JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 4, October 1995

CHERN-SCHWARTZ-MACPHERSON CLASSES AND THE EULER CHARACTERISTIC OF DEGENERACY LOCI AND SPECIAL DIVISORS

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Dedicated to Professor F. Hirzebruch

INTRODUCTION

The aim of this paper is to study the topological Euler-Poincaré characteristic (Euler characteristic, for short) of degeneracy loci associated with various bundle homomorphisms. Recall that for a given holomorphic morphism $\varphi: F \to E$ of vector bundles on a (possibly singular) analytic variety X the *r-th degeneracy locus* is the set

$$D_r(\varphi) = \{x \in X \mid \operatorname{rank} \varphi(x) \leq r\}.$$

This concept overlaps a large family of interesting varieties (for example, the varieties of special divisors studied in Section 3).

Several authors have worked out explicit formulas for the Euler characteristic of $D_r(\varphi)$ in terms of different cohomological and numerical invariants under the assumption that X is nonsingular and φ is appropriately "general".

For instance, if φ is a section of a vector bundle, then the formulas for the Euler characteristic $\chi(D_0(\varphi))$ were given by Hirzebruch [H] and Navarro-Aznar [N].

If $D_r(\varphi)$ is a curve or a surface in a nonsingular X, some explicit formulas for $\chi(D_r(\varphi))$ were given by Harris and Tu [H-T] in terms of the Chern classes of E, F and X, but under the extra assumption $D_{r-1}(\varphi) = \varnothing$ (which implies that $D_r(\varphi)$ is nonsingular). In loc.cit. the authors also posed the problem of finding a general formula for $\chi(D_r(\varphi))$—if such exists!—under the assumption $D_{r-1}(\varphi) = \varnothing$ or, even stronger, without this assumption.

The first problem was solved positively by the second named author [Pr1, Proposition 5.7], by the use of *polynomials universally supported on degeneracy*

Received by the editors May 20, 1993 and, in revised form, March 29, 1994.

1991 *Mathematics Subject Classification*. Primary 14C25, 14H40, 32S20, 57R20; Secondary 14N10, 55R40.

The first author's research was started during the author's stay at the Max-Planck-Institut für Mathematik (supported by the Alexander von Humboldt Stiftung) and finished at the University of Sydney.

The second author's research was started during the author's visit to the U.B.C. in Vancouver (supported by a grant of the N.S.E.R.C. of Canada) and finished during the author's stay at the Max-Planck-Institut für Mathematik as a fellow of the Alexander von Humboldt Stiftung. While preparing this paper, the author was partially supported by KBN grant No. 2 P301002 05.

loci—the technique invented and developed in loc.cit.; this result was a starting point of the present work.

We refer the reader to [P-P1] for more information of both mathematical and historical nature concerning these topics.

In the present paper, in Section 2, we give an explicit formula (Theorem 2.10) for the Euler characteristic $\chi(D_r(\varphi))$ only under the assumption that φ is an *r-general* holomorphic morphism of vector bundles on a possibly singular variety. Maybe the most transparent definition of the r-generality of a morphism φ over a nonsingular pure-dimensional X is given by imposing the conditions: the subset $D_k(\varphi) \setminus D_{k-1}(\varphi)$ is nonsingular of pure dimension $\dim X - (\operatorname{rank} F - k)(\operatorname{rank} E - k)$ for every $k = 0, 1, \ldots, r$. Thus, the "stronger version" of Harris and Tu's problem is here positively solved. In fact, we prove a more general result because we compute (in Theorem 2.1) the image of the whole Chern-Schwartz-MacPherson class of $D_r(\varphi)$ in the homology of X (the formula for $\chi(D_r(\varphi))$ then results by taking the degree of the 0-dimensional component of this image).

The key point of our argument is to pass first to a certain desingularization of $D_r(\varphi)$ and calculate explicitly the image of the homology dual to its Chern class in the homology of X. To this end, by using some algebra (of symmetric polynomials and Gysin push forwards), we show that this image has the form $P \cap c_*(X)$, where $P = P(\{c_i\}, \{c'_j\})$ is a polynomial universally supported on the r-th degeneracy locus and not universally supported on the $(r-1)$-th one, specialized by setting $c_i := c_i(E)$, $c'_j := c_j(F)$. Thus "morally", without changing the result of the computation, we can assume that $D_{r-1}(\varphi) = \varnothing$. But then the desingularization equals $D_r(\varphi)$ and the wanted class is known by [Pr1, Proposition 5.7] quoted above. Precise arguments require some rather detailed information about the algebraic structure of the ideal of all polynomials universally supported on the r-th degeneracy locus which is a part of the theory developed in [Pr1] and [Pr2].

Secondly, stratifying $D_r(\varphi)$ by the subsets where the rank of φ is constant, the desingularization turns out to be a Grassmannian bundle over each stratum. This leads to an equation with the known $H_*(X)$-image of the Chern class of the desingularization on the one hand, and a linear combination of the unknown $H_*(X)$-images of the Chern-Schwartz-MacPherson classes of $D_k(\varphi)$ $(k \leq r)$ on the other hand. By varying r, this leads to a system of linear equations in the unknown $H_*(X)$-images of the Chern-Schwartz-MacPherson classes of $D_r(\varphi)$ (and with known coefficients). Solving this system of equations with the help of some algebra of binomial numbers, we get the looked for formula. Our method enables us to write the desired formula in a clear and compact algebro-geometric form as a sum of polynomials universally supported on the subsequent degeneracy loci $D_r(\varphi) \supset D_{r-1}(\varphi) \supset \ldots \supset D_0(\varphi)$.

We need here a theory of Chern classes for singular varieties. There are several ways of extending Chern classes to this case. The approach which we find "the best suited" for computing the Euler characteristic was given by R.D. MacPherson [McP], and in a different form by M. H. Schwartz [S]. The functorial approach of MacPherson is especially useful for the purposes of this paper. We recall it in Section 1 where we also establish some simple and useful prop-

erties of Chern-Schwartz-MacPherson classes that we need in the proof of our main Theorem 2.1.

As a by-product of our considerations, we also get a formula for the Intersection Homology-Euler characteristic of $D_r(\varphi)$ associated with an r-general morphism φ (Theorem 2.12).

As an example of an application of our formula, we compute the Euler characteristic of the Brill-Noether loci $W_d^r(C) = \{L \in \mathrm{Pic}^d(C) | \ h^0(C, L) > r\}$ parametrizing all complete linear series of degree d and dimension r on a general curve C (Theorem 3.4). This is done by using a presentation of $W_d^r(C)$ as a suitable degeneracy locus in the Jacobian of C due to Kempf [K] and Kleiman-Laksov [K-L] , combined with some results of Griffiths-Harris [G-H] and Gieseker [Gi]. We also give formulas for the Intersection Homology-Euler characteristic of $W_d^r(C)$ and the Euler characteristic of $G_d^r(C) = \{ \ g_d^r$'s on $C \}$—the variety parametrizing all linear series of degree d and dimension r on C.

For the formula of Theorem 2.10 to hold, it is not enough to assume only that the degeneracy locus has the expected codimension, even in the case of a section of a line bundle over a nonsingular projective variety. We investigate this case in detail in [P-P2]. The difference between the Euler characteristic of a "non-general" hypersurface and the expected polynomial in Chern classes is measured with the help of topological invariants of singularities including some generalizations of the Milnor number and the Chern-Schwartz-MacPherson classes of the closures of the strata of a Whitney stratification of the hypersurface.

Some of the results presented here were announced in [P-P1].

We thank Professor P. Deligne for a useful suggestion that the method of a preliminary version of this paper (which appeared as Section 2 of Preprint of the Max-Planck Institut für Mathematik No.90-68) should give not only a formula for $\chi(D_r(\varphi))$ but also the image of the whole Chern-Schwartz-MacPherson class of $D_r(\varphi)$ in the homology of X.

Both authors have greatly benefited from the hospitality of Professor F. Hirzebruch in the Max-Planck Institut für Mathematik during different periods of the work on this paper; we acknowledge this hospitality with gratitude. It is our pleasure to dedicate the present work to Professor F. Hirzebruch.

The second named author thanks J.B. Carrell for organizing a meeting in the U.B.C. in Vancouver (June 1989) where this research started.

Notation and conventions. In this paper we work exclusively over the field of complex numbers. By an analytic variety we mean, in this paper, a locus in a complex manifold given locally as the zeros of a finite collection of holomorphic functions.

For a (compact) analytic variety X, by $\chi(X)$ we denote its Euler characteristic defined as the alternating sum of the ranks of the singular homology groups.

For a topological space X, by $H_*(X) = \bigoplus_i H_i(X)$ we understand the Borel-Moore homology groups (see, e.g., [F, Chapter 19]) and by $H^*(X)$ the singular cohomology ring (both with integer coefficients).

For any analytic variety X, by $[X]$ we denote the fundamental class of X in $H_*(X)$.

If $f : X \to Y$ is a morphism of pure-dimensional nonsingular analytic varieties, then by $f^* : H_*(Y) \to H_*(X)$ we denote the morphism:

$$H_i(Y) = H^{2\dim Y - i}(Y) \to H^{2\dim Y - i}(X) = H_{i+2\dim X - 2\dim Y}(X).$$

For a given element $z \in H_*(X)$, X compact, by $\int_X z$ we denote the degree of the 0-dimensional component of z.

By $\dim X$ we mean always the complex dimension of X.

If E is a vector bundle on X and $f: Y \to X$ is a morphism of varieties, then E_Y denotes the pull-back bundle f^*E.

For a given vector bundle E on X, by $c_i(E) \in H^{2i}(X)$, $i = 1, \dots, \operatorname{rank} E$, we denote the i-th Chern class of E. The top Chern class of E is denoted by $c_{top}(E)$. By $s_i(E)$ we denote the i-th Segre class of E, i.e. the i-th complete symmetric polynomial in the Chern roots of E satysfying $s_i(E) = (-1)^i c_i(-E)$. (Note that this convention differs from that used in [F], where $s_i(E) = c_i(-E)$.) We assume also $s_i(E) = c_i(E) = 0$ if $i < 0$.

By $c(E) = 1 + c_1(E) + \dots + c_{top}(E)$ we denote the total Chern class of E.

