  be a Hilbert space structure on  $V$  which is invariant under  $\pi(SU(n))$ . Let  $A \in Sl(n, \mathbb{C})$  be positive definite, and let  $\{w_i\}$  be an orthonormal basis of  $V^H$ . To prove Theorem 4 in this case it suffices to show that for all  $i$

$$(17) \quad \langle \pi(A) w_i, w_i \rangle \geq 1.$$

But since  $A$  is positive definite, as one knows, we may write  $A = U^* D^* D U$ , where  $U$  is upper triangular and unipotent,  $D \in H$ , and the superscript  $*$  denotes Hermitian adjoint. But clearly  $\pi(U)w_i - w_i \in \text{Ker } P$ . Since  $\pi(D)|V^H$  is the identity, one also has  $\pi(DU)w_i - w_i \in \text{Ker } P$ . But then

$$\begin{aligned} \langle \pi(A) w_i, w_i \rangle &= \langle \pi(DU)w_i, \pi(DU)w_i \rangle \\ &\geq \langle w_i, w_i \rangle = 1. \quad \text{Q.E.D.} \end{aligned}$$

#### 5. PROOF OF THEOREM 4 WHEN $A$ IS TOTALLY POSITIVE

Assume  $A$  is totally positive. Let  $\alpha_i, i = 1, \dots, n - 1$ , be simple positive roots relative to  $(H, Sl(n, \mathbb{C}))$  such that one can choose the matrix unit  $e_{ii+1}$  to be a corresponding root vector  $e_{\alpha_i}$ . Let  $\{x_j\}$  be the corresponding canonical basis of  $V$  (see [Lu]). In particular the  $\{x_j\}$  are a weight basis. If  $g \in$

$Sl(n, \mathbb{C})$ , let  $\mu(g)$  be the matrix of  $\pi(g)$  with respect to the basis  $\{x_j\}$ . To prove the theorem it suffices to show that

$$(18) \quad \mu(A)_{jj} \geq 1 \quad \text{whenever } x_j \in V^H.$$

But now by Theorem 2 in [W] or as more clearly stated in [Lo], we can write  $A = U_- D U_+$ , where  $U_+$  is a product of elements of the form  $\exp t e_{\alpha_i}$  with  $t > 0$ ,  $U_-$  is a product of elements of the form  $\exp t e_{-\alpha_i}$  with  $t > 0$ , and  $D \in H$  has positive diagonal entries. Since elements of the form  $U_+$  are clearly stable under conjugation by elements of the form  $D$ , we can assume that  $A = U_- U_+ D$ . But then if  $x_j \in V^H$  one has

$$(19) \quad \pi(A)x_j = \pi(U_-)\pi(U_+)x_j.$$

But now by Theorem 22.1.7 in [Lu] all the entries of  $\mu(\exp t e_{\alpha_i})$  and  $\mu(\exp t e_{-\alpha_i})$  are non-negative for any  $\alpha_i$  and  $t > 0$ . Thus all the entries of  $\mu(U_-)$  and  $\mu(U_+)$  are non-negative. On the other hand since  $U_-$  and  $U_+$  are respectively lower and upper triangular unipotent, one has that

$$\mu(U_-)_{jj} = \mu(U_+)_{jj} = 1$$

for any  $j$ . But now the product of two matrices each with non-negative entries and each having diagonal entries  $\geq 1$  still has these same two properties. But then (18) follows from (19). Q.E.D.

#### REFERENCES

- [B] A. I. Barvinok, *Partition functions in optimization and computational problems*, St. Petersburg Math. J. **4** (1993), 1–49.
- [GJ] I. Goulden and D. Jackson, *Immanants of combinatorial matrices*, J. Algebra **148** (1992), 305–324.
- [H] M. Haiman, *Hecke algebra characters and immanant conjectures*, J. Amer. Math. Soc. **6** (1993), 569–595.
- [Lo] C. Loewner, *On totally positive matrices*, Math. Z. **63** (1955), 338–340.
- [Li] D. Littlewood, *The theory of group characters*, 2nd ed., Oxford Univ. Press, London and New York, 1958.
- [Lu] G. Lusztig, *Introduction to quantum groups*, Birkhauser, Boston, 1993.
- [St] J. Stembridge, *Immanants of totally positive matrices are non-negative*, Bull. London Math. Soc. **23** (1991), 422–428.
- [Sc] I. Schur, *Über endliche Gruppen und Hermitesche Formen*, Math. Z. **1** (1918), 184–207.
- [W] A. Whitney, *A reduction theorem for totally positive matrices*, J. Analyse Math. **2** (1952), 88–92.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

*E-mail address:* kostant@math.mit.edu