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MAPPINGS OF REAL ALGEBRAIC HYPERSURFACES

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0. Introduction

A holomorphic function $h(Z)$ defined in a neighborhood of a point $Z_0 \in \mathbb{C}^N$ is *algebraic* if there are polynomials $q_j(Z)$, $j = 0, \dots, k$, not all identically zero, such that $q_k(Z)h(Z)^k + \dots + q_0(Z) \equiv 0$. A holomorphic mapping is *algebraic* if all its components are algebraic. A smooth real hypersurface M in $\mathbb{C}^N$ is *algebraic* if it is contained in the zero set of a nontrivial real-valued polynomial in Z and $\bar{Z}$. We assume throughout this paper that $N > 1$.

In [W1], Webster proved the following celebrated result: If M and M' are algebraic hypersurfaces in $\mathbb{C}^N$ with nondegenerate Levi forms and if H is a biholomorphism defined in an open neighborhood of M and mapping M into M', then H is algebraic. Previously, it had long been known that if M and M' are open subsets of spheres, then the components of H are in fact rational functions (Poincaré [P], Tanaka [T]).

In this paper we go a step further and give a complete characterization of algebraic hypersurfaces in $\mathbb{C}^N$ such that any holomorphic mapping with nonvanishing Jacobian determinant between two such hypersurfaces must be algebraic. Our main result is stated in Theorem 1 below.

By a germ at p_0 of a holomorphic vector field in $\mathbb{C}^N$, we shall mean a complex vector field of the form $\sum_1^N a_j(Z)\frac{\partial}{\partial Z_j}$, where the $a_j(Z)$ are germs at p_0 of holomorphic functions. We say that a hypersurface M is *holomorphically degenerate* at a point $p_0 \in M$ if there exists a nonzero germ of a holomorphic vector field tangent to M in a neighborhood of p_0. This terminology was introduced by Stanton in [S].

Theorem 1. *Let M be a connected algebraic hypersurface contained in $\mathbb{C}^N$, and let $p_0 \in M$. If there is no point $p_1 \in M$ at which M is holomorphically degenerate, then every biholomorphism defined in a neighborhood of p_0 and mapping M to another algebraic hypersurface $M' \subset \mathbb{C}^N$ is algebraic. Conversely, if M is holomorphically degenerate at some point p_1, then M is holomorphically degenerate at every point, and there exists a nonalgebraic biholomorphism in a neighborhood of p_0 in $\mathbb{C}^N$ fixing p_0 and mapping M into itself.*

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In $\mathbb{C}^2$ every hypersurface which has everywhere degenerate Levi form is also holomorphically degenerate at every point (see Remark 6.1). It is important to note here that in higher dimensions there exist algebraic hypersurfaces with everywhere degenerate Levi form, but which are not holomorphically degenerate at any point. We give here an example of such a hypersurface due to Freeman [Fr]. We let $M \subset \mathbb{C}^3$ given by $X_1^3 + X_2^3 + X_3^3 = 0$ with $Z = (Z_1, Z_2, Z_3)$ and $X_k = \Re Z_k$, $X \neq 0$. The Levi form of M has a zero eigenvalue at every point, but there is no holomorphic vector field tangent to M in a neighborhood of any point. (See [S] and Remark 6.3.)

The following is an easy corollary of Theorem 1 and Theorem 3 of [BR3].

Corollary. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ and H a holomorphic mapping defined in a neighborhood of M in $\mathbb{C}^N$ with $H(M) \subset M'$. Assume M is not holomorphically degenerate at any point. Then H is algebraic if either one of the following conditions holds.*

(i) *The Jacobian determinant of H does not vanish identically.*

(ii) *M' does not contain any nontrivial complex variety.*

We shall connect the notion of holomorphic nondegeneracy to the condition of essential finiteness of a hypersurface at a point. As in [BJT] we say that a real analytic hypersurface M given by $\rho(Z, \bar{Z}) = 0$ near p_0 is *essentially finite* at p_0 if there is no $Z \neq p_0$ close to p_0 satisfying $\rho(Z, \zeta) = 0$ for all ζ near $\bar{Z}^0$ with $\rho(p_0, \zeta) = 0$. (See §1 for more details.) The following shows the relationship between essential finiteness and holomorphic degeneracy.

Theorem 2. *Let M be a connected real analytic hypersurface in $\mathbb{C}^N$. Then there exists $p_0 \in M$ such that M is essentially finite at p_0 if and only if M is not holomorphically degenerate at any point in M.*

Note that if M is a real analytic, connected hypersurface and the set of its essentially finite points is not empty, then it is proved in [BR2] that M is essentially finite at every point except those in a real proper subanalytic subset of M. Hence, by Theorem 2, M is generically essentially finite if and only if M is not holomorphically degenerate at any point. (Note that in [S] Stanton proved that if M is essentially finite at p_0, then M is not holomorphically degenerate at p_0; our Theorem 2 includes this result.)

The first statement in Theorem 1 is an immediate consequence of the following.

Proposition 0.1. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ with M essentially finite at p_0. If H is a biholomorphism defined in a neighborhood of p_0 in $\mathbb{C}^N$ mapping M into M', then H is algebraic.*

It should be mentioned here that Webster's result [W1] mentioned above has been extended in some cases to nondegenerate hypersurfaces of different dimensions (see e.g. Webster [W2], Forstnerič [Fo], Huang [H] and their references). See also Bedford-Bell [BB] for other results related to this work.

The paper is organized as follows. In §1 we give some notation and prove some preliminary results on algebraic holomorphic functions. Sections 2 and

3 are devoted to the proof of Proposition 0.1. Some of the techniques in §2 are adapted from those used in [BJT] and [BR1] in studying extendibility of CR mappings. In §4 we give most of the ingredients necessary for the proof of Theorem 2. In §5 we study the flow of germs of holomorphic vector fields tangent to M and show the existence of a nonalgebraic map at holomorphically degenerate points. In the last section we give first the proof of Theorem 2 and then that of Theorem 1 and its corollary. We conclude with some remarks and examples.

Results of this paper will be used to prove holomorphic extendibility of CR mappings between algebraic hypersurfaces in a forthcoming joint paper of the authors and X. Huang. Some of the theorems of the present paper can be extended to holomorphic mappings of real manifolds in $\mathbb{C}^N$ of codimension higher than one. This work is in preparation and will appear elsewhere.

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1. Preliminaries. Algebraic holomorphic functions

Let $M \subset \mathbb{C}^N$ be a real analytic hypersurface, i.e. for every $p_0 \in M$ there exists a real-valued, real analytic function ρ defined near p_0 such that M is given by $\rho(Z, \bar{Z}) = 0$ near p_0 with $d\rho \neq 0$. It will be convenient to write $N = n + 1$. Recall that by the use of the implicit function theorem (see [CM], [BJT]) we can find holomorphic coordinates (z, w), $z \in \mathbb{C}^n$, $w \in \mathbb{C}$, vanishing at p_0 such that near p_0, M is given by

$$w = Q(z, \bar{z}, \bar{w}), \tag{1.1}$$

where $Q(z, \zeta, \tau)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^{2n+1}$ and satisfies

$$Q(z, 0, \tau) \equiv Q(0, \zeta, \tau) \equiv \tau. \tag{1.2}$$

Note that (1.1) is equivalent to

$$\bar{w} = \bar{Q}(\bar{z}, z, w). \tag{1.1'}$$

It follows from the reality of ρ and (1.1) that the following identity holds for all $(z, \zeta, w) \in \mathbb{C}^{2n+1}$ near the origin:

$$Q(z, \zeta, \bar{Q}(\zeta, z, w)) \equiv w. \tag{1.3}$$

Coordinates (z, w) satisfying the above properties are called *normal coordinates* for M at p_0.