As is customary, we treat $c_i(E)$, $s_i(E)$ and polynomials in them as operators acting on $H_*(X)$ (via the "$\cap -$" - map).

For a nonsingular variety X, we denote by TX the tangent bundle of X; we write $c_i(X)$ instead of $c_i(TX)$ and assume $c_i(X) = 0$ for $i < 0$.

By a *partition* we mean a sequence of integers $I = (i_1, \dots, i_k)$, where $i_1 \geq i_2 \geq \dots \geq i_k \geq 0$. We write $l(I)$ for $\operatorname{card}\{p \mid i_p \neq 0\}$, $|I|$ for $\sum i_p$, i.e. the number that is partitioned by I, $I^\sim = (j_1, j_2, \dots)$ for the conjugate partition with $j_p = \operatorname{card}\{h \mid i_h \geq p\}$ and $(i)^k$ for $(i, \dots, i)$ (k-times). More generally, if $l(I) \leq k$ and $l(J^\sim) \leq i$, then $(i)^k + I, J$ denotes the partition $(i + i_1, \dots, i + i_k, j_1, j_2, \dots)$. For two partitions I, J, we write $I \supset J$ if $i_k \geq j_k$ for each k.

1. Some properties of Chern-Schwartz-MacPherson classes

In order to prove our main formulas we need some properties of Chern-Schwartz-MacPherson classes. We are, however, unable to find convenient references and shall therefore state and prove these simple but useful results below.

Let X be a possibly singular analytic variety. Recall that X admits always a Whitney stratification (see, e.g., [G-M]).

The Chern-Schwartz-MacPherson class $c_*(X) \in H_*(X)$, in the version used in the present paper, was introduced (for an algebraic X) by R.D. MacPherson in [McP]. In fact, the approach of MacPherson defines a class $c_*(X)$ also in the Chow group of cycles on X modulo rational equivalence (see [F, Ex.19.1.7]). It is known that MacPherson's $c_*(X) \in H_*(X)$ is equal, via the Alexander isomorphism, to the M.H. Schwartz class (see [S], [B-S]) defined originally in a different way.

Let us first recall briefly MacPherson's definition. Assume that an irreducible variety X of dimension n is imbedded in a manifold M. Then the tangent bundle to the smooth part X_{reg} of X defines a section over X_{reg} of the Grassmannian bundle $G_n(TM)$. By *the Nash blowing-up* $\nu: \widetilde{X} \to X$ of X we mean

the closure $\widetilde{X}$ of the image of this section together with a map ν induced by the restriction of the projection of $G_n(TM)$ on M. We denote by $\widetilde{T}$ (or $\widetilde{T}_X$) the restriction to $\widetilde{X}$ of the tautological bundle over $G_n(TM)$. Note that $\widetilde{T}|_{\nu^{-1}(X_{reg})}$ is isomorphic to $\nu^* T(X_{reg})$. All the above data are analytically independent of the imbedding chosen since, near each point, X has a unique minimal local analytic imbedding.

The Chern-Mather class of X is defined in $H_*(X)$ by

$$c_M(X) = \nu_*(c(\widetilde{T}) \cap [\widetilde{X}]),$$

where $[\widetilde{X}]$ is the fundamental class of $\widetilde{X}$. We may define c_M for any analytic cycle $\sum n_i V_i$ of X by

$$c_M(\sum n_i V_i) = \sum n_i (incl_i)_* c_M(V_i),$$

where $incl_i$ is the inclusion of V_i in X.

In [McP], MacPherson defined *the local Euler obstruction* $Eu_X(x)$ of X at $x \in X$. The function Eu_X is constructible with integer values. Though the original definition of MacPherson is transcendental, there exists an algebraic approach to Eu_X due to González-Sprinberg and Verdier. The interested reader is referred to [Go], [L-T] or [F, Ex. 4.2.9]. We now record some well-known properties of the local Euler obstruction needed in the sequel.

Lemma 1.1. (1) *$Eu_X(x)$ is constant on the strata of (any) Whitney stratification of X.*

(2) *$Eu_X(x) = 1$ if $x \in X_{reg}$.*

(3) *Assume that X is locally imbedded in $\mathbb{C}^N$ and a nonsingular subvariety $W \subset \mathbb{C}^N$ intersects a Whitney stratification of X transversely. Then, $Eu_{W \cap X}(x) = Eu_X(x)$ for $x \in W \cap X$.*

(4) *$Eu_{X \times Y}(x, y) = Eu_X(x) \cdot Eu_Y(y)$ for $x \in X$ and $y \in Y$.*

In [McP], MacPherson defined an isomorphism T between the free abelian group of analytic cycles on an analytic variety X and the space of constructible functions with integer values on X by: $T(\sum n_i V_i) = \sum n_i Eu_{V_i}(-)$. Here, given an irreducible subvariety V of X, we understand by $Eu_V(-)$ the constructible function on X which is equal to the above mentioned Eu_V on V, and zero otherwise. Let us call $T^{-1}(\mathbf{1}_X)$ *the Schwartz-MacPherson cycle of* X. Equivalently, this is the cycle $\sum n_i V_i$ characterized by the property $\sum n_i Eu_{V_i}(x) = 1$ for every $x \in X$. *The Chern-Schwartz-MacPherson class of* X is defined in $H_*(X)$ by

$$c_*(X) = c_M(T^{-1}(\mathbf{1}_X)),$$

and satisfies good functorial properties (see [McP] or [F, Ex.19.1.7]). Of course, for a nonsingular X, we have $c_*(X) = c(TX) \cap [X]$. Moreover, the following generalization of Hopf's theorem or the Gauss-Bonnet formula for compact manifolds (that is exposed e.g. in [H, pp. 70-71]) now holds for the possibly singular compact analytic variety X:

$$\chi(X) = \int_X c_*(X). \tag{1.1}$$

Let us fix a Whitney stratification $\mathscr{X}$ of X. Let E be a holomorphic vector bundle on X and let Z be the variety of zeros of a holomorphic section s of E. Assume that s intersects, on each stratum of $\mathscr{X}$, the zero section of E transversely. Let $\iota: Z \to X$ be the inclusion.

We now record the following easy consequences of the properties of Whitney stratifications, Lemma 1.1 and the definition of the Schwartz-MacPherson cycle.

Lemma 1.2. *Let E, X, $\mathscr{X}$ and Z be as above. Then:*

(1) *$\mathscr{X}$ induces a Whitney stratification of Z whose strata are of the form $S \cap Z$ for $S \in \mathscr{X}$.*

(2) *The Nash blowing-up of Z equals $\nu_Z : \tilde{Z} = \nu^{-1}(Z) \to Z$, where ν_Z is the restriction of ν to $\tilde{Z}$:*

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\iota}} & \tilde{X} \\ {\scriptstyle \nu_Z}\downarrow & & {\scriptstyle \nu}\downarrow \\ Z & \xrightarrow{\iota} & X \end{array}$$

Moreover, on $\tilde{Z}$ we have an exact sequence of vector bundles

$$0 \to \tilde{T}_Z \to \tilde{T}_X|_{\tilde{Z}} \to \nu_Z^*(E|_Z) \to 0. \tag{1.2}$$

(3) *If $\sum n_i V_i$ is the Schwartz-MacPherson cycle for X, then $\sum n_i V_i \cap Z$ is the one for Z.*

It follows from (1.2) that $c(\tilde{T}_Z) = \tilde{\iota}^*(\nu^* c(E)^{-1} \cdot c(\tilde{T}_X))$, and consequently we have

$$\begin{aligned} \iota_* c_M(Z) &= \iota_*(\nu_Z)_*(c(\tilde{T}_Z) \cap [\tilde{Z}]) = \iota_*(\nu_Z)_*\left(\tilde{\iota}^*(\nu^* c(E)^{-1} \cdot c(\tilde{T}_X)) \cap [\tilde{Z}]\right) \\ &= \nu_* \tilde{\iota}_*\left(\tilde{\iota}^*(\nu^* c(E)^{-1} \cdot c(\tilde{T}_X)) \cap [\tilde{Z}]\right) = \nu_*(\nu^* c(E)^{-1} \cdot c(\tilde{T}_X) \cap \tilde{\iota}_*[\tilde{Z}]) \\ &= \nu_*(\nu^* c(E)^{-1} \cdot \nu^* c_{top}(E) \cdot c(\tilde{T}_X) \cap [\tilde{X}]) = c(E)^{-1} \cdot c_{top}(E) \cap c_M(X). \end{aligned}$$

Assume that Y is a subvariety of X given by a union of strata of $\mathscr{X}$. Then, since Z intersects $\mathscr{X}$ transversely, by the same argument as above for the inclusion $\iota' : Z \cap Y \to Y$, we have

$$\iota'_* c_M(Z \cap Y) = c(E)^{-1} \cdot c_{top}(E) \cap c_M(Y).$$

It is known (see [L-T]) that the Schwartz-MacPherson cycle on X is a $\mathbb{Z}$-combination of the closures of strata (which are the unions of strata) of a Whitney stratification of X. Therefore, using this information, the above definition of the Chern-Schwartz-MacPherson class and Lemma 1.2(3), we now infer the following formula.

Proposition 1.3. *Let X, E and $\iota: Z \to X$ be as above. Then*

$$\iota_*(c_*(Z)) = c(E)^{-1} \cdot c_{top}(E) \cap c_*(X).$$

In particular, for a compact analytic variety X,

$$\chi(Z) = \int_X c(E)^{-1} \cdot c_{top}(E) \cap c_*(X).$$

We pass now to another formula. For a given vector bundle E on X, let $\pi: G_r(E) \to X$ denote the Grassmannian bundle parametrizing all rank r subbundles of E. The variety $G = G_r(E)$ is equipped with the tautological sequence

$$0 \to R \to E_G \to Q \to 0\,,$$

where rank $R = r$.

