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SUPPORT VARIETIES FOR INFINITESIMAL GROUP SCHEMES

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The representation theory of a connected smooth affine group scheme over a field k of characteristic $p > 0$ is faithfully captured by that of its family of Frobenius kernels. Such Frobenius kernels are examples of infinitesimal group schemes, affine group schemes G whose coordinate (Hopf) algebra $k[G]$ is a finite-dimensional local k-algebra. This paper presents a study of the cohomology algebra $H^*(G,k)$ of an arbitrary infinitesimal group scheme over k.

We provide a geometric determination of the "cohomological support variety" $|G| \equiv \mathit{Spec}\ H^{ev}(G,k)$ analogous to that given by D. Quillen for the cohomology of finite groups [Q]. We further study finite-dimensional rational G-modules M for arbitrary infinitesimal group schemes G over k. In a manner initiated by J. Alperin and L. Evens [A-E] and J. Carlson [C1] for finite groups, we consider the variety $|G|_M \subset |G|$ of the ideal $I_M = \mathit{ker}\{H^{ev}(G,k) \to \mathit{Ext}^*_G(M,M)\}$ and provide a geometric description of this variety which is analogous to that given by G. Avrunin and L. Scott for finite-dimensional modules for finite groups [A-S].

This paper is a continuation of our recent work establishing the finite generation of $H^*(G,k)$ [F-S] and investigating the infinitesimal 1-parameter subgroups of G [S-F-B]. Earlier work of E. Friedlander and B. Parshall [FP1], [FP2], [FP3], [FP4] and J. Jantzen [J1] concerning the cohomology of restricted Lie algebras are forerunners of the results presented here: finite-dimensional restricted Lie algebras are in 1-1 correspondence with infinitesimal group schemes of height ≤ 1. Our main theorems (Theorems 5.2 and 6.7 below) when restricted to infinitesimal group schemes of height ≤ 1 significantly strengthen previously known cohomological information for restricted Lie algebras.

An interesting aspect of our work is the extent to which infinitesimal 1-parameter subgroups $\nu : \mathbb{G}_{a(r)} \to G$ for infinitesimal group schemes G of height $\leq r$ play the role of elementary abelian p-subgroups (and their generalizations, shifted subgroups) for finite groups. Indeed, much of our effort is dedicated to proving that cohomology classes are detected (modulo nilpotence) by such 1-parameter subgroups. This is first done in §2 for unipotent infinitesimal group schemes, using an induction argument made possible by a structure theorem presented in §1. This structure theorem is the analogue in our context of a theorem of J.-P. Serre characterizing elementary abelian p-groups [S].

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The proof of the detection theorem for arbitrary infinitesimal group schemes over k relies upon a generalization of a spectral sequence introduced by H. Andersen and J. Jantzen [A-J] which presents the cohomology of an infinitesimal kernel $G_{(r)}$ of a reductive algebraic group in terms of the cohomology of the infinitesimal kernel of a Borel subgroup. Our generalized spectral sequence is presented in §3, enabling the proof in §4 of the general detection theorem (Theorem 4.3).

The detection theorem demonstrates the essential injectivity of the natural map considered in [S-F-B]

$$\psi : H^{ev}(G,k) \to k[V_r(G)],$$

where $V_r(G)$ is the scheme of infinitesimal 1-parameter subgroup schemes $\nu : \mathbb{G}_{a(r)} \to G$ of an infinitesimal group scheme G over k of height $\leq r$. The essential surjectivity of ψ (more precisely, surjectivity onto p^r-th powers) is a main result of [S-F-B]. This is formalized in Theorem 5.2 which presents a geometric, noncohomological description of the cohomological support variety $|G|$ of G. Corollary 6.8 gives a similarly geometric, noncohomological identification of $|G|_M \subset |G|$ for any finite-dimensional rational G-module M. We conclude in §7 with a few applications of these descriptions, applications analogous to results obtained previously for the cohomology of finite groups.

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§1. 1-parameter cohomomorphisms

In this section, we investigate properties associated to homomorphisms $G \to \mathbb{G}_{a(s)}$, homomorphisms which we call 1-parameter cohomomorphisms. Even though the main theorems of this paper concern the interpretation of the cohomology of an (infinitesimal) group scheme G in terms of 1-parameter homomorphisms $\mathbb{G}_{a(r)} \to G$, such 1-parameter cohomomorphisms arise in our inductive analysis of a unipotent group scheme G.

We remind the reader that an affine group scheme is said to be unipotent if it admits an embedding as a closed subgroup scheme of some U_N, the group scheme of strictly upper triangular matrices of GL_N.

Lemma 1.1. *Let G be an affine group scheme over k.*

(a) *$H^1(G,k) = Hom_{Gr/k}(G, \mathbb{G}_a)$ (as abelian groups).*

(b) *If G is infinitesimal of height $\leq r$, then $H^1(G,k) = Hom_{Gr/k}(G, \mathbb{G}_{a(r)})$.*

(c) *$Hom_{Gr/k}(G, \mathbb{G}_{a(1)}) = ker\{F : H^1(G,k) \to H^1(G,k)\}$, where F is induced by the Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$ using the identification of* (a).

(d) *If G is a nontrivial unipotent group scheme, then $H^1(G,k) \neq 0$.*

Proof. Part (a) can be seen directly from the standard cobar resolution used to compute $H^*(G,k)$. Part (b) then follows formally from (a). Part (c) follows from the short exact sequence $0 \to \mathbb{G}_{a(1)} \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0$. Finally, part (d) is well known (see, for example, [D-G], IV.2.2.1) and can easily be proved as follows. Embed G as a closed subgroup of some $U_N \subset GL_N$. Denote by $X^{i,j}$ $(1 \leq i,j \leq N)$ the standard coordinate functions on GL_N. Find an integer $n > 0$ such that all functions $X^{i,j}$ with $0 < j-i < n$ vanish on G and such that there exists some X^{i_0,j_0} with $j_0 - i_0 = n$ which is nontrivial on G. Then $X^{i_0,j_0} : G \to \mathbb{G}_a$ is a nontrivial group homomorphism. □

We shall employ the following familiar result.

Lemma 1.2. *Let $\psi : G \to H$ be a surjective homomorphism of unipotent group schemes. If $\psi^* : H^1(H,k) \to H^1(G,k)$ is an isomorphism and if $\psi^* : H^2(H,k) \to H^2(G,k)$ is injective, then ψ is an isomorphism.*

Proof. The Hochschild-Serre spectral sequence associated to the extension of group schemes

$$1 \to N \to G \xrightarrow{\psi} H \to 1$$

gives the exact sequence

$$0 \to H^1(H,k) \xrightarrow{\psi^*} H^1(G,k) \to H^1(N,k)^H \to H^2(H,k) \xrightarrow{\psi^*} H^2(G,k).$$

Thus, our hypotheses imply that $H^1(N,k)^H = 0$. Since H is unipotent, we conclude that $H^1(N,k) = 0$ and thus by Lemma 1.1(d) that N is the trivial group scheme. □

For future reference, we recall the following computation of the cohomology of $\mathbb{G}_a$ and its infinitesimal subgroups $\mathbb{G}_{a(r)}$.

Theorem 1.3 ([CPSvdK])**.** (1) *Assume that $p \neq 2$. Then the cohomology algebra $H^*(\mathbb{G}_a,k)$ is a tensor product of a polynomial algebra $k[x_1,x_2,...]$ in generators x_i of degree 2 and an exterior algebra $\Lambda(\lambda_1,\lambda_2,...)$ in generators λ_i of degree one. If $p = 2$, then $H^*(\mathbb{G}_a,k) = k[\lambda_1,\lambda_2,...]$ is a polynomial algebra in generators λ_i of degree 1; in this case, we set $x_i = \lambda_i^2$.*

(2) *Let $F : \mathbb{G}_a \to \mathbb{G}_a$ denote the Frobenius endomorphism. Then $F^*(x_i) = x_{i+1}$, $F^*(\lambda_i) = \lambda_{i+1}$.*

(3) *Let s be an element of k and use the same notation s for the endomorphism (multiplication by s) of $\mathbb{G}_a$. Then $s^*(x_i) = s^{p^i} x_i$, $s^*(\lambda_i) = s^{p^{i-1}}\lambda_i$.*

(4) *Restriction of x_i and λ_i to $\mathbb{G}_{a(r)}$ is trivial for $i > r$. Denoting the restrictions of x_i and λ_i (for $i \leq r$) to $\mathbb{G}_{a(r)}$ by the same letter we have*

$$\begin{aligned} H^*(\mathbb{G}_{a(r)},k) =& k[x_1,...,x_r] \otimes \Lambda(\lambda_1,...,\lambda_r), && p \neq 2, \\ H^*(\mathbb{G}_{a(r)},k) =& k[\lambda_1,...,\lambda_r], && p = 2. \end{aligned}$$

Lemma 1.4. *Denote by $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the addition homomorphism. Then*

(a) $m^*(\lambda_i) = \lambda_i \otimes 1 + 1 \otimes \lambda_i$.

(b) $m^*(x_i) = x_i \otimes 1 + 1 \otimes x_i$.

Proof . The first statement is evident. In proving the second one it suffices (in view of Theorem 1.3(2) to consider the case $i = 1$. Moreover we may assume, extending scalars if necessary, that the field k is infinite. Theorem 1.3(3) shows that the element $m^*(x_1) - x_1 \otimes 1 - 1 \otimes x_1 \in H^1(\mathbb{G}_a,k) \otimes H^1(\mathbb{G}_a,k)$ has weight p with respect to the diagonal action of $\mathbb{G}_m$. On the other hand, the weights appearing in $H^1(\mathbb{G}_a,k) \otimes H^1(\mathbb{G}_a,k)$ are of the form $p^i + p^j$ with $0 \leq i,j$. This implies immediately that the above element is trivial unless $p = 2$. However in the case $p = 2$ our statement is trivial since in this case $x_1 = \lambda_1^2$. □

Remark 1.4.1. Alternatively in case $p \neq 2$ one could use the formula $x_i = -\beta\mathcal{P}_0(\lambda_i)$ [M] and the properties of the Steenrod operations.

Corollary 1.5. *For any $\phi : G \to \mathbb{G}_{a(1)}$, set*

$$x_\phi \equiv \phi^*(x_1) \in H^2(G,k).$$

Then

(a) $x_{t\phi} = t^p x_\phi$ *for any* $t \in k$.
(b) $x_{\phi+\phi'} = x_\phi + x_{\phi'}$ *for any* $\phi, \phi' : G \to \mathbb{G}_{a(1)}$.

Proof. Part (a) follows immediately from Theorem 1.3(3). Part (b) follows from Lemma 1.4(b). □

The proof of the following theorem is inspired by the original argument of J.-P. Serre characterizing elementary abelian p-groups in terms of their cohomology algebras [S].

Theorem 1.6. *Let G be an infinitesimal unipotent group scheme of height $\leq r$ with the property that $Hom_{Gr/k}(G, \mathbb{G}_{a(1)})$ is 1-dimensional spanned by some $\phi : G \to \mathbb{G}_{a(1)}$. Then either $G = \mathbb{G}_{a(s)}$ for some $s \leq r$ or $x_\phi \in H^2(G,k)$ is nilpotent.*

Proof. Let s be the least integer such that $F^s : H^1(G,k) \to H^1(G,k)$ is 0. Then ϕ can be written as $\phi = F^{s-1}(\psi)$ for some $\psi \in H^1(G,k)$. Moreover it's clear that $s \leq r$ and the elements $\psi, F(\psi), ..., F^{s-1}(\psi) = \phi$ form a basis of $H^1(G,k)$. The homomorphism $\psi : G \to \mathbb{G}_{a(s)}$ induces an isomorphism $\psi^* : H^1(\mathbb{G}_{a(s)}, k) \to H^1(G,k)$: the basis $\lambda_1, \lambda_2, \ldots, \lambda_s$ is mapped to the basis $\psi, F(\psi), \ldots, F^{s-1}(\psi)$. Lemma 1.2 implies that either ψ is an isomorphism or the ideal

$$I = Ker\{\psi^* : H^*(\mathbb{G}_{a(s)}, k) \to H^*(G,k)\}$$

contains a nonzero element of degree 2. Since ψ^* is a map of modules for the Steenrod algebra [D], [P], I is stable with respect to action of the Steenrod operations $\mathcal{P}^i, \beta\mathcal{P}^i$ for $p \neq 2$ and Sq^i for $p = 2$.

Observe that $x_\phi = \psi^*(x_s) \in H^2(G,k)$. Thus, the theorem follows from the following proposition.

Proposition 1.7. *Let $I \subset H^*(\mathbb{G}_{a(s)}, k)$ be a k-submodule stable with respect to the action of the Steenrod operations $\mathcal{P}^i, \beta\mathcal{P}^i$ for $p \neq 2$ and Sq^i for $p = 2$. If I contains a nonzero element of degree 2, then some power of x_s lies in I.*

Proof. We first assume that $p \neq 2$. Here is a tabulation of the actions of $\mathcal{P}^i, \beta\mathcal{P}^i$ on the generators $x_1, \ldots, x_s, \lambda_1, \ldots, \lambda_s$.

(a) $\mathcal{P}^0(x_i) = F^*(x_i) = x_{i+1}$; $\mathcal{P}^0(\lambda_i) = F^*(\lambda_i) = \lambda_{i+1}$.
(b) $\mathcal{P}^j(x_i) = 0,\ j > 1,\ \beta\mathcal{P}^j(x_i) = 0,\ j \geq 1$; $\mathcal{P}^j(\lambda_i) = 0 = \beta\mathcal{P}^j(\lambda_i), j \geq 1$.
(c) $\mathcal{P}^1(x_i) = x_i^p$.
(d) $\beta\mathcal{P}^0(\lambda_i) = -x_i$; $\beta\mathcal{P}^0(x_i) = 0$.
(e) $\mathcal{P}^m(x_s^k) = x_s^{pk}$ if $m = k$; $\mathcal{P}^m(x_s^k) = 0$ if $m \neq k$.

Here (a) is a direct consequence of the usual construction of Steenrod operations [M], (c) is a defining axiom of the Steenrod algebra action, and (b) follows from the dimension axiom. The first part of (d) is proved in [M] using the theory of Massey products, whereas the second statement follows by applying the Adem relation. Finally, (e) follows from the Cartan formula using induction on m.

Let $x = \sum_{i<j} a_{i,j}\lambda_i\lambda_j + \sum_b a_b x_b$ be a nonzero element in $I \cap H^2(\mathbb{G}_{a(s)}, k)$. Assume first that all coefficients $a_{i,j}$ are 0. Find the smallest ℓ such that $a_\ell \neq 0$. Applying $(\mathcal{P}^0)^{s-\ell} = F^{*s-\ell}$ to x, we conclude that $x_s = a_\ell^{-p^{s-\ell}}(\mathcal{P}^0)^{s-\ell}(x) \in I$.