We associate to M the complex hypersurface $\mathscr{M}$ in $\mathbb{C}^{2N}$ locally defined near $(p_0, \bar{p}_0)$ by

$$\mathscr{M} = \{(Z, \zeta) : \rho(Z, \zeta) = 0\}, \tag{1.4}$$

where $\rho(Z, \bar{Z})$ is the defining function for M near p_0 as above. We define the germ of an analytic subset $\mathscr{V}_{p_0} \subset \mathbb{C}^N$ through p_0 by

$$\mathscr{V}_{p_0} = \{Z : \rho(Z, \zeta) = 0 \text{ for all } \zeta \text{ near } \bar{p}_0 \text{ with } \rho(p_0, \zeta) = 0\}. \tag{1.5}$$

Note in fact that $\mathscr{V}_{p_0} \subset M$. Then M is called *essentially finite* at p_0 if $\mathscr{V}_{p_0} = \{p_0\}$. In particular, if (z, w) are normal coordinates, then

$$\mathscr{V}_0 = \{(z, 0) : \bar{Q}(\zeta, z, 0) = 0 \text{ for all } \zeta \in \mathbb{C}^n\}. \tag{1.6}$$

The manifold $\mathscr{M}$ of (1.4) has been extensively used since the work of Segre, and analytic subsets similar to (1.5) have previously appeared in the work of Webster [W1], Diederich-Webster [DW], Diederich-Fornaess [DF] and others.

As in [BR1], we observe that by the Rückert Nullstellensatz (see e.g. [GR]), the condition that M is essentially finite at 0 can be formulated more algebraically. We write

$$\bar{Q}(\zeta, z, 0) = \sum_{\alpha} \bar{Q}_\alpha(z)\zeta^\alpha. \tag{1.7}$$

Let $\mathscr{J}$ be the ideal generated in $\mathbb{C}\{z\}$, the ring of convergent power series in n variables, by all the $\bar{Q}_\alpha(z)$. Then M is essentially finite at 0 if and only if $\mathscr{J}$ is of finite codimension in $\mathbb{C}\{z\}$, i.e. $\mathbb{C}\{z\}/\mathscr{J}$ is a finite-dimensional vector space over $\mathbb{C}$.

We denote by $\mathscr{A}_N$ the subring of $\mathbb{C}\{Z\}$ consisting of all convergent series which are algebraic functions. We shall use the following elementary facts about germs of algebraic holomorphic functions.

Lemma 1.8. (i) (*Weierstrass Preparation Theorem for algebraic holomorphic functions.*) *Let $f, g \in \mathscr{A}_N$ and assume $\partial^J_{Z_1} f(0) \neq 0$ with J minimal. Then*

$$g(Z) = q(Z)f(Z) + \sum_{j=0}^{J-1} a_j(Z')Z_1^j \tag{1.9}$$

with $q(Z) \in \mathscr{A}_N$, $a_j(Z') \in \mathscr{A}_{N-1}$, $a_j(0) = 0$, $Z' = (Z_2, \dots, Z_N)$. In particular, $f \in \mathscr{A}_N$ can be written uniquely in the form

$$f(Z) = U(Z)(Z_1^J + \sum_{j=0}^{J-1} b_j(Z')Z_1^j), \tag{1.10}$$

with $U \in \mathscr{A}_N$, $U(0) \neq 0$, and $b_j \in \mathscr{A}_{N-1}$ with $b_j(0) = 0$.

(ii) (*Newton's theorem for algebraic holomorphic functions.*) *Let*

$$P(Z, X) = X^J + \sum_{j=0}^{J-1} c_j(Z)X^j$$

with $c_j \in \mathscr{A}_N$, $c_j(0) = 0$, and $K(u, Z) \in \mathscr{A}_{J+N}$. Assume that $u \mapsto K(u, Z)$ is a symmetric function in $(u_1, \dots, u_J)$. If $\rho_1(Z), \dots, \rho_J(Z)$ are the roots of $P(Z, X) = 0$, then $K(\rho_1(Z), \dots, \rho_J(Z), Z)$ is in $\mathscr{A}_N$.

(iii) (*Transitivity property for algebraic holomorphic functions.*) *Let $R(Z, X) = \sum_{j=0}^J d_j(Z)X^j$, with $d_j(Z) \in \mathscr{A}_N$, $d_J \not\equiv 0$. If $f \in \mathbb{C}\{Z\}$ satisfies the equation $R(Z, f(Z)) \equiv 0$, then $f \in \mathscr{A}_N$.*

(iv) (*Composition of algebraic functions.*) *If* $f \in \mathscr{A}_K$, $g_j \in \mathscr{A}_N$, $j = 1, \dots, K$, *and* $g_j(0) = 0$, *then* $f(g_1(Z), \dots, g_K(Z)) \in \mathscr{A}_N$.

The proof of Lemma 1.8 uses standard arguments. See e.g. [BM]. In particular, it follows from Lemma 1.8(i) that $\mathscr{A}_N$ is a Noetherian ring.

We shall also need a version of the Rückert Nullstellensatz for $\mathscr{A}_N$. The idea of the proof of the following lemma was suggested to us by M. Artin.

Lemma 1.11. *Let* $f_1, \dots, f_K \in \mathscr{A}_N$, *with* $f_j(0) = 0$, *for* $j = 1, \dots, K$. *Suppose that the germ of the common zeros of the* f_j *is* $\{0\}$. *If* $\mathscr{I}$ *is the ideal in* $\mathscr{A}_N$ *generated by the* f_j, *then* $\mathscr{A}_N/\mathscr{I}$ *is a finite-dimensional vector space over* $\mathbb{C}$. *Equivalently, there exists a positive integer* p *and* $b_{jk} \in \mathscr{A}_N$ *so that*

$$Z_k^p = \sum_{j=1}^{K} b_{jk}(Z) f_j(Z), \quad k = 1, \dots, N. \tag{1.12}$$

Proof. Under the assumptions of the lemma, by the Rückert Nullstellensatz for analytic functions (see e.g. [GR]), one can find holomorphic functions $b_{jk}(Z)$ so that (1.12) is satisfied. We must show that we can choose these functions (which are not uniquely determined) to be holomorphic algebraic. Since f_j is algebraic, f_j satisfies a polynomial equation (with polynomial coefficients) $P_j(Z, f_j(Z)) \equiv 0$. We consider the system of equations in the unknowns X and Y:

$$Z_k^p - \sum_{j=1}^{K} Y_{jk} X_j = 0, \quad P_i(Z, X_i) = 0, \quad k = 1, \dots, N, \quad i = 1, \dots, K. \tag{1.13}$$

By the above, the system (1.13) has a convergent power series solution: $X_i = f_i(Z)$, $Y_{jk} = b_{jk}(Z)$. Since the system has polynomial coefficients, by an approximation theorem due to Artin [A2] (see also [DL]), there is an algebraic formal power series solution $\bar{X}(Z)$, $\bar{Y}(Z)$, which approximates the given solution up to any given finite order c. Hence the $\bar{Y}_{jk}(Z)$ satisfy polynomial equations $R_{jk}(Z, \bar{Y}_{jk}(Z)) = 0$. We consider now a new system of polynomial equations consisting of those in (1.13) together with

$$R_{jk}(Z, Y_{jk}) = 0, \quad j = 1, \dots, K, \quad k = 1, \dots, N.$$

By another approximation theorem of Artin [A1], there exists a convergent solution $\hat{X}(Z)$, $\hat{Y}(Z)$, which agrees with $\bar{X}(Z)$ and $\bar{Y}(Z)$ up to order c. If c is chosen sufficiently large (so that two holomorphic solutions of $P_j(Z, X_j) = 0$ which agree up to order c must be equal), then $\hat{X}_i(Z) = f_i(Z)$, the $\hat{Y}_{jk}(Z)$, are holomorphic algebraic, and we may take them for $b_{jk}(Z)$ in (1.12). This completes the proof of the lemma. □

Remark 1.14. Note that if the hypersurface M is algebraic, then the function Q in (1.1), as well as the functions $\bar{Q}_\alpha$ in (1.7), are holomorphic algebraic. Indeed, Q is obtained by the use of the implicit function theorem, so that we may apply (1.10) with $J = 1$; the $\bar{Q}_\alpha$ are obtained by taking derivatives.

