

## PRODUCTS OF CYCLES AND THE TODD CLASS OF A TORIC VARIETY

JAMES E. POMMERSHEIM

### 1. INTRODUCTION

The purpose of this paper is to show that the Todd class of a simplicial toric variety has a canonical expression in terms of products of torus-invariant divisors. The coefficients in this expression, which are generalizations of the classical Dedekind sum, are shown to satisfy a reciprocity relation which characterizes them uniquely. We achieve these results by giving an explicit formula for the push-forward of a product of cycles under a proper birational map of simplicial toric varieties.

Since the introduction of toric varieties in the 1970s, finding formulas for their Todd class has been an interesting and important problem. This is partly due to a well-known application of the Riemann-Roch theorem which allows a formula for the Todd class of a toric variety to be translated directly into a formula for the number of lattice points in a lattice polytope (cf. [Dan]). An example of this application is contained in [Pom], where a formula for the Todd class of a toric variety in terms of Dedekind sums is used to obtain new lattice point formulas.

Danilov [Dan] posed the problem of finding a formula for the Todd class of a toric variety in terms of the orbit closures under the torus action. Specifically, he asked if it is possible, given a lattice, to assign a rational number to each cone in the lattice such that given any fan in the lattice, the Todd class of the corresponding toric variety equals the sum of the orbit closures with coefficients given by these assigned rational numbers. Morelli [Mor] showed that such an assignment is indeed possible in a natural way if the coefficients, instead of being rational numbers, are allowed to take values in the field of rational functions on a Grassmannian of linear subspaces of the lattice. However, if it is required that the coefficients be rational numbers invariant under lattice automorphisms, such an assignment is clearly impossible. For example, the nonsingular cone  $\sigma$  in  $\mathbb{Z}^2$  generated by  $(1, 0)$  and  $(0, 1)$  when subdivided by the ray through  $(1, 1)$  yields two cones  $\sigma_1$  and  $\sigma_2$  which are both lattice equivalent to  $\sigma$ . By additivity, a consequence of the fact that the Todd class pushes forward, we deduce that the coefficient assigned to  $\sigma$  must equal 0, which is absurd.

In this paper, we show that there is a canonical expression for the Todd class of a simplicial toric variety in terms of *products* of the torus invariant divisors. Furthermore, this expression is invariant under lattice automorphisms. That is, the coefficient of each product depends only on the set of rays with multiplicities

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corresponding to the divisors occurring in the product. To make this precise, we have

**Theorem 1.** *Given a lattice  $N$ , there exists a canonical function  $f$  assigning to each tuple  $(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  of rays with multiplicities  $a_i \in \mathbb{N}$  a rational number  $f(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  such that*

- (1)  *$f$  is invariant under lattice automorphisms, and*
- (2) *for any complete, simplicial fan  $\Sigma$  in  $N$ ,*

$$Td^i X_\Sigma = \sum f(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \dots F_{\rho_k}^{a_k},$$

*the sum being taken over all tuples of rays  $\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}$  with each  $\rho_j \in \Sigma^{(1)}$  and  $a_1 + \dots + a_k = i$ .*

Here  $F_\rho$  represents the class in  $A^1 X_\Sigma$  of the invariant divisor corresponding to the ray  $\rho$ , and  $Td^i X_\Sigma$  represents the codimension- $i$  part of the Todd class as in [Ful].

The key ingredient in the proof of this theorem is the behavior of products of cycles under proper birational maps of toric varieties. Given such a map, we show that there is a canonical way to express the push-forward of a product of cycles in terms of products of cycles, with coefficients that depend only on local data in the corresponding fans. The following theorem gives an explicit formula for the push-forward of any product in the case that a single ray is added.

For a simplicial cone  $\sigma = \langle \rho_1, \dots, \rho_d \rangle$  of dimension  $d$ , we use the notation  $\text{mult } \sigma$  to denote the multiplicity of  $\sigma$ . This is the index of  $\mathbb{Z}\rho_1 + \dots + \mathbb{Z}\rho_d$  in the  $d$ -plane of  $N$  which contains  $\sigma$ .

**Theorem 2.** *Let  $\Sigma$  be a complete simplicial fan in a lattice  $N$  and let  $\rho_1, \dots, \rho_d$  be rays of  $\Sigma$  generating a  $d$ -dimensional cone  $\sigma \in \Sigma$ . Let  $\rho_0$  be a ray in the interior of  $\sigma$ , and let  $\Sigma'$  be the fan obtained from  $\Sigma$  by adding the ray  $\rho_0$ . Let  $\pi : X_{\Sigma'} \rightarrow X_\Sigma$  be the induced proper birational map of toric varieties. We use  $E_i \in A^1 X_{\Sigma'}$  and  $F_i \in A^1 X_\Sigma$  to denote the classes of the divisors corresponding to the rays  $\rho_i$ , with  $F_0 = 0$ . Let  $m_0 = \text{mult } \sigma$  and  $m_i = \text{mult } \langle \rho_0, \dots, \hat{\rho}_i, \dots, \rho_d \rangle$ . Then*

- (A) *For any integers  $r_0, \dots, r_d \geq 0$ , we have*

$$\frac{\pi_* E_0^{r_0} \dots E_d^{r_d}}{m_0^{r_0} \dots m_d^{r_d}} = \frac{F_0^{r_0} \dots F_d^{r_d}}{m_0^{r_0} \dots m_d^{r_d}} + \sum \frac{(-1)^{r_0+1} \binom{r_1-1}{t_1-1} \dots \binom{r_d-1}{t_d-1} F_1^{t_1} \dots F_d^{t_d}}{m_1^{t_1} \dots m_d^{t_d}},$$

*where the sum is taken over all positive integers  $t_1, \dots, t_d$  such that  $t_1 + \dots + t_d = r_0 + \dots + r_d$ . Here we take  $\binom{-1}{n} = (-1)^n$  when  $n \geq 0$ , and  $\binom{-1}{-1} = 0$ .*