If some coefficient $a_{i,j} \neq 0$, then by applying to x an appropriate power of $\mathcal{P}^0$ we may assume that $a_{i,j} = 0$ for $j < s$; thus,

$$x = \sum_{i<s} a_{i,s}\lambda_i\lambda_s + \sum_b a_b x_b.$$

Now apply the operation $\beta\mathcal{P}^1\beta\mathcal{P}^0$. Using the Cartan formula

$$\beta\mathcal{P}^i(u \cdot v) = \sum_{j=0}^{i} \beta\mathcal{P}^j(u) \cdot \mathcal{P}^{i-j}(v) + (-1)^{dimu}\mathcal{P}^j(u) \cdot \beta\mathcal{P}^{i-j}(v)$$

and the above tabulation, we easily compute that

$$-\beta\mathcal{P}^1\beta\mathcal{P}^0(x) = \sum_{i<s} a_{i,s}^{p^2} x_{i+1} x_s^p.$$

Using the Cartan formula and (e) above, we conclude that $\mathcal{P}^p$ applied to the right hand side of the above equality yields

$$\mathcal{P}^p((\sum_{i<s} a_{i,s}^{p^2} x_{i+1})x_s^p) = \mathcal{P}^0(\sum_{i<s} a_{i,s}^{p^2} x_{i+1}) \cdot \mathcal{P}^p(x_s^p) = (\sum_{i<s} a_{i,s}^{p^3} x_{i+2}) \cdot x_s^{p^2}.$$

Successively applying the operations $\mathcal{P}^{p^2}, \mathcal{P}^{p^3}, \ldots$, we conclude that $x_s^{p^{s-i}+1} \in I$ provided that i is the least integer for which $a_{i,s} \neq 0$.

For $p = 2$, the proof has exactly the same form with $\mathcal{P}^{p^k}$ replaced by $Sq^{2^{k+1}}$ and $\beta\mathcal{P}^{p^k}$ replaced by $Sq^{2^{k+1}+1}$. □

§2. The detection theorem for unipotent infinitesimal group schemes

We prove in Theorem 2.5 below for G/k a unipotent infinitesimal group scheme of height $\leq r$ and $z \in H^*(G,k)$ that z is nilpotent provided that its restrictions via all 1-parameter homomorphims $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ are nilpotent where K is an arbitrary field extension of k. We prove this theorem for the cohomology algebra $H^*(G,\Lambda)$, where Λ is an associative unital rational G-algebra. Consideration of such Λ of the form $Ind_G^{GL_{n(r)}} k$ will be required in our proof (in §4) of the detection theorem for $H^*(G,k)$ for arbitrary infinitesimal group schemes G.

Throughout this section, G will denote an infinitesimal group scheme over k and Λ an associative, unital rational G-algebra (i.e., Λ has structures of an associative unital k-algebra and of a rational G-module which are compatible in the sense that $1_\Lambda \in \Lambda^G$ and the multiplication map $\Lambda \otimes_k \Lambda \to \Lambda$ is a homomorphism of rational G-modules).

The following lemma is an explicit statement of the usual graded commutativity of the cohomology of a Hopf algebra.

Lemma 2.1. *$H^*(G,\Lambda)$ is a graded $H^*(G,k)$-algebra with the property that*

$$\rho_\Lambda(x) \cdot y = (-1)^{nm} y \cdot \rho_\Lambda(x) \ , \quad x \in H^n(G,k), y \in H^m(G,\Lambda),$$

where $\rho_\Lambda : H^*(G,k) \to H^*(G,\Lambda)$ is induced by the unit $1_\Lambda : k \to \Lambda$. In particular, $\rho_\Lambda(H^{ev}(G,k))$ is contained in the center of $H^*(G,\Lambda)$.

We next give an explicit statement of the multiplicative nature of the Hochschild-Serre spectral sequence (cf. [E]).

Lemma 2.2. *Let*

$$1 \to N \to G \to H \to 1$$

be an extension of affine group schemes. Then the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(H, H^q(N,\Lambda)) \Longrightarrow H^{p+q}(G,\Lambda)$$

has a natural algebra structure. In particular,

$$F^i(H^n(G,\Lambda)) \cdot F^j(H^m(G,\Lambda)) \subset F^{i+j}(H^{n+m}(G,\Lambda))$$

where $F^i(H^n(G,\Lambda))$ *is the (decreasing) filtration associated to this spectral sequence.*

The following proposition is a reformulation of the "Quillen-Venkov Lemma" [Q-V] in our context.

Proposition 2.3. *Let* $f : G \to \mathbb{G}_{a(1)}$ *be a nontrivial homomorphism of affine group schemes, and let* $z \in H^n(G,\Lambda)$ *satisfy* $z_{|Ker\ f} = 0$*. Then* z^2 *is divisible by* $\rho_\Lambda(x_f) \in H^2(G,\Lambda)$*.*

Proof. Denote $Kerf$ by N and consider the Hochschild-Serre spectral sequence for the group scheme extension

$$1 \to N \to G \xrightarrow{f} \mathbb{G}_{a(1)} \to 1,$$

$$E_2^{p,q} = H^p(\mathbb{G}_{a(1)}, H^q(N,\Lambda)) \Longrightarrow H^{p+q}(G,\Lambda).$$

This spectral sequence admits an action of $H^*(\mathbb{G}_{a(1)}, k)$ compatible with the action of $H^*(\mathbb{G}_{a(1)}, k)$ on $H^*(G,\Lambda)$.

The periodicity of the cohomology of $\mathbb{G}_{a(1)}$ with respect to multiplication by $x_1 \in H^2(\mathbb{G}_{a(1)}, k)$ shows that the natural map $x_1 : E_2^{p,q} \to E_2^{p+2,q}$ is surjective for $p \geq 0$ and bijective for $p \geq 1$. An easy induction on r shows further that $x_1 : E_r^{p,q} \to E_r^{p+2,q}$ is surjective for $p \geq 0$ and bijective for $p \geq r-1$. Thus $x_1 : E_\infty^{p,q} \to E_\infty^{p+2,q}$ is surjective for all $p \geq 0$. In other words,

$$F^{p+2}(H^{n+2}(G,\Lambda)) = \rho_\Lambda(x_f) \cdot F^p(H^n(G,\Lambda)) + F^{p+3}(H^{n+2}(G,\Lambda)).$$

Using descending induction on p, we conclude that for $p \geq 0$

$$F^{p+2}(H^{n+2}(G,\Lambda)) = \rho_\Lambda(x_f) \cdot F^p(H^n(G,\Lambda)).$$

Finally, the kernel of $H^n(G,\Lambda) \to H^n(N,\Lambda)$ coincides with $F^1(H^n(G,\Lambda))$, so that $z \in F^1(H^n(G,\Lambda))$. Lemma 2.2 implies now that

$$z^2 \in F^2(H^{2n}(G,\Lambda)) = \rho_\Lambda(x_f) \cdot H^{2n-2}(G,\Lambda).$$

□

Lemma 2.4. *Assume that* $r \geq s$ *and consider the canonical projection* $\mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$*. For any rational* $\mathbb{G}_{a(s)}$*-module* M*, the induced map in cohomology* $H^*(\mathbb{G}_{a(s)}, M) \to H^*(\mathbb{G}_{a(r)}, M)$ *is injective.*

Proof. (See also [J], II.10.14.) Let $u_i \in k[\mathbb{G}_{a(r)}]^\# = (k[t]/t^{p^r})^\#$ $(0 \leq i \leq r-1)$ be a linear function defined by $u_i(t^j) = \delta_{p^i,j}$. The algebra $k[\mathbb{G}_{a(r)}]^\#$ is equal to $k[u_0, ..., u_{r-1}]/(u_0^p, ..., u_{r-1}^p)$ (cf. [S-F-B], 1.4). In the same way, $k[\mathbb{G}_{a(s)}]^\# = k[u_0, ..., u_{s-1}]/(u_0^p, ..., u_{s-1}^p)$. The homomorphism $k[\mathbb{G}_{a(r)}]^\# \to k[\mathbb{G}_{a(s)}]^\#$ takes u_i to u_{i-r+s} if $i \geq r-s$ and to zero otherwise. The statement follows since there is an evident splitting of k-algebras $k[\mathbb{G}_{a(s)}]^\# \to k[\mathbb{G}_{a(r)}]^\#$ (which is not a homomorphism of Hopf algebras). □

Theorem 2.5. *Let G be a unipotent infinitesimal group scheme over k of height $\leq r$ and let Λ be an associative unital rational G-algebra. If $z \in H^n(G, \Lambda)$ satisfies the property that for any field extension K/k and any group scheme homomorphism $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ over K the cohomology class $\nu^*(z_K) \in H^n(\mathbb{G}_{a(r)} \otimes_k K, \Lambda_K)$ is nilpotent, then z is itself nilpotent.*

Proof. Observe that if $z_L \in H^n(G_L, \Lambda_L)$ is nilpotent for some field extension L/k, then z is itself nilpotent. Thus we may assume, extending scalars, that the field k is algebraically closed.

The theorem is trivially valid if $dim_k k[G] = 1$. Proceeding by induction on $dim_k k[G]$, we may assume that the theorem has been proved for all unipotent infinitesimal group schemes H over fields K/k with $dim_K K[H] < dim_k k[G]$. For notational convenience, we shall abuse notation in the remainder of this proof by writing x_ϕ for $\rho_\Lambda(x_\phi)$.

Assume first that $dim_k Hom_{Gr/k}(G, \mathbb{G}_{a(1)}) = 1$ and let $\phi : G \to \mathbb{G}_{a(1)}$ be a nontrivial homomorphism. If $G = \mathbb{G}_{a(s)}$ for some $s \leq r$, then the theorem follows immediately by employing the canonical projection $\nu : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$—see Lemma 2.4. If G is not isomorphic to $\mathbb{G}_{a(s)}$ for any $s \leq r$, Theorem 1.6 implies that $x_\phi \in H^2(G, k)$ is nilpotent. Let $H = Ker\ \phi \subset G$. By induction, $z_{|H} \in H^n(H, \Lambda)$ is nilpotent; say $(z_{|H})^N = 0 \in H^{nN}(H, \Lambda)$. Proposition 2.3 now implies that x_ϕ divides z^{2N}, so that z is nilpotent.

Assume now that $dim_k Hom_{Gr/k}(G, \mathbb{G}_{a(1)}) > 1$ and let $\phi_1, \phi_2 : G \to \mathbb{G}_{a(1)}$ be linearly independent homomorphisms. Set ϕ equal to

$$\phi \equiv Y_1^{1/p}\phi_1 + Y_2^{1/p}\phi_2 : G \otimes_k K \to \mathbb{G}_{a(1)} \otimes_k K$$

where $K = k(Y_1^{1/p}, Y_2^{1/p})$ is a finite purely inseparable extension of the rational function field $k(Y_1, Y_2)$. Our induction hypothesis implies that there exists $N > 0$ such that $(z_K^N)_{|Ker\ \phi} = 0$ and also $(z^N)_{|Ker\ \phi_i} = 0$. Proposition 2.3 shows now that x_ϕ divides z_K^{2N} and also x_{ϕ_i} divides z^{2N}.

We write $z_K^{2N} \in H^{2nN}(G \otimes_k K, \Lambda_K)$ as

$$z_K^{2N} = x_\phi \cdot (\sum_{i=1}^m a_i v_i) = \sum_{i=1}^m a_i \cdot \{(Y_1 x_{\phi_1} + Y_2 x_{\phi_2}) \cdot v_i\}$$

and write

$$z^{2N} = x_{\phi_1} \cdot (\sum_{i=1}^m b_i v_i)$$

for some elements $a_i \in K$, $b_i \in k$ and some linearly independent elements $v_1, \ldots, v_m \in H^{2nN-2}(G, \Lambda)$. We may consider the first of the above formulas as a system of linear equations in variables $a_1, ..., a_m$ with coefficients in the field $k(Y_1, Y_2)$. Since this system has a solution in K, it really has a solution in $k(Y_1, Y_2)$ as well (i.e. the

a_i may be chosen in $k(Y_1, Y_2)$). Eliminating denominators, we conclude that there exist polynomials $P_1, ..., P_m, Q \in k[Y_1, Y_2]$, $Q \neq 0$, such that

(2.5.1)

$$Q(Y_1, Y_2) \cdot z^{2N} = \sum_{i=1}^{m} P_i(Y_1, Y_2) \cdot \{(Y_1 x_{\phi_1} + Y_2 x_{\phi_2}) \cdot v_i\} \in H^{2nN}(G, \Lambda) \otimes_k k[Y_1, Y_2].$$

Replacing P_i and Q by their appropriate homogeneous components we may even assume that P_i and Q are homogeneous and $deg\ P_i + 1 = deg\ Q$ for all i.

Define the k-vector space

$$W = \{w \in H^{2nN}(G, \Lambda) : z^\ell w = 0 \text{ for some } \ell > 0\}.$$

To prove that z is nilpotent, it suffices to show that $z^{2N} \in W$. Since z^{2N} is a linear combination of $x_{\phi_1} \cdot v_i$, it suffices to show that $x_{\phi_1} \cdot v_i \in W$ for all $i = 1, ..., m$.

If $x_{\phi_1} \cdot v, x_{\phi_2} \cdot v$ are linearly dependent modulo W for some $v \in H^{2nN-2}(G, \Lambda)$, then

$$t_1 x_{\phi_1} \cdot v + t_2 x_{\phi_2} \cdot v = (x_{t_1^{1/p}\phi_1 + t_2^{1/p}\phi_2}) \cdot v \in W$$

for some $0 \neq (t_1, t_2) \in k^2$. Applying our induction hypothesis to $Ker\ \psi$, where

$$\psi = t_1^{1/p}\phi_1 + t_2^{1/p}\phi_2 : G \to \mathbb{G}_{a(1)},$$

and using Proposition 2.3 we conclude that x_ψ divides an appropriate power of z. Since $x_\psi \cdot v \in W$, we conclude further that $z^\ell \cdot v = 0$ for some $\ell > 0$ and hence $x_{\phi_1} \cdot v, x_{\phi_2} \cdot v \in W$. In other words, if $x_{\phi_1} \cdot v, x_{\phi_2} \cdot v$ are linearly dependent modulo W, then both $x_{\phi_1} \cdot v$ and $x_{\phi_2} \cdot v$ belong to W.