2. Proof of Proposition 0.1. Part I

In this section we shall begin the proof of Proposition 0.1, which will be completed in §3. Assume M, M', p_0 and H are as in the assumptions of Proposition 0.1. We choose normal coordinates (z, w) for M, vanishing at p_0, and normal coordinates (z', w') for M' vanishing at $H(p_0)$ as in §1. We write the mapping $H = (f, g)$ with $z' = f(z, w)$ and $w' = g(z, w)$. We assume that M is given by (1.1) and M' is given by $w' = Q'(z', \bar{z}', \bar{w}')$. Since $H(M) \subset M'$, we have for $(z, w) \in M$ in a neighborhood of 0,

$$\bar{g}(\bar{z}, \bar{w}) = \bar{Q}'(\bar{f}(\bar{z}, \bar{w}), f(z, w), g(z, w)).$$

Since the manifold $\mathscr{M}$ introduced in §1 (see (1.4)) is given by $\tau = \bar{Q}(\zeta, z, w)$ for $(z, \zeta, w, \tau) \in \mathbb{C}^{2N}$, and a similar equation for $\mathscr{M}'$, it follows from the above that we have for $(z, w, \zeta, \tau) \in \mathscr{M}$

$$\bar{g}(\zeta, \tau) = \bar{Q}'(\bar{f}(\zeta, \tau), f(z, w), g(z, w)). \tag{2.1}$$

We now introduce the following holomorphic vector fields which span the tangent space to $\mathscr{M}$:

$$\mathscr{L}_j = \frac{\partial}{\partial \zeta_j} + \bar{Q}_{\zeta_j}(\zeta, z, w)\frac{\partial}{\partial \tau}, \quad j = 1, \dots, n, \tag{2.2}$$

$$\tilde{\mathscr{L}}_j = \frac{\partial}{\partial z_j} + Q_{z_j}(z, \zeta, \tau)\frac{\partial}{\partial w}, \quad j = 1, \dots, n, \tag{2.3}$$

$$\mathscr{K} = \frac{\partial}{\partial w} + \bar{Q}_w(\zeta, z, w)\frac{\partial}{\partial \tau}. \tag{2.4}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we shall write $\mathscr{L}^\alpha = \mathscr{L}_1^{\alpha_1} \dots \mathscr{L}_n^{\alpha_n}$. (Note that the $\mathscr{L}_j$ commute with each other.) By Remark 1.14, since M and M' are algebraic, the functions Q and Q' are holomorphic algebraic.

Proposition 2.5. *There exists a finite set S of multi-indices in $\mathbb{Z}_+^n$ such that for each j, $1 \le j \le n$, there exist a positive integer N_j and algebraic functions $a_{jk}(u_p^\gamma, v^\beta)$, $0 \le k \le N_{j-1}$, $1 \le p \le n$, $\beta, \gamma \in S$, holomorphic near $u_{p,0}^\gamma = \mathscr{L}^\gamma \bar{f}_p(0)$ and $v_0^\beta = 0$, such that for $(z, w, \zeta, \tau) \in \mathscr{M}$ the following holds:*

$$f_j^{N_j}(z, w) + \sum_{k=0}^{N_j - 1} a_{jk}(\mathscr{L}^\gamma \bar{f}_p, \mathscr{L}^\beta \bar{g}) f_j^k(z, w) \equiv 0, \qquad j = 1, \dots, n, \tag{2.6}$$

where $\mathscr{L}^\gamma \bar{f}$, $\mathscr{L}^\beta \bar{g}$ are evaluated at (ζ, τ).

Proof of Proposition 2.5. We may write the defining equation of $\mathscr{M}'$ in the form

$$\tau = \bar{Q}'(\zeta', z', w') = \bar{Q}'(\zeta', z', 0) + w'\bar{P}'(\zeta', z', w'), \tag{2.7}$$

where P' is a holomorphic algebraic function in a neighborhood of 0 in $\mathbb{C}^{2n+1}$. In what follows we write f for $f(z, w)$, $\bar{f}$ for $\bar{f}(\zeta, \tau)$, with (z, w, ζ, τ)

assumed to be in $\mathscr{M}$. We also use similar notation for g and $\bar{g}$. By (2.1) and (2.7) we have

$$\bar{g} = \bar{Q}'(\bar{f}, f, g) = \bar{Q}'(\bar{f}, f, 0) + g\bar{P}'(\bar{f}, f, g). \tag{2.8}$$

As in (1.7) we write

$$\bar{Q}'(\zeta', z', 0) = \sum_{\alpha} \bar{Q}'_{\alpha}(z')\zeta'^{\alpha}. \tag{2.9}$$

Since M' is essentially finite at 0 (because M is), by the Noetherian Theorem there exists $\ell > 0$ such that $\{\bar{Q}_{\alpha}(z') : |\alpha| \leq \ell\}$ have no common zeros, other than 0 itself, near 0. For every multi-index β with $|\beta| \leq \ell$ we define $r_{\beta}(z')$ by

$$r_{\beta}(z') = \sum_{\alpha} \bar{Q}'_{\alpha}(z')[\mathscr{L}^{\beta}\bar{f}^{\alpha}](0) = \sum_{|\alpha|\leq\ell} \bar{Q}'_{\alpha}(z')[\mathscr{L}^{\beta}\bar{f}^{\alpha}](0). \tag{2.10}$$

Note that the second equality holds since $f(0) = 0$ and hence $[\mathscr{L}^{\beta}\bar{f}^{\alpha}](0) = 0$ for $|\alpha| > |\beta|$.

We shall use the following to calculate $[\mathscr{L}^{\beta}\bar{f}^{\alpha}](0)$.

Lemma 2.11. *For any multi-index β, and any germ of holomorphic function $J(z, w, \zeta, \tau)$ at 0 in $\mathbb{C}^{2N}$, we have*

$$\left(\frac{\partial}{\partial\zeta}\right)^{\beta} J(0) = \mathscr{L}^{\beta} J(0). \tag{2.12}$$

Proof. The lemma follows immediately from the form of the $\mathscr{L}_j$ given by (2.2) and from the normality of the coordinates as in (1.2). □

From Lemma 2.11 we have $r_{\beta}(z') = \sum_{\alpha} \bar{Q}'_{\alpha}(z')[(\frac{\partial}{\partial\zeta})^{\beta}\bar{f}^{\alpha}](0)$. Since $\frac{\partial g}{\partial z}(0) = 0$ (which follows from (2.1) and the normality of the coordinates), and H is a biholomorphism at 0, it follows that the Jacobian matrix $\frac{\partial f}{\partial z}$ is invertible at 0. From these we easily obtain the following.

Lemma 2.13. *The linear span over $\mathbb{C}$ of the convergent power series, $\bar{Q}_{\alpha}(z')$, $|\alpha| \leq \ell$, is the same as that of the $r_{\beta}(z')$, $|\beta| \leq \ell$, where $r_{\beta}(z')$ is defined by* (2.10).

