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THE L^2 $\bar{\partial}$-METHOD, WEAK LEFSCHETZ THEOREMS, AND THE TOPOLOGY OF KÄHLER MANIFOLDS

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0. Introduction

In [No], Nori studied the fundamental group of complements of nodal curves with ample normal bundle in smooth projective surfaces. The main tool was the following weak Lefschetz theorem:

Theorem (Nori). *Suppose $\Phi : U \to X$ is a local biholomorphism from a connected complex manifold U into a connected smooth projective variety X of dimension at least 2, and U contains a connected effective divisor Y with compact support and ample normal bundle. Then, for every Zariski open subset Z of X, the image of $\pi_1(\Phi^{-1}(Z))$ in $\pi_1(Z)$ is of finite index.*

For X a surface, he obtained sharp bounds for the index using the Hodge index theorem. A striking corollary of this result is the following:

Corollary (Nori). *If X and Y are connected smooth projective varieties with*

$$\dim X = \dim Y + 1 > 1,$$

and $f : Y \to X$ is a holomorphic immersion with ample normal bundle, then the image of $\pi_1(Y)$ in $\pi_1(X)$ is of finite index.

Nori's proof of these results depends heavily on deformations. The first step is to show that a large multiple of the divisor Y in the theorem moves in a family in which the general member is irreducible and meets Y and the union of these members contains an open subset of U. Unfortunately, moving arguments do not seem to apply in the higher codimensional case, because Fulton and Lazarsfeld [FL2] have observed that for a certain smooth projective 4-fold and a smooth surface Y in X with ample normal bundle constructed by Gieseker, no multiple of Y in X moves. Given the existence of a sufficiently large number of deformations, the rest of the proof of Nori's weak Lefschetz theorem has been streamlined by Campana [C1], and Kollár [K]. In [NR], another proof of Nori's theorem was given when $Z = X$ using harmonic functions, but it was the same in spirit as the earlier arguments. A survey on Lefschetz type theorems can be found in Fulton [F].

In this paper we introduce a new approach which avoids moving arguments and which gives much stronger results. In particular, the new approach allows one to

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address the case of higher codimension. Before giving precise statements, we recall some terminology. Let Y be a complex analytic subspace of complex space U. We denote the structure sheaf of U by $\mathcal{O}_U$ and the ideal sheaf of Y in U by $\mathcal{I}_Y$. The *formal completion* $\widehat{U}$ *of* U *with respect to* Y is the ringed space

$$(\widehat{U}, \mathcal{O}_{\widehat{U}}) = (Y, \lim_{\leftarrow} \mathcal{O}_U/\mathcal{I}_Y^n).$$

If $\mathcal{F}$ is an analytic sheaf on U we denote by $\widehat{\mathcal{F}}$ the associated analytic sheaf on $\widehat{U}$ given by

$$\widehat{\mathcal{F}} = \lim_{\leftarrow}(\mathcal{F} \otimes \mathcal{O}_U/\mathcal{I}_Y^n).$$

If $\mathcal{F}$ is coherent, then $\widehat{\mathcal{F}}$ is also coherent over $\mathcal{O}_{\widehat{U}}$. The main result is the following generalization of Nori's weak Lefschetz theorem:

Theorem 0.1. *Suppose $\Phi : U \to X$ is a holomorphic map from a connected complex manifold U into a connected smooth projective variety X of dimension at least 2 which is a submersion at some point. Let $Y \subset U$ be a connected compact analytic subspace such that $\dim H^0(\widehat{U}, \widehat{\mathcal{L}}) < \infty$ for every locally free analytic sheaf $\mathcal{L}$ on U. Then, for every Zariski open subset Z of X, the image of $\pi_1(\Phi^{-1}(Z))$ in $\pi_1(Z)$ is of finite index.*

Remarks. 1. For example, by a theorem of Hartshorne [H] (and Grothendieck [Gr]), $H^0(\widehat{U}, \widehat{\mathcal{L}})$ is finite dimensional when Y is a connected compact analytic subspace which is locally a complete intersection and which has ample normal bundle (or even k-ample normal bundle in the sense of Sommese [So] where $k = \dim Y - 1$).

2. Theorem 0.1 also holds for U irreducible and reduced and X normal and projective. Moreover, as will be shown in Sect. 3 (Corollary 3.4), in the smooth case one only needs finite dimensionality for $\mathcal{L}$ the analytic pullback of an invertible sheaf on X.

3. As a consequence of Theorem 0.1, one can remove the dimension restriction on the subspace Y in the corollary to Nori's theorem. More precisely, we get the following:

Corollary 0.2. *If X and Y are connected smooth projective varieties of positive dimension and $f : Y \to X$ is a holomorphic immersion with ample normal bundle, then the image of $\pi_1(Y)$ in $\pi_1(X)$ is of finite index.*

Hironaka and Matsumura [HM] proved the analogous result for algebraic fundamental groups when f is an inclusion with ample normal bundle. However, the result for topological fundamental groups (as stated in the above corollary) is new (provided $\dim X > \dim Y + 1$) even for f an inclusion. Moreover, simple examples show that, if $\dim X > \dim Y + 1$, then, even if f is an inclusion (with ample normal bundle), the map $\pi_1(Y) \to \pi_1(X)$ is not necessarily surjective.

The idea of the proof of Theorem 0.1 is to form a covering space $\widetilde{Z} \to Z$ with fundamental group equal to the image G of $\pi_1(\Phi^{-1}(Z))$ and then construct L^2 holomorphic sections of a suitable line bundle which separate the sheets of the covering. This construction is a standard application of the L^2 $\bar{\partial}$-method (Andreotti-Vesentini [AV], Hörmander [Ho], Skoda [Sk], Demailly [D1]). Pulling these sections back to $\Phi^{-1}(Z)$ by a lifting of Φ, the finite dimensionality of the space of holomorphic sections on the formal completion gives a bound on the dimension of the space of sections on $\widetilde{Z}$ and hence a bound on the degree of the covering space (i.e., on the index of G).

Remark. Campana [C2] has independently applied L^2-methods to study exceptional curves on coverings of surfaces.

The second main result of this paper generalizes a theorem of Burns [B] which states that a quotient of the unit ball in $\mathbb{C}^n$ ($n \geq 3$) by a discrete group of automorphisms which has a strongly pseudoconvex boundary component has only finitely many ends. The main tools are a theorem of Lempert on the compactification of a pseudoconvex boundary from the pseudoconcave side [L], a finiteness theorem of Andreotti for pseudoconcave manifolds [A], and the L^2 Riemann-Roch inequality of Nadel and Tsuji [NT]. The precise statement is as follows:

Theorem 0.3. *If a complete Hermitian manifold X of (complex) dimension at least 3 has a strongly pseudoconvex end and* $\operatorname{Ricci}(X) \leq -C$ *for some positive constant C, then, away from the strongly pseudoconvex end, the manifold has finite volume.*

As in the proof of Theorem 0.1, the idea is to apply finite dimensionality of the space of holomorphic sections of a line bundle. By Lempert's theorem, one can cap off the strongly pseudoconvex end by a domain in a smooth projective variety. Andreotti's finiteness theorem applied to the resulting pseudoconcave manifold gives finite dimensionality of the space of holomorphic sections of a suitable line bundle. Finally, the L^2 Riemann-Roch inequality of Nadel and Tsuji gives a (finite) upper bound for the volume in terms of the dimension of this space of sections.

Remark. One natural question which arises is might there be an improved version of the L^2 Riemann-Roch inequality which would give improved bounds for the volume in Theorem 0.3 as well as the index in Theorem 0.1? Also, for X a surface in the corollary to Nori's weak Lefschetz theorem, Nori [No] found bounds for the index in terms of certain intersection numbers. It is therefore natural to look for analogous bounds in more general cases.

Sect. 1 begins with a proof of Theorem 0.1 in the case where Φ is a local biholomorphism. The main idea of the new approach is easy to see in this context, and, although a few technicalities arise in the general case, the proof is essentially the same. The proof of Theorem 0.1 is then given. The required result from the L^2 $\bar{\partial}$-method is discussed in Sect. 2. Further generalizations of the weak Lefschetz theorem for X not necessarily projective are considered in Sect. 3. Theorem 0.3 is proved in Sect. 4, which may be read independently of Sects. 1–3.

1. Weak Lefschetz theorems for a projective variety

This section contains the proof of Theorem 0.1. We first prove the theorem for the case of a local biholomorphism. This is a direct generalization to immersed complex spaces of *arbitrary* codimension (Nori proved the theorem stated below for Y an ample divisor in U). More general versions will be stated later. Aside from a few minor technical problems, however, the proofs of all of the generalizations are the same in spirit as the proof of this special case.

Theorem 1.1. *Let U be a connected complex manifold, let X be a connected smooth projective variety of dimension $n > 1$, let $\Phi : U \to X$ be a holomorphic map, let Y be a connected compact analytic subspace (not necessarily reduced) of U, and let $\widehat{U}$ be the formal completion of U with respect to Y. Assume that*

(i) *Φ is locally biholomorphic, and*

(ii) $\dim H^0(\widehat{U}, \widehat{\mathcal{O}(\Phi^* L)}) < \infty$ *for every holomorphic line bundle* L *on* X.