Observe that since $\pi: G \to X$ is a locally trivial fibration with a nonsingular fiber, we have by Lemma 1.1:

Lemma 1.4. (1) *$\mathscr{X}$ induces a Whitney stratification of G whose strata are of the form $\pi^{-1}(S)$ for $S \in \mathscr{X}$.*

(2) *The Nash blowing-up $\nu_G: \widetilde{G} \to G$ of G is equal to the fibre product of ν and π:*

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\nu_G"'] & \widetilde{X} \arrow[d, "\nu"'] \\ G \arrow[r, "\pi"] & X \end{tikzcd} \tag{1.3}$$

(3) *For any $z \in G$, $Eu_G(z) = Eu_X(\pi(z))$ and if $\sum n_i V_i$ is the Schwartz-MacPherson cycle for X, then $\sum n_i \pi^{-1}(V_i)$ is the one for G.*

It follows from Lemma 1.4(2) that $[\widetilde{T}_G] = [\widetilde{\pi}^* \widetilde{T}_X] + [\nu_G^*(R^\vee \otimes Q)]$ and consequently we have

$$\begin{aligned} c_M(G) &= (\nu_G)_*\big(c(\widetilde{T}_G) \cap [\widetilde{G}]\big) = (\nu_G)_*\Big(\nu_G^*\big(c(R^\vee \otimes Q) + \widetilde{\pi}^* c(\widetilde{T}_X)\big) \cap [\widetilde{G}]\Big) \\ &= c(R^\vee \otimes Q) \cap (\nu_G)_*\big(\widetilde{\pi}^* c(\widetilde{T}_X) \cap [\widetilde{G}]\big) \\ &= c(R^\vee \otimes Q) \cap (\nu_G)_*\big(\widetilde{\pi}^* c(\widetilde{T}_X) \cap \widetilde{\pi}^*[\widetilde{X}]\big) \\ &= c(R^\vee \otimes Q) \cap (\nu_G)_*\, \widetilde{\pi}^*\big(c(\widetilde{T}_X) \cap [\widetilde{X}]\big) \\ &= c(R^\vee \otimes Q) \cap \pi^* \nu_*\big(c(\widetilde{T}_X) \cap [\widetilde{X}]\big) = c(R^\vee \otimes Q) \cap \pi^* c_M(X) \end{aligned}$$

by the equality $(\nu_G)_* \circ \widetilde{\pi}^* = \pi^* \circ \nu_*$ which is a consequence of the fibre square (1.3).

Let Y be a subvariety of X given by a union of strata of $\mathscr{X}$. By the same argument as above we get

$$c_M(\pi^{-1} Y) = c(R^\vee \otimes Q) \cap \pi^* c_M(Y)\,,$$

where R and Q denote now the restrictions of the tautological bundles on G to $\pi^{-1} Y$, for brevity.

Hence, arguing as in the proof of Proposition 1.3 and using Lemma 1.4(3), we obtain the following formula.

Proposition 1.5. *Let X, E, $\pi: G \to X$, R and Q be as above. Then*

$$c_*(G) = c(R^\vee \otimes Q) \cap \pi^* c_*(X)\,.$$

For a given analytic variety X, denote by $F(X)$ the group of constructible functions on X (with integer values). $F(X)$ is a free abelian group generated

by characteristic functions associated with irreducible subvarieties of X; given such a subvariety $V \subset X$, we define its characteristic function $\mathbf{1}_V$ by $\mathbf{1}_V(x) = 1$ if $x \in V$ and zero otherwise. With every proper morphism of analytic varieties $f: X \to Y$, we associate a homomorphism of groups $f_*^F: F(X) \to F(Y)$ defined on the generators of $F(X)$ by

$$f_*^F(\mathbf{1}_V)(y) = \chi(f^{-1}(y) \cap V),$$

where χ denotes the Euler characteristic. This makes F a covariant functor.

Let $f: X \to Y$ be a proper morphism of analytic varieties. Let $\mathscr{S} = \{S_\alpha\}$ be a stratification of Y such that the function $f_*^F(\mathbf{1}_X)$ is constant along each stratum S_α. In other words, for every α there exists an integer χ_α such that $\chi(\eta^{-1}(x)) = \chi_\alpha$ for every $x \in S_\alpha$. (Note that all χ_α are finite because f is proper.) Moreover, assume that there is a unique top-dimensional stratum of $\mathscr{S}$, denoted S_0.

Proposition 1.6. *In the above situation, there exists a unique family of integers* $\{d_\alpha\}$ *such that*

$$f_*^F(\mathbf{1}_X) = \sum_\alpha d_\alpha \mathbf{1}_{S_\alpha}.$$

Proof. We set $d_0 = \chi_0$. Assume that d_β has been defined for every β such that $\operatorname{codim}_Y S_\beta < c$. We then define, for every α such that $\operatorname{codim}_Y S_\alpha = c$,

$$d_\alpha := \chi_\alpha - \sum d_\beta,$$

where the sum is over all β such that $S_\alpha \subset \overline{S}_\beta$. It is easy to check that the so-defined family of integers $\{d_\alpha\}$ satisfies the assertion. □

2. Degeneracy loci

Let X be an analytic variety. Let us fix a Whitney stratification $\mathscr{X}$ of X. Let $\varphi: F \to E$ be a holomorphic morphism of vector bundles on X of respective ranks m and n. In order to state our main result we need an appropriate notion of the generality of the morphism. For a nonnegative integer k, let $\underline{D}_k \subset \underline{\mathrm{Hom}}(F, E)$ denote the k-th universal (tautological) degeneracy locus (the fibre of $\underline{D}_k$ over $x \in X$ is equal to $\{f \in \mathrm{Hom}(F(x), E(x)) \mid \operatorname{rank} f \leq k\}$). We say that φ is *r-general* if the section $s_\varphi: X \to \underline{\mathrm{Hom}}(F, E)$ induced by φ intersects, on each stratum of $\mathscr{X}$, the subset $\underline{D}_k \setminus \underline{D}_{k-1}$ transversely for every $k = 0, 1, \dots, r$. For a pure-dimensional, nonsingular X, this condition can be expressed in a more transparent way (see Lemma 2.9(2)): a morphism φ is r-general iff for every $k = 0, 1, \dots, r$, the subset $D_k(\varphi) \setminus D_{k-1}(\varphi)$ is nonsingular of pure dimension $\dim X - (m-k)(n-k)$ (here, $D_{-1}(\varphi) = \varnothing$).

To state the main result of this paper we need some definitions.

Given two vector bundles E, F (of fixed ranks n and m) on X and a partition $I = (i_1, \dots, i_k)$, we define in $H^{2|I|}(X)$ the class

$$s_I(E - F) := \operatorname{Det}\left[s_{i_p - p + q}(E - F)\right]_{1 \leq p, q \leq k},$$

where

$$s_i(E-F) := \sum_{p=0}^{i} (-1)^{i-p} s_p(E) c_{i-p}(F).$$

In particular, if $F = 0$, then $s_I(E) = \operatorname{Det}\left[s_{i_p-p+q}(E)\right]_{1\le p,q\le k}$; if $E = 0$, then $s_I(-F) = (-1)^{|I|} s_{I^\sim}(F)$.

Let $m \wedge n$ denote the minimum of m and n.

We now define the following element in $H_*(X)$. We set

$$\Psi(k) := P_k(E, F) \cap c_*(X),$$

where

$$P_k(E, F) := \sum (-1)^{|I|+|J|} D^k_{I,J} s_{(m-k)^{n-k}+I, J^\sim}(E-F).$$

Here, the sum is over all partitions I, J such that $l(I) \le m \wedge n - k$, $l(J) \le m \wedge n - k$ and

$$D^k_{I,J} := \operatorname{Det}\left[\binom{i_p + j_q + m + n - 2k - p - q}{i_p + n - k - p}\right]_{1\le p,q\le m\wedge n-k}.$$

Observe that this determinant depends only on $m-k$, $n-k$ and I, J. This will be reflected in the notation $D^{m-k,n-k}_{I,J}$ for this determinant, used in Section 3.

The following formula gives an explicit expression for the image of the Chern-Schwartz-MacPherson class of $D_r(\varphi)$ in the homology of X. Let $\iota: D_r(\varphi) \to X$ denote the inclusion.

Theorem 2.1. *If φ is r-general, then one has in $H_*(X)$*

$$\iota_*(c_*(D_r(\varphi))) = \sum_{k=0}^{r} (-1)^k \binom{m \wedge n - r + k - 1}{k} \Psi(r-k).$$

We refer the reader to the end of this section for some examples illustrating the theorem.

The proof of Theorem 2.1 requires several preliminary definitions and results. Following [Pr1] we say that a polynomial $P(c_1, \dots, c_n, c'_1, \dots, c'_m)$, where $\{c_i\}$, $\{c'_j\}$ are independent variables, *is universally supported on the r-th degeneracy locus* if for every variety X and every morphism $\varphi: F \to E$ of vector bundles on X with rank $F = m$, rank $E = n$ and every $\alpha \in H_*(X)$, we have

$$P(c_1(E), \dots, c_n(E), c_1(F), \dots, c_m(F)) \cap \alpha \in \operatorname{Im}(\iota_*).$$

Here, $\iota_* : H_*(D_r(\varphi)) \to H_*(X)$ denotes the induced morphism of the homology groups. The set of polynomials universally supported on the r-th degeneracy locus forms an ideal which was originally described with generators and a $\mathbb{Z}$-basis in [Pr1] for Chow homology replacing in the above definition the Borel-Moore homology. It was then shown in [P-R] that for Borel-Moore homology the analogous ideal admits exactly the same description. Let us recall a coarse description of this ideal.

Given $c_1, \dots, c_n$ as above, we define inductively ($c_i = 0$ for $i > n$),

$$s_i := s_{i-1}c_i - s_{i-2}c_2 + \cdots + (-1)^{i-1}c_i.$$

Then, we define $s_i(c., c'.)$ by the formula

$$s_i(c., c'.) := \sum (-1)^{i-p} s_p c'_{i-p}.$$

Finally, for a given partition $I = (i_1, \dots, i_k)$ we set

$$s_I(c., c'.) := \operatorname{Det}\left[s_{i_p - p + q}(c., c'.)\right]_{1 \le p, q \le k}.$$

In particular, the class $s_I(E - F)$ defined above is $s_I(c., c'.)$ with $c_i := c_i(E)$, $c'_j =: c_j(F)$.

Then the ideal in question is generated by the polynomials $s_I(c., c'.)$, where $I \supset (m-r)^{n-r}$. Observe that, in particular, $P_r(E, F)$ is a polynomial universally supported on the r-th degeneracy locus, specialized with $c_i := c_i(E)$, $c'_j := c_j(F)$.