Using the Nullstellensatz (Lemma 1.11), there exists s such that for $p = 1, \dots, n$, $z_p'^{s}$ is in the ideal generated by the $\bar{Q}_{\alpha}(z')$, $|\alpha| \leq \ell$, in the ring $\mathscr{A}_n$. That is,

$$z_p'^{s} = \sum_{|\alpha|\leq\ell} c_{\alpha,p}(z')\bar{Q}_{\alpha}(z'), \quad p = 1, \dots, n, \tag{2.14}$$

with $c_{\alpha,p}(z')$ in $\mathscr{A}_n$. Hence by Lemma 2.13 we have

$$\begin{aligned} z_p'^{s} &= \sum_{|\beta|\leq\ell} b_{\beta,p}(z')r_{\beta}(z') \\ &= \sum_{|\beta|\leq\ell} b_{\beta,p}(z') \sum_{\alpha} \bar{Q}_{\alpha}(z')[\mathscr{L}^{\beta}\bar{f}^{\alpha}](0), \quad p = 1, \dots, n. \end{aligned} \tag{2.15}$$

Note that while the $c_{\alpha,p}(z')$ in (2.14) depend only on M' (and are independent of M and H), the $b_{\beta,p}(z')$ in (2.15) depend on M' and also on the constants $\partial^\alpha f(0)$, $|\alpha| \le \ell$. More precisely, the $b_{\beta,p}(z')$ are linear combinations of the $c_{\beta,p}(z')$ with constant coefficients depending only on the $\partial^\alpha f(0)$, $|\alpha| \le \ell$. (Note that the $\bar{Q}_\alpha$ are algebraic holomorphic.) Substituting f for z' in (2.15), we obtain

$$f_p^s = \sum_{|\beta|\le\ell} b_{\beta,p}(f) \sum_\alpha \bar{Q}_\alpha(f)\mathscr{L}^\beta \bar{f}^\alpha(0), \quad p = 1, \dots, n. \tag{2.16}$$

We now apply $\mathscr{L}^\beta$ to (2.8) and use (2.9) to obtain

$$\mathscr{L}^\beta \bar{g} = \sum_\alpha \bar{Q}'_\alpha(f)\mathscr{L}^\beta \bar{f}^\alpha + g\mathscr{L}^\beta(\bar{P}'(\bar{f}, f, g)). \tag{2.17}$$

Rewriting (2.1) in the form $g = Q'(f, \bar{f}, \bar{g})$ and substituting for g in (2.17) we have

$$\mathscr{L}^\beta \bar{g} = \sum_\alpha \bar{Q}'_\alpha(f)\mathscr{L}^\beta \bar{f}^\alpha + Q'(f, \bar{f}, \bar{g})\mathscr{L}^\beta(\bar{P}'(\bar{f}, f, Q'(f, \bar{f}, \bar{g}))). \tag{2.18}$$

By taking the product of (2.18) with $b_{\beta,p}(f)$ and summing over $|\beta| \le \ell$ we obtain

$$\begin{aligned} &\sum_{|\beta|\le\ell}\sum_\alpha b_{\beta,p}(f)\bar{Q}'_\alpha(f)\mathscr{L}^\beta \bar{f}^\alpha \\ &\quad = \sum_{|\beta|\le\ell} b_{\beta,p}(f)[\mathscr{L}^\beta \bar{g} - Q'(f, \bar{f}, \bar{g})\mathscr{L}^\beta(\bar{P}'(f, \bar{f}, \bar{Q}'(\bar{f}, f, \bar{g})))]. \end{aligned} \tag{2.19}$$

Using (2.16) and (2.19), we have

$$\begin{aligned} f_p^s = &\sum_{|\beta|\le\ell}\sum_\alpha b_{\beta,p}(f)\bar{Q}'_\alpha(f)(\mathscr{L}^\beta \bar{f}^\alpha(0) - \mathscr{L}^\beta \bar{f}^\alpha) \\ &+ \sum_{|\beta|\le\ell} b_{\beta,p}(f)[\mathscr{L}^\beta \bar{g} - Q'(f, \bar{f}, \bar{g})\mathscr{L}^\beta(\bar{P}'(f, \bar{f}, \bar{Q}'(\bar{f}, f, \bar{g})))]. \end{aligned} \tag{2.20}$$

Let $u_p^\gamma = \mathscr{L}^\gamma \bar{f}_p$ and $v^\gamma = \mathscr{L}^\gamma \bar{g}$, considered as independent variables, for $1 \le p \le n$, and $\gamma \in S$, where S consists of all multi-indices $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le \ell$. We let $u_{p,0}^\gamma = \mathscr{L}^\gamma \bar{f}_p(0)$ and $v_0^\gamma = \mathscr{L}^\gamma \bar{g}(0)$. (Note that $\mathscr{L}^\gamma \bar{g}(0) = 0$ by the normality of the coordinates and (2.1).)

Since $Q'(z', 0, 0) \equiv 0$, we may rewrite (2.20) in the form

$$f_p^s + K_p(f, u, v) = 0, \quad 1 \le p < n, \tag{2.21}$$

where $K_p(z', u, v)$ is algebraic holomorphic in its arguments, $z' \in \mathbb{C}^n$, $u = (u_p^\gamma)$, $v = (v^\gamma)$ as above and

$$K_p(z', u_0, v_0) \equiv 0, \tag{2.22}$$

with $u_0 = (u_{p,0}^\gamma)$, $v_0 = (v_0^\gamma)$. The rest of the proof of Proposition 2.5 will be completed by the following.

Lemma 2.23. *Let $\omega_0 \in \mathbb{C}^r$, and $K_p(Z, \omega)$, $p = 1, \dots, n$, be holomorphic algebraic functions in a neighborhood of $(0, \omega_0) \in \mathbb{C}^{n+r}$ satisfying $K_p(Z, \omega_0) \equiv 0$, $p = 1, \dots, n$. Then given positive integers $N_1, \dots, N_n$ there exist positive integers $N_1^*, \dots, N_n^*$ and holomorphic algebraic functions $d_{jp}(\omega)$ defined near ω_0 for $1 \le p \le n$, $0 \le j \le N_p^* - 1$, with $d_{jp}(\omega_0) = 0$, such that if (Z, ω) is near $(0, \omega_0)$ and satisfies the system of equations*

$$Z_p^{N_p} + K_p(Z, \omega) = 0, \quad 1 \le p \le n, \tag{2.24}$$

then (Z, ω) also satisfies the system

$$Z_p^{N_p^*} + \sum_{j=0}^{N_p^*-1} d_{jp}(\omega) Z_p^j = 0, \quad p = 1, \dots, n. \tag{2.25}$$

Proof. We reason by induction on n. For $n = 1$, we obtain (2.25) with $N_1^* = N_1$ by applying the Weierstrass Preparation Theorem (see Lemma 1.8(i)) to (2.24). We shall now show how to reduce the case of n to that of $n - 1$.

We use (2.24) with $p = 1$ and apply equality (1.10). Hence there exist algebraic holomorphic functions $c_j(Z_2, \dots, Z_n, \omega)$, $j = 0, \dots, N_1 - 1$, with $c_j(Z_2, \dots, Z_n, \omega_0) \equiv 0$ so that (2.24) (with $p = 1$) is equivalent to

$$Z_1^{N_1} + \sum_{j=0}^{N_1-1} c_j(Z_2, \dots, Z_n, \omega) Z_1^j = 0. \tag{2.26}$$

Let $\rho_j(Z_2, \dots, Z_n, \omega)$, $j = 1, \dots, N_1$, be the roots (counted with multiplicity) in Z_1 of the polynomial (2.26). Then replace Z_1 by ρ_j, $j = 1, \dots, N_1$, in (2.24) with $p = 2, \dots, n$. Taking products over j we obtain

$$\prod_{j=1}^{N_1} [Z_p^{N_p} + K_p(\rho_j(Z_2, \dots, Z_n, \omega), Z_2, \dots, Z_n, \omega)] = 0, \quad p = 2, \dots, n. \tag{2.27}$$

Since the left-hand side of (2.27) is a symmetric algebraic holomorphic function of $\rho_1, \dots, \rho_{N_1}$, by Lemma 1.8(ii) it is an algebraic holomorphic function of the elementary symmetric functions of the ρ_j, i.e., of the coefficients c_j in (2.26). Therefore we may rewrite (2.27) in the form

$$H_p(Z_2, \dots, Z_n, \omega) = 0, \quad p = 2, \dots, n, \tag{2.28}$$

where H_p is holomorphic algebraic in its arguments and $H_p(Z_2, \dots, Z_n, \omega_0) \equiv Z_p^{N_p N_1}$ by the hypothesis on K_p. Therefore, (2.28) is a system of the form (2.24) with n replaced by $n - 1$ and N_p by $N_p N_1$. By induction, we obtain (2.25) for $p = 2, \dots, n$. It remains only to show that (2.25) holds for $p = 1$. For this, we start with (2.26) and replace each Z_j, $j = 2, \dots, n$, by one of the roots (counted with multiplicity) of (2.25), $p = 2, \dots, n$. Taking a product, similar to (2.27), over all possible expressions so obtained and again using symmetry and Lemma 1.8(ii), we obtain an equation of the form $H_1(Z_1, \omega) = 0$,

where H_1 is holomorphic and $H_1(Z_1, \omega_0) \equiv Z_1^{N_1^*}$ for some integer N_1^*. Applying once more the Weierstrass Preparation Theorem (Lemma 1.8(i)) yields (2.25) with $p = 1$. $\square$

Remark 2.29. An inspection of the proof of Proposition 2.5 shows that the functions a_{jk} in (2.6) depend only on M' and on the finite number of constants $\partial^\alpha f(0)$, $\alpha \in S$, where S is as in the statement of the proposition.