Then there is a positive constant b *depending only on the mapping* $\Phi : U \to X$ *and the subspace* $Y \subset U$ *such that, if* $R \subset X$ *is a nowhere dense analytic subset of* X *and* V *is a connected neighborhood of* Y *in* U*, then the image* G *of* $\pi_1(V \setminus \Phi^{-1}(R)) \to \pi_1(X \setminus R)$ *is of index at most* b *in* $\pi_1(X \setminus R)$*. Moreover, if* $\Phi(Y) \cap R = \emptyset$*, then the image of* $\pi_1(Y) \to \pi_1(X \setminus R)$ *is also of index at most* b*.*

Proof. Given R,V, and G as in the statement of the theorem, let $S = \Phi^{-1}(R)$, let $M = X \setminus R$, let $W = V \setminus S$, and let $\pi : \widetilde{M} \to M$ be a connected covering space with $\pi_*(\pi_1(\widetilde{M})) = G$. Thus $\pi : \widetilde{M} \to M$ has degree $d = [\pi_1(M) : G]$ and we have the following commutative diagram of holomorphic mappings:

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ W = V \setminus S \subset V \arrow[ur, "\tilde{\Phi}"] \arrow[rr, "\Phi"'] & & X \supset X \setminus R = M \end{tikzcd}$$

Since X is projective, there exists a Hermitian holomorphic line bundle (L, h) with positive curvature and a Kähler metric g on X. As will be shown in Sect. 2 (see Corollary 2.3), the L^2 $\bar{\partial}$-method, in the form given by Skoda [Sk] and Demailly [D1], enables one to prove that there is a positive integer ν independent of R and V such that

$$d \leq \dim H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(L^\nu \otimes K_M))),$$

where K_M is the canonical bundle on M and $H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(L^\nu \otimes K_M)))$ is the space of holomorphic sections of $\pi^*(L^\nu \otimes K_M)$ which are in L^2 with respect to the Hermitian metrics $\pi^*(h \otimes g^*)$ on $\pi^*(L^\nu \otimes K_M)$ and $\pi^* g$ on $\widetilde{M}$. If s is a section in $H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(L^\nu \otimes K_M)))$, then $\tilde{\Phi}^* s$ is a holomorphic section of $\Phi^*(L^\nu \otimes K_X)$ on W. Given a point $x_0 \in S \cap V$, Φ maps a neighborhood Q of x_0 in V biholomorphically onto $\Phi(Q) \subset X$. Hence $\tilde{\Phi}$ maps $Q \setminus S$ biholomorphically onto its image in $\widetilde{M}$ and, therefore, $\tilde{\Phi}^* s$ is in L^2 on $Q \setminus S$ with respect to the Hermitian metrics $\Phi^*(h \otimes g^*)$ in $\Phi^*(L^\nu \otimes K_X)$ and $\Phi^* g$ on U. Since these metrics are defined over the entire set U and a square integrable function which is holomorphic outside a nowhere dense analytic set in a manifold extends holomorphically past the analytic set, $\tilde{\Phi}^* s$ extends to a holomorphic section of $\Phi^*(L^\nu \otimes K_X)$ on V. Therefore

$$d \leq \dim H^0(V, \mathcal{O}(\Phi^*(L^\nu \otimes K_X))).$$

On the other hand, by a general fact about formal completions, if $\mathcal{F}$ is a coherent analytic sheaf on V, then the kernel of the mapping

$$H^0(V, \mathcal{F}) \to H^0(\widehat{V}, \widehat{\mathcal{F}}) = H^0(\widehat{U}, \widehat{\mathcal{F}})$$

consists of all of the sections of $\mathcal{F}$ on V which vanish on a neighborhood of Y in V (see [BS, Proposition VI.2.7]). In particular, if $\mathcal{F}$ is locally free, then this mapping is injective. Therefore, taking $\mathcal{F} = \mathcal{O}(\Phi^*(L^\nu \otimes K_X))$, we get

$$d \leq \dim H^0(V, \mathcal{F}) \leq \dim H^0(\widehat{U}, \widehat{\mathcal{F}}) < \infty.$$

Thus $b = \dim H^0(\widehat{U}, \widehat{\mathcal{F}})$ is a uniform bound for d independent of R and V.

Finally, if $\Phi(Y) \cap R = \emptyset$, then we may choose the neighborhood V so that $V \subset U \setminus S$ and the map $\pi_1(Y) \to \pi_1(V)$ is a surjective isomorphism. Hence the image of $\pi_1(Y)$ in $\pi_1(M)$ is equal to the image of $\pi_1(V) = \pi_1(V \setminus S)$ and therefore is of index at most b. $\square$

We now consider generalizations. If in the above theorem one assumes only that Φ is a generic submersion (or a generic local biholomorphism), then a slight technical problem arises. While (as one may easily check) the section $\tilde{\Phi}^* s$ of $\Phi^*(L^\nu \otimes K_X)$ extends holomorphically past S near points at which Φ is submersive, $\tilde{\Phi}^* s$ need not extend near points where $\operatorname{rank} \Phi_* < n$. However, as we will see, $\tilde{\Phi}^* s$ does extend as a holomorphic n-form with values in $\Phi^* L^\nu$. A simple illustration is given by

$$U = \Delta \ni z \overset{\Phi}{\mapsto} \zeta = z^2 \in \Delta, \quad \widetilde{M} = \Delta^* \ni z \overset{\pi}{\mapsto} \zeta = z^2 \in \Delta^* = M, \quad \text{and } s = z^{-1} \pi^* d\zeta.$$

Here, $\tilde{\Phi}^* s$ does not extend as a section of the pullback of the canonical bundle, but the corresponding holomorphic 1-form $2dz$ does extend.

In fact, by passing to desingularizations, one also gets this extension property for U and X singular. Given an irreducible reduced complex space A and a positive integer n, we denote by Ω^n_A the coherent analytic sheaf on A obtained by forming a desingularization $\check{A} \to A$ of A and taking the direct image of $\Omega^n_{\check{A}}$. By the following lemma, this sheaf is independent of the choice of the desingularization.

Lemma 1.2 (Grauert and Riemenschneider [GR, Sect. 2.1])**.** *Let A be an irreducible reduced complex space of dimension m and let n be a positive integer. Suppose that, for $i = 1, 2$, B_i is a connected complex manifold of dimension m and $\Psi_i : B_i \to A$ is a proper modification. Then $(\Psi_1)_* \Omega^n_{B_1} = (\Psi_2)_* \Omega^n_{B_2}$.*

The proof is similar to the proof for $\dim A = n$ given in [GR]. The main point is that if A is smooth, then Ψ_1 is biholomorphic outside an analytic set of codimension at least 2 in A. For the general case, one passes to a common proper modification of B_1 and B_2.

We may now state the extension property as follows:

Lemma 1.3. *Let $\Phi : U \to X$ be a holomorphic mapping of irreducible reduced complex spaces U and X of dimensions m and n, respectively, such that $\Phi(U)$ has nonempty interior. Suppose*

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ W = U \setminus S \subset U \arrow[ur, "\tilde{\Phi}"] \arrow[rr, "\Phi"'] & & X \supset X \setminus R = M \end{tikzcd}$$

is a commutative diagram of holomorphic mappings where $R \subset X$ is a nowhere dense analytic subset which contains X_{sing}, $S = \Phi^{-1}(R)$, and $\pi : \widetilde{M} \to M$ is a connected holomorphic covering space. Let L be a holomorphic line bundle on X and let θ be a holomorphic n-form with values in $\pi^* L$ on $\widetilde{M}$ which is in L^2 with respect to the liftings of a Hermitian metric h in L on X and a Hermitian metric g on M. Then the pullback $(\tilde{\Phi}|_{W_{\text{reg}}})^* \theta$ of θ to a holomorphic n-form with values in $\Phi^* L$ on W_{reg} extends to a (unique) section in $H^0(U, \mathcal{O}(\Phi^* L) \otimes \Omega^n_U)$.

The proof uses standard methods but will be postponed until the end of this section (see also Sakai [S]). We may now apply the argument given in the proof of Theorem 1.1 to get Theorem 0.1 of the introduction. In fact, we get the following:

Theorem 1.4. *Let U be an irreducible reduced complex space, let X be a connected normal projective variety of dimension $n > 1$, let $\Phi : U \to X$ be a holomorphic map, let Y be a connected compact analytic subspace (not necessarily reduced) of U, and let $\widehat{U}$ be the formal completion of U with respect to Y. Assume that*

(i) *$\Phi(U)$ has nonempty interior, and*
(ii) *$\dim H^0(\widehat{U}, \widehat{\mathcal{O}(\Phi^* L)} \otimes \widehat{\Omega^n_U}) < \infty$ for every holomorphic line bundle L on X.*

Then there exists a positive constant b depending only on the mapping $\Phi : U \to X$ and the subspace Y such that , if $R \subset X$ is a nowhere dense analytic subset and V is a connected neighborhood of Y in U, then the image of $\pi_1(V \setminus \Phi^{-1}(R)) \to \pi_1(X \setminus R)$ is of index at most b. Moreover, if $\Phi(Y) \cap R = \emptyset$, then the image of $\pi_1(Y)$ in $\pi_1(X \setminus R)$ is also of index at most b.