Modifying the linearly independent set $v_1, \ldots, v_m$ of $H^{2nN-2}(G, \Lambda)$, we may assume that $x_{\phi_1} \cdot v_1, \ldots, x_{\phi_1} \cdot v_j$ are linearly independent modulo W whereas $x_{\phi_1} \cdot v_{j+1}, \ldots, x_{\phi_1} \cdot v_m$ belong to W (and hence $x_{\phi_2} \cdot v_{j+1}, ..., x_{\phi_2} \cdot v_m \in W$). We want to show that $j = 0$. Assume to the contrary that $j > 0$. Reducing the equation (2.5.1) modulo W we obtain the following equation in $H^{2nN}(G, \Lambda)/W \otimes_k k[Y_1, Y_2]$:

$$(2.5.2) \qquad Q \cdot \overline{z}^{2N} = \sum_{i=1}^{j} P_i \cdot \{Y_1(\overline{x_{\phi_1} \cdot v_i}) + Y_2(\overline{x_{\phi_2} \cdot v_i})\}.$$

Dividing both sides of (2.5.2) by the greatest common divisor of $Q, P_1, \ldots, P_j$, we may assume that $g.c.d.(Q, P_1, \ldots, P_j) = 1$. If $deg\ Q = 0$ so that each $P_i = 0$, then $z^{2N} \in W$; this implies that z is nilpotent and hence $W = H^{2nN}(G, \Lambda)$, thereby contradicting the hypothesis that $j > 0$. If $deg\ Q > 0$, we use the fact that k is algebraically closed to find $0 \neq (t_1, t_2) \in k^2$ such that $Q(t_1, t_2) = 0$ and $P_i(t_1, t_2) \neq 0$ for some i $(1 \leq i \leq j)$. Then

$$\sum_{i=1}^{j} P_i(t_1, t_2)\{t_1 \overline{x_{\phi_1} \cdot v_i} + t_2 \overline{x_{\phi_2} \cdot v_i}\} = t_1 \overline{x_{\phi_1} \cdot v} + t_2 \overline{x_{\phi_2} \cdot v} = 0 \in H^{2nN}(G, \Lambda)/W$$

where $v = \sum_{i=1}^{j} P_i(t_1, t_2) v_i$. As seen above, this implies that $x_{\phi_1} \cdot v \in W$. On the other hand, this contradicts the assumed linear independence of $x_{\phi_1} \cdot v_1, \ldots, x_{\phi_1} \cdot v_j$ modulo W. $\square$

§3. A spectral sequence for Frobenius kernels

In [A-J], H. Andersen and J. Jantzen introduced a spectral sequence for a reductive (smooth) group scheme G and a rational G-module M,

$$E_2^{p,q} = H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^q(B_{(r)}, M))) \Longrightarrow H^{p+q}(G_{(r)}, M),$$

which relates the cohomology of the Frobenius kernel $G_{(r)}$ to that of the Frobenius kernel of the Borel subgroup $B \subset G$. As we observe below, this immediately generalizes to arbitrary smooth group schemes. The purpose of this section is to further generalize this spectral sequence so that it applies to rational $G_{(r)}$-modules M which are not necessarily modules for G.

Let G/k be a reduced (= smooth) affine group scheme. We recall that when k is algebraically closed the Borel subgroup $B \subset G$ is defined to be a closed, reduced, connected, solvable subgroup scheme of maximal possible dimension. In general, we will say that $B \subset G$ is a Borel subgroup iff $B \otimes_k \overline{k}$ is a Borel subgroup of $G \otimes_k \overline{k}$ (where $\overline{k}$ denotes the algebraic closure of k). Note that when k is not algebraically closed not every reduced group scheme has Borel subgroups. Throughout this section (if not specified otherwise) G will denote a connected reduced affine group scheme over k and B will denote its Borel subgroup (thus we assume that Borel subgroups in G exist).

The following proposition is a straightforward generalization of a fundamental theorem of E. Cline, B. Parshall, L. Scott, and W. van der Kallen [CPSvdK] (see also [K]).

Proposition 3.1. *For any rational G-module M, the restriction map*

$$H^*(G, M) \to H^*(B, M)$$

is an isomorphism.

Proof. Since rational cohomology commutes with flat base change, it suffices to assume that k is algebraically closed. If G is semisimple, then our statement is precisely the theorem proved in [CPSvdK]. In the general case, let $R(G)$ denote the radical of G and compare the two Hochschild-Serre spectral sequences

$$E_2^{p,q} = H^p(G/R(G), H^q(R(G), M)) \Longrightarrow H^{p+q}(G, M),$$

$$E_2^{p,q} = H^p(B/R(G), H^q(R(G), M)) \Longrightarrow H^{p+q}(B, M).$$

The proposition now follows since $B/R(G)$ is a Borel subgroup of the semisimple group $G/R(G)$. □

Proposition 3.2. *Let G be an arbitrary affine group scheme over k. Let further $H \subset G$ be a closed subgroup scheme, and let $X = G/H$ denote the quotient scheme with quotient map $p : G \to X$. Then the functor $\mathcal{M} \mapsto \Gamma(G, p^*\mathcal{M})$ defines an equivalence of categories between the category of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$ and the category of k-vector spaces M provided with the structures of a left $k[G]$-module and of a rational H-module which are compatible in the sense that the multiplication map $k[G] \otimes M \to M$ is a homomorphism of rational H-modules (H acts on $k[G]$ via the right regular representation).*

Proof. Since $p : G \to X$ is a faithfully flat, quasi-compact morphism, the theory of faithfully flat descent [SGA1] tells us that the functor p^* is an equivalence of the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathcal{O}_G$-modules $\mathcal{N}$ equipped with descent data: an isomorphism $\psi : q_1^*(\mathcal{N}) \xrightarrow{\sim} q_2^*(\mathcal{N})$ of $\mathcal{O}_{G\times_X G}$-modules (where $q_1, q_2 : G \times_X G \to G$ are the projections) satisfying the usual compatibility property: $q_{23}^*(\psi)q_{12}^*(\psi) = q_{13}^*(\psi)$. Moreover, since the scheme G is affine, the category of quasi-coherent $\mathcal{O}_G$-modules is equivalent (via the functor $M \mapsto \widetilde{M}$) to the category of left $k[G]$-modules.

The cartesian square (cf. [D-G], III.1.2.4)

$$\begin{array}{ccc} G \times H & \xrightarrow{pr_1} & G \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{p} & X \end{array}$$

enables us to identify $q_1, q_2 : G \times_X G \to G$ with $m, pr_1 : G \times H \to G$. This shows that descent data on $\widetilde{M}$ is equivalent to the data of a homomorphism

$$\phi : M \to M \otimes k[H]$$

satisfying the following properties:

(a) ϕ is a homomorphism of $k[G]$-modules provided that $M \otimes k[H]$ is made into a $k[G]$-module via the homomorphism $m^* : k[G] \to k[G] \otimes k[H]$.

(b) The induced homomorphism of $k[G] \otimes k[H]$-modules

$$(k[G] \otimes k[H]) \otimes_{m^*} M \to M \otimes k[H]$$

is an isomorphism.

(c) The following diagram (in which Δ denotes the diagonal map of the Hopf algebra $k[H]$) commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \otimes k[H] \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle 1_M \otimes \Delta} \\ M \otimes k[H] & \xrightarrow{\phi \otimes 1_{k[H]}} & M \otimes k[H] \otimes k[H]. \end{array}$$

Finally it is not hard to see that conditions (a), (b), (c) are equivalent to the fact that ϕ gives M a structure of a rational H-module compatible with its structure as a $k[G]$-module. □

Remark 3.3. Let N be a rational H-module. The $k[G]$-module $k[G] \otimes_k N$ has a natural structure of a rational H-module (H acts in the given way on N and right regularly on $k[G]$). This H-module structure is clearly compatible with the $k[G]$-module structure in the sense described above and hence we get a quasi-coherent sheaf on G/H which we'll denote by $\mathcal{L}(N)$. The theory of faithfully flat descent shows that $\mathcal{L}(N)$ is a locally free $\mathcal{O}_{G/H}$-module and it is coherent if and only if N is finite dimensional. It's easy to see that this construction gives the same result as the one used in [J].

Proposition 3.4. *Assume the notation and conventions of 3.2. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, the Zariski cohomology $H^*_{Zar}(X, \mathcal{M})$ is naturally isomorphic to the rational cohomology $H^*(H, \Gamma(G, p^*(\mathcal{M})))$.*

Proof. It is well known (cf. [Mi], III.3.7) that Zariski cohomology with coefficients in a quasi-coherent sheaf coincides with fppf-cohomology. Starting with the fppf-covering $p : G \to X$ of a scheme X, we may construct its nerve (i.e. the simplicial scheme $X_\bullet$ with $X_n = G \times_X \dots \times_X G$ ($n+1$ times) and the standard face and degeneracy operators). In a familiar way, this nerve determines a spectral sequence

$$E_2^{pq} = H^p_{fppf}(X_q, \mathcal{M}) = H^p_{Zar}(X_q, \mathcal{M}) \Longrightarrow H^{p+q}_{fppf}(X, \mathcal{M}) = H^{p+q}_{Zar}(X, \mathcal{M}).$$

The scheme X_q may be identified with $G \times H^q$ and in particular is affine. Thus the above spectral sequence degenerates, thereby identifying $H^*_{Zar}(X, M)$ with the cohomology of the complex

$$\Gamma(G, p^*(\mathcal{M})) \to \Gamma(G, p^*(\mathcal{M})) \otimes k[H] \to \Gamma(G, p^*(\mathcal{M})) \otimes k[H]^{\otimes 2} \to \dots. \tag{3.4.1}$$

Finally (3.4.1) is the Hochschild complex computing the rational cohomology groups of H with coefficients in $\Gamma(G, p^*(\mathcal{M}))$. □

Let $F^r : G \to G^{(r)}$ be the r-th power of the Frobenius map, where $G^{(r)}$ is the r-th Frobenius twist of G (see, for example, [F-S], §1). The kernel of F^r, $G_{(r)}$, is an infinitesimal group scheme of height r and F^r induces an isomorphism

$$F^r : G/G_{(r)} \xrightarrow{\sim} G^{(r)}.$$

Proposition 3.5. *For any rational $G_{(r)}$-module M and any $q \geq 0$, the cohomology group $H^q(B_{(r)}, Ind^G_{G_{(r)}}(M))$ has the natural structures of a rational $B^{(r)}$-module and of a $k[G^{(r)}]$-module which satisfy the compatibility condition of 3.2.*

Proof. Recall that the induced module $Ind^G_{G_{(r)}}(M)$ is defined to be $(k[G] \otimes M)^{G_{(r)}}$, where $G_{(r)}$ acts via the given action on M and via the right regular representation on $k[G]$, and where $Ind^G_{G_{(r)}}(M)$ is given the structure of a rational G-module via the left regular representation of G on $k[G]$ and the trivial action on M. For any $q \geq 0$, $H^q(B_{(r)}, Ind^G_{G_{(r)}}(M))$ has the structure of a rational B-module on which $B_{(r)} \subset B$ acts trivially, hence the structure of a rational $B^{(r)}$-module. Moreover, the action of $k[G^{(r)}] = k[G]^{G_{(r)}}$ on $Ind^G_{G_{(r)}}(M)$ (by $B_{(r)}$-linear transformations) determines an action on $H^q(B_{(r)}, Ind^G_{G_{(r)}}(M))$. One readily verifies that the multiplication map

$$k[G^{(r)}] \otimes H^q(B_{(r)}, Ind^G_{G_{(r)}}(M)) \to H^q(B_{(r)}, Ind^G_{G_{(r)}}(M))$$

is a homomorphism of rational $B^{(r)}$-modules provided that $B^{(r)}$ acts on $k[G^{(r)}]$ via the left regular representation. To get the compatibility we need to change the $k[G^{(r)}]$-module structure using the automorphism of $k[G^{(r)}]$ induced by the automorphism of the scheme $G^{(r)}$:

$$\sigma : G^{(r)} \to G^{(r)} : \ \sigma(x) = x^{-1}.$$

□

Definition 3.5.1. We denote by $\mathcal{H}^q(\mathcal{B}_{(r)}, M)$ the quasi-coherent sheaf on $X^{(r)} = G^{(r)}/B^{(r)}$ associated in view of Proposition 3.2 to $H^q(B_{(r)}, Ind^G_{G_{(r)}}(M))$ with the above structures.

Theorem 3.6. *For any rational $G_{(r)}$-module M there is a natural spectral sequence*

$$E_2^{p,q} = H^p(X^{(r)}, \mathcal{H}^q(\mathcal{B}_{(r)}, M)) \Longrightarrow H^{p+q}(G_{(r)}, M)$$

where $X^{(r)} = G^{(r)}/B^{(r)}$.

Proof. Following [FP4], we consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(B^{(r)}, H^q(B_{(r)}, Ind_{G_{(r)}}^G(M))) \Longrightarrow H^{p+q}(B, Ind_{G_{(r)}}^G(M)).$$

By Proposition 3.1, the abutment may be identified with $H^{p+q}(G, Ind_{G_{(r)}}^G(M))$. Since $G/G_{(r)} = G^{(r)}$ is affine, this can be identified with $H^{p+q}(G_{(r)}, M)$ (cf. [J], I.5.12). Proposition 3.4 implies that the $E_2^{p,q}$ term can be identified as

$$H^p(B^{(r)}, H^q(B_{(r)}, Ind_{G_{(r)}}^G(M))) = H^p(X^{(r)}, \mathcal{H}^q(\mathcal{B}_{(r)}, M)).$$

□

Remark 3.6.1. If M in 3.6 is the restriction of a rational G-module (this is the case considered in [A-J]), then the tensor identity (see [J]) shows that the quasi-coherent sheaf $\mathcal{H}^q(\mathcal{B}_{(r)}, M)$ is canonically isomorphic to the locally free sheaf $\mathcal{L}(H^q(B_{(r)}, M))$.

§4. The detection theorem for infinitesimal group schemes

In this section, we extend Theorem 2.5 to an arbitrary infinitesimal group scheme. Our strategy is to use the spectral sequence of the previous section, valid for Frobenius kernels of smooth group schemes, to reduce the detection theorem for Frobenius kernels to the unipotent case. We then extend the validity of the detection theorem to arbitrary infinitesimal group schemes G by embedding G in some Frobenius kernel $GL_{n(r)}$ and identifying $H^*(G, \Lambda)$ with $H^*(GL_{n(r)}, Ind_G^{GL_{n(r)}}(\Lambda))$.

We begin with the detection theorem for Frobenius kernels.

Theorem 4.1. *Let G denote a connected, smooth affine group scheme over k and assume that there exists a Borel subgroup $B \subset G$. Let Λ be an associative unital rational $G_{(r)}$-algebra. If $z \in H^n(G_{(r)}, \Lambda)$ satisfies the property that for any field extension K/k and any group scheme homomorphism $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ (over K) the cohomology class $\nu^*(z_K) \in H^n(\mathbb{G}_{a(r)} \otimes_k K, \Lambda_K)$ is nilpotent, then z is itself nilpotent.*

Proof. By [D-G], I.4.6.9, since G has a k-rational point (namely, the identity), $G \otimes_k \overline{k}$ is also connected. Thus, as in the proof of Proposition 3.1, it suffices to assume that $k = \overline{k}$. Consider the spectral sequence of Theorem 3.6:

$$H^p(X^{(r)}, \mathcal{H}^q(\mathcal{B}_{(r)}, \Lambda)) \Longrightarrow H^{p+q}(G_{(r)}, \Lambda).$$

This spectral sequence appears as the Hochschild-Serre spectral sequence corresponding to the group extension

$$1 \to B_{(r)} \to B \to B^{(r)} \to 1.$$

By Lemma 2.2,

$$F^i(H^n(G_{(r)}, \Lambda)) \cdot F^j(H^m(G_{(r)}, \Lambda)) \subset F^{i+j}(H^{n+m}(G_{(r)}, \Lambda)).$$

On the other hand $F^i(H^*(G_{(r)}, \Lambda)) = 0$ for $i > dim\ X$. This shows that the ideal $F^1(H^*(G_{(r)}, \Lambda))$ is nilpotent and hence it suffices to show that the image of z in $H^0(X^{(r)}, \mathcal{H}^n(\mathcal{B}_{(r)}, \Lambda))$ is nilpotent. The k-algebra $H^0(X^{(r)}, \mathcal{H}^*(\mathcal{B}_{(r)}, \Lambda)) = H^0(B^{(r)}, H^*(B_{(r)}, Ind\ \Lambda))$ is a subalgebra of the $k[G^{(r)}]$-algebra $H^*(B_{(r)}, Ind\ \Lambda)$ (here and in the sequel we abbreviate the notation $Ind_{G_{(r)}}^G$ to Ind). Thus it suffices to show that the image $\tilde{z}$ of z in $H^*(B_{(r)}, Ind\ \Lambda)$ is nilpotent.