- (B) *If  $\gamma \in \Sigma'^{(1)}$ , and  $\gamma \neq \rho_i$  ( $i = 0, \dots, d$ ), then for any  $p \in A^* X_{\Sigma'}$ ,*

$$\pi_*(E_\gamma \cdot p) = F_\gamma \cdot \pi_* p.$$

The idea behind the proof of Theorem 1 is as follows: We start with the well-known fact that for nonsingular  $X_\Sigma$ , the Chern classes are given by

$$c^i X_\Sigma = \sum_{\sigma \in \Sigma^{(i)}} F_\sigma$$

(cf. [Dan, p. 114]). Taking the Todd polynomials (as defined in [Hir]) in these Chern classes yields an expression for the Todd classes of a nonsingular toric variety in

terms of products:

$$(*) \quad Td^i X_\Sigma = \sum \lambda_{a_1} \cdots \lambda_{a_d} F_{\rho_1}^{a_1} \cdots F_{\rho_d}^{a_d},$$

where  $\lambda_i \in \mathbb{Q}$  is given by  $\frac{t}{1 - e^{-t}} = \sum_{i=0}^\infty \lambda_i t^i$ , and the sum is taken over all tuples  $\rho_1^{(a_1)}, \dots, \rho_d^{(a_d)}$  such that each  $\rho_i \in \Sigma^{(1)}$ , each  $a_i > 0$ , and  $a_1 + \cdots + a_n = i$ . For singular  $X_\Sigma$ , we may always find a sequence of subdivisions

$$\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_l = \Sigma_{\text{ns}},$$

where  $\Sigma_{\text{ns}}$  is nonsingular and each  $\Sigma_i$  is obtained from  $\Sigma_{i-1}$  by adding a single ray, as in Theorem 2. We then have maps

$$X_{\Sigma_{\text{ns}}} \longrightarrow \cdots \longrightarrow X_{\Sigma_1} \longrightarrow X_\Sigma.$$

We may therefore express  $Td X_\Sigma$  by taking the expression for  $Td X_{\Sigma_{\text{ns}}}$  in equation (\*) and pushing forward these products using the formulas of Theorem 2. It then remains only to check that the expression for  $Td X_\Sigma$  obtained in this way is independent of the resolution chosen.

Finally, we show that the coefficients  $f$  of Theorem 1 satisfy a certain reciprocity relation which characterizes them uniquely.

**Theorem 3.** *The function  $f$  of Theorem 1 is uniquely determined by the following properties:*

- (1) Define  $\lambda_i \in \mathbb{Q}$  by  $\frac{t}{1 - e^{-t}} = \sum_{i=0}^\infty \lambda_i t^i$ . Then for any rays  $\rho_1, \dots, \rho_d$  which form a nonsingular cone (that is, the primitive elements of the lattice  $N$  which lie on the rays  $\rho_1, \dots, \rho_d$  form part of a basis of  $N$ ), we have

$$f(\rho_1^{(a_1)}, \dots, \rho_d^{(a_d)}) = \lambda_{a_1} \cdots \lambda_{a_d}.$$

- (2) Let  $\rho_1, \dots, \rho_k$  be the rays of a  $k$ -dimensional simplicial cone in a lattice  $N$  and let  $\rho_0$  be a ray in the interior of  $\langle \rho_1, \dots, \rho_d \rangle$  ( $d \leq k$ ), as in Theorem 2. Let the  $m_i$  ( $i = 0, \dots, d$ ) be as in Theorem 2, and fix integers  $a_1, \dots, a_k \geq 1$ . Then

$$\sum (-1)^{b_0+1} \binom{b_1-1}{a_1-1} \cdots \binom{b_d-1}{a_d-1} m_0^{b_0} \cdots m_d^{b_d} f(\rho_0^{(b_0)}, \rho_1^{(b_1)}, \dots, \rho_d^{(b_d)}, \rho_{d+1}^{(a_{d+1})}, \dots, \rho_k^{(a_k)}) = 0,$$

where the sum is taken over all nonnegative integers  $b_0, \dots, b_d$  such that  $b_0 + \cdots + b_d = a_1 + \cdots + a_d$ .

The fact that the above properties characterize  $f$  follows from the fact that any fan may be desingularized by a sequence of subdivisions of the type involved in (2) of the above theorem (cf. [Dan]). The coefficients  $f$  for the cones of this desingularized fan are then given by (1). Thus the above theorem determines an algorithm for computing the coefficients of the Todd class which appear in Theorem 1.

In the final section of this paper, we use Theorem 3 to obtain an expression for the codimension three part of the Todd class of a simplicial toric variety in terms of the classical Dedekind sum. By the standard application of Riemann-Roch, one may obtain an expression for the degree  $n - 3$  term in the Ehrhart polynomial of a simple polytope in terms of Dedekind sums. Such a formula may also be

obtained from the codimension two formula of [Pom] via the inversion formula for the Ehrhart polynomial, a consequence of Serre duality for toric varieties (cf. [Dan, p.135]). Kantor and Khovanskii point this out in the context of their combinatorial Riemann-Roch theorem in [K-K]. Specifically, they indicate how one may obtain a formula for the degree  $n - 3$  term in the Ehrhart polynomial of a 4-dimensional polytope if one knows a formula for the degree  $n - 2$  coefficient.

The theorems of this paper are natural generalizations of the results in [Pom]. We make these connections explicit. Theorem 2 of [Pom] states that if  $d$  denotes the codimension of the singular locus of a complete simplicial toric variety  $X_\Sigma$ , then  $Td^d X_\Sigma$  may be expressed as the mock Todd class,  $TD^d X_\Sigma$ , which is simply the codimension  $d$  part of the formal product  $(*)$ , plus local contributions from the singular orbit closures of codimension  $d$ :

$$Td^d X_\Sigma = TD^d X_\Sigma + \sum_{\tau \in \Sigma^{(d)}} t(\tau) F_\tau,$$

where  $t(\tau)$  is an invariant of the cone  $\tau$ . The function  $t$  and this expression are special cases of the function  $f$  and Theorem 1. Indeed, one easily sees that

$$t(\tau) = \frac{1}{\text{mult } \tau} \left( f(\rho_1^{(1)}, \dots, \rho_d^{(1)}) - \frac{1}{2^d} \right),$$

where  $\tau = \langle \rho_1, \dots, \rho_d \rangle$  is any  $d$ -dimensional simplicial cone with nonsingular  $(d-1)$ -dimensional faces. The case  $d = 2$  is Theorem 3 of [Pom], which gives an expression for the codimension-two part of the Todd class of a toric variety. This theorem says exactly that  $f(\rho_1^{(1)}, \rho_2^{(1)})$  reduces to the classical Dedekind sum:

$$f(\rho_1^{(1)}, \rho_2^{(1)}) = q \left( s(p, q) + \frac{1}{4} \right),$$

whenever the cone  $\langle \rho_1, \rho_2 \rangle$  is lattice-equivalent to the cone  $\langle (1, 0), (p, q) \rangle$  in  $\mathbb{Z}^2$ . Furthermore, Theorem 3 of this paper, when applied in the case  $k = d = 2$  and  $a_1 = a_2 = 1$ , yields Theorem 7 of [Pom], which expresses the sum of two arbitrary Dedekind sums in terms of a single Dedekind sum. This relation is a generalization of Rademacher's three-term reciprocity relation for Dedekind sums [Ra].

