

DOUBLE BRUHAT CELLS AND TOTAL POSITIVITY

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0. INTRODUCTION

This paper continues the algebraic study of total positivity in semisimple algebraic groups undertaken in [3, 4]. Traditional theory of total positivity, pioneered in the 1930s by Gantmacher, Krein, and Schoenberg, studies matrices whose all minors are nonnegative. Recently, G. Lusztig [18] extended this classical subject by introducing the totally nonnegative variety $G_{\geq 0}$ in an arbitrary reductive group G . (Lusztig's study was motivated by surprising connections he discovered between total positivity and his theory of canonical bases for quantum groups.) The main object of study in [3, 4] was the structure of the intersection $G_{\geq 0} \cap N$, where N is a maximal unipotent subgroup in G . In this paper we extend the results of [3, 4] to the whole variety $G_{\geq 0}$.

It turns out that the natural geometric framework for the study of $G_{\geq 0}$ is provided by the decomposition of G into the disjoint union of *double Bruhat cells* $G^{u,v} = BuB \cap B_-vB_-$; here B and B_- are two opposite Borel subgroups in G , and u and v belong to the Weyl group W of G . We believe these double cells to be a very interesting object of study in their own right. The term “cells” might be misleading: in fact, the topology of $G^{u,v}$ is in general quite nontrivial. (In some special cases, the “real part” of $G^{u,v}$ was studied in [21, 22]. V. Deodhar [9] studied the intersections $BuB \cap B_-vB$ whose properties are very different from those of $G^{u,v}$.)

We study a family of birational parametrizations of $G^{u,v}$, one for each reduced expression \mathbf{i} of the element (u, v) in the Coxeter group $W \times W$. Every such parametrization can be thought of as a system of local coordinates in $G^{u,v}$. We call these coordinates the *factorization parameters* associated to \mathbf{i} . They are obtained by expressing a generic element $x \in G^{u,v}$ as an element of the maximal torus $H = B \cap B_-$ multiplied by the product of elements of various one-parameter subgroups in G associated with simple roots and their negatives; the reduced expression \mathbf{i} prescribes the order of factors in this product. The main technical result of this paper (Theorem 1.9) is an explicit formula for these factorization parameters as rational functions on the double Bruhat cell $G^{u,v}$.

Theorem 1.9 is formulated in terms of a special family of regular functions $\Delta_{\gamma,\delta}$ on the group G . These functions are suitably normalized matrix coefficients corresponding to pairs of extremal weights (γ, δ) in some fundamental representation of G . We believe these functions are very interesting subjects that merit further study. For the type A , they specialize to the minors of a matrix, and their properties are of course developed in great detail. It would be very interesting to extend the main body of the classical theory of determinantal identities to the family of functions $\Delta_{\gamma,\delta}$. In this paper, we make the first steps in this direction (see especially Theorems 1.16 and 1.17 below).

As in [3, 4], the main algebraic relations involving factorization parameters and generalized minors $\Delta_{\gamma,\delta}$ can be written in a “subtraction-free” form, and thus the theory can be developed over an arbitrary *semifield*. The readers familiar with [3, 4] will have no trouble extending the corresponding results there (cf. [3, Section 2]) to the more general context of this paper. We do not pursue this path here since at the moment we have not developed applications of this more general setup.

Returning to total positivity, our explicit formulas for factorization parameters allow us to obtain a family of *total positivity criteria*, each of which efficiently tests whether a given element x from an arbitrary double Bruhat cell $G^{u,v}$ is totally

nonnegative. More specifically, each of our criteria consists in verifying whether x satisfies a system of inequalities of the form $\Delta_{\gamma,\delta}(x) > 0$, the number of these inequalities being equal to the dimension of $G^{u,v}$ (see Theorem 1.11).

In Section 1 we give precise formulations of our main results. Their proofs are given in Sections 2 and 3.

The last Section 4 contains applications of our theory to the case of the *general linear group*. The case $G = GL_n$ is treated separately for a number of reasons. First, GL_n is not a semisimple group (although everything reduces easily to SL_n). Furthermore, the questions that we consider become some very natural linear-algebraic questions whose understanding does not require any Lie-theoretic background. For instance, the factorization parameters become the parameters in factorizations of a square matrix into the smallest possible number of elementary Jacobi matrices. Our main results seem to be new even in this case. In Section 4, we tried to present them in an elementary form, making this section as self-contained as possible. Last but not least, our results in the GL_n case have a particularly transparent formulation in the language of pseudoline arrangements.