Denote by $I^\bullet$ the standard $G_{(r)}$-injective resolution of Λ: $I^m = \Lambda\otimes k[G_{(r)}]^{\otimes(m+1)}$ (here $G_{(r)}$ acts on I^m via the right regular representation on the last tensor factor). Note that $I^\bullet$ is naturally a differential graded k-algebra. The differential graded $k[G^{(r)}]$-algebra $J^\bullet = (Ind\ I^\bullet)^{B_{(r)}}$ satisfies $H^*(J^\bullet) = H^*(B_{(r)}, Ind\ \Lambda)$. Let g be any point of G and let $g^{(r)}$ be the image of g in $G^{(r)}$. Consider the differential graded $k(g^{(r)})$-algebra $J^\bullet \otimes_{k[G^{(r)}]} k(g^{(r)})$ and tensor it over $k(g^{(r)})$ with $k(g)$. To understand the structure of the resulting differential graded $k(g)$-algebra assume first that the point g is k-rational.

For any rational $G_{(r)}$-module M, there is a canonical $G_{(r)}$-equivariant homomorphism $\epsilon_M : Ind\ M \to M$: $f\otimes m \mapsto f(1)\cdot m$. This homomorphism is clearly $k[G^{(r)}]$-linear if we make M into a $k[G^{(r)}]$-module via the evaluation at $1\in G^{(r)}(k)$ map $k[G^{(r)}]\to k$. Consider further the composition

$$Ind\ M \xrightarrow{g^{-1}} Ind\ M \xrightarrow{\epsilon_M} M.$$

This homomorphism takes $(Ind\ M)^{B_{(r)}} \subset Ind\ M$ to $M^{g^{-1}B_{(r)}g} \subset M$. Furthermore, this homomorphism is $k[G^{(r)}]$-linear if we make M into a $k[G^{(r)}]$-module via the evaluation at $g^{(r)} \in G^{(r)}(k)$ map. Thus we get a natural homomorphism

$$\theta_M : ((Ind\ M)^{B_{(r)}} \otimes_{k[G^{(r)}]} k(g^{(r)})) \otimes_{k(g^{(r)})} k(g) \to M^{g^{-1}B_{(r)}g}.$$

The homomorphism $\theta_{k[G_{(r)}]}$ is an isomorphism as one sees from the following Cartesian square:

$$\begin{array}{ccc} g^{-1}B_{(r)}g\backslash G_{(r)} & \longrightarrow & Spec\ k \\ {\scriptstyle x\mapsto gx}\big\downarrow & & {\scriptstyle g^{(r)}}\big\downarrow \\ B_{(r)}\backslash G & \longrightarrow & G_{(r)}\backslash G = G^{(r)} \end{array} \tag{4.1.1}$$

(To check that (4.1.1) is Cartesian, one realizes this square as the sheafification in the fppf topology of a pull-back square of presheaves of sets of the same form in which the indicated quotients are quotients as presheaves.) Hence, θ_M is an isomorphism for any injective $G_{(r)}$-module M. This implies that

$$(J^\bullet \otimes_{k[G^{(r)}]} k(g^{(r)})) \otimes_{k(g^{(r)})} k(g) = (I^\bullet)^{g^{-1}B_{(r)}g}.$$

Using extension of scalars, one sees easily that the same kind of an answer holds for any point $g\in G$. More precisely, we have the following general formula:

$$(J^\bullet \otimes_{k[G^{(r)}]} k(g^{(r)})) \otimes_{k(g^{(r)})} k(g) = (I^\bullet \otimes_k k(g))^{g^{-1}(B_{(r)}\otimes_k k(g))g}.$$

In this formula, we use the same notation g for the $k(g)$-rational point determined by the point $g\in G$. This formula implies, in particular, that

$$\begin{aligned} H^*(J^\bullet \otimes_{k[G^{(r)}]} k(g)) =& H^*(g^{-1}(B_{(r)}\otimes_k k(g))g, \Lambda\otimes_k k(g)) \\ & \subset H^*(g^{-1}(U_{(r)}\otimes_k k(g))g, \Lambda\otimes_k k(g)). \end{aligned}$$

A direct computation shows that the image of $\tilde z$ in this cohomology algebra coincides with the restriction of $z_{k(g)} \in H^n(G_{(r)}\otimes_k k(g), \Lambda_{k(g)})$ to $g^{-1}(U_{(r)}\otimes_k k(g))g$. Thus, we get a cohomology class of the unipotent group $g^{-1}(U_{(r)}\otimes_k k(g))g$ with coefficients in the associative unital rational algebra $\Lambda\otimes_k k(g)$ which restricts nilpotently (according to our assumptions on z) to all 1-parameter subgroups of $g^{-1}(U_{(r)}\otimes_k k(g))g$. Theorem 2.5 shows that this cohomology class is nilpotent. Since extension of scalars from $k(g^{(r)})$ to $k(g)$ gives an injective map in cohomology, we conclude that the image of $\tilde{z}$ in the cohomology of $J^\bullet \otimes_{k[G^{(r)}]} k(x)$ is nilpotent for each point $x \in G^{(r)}$. Finally we can use the following result.

Proposition 4.2. *Let A be a commutative noetherian ring of finite Krull dimension and let $J^\bullet$ be an associative unital differential graded A-algebra. Assume that $z \in H^n(J^\bullet)$ is a cohomology class such that for each prime ideal $\mu \in Spec\ A$ the image of z in $H^*(J^\bullet \otimes_A k(\mu))$ is nilpotent. Then z is nilpotent.*

Proof. We proceed by induction on $d = dim\ A$. Assume first that $d = 0$ (i.e., A is an artinian ring). Since every artinian ring is (canonically) a product of local artinian rings, $A = \prod A_i$ and similarly $J^\bullet = \prod J_i^\bullet$, where $J_i^\bullet$ is a differential graded A_i-algebra, we may further assume that A is local. In this case, we use an additional induction on the length of the maximal ideal $\mathfrak{M}(A)$ of A. If this length is equal to zero, then A is a field and there is nothing to prove. Otherwise, choose any nonzero element $t \in \mathfrak{M}(A)$. Applying the induction hypothesis to A/tA and a differential graded A/tA-algebra $J^\bullet/tJ^\bullet$, we see that there exists $N > 0$ such that the image of z^N in $J^\bullet/tJ^\bullet$ is a boundary. Replacing z by z^N we may assume that z is a cohomology class of a cycle of the form tu ($u \in J^n$). Since t is nilpotent the statement is clear in this case.

Assume now that $d > 0$ and the statement has been proved already for differential graded algebras over commutative rings of dimension $< d$. Consider the following chain of ideals in A:

$$Ann_A z \subset Ann_A z^2 \subset Ann_A z^3 \subset ....$$

Since A is noetherian this chain eventually stabilizes. Replacing z by its appropriate power we may assume in the sequel that

$$Ann_A z = Ann_A z^2 = ...$$

Denote this ideal by I. Let $\mu \in Spec\ A$ be a prime ideal of height $< d$. Applying the induction hypothesis to A_μ and $J_\mu^\bullet$ we conclude that the image of z in $H^*(J_\mu^\bullet) = H^*(J^\bullet)_\mu$ is nilpotent which implies readily that $I \not\subset \mu$. This shows that the only prime ideals which could possibly contain I are maximal ideals of height d. Since $d > 0$ we can find $t \in I$ such that $dim\ A/tA < d$. Applying the induction hypothesis to A/tA and $J^\bullet/tJ^\bullet$, we conclude that (after replacing z by an appropriate power of z) we may assume that z is a cohomology class of a cycle of the form tu with $u \in J^n$. Note that $t \cdot du = 0$ and hence $t \cdot d(u^2) = t \cdot (u \cup du + du \cup u) = 0$. Thus tu^2 is a cycle. Denoting the cohomology class of this cycle by y we obtain the equality $z^2 = ty$ and hence the vanishing $z^3 = z \cup z^2 = z \cup ty = tz \cup y = 0$. $\square$

Remark 4.2.1. The proof of Theorem 4.1 can be much simplified provided that one assumes that Λ is a rational G-algebra. In this case it suffices to show that the image of $z \in H^n(G_{(r)}, \Lambda)$ in $H^0(X^{(r)}, \mathcal{L}(H^n(B_{(r)}, \Lambda)))$ is nilpotent. One checks readily that the fiber of the vector bundle $\mathcal{L}(H^n(B_{(r)}, \Lambda))$ at a rational point $x^{(r)} = g^{(r)}B^{(r)} \in X^{(r)}$ ($g \in G(k)$) coincides with $H^n((B^g)_{(r)}, \Lambda)$. Furthermore the image of $z \in H^n(G_{(r)}, \Lambda)$ under the composition

$$H^n(G_{(r)}, \Lambda) \xrightarrow{\rho} H^0(X^{(r)}, \mathcal{L}(H^n(B_{(r)}, \Lambda))) \to H^n((B^g)_{(r)}, \Lambda)$$

coincides with $z_{|(B^g)_{(r)}}$. Now the proof is easily concluded using Theorem 2.5 and induction on the dimension of the support of the section

$$\rho(z) \in H^0(X^{(r)}, \mathcal{L}(H^n(B_{(r)}, \Lambda))).$$

In [S-F-B], we introduced for any affine group scheme G/k an affine scheme $V_r(G)$ of finite type over k, whose A-points for any commutative k-algebra A are in one-to-one correspondence with group scheme homomorphisms (over A) $\mathbb{G}_{a(r)} \otimes_k A \to G \otimes_k A$. Let $s \in V_r(G)$ be a point. This point defines a canonical $k(s)$-point of $V_r(G)$ and hence an associated group scheme homomorphism over $k(s)$ which we denote

$$\nu_s : \mathbb{G}_{a(r)} \otimes_k k(s) \to G \otimes_k k(s).$$

Note that if K/k is a field extension and $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ is a group scheme homomorphism, then this data defines a point $s \in V_r(G)$ and a field embedding $k(s) \hookrightarrow K$ such that ν is obtained from ν_s by extending scalars from $k(s)$ to K.

Theorem 4.3. *Let G be an infinitesimal group scheme of height $\leq r$ over k and let Λ be an associative unital rational G-algebra. Then the following conditions on a cohomology class $z \in H^n(G, \Lambda)$ are equivalent:*

(a) *z is nilpotent.*

(b) *For every field extension K/k and every group scheme homomorphism over K, $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$, the cohomology class $\nu^*(z_K) \in H^n(\mathbb{G}_{a(r)} \otimes_k K, \Lambda_K)$ is nilpotent.*

(c) *The cohomology class $\nu_s^*(z_{k(s)}) \in H^n(\mathbb{G}_{a(r)} \otimes_k k(s), \Lambda_{k(s)})$ is nilpotent, for every point $s \in V_r(G)$.*

Proof. The equivalence of conditions (b) and (c) is obvious from the discussion above. Clearly (a) implies (b). We now proceed to prove the converse.

Embed G in some $G' = GL_{n(r)}$ and set $\Lambda' = Ind_G^{G'}(\Lambda)$. Thus, Λ' is an associative, unital rational G'-algebra. Moreover, $H^*(G, \Lambda) = H^*(G', \Lambda')$ as graded algebras. By Theorem 4.1, it suffices to show that the element $z' \in H^n(G', \Lambda')$ which corresponds to z under the above identification satisfies the condition that $\nu'^*(z') \in H^n(\mathbb{G}_{a(r)} \otimes_k K, \Lambda'_K)$ is nilpotent for every field extension K/k and every group scheme homomorphism $\nu' : \mathbb{G}_{a(r)} \otimes_k K \to G' \otimes_k K$. We consider the cartesian square

$$\begin{array}{ccc} \mathbb{G}_{a(s)} \otimes_k K & \xrightarrow{\nu} & G \otimes_k K \\ \downarrow & & \downarrow \\ \mathbb{G}_{a(r)} \otimes_k K & \xrightarrow{\nu'} & G' \otimes_k K \end{array} \tag{4.3.1}$$

where the upper left corner is of the indicated form (with $s \leq r$) because it is a K-subgroup of $\mathbb{G}_{a(r)} \otimes_k K$. For notational convenience, we re-write (4.3.1) as

$$\begin{array}{ccc} H & \xrightarrow{\nu} & G_K \\ \downarrow & & \downarrow \\ H' & \xrightarrow{\nu'} & G'_K. \end{array}$$

Consider the homomorphism of rational H'-modules

$$\theta : \Lambda'_K = Ind_{G_K}^{G'_K}(\Lambda_K) \to Ind_H^{H'}(\Lambda_K)$$

determined by the adjointness of Ind and Res. This homomorphism is surjective. To prove this consider the following commutative diagram of schemes:

$$\begin{array}{ccc} H' & \xrightarrow{\nu'} & G'_K \\ {\scriptstyle p_H}\downarrow & & {\scriptstyle p_G}\downarrow \\ H'/H & \xrightarrow{\overline{\nu'}} & G'_K/G_K. \end{array}$$

According to our definitions, $Ind_{G_K}^{G'_K}(\Lambda_K) = \Gamma(G'_K/G_K, \mathcal{L}(\Lambda_K))$ and $Ind_H^{H'}(\Lambda_K) = \Gamma(H'/H, \mathcal{L}(\Lambda_K))$ (cf. 3.2 and 3.3). Observe that

$$\begin{aligned}(p_H)^*(\overline{\nu'}^*(\mathcal{L}(\Lambda_K))) &= \nu'^*(p_G^*(\mathcal{L}(\Lambda_K))) = \nu'^*(K[\widetilde{G'] \otimes_K} \Lambda_K) \\ &= K[\widetilde{H'] \otimes_K} \Lambda_K = (p_H)^*(\mathcal{L}(\Lambda_K)).\end{aligned}$$

This together with Proposition 3.2 shows that $\overline{\nu'}^*(\mathcal{L}(\Lambda_K)) = \mathcal{L}(\Lambda_K)$. Now, it suffices to observe that $\overline{\nu'}$ is a closed embedding of affine schemes.