Cappell and Shaneson [C-S] have announced an extension of the program of [Pom], in which they use facts about L-classes of singular spaces to obtain an explicit formula for the Todd class of a simplicial toric variety which can be seen to satisfy the conditions of Theorem 1. Showing that the expressions of [C-S] coincide with the Todd class formulas of this paper is equivalent to verifying that the exponential sums involved in the formula of Cappell-Shaneson satisfy the reciprocity relations of Theorem 3. This would be of considerable number-theoretic interest. For example, as noted above, even in codimension two, Theorem 3 leads to new non-trivial number theory. In higher dimensions, the formula of [C-S] predicts that the function  $t(\tau)$ , which is a very restricted version of the function  $f$ , coincides with Zagier's higher-dimensional Dedekind sums [Zag]. Zagier's reciprocity law for these sums is then equivalent to a very special case of Theorem 3. (Namely, taking  $k = d$ , all  $a_i = 1$ , letting  $\rho_1, \dots, \rho_d$  be the standard basis of  $\mathbb{Z}^d$ , and  $\rho_0 = (q_1, \dots, q_d)$  with  $\gcd(q_i, q_j) = 1$  yields Zagier's reciprocity relations.) Thus it seems reasonable to expect that Theorem 3 in full generality would lead to many interesting results about exponential sums.

It should also be pointed out that Theorem 1 may be interpreted as saying that there is a canonical lifting of the Todd class of a simplicial toric variety to the equivariant cohomology ring. The coefficients of this lifting may be computed in terms of local combinatorial data by using the relations of Theorem 3. In this context, Theorem 2 is a local formula for the push-forward maps on the equivariant cohomology of a simplicial toric variety.

### 2. BACKGROUND ON TORIC VARIETIES

For the necessary background on toric varieties, we refer the reader to [Pom, p. 4–5], which is a summary of basic facts which we will use freely without repeating here. These facts are collected from the survey article [Dan] and the book [Oda], which provide complete definitions and proofs. The reader will also find a very readable introduction to the subject of toric varieties in [Ful2].

It will be useful to point out a few pieces of notation that we will use throughout.

We will fix a lattice  $N$  of dimension  $n$ , and let  $M = \text{Hom}(N, \mathbb{Z})$  be the dual lattice with  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$  the natural pairing. If  $\Sigma$  is a fan in  $N$ ,  $X_\Sigma$  will be the associated toric variety, and for each cone  $\sigma \in \Sigma$ ,  $V(\sigma)$  will denote the closed subvariety of  $X_\Sigma$  corresponding to  $\sigma$ . We will use  $A^*X_\Sigma$  to denote the Chow ring of  $X_\Sigma$  with *rational* coefficients.  $F_\sigma$  will denote the class of  $V(\sigma)$  in  $A^*X_\Sigma$ .

If  $\Sigma'$  is a subdivision of  $\Sigma$ , with  $\pi : X_{\Sigma'} \rightarrow X_\Sigma$  the induced map of toric varieties, then whenever  $\sigma' \in \Sigma'$  and  $\sigma \in \Sigma$ , we will use  $E_{\sigma'}$  for the class of  $V(\sigma')$  in  $A^*X_{\Sigma'}$  and  $F_\sigma$  for the class of  $V(\sigma)$  in  $A^*X_\Sigma$ .

### 3. THE PUSH-FORWARD OF PRODUCTS

In this section we prove Theorem 2, which describes how to push a product of cycles forward under a proper birational map of toric varieties.

We will prove Part B first. Let  $\pi : X_{\Sigma'} \rightarrow X_\Sigma$  be as in the statement of the theorem, and let  $\gamma$  be distinct from  $\rho_0, \dots, \rho_d$ . We must show

$$\pi_*(E_\gamma p) = F_\gamma(\pi_* p)$$

for any  $p \in A^*X_{\Sigma'}$ .

Since  $\Sigma$  is simplicial, the function which takes the value 1 on  $\gamma$  and 0 on all other rays of  $\Sigma$  may be extended to a continuous piecewise linear function  $s$  on  $N$  with values in  $\mathbb{Q}$ . Some integral multiple of  $s$  takes all values in  $\mathbb{Z}$ , and hence defines a Cartier divisor on  $X_\Sigma$ . Hence, we may consider  $F_\gamma$  as an element of  $\text{Pic}(X_\Sigma) \otimes \mathbb{Q}$ . The pull-back  $\pi^*F_\gamma$  is obtained by considering  $s$  as a support function on  $\Sigma'$ . Since  $s$  vanishes on  $\rho_1, \dots, \rho_d$ , it vanishes also on  $\rho_0$  by linearity. Thus  $s$  vanishes at all rays of  $\Sigma'$  except  $\gamma$ , and we obtain

$$\pi^*F_\gamma = E_\gamma$$

in  $\text{Pic}(X_{\Sigma'}) \otimes \mathbb{Q}$ .

One now sees from the projection formula that

$$\pi_*(E_\gamma p) = F_\gamma(\pi_* p),$$

which proves Part B of Theorem 2.

To prove Part A, we will need the following lemma which allows us to compute push-forwards of products inductively:

**Lemma.** *With the notations of Theorem 2, let  $p$  be any element of  $A^*X_{\Sigma'}$ , and let  $1 \leq i, j \leq d$ , with  $i \neq j$ . Then*

$$(1) \quad \pi_*(E_i p) = \frac{m_i}{m_j} \pi_*(E_j p) + \left(F_i - \frac{m_i}{m_j} F_j\right) \pi_* p,$$

$$(2) \quad \pi_*(E_0 p) = -\frac{m_0}{m_j} \pi_*(E_j p) + \left(\frac{m_0}{m_j} F_j\right) \pi_* p.$$

*Proof.* Let  $1 \leq j \leq d$ . If  $s$  is the continuous piecewise linear support function on  $\Sigma$  which takes the value 1 at  $\rho_j$  and 0 at all other rays of  $\Sigma$ , then an easy lattice computation shows that

$$s(\rho_0) = \frac{m_j}{m_0}.$$

Hence it follows that

$$\pi^*(F_j) = E_j + \frac{m_j}{m_0} E_0.$$

As before, we regard this as an equation in  $\text{Pic}(X_\Sigma) \otimes \mathbb{Q}$ .