Proof. Given R and V as in the statement of the theorem, we get a commutative diagram

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ W = V \setminus S \subset V \arrow[ur, "\tilde{\Phi}"] \arrow[rr, "\Phi"'] & & X \supset X \setminus R = M \end{tikzcd}$$

as in the proof of Theorem 1.1. Since X is normal, the map $\pi_1(M \setminus X_{\text{sing}}) \to \pi_1(M)$ is surjective. Therefore, by replacing R by $R \cup X_{\text{sing}}$, we may assume that $X_{\text{sing}} \subset R$; i.e., that M is a complete Kähler manifold. If $s \in H^0_{L^2}(\widetilde{M}, \mathcal{O}((\pi^* L^\nu) \otimes K_{\widetilde{M}}))$ for some ν (with respect to metrics lifted from the base), then, by Lemma 1.3, the pullback to W_{reg} *as a holomorphic n-form with values in* $\Phi^* L^\nu$ extends to a unique section in $H^0(V, \mathcal{O}(\Phi^* L^\nu) \otimes \Omega^n_U)$. By applying Corollary 2.3 as in the proof of Theorem 1.1, one now gets the required bound on the index. □

A finiteness theorem of Hartshorne [H, Theorem III.4.1] and Grothendieck [Gr] and the above theorem together imply immediately that, in Nori's weak Lefschetz theorem, one may take the mapping to be a generic submersion and the subvariety to be of arbitrary codimension. More precisely, we have the following:

Corollary 1.5. *Let U be a connected complex manifold, let X be a connected normal projective variety of dimension $n > 1$, let $\Phi : U \to X$ be a holomorphic map, and let Y be a positive dimensional connected compact analytic subspace (not necessarily reduced) of U. Assume that*

(i) *$\Phi(U)$ has nonempty interior,*
(ii) *Y is locally a complete intersection in U, and*
(iii) *the normal bundle $N_{Y/U}$ is ample.*

Then there is a positive constant b depending only on the mapping $\Phi : U \to X$ and the subspace $Y \subset U$ such that, if Z is a nonempty Zariski open subset of X and V is a connected neighborhood of Y in U, then the image of $\pi_1(V \cap \Phi^{-1}(Z)) \to \pi_1(Z)$

is of index at most b in $\pi_1(Z)$. Moreover, if $\Phi(Y) \subset Z$, then the image of $\pi_1(Y)$ in $\pi_1(Z)$ is also of index at most b.

Remarks. 1. The approach of considering sections of vector bundles on formal completions fits well with Grothendieck's approach to Lefschetz theorems for the algebraic fundamental group [Gr] (see also [H]). In a sense, the results of this paper (for the topological fundamental group) are extensions of Grothendieck's Lefschetz theorems.

2. Further generalizations in which X is not necessarily projective will be stated and proved in Sect. 3. A slightly more precise bound for the index in terms of the dimension of a space of sections will also be obtained.

We conclude this section with the proof of the extension property.

Proof of Lemma 1.3. We first observe that we may assume that U and X are smooth and that R is a divisor with normal crossings by passing to desingularizations. More precisely, we may form a commutative diagram

$$\begin{tikzcd} U \times_X X' \arrow[r, "\mathrm{pr}_{X'}"] \arrow[d, "\mathrm{pr}_U"'] & X' \arrow[d, "\beta"] \\ U \arrow[r, "\Phi"'] & X \end{tikzcd}$$

where X' is a connected complex manifold, $R' = \beta^{-1}(R)$ of R is a divisor with normal crossings, and $\beta : X' \to X$ is a proper modification which maps $M' = X' \setminus R'$ biholomorphically onto $M = X \setminus R$. Since $\mathrm{pr}_U^{-1}(W) = W \times_M M'$ is just the graph of the restriction of Φ to a mapping $W \to M' = M$ and U is irreducible, $\mathrm{pr}_U^{-1}(W)$ is an open irreducible subset of $U \times_X X'$ which is mapped isomorphically onto W. In particular, $\mathrm{pr}_U^{-1}(W)$ lies in a unique irreducible component C of $U \times_X X'$; and, since pr_U is a proper mapping, we must have $\mathrm{pr}_U(C) = U$. Passing to a desingularization of C, we get a commutative diagram of holomorphic mappings

$$\begin{tikzcd} U' \arrow[r, "\Phi'"] \arrow[d, "\alpha"'] & X' \arrow[d, "\beta"] \\ U \arrow[r, "\Phi"'] & X \end{tikzcd}$$

where U' is a connected complex manifold of dimension m, $\alpha : U' \to U$ is a proper modification, $S' \equiv \alpha^{-1}(S) = (\Phi')^{-1}(R')$, and, if $W' = U' \setminus S' = \alpha^{-1}(W)$, then α maps the set $W' \setminus \alpha^{-1}(U_{\mathrm{sing}})$ biholomorphically onto W_{reg}. We also get a connected covering space $\pi' = (\beta|_{M'})^{-1} \circ \pi : \widetilde{M} \to M'$ and a lifting $\tilde{\Phi}' = \tilde{\Phi} \circ (\alpha|_{W'}) : W' \to \widetilde{M}$ of $\Phi'|_{W'}$.

Therefore, if $L' = \beta^* L$, then θ is a holomorphic n-form with values in $\pi^* L = (\pi')^* L'$ which is in L^2 with respect to the metrics $\pi^* h = (\pi')^* \beta^* h$ in $(\pi')^* L'$ and

$\pi^* g = (\pi')^* \beta^* g$ on $\widetilde{M}$. Suppose the pullback of θ to W' extends to a section

$$\eta \in H^0(U', \mathcal{O}((\Phi')^* L') \otimes \Omega^n_{U'}).$$

Since $(\Phi')^* L' = \alpha^* \Phi^* L$ and $\alpha : U' \to U$ is a proper modification, we have (by the definition of Ω^n_U and Lemma 1.2)

$$\alpha_* \big(\mathcal{O}((\Phi')^* L') \otimes \Omega^n_{U'}\big) = \mathcal{O}(\Phi^* L) \otimes \Omega^n_U.$$

Hence η determines an extension of $(\tilde{\Phi}|_{W_{\mathrm{reg}}})^* \theta$ to a section in $H^0(U, \mathcal{O}(\Phi^* L) \otimes \Omega^n_U)$. Thus we may assume that U and X are smooth and that R is a divisor with normal crossings in X. In particular, $S = \Phi^{-1}(R)$ is a divisor in U.

Since the lemma is entirely local, it suffices to extend the section near each point $x_0 \in S$ and we may assume that $U = \Delta^m$ is the unit polydisk centered at $x_0 = 0$ in $\mathbb{C}^m$, that $X = \Delta^n$ is the unit polydisk centered at $\Phi(x_0) = 0$ in $\mathbb{C}^n$, that L is the trivial line bundle with the trivial metric on X (since all metrics are comparable on relatively compact subsets), and that g is the restriction of the Euclidean metric $g_{\mathbb{C}^n}$ to M (since the L^2 condition on forms of type $(n, 0)$ is independent of the choice of the metric on an n-dimensional manifold). We denote the coordinates in $\mathbb{C}^m$ by $z = (z_1, \ldots, z_m)$, the coordinates in $\mathbb{C}^n$ by $\zeta = (\zeta_1, \ldots, \zeta_n)$, and the coordinate functions of the mapping by $\Phi = (\Phi_1, \ldots, \Phi_n)$. Thus $\theta = f d(\zeta_1 \circ \pi) \wedge \cdots \wedge d(\zeta_n \circ \pi)$ for some holomorphic function f which is square integrable on $\widetilde{M}$ with respect to $\pi^* g_{\mathbb{C}^n}$ and $\tilde{\Phi}^* \theta = (f \circ \tilde{\Phi}) d\Phi_1 \wedge \cdots \wedge d\Phi_n$ on W. Since holomorphic sections extend past analytic sets of codimension at least 2, we may assume that $x_0 \in S_{\mathrm{reg}}$ and hence that S is the zero set of z_1. Since R is a divisor with normal crossings, we may also assume that R is the zero set of $\zeta_1 \cdots \zeta_k$. Finally, if $\{\, x \in S \mid \Phi_j(x) = 0 \,\}$ is nowhere dense in S for some j, then, again, it suffices to consider a point x_0 which avoids this zero set. Thus we may assume that

$$S = \Phi^{-1}(R) = \{\Phi_j = 0\} \text{ for } j = 1, \ldots, k.$$

We now show that $(\Phi_1 \cdots \Phi_k) \cdot (f \circ \tilde{\Phi})$ extends to a holomorphic function which vanishes along S. If $x = (x_1, \ldots, x_m)$ is a point in $W = U \setminus S = \Delta^* \times \Delta^{m-1}$ near $x_0 = 0$ and $y = \Phi(x) = (y_1, \ldots, y_n)$, then we have $r_j = |y_j| < 1/2$ for $j = 1, \ldots, n$ and $r_j > 0$ for $j = 1, \ldots, k$. Thus the polydisk

$$P = \Delta(y_1; r_1) \times \cdots \times \Delta(y_k; r_k) \times \Delta(y_{k+1}; 1/2) \times \Delta(y_n; 1/2)$$

centered at y is contained in $M = (\Delta^*)^k \times \Delta^{n-k}$ and is therefore evenly covered by $\pi : \widetilde{M} \to M$. Hence π maps the component $\widetilde{P}$ of $\pi^{-1}(P)$ containing $\tilde{y} = \tilde{\Phi}(x)$ isomorphically onto P. The L^∞/L^2-estimate now gives

$$|f(\tilde{y})|^2 \leq (\mathrm{vol}(\widetilde{P}))^{-1} \int_{\widetilde{P}} |f|^2 \, dV_{\pi^* g_{\mathbb{C}^n}}.$$

As x approaches a point x_1 in S near x_0, $\mathrm{vol}(\widetilde{P}) = \mathrm{vol}(P)$ will approach 0. Therefore, after multiplying both sides of the above inequality by $(r_1 \cdots r_k)^2$ we get, since $|f|^2$ is integrable on $\widetilde{M}$,

$$|\Phi_1(x) \cdots \Phi_k(x) f(\tilde{\Phi}(x))|^2 = (r_1 \cdots r_k)^2 |f(\tilde{y})|^2 \leq \pi^{-n} 4^{(n-k)} \int_{\widetilde{P}} |f|^2 \, dV_{\pi^* g_{\mathbb{C}^n}} \to 0$$

and the claim follows.