We now record:

Proposition 2.2. (i) *No nonzero* $\mathbb{Z}[c_1, \dots, c_n]$*-combination of the* $s_I(c., c'.)$ *with* $I \not\supset (m-r)^{n-r}$ *is universally supported on the r-th degeneracy locus.*

(ii) *There exist nonsingular varieties* $X_{v,w}$, *vector bundles* $E_{v,w}$, $F_{v,w}$ *and vector bundle homomorphisms* $\varphi_{v,w}$ *depending on a pair of positive integers such that*:

1. *The Chern classes of* $E_{v,w}$ *and* $F_{v,w}$ *are algebraically independent if* $v, w \to \infty$.
2. *Setting* $D_r(v, w) = D_r(\varphi_{v,w})$, $\iota_{v,w} : D_r(v, w) \to X_{v,w}$ *for the inclusion and letting* $v, w \to \infty$, *the image* $\operatorname{Im}(\iota_{v,w})_*$ *of* $H_*(D_r(v, w))$ *considered in* $H^*(X_{v,w})$ *is equal to the ideal of polynomials universally supported on the r-th degeneracy locus, specialized by setting* $c_i = c_i(E_{v,w})$, $c'_j = c_j(F_{v,w})$.

Proof. (i) This assertion is a consequence of [Pr2, Theorem 5.3(i)] and its proof combined with the Borel-Moore homology version of the main Theorem 3.4 of [Pr1], given in [P-R].

(ii) We use here the construction given before Lemma 2.5 in [P-R]. We now recall briefly this construction (and refer the reader to [P-R] for details). Let V, W be complex vector spaces of dimension $v = \dim V$ and $w = \dim W$. Let $G^m = G^m(W)$ be the Grassmannian parametrizing m-quotients of W and $G_n = G_n(V)$ be the Grassmannian parametrizing n-subspaces of V. Denote by $\mathscr{Q}$ the tautological rank m quotient bundle on G^m and by $\mathscr{R}$ the tautological rank n (sub)bundle on G_n. We define $X_{v,w}$ to be the total space of the Grassmannian bundle

$$G_m(\mathscr{Q}_{G^m \times G_n} \oplus \mathscr{R}_{G^m \times G_n})$$

over $G^m \times G_n$. The variety $X_{v,w}$ is endowed with the tautological rank m (sub)bundle $\mathscr{S} \subset (\mathscr{Q} \oplus \mathscr{R})_{X_{v,w}}$. We put $F_{v,w} = \mathscr{S}$, $E_{v,w} = \mathscr{R}_{X_{v,w}}$, and define $\varphi_{v,w}$ as the composite:

$$F_{v,w} = \mathscr{S} \hookrightarrow (\mathscr{Q} \oplus \mathscr{R})_{X_{v,w}} \xrightarrow{\mathrm{pr}_2} E_{v,w} = \mathscr{R}_{X_{v,w}}.$$

Finally, we set $D_r(v, w) = D_r(\varphi_{v,w})$. It is proved in loc.cit. that properties 1 and 2 hold true. □

Fix now φ and write $D_r = D_r(\varphi)$ for brevity. We will need, in the sequel, the following property of $\Psi(r)$ which stems implicitly from [Pr1]. Let $\pi_E: G_r(E) \to X$ (resp. $\pi_F: G^r(F) \to X$) be the Grassmannian bundle parametrizing r-subbundles of E (resp. r-quotients of F). Moreover, let

$$0 \to R_E^{(r)} \to E_{G_r(E)} \to Q_E^{(n-r)} \to 0,$$

$$0 \to R_F^{(m-r)} \to F_{G^r(F)} \to Q_F^{(r)} \to 0$$

be the tautological sequences on $G_r(E)$ and $G^r(F)$. Consider the following fibre product of Grassmannian bundles:

$$\tau: GG := G^r(F) \times_X G_r(E) \xrightarrow{\pi_F \times 1} G_r(E) \xrightarrow{\pi_E} X.$$

The morphism φ induces the section s_φ of $\underline{\mathrm{Hom}}(F, E)$ and thus the section $\bar{s}_\varphi$ of $H = \underline{\mathrm{Hom}}(F, E)_{GG}/\underline{\mathrm{Hom}}(Q_F, R_E)$. Let Y be the variety of zeros of $\bar{s}_\varphi$. Denote by ρ the restriction of τ to Y. It factorizes through D_r: $\tau \circ l = \iota \circ \rho$, where $l : Y \to GG$ is the inclusion. Let $k: D_r \setminus D_{r-1} \to D_r$ be the inclusion and let K (resp. C) be the kernel (resp. cokernel) bundle of φ restricted to $D_r \setminus D_{r-1}$.

Lemma 2.3. (i) *Assume that X is pure-dimensional nonsingular and Y is nonsingular of pure codimension $mn - r^2$ in GG. We set in $H_*(D_r)$:*

$$a = \rho_*\Big(c(-(R_F^\vee \otimes Q_E)|_Y) \cap (\tau l)^* c_*(X)\Big).$$

Then $\iota_*(a) = \Psi(r)$ and $k^*(a) = c(-K^\vee \otimes C) \cap (\iota k)^* c_*(X)$.

(ii) *The variety Y associated with the morphism $\varphi_{v,w}$ from Proposition* 2.2 *satisfies the assumptions of* (i).

Proof. (i) Note that $(\tau l)^*$ makes sense because X and Y are nonsingular. Also, since $D_r \setminus D_{r-1}$ is isomorphic to $\rho^{-1}(D_r \setminus D_{r-1})$ which is an open subset of a nonsingular Y, $\iota k : D_r \setminus D_{r-1} \to X$ is a morphism of nonsingular varieties, so $(\iota k)^*$ makes sense.

We now prove the first assertion. It follows from the assumptions that $l_*[Y] = c_{top}(H) \cap [GG]$. This equation combined with $\iota_* \rho_* = \tau_* l_*$ and the projection formula yields:

$$\begin{aligned} \iota_*(a) &= (\tau l)_*\Big(c(-(R_F^\vee \otimes Q_E)|_Y) \cap (\tau l)^* c_*(X)\Big) \\ &= \tau_*\big(c(-R_F^\vee \otimes Q_E) \cdot c_{top}(H) \cap \tau^* c_*(X)\big). \end{aligned}$$

The assertion now follows from a calculation analogous to the one in [Pr1, Proposition 5.7]; the only difference being the use of Lemma 3.1 from loc.cit. instead of Lemma 5.1 from loc.cit.

The second assertion is immediate as $R_F|_Y$ restricts (via k) to K and $Q_E|_Y$ restricts to C.

(ii) We must check that Y associated with $\varphi_{v,w}$ is a nonsingular variety of pure codimension $mn - r^2$ in GG. Consider the standard coordinate bundle of $X_{v,w}$:

$$U = \underline{\mathrm{Hom}}(\mathscr{Q}_{G^m \times G_n}, \mathscr{R}_{G^m \times G_n}),$$

in the notation of Proposition 2.2(ii). We can identify GG_U with the total space of the vector bundle:

$$\underline{\mathrm{Hom}}(\mathscr{Q}_{Fl^{m,r} \times Fl_{r,n}}, \mathscr{R}_{Fl^{m,r} \times Fl_{r,n}}),$$

where $Fl^{m,r}$ (resp. $Fl_{r,n}$) is the Flag variety parametrizing the (rank m, rank r)-flags of quotients of W (resp. (rank r, rank n)-flags of subspaces of V). Under this identification $Y_U \subset GG_U$ becomes the subbundle

$$\underline{\mathrm{Hom}}(\mathscr{Q}'_{Fl^{m,r} \times Fl_{r,n}}, \mathscr{R}'_{Fl^{m,r} \times Fl_{r,n}}),$$

where $\mathscr{Q}'$ (resp. $\mathscr{R}'$) is the tautological rank r bundle on $Fl^{m,r}$ (resp. on $Fl_{r,n}$). This implies easily that Y satisfies the assumptions of (i). □

At the end of the list of preliminary results we record the following consequence of the Littlewood-Richardson rule for the multiplication of the $s_I(F)$'s.

Lemma 2.4. *Let I, J be two partitions such that $l(I^\sim) \leq a$, $l(J) \leq b$. Then the nonzero coefficients γ_K occurring in the decomposition*

$$s_I(F) \cdot s_J(F) = \sum \gamma_K s_K(F) \quad (\gamma_K \in \mathbb{Z})$$

are indexed by partitions $K \not\supset (a+1)^{b+1}$.

Proof. We use the terminology and formulation of the quoted rule as in [M, I.9]. Recall that the diagrams of K for which $\gamma_K \neq 0$ are obtained by adding to the diagram of I the boxes coming from the diagram of J according to certain rules. One of these rules implies that the number of new boxes added in a single column cannot be greater than $l(J)$. Our assertion now follows from the observation that the $(a+1)$-th column of the diagram of K, for which $\gamma_K \neq 0$, cannot contain $b+1$ boxes because $i_1 < a+1$ and $l(J) \leq b$. □

Assume now that $m \geq n$ (we can assume this without loss of generality through replacing φ by its dual, if necessary).

Consider the following geometric construction. Fix a morphism φ and consider the variety $Z_r = Z_r(\varphi)$ ($r = 0, 1, \ldots, n$) defined by:

$$\begin{array}{ccccc} Z_r = Z_r(\varphi) = Zeros(F_G \xrightarrow{\varphi_G} E_G \to Q) & \xrightarrow{j} & G = G_r(E) \\ \downarrow \eta & & \downarrow \pi \\ D_r & \xrightarrow{\iota} & X \end{array} \tag{2.1}$$

where $G_r(E)$ is the Grassmannian bundle of r-subbundles of E and Q is the $(n-r)$-bundle appearing in the exact (tautological) sequence $0 \to R \to E_G \to Q \to 0$ on G. In particular, for the construction given in Proposition 2.2(ii), we define $Z_{v,w} = Z(\varphi_{v,w})$.

The key information for the purposes of this section is contained in:

Proposition 2.5. *Assume that $\varphi : F \to E$ is a holomorphic homomorphism of vector bundles over an analytic variety such that the section of $\underline{\mathrm{Hom}}(F_G, Q)$ associated with the morphism $F_G \xrightarrow{\varphi_G} E_G \to Q$ intersects the zero section transversely. Then $\pi_* j_*(c_*(Z_r)) = \Psi(r)$.*

Proof. Our proof is more conceptual than computational, is divided into several steps and relies heavily on several formulas that were used in [Pr1]. We refer the reader to loc.cit. for the precise source-references.