3. End of the proof of Proposition 0.1

In this section we complete the proof of Proposition 0.1. We assume throughout this section that (z, w) and (z', w') are normal coordinates for M and M' respectively. We write $H = (f, g)$ as in §2, and shall make use of Proposition 2.5.

We begin with the following.

Lemma 3.1. *Under the assumptions of Theorem* 1, *for every integer* q *the mapping* $z \mapsto \frac{\partial^q}{\partial w^q} H(z, 0)$ *is holomorphic algebraic in a neighborhood of* 0 *in* $\mathbb{C}^n$.

Proof. We begin with the polynomial identities (2.6) of Proposition 2.5. We note first that for $z \in \mathbb{C}^n$ close to 0, the point $(z, w, \zeta, \tau) = (z, 0, 0, 0)$ is in $\mathscr{M}$, since $Q(z, 0, 0) \equiv 0$. Since the coefficients of $\mathscr{L}_j$ given by (2.2) are algebraic holomorphic, for any holomorphic function $J(\zeta, \tau)$, the functions $(z, w) \mapsto (\mathscr{L}^\gamma J(\zeta, \tau))|_{\zeta=0, \tau=0}$ are algebraic holomorphic. Evaluating (2.6) at $(z, 0, 0, 0)$, we have

$$(3.2) \qquad f_j^{N_j}(z, 0) + \sum_{k=0}^{N_j - 1} a_{jk}((\mathscr{L}^\gamma \bar{f}_p)|_{\zeta=0, \tau=0}, (\mathscr{L}^\beta \bar{g})|_{\zeta=0, \tau=0}) f_j^k(z, 0) \equiv 0.$$

Since the a_{jk} are algebraic holomorphic, and by the above comments the functions $(z, w) \mapsto \mathscr{L}^\gamma \bar{f}_p|_{\zeta=0, \tau=0}$ and $(z, w) \mapsto \mathscr{L}^\beta \bar{g}_p|_{\zeta=0, \tau=0}$ are algebraic holomorphic, it follows by Lemma 1.8(iv) that the coefficients of the $f_j^k(z, 0)$ in (3.2) above are all algebraic holomorphic. Hence by (3.2) each $z \mapsto f_j(z, 0)$ satisfies a monic polynomial with algebraic holomorphic coefficients. By Lemma 1.8(iii), we may conclude that each $f_j(z, 0)$ is algebraic. This proves the lemma for $q = 0$, since $g(z, 0) \equiv 0$, by (2.1) and the normality of the coordinates.

In order to prove the lemma for $q > 0$, we shall need the following algebraic result.

Lemma 3.3. *Let* $\mathscr{A}_n\{w\}$ *be the ring of germs of holomorphic functions* $\sum_j a_j(z) w^j$ *with coefficients* $a_j(z)$ *in* $\mathscr{A}_n$. *Then any element of* $\mathbb{C}\{z, w\}$ *which is algebraic over* $\mathscr{A}_n\{w\}$ *is already in* $\mathscr{A}_n\{w\}$. *That is, if* $h(z, w) \in \mathbb{C}\{z, w\}$ *satisfies a nontrivial polynomial equation of the form*

$$(3.4) \qquad \sum_{j=0}^{K} c_j(z, w) h^j(z, w) \equiv 0, \quad c_j(z, w) \in \mathscr{A}_n\{w\},$$

then $h(z, w) \in \mathscr{A}_n\{w\}$.

Proof of Lemma 3.3. We may assume that the polynomial in (3.4) is irreducible in $\mathscr{A}_n\{w\}[X]$. Putting $w = 0$ in (3.4), we have $\sum c_j(z, 0)h^j(z, 0) \equiv 0$. Note that $\sum |c_j(z, 0)| \not\equiv 0$, since otherwise w would be a factor of each coefficient of the polynomial given by (3.4), contradicting irreducibility. This proves that $h(z, 0)$ is algebraic over $\mathscr{A}_n$ and hence algebraic over $\mathscr{A}_n\{w\}$. Therefore, $h(z, w) - h(z, 0) = wk(z, w)$ is also algebraic over $\mathscr{A}_n\{w\}$, i.e., there exist $b_j(z, w) \in \mathscr{A}_n\{w\}$ not all identically 0 such that

$$\sum_{j=0}^{N_1} b_j(z, w)w^j k(z, w)^j \equiv 0.$$

This proves $k(z, w)$ is also algebraic over $\mathscr{A}_n\{w\}$, which implies $k(z, 0)$ is algebraic over $\mathscr{A}_n$. Repeating the argument for $h(z, w) - h(z, 0) - w\frac{\partial}{\partial w}h(z, 0)$, etc., we obtain $\frac{\partial^k h}{\partial w^k}(z, 0)$ is algebraic over $\mathscr{A}_n$ for $k = 0, 1, \dots$. This completes the proof of Lemma 3.3. □

We return to the proof of Lemma 3.1. We restrict to the submanifold of $\mathscr{M}$ given by $\zeta = 0$ and $\tau = w$. The vector field $\mathscr{K}$ given by (2.4) then becomes $2\frac{\partial}{\partial w}$, and the polynomial identity (2.6) of Proposition 2.5 becomes

$$f_j^{N_j}(z, w) + \sum_{k=0}^{N_j - 1} a_{jk}(\mathscr{L}^\gamma \bar{f}_p(0, w), \mathscr{L}^\beta \bar{g}(0, w))f_j^k(z, w) \equiv 0, \tag{3.5}$$

which holds identically in z and w. (Note that the coefficients of the operators $\mathscr{L}^\gamma$ are algebraic holomorphic functions of z and w.) Hence the equation (3.5) is of the form (3.4) with $h(z, w) = f_j(z, w)$. From Lemma 3.3, we conclude $f_j(z, w) \in \mathscr{A}_n\{w\}$ for $j = 1, \dots, n$. In particular, the mappings $z \mapsto \frac{\partial^k}{\partial w^k} f_j(z, 0)$ are holomorphic algebraic for all k. To complete the proof of Lemma 3.1, we must show the same is true for $g(z, w)$. From (2.1) we have

$$g(z, w) = Q'(f(z, w), \bar{f}(\zeta, \tau), \bar{g}(\zeta, \tau)), \tag{3.6}$$

for $(z, w, \zeta, \tau) \in \mathscr{M}$. In (3.6) we set $\zeta = 0$, $\tau = w$, and differentiate in w. We obtain

$$\begin{aligned} g_w(z, w) = {} & Q'_{z'}(f(z, w), \bar{f}(0, w), \bar{g}(0, w))f_w(z, w) \\ & + Q'_{\zeta'}(f(z, w), \bar{f}(0, w), \bar{g}(0, w))\bar{f}_\tau(0, w) \\ & + Q'_{\tau'}(f(z, w), \bar{f}(0, w), \bar{g}(0, w))\bar{g}_\tau(0, w). \end{aligned} \tag{3.7}$$