Now by the projection formula,

$$\pi_*\left(\left(E_j + \frac{m_j}{m_0} E_0\right)p\right) = F_j(\pi_* p),$$

from which Part (2) of the lemma follows.

We also see that

$$\begin{aligned} \pi^*\left(F_i - \frac{m_i}{m_j} F_j\right) &= \left(E_i + \frac{m_i}{m_0} E_0\right) - \frac{m_i}{m_j} \left(E_j + \frac{m_j}{m_0} E_0\right) \\ &= E_i - \frac{m_i}{m_j} E_j. \end{aligned}$$

Hence, by the projection formula,

$$\pi_*\left(\left(E_i - \frac{m_i}{m_j} E_j\right)p\right) = \left(F_i - \frac{m_i}{m_j} F_j\right) \pi_* p,$$

which yields Part (1) of the lemma.

We now prove Theorem 2, Part A, beginning with two remarks:

- (1) If all  $\rho_j > 0$ ,  $j = 1, \dots, d$ , then the left-hand side vanishes since  $\langle \rho_1, \dots, \rho_d \rangle \notin \Sigma'$ . Also, the sum on the right-hand side is easily seen to vanish — all terms are 0 unless  $\rho_0 = 0$ , in which case there are two cancelling terms. Thus in what follows, we may always assume that some  $\rho_j = 0$  with  $1 \leq j \leq d$ .
- (2) Another easy case is when  $e = \sum_{i=0}^d r_i < d$ . In this case, each product of binomial coefficients appearing on the right-hand side vanishes, and so the equation asserts that  $\pi_*$  is multiplicative. But this follows immediately from [Pom, p. 6]. The proof is an easy exercise in intersection theory if we remember that  $\pi$  is an isomorphism except above  $V(\sigma)$ .

Let  $e = \sum_{i=0}^d r_i$  be the degree of the monomial involved, and let  $k = \#\{i | r_i > 0, i = 0, \dots, d\}$ . As before we induct on  $e - k$ , which represents the number of coincidences in the monomial. We distinguish the cases  $r_0 = 0$  and  $r_0 > 0$ . They will be handled respectively by Parts (1) and (2) of the lemma.

Suppose first that  $r_0 = 0$ . By Remark 1, it follows that there exists  $j$  ( $j = 1, \dots, d$ ) such that  $r_j = 0$ . Thus  $k \leq d - 1$ . So if all  $r_i \leq 1$ , then  $e \leq d - 1$ , and we are done by Remark 2. Otherwise there exists some  $i$  ( $i = 1, \dots, d$ ) such that

$r_i > 1$ . Let  $p$  denote the product  $E_0^{r_0} \cdots E_d^{r_d}$  with the exponent of  $E_i^{r_i}$  diminished by one. Part (1) of the lemma tells us that

$$\pi_*(E_0^{r_0} \cdots E_d^{r_d}) = \pi_*(E_i p) = \frac{m_i}{m_j} \pi_*(E_j p) + \left(F_i - \frac{m_i}{m_j} F_j\right) \pi_* p.$$

Note however that both  $\pi_*(E_j p)$  and  $\pi_* p$  are known by induction. In this way we obtain an expression for  $\pi_*(E_0^{r_0} \cdots E_d^{r_d})$ , and a straightforward calculation shows that the expression so obtained matches the right-hand side of the theorem. We omit the calculation which requires nothing more than the identity  $\binom{p}{q} = \binom{p-1}{q} + \binom{p-1}{q-1}$ .

Now assume  $r_0 > 0$ . As before, choose  $j$  such that  $r_j = 0$ . Part (2) of the lemma then reduces the computation of  $\pi_*(E_0^{r_0} \cdots E_d^{r_d})$  to that of  $\pi_*(E_0^{r_0-1} \cdots E_d^{r_d})$  and  $\pi_*(E_0^{r_0-1} \cdots E_d^{r_d} \cdot E_j)$ . Again we omit the computation.

#### 4. AN EXPRESSION FOR THE TODD CLASS

In this section, we prove Theorem 1, which gives a canonical expression for the Todd class of a simplicial toric variety in terms of products of torus-invariant divisors. We also see that the coefficients in this expression satisfy the reciprocity relation of Theorem 3.

To prove Theorem 1, we follow the sketch outlined in the introduction. We begin with a complete simplicial fan  $\Sigma$ , and let

$$\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_l = \Sigma_{\text{ns}}$$

be a sequence of fans, where  $\Sigma_{\text{ns}}$  is nonsingular and each  $\Sigma_i$  is obtained from  $\Sigma_{i-1}$  by adding a single ray, as in Theorem 2. For each tuple  $\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}$  of rays of  $\Sigma$  with multiplicities  $a_i \geq 0$  we define a rational number  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  as follows:

First, we always take  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = 0$  if those  $\rho_i$  such that  $a_i > 0$  do not form a cone of  $\Sigma$ .

If  $l = 0$ , so that  $\Sigma = \Sigma_{\text{ns}}$  is nonsingular, then set

$$f_{\Sigma}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = \lambda_{a_1} \cdots \lambda_{a_k}.$$

Otherwise, we define  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}$  inductively in terms of  $f_{\Sigma_1, \dots, \Sigma_{\text{ns}}}$ . To fix notation, suppose that  $\Sigma_1$  is obtained from  $\Sigma$  by adding the ray  $\rho_0$  in the interior of the cone  $\langle \rho_1, \dots, \rho_d \rangle \in \Sigma$ , and let  $m_0, \dots, m_d$  be the multiplicities as in Theorem 2. Let  $\rho_{d+1}, \dots, \rho_k \in \Sigma^{(1)}$  be distinct rays, all distinct from  $\rho_1, \dots, \rho_d$ , and let  $a_1, \dots, a_k \geq 0$ . If  $a_i = 0$  for any  $i = 1, \dots, d$ , then set

$$f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f_{\Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}).$$

Otherwise we define  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  to equal

$$(*) \quad \sum \frac{1}{m_1^{a_1} \cdots m_d^{a_d}} (-1)^{b_0+1} \binom{b_1-1}{a_1-1} \cdots \binom{b_d-1}{a_d-1} m_0^{b_0} \cdots m_d^{b_d} \\ \times f_{\Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_0^{(b_0)}, \rho_1^{(b_1)}, \dots, \rho_d^{(b_d)}, \rho_{d+1}^{(a_{d+1})}, \dots, \rho_k^{(a_k)}),$$

where the sum is taken over all nonnegative integers  $b_0, \dots, b_d$  such that  $b_0 + \cdots + b_d = a_1 + \cdots + a_d$ .