Step 1. We claim that the following identity holds:

$$\pi_* j_*(c_*(Z_r)) = \pi_*\Big(s_{(m)^{n-r}}(Q - F_G)c(R^\vee \otimes Q - F_G^\vee \otimes Q) \cap \pi^* c_*(X)\Big). \tag{2.2}$$

By Proposition 1.3 we get

$$\begin{aligned} j_*(c_*(Z_r)) &= c(F_G^\vee \otimes Q)^{-1} c_{top}(F_G^\vee \otimes Q) \cap c_*(G) \\ &= s_{(m)^{n-r}}(Q - F_G)c(F_G^\vee \otimes Q)^{-1} \cap c_*(G), \end{aligned}$$

where the last equality follows from the well-known decomposition of the resultant into Schur polynomials (loc.cit., Lemma 1.2). Combining this with the formula for $c_*(G)$ from Proposition 1.5, the identity (2.2) follows.

It follows from the formula for Gysin push-forward in a Grassmannian bundle (loc.cit., Proposition 2.2) that equation (2.2) can be rewritten in the form

$$\pi_* j_*(c_*(Z_r)) = \widetilde{P}_r(E, F) \cap c_*(X), \tag{2.3}$$

where $\widetilde{P}_r(E, F)$ is a certain universal polynomial expression (with integer coefficients) in the Chern classes of F and E. Moreover, it is clear from its definition that $\widetilde{P}_r(E, F)$ is a polynomial universally supported on the r-th degeneracy locus, specialized by setting $c_i := c_i(E)$, $c'_j := c_j(F)$.

Step 2. We claim that it is sufficient to work with $\varphi = \varphi_{v,w}$ in the notation of Proposition 2.2. Indeed, $\varphi = \varphi_{v,w}$ satisfies the assumptions of Proposition 2.5. This follows, e.g., from the fact that $Z_r(v, w)$ is of pure dimension $\dim G - m(n-r)$ and nonsingular (see [P-R, Lemma 2.6]).

Moreover, if we write

$$\widetilde{P}_r(E, F) = \sum \alpha_{IJ}\, s_I(E) \cdot s_J(F),$$

then for $v, w \gg 0$ the coefficients $\alpha_{IJ} \in \mathbb{Z}$, computed in the above situation for $E = E_{v,w}$, $F = F_{v,w}$, are the same as the wanted universal ones. This follows from the property that the Chern classes of $E_{v,w}$ and $F_{v,w}$ are algebraically independent if $v, w \to \infty$.

Express $\widetilde{P}_r(E, F)$ as

$$\widetilde{P}_r(E, F) = \sum_I \alpha_I(E) s_I(E - F), \tag{2.4}$$

where the sum is taken over partitions, and the $\alpha_I(E)$ depend only on $c.(E)$ and do not depend on $c.(F)$ (This is possible by the linearity formula [Pr1, Formula 4].)

Step 3. We claim that $I \not\supset (m-r+1)^{n-r+1}$ if $\alpha_I(E) \neq 0$. To prove it, let us look at (2.2) and analyse for which partitions L the following property holds: if

$$\widetilde{P}_r(E, F) = \sum_L \beta_L(E) s_L(F),$$

where $\beta_L(E) \in \mathbb{Z}[c.(E)]$, then $\beta_L(E) \neq 0$.

Note first that every class $s_I(F_G)$ appearing in the decomposition of $s_{(m)^{n-r}}(Q-F_G)$, as a $\mathbb{Z}$-combination of the products of the form $s_K(Q)\cdot s_I(F_G)$, satisfies

$$l(I^{\sim}) \leq rk\, Q = n-r$$

(loc.cit., Formula (2)). Moreover, every $s_J(F_G)$ appearing in the decomposition of $c_i(-F_G^{\vee} \otimes Q) = (-1)^i s_i(F_G^{\vee} \otimes Q)$, as a $\mathbb{Z}$-combination of the products of the form $s_L(Q)\cdot s_J(F_G)$, satisfies $l(J) \leq n-r$ (loc.cit., Lemma 5.6). But, by Lemma 2.4, if

$$l(I^{\sim}) \leq n-r \text{ and } l(J) \leq n-r \leq m-r,$$

then the nonzero γ_K occurring in

$$s_I(F)\cdot s_J(F) = \sum_K \gamma_K s_K(F) \quad (\gamma_K \in \mathbb{Z})$$

are indexed by partitions $K \not\supset (n-r+1)^{m-r+1}$. Consequently, using the property that $s_K(-F) = \pm s_{K^{\sim}}(F)$ and the linearity formula decomposing $s_I(E-F)$ as a $\mathbb{Z}$-combination of the products of the form $s_M(E)\cdot s_N(F)$, we infer that if in (2.4) $\alpha_I(E) \neq 0$, then $I \not\supset (m-r+1)^{n-r+1}$, as claimed.

Step 4. We claim that $\pi_* j_*(c_*(Z_r)) = P_r(E, F) \cap c_*(X)$. It is enough to show that $\widetilde{P}_r(E, F) = P_r(E, F)$. We know that it is sufficient to prove this assertion for the variety $X_{v,w}$ equipped with the morphism $\varphi_{v,w} : F_{v,w} \to E_{v,w}$ (see Proposition 2.2(ii)) by letting $v, w \to \infty$. We write $X = X_{v,w}$, $D_r = D_r(v, w)$, $Z_r = Z_r(v, w)$, $E = E_{v,w}$ and $F = F_{v,w}$ for brevity.

We have the commutative diagram

$$\begin{array}{ccc}
H_*(Z_r) & \xrightarrow{k_2^*} & H_*(Z_r \setminus \pi^{-1}(D_{r-1})) \\
\downarrow{\scriptstyle \eta_*} & & {\scriptstyle \eta'_*}\downarrow{\scriptstyle \simeq} \\
H_*(D_r) & \xrightarrow{k^*} & H_*(D_r \setminus D_{r-1}) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota'_*} \\
H_*(X) & \xrightarrow{k_1^*} & H_*(X \setminus D_{r-1})
\end{array}$$

where ι', k, k_1, k_2 are the inclusions and η' is the restriction of η.

We now record:

Claim: $k_1^*(P_r(E, F) - \widetilde{P}_r(E, F)) = 0$.

To prove the Claim let K (resp. C) be the kernel (resp. cokernel) bundle of φ restricted to $D_r \setminus D_{r-1}$. We have

$$\begin{aligned} k^*\eta_*(c_*(Z_r)) &= \eta'_* k_2^* \left(c((R^\vee \otimes Q)_{Z_r} - (F_G^\vee \otimes Q)_{Z_r}) \cap (\iota\eta)^* c_*(X) \right) \\ &= c(-K^\vee \otimes C) \cap (\iota k)^* c_*(X) . \end{aligned}$$

On the other hand, it follows from Lemma 2.3 that $\Psi(r) \in H_*(X)$ is the image by ι_* of an element $a \in H_*(D_r)$ satisfying the property

$$k^*(a) = c(-K^\vee \otimes C) \cap (\iota k)^* c_*(X).$$

Since $\pi_* j_* = \iota_* \eta_*$, these two equalities give

$$k_1^*(\Psi(r) - \pi_* j_*(c_*(Z_r))) = 0 .$$

Interpreting k_1^* as the ring homomorphism of the corresponding cohomology rings, we then infer

$$k_1^*(\Psi(r) - \pi_* j_*(c_*(Z_r))) = k_1^*(P_r(E, F) - \widetilde{P}_r(E, F)) \cdot k_1^*(c_*(X)) = 0.$$

Since

$$k_1^*(c_*(X)) = 1 + (\text{elements of higher degree in } H^*(X \setminus D_{r-1}))$$

is a non-zero-divisor in $H^*(X \setminus D_{r-1})$, our Claim follows.

By [F, Chapter 19] we have exact sequences:

$$H_{2i}(D_{r-1}) \xrightarrow{\bar{\iota}_*} H_{2i}(X) \xrightarrow{k_1^*} H_{2i}(X \setminus D_{r-1}),$$

where $i \geq 0$ and $\bar{\iota} : D_{r-1} \to X$ is the inclusion. Hence, we infer by the Claim that $P_r(E, F) - \widetilde{P}_r(E, F)$ is contained in $\mathrm{Im}(\bar{\iota}_*)$. We thus conclude, using Proposition 2.2(ii), that $P_r(E, F) - \widetilde{P}_r(E, F)$ is a polynomial universally supported on the $(r-1)$-th degeneracy locus, specialized by setting $c_i := c_i(E)$ and $c'_j := c_j(F)$.

By Step 3 we know that $P_r(E, F) - \widetilde{P}_r(E, F)$ is a $\mathbb{Z}[c.(E)]$-combination of the $s_I(E-F)$, where $I \not\supset (m-r+1)^{n-r+1}$. By virtue of Proposition 2.2(i) with r replaced by $r-1$, this forces the equality $P_r(E, F) = \widetilde{P}_r(E, F)$, as desired.

The proposition has been proved. □

Remark. The above proof is given in the Borel-Moore homology framework which is the original setup for the Chern-Schwartz-MacPherson classes. A similar proof can be given for the Chern-Schwartz-MacPherson classes defined in the Chow groups $A_*(-)$ of cycles modulo rational equivalence (see [F, Ex.19.1.7]). In fact, all formulas from Section 1 that were used in the above proof hold in the Chow groups setup. Then the above proof goes through mutatis mutandis (with the Chow groups replacing the Borel-Moore homology groups and the Chow rings replacing the cohomology rings). The specialization construction from Proposition 2.2(ii) works well; however, it can be replaced by the construction (13) from [Pr1] which also does the job.

In the next lemma, η_*^F will denote the group homomorphism from $F(Z_r)$ to $F(D_r)$ defined in Section 1, where $F(Z_r)$ (resp. $F(D_r)$) stands for the free abelian group of constructible functions on Z_r (resp. D_r).