We rewrite θ (as a map of K-algebras) as

$$\theta : (K[G'] \otimes_K \Lambda_K)^{G_K} = Ind_{G_K}^{G'_K}(\Lambda_K) \to Ind_H^{H'}(\Lambda_K) = (K[H'] \otimes_K \Lambda_K)^H.$$

Since $Ker\{K[G'] \to K[H']\}$ is contained in the maximal ideal of the local artinian ring $K[G']$, it is nilpotent. From this we conclude immediately that $I = Ker\ \theta$ is also nilpotent. Now consider the following portion of the exact cohomology sequence:

$$H^n(H', I) \to H^n(H', \Lambda'_K) \to H^n(H', Ind_H^{H'}(\Lambda_K)) = H^n(H, \Lambda_K).$$

To prove that $\nu'^*(z'_K) \in H^n(H', \Lambda'_K)$ is nilpotent, we first observe that its image in $H^n(H, \Lambda_K)$ is nilpotent by our original hypothesis on z and Lemma 2.4, for this image equals $\nu^*(z_K)$. Thus, replacing z by its appropriate power, we may assume that $\nu'^*(z'_K) \in H^n(H', I)$. Since I is nilpotent, the ring without unit $H^*(H', I)$ is also nilpotent and thus $\nu'^*(z'_K)$ is nilpotent. $\square$

Since $Ext^*_G(M, M) = H^*(G, End_k(M))$ for a rational G-module M and as $End_k(M)_K = End_K(M_K)$ if M is finite dimensional, Theorem 4.3 has the following immediate corollary.

Corollary 4.4. *Let G be an infinitesimal group scheme of height $\leq r$ over k, and let M be a finite-dimensional rational G-module. Then the following conditions on an extension class $e \in Ext^n_G(M, M)$ are equivalent:*

(a) *e is nilpotent.*

(b) *For every field extension K/k and every group scheme homomorphism over K, $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$, the extension class $\nu^*(e_K) \in Ext^n_{\mathbb{G}_{a(r)} \otimes_k K}(M_K, M_K)$ is nilpotent.*

(c) *The extension class $\nu_s^*(e_{k(s)}) \in Ext^n_{\mathbb{G}_{a(r)} \otimes_k k(s)}(M_{k(s)}, M_{k(s)})$ is nilpotent, for every point $s \in V_r(G)$.*

§5. Support schemes

In [S-F-B], we introduced and analyzed a natural ring homomorphism $\psi : H^{ev}(G, k) \to k[V_r(G)]$ and the associated morphism of schemes $\Psi : V_r(G) \to$

$Spec\ H^{ev}(G,k)$. In the special case $G = GL_{n(r)}$, we identified the composition

$$V_r(GL_{n(r)}) \xrightarrow{\Psi} Spec\ H^{ev}(GL_{n(r)},k) \xrightarrow{\Phi} V_r(GL_{n(r)})$$

as the r-th Frobenius twist morphism (cf. [S-F-B], 5.2) by identifying ψ applied to the universal classes $e_i^{(r-i)} \in H^{2p^{r-i}}(GL_{n(r)}, gl_n^{(r)})$ introduced in [F-S]. This result implies immediately that for any infinitesimal group scheme G of height $\leq r$ the morphism Ψ is finite and universally injective. As we shall see in this section, our detection theorem together with this result enables us to demonstrate that

$$\Psi : V_r(G) \to |G| \equiv Spec\ H^{ev}(G,k)$$

is a (universal) homeomorphism.

Throughout this section, r will denote a fixed positive integer and G will denote an infinitesmal group scheme over k of height $\leq r$.

The following proposition is a somewhat sharper version of Theorem 4.3 in the case of the cohomology algebra with trivial coefficients.

Proposition 5.1. *For any $z \in H^{2n}(G,k)$, z is nilpotent whenever $\psi(z) \in k[V_r(G)]$ is nilpotent.*

Proof. Consider some $z \in H^{2n}(G,k)$ with $\psi(z)$ nilpotent. By Theorem 4.3, it suffices to prove for every field extension K/k and every group scheme homomorphism over K, $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$, that $\nu^*(z_K) \in H^{2n}(\mathbb{G}_{a(r)} \otimes_k K, K)$ is nilpotent. Furthermore, it suffices to restrict attention to fields K which are algebraically closed. Write

$$\overline{\nu^*(z_K)} = \sum_{i_1+\ldots+i_r=n} a_{i_1,\ldots,i_r} x_1^{i_1} \cdot \ldots \cdot x_r^{i_r} \in K[x_1,\ldots,x_r] = H^*(\mathbb{G}_{a(r)} \otimes_k K, K)_{red}.$$

According to definitions (see [S-F-B]), the coefficient $a_{0,\ldots,0,n}$ coincides with $\psi(z)(s)$, where $s \in V_r(G)$ is the point defined by ν. Since $\psi(z)$ is nilpotent, its value at each point is trivial and hence $a_{0,\ldots,0,n} = 0$.

Let $\underline{c}$ be an arbitrary r-tuple in the algebraically closed field K. Let $\gamma_{\underline{c}} : \mathbb{G}_{a(r)} \otimes_k K \to \mathbb{G}_{a(r)} \otimes_k K$ denote the composition

$$\mathbb{G}_{a(r)} \otimes_k K \xrightarrow{\Delta} (\mathbb{G}_{a(r)} \otimes_k K)^{\times r} \xrightarrow{c_1, c_2\circ F, \ldots, c_r \circ F^{r-1}} (\mathbb{G}_{a(r)} \otimes_k K)^{\times r} \xrightarrow{m} \mathbb{G}_{a(r)} \otimes_k K$$

where m denotes the group operation (addition) of $\mathbb{G}_{a(r)} \otimes_k K$. We easily verify that

$$\gamma_{\underline{c}}^*(x_i) = c_1^{p^i} x_i + c_2^{p^i} x_{i+1} + \cdots + c_{r-i+1}^{p^i} x_r$$

using 1.3(2), 1.3(3) and 1.4. Let $\nu' = \nu \circ \gamma_{\underline{c}} : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$. The coefficient of x_r^n in $\nu'^*(z) \in H^{2n}(\mathbb{G}_{a(r)} \otimes_k K, K)$ equals

$$\sum_{i_1+\ldots+i_r=n} a_{i_1,\ldots,i_r} (c_r^p)^{i_1} \cdot \ldots \cdot (c_1^{p^r})^{i_r}$$

which equals 0 by our assumption on z. We conclude that each $a_{i_1,\ldots,i_r}$ equals 0 and thus that $\overline{\nu^*(z)} = 0$ as required. $\square$

As shown in [S-F-B], 5.5, the image of the homomorphism $\psi : H^{ev}(G,k) \to k[V_r(G)]$ contains the p^r-th power of each element of $k[V_r(G)]$. This, together with Proposition 5.1, immediately implies the following theorem.

Theorem 5.2. *Let G be an infinitesimal group scheme of height $\leq r$. Then the kernel of the canonical homomorphism*

$$\psi : H^{ev}(G,k) \to k[V_r(G)]$$

is nilpotent and its image contains the p^r-th power of each element of $k[V_r(G)]$. Consequently, the associated morphism of schemes

$$\Psi : V_r(G) \to |G| = Spec\ H^{ev}(G,k)$$

is a finite universal homeomorphism.

Proposition 5.3. *Let G be a closed subgroup of $GL_{n(r)}$. Then the homomorphism*

$$Res \circ \phi : S^*(\bigoplus_{i=1}^{r} gl_n^{\#(r)}[2p^{i-1}]) \to H^{ev}(G,k)_{red}$$

induced by the universal cohomology classes $e_1^{(r-1)}, \dots, e_r \in H^*(GL_{n(r)}, gl_n^{(r)})$ has image containing all p^r-th powers. Consequently the associated morphism of schemes

$$|G| \to |GL_{n(r)}| \xrightarrow{\Phi} V_r(GL_{n(r)}) \subset gl_n^{\times r}$$

is finite and (universally) injective.

Proof. Recall that the closed embedding $G \hookrightarrow GL_{n(r)}$ defines a closed embedding $V_r(G) \hookrightarrow V_r(GL_n)$ (cf. [S-F-B], 1.5), so that the induced map on coordinate algebras $k[V_r(GL_{n(r)})] \to k[V_r(G)]$ is surjective. Consider the following commutative diagram:

$$\begin{array}{ccccc}
k[V_r(GL_{n(r)})] & \xrightarrow{\overline{\phi}} & H^{ev}(GL_{n(r)},k) & \xrightarrow{\psi} & k[V_r(GL_{n(r)})] \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle Res}\big\downarrow & & \big\downarrow \\
k[V_r(GL_{n(r)})] & \xrightarrow{Res\circ\overline{\phi}} & H^{ev}(G,k) & \xrightarrow{\psi} & k[V_r(G)].
\end{array}$$

For any cohomology class $z \in H^{ev}(G,k)$, consider $\psi(z) \in k[V_r(G)]$ and lift it to a (homogeneous) element $y' \in k[V_r(GL_{n(r)})] = k \otimes_{\mathbb{F}_p} \mathbb{F}_p[V_r(GL_{n(r)})]$. Applying to k-coefficients of y' the r-th power of the Frobenius endomorphism, we get an element $y \in k[V_r(GL_{n(r)})]$ whose r-th Frobenius twist is y'^{p^r}. Commutativity of the above diagram and the fact (discussed above) that the composition of the homomorphisms of the top row is the r-th Frobenius twist immediately implies that the image of $z^{p^r} - Res(\overline{\phi}(y))$ in $k[V_r(G)]$ is trivial and hence (by Proposition 5.1) $z^{p^r} - Res(\overline{\phi}(y))$ is nilpotent. □

Since the map $Res \circ \phi$ of Proposition 5.3 factors through $H^{ev}(GL_{n(r)},k)$, we conclude that the restriction map

$$H^{ev}(GL_{n(r)},k)_{red} \to H^{ev}(G,k)_{red}$$

has image containing all p^r-th powers. This immediately implies the following.

Corollary 5.4. *Let G be an infinitesimal group scheme of height $\leq r$ and let H be a closed subgroup scheme of G. Then the homomorphism*

$$H^{ev}(G,k)_{red} \xrightarrow{Res} H^{ev}(H,k)_{red}$$

has image containing all p^r-th powers. Consequently, for each $z \in H^{ev}(H,k)$ there exists $N \geq 0$ such that z^{p^N} may be lifted to $H^{ev}(G,k)$.

Corollary 5.4.1. *In conditions and notations of Corollary 5.4, the associated morphism of support schemes $|H| \to |G|$ is finite and universally injective.*

Remark 5.5. As seen above, the morphism $\Phi : |GL_{n(r)}| \to V_r(GL_{n(r)})$ induced by $\overline{\phi}$ is a universal homeomorphism, the topological inverse to Ψ. Taken in conjunction with Theorem 5.2, this implies that Φ restricts to a topological homeomorphism from $|G|$ to $V_r(G)^{(r)}$. In [S-F-B], we asked whether $\Phi(|G|) \subset V_r(GL_{n(r)})$ is contained scheme-theoretically in $V_r(G)^{(r)}$ and whether the morphism $\Phi_G : |G| \to V_r(G)^{(r)}$ which would then be determined is independent of the embedding $G \hookrightarrow GL_{n(r)}$. We see that these questions have an affirmative answer if weakened to set-theoretic statements.

§6. Support varieties for modules

The purpose of this section is to give a noncohomological, geometric interpretation of the cohomological support variety $|G|_M$ of a finite-dimensional, rational G-module M for an infinitesimal group scheme G . Indeed, Theorem 6.7 presents the appropriate analogue of Proposition 5.1 with k replaced by M; this easily implies in Corollary 6.8 a homeomorphism between $|G|_M$ and the noncohomological $V_r(G)_M \subset V_r(G)$. The key ingredient in our analysis is the detection result of Theorem 4.3 for $H^*(G,\Lambda)$, where $\Lambda = End_k(M)$. This, however, is not sufficient, for we must carefully analyze the special case $G = \mathbb{G}_{a(r)}$.

Throughout this section, G will denote an infinitesimal group scheme of height $\leq r$ and M a finite-dimensional rational G-module. Moreover, Λ will denote $End_k(M) = M \otimes M^{\#}$, a finite-dimensional, associative, unital rational G-algebra.

We recall that the (cohomological) support variety of M (denoted $|G|_M$) is defined as the Zariski closed subset of $|G| = Spec\ H^{ev}(G,k)$ defined by the ideal $I_M = Ker(H^{ev}(G,k) \xrightarrow{\rho_\Lambda} H^{ev}(G,\Lambda))$. (In particular, $|G|_M$ may be considered as a reduced k-scheme.)

We begin by introducing a closed subset $V_r(G)_M \subset V_r(G)$ which we shall show maps homeomorphically onto $|G|_M$.

Proposition 6.1. *Let G be an infinitesimal group scheme over k and M a finite-dimensional rational G-module. Then*

$$V_r(G)_M = \{s \in V_r(G) : \ M \otimes_k k(s) \textit{ is not projective over the subalgebra } \\ k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[u_0,...,u_{r-1}]/(u_0^p,...,u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^{\#}\}$$

is a Zariski closed conical subset in $V_r(G)$.

Proof. Let A denote the coordinate algebra $k[V_r(G)]$ and consider the canonical morphism of group schemes over A, $\mathbb{G}_{a(r)} \otimes_k A \to G \otimes_k A$. Using this homomorphism, we make M_A into a rational module over $\mathbb{G}_{a(r)} \otimes_k A$, i.e. into a module over $A[\mathbb{G}_{a(r)}]^{\#} = A[u_0,...,u_{r-1}]/(u_0^p,...,u_{r-1}^p)$. Observe that M_A is evidently a free A-module. Using this fact, one checks easily that if $M \otimes_k k(s)$ is projective (= free) over $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^{\#}$ for some point $s \in V_r(G)$, then $M \otimes_k A_s$ is a free $A_s[u_{r-1}]/u_{r-1}^p$-module. Here A_s denotes the localization of A at a prime ideal μ_s corresponding to s. Since the rank of $M \otimes_k A$ is finite, this implies immediately the existence of $a \in A \setminus \mu_s$ such that $M \otimes_k A_a$ is a free $A_a[u_{r-1}]/u_{r-1}^p$-module. Hence, for all points s' of the principal open set $Spec\ A_a \subset Spec\ A$, the module $M \otimes_k k(s')$ is free over $k(s')[u_{r-1}]/u_{r-1}^p$.