For each $j = 1, \dots, k$, we have $\Phi_j = z_1^{\mu_j} h_j$ where $\mu_j = \mathrm{ord}_S \Phi_j$ and h_j is a unit. Therefore, setting $\mu = \mu_1 + \cdots + \mu_k$ and $\psi = d\Phi_{k+1} \wedge \cdots \wedge d\Phi_n$, we get

$$d\Phi_1 \wedge \cdots \wedge d\Phi_n = z_1^{\mu} dh_1 \wedge \cdots \wedge dh_k \wedge \psi + z_1^{\mu-1} \sum_{j=1}^{k} \mu_j h_j (-1)^{j-1} dz_1 \wedge dh_1 \wedge \cdots \wedge \widehat{dh_j} \wedge \cdots \wedge dh_k \wedge \psi.$$

Since $z_1^{\mu}(f \circ \tilde{\Phi})$ extends to a holomorphic function which vanishes along S, it follows that the n-form $\tilde{\Phi}^* \theta = (f \circ \tilde{\Phi}) d\Phi_1 \wedge \cdots \wedge d\Phi_n$ also extends holomorphically as claimed. □

2. Results from the L^2 $\bar{\partial}$-method

As described in Sect. 1, the proofs of the weak Lefschetz theorems rely on a consequence of the L^2 $\bar{\partial}$-method (Andreotti-Vesentini [AV], Hörmander [Ho], Skoda [Sk], Demailly [D1]) which will be described in this section.

Given a real-valued function φ of class C^2 on a complex manifold M of dimension n, the *Levi form* $\mathcal{L}(\varphi)$ of φ is the Hermitian tensor defined by

$$\mathcal{L}(\varphi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

in local holomorphic coordinates $(z_1, \dots, z_n)$. The function φ is said to be plurisubharmonic if $\mathcal{L}(\varphi) \geq 0$ and strictly plurisubharmonic if $\mathcal{L}(\varphi) > 0$. If (L, h) is a Hermitian holomorphic line bundle on a complex manifold M, then the *curvature tensor* $\mathcal{C}(L, h)$ of (L, h) is given by

$$\mathcal{C}(L, h) = \mathcal{L}(-\log |s|^2)$$

for any nonvanishing local holomorphic section s of L. We will need the following special case of a theorem of Demailly [D1, Theorem 5.1] concerning the $\bar{\partial}$-method for singular metrics with semipositive curvature.

Theorem 2.1 (Demailly)**.** *Let (E, h) be a Hermitian holomorphic line bundle with semipositive curvature (i.e., $\mathcal{C}(L, h) \geq 0$) on a complete Kähler manifold (M, g) of dimension n. Suppose $\varphi : M \to [-\infty, 0]$ is a function which is of class C^∞ outside a discrete subset S of M and, near each point $p \in S$, $\varphi(z) = A_p \log |z|^2$ where A_p is a positive constant and $z = (z_1, \dots, z_n)$ are local holomorphic coordinates centered at p. Assume that $\mathcal{C}(E, he^{-\varphi}) = \mathcal{C}(E, h) + \mathcal{L}(\varphi) \geq 0$ on $M \setminus S$ (and hence on M as the curvature of a singular metric) and let $\lambda : M \to [0, 1]$ be a continuous function such that $\mathcal{C}(E, h) + \mathcal{L}(\varphi) \geq \lambda g$ on $M \setminus S$. Then, for every C^∞ form θ of type $(n, 1)$ with values in E on M which satisfies*

$$\bar{\partial}\theta = 0 \quad \text{and} \quad \int_M \lambda^{-1} |\theta|^2_{h \otimes g^*} e^{-\varphi} \, dV_g < \infty,$$

there exists a C^∞ form η of type $(n, 0)$ with values in L on M such that

$$\bar{\partial}\eta = \theta \quad \text{and} \quad \int_M |\eta|^2_{h \otimes g^*} e^{-\varphi} \, dV_g \leq \int_M \lambda^{-1} |\theta|^2_{h \otimes g^*} e^{-\varphi} \, dV_g.$$

Remark. Demailly's theorem is much stronger than the above special case. This special case also follows from Theorem 4.1 of [D1], since one can approximate $\varphi(z)$ by functions which locally have the form $A_p \log(|z|^2 + \epsilon)$ near the nonsmooth points; or one can complete the metric on $M \setminus S$.

A well-known technique for producing sections with prescribed values on a discrete set gives the following:

Theorem 2.2. *Suppose (L,h) is a Hermitian holomorphic line bundle on an irreducible reduced complex space X of dimension n and the curvature of h is semipositive on X and positive at some point in X. Then there exist a positive integer ν_0 and a positive constant c_0 which depend only on X and $\mathcal{C}(L,h)$ and which have the following property. If ν is an integer with $\nu \geq \nu_0$, R is a nowhere dense analytic subset of X whose complement $M = X \setminus R$ is smooth and admits a complete Kähler metric, (F,k) is a Hermitian holomorphic line bundle on X with semipositive curvature, $E_\nu = L^\nu \otimes F$, and $\pi : \widetilde{M} \to M$ is a connected covering space of degree d $(1 \leq d \leq \infty)$, then*

$$c_0 \nu^n d \leq \dim H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(E_\nu \otimes K_M))).$$

The L^2 condition is taken with respect to the Hermitian metric $\pi^*(h^\nu \otimes k)$ in $\pi^* E_\nu$ and, for any choice of a Hermitian metric g on $\widetilde{M}$, with respect to the Hermitian metric g^* in $K_{\widetilde{M}} = \pi^* K_M$ and g on $\widetilde{M}$ (the L^2-norm of an $(n,0)$-form does not depend on the choice of the metric on the manifold).

Remarks. 1. The curvature condition on L means that if s is a nonvanishing holomorphic section of L on an open set W, then $-\log |s|_h^2$ is plurisubharmonic and, for some choice of $W \neq \emptyset$, $-\log |s|_h^2$ is strictly plurisubharmonic.

2. The proof will also show that

$$\begin{aligned} c_0 \nu^n d &\leq c_0 \nu^n (d-1) + \dim H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X) \\ &\leq \dim \Big(H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(E_\nu \otimes K_M))) + \pi^* H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X) \Big), \end{aligned}$$

where the sum in the last expression takes place in $H^0(\widetilde{M}, \mathcal{O}(\pi^*(E_\nu \otimes K_M)))$.

3. By a theorem of Demailly [D1], $M = X \setminus R$ admits a complete Kähler metric if, for example, X is a complete Kähler manifold and R is a compact analytic subset. In particular, any smooth quasiprojective variety admits a complete Kähler metric. Thus we get as a special case the following:

Corollary 2.3. *Suppose (L,h) is a positive Hermitian holomorphic line bundle on an irreducible reduced projective variety X of dimension n. Then there exists a positive integer ν_0 which depends on X and $\mathcal{C}(L,h)$ and which has the following property. If ν is an integer with $\nu \geq \nu_0$, R is a nowhere dense analytic subset of X with smooth complement $M = X \setminus R$, and $\pi : \widetilde{M} \to M$ is a connected covering space of degree d, then*

$$d \leq \dim H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(L^\nu \otimes K_M))).$$

The L^2 condition is taken with respect to the Hermitian metric $\pi^* h^\nu$ in $\pi^* L^\nu$ and, for any choice of a Hermitian metric g on $\widetilde{M}$, with respect to the Hermitian metric g^* in $K_{\widetilde{M}} = \pi^* K_M$ and g on $\widetilde{M}$.

Proof of Theorem 2.2. By hypothesis, $\mathcal{C}(L,h) \geq 0$ on X and $\mathcal{C}(L,h) > 0$ on some relatively compact open subset W of X. We may assume that $W \subset X_{\text{reg}}$ and that there exist holomorphic coordinates $z = (z_1, \ldots, z_n)$ with $|z| < 1/2$ on W. Fix a nonempty relatively compact open subset V of W and a C^∞ function ρ with

compact support in W such that $\rho \equiv 1$ on a neighborhood of $\overline{V}$, and, for each point $p \in V$, let φ_p be the function on X defined by

$$\varphi_p(x) = \begin{cases} \rho(x)\log(|z(x) - z(p)|^2) & \text{if } x \in W, \\ 0 & \text{if } x \in X \setminus W. \end{cases}$$

Then $\operatorname{supp}\varphi_p = \operatorname{supp}\rho \subset W$, $\varphi_p = \log(|z - z(p)|^2)$ (a plurisubharmonic function) on V, and there is a positive constant a_0 which does not depend on p such that $a_0\mathcal{C}(L,h) + \mathcal{L}(\varphi_p)$ is semipositive on $X \setminus \{p\}$ and positive on $W \setminus \{p\}$. Fix an integer $\nu_0 > na_0$ (later, we will also choose c_0 to depend only on a_0 and n).

Let $X_{\text{sing}} \subset R \subset X$, $\pi : \widetilde{M} \to M = X \setminus R$, and d be as in the statement of the theorem, and fix a point p in the nonempty open set $V \setminus R$. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, a nonnegative integer ν, and a C^∞ section s of $E_\nu \otimes K_X = L^\nu \otimes F \otimes K_X$ on a neighborhood of p, we denote by $\partial^{|\alpha|}s/\partial z^\alpha$ the corresponding multiple derivative of s with respect to some fixed trivialization in L and F on a neighborhood of p and the trivialization in K_X induced by the holomorphic coordinates $z = (z_1, \dots, z_n)$ on W. Similarly, if s is a C^∞ section of $\pi^*(E_\nu \otimes K_M) = \pi^*E_\nu \otimes K_{\widetilde{M}}$ on a neighborhood of a point $q \in \pi^{-1}(p)$, then we denote by $\partial^{|\alpha|}s/\partial z^\alpha$ the corresponding multiple derivative of s with respect to the trivialization and local coordinates lifted from X.