The criteria of Theorem 1.11 lead us to new solutions of the classical problem of efficiently testing whether a given $n \times n$ matrix is totally positive, i.e., has all minors > 0 . This problem has a long history. The ground was broken in 1912 by M. Fekete [10] who proved that positivity of all solid minors, i.e., those formed by several consecutive rows and several consecutive columns, is sufficient for total positivity. It took a while before it was realized that Fekete's criterion was far from optimal: as shown in [13], it is enough to check the positivity of those solid minors that involve the first row or the first column of the matrix (this result can also be derived from [8]). Note that the number of such minors is n^2 while the total number of minors is $\binom{2n}{n} - 1$. One can show that at least n^2 minors are needed to characterize total positivity; thus the criterion reproduced above is "minimal". Theorem 4.13 (a specialization of Theorem 1.11) includes this criterion into a family of minimal total positivity criteria associated to shuffles of two reduced words for the permutation $w_\circ = n \ n-1 \cdots 2 \ 1$. For example, for $n = 3$ we obtain 34 different criteria shown in Figure 8 at the end of the paper.

1. MAIN RESULTS

1.1. Semisimple groups. We begin by introducing general terminology and notation (mostly standard) for semisimple Lie groups and algebras (cf., e.g., [23]). Let \mathfrak{g} be a *semisimple complex Lie algebra* of rank r with the *Cartan decomposition* $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let e_i, h_i, f_i , for $i = 1, \dots, r$, be the standard generators of \mathfrak{g} , and let $A = (a_{ij})$ be the *Cartan matrix*. Thus $a_{ij} = \alpha_j(h_i)$, where $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ are the *simple roots* of \mathfrak{g} . Let G be a simply connected complex Lie group with the Lie algebra \mathfrak{g} . Let N_-, H and N be closed subgroups of G with Lie algebras $\mathfrak{n}_-, \mathfrak{h}$ and \mathfrak{n} , respectively. Thus H is a *maximal torus*, and N and N_- are two opposite *maximal unipotent subgroups* of G . Let $B_- = HN_-$ and $B = HN$ be the corresponding pair of opposite *Borel subgroups*. For $i = 1, \dots, r$ and $t \in \mathbb{C}$, we write

$$(1.1) \quad x_i(t) = \exp(te_i), \quad x_{\bar{i}}(t) = \exp(tf_i),$$

so that $t \mapsto x_i(t)$ (resp. $t \mapsto x_{\bar{i}}(t)$) is a one-parameter subgroup in N (resp. in N_-). We prefer the notation $x_{\bar{i}}(t)$ to the usual $y_i(t)$, for reasons that will become clear later. It will be convenient to denote $[1, r] = \{1, \dots, r\}$ and $[\bar{1}, \bar{r}] = \{\bar{1}, \dots, \bar{r}\}$.

The *weight lattice* P is the set of all weights $\gamma \in \mathfrak{h}^*$ such that $\gamma(h_i) \in \mathbb{Z}$ for all i . The group P has a \mathbb{Z} -basis formed by the *fundamental weights* $\omega_1, \dots, \omega_r$ defined by $\omega_i(h_j) = \delta_{ij}$. Every weight $\gamma \in P$ gives rise to a multiplicative character $a \mapsto a^\gamma$ of the maximal torus H ; this character is given by $\exp(h)^\gamma = e^{\gamma(h)}$ ($h \in \mathfrak{h}$).

The *Weyl group* W of G is defined by $W = \text{Norm}_G(H)/H$. The action of W on H by conjugation gives rise to the action of W on the weight lattice P given by

$$(1.2) \quad a^{w(\gamma)} = (w^{-1}aw)^\gamma \quad (w \in W, a \in H, \gamma \in P) .$$

As usual, we identify W with the corresponding group of linear transformations of \mathfrak{h}^* . The group W is a *Coxeter group* generated by *simple reflections* s_1, \dots, s_r given by $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i$, for $\gamma \in \mathfrak{h}^*$.

A *reduced word* for $w \in W$ is a sequence of indices $\mathbf{i} = (i_1, \dots, i_m)$ of shortest possible length m such that $w = s_{i_1} \cdots s_{i_m}$. The number m is denoted by $\ell(w)$ and is called the *length* of w . The set of reduced words for w will be denoted by $R(w)$. The Weyl group W has the unique element w_o of maximal length, and $\ell(w_o) = \ell(w) + \ell(w^{-1}w_o)$ for any $w \in W$.