Let $V_r(G)_M$ also denote the closed reduced subscheme of $V_r(G)$ associated to this closed subset. Then field valued points $Spec\ K \to V_r(G)_M$ are in one-to-one correspondence with group scheme homomorphisms $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ for which M_K is not projective over $K[u_{r-1}]/u_{r-1}^p \subset K[\mathbb{G}_{a(r)}]^\#$. To prove that $V_r(G)_M$ is conical, it suffices to show that if $s \in V_r(G)_M(K)$ is a field valued point and $c \in K$, then $c \cdot s \in V_r(G)_M(K)$. For $c = 0$, this is evident since the point $0 \cdot s \in V_r(G)(K)$ corresponds to the trivial group scheme homomorphism $\mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$. Assume next that $c \neq 0$. The morphism $\nu_{c\cdot s} : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ is given by the composition $\mathbb{G}_{a(r)} \otimes_k K \xrightarrow{c} \mathbb{G}_{a(r)} \otimes_k K \xrightarrow{\nu_s} G$. One checks immediately that the homomorphism $c_* : K[\mathbb{G}_{a(r)}]^\# \to K[\mathbb{G}_{a(r)}]^\#$ takes u_i to $c^{p^i}u_i$. Thus a $K[u_{r-1}]/u_{r-1}^p$-module structure on M, corresponding to the point $c \cdot s$, is obtained from the module structure corresponding to s via the ring homomorphism

$$K[u_{r-1}]/u_{r-1}^p \xrightarrow{u_{r-1}\mapsto c^{p^{r-1}}u_{r-1}} K[u_{r-1}]/u_{r-1}^p.$$

Since this ring homomorphism is clearly an isomorphism, the statement follows. □

In the unipotent case, one can give a slightly different description of the support of a module using the following lemma.

Lemma 6.2. *Let M be a finite-dimensional rational module over an infinitesimal group scheme G. Denote by J_M the annihilator of the left $H^{ev}(G,k)$-module $H^*(G,M)$. Then*

(a) $I_M \subset J_M$.

(b) *If G is unipotent, then $\sqrt{I_M} = \sqrt{J_M}$.*

Proof. We have a natural left action of $H^{ev}(G,\Lambda)$ on $H^*(G,M)$ and the action of $H^{ev}(G,k)$ on $H^*(G,M)$ is obtained from this one via the homomorphism $\rho_\Lambda : H^{ev}(G,k) \to H^{ev}(G,\Lambda)$. This makes the first statement obvious.

Assume now that G is unipotent. In this case $M^\#$ admits a filtration of length $d = dim_k M$ with trivial one-dimensional factors. Assuming now that $z \in H^{2n}(G,k)$ annihilates $H^*(G,M)$ we conclude easily that z^d annihilates $H^*(G, M \otimes M^\#) = H^*(G,\Lambda)$, i.e. $z^d \in I_M$. □

The following proposition is a special case (namely, G is assumed to have height 1) of our main Theorem 6.7.

Proposition 6.3. *Let G be an infinitesimal group scheme of height 1, and let M be a finite-dimensional rational G-module. The following conditions on a cohomology class $z \in H^{2n}(G,k)$ are equivalent:*

(a) $z \in \sqrt{I_M}$.

(b) *The function $\psi(z) \in k[V_1(G)]$ takes zero value at all points $s \in V_1(G)_M$.*

Proof. The statement is trivial for zero-dimensional classes, so we assume that $n > 0$. By definition $z \in \sqrt{I_M}$ if and only if $\rho(z) \in H^{2n}(G,\Lambda)$ is nilpotent. By Theorem 4.3, $\rho(z)$ is nilpotent if and only if $\nu_s^*(\rho(z)_{k(s)}) = \rho(\nu_s^*(z_{k(s)})) \in H^{2n}(\mathbb{G}_{a(1)} \otimes_k k(s), \Lambda_{k(s)})$ is nilpotent for every point $s \in V_1(G)$. Lemma 6.2 shows further that this last condition is equivalent to the condition that for every $s \in V_1(G)$ a sufficiently high power of $\nu_s^*(z_{k(s)})$ annihilates $H^*(\mathbb{G}_{a(1)} \otimes_k k(s), M_{k(s)})$. By definition, $\nu_s^*(z_{k(s)}) = \psi(z)(s) \cdot x_1^n$. Thus the previous condition means that either $\psi(z)(s) = 0$ or a sufficiently high power of $x_1 \in H^2(\mathbb{G}_{a(1)} \otimes_k k(s), k(s))$ annihilates $H^*(\mathbb{G}_{a(1)} \otimes_k k(s), M_{k(s)})$. Since multiplication by x_1 defines periodicity in the cohomology of any rational $\mathbb{G}_{a(1)} \otimes_k k(s)$-module, either $\psi(z)(s) = 0$ or $H^i(\mathbb{G}_{a(1)} \otimes_k k(s), M_{k(s)}) = 0$ for $i \geq 2$ so that $M_{k(s)}$ is a projective $k(s)[\mathbb{G}_{a(1)}]^\# = k(s)[u_1]/u_1^p$-module. □

Proposition 6.3 admits the following formulation in terms of support varieties.

Corollary 6.3.1. *Let M be a finite-dimensional rational module over an infinitesimal group scheme G of height one. Then*

$$\Psi^{-1}(|G|_M) = V_1(G)_M.$$

Proof. Let $k[V_1(G)_M]$ denote the coordinate algebra of the reduced closed subscheme of $V_1(G)$ associated to the closed subset $V_1(G)_M \subset V_1(G)$, and let $\mathcal{I}_M$ denote the kernel of the surjective homomorphism $k[V_1(G)] \to k[V_1(G)_M]$. Proposition 6.3 asserts that $\psi^{-1}(\mathcal{I}_M) = \sqrt{I_M}$. Since $\Psi : V_1(G) \to |G|$ is the map induced by ψ, we conclude that $\Psi^{-1}(|G|_M) = V_1(G)_M$. □

To compute the support varieties for modules over $\mathbb{G}_{a(r)}$ we need the following lemma. A very similar result has been proved by J. Carlson in the context of elementary abelian groups (cf. [C1], 6.4).

Lemma 6.4. *Let M be a finite-dimensional vector space over k equipped with an r-tuple of commuting p-nilpotent endomorphisms $\alpha_1, ..., \alpha_r$. Furthermore, let $f \in k[X_1, ..., X_r]$ be a polynomial without constant and linear terms. Set $\tilde{\alpha}_1 = \alpha_1 + f(\alpha_1, ..., \alpha_r)$. If M is a projective $k[u]/u^p$-module with the action of u being given by α_1, then it's also projective over $k[u]/u^p$ with the action of u being given by $\tilde{\alpha}_1$ (and vice versa).*

Proof. The general case is immediately reduced to the case $\tilde{\alpha}_1 = \alpha_1 + \alpha_2\alpha_3$ and if $p \neq 2$ is reduced further to the case $\tilde{\alpha}_1 = \alpha_1 + \alpha_2^2$. We shall assume that $p \neq 2$; the case $p = 2$ may be treated similarly. We use the notation $H^i(\alpha_1, M)$ for the cohomology of $k[u]/u^p$ with coefficients in M, where u acts on M via α_1. To verify that M is projective over $k[\alpha_1 + \alpha_2^2]$ it suffices to check that

$$Ker(\alpha_1 + \alpha_2^2)/Im(\alpha_1 + \alpha_2^2)^{p-1} = H^2(\alpha_1 + \alpha_2^2, M)$$

is a trivial vector space. Furthermore, since the action of α_2 on $H^2(\alpha_1 + \alpha_2^2, M)$ is nilpotent it suffices to show that the homomorphism

$$\alpha_2 : H^2(\alpha_1 + \alpha_2^2, M) \to H^2(\alpha_1 + \alpha_2^2, M)$$

is injective. So let $x \in Ker(\alpha_1 + \alpha_2^2)$ be an element such that

$$\alpha_2(x) = (\alpha_1 + \alpha_2^2)^{p-1}(y)$$

for some $y \in M$. Note the formula

$$\text{(6.4.1)} \qquad (X + Y)^{p-1} = \frac{X^p + Y^p}{X + Y} = X^{p-1} - X^{p-2}Y + ... - XY^{p-2} + Y^{p-1}.$$

Using this formula, we get

$$(\alpha_1 + \alpha_2^2)^{p-1} = \sum_{i=[\frac{p}{2}]}^{p-1} (-1)^i \alpha_1^i \alpha_2^{2(p-1-i)}$$

and further

$$\alpha_1(x) = -\alpha_2^2(x) = -\alpha_2(\sum_{i=[\frac{p}{2}]}^{p-1}(-1)^i\alpha_1^i\alpha_2^{2(p-1-i)}y)$$
$$= \alpha_1(\sum_{i=[\frac{p}{2}]}^{p-1}(-1)^{i-1}\alpha_1^{i-1}\alpha_2^{2(p-1-i)+1}y).$$

Thus $x - \sum_{i=[\frac{p}{2}]}^{p-1}(-1)^{i-1}\alpha_1^{i-1}\alpha_2^{2(p-1-i)+1}y \in Ker\ \alpha_1 = Im\ \alpha_1^{p-1}$ and hence

$$x = \sum_{i=[\frac{p}{2}]}^{p-1}(-1)^{i-1}\alpha_1^{i-1}\alpha_2^{2(p-1-i)+1}y + \alpha_1^{p-1}z$$

for some $z \in M$. Applying α_2 to this formula and taking into account that $\alpha_2(x) = (\alpha_1+\alpha_2^2)^{p-1}(y)$, we conclude

$$\sum_{i=[\frac{p}{2}]}^{p-1}(-1)^i\alpha_1^i\alpha_2^{2(p-1-i)}y = \sum_{i=[\frac{p}{2}]-1}^{p-2}(-1)^i\alpha_1^i\alpha_2^{2(p-1-i)}y + \alpha_2\alpha_1^{p-1}z.$$

The term corresponding to $i = [\frac{p}{2}] - 1$ on the right vanishes and the above formula simplifies to $\alpha_1^{p-1}y = \alpha_1^{p-1}\alpha_2 z$. Hence $y - \alpha_2(z) \in Ker\ \alpha_1^{p-1} = Im\ \alpha_1$; in other words, $y = \alpha_2 z + \alpha_1 t$ for some $t \in M$. Now, one checks easily that $x = (\alpha_1+\alpha_2^2)^{p-1}(z - \alpha_2 t)$ and hence $\overline{x} = 0 \in H^2(\alpha_1+\alpha_2^2, M)$. □

Proposition 6.5. *Let M be a finite-dimensional rational $\mathbb{G}_{a(r)}$-module. Then*

$$\Psi^{-1}(|\mathbb{G}_{a(r)}|_M) = V_r(\mathbb{G}_{a(r)})_M.$$

Proof. Since $k[\mathbb{G}_{a(r)}]^\# = k[u_0,...,u_{r-1}]/(u_0^p,...,u_{r-1}^p) = k[\mathbb{G}_{a(1)}^{\times r}]^\#$, the category of rational $\mathbb{G}_{a(r)}$-modules is equivalent to the category of rational $\mathbb{G}_{a(1)}^{\times r}$-modules. For any rational $\mathbb{G}_{a(r)}$-module M, we shall denote by $\widetilde{M}$ the same module considered as a rational $\mathbb{G}_{a(1)}^{\times r}$-module. Note that for any M we have a natural isomorphism $H^*(\mathbb{G}_{a(r)}, M) = H^*(\mathbb{G}_{a(1)}^{\times r}, \widetilde{M})$. Moreover the isomorphism $H^*(\mathbb{G}_{a(r)}, k) = H^*(\mathbb{G}_{a(1)}^{\times r}, k)$ is an isomorphism of k-algebras and the isomorphism $H^*(\mathbb{G}_{a(r)}, M) = H^*(\mathbb{G}_{a(1)}^{\times r}, \widetilde{M})$ is compatible with the action of $H^*(\mathbb{G}_{a(r)}, k) = H^*(\mathbb{G}_{a(1)}^{\times r}, k)$ on both sides. Thus we get a natural isomorphism of support schemes $|\mathbb{G}_{a(r)}| \xrightarrow{\sim} |\mathbb{G}_{a(1)}^{\times r}|$ and this isomorphism takes $|\mathbb{G}_{a(r)}|_M$ onto $|\mathbb{G}_{a(1)}^{\times r}|_{\widetilde{M}}$ for each M.

In [S-F-B], §1, we exhibited an identification of the scheme $V_r(\mathbb{G}_{a(r)})$ with the affine space $\mathbb{A}^r$. This identification takes a point with coordinates $(a_0,...,a_{r-1})$ to the endomorphism ν_a of $\mathbb{G}_{a(r)}$ given by the additive polynomial $a_0T + a_1T^p + ... + a_{r-1}T^{p^{r-1}}$. With this identification,

$$\nu_a^*(x_i) = a_0^{p^i}x_i + \ldots + a_{r-i}^{p^i}x_r \in H^2(\mathbb{G}_{a(r)}, k).$$

We can also identify $\mathbb{A}^r$ with $V_1(\mathbb{G}_{a(1)}^{\times r})$, this time identifying a point $(b_0,...,b_{r-1})$ with a morphism $\mathbb{G}_{a(1)} \to \mathbb{G}_{a(1)}^{\times r}$ $\quad (t \mapsto (b_0t,...,b_{r-1}t))$. The map $h : V_r(\mathbb{G}_{a(r)}) \to$

$V_1(\mathbb{G}_{a(1)}^{\times r})$ making the square

$$\begin{array}{ccc} \mathbb{A}^r = V_r(\mathbb{G}_{a(r)}) & \xrightarrow{\Psi_{\mathbb{G}_{a(r)}}} & |\mathbb{G}_{a(r)}| \\ {\scriptstyle h}\big\downarrow & & \cong\big\downarrow \\ \mathbb{A}^r = V_1(\mathbb{G}_{a(1)}^{\times r}) & \xrightarrow{\Psi_{\mathbb{G}_{a(1)}^{\times r}}} & |\mathbb{G}_{a(1)}^{\times r}| \end{array}$$

commute is given by the formula $h(a_0, \ldots, a_{r-1}) = (a_{r-1}, ..., a_1^{p^{r-2}}, a_0^{p^{r-1}})$, where $|\mathbb{G}_{a(r)}| \overset{\simeq}{=} |\mathbb{G}_{a(1)}^{\times r}|$ is the isomorphism described above. The discussion above and Corollary 6.3.1 show that

$$\begin{aligned} \Psi^{-1}_{\mathbb{G}_{a(r)}}(|\mathbb{G}_{a(r)}|_M) &= h^{-1}(V_1(\mathbb{G}_{a(1)}^{\times r})_{\widetilde{M}}) \\ &= \{(a_0, ..., a_{r-1}) : M \text{ is not projective over } k[u]/u^p, \\ &\qquad \text{with the action of } u \text{ on } M \text{ being given by the formula} \\ &\qquad\qquad u \cdot m = (a_{r-1}u_0 + a_{r-2}^p u_1 + ... + a_0^{p^{r-1}} u_{r-1}) \cdot m\}. \end{aligned}$$

We want to show that the set we get on the right of the above formula equals $V_r(\mathbb{G}_{a(r)})_M$. Consider the action of $k[u]/u^p = k[u_{r-1}]/u_{r-1}^p$ on M, corresponding to the morphism $\nu_a : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$. A straightforward computation shows that