It then follows from Theorem 2 that for the map  $\pi : X_{\Sigma_1} \longrightarrow X_{\Sigma}$ ,

$$\begin{aligned} \pi_* \left( \sum f_{\Sigma_1, \dots, \Sigma_{\text{ns}}}(\beta_1^{(a_1)}, \dots, \beta_r^{(a_r)}) E_{\beta_1}^{a_1} \cdots E_{\beta_r}^{a_r} \right) \\ = \sum f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r}, \end{aligned}$$

where the sums are taken over all positive integers  $r$ , all rays  $\beta_1, \dots, \beta_r$  of  $\Sigma_1$ , all rays  $\rho_1, \dots, \rho_r$  of  $\Sigma$ , and all positive integers  $a_i$ . Hence it is clear by induction that

$$Td X_{\Sigma} = \sum f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r}.$$

We must show that  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  is independent of the chosen sequence of subdivisions  $\Sigma_1, \dots, \Sigma_{\text{ns}}$ . First we show that  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  depends only on  $\Sigma$ ,  $\Sigma_{\text{ns}}$ , and  $\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}$ . That is, it does not depend on the intermediate subdivisions  $\Sigma_1, \dots, \Sigma_{l-1}$ . To do this, let  $\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}$  be another sequence of subdivisions ending in the same nonsingular fan  $\Sigma_{\text{ns}}$ . Let  $\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}$  be any tuple of rays of  $\Sigma$ . We will show by induction on the number of nonzero exponents  $a_i$  that

$$f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f_{\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}).$$

Observe that by definition these numbers depend only on those cones of the subdivisions  $\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}$  and  $\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}$  which lie within the cone  $\sigma = \langle \rho_1, \dots, \rho_k \rangle$ . Thus in the sequence of fans  $\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}$ , let us skip every step in which the added ray lies outside  $\sigma$ . This determines a new sequence of fans  $\Sigma, \Gamma_1, \dots, \Gamma$ , where at each stage the single ray added is contained in  $\sigma$ . By construction  $\Gamma$  and  $\Sigma_{\text{ns}}$  agree within  $\sigma$ . Thus while  $\Gamma$  may be singular outside  $\sigma$ , all cones of  $\Gamma$  which are contained in  $\sigma$  are nonsingular. So we may choose a desingularization  $\Gamma, \Delta_1, \dots, \Delta_{\text{ns}}$  where at each step a single ray is added *outside*  $\sigma$ .

Similarly, by omitting the addition of rays outside  $\sigma$ , the sequence  $\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}$  determines a sequence  $\Sigma, \Gamma'_1, \dots, \Gamma$ , with all added rays contained in  $\sigma$ . It is clear that this sequence ends in the same fan  $\Gamma$ .

We then have

$$\begin{aligned} (**) \quad f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) &= f_{\Sigma, \Gamma_1, \dots, \Gamma, \Delta_1, \dots, \Delta_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) \\ \text{and } f_{\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) &= f_{\Sigma, \Gamma'_1, \dots, \Gamma, \Delta_1, \dots, \Delta_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}). \end{aligned}$$

Now consider

$$\alpha = \sum f_{\Gamma, \Delta_1, \dots, \Delta_{\text{ns}}}(\beta_1^{(a_1)}, \dots, \beta_r^{(a_r)}) E_{\beta_1}^{a_1} \cdots E_{\beta_r}^{a_r} \in A^* X_{\Gamma}$$

summed over all tuples of rays of  $\Gamma$  with each  $\beta_i \subset \sigma$  and multiplicities  $a_i \geq 0$ .

Pushing forward these products under the map  $\pi : X_{\Gamma} \longrightarrow X_{\Sigma}$ , it follows from Theorem 2 that

$$\begin{aligned} \pi_*(\alpha) &= \sum f_{\Sigma, \Gamma_1, \dots, \Gamma, \Delta_1, \dots, \Delta_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r} \\ &= \sum f_{\Sigma, \Gamma'_1, \dots, \Gamma, \Delta_1, \dots, \Delta_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r}, \end{aligned}$$

where the sums are taken over all  $a_i \geq 0$ . Note that each term we get downstairs involves only the rays of  $\sigma$  since all rays upstairs are contained in  $\sigma$  and all added rays in the sequences  $\Sigma, \Gamma_1, \dots, \Gamma$  and  $\Sigma, \Gamma'_1, \dots, \Gamma$  also lie within  $\sigma$ .



In the above equation, those terms for which some  $a_i = 0$  cancel by induction. In light of (\*\*), we thus obtain

$$\begin{aligned} \sum f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r} \\ = \sum f_{\Sigma, \Sigma'_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) F_{\rho_1}^{a_1} \cdots F_{\rho_r}^{a_r}, \end{aligned}$$

where this time the sum is taken over all  $a_i > 0$ .

Once again taking into account that the computation of the coefficients above depends only on the parts of the above fans lying in  $\sigma$ , the following proposition implies that coefficients on the right-hand side match those on the left:

**Proposition.** *Let  $\sigma = \langle \rho_1, \dots, \rho_d \rangle$  be a  $d$ -dimensional simplicial cone in an  $n$ -dimensional lattice  $N$ . Let  $P$  be a homogeneous polynomial of degree  $n$  in the rays  $\rho_1, \dots, \rho_d$  which is divisible by  $\rho_i$  for all  $i = 1, \dots, d$ . Suppose that for any complete, simplicial fan  $\Sigma$  in  $N$  which contains the cone  $\sigma$ , the polynomial  $P$  represents 0 in  $A^*X_\Sigma$ . Then  $P = 0$ .*

The proof amounts to constructing a collection of toric varieties which separate polynomials in the rays of  $\sigma$ . This is done in the Appendix.

We have shown to this point that  $f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  is independent of the intermediate subdivisions and depends only on the final subdivision  $\Sigma_{\text{ns}}$ . This means that for each tuple  $\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}$  of rays of  $\Sigma$  and each nonsingular subdivision  $\Sigma_{\text{ns}}$ , there is a number  $f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  which may be defined by

$$f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}),$$

where the intermediate subdivisions are chosen arbitrarily. We will now finish the proof of Theorem 1 by showing that  $f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  is independent of  $\Sigma_{\text{ns}}$ .