We will now apply the $\bar\partial$-method to show that if $\nu \geq \nu_0$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with $|\alpha| = \sum \alpha_j \leq (\nu/a_0) - n$, and $q \in \pi^{-1}(p)$, then there exists a section

$$s \in H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*(E_\nu \otimes K_M))) = H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^*E_\nu \otimes K_{\widetilde{M}}))$$

such that, for every multi-index β with $|\beta| \leq (\nu/a_0) - n$ and for every point $r \in \pi^{-1}(p)$, we have

$$\frac{\partial^{|\beta|}s}{\partial z^\alpha}(r) = \begin{cases} 1 & \text{if } \beta = \alpha \text{ and } r = q, \\ 0 & \text{otherwise.} \end{cases}$$

By hypothesis, there exists a complete Kähler metric g on $M = X \setminus R$. Let $\widetilde{E}_\nu = \pi^*E_\nu$ for each ν, let $\tilde{h} = \pi^*h$, let $\tilde{k} = \pi^*k$, let $\tilde{g} = \pi^*g$, and let $\tilde{\varphi}_p = \pi^*\varphi_p$. We may choose a relatively compact neighborhood U of q in $\pi^{-1}(V) \setminus (\pi^{-1}(p) \setminus \{q\})$ and a C^∞ section u of $\widetilde{E}_\nu \otimes K_{\widetilde{M}}$ with compact support in U such that u is holomorphic on a neighborhood of q in $\widetilde{M}$ and, for every multi-index β,

$$\frac{\partial^{|\beta|}u}{\partial z^\alpha}(q) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases}$$

Hence the form $\theta = \bar\partial u$ is a C^∞ $\bar\partial$-closed $(n,1)$-form with values in $\widetilde{E}_\nu$ and the support of θ is a compact subset of $U \setminus \pi^{-1}(p)$ (since $\bar\partial u = 0$ near q). By construction, there is also a continuous function $\lambda : \widetilde{M} \to [0,1]$ such that $\lambda > 0$ on $\pi^{-1}(V)$ and

$$\mathcal{C}\left(\widetilde{E}_\nu, \exp\left(-\frac{\nu}{a_0}\tilde{\varphi}_p\right)\tilde{h}^\nu \otimes \tilde{k}\right) = \nu\mathcal{C}(\pi^*L, \tilde{h}) + \mathcal{C}(\pi^*F, \tilde{k}) + \frac{\nu}{a_0}\mathcal{L}(\tilde{\varphi}_p) \geq \lambda\tilde{g}$$

on $\widetilde{M} \setminus \pi^{-1}(p)$. Moreover,

$$\int_{\widetilde{M}} \lambda^{-1}|\theta|^2_{\tilde{h}^\nu \otimes \tilde{k} \otimes \tilde{g}^*} e^{-\frac{\nu}{a_0}\tilde{\varphi}_p}\, dV_{\tilde{g}} < \infty,$$

because θ has compact support in $U \setminus \pi^{-1}(p)$, $\lambda > 0$ on U, and $\tilde{\varphi}_p$ is smooth on $U \setminus \{q\} = U \setminus \pi^{-1}(p)$. Applying Demailly's theorem (Theorem 2.1), one gets a C^∞

form η of type $(n,0)$ with values in $\widetilde{E}_\nu$ on $\widetilde{M}$ such that

$$\bar\partial\eta = \theta \quad \text{and} \quad \int_{\widetilde{M}} |\eta|^2_{\tilde{h}^\nu\otimes\tilde{k}\otimes\tilde{g}^*} e^{-\frac{\nu}{a_0}\tilde{\varphi}_p}\, dV_{\tilde{g}} < \infty.$$

In particular, the $(n,0)$-form $s = u - \eta$ is in $H^0_{L^2}(\widetilde{M}, \mathcal{O}(\widetilde{E}_\nu \otimes K_{\widetilde{M}}))$ because $\bar\partial s = 0$ and $\tilde{\varphi}_p \le 0$. Since u is holomorphic near each point $r \in \pi^{-1}(p)$, so is η. Moreover, in suitable local holomorphic coordinates $w = (w_1, \dots, w_n)$ centered at r in a neighborhood Q, we have $\tilde{\varphi}_p(w) = \log|w|^2$ and hence

$$\int_Q |\eta|^2 |w|^{-2\nu/a_0}\, dV < \infty,$$

where the notation for the metrics has been suppressed. Therefore η vanishes at r to an order greater than $(\nu/a_0) - n$. Thus, if β is a multi-index with $|\beta| \le (\nu/a_0) - n$, then $\partial^{|\beta|}\eta/z^\beta$ vanishes at each point $r \in \pi^{-1}(p)$ and the claim follows.

The claim implies that if, for each $c \ge 0$, $b_c = \binom{[c]+n}{n}$ denotes the number of multi-indices α satisfying $|\alpha| \le c$, then we have

$$\dim H^0_{L^2}(\widetilde{M}, \mathcal{O}(\widetilde{E}_\nu \otimes K_{\widetilde{M}})) \ge b_{(\nu/a_0)-n} \cdot d$$

for each integer $\nu \ge \nu_0$. It is easy to see that $b_{(\nu/a_0)-n} \ge c_0\nu^n$ for some positive constant c_0 depending only on a_0 and n and the theorem now follows. □

Remark. To obtain the inequalities given in 2 of the previous remark, we fix a point $r \in \pi^{-1}(p)$ and, for each point $q \in \pi^{-1}(p) \setminus \{r\}$ and each multi-index α with $|\alpha| \le (\nu/a_0) - n$, we form a section in $H^0_{L^2}(\widetilde{M}, \mathcal{O}(\widetilde{E}_\nu \otimes K_{\widetilde{M}}))$ as in the above proof. We then get a collection of $b_{(\nu/a_0)-n} \cdot (d-1)$ linearly independent sections and the span of this collection meets $\pi^* H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X)$ only in the zero section. Therefore

$$\begin{aligned} c_0\nu^n(d-1) &+ \dim H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X) \\ &\le \dim\Big(H^0_{L^2}(\widetilde{M}, \mathcal{O}(\widetilde{E}_\nu \otimes K_{\widetilde{M}})) + \pi^* H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X) \Big). \end{aligned}$$

Finally, observe that if we take $\widetilde{M} = M$ and $s \in H^0_{L^2}(M, \mathcal{O}(E_\nu \otimes K_M))$, then, since the L^2 condition in the canonical bundle is independent of the choice of the metric on the base manifold (provided one also takes the associated metric in the canonical bundle), the pullback s' of s to a desingularization X' of X is locally in L^2 with respect to a metric on X'. Therefore s' extends to a section in $H^0(X', \Omega^n_{X'})$ and hence s extends to a section in $H^0(X, \Omega^n_X)$. It follows that

$$c_0\nu^n \le \dim H^0(X, \mathcal{O}(E_\nu) \otimes \Omega^n_X).$$

Thus we get all of the desired inequalities.

3. Further generalizations of the weak Lefschetz theorem

Theorem 2.2 and the arguments given in the proofs of Theorem 1.1 and Theorem 1.4 now give the following generalization:

Theorem 3.1. *Let U be an irreducible reduced complex space, let X be a connected normal complex space of dimension $n > 1$, let $\Phi : U \to X$ be a holomorphic mapping, and let (L, h) be a Hermitian holomorphic line bundle on X. Assume*

that

(i) $\Phi(U)$ *has nonempty interior, and*
(ii) *the curvature of* (L,h) *is semipositive everywhere on* X *and positive at some point in* X.

Then there exist a positive integer ν_0 *and a positive constant* c_0 *which depend only on* X *and (the curvature of)* (L,h) *such that, if* R *is a nowhere dense analytic subset of* X *whose complement* $X_{\mathrm{reg}} \setminus R$ *in* X_{reg} *admits a complete Kähler metric,* V *is a (nonempty) domain in* U*, and* ν *is an integer with* $\nu \geq \nu_0$*, then*

$$c_0 \nu^n d \leq \dim H^0(V, \mathcal{O}(\Phi^* L^\nu) \otimes \Omega_U^n),$$

where d *is the index of the image* G *of* $\pi_1(V \setminus \Phi^{-1}(R)) \to \pi_1(X \setminus R)$*. In particular, if* $H^0(V, \mathcal{O}(\Phi^*(L^\nu)) \otimes \Omega_U^n)$ *is finite dimensional for some choice of a sufficiently large* ν*, then* G *is of finite index.*

Next, we show that for U and X smooth, there exists a bound on the index in terms of the dimension of the space of sections of an *invertible* sheaf. We first prove an elementary fact which relates sections of the pullback of the canonical bundle to holomorphic n-forms (see also Sakai [S]).