(6.5.1)
$$(\nu_a)_*(u_{r-1}) = \sum_{i=0}^{p^r-1} \sum_{\substack{i_0+i_1+...+i_{r-1}=i \\ i_0+pi_1+...+p^{r-1}i_{r-1}=p^{r-1}}} (i_0, ..., i_{r-1})(a_0^{i_0} \cdot ... \cdot a_{r-1}^{i_{r-1}}) \cdot v_i$$

where $(i_0, \ldots, i_{r-1}) = \frac{(i_1+\cdots+i_{r-1})!}{i_0!\cdots i_{r-1}!}$ and $v_0, ..., v_{p^r-1}$ is the standard basis of $k[\mathbb{G}_{a(r)}]^{\#}$. We proved in [S-F-B] the following formula for v_i:

$$v_i = \frac{u_0^{i^{(0)}} \cdot ... \cdot u_{r-1}^{i^{(r-1)}}}{i^{(0)}! \cdot ... \cdot i^{(r-1)}!} \tag{6.5.2}$$

where $i = i^{(0)} + i^{(1)}p + ... + i^{(r-1)}p^{r-1} \quad (0 \le i^{(j)} \le p-1)$ is the p-adic expansion of i. This formula shows that each v_i is a monomial in $u_0, ..., u_{r-1}$ of degree $s(i) =$ sum of digits of i. It is clear that the coefficient at $v_0 = 1$ in (6.5.1) is equal to zero. Next we look at coefficients at $u_0 = v_1, u_1 = v_p, ..., u_{r-1} = v_{p^{r-1}}$. Assume that $i_0 + ... + i_{r-1} = p^l, i_0 + pi_1 + ... + p^{r-1}i_{r-1} = p^{r-1}$. Consider the p-adic expansions of $i_0, ..., i_{r-1}$:

$$i_0 = i_0^{(0)} + i_0^{(1)}p + ... + i_0^{(r-1)}p^{r-1}, \ \ldots, i_{r-1} = i_{r-1}^{(0)} + i_{r-1}^{(1)}p + ... + i_{r-1}^{(r-1)}p^{r-1}.$$

One checks easily that $(i_0, ..., i_{r-1}) \not\equiv 0 \mod p \Longleftrightarrow i_0^{(j)} + ... + i_{r-1}^{(j)} \le p-1$ for all $0 \le j \le r-1$. Thus, nonzero terms in the coefficient at u_l correspond to those $i_0, ..., i_{r-1}$ for which

$$p^l = i_0 + ... + i_{r-1} = (i_0^{(0)} + ... + i_{r-1}^{(0)}) + ... + (i_0^{(r-1)} + ... + i_{r-1}^{(r-1)})p^{r-1}$$

is the p-adic expansion of p^l. This implies readily that all i_j exept one has to be zero and the only nonzero one equals p^l. Looking finally at the equation $i_0 + i_1p +$

$\ldots + i_{r-1}p^{r-1} = p^{r-1}$ we conclude that $i_{r-l-1} = p^l,\ i_j = 0\ (j \neq r-l-1)$. All in all the coefficient at u_l equals $a_{r-l-1}^{p^l}$. Thus, the linear term in (6.5.1) equals

$$a_{r-1}u_0 + a_{r-2}^p u_1 + \ldots + a_0^{p^{r-1}} u_{r-1}.$$

It is easy to check that for $r = 2$ there are no nonlinear terms in (6.5.1). However for $r \geq 3$ nonlinear terms really appear in (6.5.1). For example for $r = 3$ the formula looks as follows:

$$(\nu_a)_*(u_2) = a_2u_0 + a_1^p u_1 + a_0^{p^2} u_2 + \sum_{i=1}^{p-1} a_0^{p(p-i)} a_1^i \cdot u_0^i u_1^{p-i}/(i!(p-i)!).$$

Nevertheless the proposition follows using Lemma 6.4. □

The following lemma follows immediately from the definitions of $|G|_M$ and $V_r(G)_M$.

Lemma 6.6. *Let $f : H \to G$ be a homomorphism of infinitesimal group schemes of height $\leq r$, and let M be a finite-dimensional, rational G-module (considered also as a rational H-module via f). Denote by $f_* : V_r(H) \to V_r(G)$ and $f_* : |H| \to |G|$ the associated morphisms of schemes. Then*

$$|H|_M \subset f_*^{-1}|G|_M, \quad V_r(H)_M = f_*^{-1}(V_r(G)_M).$$

Theorem 6.7. *Let G be an infinitesimal group scheme of height $\leq r$, and let M be a finite-dimensional rational G-module. Let further $z \in H^{2n}(G,k)$ be a cohomology class. The following conditions are equivalent:*

(a) $z \in \sqrt{I_M}$.

(b) *The function $\psi(z) \in k[V_r(G)]$ takes zero value at all points $s \in V_r(G)_M$.*

Proof. Assume first that $z \in \sqrt{I_M}$ and let s be any point of $V_r(G)_M$. As usual we'll use the notation ν_s for the group scheme homomorphism $\mathbb{G}_{a(r)} \otimes_k k(s) \to G \otimes_k k(s)$ corresponding to s. The cohomology class $\nu_s^*(z_{k(s)})$ obviously belongs to $\sqrt{I_{M,s}}$, where $I_{M,s} = Ker\{H^*(\mathbb{G}_{a(r)} \otimes_k k(s), k(s)) \to H^*(\mathbb{G}_{a(r)} \otimes_k k(s), \Lambda_{k(s)})\}$. Proposition 6.5 implies now that the function $\nu_s^*(\psi(z)_{k(s)}) = \psi(\nu_s^*(z_{k(s)})) \in k(s)[V_r(G)]$ takes zero value at all points of $V_r(\mathbb{G}_{a(r)} \otimes_k k(s))_{M_{k(s)}}$ and, in particular, at the unit point $1 \in V_r(\mathbb{G}_{a(r)} \otimes_k k(s))$ of the monoid scheme $V_r(\mathbb{G}_{a(r)} \otimes_k k(s))$. However $\nu_s^*(\psi(z)_{k(s)})(1) = \psi(z)(s) \in k(s)$.

Assume now that $\psi(z)$ takes zero value at all points of $V_r(G)_M$. Let s be any point of $V_r(G)$. Using Lemma 6.6 we conclude immediately that the function $\psi(\nu_s^*(z_{k(s)})) = \nu_s^*(\psi(z)_{k(s)})$ vanishes at all points of $V_r(\mathbb{G}_{a(r)} \otimes_k k(s))_{M_{k(s)}}$. Proposition 6.5 implies that $\nu_s^*(z_{k(s)}) \in \sqrt{I_{M,s}}$. In other words, the cohomology class $\nu_s^*(\rho(z)_{k(s)}) \in H^{2n}(\mathbb{G}_{a(r)} \otimes_k k(s), \Lambda_{k(s)})$ is nilpotent. Since this holds for all points $s \in V_r(G)$ we conclude from Theorem 4.3 that $\rho(z)$ is nilpotent, i.e. $z \in \sqrt{I_M}$. □

The following corollary follows from Theorem 6.7 exactly as Corollary 6.3.1 follows from Proposition 6.3.

Corollary 6.8. *Assuming the conditions and notation of Theorem 6.7,*

$$\Psi^{-1}(|G|_M) = V_r(G)_M.$$

In other words, Ψ induces a homeomorphism $\Psi : V_r(G)_M \xrightarrow{\sim} |G|_M$.

Example 6.9. Let G be an infinitesimal algebraic group scheme, and let M be a finite-dimensional rational G-module. If M has dimension not divisible by p, then $|G|_M = |G|$. This can be seen immediately using Corollary 6.8 and the observation that a necessary condition for a finite-dimensional module over $k[u]/u^p$ to be projective is that the dimension of the module must be divisible by p.

§7. Applications

In this section, we provide various applications of Theorems 5.2 and 6.7 closely related to analogous results for finite groups proved by J. Carlson (cf. [C2]). These applications are based upon the fact that our theorems provide a noncohomological, geometric description of cohomological support varieties.

Proposition 7.1. *Let $f : H \to G$ be a homomorphism of infinitesimal group schemes, and let M be a finite-dimensional rational G-module (considered also as a rational H-module via f). Then*

$$f_*^{-1}(|G|_M) = |H|_M.$$

Proof. Assume that both H and G are of height $\leq r$. The statement follows from the following commutative square, whose vertical arrows are homeomorphisms, and Lemma 6.6:

$$\begin{tikzcd} V_r(H) \arrow[r, "f_*"] \arrow[d, "\Psi_G"'] & V_r(G) \arrow[d, "\Psi_H"'] \\ {|H|} \arrow[r, "f_*"] & {|G|} \end{tikzcd}$$

□

Theorem 7.2. *Let G be an infinitesimal group scheme, and let M, N be finite-dimensional rational G-modules. Then*

$$|G|_{M\otimes N} = |G|_M \cap |G|_N.$$

Proof. Using the Künneth Theorem, one sees easily that if G and H are infinitesimal group schemes, then the canonical morphism $((pr_1)_*, (pr_2)_*) : |G\times H| \to |G|\times|H|$ is a finite universal homeomorphism. Moreover if M (resp. N) is a rational G-module (resp. H-module), then

$$|G \times H|_{M\tilde{\otimes} N} = ((pr_1)_*, (pr_2)_*)^{-1}(|G|_M \times |H|_N)$$

where $M\tilde{\otimes}N = pr_1^*M \otimes pr_2^*N$ is the external tensor product. Applying this remark in the case $G = H$ and applying Proposition 7.1 to $\Delta : G \to G \times G$, we conclude

$$\begin{aligned} |G|_{M\otimes N} &= \Delta_*^{-1}(|G \times G|_{M\tilde{\otimes}N}) = \Delta_*^{-1}((pr_1)_*^{-1}(|G|_M) \cap (pr_2)_*^{-1}(|G|_N)) \\ &= |G|_M \cap |G|_N. \end{aligned}$$

□

For our next application, we require the following two lemmas.

Lemma 7.3. *Let G be an infinitesimal group scheme of height $\leq r$, and let $\nu : \mathbb{G}_{a(r)} \to G$ be a nontrivial group scheme homomorphism. Consider the associated homomorphism of k-algebras*

$$k[u_{r-1}]/u_{r-1}^p \hookrightarrow k[u_0, ..., u_{r-1}]/(u_0^p, ..., u_{r-1}^p) = k[\mathbb{G}_{a(r)}]^\# \xrightarrow{\nu_*} k[G]^\#.$$

This homomorphism makes $k[G]^\#$ into a projective (left) $k[u_{r-1}]/u_{r-1}^p$-module.

Proof. The homomorphism ν may be decomposed as

$$\mathbb{G}_{a(r)} \xrightarrow{p} \mathbb{G}_{a(s)} \hookrightarrow G$$

where $1 \leq s \leq r$, the first arrow is the standard projection and the second arrow is the closed embedding. Since the quotient scheme $G/\mathbb{G}_{a(s)}$ is affine we conclude (cf. [J]) that $k[G]$ is an injective rational $\mathbb{G}_{a(s)}$-module. The functor $(-)^{\#}$ sending a k-vector space to its k-linear dual defines an anti-equivalence between the category of finite-dimensional rational $\mathbb{G}_{a(s)}$-modules and the category of finite-dimensional $k[\mathbb{G}_{a(s)}]^{\#}$-modules, which shows that $k[G]^{\#}$ is a projective $k[\mathbb{G}_{a(s)}]^{\#}$-module. Consider finally the composition

$$k[u_{r-1}]/u_{r-1}^p \hookrightarrow k[\mathbb{G}_{a(r)}]^{\#} \xrightarrow{p_*} k[\mathbb{G}_{a(s)}]^{\#}.$$

Since $p_*(u_{r-1}) = u_{s-1}$ we conclude that $k[\mathbb{G}_{a(s)}]^{\#} = k[u_0, ..., u_{s-1}]/(u_0^p, ..., u_{s-1}^p)$ is a projective $k[u_{r-1}]/u_{r-1}^p$-module, which concludes the proof. □

The following lemma is closely related to lemmas proven by J. Carlson in [C2].

Lemma 7.4. *Let $x \in H^{2n}(k[u]/u^p, k)$ be a cohomology class. Consider a $k[u]/u^p$-projective resolution*

$$0 \leftarrow k \leftarrow P_0 \xleftarrow{d} P_1 \leftarrow \ \ldots$$

of k and choose a representing cocycle $f : P_{2n} \to k$ for x. Finally set $\overline{P}_{2n} = P_{2n}/d(P_{2n+1})$, $M = Ker\{f : \overline{P}_{2n} \to k\}$. The following conditions are equivalent:

(a) *$x \neq 0$.*

(b) *M is a projective $k[u]/u^p$-module.*

Proof. Observe that we have a long exact sequence with all terms except the first and the last one projective:

$$0 \leftarrow k \leftarrow P_0 \xleftarrow{d} \ \ldots \ \xleftarrow{d} P_{2n-1} \xleftarrow{\overline{d}} \overline{P}_{2n} \leftarrow 0.$$

Applying the same construction to the standard periodic resolution $Q_\bullet$ of k, we get another long exact sequence of the same kind:

$$0 \leftarrow k \leftarrow Q_0 \leftarrow \ \ldots \ \leftarrow Q_{2n-1} \leftarrow \overline{Q}_{2n} = k \leftarrow 0.$$

By Schanuel's Lemma (cf. [B]),

$$\overline{P}_{2n} \oplus \bigoplus_{i=0}^{n-1} P_{2i} \oplus \bigoplus_{i=0}^{n-1} Q_{2i+1} \cong \overline{Q}_{2n} \oplus \bigoplus_{i=0}^{n-1} Q_{2i} \oplus \bigoplus_{i=0}^{n-1} P_{2i+1}.$$

Using now the Krull-Schmidt Theorem, we conclude that $\overline{P}_{2n}$ is of the form

$$\overline{P}_{2n} = k \oplus P$$

where P is a projective (= free) $k[u]/u^p$-module. If $f_{|k} \neq 0$, then $f : \overline{P}_{2n} \to k$ splits and hence $M \cong P$ is projective. On the other hand, if $f_{|k} = 0$, then the trivial $k[u]/u^p$-module k is a direct summand in $M = Ker\{\overline{P}_{2n} \to k\}$ and hence M is not projective. Thus, condition (b) is equivalent to the condition $f_{|k} \neq 0$ (which we designate as condition (c)). Observe that each $k[u]/u^p$-linear homomorphism $P_{2n-1} \to k$ restricts trivially to the maximal trivial submodule $Ker\{u : P_{2n-1} \to P_{2n-1}\}$ of P_{2n-1}. Hence if $f_{|k} \neq 0$, then f cannot be extended to a $k[u]/u^p$-linear homomorphism $P_{2n-1} \to k$, i.e. $x \neq 0$. Assume, on the other hand, that $f_{|k} = 0$. Since every projective $k[u]/u^p$-module is also injective, a projective submodule $P \subset \overline{P}_{2n} \subset P_{2n-1}$ is a direct summand and hence we may extend $f_{|P}$ to a $k[u]/u^p$-linear homomorphism $g : P_{2n-1} \to k$. Since $f_{|k} = 0 = g_{|k}$ we conclude that $f = g_{|\overline{P}_{2n}}$ and hence $x = 0$. Thus condition (a) is also equivalent to (c). $\square$

Theorem 7.5. *Let G be an infinitesimal group scheme of height $\leq r$. Then $W \subset |G|$ is the support $|G|_M$ of some finite-dimensional rational G-module M if and only if W is a closed, conical subset of $|G|$.*

Proof. The necessity of the conditions that W be closed and conical is obvious (since the ideal I_M is homogeneous).