Suppose that  $\Sigma'_{\text{ns}}$  is another nonsingular subdivision of  $\Sigma$ . We aim to show that

$$f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f(\Sigma'_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}).$$

To do this, we invoke the following result of R. Morelli:

**Theorem** (Morelli, [Mor2]). *If  $\Delta$  and  $\Delta'$  are two nonsingular fans with the same support, then there exists a sequence of smooth toric varieties*

$$X_\Delta = X_{\Delta_0}, X_{\Delta_1}, \dots, X_{\Delta_l} = X_{\Delta'}$$

where each  $X_{\Delta_{i+1}}$  is obtained from  $X_{\Delta_i}$  by an equivariant blow-up or blow-down along an orbit closure.

Since the computation of  $f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  and  $f(\Sigma'_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$  depends only on the portions of  $\Sigma_{\text{ns}}$  and  $\Sigma'_{\text{ns}}$  which lie within the cone  $\sigma = \langle \rho_1, \dots, \rho_d \rangle$ , we consider the fans  $\Delta = \Sigma_{\text{ns}} \cap \sigma$  and  $\Delta' = \Sigma'_{\text{ns}} \cap \sigma$ . By Morelli's theorem applied to these two fans, it suffices to check that

$$f(\Sigma_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f(\Sigma'_{\text{ns}}; \rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$$

in the case where  $\Sigma_{\text{ns}}$  is obtained from  $\Sigma'_{\text{ns}}$  by the addition of a single ray. By independence of the intermediate subdivisions it suffices to show

$$f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f_{\Sigma, \Sigma_1, \dots, \Sigma_{\text{ns}}, \Sigma'_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}).$$

Because the right-hand side is ultimately computed in terms of the values of  $f_{\Sigma_{\text{ns}}, \Sigma'_{\text{ns}}}$ , and the left-hand side in terms of the values of  $f_{\Sigma_{\text{ns}}}$ , it is enough to show that

$$f_{\Sigma_{\text{ns}}, \Sigma'_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)}) = f_{\Sigma_{\text{ns}}}(\rho_1^{(a_1)}, \dots, \rho_k^{(a_k)})$$

for any cone  $\sigma = \langle \rho_1, \dots, \rho_r \rangle \in \Sigma_{\text{ns}}$ .

Looking at equation (\*), and using the definition of  $f$  for nonsingular cones, we are reduced to the following identity:

**Lemma.** *Given integers  $a_1, \dots, a_d > 0$ ,*

$$\sum (-1)^{b_0+1} \binom{b_1-1}{a_1-1} \cdots \binom{b_d-1}{a_d-1} \lambda_{b_0} \cdots \lambda_{b_d} = 0,$$

where the sum is taken over all  $b_0, \dots, b_d \geq 0$  such that  $b_0 + \cdots + b_d = a_0 + \cdots + a_d$ .

*Proof.* The expression above equals the coefficient of  $t^{a_1+\cdots+a_d}$  in

$$\left( \sum_{b_0=0}^{\infty} (-1)^{b_0+1} \lambda_{b_0} t^{b_0} \right) \left( \sum_{b_1=0}^{\infty} \binom{b_1-1}{a_1-1} \lambda_{b_1} t^{b_1} \right) \cdots \left( \sum_{b_d=0}^{\infty} \binom{b_d-1}{a_d-1} \lambda_{b_d} t^{b_d} \right).$$

Setting  $h(t) = \frac{1}{1-e^{-t}} = \sum_{b=0}^{\infty} \lambda_b t^{b-1}$ , one easily calculates the derivatives

$$h^{(a-1)}(t) = \frac{(a-1)!}{t^a} \left( \sum_{b=0}^{\infty} \binom{b-1}{a-1} \lambda_b t^b \right),$$

so that we are reduced to finding the coefficient of  $t^{a_1+\cdots+a_d}$  in

$$\left( \frac{t}{1-e^t} \right) \left( \frac{t^{a_1}}{(a_1-1)!} h^{(a_1-1)}(t) \right) \cdots \left( \frac{t^{a_d}}{(a_d-1)!} h^{(a_d-1)}(t) \right).$$

By Cauchy’s Integral Formula, the desired coefficient  $C$  satisfies

$$(a_1-1)! \cdots (a_d-1)! \cdot C = \frac{1}{2\pi i} \int \frac{1}{1-e^t} h^{(a_1-1)}(t) \cdots h^{(a_d-1)}(t) dt,$$

where we integrate around a circle centered at the origin. With the substitution  $u = 1 - e^{-t}$ , the integral becomes

$$- \int u^{-1} \frac{d^{(a_1-1)}}{dt^{(a_1-1)}}(u^{-1}) \cdots \frac{d^{(a_d-1)}}{dt^{(a_d-1)}}(u^{-1}) du.$$

However, one easily verifies that  $\frac{d^{(a-1)}}{dt^{(a-1)}}(u^{-1})$ , when expressed in terms of  $u$ , contains only terms with negative exponents. Hence, the integrand contains no terms of exponent greater than  $-(d+1)$ . In particular, there is no  $u^{-1}$  term, so the integral vanishes, and the lemma is established.

### 5. CODIMENSION THREE

In this section, we illustrate Theorem 3 by using it to give a formula for the codimension three part of the Todd class of a toric variety.

If  $\tau$  is a two-dimensional cone in a lattice, then  $\tau$  determines a Dedekind sum as follows [Pom, p. 12]: For some relatively prime integers  $p$  and  $q$ ,  $\tau$  is lattice-equivalent to the cone  $\langle (1, 0), (p, q) \rangle$  in  $\mathbb{Z}^2$ . We define the Dedekind sum  $s(\tau)$  associated to  $\tau$  as the classical Dedekind sum  $s(p, q)$ . The following formula determines

the codimension-two part of the Todd class of a toric variety:

$$f(\rho_1^{(1)}, \rho_2^{(1)}) = \text{mult}(\langle \rho_1, \rho_2 \rangle) \left( s(\langle \rho_1, \rho_2 \rangle) + \frac{1}{4} \right).$$

This is a restatement of [Pom, Theorem 3].

Now let  $\sigma = \langle \rho_1, \rho_2, \rho_3 \rangle$  be a three-dimensional simplicial cone in a lattice  $N$ . For  $\{i, j, k\} = \{1, 2, 3\}$ , let  $s_{jk}^i$  be the Dedekind sum associated to the two-dimensional cone  $\sigma_{jk}^i$  which is the image of  $\sigma$  in the lattice  $N/\mathbb{Z}\rho_i$ .