Lemma 3.2. *Suppose* $\Phi = (\Phi_1, \ldots, \Phi_n) : \Delta^m \to \mathbb{C}^n$ *is a holomorphic mapping and* Φ_* *has rank* n *at each point in* $\Delta^* \times \Delta^{m-1}$*. We denote the coordinates in* $\mathbb{C}^m$ *and the coordinates in* $\mathbb{C}^n$ *by* $z = (z_1, \ldots, z_m)$ *and* $\zeta = (\zeta_1, \ldots, \zeta_n)$*, respectively. Let* $l \geq 0$ *be the order of vanishing of the holomorphic* n*-form* $d\Phi_1 \wedge \cdots \wedge d\Phi_n$ *along* $\{0\} \times \Delta^{m-1}$*. Then the mapping* $\mathcal{O}(\Phi^* K_{\mathbb{C}^n}) \to \Omega^n_{\Delta^m}$ *given by*

$$s = f\Phi^*(d\zeta_1 \wedge \cdots \wedge d\zeta_n) \mapsto f z_1^{-l} d\Phi_1 \wedge \cdots \wedge d\Phi_n$$

maps $\mathcal{O}(\Phi^* K_{\mathbb{C}^n})$ *isomorphically onto the sheaf of holomorphic* n*-forms* θ *such that* $\theta_z \in \mathbb{C}(d\Phi_1 \wedge \cdots \wedge d\Phi_n)_z$ *for each point* $z \in \Delta^* \times \Delta^{m-1}$ *at which* θ *is defined.*

Remark. Here $\Phi^*(d\zeta_1 \wedge \cdots \wedge d\zeta_n)$ denotes the pullback of $d\zeta_1 \wedge \cdots \wedge d\zeta_n$ as a section of $\Phi^* K_{\mathbb{C}^n}$, while $d\Phi_1 \wedge \cdots \wedge d\Phi_n$ is the pullback as a form of type $(n,0)$.

Proof. Clearly, $z_1^{-l} d\Phi_1 \wedge \cdots \wedge d\Phi_n$ is a holomorphic n-form on Δ^m, so we get an injective mapping as described above. Conversely, suppose θ is a holomorphic n-form on an open set $V \subset \Delta^m$ and there exists a holomorphic function h on $V \cap (\Delta^* \times \Delta^{m-1})$ with $\theta = h d\Phi_1 \wedge \cdots \wedge d\Phi_n$ on $V \cap (\Delta^* \times \Delta^{m-1})$. We have

$$\theta = \sum{}' \theta_I dz_I \text{ on } V \quad \text{and} \quad d\Phi_1 \wedge \cdots \wedge d\Phi_n = \sum{}' \beta_I dz_I \text{ on } \Delta^m,$$

where $\sum'$ denotes the sum over increasing multi-indices. In particular,

$$l = \min_I (\mathrm{ord}_{\{0\} \times \Delta^{m-1}} \beta_I) = \mathrm{ord}_{\{0\} \times \Delta^{m-1}} \beta_{I_0}$$

for some multi-index I_0, and, for each nonzero coefficient β_I, we have $h = \theta_I / \beta_I$ on $V \cap (\Delta^* \times \Delta^{m-1})$. Therefore h is a meromorphic function on V with pole set contained in $\{0\} \times \Delta^{m-1}$ and $z_1^l h = \theta_{I_0} / (\beta_{I_0} / z_1^l)$. Since the intersection of the zero set of β_{I_0} / z_1^l and $\{0\} \times \Delta^{m-1}$ has codimension at least 2 in Δ^m and since the pole set of $z_1^l h$ lies in this intersection, the pole set must be empty. Therefore $z_1^l h$ is holomorphic on V and the holomorphic section $s = z_1^l h \Phi^*(d\zeta_1 \wedge \cdots \wedge d\zeta_n)$ of $\Phi^* K_{\mathbb{C}^n}$ maps to θ (on $V \cap (\Delta^* \times \Delta^{m-1})$ and hence on V). □

Theorem 3.3. *Let* U *and* X *be connected complex manifolds of dimensions* m *and* $n > 1$*, respectively, let* $\Phi : U \to X$ *be a holomorphic mapping, and let* (L,h) *be a*

Hermitian holomorphic line bundle on X. Assume that

(i) *Φ has rank n at some point (i.e., $\Phi(U)$ has nonempty interior), and*
(ii) *the curvature of (L,h) is semipositive everywhere on X and positive at some point in X.*

Then there exist a positive integer ν_0 and a positive constant c_0 which depend only on X and (the curvature of) (L,h) and there exists an effective divisor D_0 in U which depends only on the mapping $\Phi : U \to X$ such that, if R is a nowhere dense analytic subset of X whose complement $X \setminus R$ admits a complete Kähler metric, (F,k) is a Hermitian holomorphic line bundle on X with semipositive curvature, V is a (nonempty) domain in U, ν is an integer with $\nu \geq \nu_0$, $E_\nu = L^\nu \otimes F$, and d is the index of the image G of $\pi_1(V \setminus \Phi^{-1}(R)) \to \pi_1(X \setminus R)$, then we have the estimates

$$\begin{aligned} c_0\nu^n d &\leq c_0\nu^n(d-1) + \dim H^0(X, \mathcal{O}(E_\nu \otimes K_X)) \\ &\leq \dim H^0(V, \mathcal{O}(\Phi^*(E_\nu \otimes K_X) \otimes [D_0])) \\ &\leq \dim H^0(V, \mathcal{O}(\Phi^* E_\nu) \otimes \Omega_U^n). \end{aligned} \tag{1}$$

Remarks. 1. If, for some positive integer k, $L^k \otimes K_X^{-1}$ is semipositive, then we may take $F = L^k \otimes K_X^{-1}$ and we get estimates which do not involve the canonical bundle K_X.

2. The divisor D_0 which will be constructed is probably not the optimal choice.

Proof of Theorem 3.3. Guided by Lemma 3.2, we first describe D_0. The set

$$B = \{\, x \in U \mid \operatorname{rank}(\Phi_*)_x < n \,\}$$

is a nowhere dense analytic subset of U. Let $\{A_i\}$ be the collection of all of the irreducible components of B of dimension $m-1$ whose image $\Phi(A_i)$ lies in some nowhere dense analytic subset of X, let $A = \bigcup_i A_i$, and, for each i, let l_i be the minimal order of vanishing along A_i of the $(n \times n)$-minor determinants of Φ_*. In other words, if $\Phi = (\Phi_1, \ldots, \Phi_n)$ and $d\Phi_1 \wedge \cdots \wedge d\Phi_n = \sum' a_J dz_J$ with respect to local coordinates $(z_1, \ldots, z_m)$ near $x_0 \in A_i$ in U and $(\zeta_1, \ldots, \zeta_n)$ near $\Phi(x_0)$ in X, then

$$l_i = \min_J (\operatorname{ord}_{A_i} a_J).$$

We define

$$D_0 = \sum l_i A_i.$$

Given a holomorphic line bundle E on X, Lemma 3.2 implies that we have an injective linear mapping

$$H^0(U, \mathcal{O}((\Phi^*(E \otimes K_X)) \otimes [D_0])) \to H^0(U, \mathcal{O}(\Phi^* E) \otimes \Omega_U^n)$$

given as follows. Let t be a global defining section for $[D_0]$ on U. To each section $s \in H^0(U, \mathcal{O}((\Phi^*(E \otimes K_X)) \otimes [D_0]))$, we may associate a holomorphic n-form θ with values in $(\Phi^* E) \otimes [D_0]$ and θ/t is a holomorphic n-form on $U \setminus A$ with values in $\Phi^* E$. But the lemma implies that, near points of $A \setminus B_{\text{sing}}$, θ/t extends holomorphically past A. Thus θ/t extends to a holomorphic n-form on $U \setminus (A \cap B_{\text{sing}})$ with values in $\Phi^* E$. Since $\operatorname{codim} B_{\text{sing}} \geq 2$, θ/t extends holomorphically to the entire manifold U. Thus we get a mapping $s \mapsto \theta/t$ (similarly, this mapping surjects onto the space of holomorphic n-forms with values in $\Phi^* E$ whose restriction to $U \setminus A$ comes from a section of $\Phi^*(E \otimes K_X)$). In particular, the third of the inequalities in (1) holds.

Let

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ W = V \setminus S \subset V \arrow[ur, "\tilde{\Phi}"] \arrow[rr, "\Phi"'] & & X \supset X \setminus R = M \end{tikzcd}$$

be a commutative diagram as in the proof of Theorem 1.1. We will show that if $s \in H^0_{L^2}(\widetilde{M}, \mathcal{O}((\pi^* E_\nu) \otimes K_{\widetilde{M}}))$ for some ν (with respect to metrics lifted from the base), then $(\tilde{\Phi}^* s) \otimes t$ extends to a unique holomorphic section of $(\Phi^*(E_\nu \otimes K_X)) \otimes [D_0]$ on V. By Lemma 1.3, the pullback of s *as an E_ν-valued holomorphic n-form* extends to a $\Phi^* E_\nu$-valued holomorphic n-form on V. In particular, $\tilde{\Phi}^* s$ extends holomorphically as a section of $\Phi^*(E_\nu \otimes K_M)$ near each point at which Φ_* is of maximal rank (by Lemma 3.2 with $l = 0$). Moreover, $\tilde{\Phi}^* s$ extends holomorphically past analytic sets of codimension at least 2. Therefore, it suffices to show that $(\tilde{\Phi}^* s) \otimes t$ extends holomorphically near each point $x_0 \in S_{\mathrm{reg}} \cap B_{\mathrm{reg}}$ at which $S \cap B$ is of dimension $m - 1$. An irreducible component of B containing such a point x_0 must also be an irreducible component of $S = \Phi^{-1}(R)$ and must therefore be one of the irreducible components A_i of the support A of D_0. Since the pullback of s as an E_ν-valued holomorphic n-form extends to V, Lemma 3.2 and the definition of D_0 and t now imply the claim.