To prove the converse, it suffices in view of Theorem 7.2 to show that the zero locus $Z(x) \subset |G|$ of any cohomology class $x \in H^{2n}(G, k)$ may be realized in the form $|G|_M$ for an appropriate M. We proceed as in [C2]. Let

$$0 \leftarrow k \leftarrow P_0 \xleftarrow{d} P_1 \xleftarrow{d} \ \ldots$$

be a resolution of k by finitely generated projective $k[G]^{\#}$-modules. Choose a representing cocycle $f : P_{2n} \to k$ for x and set, $\overline{P}_{2n} = P_{2n}/d(P_{2n+1})$, $M = Ker\{f : \overline{P}_{2n} \to k\}$. Modifying the resolution $P_\bullet$ if necessary, we may assume that $M \neq 0$. In this case, we claim that

$$|G|_M = Z(x).$$

In view of Theorem 6.7, it suffices to check the equality

$$V_r(G)_M = \Psi^{-1}(Z(x)) = Z(\psi(x)). \tag{7.5.1}$$

Considering both sides of (7.5.1) as closed reduced subschemes of $V_r(G)$, it suffices to establish that they have the same field valued points. This amounts to showing that if K/k is a field extension and $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ is a group scheme homomorphism, then the following conditions are equivalent:

(a) M_K is not a projective $K[u_{r-1}]/u_{r-1}^p$-module.

(b) The pullback of $x_K \in H^{2n}(G \otimes_k K, K)$ via the K-algebra homomorphism

$$K[u_{r-1}]/u_{r-1}^p \hookrightarrow K[\mathbb{G}_{a(r)}]^{\#} \xrightarrow{\nu_*} K[G]^{\#}$$

is trivial in $H^{2n}(K[u_{r-1}]/u_{r-1}^p, K)$. If ν is the trivial homomorphism, then both conditions are satisfied (here we need the assumption $M \neq 0$). If, on the other hand, ν is nontrivial, then Lemma 7.3 shows that $P_\bullet \otimes_k K$ is a $K[u_{r-1}]/u_{r-1}^p$-projective resolution of K, so that the equivalence of (a) and (b) follows from Lemma 7.4. $\square$

Our next application provides retrospective motivation for the detection theorems we have presented. Recall that if G is a simple smooth group scheme with Borel subgroup $B \subset G$ and maximal torus $T \subset B$, then the group scheme $TG_{(r)}$ is defined to fit in the following cartesian square:

$$\begin{array}{ccc} TG_{(r)} & \longrightarrow & T \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow{F^r} & G \end{array}$$

The "Main Theorem" of [CPS] asserts for a simple smooth group scheme G over an algebraically closed field k that a finite-dimensional rational $TG_{(r)}$-module M is injective if and only if the restrictions of M to root subgroups $U_{\alpha(r)} \subset G_{(r)}$ are injective.

Proposition 7.6. *Let G be an infinitesimal group scheme of height $\leq r$ over an algebraically closed field k. Let further M be a finite-dimensional rational G-module. Assume that whenever $H \subset G$ is a subgroup scheme isomorphic to $\mathbb{G}_{a(s)}$ (with $s \leq r$) the restriction of M to H is an injective rational H-module. Then M is injective as a rational G-module.*

Proof. Let $x \in V_r(G)$ be a k-rational point different from the origin, and let $\nu_x : \mathbb{G}_{a(r)} \to G$ be the corresponding (nontrivial) group scheme homomorphism (over k). Denote the image of ν_x by H. Clearly $H \overset{\sim}{=} \mathbb{G}_{a(s)}$ for some $s \leq r$. Our assumption implies that M is an injective (= projective) $k[H]^{\#}$-module. Using Lemma 7.3 we conclude immediately that M is also projective over $k[u_{r-1}]/u_{r-1}^p$, i.e. $x \notin V_r(G)_M$. Since rational points are dense in any k-scheme of finite type, we conclude that either $V_r(G)_M$ is empty (in which case $M = 0$) or consists of the origin only. In the latter case we conclude from Theorem 6.7 that $\sqrt{I_M}$ coincides with the augmentation ideal of $H^{ev}(G,k)$. The $H^{ev}(G,k)$-module $Ext_G^*(M,M) = H^*(G, End_k(M))$ is finitely generated according to [F-S] and is killed by a sufficiently high power of the augmentation ideal. This shows that there exists an integer $N > 0$ such that $Ext_G^i(M,M) = 0$ for $i \geq N$. All we have to do now is to use the following result.

Lemma 7.6.1. *Let M be a finite-dimensional rational module over an infinitesimal group scheme G. Assume that there exists an integer N such that $Ext_G^i(M,M) = 0$ for $i \geq N$. Then M is injective (= projective).*

Proof. Denote by L_j $(j = 1, ..., n)$ the simple G-modules. For each j the Ext-group $Ext_G^*(L_j, M)$ has a natural structure of a left module over the Yoneda algebra $Ext_G^*(M,M)$. Moreover the action of $H^{ev}(G,k)$ on $Ext_G^*(L_j,M)$ is the composition of the above action and the homomorphism $\rho : H^{ev}(G,k) \to Ext_G^*(M,M)$. Our condition implies now that the finitely generated $H^{ev}(G,k)$-module $Ext_G^*(L_j,M) = H^*(G, L_j^{\#} \otimes M)$ is killed by a sufficiently high power of the augmentation ideal of $H^{ev}(G,k)$ and hence $Ext_G^i(L_j,M) = 0$ for all sufficiently high i. Thus, increasing N if necessary, we may assume that $Ext_G^i(L_j,M) = 0$ for all j and all $i \geq N$. An obvious induction shows further that $Ext_G^i(L,M) = 0$ for $i \geq N$ and all finite-dimensional modules L. Consider finally an injective resolution of M

$$0 \to M \to I^0 \to I^1 \to ... \to I^N \to ...$$

consisting of finite-dimensional modules and denote by Z^i the submodule of i-cocycles. We prove that all Z^i are injective (= projective) by decreasing induction on i. Assume first that $i \geq N$. The embedding $Z^i \hookrightarrow I^i$ defines an element in $Ext_G^i(Z^i, M)$ which has to be trivial since $i \geq N$. The triviality of this element means that the projection $I^{i-1} \xrightarrow{d} Z^i$ splits and hence Z^i is injective. To make the induction step we use the crucial fact that injectives and projectives are the same in our situation. Thus injectivity of Z^i implies that the exact sequence

$$0 \to Z^{i-1} \to I^{i-1} \xrightarrow{d} Z^i \to 0$$

splits and hence Z^{i-1} is injective as well. □

The proof of the following theorem is merely a repetition of the module-theoretic arguments given by Carlson in [C2].

Theorem 7.7 (cf. [C2], Thm1$'$). *Let G be an infinitesimal group scheme of height $\leq r$, and let M be a finite-dimensional rational G-module. If $V_r(G)_M$ can be written as a union of closed subsets, $V_r(G)_M = V_1 \cup V_2$, with $V_1 \cap V_2 = \{0\}$ and $dim V_1, dim V_2 \geq 1$, then M can be written as $M_1 \oplus M_2$ with $V_r(G)_{M_i} = V_i$.*

We conclude the paper with a few explicit computations of support varieties for modules over $SL_{2(r)}$. Let $H^0(\lambda) \simeq Ind_{B_2}^{SL_2}(\lambda)$ denote the induced module. Here $B_2 \subset SL_2$ is the subgroup of lower triangular matrices, λ is a dominant weight of SL_2 and we use the same notation λ for the one-dimensional rational B_2-module with character λ. Let further $L(\lambda) \subset H^0(\lambda)$ denote the socle of $H^0(\lambda)$, thus $L(\lambda)$ is a simple SL_2-module of heighest weight λ (cf. [J]). Recall also that dominant weights of SL_2 are of the form $\lambda = n\rho = n \cdot \alpha/2$, where $n \in \mathbb{N}, \rho = \alpha/2$ and α is the simple root of SL_2.

In view of Corollary 6.8 to identify the variety $|SL_{2(r)}|_M$ is the same as to identify the variety $V_r(SL_{2(r)})_M \subset V_r(SL_{2(r)})$ (here M is any finite-dimensional rational $SL_{2(r)}$-module). Recall also that for any field $k/\mathbb{F}_p$ the k-points of $V_r(SL_{2(r)})$ are given by r-tuples of commuting p-nilpotent matrices in $sl_2(k)$ (cf. [S-F-B], §1). Denote by $N(k) \subset sl_2(k)$ the subset of p-nilpotent matrices.

Proposition 7.8. *Let $\lambda = n\rho$ be a dominant weight of SL_2. Consider the p-adic expansion of n: $n = n_0 + n_1 p + ... + n_q p^q$ $(0 \leq n_i < p)$. Then for all $r \geq 1$ and all fields $k/\mathbb{F}_p$,*

(a) $|SL_{2(r)}|_{L(\lambda)}(k) = \{(\alpha_0, \ldots, \alpha_{r-1}) \in N(k)^r : [\alpha_i, \alpha_j] = 0\ \forall i, j$ *and* $\alpha_{r-i-1} = 0$ *if* $n_i = p-1\}$.

(b) $|SL_{2(r)}|_{H^0(\lambda)}(k) = \{(\alpha_0, \alpha_1, \ldots, \alpha_{r-s-1}, 0, \ldots, 0) \in N(k)^r : [\alpha_i, \alpha_j] = 0\ \forall i, j$ *and $s \geq 0$ is the least integer such that $n_s \neq p-1$*$\}$.

To verify Proposition 7.8, we require the following sublemma.

Sublemma 7.9. *Let St_s denote the Steinberg module $H^0((p^s-1)\rho) \simeq S^{p^s-1}(V)$ for each $s \geq 1$ (where V is the natural two-dimensional module on which SL_2 acts).*

(a) *For all $1 \leq s \leq r$, $|SL_{2(r)}|_{St_s}(k) = \{(\alpha_0, \ldots, \alpha_{r-s-1}, 0, \ldots, 0) \in N(k)^r : [\alpha_i, \alpha_j] = 0\}$.*

(b) *For all $0 \leq i \leq r-1$, $|SL_{2(r)}|_{St_1^{(i)}}(k) = \{(\alpha_0, \ldots, \alpha_{r-i-2}, 0, \alpha_{r-i}, \ldots, \alpha_{r-1}) \in N(k)^r : [\alpha_i, \alpha_j] = 0\}$.*

Proof (of the sublemma). For any field $k/\mathbb{F}_p$ the group $SL_2(k)$ acts by conjugation on $V_r(SL_{2(r)})(k) = \{(\alpha_0, ..., \alpha_{r-1}) \in N(k)^r : [\alpha_i, \alpha_j] = 0\ \forall\ i, j\}$ and for any SL_2-module M the subset $V_r(SL_{2(r)})_M(k)$ is clearly invariant under this action. Given $\underline{\alpha} \in V_r(SL_{2(r)})(k)$, there exists $g \in SL_2(k)$ such that $g\alpha_i g^{-1}$ is upper triangular for all i. Thus we may assume that $\alpha_i = a_i E_{12}$, $0 \leq i \leq r-1$. The homomorphism $\exp_{\underline{\alpha}} : \mathbb{G}_a \to SL_2$ is given by the composition

$$\mathbb{G}_{a(r)} \xrightarrow{p_a:\ t \mapsto a_o t + a_1 t^p + \ldots + a_{r-1} t^{p^{r-1}}} \mathbb{G}_{a(r)} \hookrightarrow \mathbb{G}_a \hookrightarrow SL_2$$

where the last arrow is the standard embedding of $\mathbb{G}_a$ into SL_2 as a subgroup of (strictly) upper triangular matrices. As in the proof of Proposition 6.5, to determine whether $\underline{\alpha}$ is in $V_r(SL_{2(r)})_{St_s}$ we must consider the action of $u_{\underline{\alpha}} = a_{r-1}u_0 + \cdots + a_0^{p^{r-1}} u_{r-1}$ on St_s. By direct calculation, one sees that $u_s, u_{s+1}, \ldots, u_{r-1}$ all act trivially on St_s, so that we are reduced to the action of $a_{r-1}u_0 + \cdots + a_{r-s}^{p^{s-1}} u_{s-1}$. On the other hand, St_s is projective over $SL_{2(s)}$, and so St_s is projective over $k[u_{\underline{\alpha}}]/u_{\underline{\alpha}}^p$

if and only if at least one of $a_{r-s}, \ldots, a_{r-1}$ is nonzero—see Lemma 7.3. In other words, $V_r(SL_{2(r)})_{St_s} = \{(\alpha_0, ..., \alpha_{r-1}) \in V_r(SL_{2(r)}) : \ \alpha_{r-s} = ... = \alpha_{r-1} = 0\}$ from which 7.9(a) follows. The proof of 7.9(b) is analogous, once one observes that only u_i acts nontrivially on $St_1^{(i)}$. □

Proof (of Proposition 7.8). By Steinberg's twisted tensor product theorem, $L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_q)^{(q)}$ with $\lambda_i = n_i\rho$, and so by Theorem 7.2,

$$V_r(SL_{2(r)})_{L(\lambda)} = \bigcap_{i=1}^{q} V_r(SL_{2(r)})_{L(\lambda_i)^{(i)}}.$$

If $n_i \neq p-1$, $dim\ L(\lambda_i)^{(i)} = n_i + 1 < p$ and hence $V_r(SL_{2(r)})_{L(\lambda_i)^{(i)}} = V_r(SL_{2(r)})$ by Example 6.9. On the other hand, if $n_i = p-1$, then $L(\lambda_i)^{(i)} = St_1^{(i)}$. Applying 7.9(b) and combining the two cases, we conclude 7.8(a).

Note that if $\lambda_0 \neq p-1$, then the dimension of $H^0(\lambda)$ $(= n+1$ for $\lambda = n\rho)$ is not divisible by p. Let s be the least integer such that $n_s \neq p-1$. Then

$$\lambda = (p^s - 1)\rho + p^s n_s \rho + \cdots + p^q n_q \rho \equiv (p^s - 1)\rho + p^s \lambda'$$

and so by [J], II.3.19, $H^0(\lambda) = H^0((p^s-1)\rho + (\lambda')^{(s)}) \simeq St_s \otimes H^0(\lambda')^{(s)}$. By assumption on s, $H^0(\lambda')^{(s)}$ has dimension not divisible by p and so applying Example 6.9 to $H^0(\lambda')^{(s)}$, 7.9(a) to St_s, and Theorem 7.2, we conclude 7.8(b). □

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