Lemma 2.6. *We have in* $F(D_r)$

$$\eta_*^F(\mathbf{1}_{Z_r}) = \sum_{k=0}^{r} \binom{n-r+k-1}{k} \mathbf{1}_{D_{r-k}}.$$

Proof. The family $\{D_k \setminus D_{k-1}\}_{0 \le k \le r}$ (where $D_{-1} = \varnothing$) provides a stratification of D_r such that $\eta_*^F(\mathbf{1}_{Z_r})$ is constant on strata. The fiber of η over any point of $D_k \setminus D_{k-1}$ is the Grassmannian $G_{r-k}(\mathbb{C}^{n-k})$. Since its Euler characteristic is $\binom{n-k}{r-k}$, the proof of Proposition 1.6 shows that

$$\eta_*^F(\mathbf{1}_{Z_r}) = \sum_{k=0}^{r} d_k \mathbf{1}_{D_{r-k}},$$

where $d_0 = 1$ and, by induction on k, we have:

$$d_k = \binom{n-r+k}{k} - \sum_{i=0}^{k-1} d_i = \binom{n-r+k}{k} - \sum_{i=0}^{k-1} \binom{n-r+i-1}{i}$$

$$= \binom{n-r+k}{k} - \binom{n-r+k-1}{k-1} = \binom{n-r+k-1}{k}. \quad \square$$

Corollary 2.7. *Denoting by* $\iota_k : D_k \to X$ *the inclusion, we have in* $H_*(X)$,

$$(\pi j)_*(c_*(Z_r)) = \sum_{k=0}^{r} \binom{n-r+k-1}{k} (\iota_{r-k})_*(c_*(D_{r-k})).$$

Lemma 2.8. *For all positive integers* a, k, *the following equality holds*:

$$\binom{a+k}{k} = \sum_{p=1}^{k} (-1)^{p-1} \binom{a+p}{p} \binom{a+k}{k-p}.$$

Proof. The assertion is a consequence of the following two equalities:

$$\binom{a+k}{k} \left[\sum_{p=0}^{k} (-1)^p \binom{k}{p} \right] = 0$$

and

$$\binom{a+k}{k} \binom{k}{p} = \binom{a+p}{p} \binom{a+k}{k-p},$$

where $p = 1, \dots, k$, a verification of the latter being straightforward. $\square$

In order to prove our main Theorem 2.1 we need some properties of D_r and Z_r. We are, however, unable to find convenient references and shall therefore state and prove these simple but useful results in the next lemma. In this lemma and in a forthcoming proof of Theorem 2.1 we write $D_k^0 = D_k \setminus D_{k-1}$, $G_k :=$

$G_k(E)$; π_k denotes the projection $G_k \to X$ and Q_k means the tautological quotient bundle on G_k. Moreover, $s_k : G_k \to H_k := \underline{\mathrm{Hom}}(F_{G_k}, Q_k)$ is the section associated with the morphism $F_{G_k} \xrightarrow{\varphi_{G_k}} E_{G_k} \to Q_k$.

Lemma 2.9. (1) *The following two conditions are equivalent*:

(i) *The section* s_φ *intersects* $\underline{D}_k \setminus \underline{D}_{k-1}$ *transversely.*

(ii) *The section* s_k *intersects the zero section of* H_k *transversely on* $\pi_k^{-1}(D_k^0)$.

(2) *Assume that* X *is pure-dimensional and nonsingular. Then the section* s_φ *intersects* $\underline{D}_k \setminus \underline{D}_{k-1}$ *transversely iff* D_k^0 *is nonsingular of pure dimension* $\dim X - (m-k)(n-k)$. *In particular,* φ *is r-general iff* D_k^0 *is nonsingular of pure dimension* $\dim X - (m-k)(n-k)$ *for all* $k = 0, 1, \ldots, r$ $(D_{-1} = \varnothing)$.

Proof. (1) Let us examine when s_φ intersects $\underline{D}_k \setminus \underline{D}_{k-1}$ transversely at $x \in D_k^0$. The problem being local, we can proceed in an open neighbourhood U of x and assume that: $F = F_1 \oplus F_2$, $E = E_1 \oplus E_2$ are trivial with rank $F_1 =$ rank $E_1 = k$, say; φ is given by an isomorphism between F_1 and E_1 (we can choose bases in F_1 and E_1 such that this isomorphism is given by the identity matrix over U); the maps between F_1 and E_2, F_2 and E_1 are zero; finally, assume that the value of the map φ between F_2 and E_2 is given by a matrix A of order $(n-k)\times(m-k)$. Then s_φ intersects $\underline{D}_k \setminus \underline{D}_{k-1}$ transversely at x iff the section of $\underline{\mathrm{Hom}}(F_2, E_2)$ determined by A intersects its zero section transversely at x. The latter condition is expressed, equivalently, by the property that the entries of A form a part of a regular system of parameters in the local ring of x.

On the other hand, we now calculate the value of the composition morphism:

$$F_{G_k} \xrightarrow{\varphi_{G_k}} E_{G_k} \to Q_k$$

over the open subset of the form $U \times$ (*standard Grassmannian chart*), where U is the above neighbourhood of $x \in D_k^0$. It suffices to examine when the section s_k of H_k intersects its zero section transversely at the point (x, y), where y belongs to the Grassmannian chart such that over the above open subset, the map $E_{G_k} \to Q_k$ is given as follows. The entries of the $((n-k)\times k)$-matrix of $E_1 \to E_2$ are the indeterminates $\{z_\alpha\}$ ("Grassmannian chart coordinates") and $E_2 \to E_2$ is the identity morphism. As the result of performing the above composition we get an $((n-k)\times m)$-matrix whose $((n-k)\times k)$-submatrix corresponding to $F_1 \to E_2$ has entries $\{z_\alpha\}$ and $((n-k)\times(m-k))$-submatrix corresponding to $F_2 \to E_2$ equals A. Then the section s_k of H_k intersects its zero section transversely at (x, y) iff $\{z_\alpha\}$ and the entries of A form a part of a regular system of parameters in the local ring of (x, y). The latter condition is expressed, equivalently, by the property that the entries of A form a part of a regular system of parameters in the local ring of x.

Comparison of the results of these two reasonings yields assertion (1).

(2) Pick the open neighbourhood U of a given point x in D_k^0 and the matrix A as in the proof of (1) above. Then, by a well-known property of commutative algebra, D_k^0 is nonsingular of pure dimension $\dim X - (m-k)(n-k)$ iff, for

every point x of D_0^k, the entries of A form a part of a regular system of parameters in the local ring of x. But the latter condition is also equivalent to the fact that s_φ intersects $\underline{D}_k \setminus \underline{D}_{k-1}$ transversely (see the proof of (1) above). This proves assertion (2). □

Proof of Theorem 2.1. We want to prove that for an r-general morphism φ, where $r = 0, 1, \dots, n$, the following equality holds (recall that we have assumed $m \geq n$):

$$\iota_*(c_*(D_r)) = \sum_{k=0}^{r} (-1)^k \binom{n-r+k-1}{k} \Psi(r-k).$$

At first, for $r = 0$ the formula reads: $\iota_*(c_*(D_0)) = \Psi(0)$. We mimic the proof of [Pr1, Proposition 5.7]. By virtue of Proposition 1.3 it suffices to show that

$$c(F^\vee \otimes E)^{-1} c_{top}(F^\vee \otimes E) = \sum (-1)^{|I|+|J|} D^0_{I,J}\, s_{(m)^n+I,J^\sim}(E-F),$$

with the sum over partitions I, J of length $\leq n$.

The above equality is a direct consequence of the following three formulas: the well-known decomposition of the resultant into Schur polynomials, a factorization formula

$$s_I E\, s_J(F^\vee) s_{(m)^n}(E-F) = s_{(m)^n+I,J^\sim}(E-F)$$

and

$$c(F^\vee \otimes E)^{-1} = \sum_i (-1)^i s_i(F^\vee \otimes E) = \sum (-1)^{|I|+|J|} D^0_{I,J} s_I(E)\, s_J(F^\vee),$$

with the sum over partitions I, J of length $\leq n$.

These formulas were used in similar calculations in [Pr1] (see Lemmas 1.2, 1.1 and 5.6 in loc.cit.) where we refer the reader for the precise source-references.

Suppose now that the formula is correct for every $k \leq r-1$. We have by Corollary 2.7

$$\iota_*(c_*(D_r)) = (\pi_r j)_*(c_*(Z_r)) - \sum_{p=1}^{r} \binom{n-r+p-1}{p} (\iota_{r-p})_*(c_*(D_{r-p})).$$

Since φ is r-general, φ is also k-general for $k < r$. We can thus use the induction assumption with respect to $(\iota_{r-p})_*(c_*(D_{r-p}))$, $p = 1, \dots, r$. Moreover, the assumption of Proposition 2.5 is satisfied. Indeed, it suffices to show that the section s_r intersects the zero section of H_r transversely on $\pi_r^{-1}(D_k^0)$, where $k = 0, 1, \dots, r$. For $k = r$, this follows from Lemma 2.9(1). For $k < r$, let $Fl_{k,r}(E)$ denote the Flag variety parametrizing all (rank k, rank r)-flags of subbundles of E. Let $p : Fl_{k,r}(E) \to G_k$ and $q : Fl_{k,r}(E) \to G_r$ denote the projections. By the assumption and Lemma 2.9(1), s_k intersects the zero section of H_k transversely on $\pi_k^{-1}(D_k^0)$. Hence the composition s of $p^* s_k$ with the surjection $p^* H_k \to q^* H_r$ intersects the zero section of $q^* H_r$

transversely on $p^{-1}\pi_k^{-1}(D_k^0) = q^{-1}\pi_r^{-1}(D_k^0)$. Since $s = q^*s_r$, the section s_r intersects transversely the zero section of H_r on $\pi_r^{-1}(D_k^0)$, as desired.

Finally, using Proposition 2.5 and the induction assumption, we can rewrite the above expression for $\iota_*(c_*(D_r))$ in the form

$$\begin{aligned}
&\Psi(r) - \sum_{p=1}^{r}\binom{n-r+p-1}{p}\left[\sum_{q=0}^{r-p}(-1)^q\binom{n-r+p+q-1}{q}\Psi(r-p-q)\right] \\
&= \sum_{k=0}^{r}(-1)^k\Psi(r-k)\left[\sum_{p=1}^{k}(-1)^{p-1}\binom{n-r+p-1}{p}\binom{n-r+k-1}{k-p}\right] \\
&= \sum_{k=0}^{r}(-1)^k\binom{n-r+k-1}{k}\Psi(r-k),
\end{aligned}$$

where the last equality follows from Lemma 2.8 with $a = n - r - 1$.