Clearly, if s is a holomorphic section of $E_\nu \otimes K_X$ on X, then $(\Phi^* s) \otimes t$ is a holomorphic section whose restriction to V is an extension of $(\tilde{\Phi}^* \pi^* s) \otimes t$. Thus we get an injective linear mapping of the subspace

$$\mathcal{S} = H^0_{L^2}(\widetilde{M}, \mathcal{O}((\pi^* E_\nu) \otimes K_{\widetilde{M}})) + \pi^* H^0(X, \mathcal{O}(E_\nu \otimes K_X))$$

of $H^0(\widetilde{M}, \mathcal{O}((\pi^* E_\nu) \otimes K_{\widetilde{M}}))$ into $H^0(V, \mathcal{O}((\Phi^*(E_\nu \otimes K_X)) \otimes [D_0]))$. We have, therefore,

$$\begin{aligned} \dim \mathcal{S} &\leq \dim H^0(V, \mathcal{O}(\Phi^*(E_\nu \otimes K_X) \otimes [D_0])) \\ &\leq \dim H^0(V, \mathcal{O}(\Phi^* E_\nu) \otimes \Omega^n_U). \end{aligned}$$

The second remark following Theorem 2.2 now gives the inequalities (1) for ν sufficiently large and for some constant c_0 (both depending only on (L, h) and X). $\square$

Remarks. 1. The proofs of Lemma 1.3 and Theorem 3.3 show that one can form a divisor D_R which depends on R, but which satisfies $D_R \leq D_0$ and gives a sharper estimate for the index. For example, it suffices to include only those irreducible components A_i which are contained in $S = \Phi^{-1}(R)$, so one may choose D_R to have support contained in S. Moreover, the proof of Lemma 1.3 shows that if $\Phi(A_i)$ contains a point $p \in R$ at which R is a divisor with normal crossings and u is a defining function for R near p, then one may take the coefficient of A_i to be $-1 + \mathrm{ord}_{A_i}(u \circ \Phi)$. The proof also shows that, by choosing p so that this coefficient is minimal, we get $D_R \leq D_0$.

2. Similarly, for U a normal neighborhood of a compact complex space Y and X a smooth projective variety, one can find a uniform bound on the index in terms of the dimension of a space of sections of a line bundle pulled back from X as in Theorem 1.1. More precisely, we have the following:

Corollary 3.4. *Let $\Phi : U \to X$ be a holomorphic mapping of a connected normal complex space U into a connected smooth projective variety X of dimension $n > 1$, let Y be a connected compact analytic subspace (not necessarily reduced) of U, and let $\widehat{U}$ be the formal completion of U with respect to Y. Assume that*

(i) *$\Phi(U)$ has nonempty interior, and*

(ii) *$\dim H^0(\widehat{U}, \widehat{\mathcal{O}(\Phi^* L)}) < \infty$ for every holomorphic line bundle L on X.*

Then there is a positive constant b depending only on the mapping $\Phi : U \to X$ and the subspace $Y \subset U$ such that, if $R \subset X$ is a nowhere dense analytic subset of X and V is a connected neighborhood of Y in U, then the image G of $\pi_1(V \setminus \Phi^{-1}(R)) \to \pi_1(X \setminus R)$ is of index at most b in $\pi_1(X \setminus R)$.

Sketch of the proof. First suppose U is smooth and let $D_0 = \sum l_i A_i$ be the associated divisor in U as in the proof of Theorem 3.3. By construction, each of the sets $\Phi(A_i)$ is contained in some nowhere dense analytic subset of X. By replacing U by a relatively compact neighborhood of Y, we may assume that there is a nowhere dense analytic subset C in X which contains all of these sets and that the collection of coefficients $\{l_i\}$ is bounded. Hence we may choose a positive holomorphic line bundle L on X and a holomorphic section t of L such that the divisor D_1 of the section $\Phi^* t$ satisfies $D_1 \geq D_0$.

Now let R, V, and G be as in the statement of the corollary and let

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ W = V \setminus S \subset V \arrow[ur, "\tilde{\Phi}"] \arrow[rr, "\Phi"'] & & X \supset X \setminus R = M \end{tikzcd}$$

be a commutative diagram as in the proof of Theorem 1.1. By the proof of Theorem 3.3 and the above remarks, if $s \in H^0_{L^2}(\widetilde{M}, \mathcal{O}(\pi^* L \otimes K_{\widetilde{M}}))$, then $(\tilde{\Phi}^* s) \otimes (\Phi^* t)$ extends to a holomorphic section of $\Phi^*(L^2 \otimes K_X)$ on V.

If U is connected and normal (but not necessarily smooth), then we may form a desingularization $\alpha : U' \to U$ of U and a commutative diagram:

$$\begin{tikzcd} U' \arrow[d, "\alpha"'] \arrow[dr, "\Phi'"] & \\ U \arrow[r, "\Phi"'] & X \end{tikzcd}$$

We may associate to $\Phi' : U' \to X$ (after shrinking U) a line bundle L and a section t as above, and we get the extension property for pullbacks of L^2 sections as described. On the other hand, U is normal, so $\alpha_* \mathcal{O}((\Phi')^*(L^2 \otimes K_X)) = \mathcal{O}(\Phi^*(L^2 \otimes K_X))$. Therefore the extension property also holds in U, and the usual argument now applies. $\square$

We close this section by observing that Theorem 3.1 has immediate consequences for pseudoconcave spaces. An open subset Ω of a complex space X is said to have

pseudoconcave boundary in the sense of Andreotti [A] if each point $x_0 \in \partial\Omega$ admits a fundamental system of neighborhoods W in X such that x_0 is an interior point of

$$(\widehat{W \cap \Omega})_X = \{\, x \in X \mid |f(x)| \leq \sup_{W\cap\Omega} |f| \quad \forall\, f \in \mathcal{O}(X)\,\}.$$

For example, by Proposition 10 of [A], if each irreducible component of X has dimension at least $k > 1$ and, for each point $x_0 \in \partial\Omega$, there is a C^∞ $(k-1)$-convex function φ on a neighborhood W of x_0 in X such that

$$\Omega \cap W = \{\, x \in W \mid \varphi(x) > 0\,\},$$

then Ω has pseudoconcave boundary in the sense of Andreotti. A connected complex space X is said to be *pseudoconcave in the sense of Andreotti* [A] if there exists a nonempty relatively compact open subset Ω which has pseudoconcave boundary in the sense of Andreotti and which meets each irreducible component of X. By a finiteness theorem of Andreotti [A, Theorem 1], if $\mathcal{F}$ is a torsion-free coherent analytic sheaf on a locally irreducible connected complex space X and X is pseudoconcave in the sense of Andreotti, then $\dim H^0(X, \mathcal{F}) < \infty$ (the case in which X admits a C^∞ $(k-1)$-convex exhaustion function, where the dimension of X is at least $k > 1$ at each point, is due to Andreotti and Grauert [AG]). Theorem 3.1 and Andreotti's finiteness theorem together give the following:

Corollary 3.5. *Let U be an irreducible reduced complex space, let X be a connected normal projective variety of dimension $n > 1$, and let $\Phi : U \to X$ be a holomorphic mapping. Assume that $\Phi(U)$ has nonempty interior and that U is pseudoconcave in the sense of Andreotti. Then there is a positive constant b depending only on the mapping $\Phi : U \to X$ such that, if Z is a nonempty Zariski open subset of X, then the image of $\pi_1(\Phi^{-1}(Z)) \to \pi_1(Z)$ is of index at most b in $\pi_1(Z)$.*

Remarks. 1. Clearly, Theorem 3.1 also gives a version of the above theorem in which X is not necessarily projective.

2. There are many results concerning when a compact analytic subset Y of an m-dimensional complex space U admits a strongly $(m-1)$-concave neighborhood, and hence when one may apply Andreotti's [A] (or Andreotti and Grauert's [AG]) finiteness theorem as above. For example, Okonek [O] proved that Y admits a fundamental system of such neighborhoods if $N_{Y/U}$ is Finsler-q-positive, where $q = \dim Y$.

4. Burns' theorem

The goal of this section is the following theorem:

Theorem 4.1. *Let (X, g) be a connected complete Hermitian manifold and let $M \subset X$ be a domain with nonempty smooth compact boundary ∂M in X. Assume that*

(i) *M is strongly pseudoconvex at each point of ∂M;*
(ii) *there exists a Hermitian metric a in K_M and a constant $c > 0$ such that $\mathcal{C}(K_M, a) \geq cg$ on M; and*
(iii) *X has dimension $n \geq 3$.*

Then $\mathrm{vol}_g(M) < \infty$.

Remarks. 1. Since M admits a complete Kähler metric, ∂M is necessarily connected (see, for example, Proposition 4.4 below).

2. If, for example, the Ricci curvature of g is bounded above by $-c$ on M, then the associated metric $a = g^*$ in K_M satisfies the condition (ii) since

$$\mathcal{C}(K_M, g^*) = -\mathrm{Ric}\,(g) \geq cg.$$

3. Clearly, it is not necessary to assume that M is a domain in some larger manifold X. The conclusion also holds if $M = X$ and M admits a C^∞ function which, along some end, is strictly plurisubharmonic and exhaustive; since one can then replace M by a suitable sublevel set of the function. It will, however, be more convenient to have Theorem 4.1 stated for a domain as above.

4. As in the proofs of the weak Lefschetz theorems, the idea is to apply finite dimensionality of a space of holomorphic sections of a line bundle to obtain a result about the manifold.

Theorem 4.1 and an analysis of the thick-thin decomposition as in [BGS] together give as a conclusion the following theorem:

Theorem 4.2 (Burns [B]). *Let Γ be a torsion-free discrete group of automorphisms of the unit ball B in $\mathbb{C}^n$ with $n \geq 3$ and let $M = \Gamma \setminus B$. Assume that the limit set Λ is a proper subset of ∂B and that the quotient $\Gamma \setminus ((\partial B) \setminus \Lambda)$ has a compact component A. Then M has only finitely many ends; all of which, except for the (unique) end corresponding to A, are cusps. In fact, M is diffeomorphic to a compact manifold with boundary.*

Remark. By applying a theorem of Lempert [L] as in the proof of Theorem 4.1 below and the argument given by Siu and Yau [SY], one can close up the cusps projectively. In other words, $M \cong \Omega \setminus D$, where Ω is a strongly pseudoconvex domain in a smooth projective variety and D is a (compact) divisor contained in Ω. The boundary component A corresponds to $\partial\Omega$.