Taking $w = 0$ and using the fact that $f(z, 0)$ and $f_w(z, 0)$ are algebraic holomorphic, as are the partial derivatives of Q', we then obtain that the mapping $z \mapsto g_w(z, 0)$ is also holomorphic algebraic. Repeated differentiation of (3.7) with respect to w yields similarly that the mappings $z \mapsto g_{w^k}(z, 0)$ are algebraic holomorphic, which completes the proof of Lemma 3.1. □

Proof of Proposition 0.1. We begin with (2.6) in which we take $\tau = 0$ and substitute $Q(z, \zeta, 0)$ for w to obtain

$$(3.8) \qquad f_j^{N_j}(z, Q(z, \zeta, 0)) + \sum_{k=0}^{N_j-1} a_{jk}(\mathscr{L}^\gamma \bar{f}_p, \mathscr{L}^\beta \bar{g}) f_j^k(z, Q(z, \zeta, 0)) \equiv 0,$$

which holds as an identity in $(z, \zeta) \in \mathbb{C}^{2n}$ near 0. Note that after this substitution the coefficients of the vector fields $\mathscr{L}_j$ are then algebraic holomorphic in (z, ζ). Since M is essentially finite, and the coordinates are taken to be normal, we conclude that the vector function $Q_\zeta(z, \zeta, 0)$ does not vanish identically. Hence we may assume there is (z^0, ζ^0) such that $Q_{\zeta_1}(z^0, \zeta^0, 0) \neq 0$. Note that (z^0, ζ^0) can be chosen arbitrarily close to 0 in $\mathbb{C}^{2n}$. Put $w^0 = Q(z^0, \zeta^0, 0)$. By the implicit function theorem (Lemma 1.8(i)), we can find an algebraic holomorphic function $\chi(z, w)$ defined near (z^0, w^0) and satisfying $\chi(z^0, w^0) = \zeta_1^0$, such that the following identity holds for (z, w) near (z^0, w^0) in $\mathbb{C}^{n+1}$:

$$(3.9) \qquad Q(z, \chi(z, w), \zeta_2^0, \dots, \zeta_n^0, 0) \equiv w.$$

We now take $\zeta = (\chi(z, w), \zeta_2^0, \dots, \zeta_n^0)$ in (3.8). After making this substitution, we consider that (z, w) are independent variables near (z^0, w^0). (Recall that τ has been set to 0 throughout this part of the proof.) We claim that after this substitution the functions

$$(3.10) \qquad (z, w) \mapsto a_{jk}(\mathscr{L}^\gamma \bar{f}_p, \mathscr{L}^\beta \bar{g})$$

are algebraic holomorphic. Indeed, as noted above, the coefficents of the $\mathscr{L}_j$ become algebraic holomorphic in (z, w), and the derivatives of $\bar{f}$ and $\bar{g}$ appearing in (3.10) are all taken at $\zeta = (\chi(z, w), \zeta_2^0, \dots, \zeta_n^0)$ and $\tau = 0$. The claim then follows from Lemma 3.1, Lemma 1.8(iv), and the fact that $\chi(z, \zeta)$ is algebraic.

From (3.8) and the observations above, we have shown that near (z^0, w^0) each $f_j(z, w)$ satisfies a polynomial equation with coefficients which are algebraic holomorphic. Hence, by the transitivity of the property of being algebraic (Lemma 1.8(iii)), each f_j satisfies a polynomial relation, with polynomial coefficients, near (z^0, w^0). Since the f_j are holomorphic in a neighborhood of 0, by analytic continuation, this relation holds in the entire connected set in which the f_j are defined. To show that $g(z, w)$ is algebraic, it suffices to take (3.6) with $\zeta = (\chi(z, w), \zeta_2^0, \dots, \zeta_n^0)$, $\tau = 0$, and to use the result just proved for $f(z, w)$. This completes the proof of Proposition 0.1. □

4. Space of meromorphic vector fields tangent to M and essential finiteness

In this section we shall always assume that M is a real analytic hypersurface in $\mathbb{C}^N$; we shall give most of the ingredients of the proof of Theorem 2 here.

We shall need some preliminary results on holomorphic vector fields tangent to M. We first introduce some notation. For $Z^0 \in \mathbb{C}^N$, we denote by $\mathscr{O}_{Z^0}$ the ring of germs of holomorphic functions at Z^0 and by $\mathscr{K}_{Z^0}$ its quotient field, i.e., the field of germs of meromorphic functions at Z^0. For $Z^0 \in M$ we denote by $\mathscr{H}_{Z^0}$ the module over $\mathscr{O}_{Z^0}$ of germs at Z^0 of holomorphic vector fields tangent to M. Similarly, we let $\mathscr{E}_{Z^0}$ be the vector space over $\mathscr{K}_{Z^0}$ consisting of all germs at Z^0 of meromorphic vector fields tangent to M, i.e.,

(4.1)

$$\mathscr{E}_{Z^0} = \{X = \sum_{k=1}^{N} a_j(Z)\frac{\partial}{\partial Z_j} : a_j \in \mathscr{K}_{Z^0} \text{ and } aX \in \mathscr{H}_{Z^0} \text{ for some } a \in \mathscr{O}_{Z^0},\ a \not\equiv 0\}.$$

Note that since $\mathscr{E}_{Z^0} \subset \mathscr{K}_{Z^0}^N$, $\mathscr{E}_{Z^0}$ is a finite-dimensional vector space over $\mathscr{K}_{Z^0}$. We observe that by definition (see §0) $\mathscr{E}_{Z^0}$ is of positive dimension if and only if M is holomorphically degenerate at Z^0.

The following proposition is one of the main results of this section.

Proposition 4.2. *For $p \in M$, let $d(p)$ be the dimension of $\mathscr{E}_p$ as a vector space over $\mathscr{K}_p$. Then the function $p \mapsto d(p)$ is constant on any connected component of M. In particular, if M is holomorphically degenerate at some point, it is holomorphically degenerate at every point.*

In order to prove Proposition 4.2, we shall need a precise local description of holomorphic vector fields tangent to M. Fix $p_0 \in M$ and let (z, w) be normal coordinates vanishing at p_0. We assume M is given by (1.1). We shall write

$$\bar{Q}(\zeta, z, w) = \sum_{\alpha} q_\alpha(z, w)\zeta^\alpha \tag{4.3}$$

for $|z|, |\zeta|, |w| < \delta$. We shall assume that δ is chosen sufficiently small such that the right-hand side of (4.3) is absolutely convergent. We let

$$V = \{(z, w) \in \mathbb{C}^{n+1} : |z| < \delta,\ |w| < \delta\}, \tag{4.4}$$

so that $M \cap V$ is given by (1.1). We have the following lemma.

Lemma 4.5. *Let (z, w) be normal coordinates as above and $(z^0, w^0) \in M \cap V$, where V is given by (4.4). If X is a germ at (z^0, w^0) of a holomorphic vector field in $\mathbb{C}^N$, then X is tangent to M if and only if*

$$X = \sum_{j=1}^{n} a_j(z, w)\frac{\partial}{\partial z_j}, \tag{4.6}$$

with a_j holomorphic in a neighborhood of (z^0, w^0), and

$$\sum_{j=1}^{n} a_j(z, w) q_{\alpha, z_j}(z, w) \equiv 0, \tag{4.7}$$

for all multi-indices α, *and* (z, w) *in a neighborhood of* (z^0, w^0), *where the* q_α *are given by* (4.3).