**Theorem 4.** *The codimension-three part of the Todd class of a simplicial toric variety is given by the following formulas:*

- (1)  $f(\rho^{(3)}) = 0,$
- (2)  $f(\rho_1^{(2)}, \rho_2) = \frac{1}{24} \left[ \text{mult} \langle \rho_1, \rho_2 \rangle \right]^2,$
- (3)  $f(\rho_1, \rho_2, \rho_3) = \text{mult} \langle \rho_1, \rho_2, \rho_3 \rangle \left( \frac{1}{8} + \frac{s_{23}^1 + s_{13}^2 + s_{12}^3}{2} \right).$

*Proof.* (1) is clear from Theorem 3, Part A as  $\lambda_3 = 0$ .

For the proof of (2), we use the formula of Theorem 3, Part B and induction on  $\text{mult} \langle \rho_1, \rho_2 \rangle$ . Choose any ray  $\rho_0$  in the interior of  $\langle \rho_1, \rho_2 \rangle$  which subdivides this cone into two cones of smaller multiplicities. This is possible by standard arguments (cf. [Dan, p. 123]). With  $k = d = 2$ ,  $a_1 = 2$ , and  $a_2 = 1$ , Theorem 3 reads:

$$m_0^2 m_2 f(\rho_0^{(2)}, \rho_2) + m_0 m_1^2 f(\rho_0, \rho_1^{(2)}) - m_0 m_2^2 f(\rho_0, \rho_2^{(2)}) - m_1^2 m_2 f(\rho_1^{(2)}, \rho_2) = 0.$$

By induction,  $f(\rho_0, \rho_1^{(2)}) = \frac{1}{24} m_2^2$ , and  $f(\rho_0^{(2)}, \rho_2) = f(\rho_0, \rho_2^{(2)}) = \frac{1}{24} m_1^2$ . Thus we obtain  $f(\rho_1^{(2)}, \rho_2) = \frac{1}{24} m_0^2$ , as desired.

To prove (3), again we subdivide using a ray  $\rho_0 \in \langle \rho_1, \rho_2, \rho_3 \rangle$  with the property that the subdivided cones have multiplicities smaller than  $\text{mult} \langle \rho_1, \rho_2, \rho_3 \rangle$ . This time, however, such a  $\rho_0$  may or may not lie in the interior of  $\langle \rho_1, \rho_2, \rho_3 \rangle$ . We will distinguish these two possibilities. We will use  $\mu_{i\dots j}$  to abbreviate  $\text{mult} \langle \rho_i, \dots, \rho_j \rangle$ , and  $M_{jk}^i$  will denote the multiplicity of the two-dimensional cone  $\sigma_{jk}^i$  which is the image of  $\langle \rho_i, \rho_j, \rho_k \rangle$  in the lattice  $N/\mathbb{Z}\rho_i$ .

First assume  $\rho_0 \in \langle \rho_1, \rho_2 \rangle$ . Theorem 3 with  $k = 3$ ,  $d = 2$ , and  $a_1 = a_2 = a_3 = 1$  says

$$\frac{f(\rho_1, \rho_2, \rho_3)}{m_0} = \frac{f(\rho_0, \rho_1, \rho_3)}{m_2} + \frac{f(\rho_0, \rho_2, \rho_3)}{m_1} - \frac{m_0^2 \mu_{03}^2 + m_1^2 \mu_{13}^2 + m_2^2 \mu_{23}^2}{24 m_0 m_1 m_2},$$

and by induction,

$$f(\rho_0, \rho_1, \rho_3) = m_2 \left( \frac{1}{8} + \frac{s_{13}^0 + s_{03}^1 + s_{01}^3}{2} \right), \quad \text{and}$$

$$f(\rho_0, \rho_2, \rho_3) = m_1 \left( \frac{1}{8} + \frac{s_{23}^0 + s_{03}^2 + s_{02}^3}{2} \right).$$

However, by [Pom, Theorem 7],

$$s_{01}^3 + s_{02}^3 = s_{12}^3 - \frac{1}{4} + \frac{1}{12} \left[ \frac{M_{01}^3}{M_{02}^3 M_{12}^3} + \frac{M_{02}^3}{M_{01}^3 M_{12}^3} + \frac{M_{12}^3}{M_{01}^3 M_{02}^3} \right],$$

$s_{13}^0 + s_{23}^0 = 0$  since  $s(-p, q) = -s(p, q)$ , and  $s_{03}^1 = s_{23}^1, s_{03}^2 = s_{13}^2$ . Taking into account the easily-verified multiplicity relation  $M_{jk}^i = \frac{\mu_{ijk}}{\mu_{ik}\mu_{jk}}$ , the desired expression

for  $f(\rho_1, \rho_2, \rho_3)$  now follows.

Next we assume that  $\rho_0$  is in the interior of  $\langle \rho_1, \rho_2, \rho_3 \rangle$ . This time we take  $k = d = 3$  and  $a_1 = a_2 = a_3 = 1$  in Theorem 3, and obtain

$$\begin{aligned} \frac{f(\rho_1, \rho_2, \rho_3)}{m_0} &= \frac{f(\rho_0, \rho_1, \rho_2)}{m_3} + \frac{f(\rho_0, \rho_1, \rho_3)}{m_2} + \frac{f(\rho_0, \rho_2, \rho_3)}{m_1} \\ &+ \frac{1}{24m_0m_1m_2m_3} \left( \sum_{i=1}^3 \mu_{0i}m_0m_i(m_i - m_0) - \sum_{1 \leq i < j \leq 3} \mu_{ij}m_im_j(m_i + m_j) \right). \end{aligned}$$

Again the right-hand side is known completely by induction, and all follows from the equations

$$\begin{aligned} s_{0i}^k + s_{0j}^k &= s_{ij}^k - \frac{1}{4} + \frac{1}{12} \left[ \frac{M_{0i}^k}{M_{0j}^k M_{ij}^k} + \frac{M_{0j}^k}{M_{0i}^k M_{ij}^k} + \frac{M_{ij}^k}{M_{0i}^k M_{0j}^k} \right], \\ s_{12}^0 + s_{13}^0 + s_{23}^0 &= \frac{1}{4} - \frac{1}{12} \left[ \frac{M_{12}^0}{M_{13}^0 M_{23}^0} + \frac{M_{13}^0}{M_{12}^0 M_{23}^0} + \frac{M_{23}^0}{M_{12}^0 M_{13}^0} \right] \end{aligned}$$

(where  $\{i, j, k\} = \{1, 2, 3\}$ ), which are consequences of Theorems 7 and 8, respectively, in [Pom].