The main tool in the proof of Theorem 4.1 is Nadel and Tsuji's [NT] L^2 version of Demailly's [D2] asymptotic Riemann-Roch inequality.

Theorem 4.3 (Nadel-Tsuji [NT]). *Suppose (X, g) is a connected complete Kähler manifold of dimension n and (L, h) is a Hermitian holomorphic line bundle on X such that*

$$\mathcal{C}(L, h) \geq cg$$

for some constant $c > 0$. Then

$$\liminf_{\nu\to\infty} \nu^{-n} \dim H^0_{L^2}(X, \mathcal{O}(K_X \otimes L^\nu)) \geq \frac{1}{n!} \int_X \big(c_1(L, h)\big)^n.$$

Remarks. 1. The Chern form $c_1(L, h)$ is the real form of type $(1, 1)$ (associated to the Hermitian tensor $\mathcal{C}(L, h)$) given by

$$c_1(L, h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |s|^2_h$$

for any local nonvanishing holomorphic section s of L.

2. As Nadel and Tsuji observed (see [NT, Lemma 2.5]), if, in particular, X is pseudoconcave in the sense of Andreotti [A] (see Sect. 3), then it follows that X has finite volume.

3. The theorem is only stated in [NT] for L the canonical bundle, but the proof of the general case is the same. The first point is that, for a smooth relatively compact domain Ω in X and for $\lambda > 0$, one has Demailly's [D2] generalization of

Weyl's asymptotic formula for the number of eigenvalues $N_\Omega(\lambda)$ less than or equal to $\nu\lambda$ for the Dirichlet problem for the Laplacian in $K_X \otimes L^\nu$:

$$\liminf_{\nu\to\infty} \nu^{-n} N_\Omega(\lambda) \geq \frac{1}{n!}\int_\Omega \big(c_1(L,h)\big)^n.$$

The second point is that for a C^∞ compactly supported form α of type $(n,1)$ with values in L^ν, the Bochner-Kodaira formula implies that

$$\|\bar\partial\alpha\|^2_{L^2} + \|\bar\partial^*\alpha\|^2_{L^2} \geq c\nu\|\alpha\|^2_{L^2}.$$

With these slight changes in mind, the proof given in [NT] goes through.

We will also apply the following Hartogs type extension property:

Proposition 4.4. *Let (X,g) be a connected complete Hermitian manifold of dimension $n>1$ and let $M\subset X$ be a domain with nonempty smooth compact strongly pseudoconvex boundary. Assume that the restriction $g|_M$ of g to M is Kähler. Suppose f is a holomorphic function on $U\cap M$ for some neighborhood U of ∂M in X. Then there exists a holomorphic function h on M such that $h=f$ near ∂M. In particular, ∂M is connected.*

Proof. We may assume that $M=\{\,x\in X \mid \varphi(x)<0\,\}$ for some C^∞ function φ on X which is strictly plurisubharmonic on a neighborhood of $X\setminus M$ in X. Since $g|_M$ is Kähler and g is complete on X, a theorem of Nakano [N] and of Demailly [D1] implies that M admits a complete Kähler metric g'. Moreover, the existence of φ implies that (M,g') admits a positive Green's function G which vanishes along ∂M. We normalize G so that, for each point $x_0\in M$,

$$\Delta_{\text{distr.}} G(\cdot,x_0) = -(2n-2)\sigma_{2n-1}\delta_{x_0},$$

where $n=\dim X$, $\sigma_{2n-1}=\mathrm{vol}(S^{2n-1})$, and δ_{x_0} is the Dirac function at x_0.

Fix a C^∞ function λ with compact support in U such that $\lambda\equiv 1$ on a neighborhood of ∂M and let α be the $\bar\partial$-closed compactly supported form of type $(0,1)$ on M given by $\alpha=\bar\partial(\lambda f)$ (extended by 0 to M). Then the function β defined by

$$\beta(x) = -\frac{1}{(2n-2)\sigma_{2n-1}}\int_M G(x,y)\bar\partial^*\alpha(y)\,dV_{g'}(y)$$

is a C^∞ bounded function with finite energy (i.e., $\int_M |\nabla\beta|^2\,dV_{g'}<\infty$), $\Delta\beta=\bar\partial^*\alpha$, and β vanishes on ∂M. Hence $\gamma\equiv\alpha-\bar\partial\beta$ is an L^2 harmonic form of type $(0,1)$ and the Gaffney theorem [G] implies that γ is closed (and coclosed). In particular, $\bar\gamma$ is a holomorphic 1-form on M and β is pluriharmonic on $W\cap M$ for some neighborhood W of $X\setminus M$ in X.

We will show that β vanishes near ∂M. Fix $a<0$ so close to 0 that φ is strictly plurisubharmonic on $V=\{\,x\in M \mid \varphi(x)>a\,\}$ and $V\subset\subset W$. If ρ is the real part or the imaginary part of β and ρ does not vanish identically near ∂M, then we may choose a nonzero regular value b of ρ contained in $\rho(V)$. Since $b\neq 0$ and ρ vanishes on ∂M, $\rho^{-1}(b)$ avoids ∂M. Thus the restriction of φ to $\rho^{-1}(b)$ assumes its maximum at some point $x_0\in V\subset W\cap M$ (with $\varphi(x_0)>a$). But the leaf L through x_0 of the foliation determined by the holomorphic 1-form $\partial\rho$ on $V\cap M$ is contained in $\rho^{-1}(b)$, so $\varphi|_L$ also assumes its maximum at x_0. Since φ is strictly plurisubharmonic on V, we have arrived at a contradiction. Therefore β vanishes near ∂M. Hence $\gamma=\alpha-\bar\partial\beta$ vanishes near ∂M and, therefore, on all of M, since $\bar\gamma$ is a holomorphic 1-form. Thus the function $h\equiv\lambda f-\beta$ is holomorphic on M (since $\bar\partial h=\gamma=0$) and equal to f near ∂M. In particular, since one can take f to

be a locally constant function which separates distinct components of ∂M, ∂M is connected. $\square$

Proof of Theorem 4.1. Since $n \geq 3$, one can apply a theorem of Rossi [R] to "fill in the holes" and obtain a connected Stein space Y with isolated singularities, a relatively compact pseudoconvex domain N in Y containing Y_{sing}, and a biholomorphic mapping $\Phi : U \to V$ of a neighborhood U of ∂M in X onto a neighborhood V of ∂N in Y such that $\Phi(U \cap M) = V \cap N$. Since N may be embedded into a Euclidean space, Proposition 4.4 implies that Φ extends to a holomorphic mapping $M \cup U \to Y$, which we also denote by Φ, and $\Phi(M) \subset N$. Next, by a theorem of Lempert [L], one can form a "cap" on N. That is, we may assume that Y is an affine algebraic variety. By forming the closure $\overline{Y}$ of Y in a projective space and desingularizing $\overline{Y}$ at infinity, we get a projective variety Z with isolated singularities such that $Z_{\text{sing}} \subset N \cup V \subset Z$. Finally, by replacing X by

$$(M \cup U) \cup (V \cup (Z \setminus \overline{N})) \Big/ x \in U \sim \Phi(x) \in V$$

and by replacing the metric g by any extension of $g|_M$ to the new manifold, we may assume that we have a holomorphic mapping $\Phi : X \to Z$ such that $\Phi(M) \subset N$ and Φ maps $(X \setminus M) \cup U$ biholomorphically onto $(Z \setminus N) \cup V$. In particular, since $X \setminus \overline{M} \subset\subset X$, it follows that X is pseudoconcave in the sense of Andreotti.

Now let H be a positive Hermitian holomorphic line bundle on Z. Then $\Phi^* H$ is semipositive on X and positive on $(X \setminus M) \cup U$. On the other hand, by shrinking M slightly and extending the Hermitian metric a, we may assume that K_X admits a Hermitian metric whose curvature is greater than or equal to cg at each point of M. It follows that if m is a sufficiently large positive integer and $L = K_X \otimes \Phi^* H^m$, then L admits a Hermitian metric h such that $\mathcal{C}(L, h) \geq cg$ on X. In particular, $g' = \mathcal{C}(L, h)$ is a complete Kähler metric on X. Therefore, by the L^2 Riemann-Roch inequality of Nadel and Tsuji (Theorem 4.3), we have, for every sufficiently large positive integer ν,

$$1 + \nu^{-n} \dim H^0_{L^2}(X, \mathcal{O}(K_X \otimes L^\nu)) \geq \frac{1}{n!} \int_X \big(c_1(L, h)\big)^n \geq c^n \pi^{-n} \int_X dV_g$$

(where the Hermitian metric in $K_X \otimes L^\nu$ is $(g')^* \otimes h^\nu$). Since, by Andreotti's finiteness theorem [A] (or by [AG]), the left-hand side is finite, we get

$$\text{vol}_g(M) \leq \text{vol}_g(X) < \infty.$$

$\square$

Remark. By a version of the L^2 Riemann-Roch inequality due to Takayama [T], it is only necessary to assume in the hypothesis (ii) that $\mathcal{C}(K_M, a) \geq cg$ outside a relatively compact neighborhood of ∂M in X.

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