Proof. It is easy to check that if X is of the form (4.6), where the a_j satisfy (4.7), then X is tangent to M. Conversely, suppose $X = \sum_{j=1}^n a_j(z, w)\frac{\partial}{\partial z_j} + b(z, w)\frac{\partial}{\partial w}$ with a_j and b holomorphic near (z^0, w^0), and assume X is tangent to M. Then we have for (z, w, ζ, τ) near $(z^0, w^0, \bar{z}^0, \bar{w}^0)$

$$X(\tau - \bar{Q}(\zeta, z, w)) = A(z, w, \zeta, \tau)[\tau - \bar{Q}(\zeta, z, w)]$$

with A holomorphic. Hence we have

$$\sum_{j=1}^n a_j(z, w)\bar{Q}_{z_j}(\zeta, z, w) + b(z, w)\bar{Q}_w(\zeta, z, w)$$
$$= -A(z, w, \zeta, \tau)[\tau - \bar{Q}(\zeta, z, w)].$$

Taking $\tau = \bar{Q}(\zeta, z, w)$ we obtain

$$\sum_{j=1}^n a_j(z, w)\bar{Q}_{z_j}(\zeta, z, w) + b(z, w)\bar{Q}_w(\zeta, z, w) \equiv 0 \tag{4.8}$$

for (z, w, ζ) near $(z^0, w^0, \bar{z}^0)$. By analytic continuation, (4.8) holds also for all ζ close to 0. Taking $\zeta = 0$ in (4.8), and using the identity $\bar{Q}(0, z, w) \equiv w$ we obtain $b(z, w) \equiv 0$. The rest of the lemma follows by making use of the expansion (4.3). □

If the $q_\alpha(z, w)$ are given by equality (4.3), for each multi-index α we write $q_{\alpha, z}(z, w)$ for the gradient with respect to z of the function $q_\alpha(z, w)$. For fixed $(z^0, w^0) \in V$, where V is given by (4.4), write $r(z^0, w^0)$ for the dimension of the span in $\mathbb{C}^n$ of all the $q_{\alpha, z}(z^0, w^0)$ as α varies in $\mathbb{Z}_+^n$. Let r be the maximum of the $r(z^0, w^0)$ for $(z^0, w^0) \in V$. Since a nontrivial holomorphic function cannot vanish in an open set of M, it follows that $r(z^0, w^0) = r$ for all $(z^0, w^0) \in V \cap M \backslash \Sigma$, where Σ is a proper analytic subset of $V \cap M$. Indeed r is determined by the nonvanishing of the determinant of a submatrix of the $q_{\alpha, z}$. Note that $0 \le r \le n$.

The proof of Proposition 4.2 will be an immediate consequence of the following lemma.

Lemma 4.9. *Let* V *and* r *be as above. Then there exist vector fields,* $X_1, \dots, X_{n-r}$, *with holomorphic coefficients in* V, *tangent to* $M \cap V$, *such that for every* $p \in M \cap V$ *the* $X_{j|p}$, $1 \le j \le n - r$, *the germs of the* X_j *at* p, *form a basis of* $\mathscr{E}_p$ *over* $\mathscr{K}_p$.

Proof. Let $\mathscr{O}(V)$ be the ring of holomorphic functions in V and $\mathscr{K}(V)$ its quotient field, i.e., the field of meromorphic functions in V. We consider the vector space $\mathscr{K}(V)^n$ over $\mathscr{K}(V)$ and the r-dimensional subspace F spanned by the $q_{\alpha, z}(z, w)$ in $\mathscr{K}(V)^n$. By Cramer's rule there exist $n - r$ linearly independent vectors in $\mathscr{K}(V)^n$ whose dot product with all the elements of F

is zero. Identifying these vectors with vector fields X_j, $1 \le j \le n-r$, in V and multiplying by a common denominator we find $n-r$ holomorphic vector fields. In view of Lemma 4.5 and the fact that $r(z, w) = r$ outside of a proper analytic subset Σ, the X_j are tangent to M and give the desired result. □

Remark 4.10. If M is an algebraic hypersurface, then an inspection of the proof of Lemma 4.9 shows that the local basis of holomorphic vector fields $X_1, \dots, X_{n-r}$ given by the lemma can be chosen to have algebraic holomorphic coefficients, since the $q_\alpha(z, w)$ given in (4.3) are algebraic holomorphic.

We will connect essential finiteness to the nonexistence of meromorphic tangent vectors. We need to describe the set $\mathscr{V}_{p_0}$ given by (1.4) in normal coordinates.

Lemma 4.11. *Let (z, w) be normal coordinates vanishing at p_0 and V given by* (4.4) *as above. For $(z^0, w^0) \in M \cap V$ we have*

$$(4.12)\quad \mathscr{V}_{(z^0, w^0)} = \{(z, w^0) : z \text{ close to } z^0,\ q_\alpha(z, w^0) = q_\alpha(z^0, w^0) \text{ for all } \alpha\}.$$

Proof. Since $\mathscr{M}$ is given by $\tau - \bar{Q}(\zeta, z, w) = 0$, in view of (1.4) the germ of $\mathscr{V}_{(z^0, w^0)}$ consists of the set of points (z, w) near (z^0, w^0) satisfying $\tau - \bar{Q}(\zeta, z, w) = 0$ for all (ζ, τ) near $(\bar{z}^0, \bar{w}^0)$ with $\tau - \bar{Q}(\zeta, z^0, w^0) = 0$. Using (4.3) and analytic continuation in ζ, we obtain (4.12). □

The following is immediate from (4.12) and the inverse mapping theorem.

Lemma 4.13. *Let V be given by* (4.4) *and let $r(z, w)$ be defined as in the comments preceding Lemma* 4.9. *If $r(z^0, w^0) = n$, with $(z^0, w^0) \in M \cap V$, then M is essentially finite at (z^0, w^0).*

Remark 4.14. Let $Z \in M$. The collection of the subspaces of $\mathbb{C}T_Z M$ consisting of $X(Z)$ for $X \in \mathscr{H}_Z$ need not form a bundle for $N > 2$. For instance, if $M \subset \mathbb{C}^3$ is given by $\Im Z_3 = |Z_1 Z_2|^2$, then the vector field $Z_1 \frac{\partial}{\partial Z_1} - Z_2 \frac{\partial}{\partial Z_2}$ spans $\mathscr{H}_Z$ over $\mathscr{O}_Z$, but vanishes at $Z_1 = Z_2 = 0$. This was noted in [S].

5. Flow of holomorphic vector fields tangent to M

In this section we study the flow of holomorphic vector fields tangent to a real analytic hypersurface M in $\mathbb{C}^N$. Let $p_0 \in M$ and assume that X is a nontrivial germ at p_0 of a holomorphic vector field tangent to M. To any such X, there is a holomorphic one-parameter group of local biholomorphisms in $\mathbb{C}^N$ sending M into M. Such a group of automorphisms is defined by the complex flow of X, i.e.,

$$(5.1)\qquad \dot{\phi}(t, Z) = X(\phi(t, Z)), \quad \phi(0, Z) = Z.$$

Then $\phi(t, Z)$ is holomorphic for $t \in \mathbb{C}$, $|t| < \epsilon$, and $Z \in V$, where V is an open neighborhood of p_0 in $\mathbb{C}^N$. For fixed t, the map $Z \mapsto \phi(t, Z)$ is a local biholomorphism preserving M, and if $X(p_0) = 0$, then $\phi(t, p_0) \equiv p_0$.

The following will be used in the proof of Theorem 2.

Proposition 5.2. *Let M be a real analytic hypersurface in $\mathbb{C}^N$ and $p_0 \in M$. Assume there exists X, a germ at p_0, of a nontrivial holomorphic vector field tangent to M. Then the germ at 0 of the holomorphic complex curve $t \mapsto \phi(t, p_0)$, where $\phi(t, p_0)$ is the flow of X starting from p_0 given by* (5.1), *is contained in $\mathscr{V}_{p_0}$ (as defined by* (1.4)).

Proof. Let $\zeta \in \mathbb{C}^N$ close to $\bar{p}_0$ be such that $\rho(p_0, \zeta) = 0$. We must show that the function $t \mapsto h(t) = \rho(\phi(t, p_0), \zeta)$ vanishes identically. If $X = \sum_{j=1}^N a_j(Z)\frac{\partial}{\partial Z_j}$, then by the definition of the flow, we have

$$\frac{dh}{dt}(t) = \sum_{j=1}^N a_j(\phi(t, p_0))\frac{\partial \rho}{\partial Z_j}(\phi(t, p_0), \zeta).$$

Since X is tangent to M, the latter must be a multiple of $h(t) = \rho(\phi(t, p_0), \zeta)$. Since $h(0) = 0$, by the uniqueness of the solution of differential equations, we conclude that $h(t) \equiv 0$. This completes the proof of the proposition. □

Proposition 5.2 shows in particular that under the assumptions of the proposition, if $X(p_0) \neq 0$, then M is not essentially finite at p_0. Indeed, in that case the curve $t \mapsto \phi(t, p_0)$ is not constant and hence $\mathscr{V}_{p_0}$ is nontrivial.