Thus the proof of Theorem 2.1 is complete. □

In particular the degree of the 0-dimensional component of the so-obtained expression for $\iota_*(c_*(D_r))$ gives an explicit answer to the problem posed by Harris and Tu [H-T].

Theorem 2.10. *If X is a compact analytic variety and φ is r-general, then*

$$\chi(D_r(\varphi)) = \int_X \sum_{k=0}^{r}(-1)^k\binom{m\wedge n-r+k-1}{k}\Psi(r-k).$$

Recall that

$$\Psi(k) = \sum(-1)^{|I|+|J|}D^k_{I,J}\,s_{(m-k)^{n-k}+I,J^\sim}(E-F)\cap c_*(X),$$

where the sum runs over all partitions I, J such that $l(I) \le m\wedge n-k$, $l(J) \le m\wedge n-k$ and $D^k_{I,J}$ is the binomial determinant defined before Theorem 2.1.

Remark 2.11. Under the assumption $D_{r-1}(\varphi) = \varnothing$, the above formula reads $\chi(D_r(\varphi)) = \int_X \Psi(r)$. This result was established in [Pr1, Proposition 5.7] as a particular case of an algorithm for computing the Chern numbers of nonsingular degeneracy loci.

As a by-product of the proof of Theorem 2.1 we get the following formula for the Intersection Homology-Euler characteristic, shortly $\chi_{IH}(-)$, of $D_r(\varphi)$.

Theorem 2.12. *If X is a nonsingular compact analytic variety and φ is r-general, then*

$$\chi_{IH}(D_r(\varphi)) = \int_X \Psi(r).$$

Proof. We can assume without loss of generality that $m \ge n$. Since, e.g., by Lemma 2.9 we have $\dim(D_k(\varphi)\setminus D_{k-1}(\varphi)) = \dim X - (m-k)(n-k)$ for every $k = 0, 1, \dots, r$, we easily show that Z_r is a *small* desingularization of $D_r(\varphi)$. The theorem now follows from a general result asserting that for every small desingularization $Z \longrightarrow D$ we have $\chi_{IH}(D) = \chi(Z)$ (see, e.g., Goresky

and MacPherson's paper *Problems and bibliography on intersection homology* in [B&al.]). □

Example 2.13. We collect here several examples illustrating Theorems 2.1 and 2.10. If φ is r-general, then

$$\operatorname{codim}_{D_r(\varphi)} D_{r-1}(\varphi) = (m-r)+(n-r)+1\,;$$

in particular this codimension is at least 3. Therefore, for $\dim D_r(\varphi) = 0, 1, 2$, the formula of Theorem 2.1 reduces to a single summand corresponding to $k=0$: $\iota_*(c_*(D_r(\varphi))) = \Psi(r)$.

If $\dim D_r(\varphi) = 0$, then

$$\Psi(r) = s_{(m-r)^{n-r}}(E-F)\cap c_*(X) = s_{(m-r)^{n-r}}(E-F)\cap [X].$$

If $\dim D_r(\varphi) = 1$, then

$$\Psi(r) = \Big(s_{(m-r)^{n-r}}(E-F) - (n-r)s_{(m-r)^{n-r},(1)}(E-F) \\ - (m-r)s_{(m-r)^{n-r}+(1)}(E-F)\Big)\cap c_*(X).$$

If $\dim D_r(\varphi) = 2$, then $\Psi(r)$ equals

$$\Big(s_{(m-r)^{n-r}}(E-F) - (n-r)s_{(m-r)^{n-r},(1)}(E-F) - (m-r)s_{(m-r)^{n-r}+(1)}(E-F) \\ + \binom{n-r+1}{2}s_{(m-r)^{n-r},(1,1)}(E-F) + \binom{m-r+1}{2}s_{(m-r)^{n-r}+(2)}(E-F) \\ + \binom{n-r}{2}s_{(m-r)^{n-r},(2)}(E-F) + \binom{m-r}{2}s_{(m-r)^{n-r}+(1,1)}(E-F) \\ + \big((m-r)(n-r)+1\big)s_{(m-r)^{n-r}+(1),(1)}(E-F)\Big)\cap c_*(X).$$

Of course, the degree of the 0-dimensional components of these expressions evaluates the Euler characteristic of $D_r(\varphi)$ (see [Pr1, Example 5.8]).

Let us look now at the "next simplest" case when, for a general morphism φ, $\dim D_r(\varphi) = 3$ and the formula of Theorem 2.1 has more than one summand. Then $D_{r-1}(\varphi) \neq \varnothing$. It follows from the above that the unique possibility for that is: $m = n = 2$, $r = 1$. Consequently, $\dim X = 4$, and $D_r(\varphi) = D_1(\varphi)$ is a hypersurface in X. Then the formula of Theorem 2.1 reads

$$\iota_*(c_*(D_1(\varphi))) = \Psi(1) - \Psi(0),$$

where the first term

$$\Psi(1) = \Big(\sum_{i+j\le 3}(-1)^{i+j}\binom{i+j}{j}s_{1+i,(1)^j}(E-F)\Big)\cap c_*(X)$$

would give $\iota_*(c_*(D_1(\varphi)))$ if $D_0(\varphi)$ were empty. The "correction term" equals

$$\Psi(0) = s_{2,2}(E-F)\cap [X] = c_{top}(F^\vee\otimes E)\cap [X].$$

In case X is nonsingular and compact, the formula

$$\chi(D_1(\varphi)) = \int_X \left(\Psi(1) - \Psi(0)\right)$$

admits the following interpretation. Let L be the line bundle $\Lambda^2 F^\vee \otimes \Lambda^2 E$. Then $D_1(\varphi)$ is the variety of zeros of the section of L given by the determinant of φ. It is easy to see that

$$\int_X \Psi(1) = \int_X c_1(L) \cdot c(L)^{-1} \cap c_*(X),$$

where $c_1(L) = c_1(E - F)$. The expression on the right-hand side is known to evaluate the Euler characteristic of the (smooth) variety of zeros associated with a general section of L. Observe $D_1(\varphi)$ is a hypersurface in X with isolated singularities $D_0(\varphi)$. Near each singular point of $D_1(\varphi)$, there exist local coordinates (x_1, x_2, x_3, x_4) such that $D_1(\varphi)$ is given (locally) by the equation $x_1x_2 - x_3x_4 = 0$. Hence the Milnor numbers of all the points in $D_0(\varphi)$ are equal to 1. Since $\int_X \Psi(0)$ evaluates the cardinality of $D_0(\varphi)$, we conclude that the above formula for $\chi(D_1(\varphi))$ expresses the following known property: the "true" Euler characteristic of a hypersurface with isolated singularities differs (up to a sign) from the one "expected" for the nonsingular case by the sum of the Milnor numbers of the singularities. We refer the reader to [P-P1,2] for a precise statement of the latter property as well as its generalization.

3. Special divisors

Let C be a nonsingular curve (over $\mathbb{C}$) of genus g. Consider the subvariety $W_d^r(C)$ of $\mathrm{Pic}^d(C)$ parametrizing complete linear series on C of degree d and dimension at least r :

$$W_d^r(C) = \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geqslant r + 1\}.$$

These varieties play a crucial role in the Brill-Noether theory of special divisors in Jacobians (we refer the interested reader to [A-C-G-H] for a fairly complete account of the theory of special divisors including a detailed treatment of the varieties W_d^r). We will assume throughout this section that d and r are integers such that $d \geq 1$, $r \geq 0$ and $g - d + r > 0$. As a matter of fact, it is shown in loc.cit, p. 204, that we can limit ourselves even to the cases $r > 0$, $g > d > 2r$. (Note that "r" and also "m" below will have here a different meaning from that in the previous section. We intend to follow, in this section, the classical notation of [A-C-G-H].)

Recall (see e.g. [A-C-G-H, p.309]) that the varieties $W_d^r(C)$ admit a presentation as degeneracy loci due to Kempf [K] and Kleiman-Laksov [K-L]. Picking an integer $m >> 0$, $W_d^r(C)$ can be presented as the locus where a certain morphism $\varphi_C : F \to E$ of bundles of respective ranks $m - g + d + 1$ and m, over $\mathrm{Pic}^{m+d}(C)$, has rank not greater than $m - g + d - r$. In other words,

$$W_d^r(C) = D_{m-g+d-r}(\varphi_C).$$

We refer the interested reader for details of this construction to [A-C-G-H, pp. 176-179 and pp. 308-309]. Furthermore, $c_i(E) = 0$ for $i > 0$ (loc.cit. p. 309) and

$$c(-F) = e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \frac{1}{3!}\theta^3 + \dots$$

(loc.cit. p.319), where θ is the class of the theta divisor in $\mathrm{Pic}^{m+d}(C)$, i.e. the translate of the theta divisor on the Jacobian of C canonically isomorphic to $\mathrm{Pic}^0(C)$. Recall that θ defines the principal polarization of the Jacobian of C and $\int \theta^g = g!$ (here and in the sequel, we identify the top-dimensional cohomology class with its dual 0-dimensional homology class).

We now invoke the following results due to Griffiths-Harris [G-H], Fulton-Lazarsfeld [F-L] and Gieseker [Gi] (which should be compared with [A-C]); see also [A-C-G-H, Chapter V]). Let

$$\rho := \rho(r) := \rho(g, d, r) := g - (r+1)(g - d + r)$$

be the Brill-Noether number.

Theorem 3.1. *Let C be a general curve of genus g. Let d and r be integers as above. Then, for $\rho > 0$, $W^r_d(C)$ is irreducible and, for any ρ, the dimension of $W^r_d(C)$ equals ρ. Moreover,* $\mathrm{Sing}\, W^r_d(C) = W^{r+1}_d(C)$.

The above theorem implies that for a general curve C, the morphism φ_C is $(m - g + d - r)$-general. We can thus apply Theorem 2.10 to compute the Euler characteristic of $W^r_d(C)$ for a general curve C. We will write in the sequel $D^{m-k,n-k}_{I,J}$ instead of $D^k_{I,J}$ for the binomial determinants defined before Theorem 2.1 (with m, n as in Section 2).