APPENDIX: PROOF OF THE PROPOSITION

*Proof.* It suffices to show that given a monomial  $\rho_1^{r_1} \dots \rho_d^{r_d}$  of degree  $n$  with all  $r_i > 0$ , there exists a fan  $\Sigma$  such that the given monomial represents a nonzero element of  $A^*X_\Sigma$ , but any other monomial of degree  $n$  represents 0 in  $A^*X_\Sigma$ . We construct such a fan.

Let  $e_1, \dots, e_d$  be a basis of the plane containing  $\sigma$  and extend this to a basis  $e_1, \dots, e_n$  of  $N$ . Let  $\gamma_i, i = d + 1, \dots, n$ , be defined by

$$(\gamma_{d+1}, \dots, \gamma_n) = (\rho_1, \dots, \rho_1, \dots, \rho_d, \dots, \rho_d),$$

where each  $\rho_i$  occurs  $r_i - 1$  times. We then define rays  $\rho_i = e_i + \gamma_i$  and  $\rho'_i = -e_i$  for  $i = d + 1, \dots, n$ . Let  $\Sigma$  be any complete simplicial fan containing the cones  $\langle \rho_1, \dots, \rho_n \rangle$  and  $\langle \rho_1, \dots, \rho_d, \rho'_{d+1}, \dots, \rho'_n \rangle$ . For a ray  $\rho \in \Sigma^{(1)}$ , we set  $F_\rho = [V(\rho)] \in A^1X_\Sigma$  and set  $F_i = F_{\rho_i}$  and  $F'_i = F_{\rho'_i}$  for short.

We must show that for integers  $a_1, \dots, a_d > 0$  whose sum is  $n$ ,

$$F_1^{a_1} \dots F_d^{a_d} \neq 0 \in A^*X_\Sigma \quad \text{iff} \quad a_i = r_i \text{ for all } i = 1, \dots, d.$$

**Claim.** *If  $\rho$  is any ray other than some  $\rho_i$  or  $\rho'_i$ , then  $\langle \rho_1, \dots, \rho_d, \rho \rangle \notin \Sigma$ .*

*Proof.* We may write

$$\rho = a_1\rho_1 + \dots + a_d\rho_d + a_{d+1}\beta_{d+1} + \dots + a_n\beta_n,$$

where for all  $i > d$ ,  $\beta_i = \rho_i$  or  $\rho'_i$ , and  $a_i \geq 0$  for all  $i > d$ . It follows that for sufficiently large  $M > 0$  ( $M > |a_i|, i = 1, \dots, d$ ),

$$\rho + M(\rho_1 + \dots + \rho_d) \in \langle \rho_1, \dots, \rho_d, \beta_{d+1}, \dots, \beta_n \rangle.$$

But it is clear that

$$\rho + M(\rho_1 + \dots + \rho_d) \in \langle \rho_1, \dots, \rho_d, \rho \rangle.$$

Under the assumption that  $\langle \rho_1, \dots, \rho_d, \rho \rangle \in \Sigma$ , the intersection of this cone with  $\langle \rho_1, \dots, \rho_d, \beta_{d+1}, \dots, \beta_n \rangle$  is their common face  $\langle \rho_1, \dots, \rho_d \rangle = \sigma$ . We conclude that  $\rho + M(\rho_1 + \dots + \rho_d) \in \sigma$ , so  $\rho$  lies in the plane containing  $\sigma$ . Hence the dimension of  $\langle \rho_1, \dots, \rho_d, \rho \rangle$  is at most  $d$  and so this cone cannot belong to the simplicial fan  $\Sigma$ . This proves the claim.

By the claim, it follows that whenever  $\rho$  is any ray other than some  $\rho_i$  or  $\rho'_i$ ,  $F_1 \cdots F_d F_\rho = 0$ . And since our given product is divisible by  $F_1 \cdots F_d$ , we can do all our computation in  $A^* X_\Sigma / (F_\rho : \rho \neq \rho_i, \rho'_i)$ .

Let  $f_1, \dots, f_n$  be the basis of  $M = \text{Hom}(N, \mathbb{Z})$  dual to  $e_1, \dots, e_n$ . For  $i = 1, \dots, d$ , let  $m_i$  be an element of the  $d$ -plane spanned by  $f_1, \dots, f_d$  satisfying for  $j = 1, \dots, n$ :

$$\langle m_i, \rho_i \rangle \neq 0, \text{ but } \langle m_i, \rho_j \rangle = 0 \text{ for any other } j \in \{1, \dots, d\}.$$

One then checks that if  $d + 1 \leq j \leq n$ , then  $\langle m_i, \rho_j \rangle = \langle m_i, \rho_i \rangle$  whenever  $1 \leq i \leq d$  and  $\gamma_j = \rho_i$ . Also,  $\langle m_i, \rho'_j \rangle = 0$  for all  $1 \leq i \leq d$  and  $d + 1 \leq j \leq n$ .

We use the  $m_i$  to obtain the following linear relations for  $i = 1, \dots, d$ :

$$F_i = - \sum_{j \in S_i} F_j,$$

where the sum is taken over the set  $S_i$  of those  $j$  ( $d + 1 \leq j \leq n$ ) such that  $\gamma_j = \rho_i$ . There are  $r_i - 1$  such  $j$ .

Also, using  $m = f_i$ ,  $i = d + 1, \dots, n$ , we obtain the linear relation

$$F_i = F'_i.$$

Furthermore, since  $\langle \rho_i, \rho'_i \rangle \notin \Sigma$ ,  $F_i F'_i = 0$ , and hence for all  $i = d + 1, \dots, n$ ,

$$F_i^2 = 0.$$

It is now quite easy to evaluate the product:

$$F_1^{a_1} \cdots F_d^{a_d} = F_1 \cdots F_d \left( - \sum_{j \in S_1} F_j \right)^{a_1-1} \cdots \left( - \sum_{j \in S_d} F_j \right)^{a_d-1}.$$

Now if  $(a_1, \dots, a_d) \neq (r_1, \dots, r_d)$ , then for some  $i$ ,  $a_i > r_i$ , in which case the factor  $\left( - \sum_{j \in S_i} F_j \right)^{a_i-1}$ , when expanded, contains no squarefree terms. Hence the whole product vanishes.

On the other hand, if  $(a_1, \dots, a_d) = (r_1, \dots, r_d)$ , then the product simplifies to

$$(-1)^{n-d} (e_1 - 1)! \cdots (e_d - 1)! F_1 \cdots F_n,$$

which is nonzero since  $F_1 \cdots F_n = \frac{1}{\text{mult } \langle \rho_1, \dots, \rho_n \rangle} [\text{pt}]$ .

This completes the proof of the proposition.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720  
*E-mail address*: jamie@math.berkeley.edu