Proposition 5.3. *Let M be an algebraic hypersurface in $\mathbb{C}^N$ and X a nontrivial germ at $p_0 \in M$ of a holomorphic vector field tangent to M with algebraic holomorphic coefficients. Then there exists $f \in \mathscr{O}_{p_0}$ and arbitrarily small t such that if $\psi(t, Z)$ is the flow of $Y = fX$, the mapping $z \mapsto \psi(t, Z)$ is a nonalgebraic local biholomorphism mapping M into itself and fixing p_0.*

Proof. If $X = \sum_{j=1}^N a_j(Z)\frac{\partial}{\partial Z_j}$ and $\phi(t, Z)$ is its flow, then by standard arguments using the local group property (see e.g. [N]), we have

$$\sum_{j=1}^N a_j(Z)\frac{\partial \phi_k}{\partial Z_j}(t, Z) = a_k(\phi(t, Z)), \quad k = 1, \dots, N. \tag{5.4}$$

After multiplying X by a nontrivial algebraic holomorphic function vanishing at p_0, if necessary, we may assume $X(p_0) = 0$. If for some arbitrarily small t the map $Z \mapsto \phi(t, Z)$ is not algebraic, there is nothing to prove. Otherwise, after renumbering if necessary, we assume $a_1 \not\equiv 0$, and let $f(Z) = e^{Z_1}$ and $Y = e^{Z_1}X$. We denote by $\psi(t, Z)$ the holomorphic flow of Y. By (5.4) for the vector field Y instead of X, and taking $k = 1$ we have

$$\sum_{j=1}^N e^{Z_1}a_j(Z)\frac{\partial \psi_1}{\partial Z_j}(t, Z) = e^{\psi_1(t, Z)}a_1(\psi(t, Z)). \tag{5.5}$$

Hence we have

$$e^{Z_1 - \psi_1(t, Z)}\sum_{j=1}^N a_j(Z)\frac{\partial \psi_1}{\partial Z_j}(t, Z) = a_1(\psi(t, Z)). \tag{5.6}$$

We claim that for some t arbitrarily small, the map $z \mapsto \psi(t, Z)$ is not algebraic. To prove this claim, we reason by contradiction. If $Z \mapsto \psi(t, Z)$ were algebraic for some fixed t, then since all the coefficients a_k are algebraic, it would follow from (5.6) that the function $Z \mapsto e^{Z_1 - \psi_1(t, Z)}$ is also algebraic. Note that $Z \mapsto Z_1 - \psi_1(t, Z)$ is algebraic and not constant (since $a_1 \not\equiv 0$). However, if $A(Z)$ is any nonconstant algebraic holomorphic function, then the function $Z \mapsto e^{A(Z)}$ cannot be algebraic. Hence we reach a contradiction, which proves the claim. □

6. Proofs of Theorems 1 and 2. Remarks

Proof of Theorem 2. Assume that M is a connected real analytic hypersurface in $\mathbb{C}^N$ and that M is essentially finite at p_0. We must show that M is not holomorphically degenerate at any point. We reason by contradiction. Assume that M is degenerate at some point. By Proposition 4.2, M is holomorphically degenerate at every point. In particular, M is holomorphically degenerate at p_0. Let X be the germ at p_0 of a nontrivial holomorphic vector field tangent to M. By Proposition 5.2, M is not essentially finite at every point p_1 such that $X(p_1) \neq 0$. If $X(p_0) \neq 0$, we immediately reach a contradiction. If $X(p_0) = 0$, there are points p_1 arbitrarily close to p_0 such that M is not essentially finite at p_1. We reach again a contradiction, since the property of being essentially finite at p is open in M (see [BR2]). This completes the proof of one implication of Theorem 2.

To prove the converse, assume now that M is not holomorphically degenerate at any point. We must show that there exists a point at which M is essentially finite. Choose any $p_0 \in M$ and let (z, w) be normal coordinates vanishing at p_0 and V given by (4.1). If r is the maximum rank in V of the $q_{\alpha, z}$ given in (4.3), then by Lemma 4.9, we have $r = n$. Hence except on a proper analytic set $\Sigma \subset M \cap V$, we have $r(z, w) = n$. On the other hand, by Lemma 4.13, M is essentially finite at points where $r(z, w) = n$. This completes the proof of Theorem 2. □

Proof of Theorem 1. Let M be a connected algebraic hypersurface contained in $\mathbb{C}^N$. If there is no point $p_1 \in M$ at which M is holomorphically degenerate, then by Theorem 2, M is essentially finite at some point Z^0. The assumptions of Proposition 0.1 are satisfied at Z^0, and we conclude that H is algebraic near Z^0 and hence everywhere in its domain of definition.

If M is holomorphically degenerate at some point p_1, then M is holomorphically degenerate at every point by Proposition 4.2. Hence there is a nontrivial germ at p_0 of a holomorphic vector field X tangent to M. By Remark 4.10, we may assume that X has algebraic coefficients. Applying Proposition 5.3, we infer the existence of a germ of a nonalgebraic local biholomorphism mapping M into itself and fixing p_0. This completes the proof of Theorem 1. □

Proof of the Corollary. Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$, and let H be a holomorphic mapping defined in a neighborhood of M in $\mathbb{C}^N$ with $H(M) \subset M'$. Assume M is not holomorphically degenerate at any point.

If the Jacobian determinant $J(Z)$ of H does not vanish identically, then the restriction of $J(Z)$ to any open set of M does not vanish identically. We can then apply Theorem 1 at any such point to conclude that H is algebraic.

On the other hand, if $J(Z) \equiv 0$ and M' contains no nontrivial complex analytic variety, we may use Theorem 3 of [BR3] to conclude that H is constant. Indeed, it suffices to apply this theorem at a point of M which is essentially finite. This completes the proof of the Corollary. □

Remark 6.1. Let M be a connected real analytic hypersurface in $\mathbb{C}^N$ and d be the integer given by Proposition 4.2. If $d = n$, then M is locally biholomorphically equivalent to $\Im Z_N = 0$. Indeed, by Lemma 4.9, we have $r = 0$, i.e., $q_\alpha(z, w) \equiv 0$, $|\alpha| \neq 0$, and $q_0(z, w) \equiv w$, where the $q_\alpha(z, w)$ are given in (4.3). In particular, a real analytic hypersurface in $\mathbb{C}^2$ is either holomorphically degenerate and hence locally equivalent to $\Im Z_2 = 0$ or M is of finite type in the sense of Kohn [K], except on a proper real analytic subset.

Remark 6.2. An algebraic holomorphic function $h(Z)$ is said to be of degree m if it satisfies a polynomial equation $P(Z, f(Z)) = 0$, where $P(Z, X)$ is an irreducible polynomial in $N+1$ variables of total degree m. An inspection of the proof of Proposition 0.1 shows that if H is as in the proposition, then the degree of its components is bounded by a constant depending only on M, M' and the points p_0 and $H(p_0)$. (See especially Remark 2.29.)

Remark 6.3. Freeman [Fr] showed that $M \subset \mathbb{C}^3/\{0\}$ given by $X_1^3+X_2^3+X_3^3 = 0$ with $Z = (Z_1, Z_2, Z_3)$ and $X_k = \Re Z_k$ has everywhere degenerate Levi form, but no local straightening. Stanton [S] showed that M is essentially finite at some points. Hence, by Theorem 2, M is not holomorphically degenerate at any point. It can also be checked directly that if X is a germ at $p_0 \in M$ of a holomorphic vector field tangent to M, then $X \equiv 0$.

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