We first record (in the next lemma m is $>> 0$, like in Kempf's and Kleiman-Laksov's construction):

Lemma 3.2. *Let E, F be the above vector bundles over $\mathrm{Pic}^{m+d}(C)$, and let $\Psi(k)$ be the element defined in Section 2. If $\rho(r) \geq 0$, then one has:*

$$\int_X \Psi(m - g + d - r) = (-1)^{\rho(r)} g! \sum D^{r+1, g-d+r}_{I,J} / h(I_{g,d,r} + I, J^{\sim}),$$

where $X = \mathrm{Pic}^{m+d}(C)$, $I_{g,d,r}$ is the partition $(r+1)^{g-d+r}$, the sum is over partitions I, J with length $\leq (r+1) \wedge (g - d + r)$ and such that $|I| + |J| = \rho(r)$. Moreover, for a partition I, $h(I)$ denotes the product of all hook lengths associated with the boxes in the Ferrers' diagram of I (see [M, *Chapter* I]).

Proof. Recall (see Section 2 and its notation) that for an irreducible nonsingular variety X, the cohomology dual to the 0-dimensional component of $\Psi(k)$ equals

$$\sum (-1)^{|I|+|J|} D^{m-k,n-k}_{I,J} s_{(m-k)^{n-k}+I, J^{\sim}}(E - F) c_{d(k)-|I|-|J|}(X),$$

where $d(k) = \dim X - (m-k)(n-k)$ and the sum is over all partitions I, J such that $l(I) \leq m \wedge n - k$, $l(J) \leq m \wedge n - k$.

By substituting our data we get:

$$\int_X \Psi(m-g+d-r)$$
$$= \int_X \sum (-1)^{|I|+|J|} D_{I,J}^{r+1,g-d+r} s_{(r+1)^{g-d+r}+I,J^\sim}(-F) c_{\rho(r)-|I|-|J|}(X).$$

Since $X = \mathrm{Pic}^{m+d}(C)$ is an abelian variety, the above summation runs over I, J of length $\leq (r+1)\wedge(g-d+r)$ and $|I|+|J| = \rho(r)$. Moreover, $c(-F) = e^\theta$ implies $s_i(-F) = \theta^i/i!$ for every $i \geq 0$. This gives

$$\int_X s_{(r+1)^{g-d+r}+I,J^\sim}(-F) = \int_X h(I_{g,r,d}+I, J^\sim)^{-1}\theta^g$$
$$= g!/h(I_{g,d,r}+I, J^\sim)$$

(for the assertions in the latter two sentences see [M, Ex.I.3.5]).

Putting this together,

$$\int_X \Psi(m-g+d-r) = (-1)^{\rho(r)} g! \sum D_{I,J}^{r+1,g-d+r}/h(I_{g,d,r}+I, J^\sim),$$

with the sum over I, J of length $\leq (r+1)\wedge(g-d+r)$, as asserted. □

For $\rho(r) \geq 0$, let $\Phi(g, d, r)$ denote the R.H.S. of the formula of Lemma 3.2. We set $\Phi(g, d, r) = 0$ if $\rho(r) < 0$.

Corollary 3.3. *For $\rho(r) \geq 0$ one has $\Phi(g, d, r) > 0$ (resp. $\Phi(g, d, r) < 0$) iff $\rho(r)$ is even (resp. $\rho(r)$ is odd).*

The fact that we have here strict inequalities is a consequence of [G-V, Corollary 2]. Indeed, this result implies the inequalities $D_{I,J}^{r+1,g-d+r} > 0$ (recall that the D's are determinants of binomial coefficients defined before Theorem 2.1).

Combining Theorems 2.10, 3.1 and Lemma 3.2 we get:

Theorem 3.4. *Assume that a curve C of genus g is general. Let d, r be integers as above and such that $\rho(r) \geq 0$. Then one has*

$$\chi(W_d^r(C)) = \sum_{k \geq r} (-1)^{k-r} \binom{k}{k-r} \Phi(g, d, k).$$

From this formula, one deduces the following corollary. If we fix g, d, r such that $\rho(r) \geq 0$ and the nonnegative numbers $\rho(r), \rho(r+1), \ldots$ change successively the parity, then $\chi(W_d^r(C)) > 0$ (resp. $\chi(W_d^r(C)) < 0$) iff $\rho(r)$ is even (resp. $\rho(r)$ is odd). Observe that the above numbers change successively the parity if $r+1$ and $r+g-d$ are of the same parity. This latter condition holds iff $g \not\equiv d \pmod 2$. Thus we get, in the situation of Theorem 3.4, the following result.

Corollary 3.5. *Assume $g \not\equiv d \pmod 2$. Then $\chi(W_d^r(C)) < 0$ (resp. $\chi(W_d^r(C)) > 0$) iff $g \equiv r \pmod 2$ (resp. $g \not\equiv r \pmod 2$).*

Example 3.6. We give here formulas for $\Phi(g, d, r)$ when $\rho = 0, 1, 2$. We set $a := r+1$, $b := g-d+r$ and $h := h((a)^b)$. We then have using Example 2.13:

- for $\rho = 0$,

$$\Phi(g, d, r) = \operatorname{card}(W_d^r(C)) = g!/h;$$

- for $\rho = 1$,

$$\Phi(g, d, r) = \frac{-2g!ab}{(a+b)h};$$

- for $\rho = 2$,

$$\Phi(g, d, r) = \frac{g!ab(2ab+1)}{(a+b-1)(a+b+1)h}.$$

Since for $\rho = 0, 1, 2$ the first sum in Theorem 3.4 is reduced to a single summand corresponding to $k = r$, the above expressions give $\chi(W_d^r(C))$ for a general curve C. The case $\rho = 0$ is the Castelnuovo formula.

The last formula of Example 2.13, applied to a general curve C of genus 4, gives

$$\chi(W_3^0(C)) = \Phi(4, 3, 0) - \Phi(4, 3, 1) = -20 - 2 = -22.$$

Finally, we record the following two consequences of the above calculations. Combining Lemma 3.2 and Theorem 2.12 we get:

Theorem 3.7. *For a general curve C and $\rho(r) \geq 0$,*

$$\chi_{IH}(W_d^r(C)) = \Phi(g, d, r).$$

Let $G_d^r(C) = \{g_d^r\text{'s on } C\}$ be the variety parametrizing linear series of degree d and dimension r on C (i.e. all series and not only the complete ones). This variety, ultimately related to $W_d^r(C)$, was widely studied in [A-C] and [A-C-G-H, Chapters IV and V]. It is known (loc.cit.) that in the notation of Section 2, $G_d^r(C) = Z_{m-g+d-r}(\varphi_C)$. Therefore, combining Proposition 2.5 and Lemma 3.2 we thus get:

Theorem 3.8. *For a general curve C and $\rho(r) \geq 0$,*

$$\chi(G_d^r(C)) = \Phi(g, d, r).$$

References

[A-C] A. Arbarello and M. Cornalba, *Su una Congettura di Petri*, Comment. Math. Helv. **56** (1981), 1–38.

[A-C-G-H] A. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves*, Vol. I, Springer-Verlag, Berlin, Heidelberg, and New York, 1984.

[B&al.] A. Borel et al., *Intersection cohomology*, Progr. Math., vol. 50, Birkhäuser, Boston, Basel, and Berlin, 1984.

[B-S] J.P. Brasselet and M.H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque **82–83** (1981), 93–147.

[F] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1984.

[F-L] W. Fulton and R. Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, Acta Math. **110** (1981), 271-283.

[G-V] I. Gessel and G. Viennot, *Binomial determinants, paths and hook length formulae*, Adv. in Math. **58** (1985), 300–321.

[Gi] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982), 251–275.

[G-M] M. Goresky and R.D. MacPherson, *Stratified Morse theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1988.

[Go] G. González-Sprinberg, *L'obstruction locale d'Euler et le théorème de MacPherson*, Astérisque **82–83** (1981), 7–32.

[G-H] P.A. Griffiths and J. Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), 233–272.

[H] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag, Berlin, Heidelberg, and New York, 1966.

[H-T] J. Harris and L. Tu, *Chern numbers of kernel and cokernel bundles*, Invent. Math. **75** (1984), 467-475.

[K] G. Kempf, *Schubert methods with an application to algebraic curves*, Publications of Mathematisch Centrum, Amsterdam, 1972.

[K-L] S.L. Kleiman and D. Laksov, *Another proof of the existence of special divisors*, Acta Math. **132** (1974), 163-176.

[L-T] Lê D.T. and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Ann. of Math. (2) **114** (1981), 457–491.

[M] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, London and New York, 1979.

[McP] R.D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423-432.

[N] V. Navarro-Aznar, *On the Chern classes and the Euler characteristic for nonsingular complete intersections*, Proc. Amer. Math. Soc. **78** (1980), 143–148.

[P-P1] A. Parusiński and P. Pragacz, *Characteristic numbers of degeneracy loci*, Enumerative Algebraic Geometry (S. L. Kleiman and A. Thorup, eds.), Contemp. Math., vol. 123, Amer. Math. Soc., Providence, RI, 1991, pp. 189–198.

[P-P2] ——, *A formula for the Euler characteristic of singular hypersurfaces*, J. Alg. Geom. **4** (1995), 337–351.

[Pr1] P. Pragacz, *Enumerative geometry of degeneracy loci*, Ann. Sci. École Norm. Sup. **21** (1988), 413–454.

[Pr2] ——, *Algebro-geometric applications of Schur S- and Q-polynomials*, Séminaire d'Algèbre Dubreil-Malliavin 1989-1990 (M.P. Malliavin, ed.), Lecture Notes in Math., vol. 1478, Springer-Verlag, Berlin, Heidelberg, and New York, 1991, pp. 130–191.

[P-R] P. Pragacz and J. Ratajski, *Polynomials homologically supported on degeneracy loci*, Preprint No. 61, Department of Mathematics, University of Bergen, 1991.

[S] M.H. Schwartz, *Cycles caractéristiques définies par une stratification d'une variété analytique complexe*, C. R. Acad. Sci. Paris **260** (1965), 3261–3264;3535–3537.

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