1.2. Factorization problem. Recall that the group G has two *Bruhat decompositions*, with respect to opposite Borel subgroups B and B_- :

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_-vB_- .$$

The *double Bruhat cells* $G^{u,v}$ are defined by

$$G^{u,v} = BuB \cap B_-vB_- ;$$

thus G is the disjoint union of all $G^{u,v}$ for $(u, v) \in W \times W$.

Theorem 1.1. *The variety $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $r + \ell(u) + \ell(v)$.*

We will study a family of birational parametrizations of $G^{u,v}$. To describe these parametrizations, we will need the following combinatorial notion.

A *double reduced word* for the elements $u, v \in W$ is a reduced word for an element (u, v) of the Coxeter group $W \times W$. To avoid confusion, we will use the indices $\bar{1}, \bar{2}, \dots, \bar{r}$ for the simple reflections in the first copy of W , and $1, 2, \dots, r$ for the second copy. A double reduced word for (u, v) is nothing but a shuffle of a reduced word for u written in the alphabet $[\bar{1}, \bar{r}]$ and a reduced word for v written in the alphabet $[1, r]$. We denote the set of double reduced words for (u, v) by $R(u, v)$.

For any sequence $\mathbf{i} = (i_1, \dots, i_m)$ of indices from the alphabet $[1, r] \cup [\bar{1}, \bar{r}]$, consider the map $x_{\mathbf{i}} : H \times \mathbb{C}^m \rightarrow G$ defined by

$$(1.3) \quad x_{\mathbf{i}}(a; t_1, \dots, t_m) = a x_{i_1}(t_1) \cdots x_{i_m}(t_m) ,$$

where we use the notation of (1.1).

Let $\mathbb{C}_{\neq 0}$ denote the set of nonzero complex numbers.

Theorem 1.2. *For any $u, v \in W$ and $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$, the map $x_{\mathbf{i}}$ restricts to a biregular isomorphism between $H \times \mathbb{C}_{\neq 0}^m$ and a Zariski open subset of the double Bruhat cell $G^{u,v}$.*

Thus $x_{\mathbf{i}}$ gives rise to a birational isomorphism between $H \times \mathbb{C}^m$ and $G^{u,v}$. We remark that this property holds if and only if \mathbf{i} is a double reduced word for (u, v) .

Theorem 1.2 tells us that for a generic element $x \in G^{u,v}$ and any $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$, there are uniquely defined *factorization parameters* a, t_1, \dots, t_m such that

$$(1.4) \quad x = a x_{i_1}(t_1) \cdots x_{i_m}(t_m) .$$

One of our main results is the solution of the following *factorization problem*: find explicit formulas for the inverse birational isomorphism $x_{\mathbf{i}}^{-1}$ between $G^{u,v}$ and $H \times \mathbb{C}^m$. In other words, we express the factorization parameters in terms of the element x and the double reduced word \mathbf{i} underlying the factorization. Our solution of the factorization problem generalizes Theorems 1.4 and 6.2 in [4] (the case of $x \in N$), which in turn generalize Theorems 1.4 and 5.4.2 in [3] (same, for type A).

1.3. Total positivity. We will apply our solution of the factorization problem to the study of total positivity. Following G. Lusztig [18], let us define *totally nonnegative elements* in G . Let $H_{>0}$ be the subgroup of H consisting of all $a \in H$ such that $a^\gamma \in \mathbb{R}_{>0}$ for any weight $\gamma \in P$. (We denote by $\mathbb{R}_{>0}$ the set of positive reals.) The set $G_{\geq 0}$ of totally nonnegative elements is, by definition, the multiplicative semigroup in G generated by $H_{>0}$ and the elements $x_i(t)$ and $x_{\bar{i}}(t)$, for $i \in [1, r]$ and $t \in \mathbb{R}_{>0}$. It is easy to see that a totally nonnegative element $x \in G$ can be represented as $x = x_{\mathbf{i}}(a; t_1, \dots, t_m)$, for some sequence \mathbf{i} , with all the t_k positive and $a \in H_{>0}$. For the type A_r (i.e., for $G = SL_n(\mathbb{C})$, $n = r + 1$), a theorem in [17], based on a result by A. Whitney [24], tells us that the above definition of total nonnegativity coincides with the usual one [2, 16]: a matrix (with determinant 1) is totally nonnegative if and only if all its minors are nonnegative.

The set $G_{\geq 0}$ is the disjoint union of the subsets $G_{>0}^{u,v}$ obtained by intersecting it with double Bruhat cells:

$$G_{>0}^{u,v} = G_{\geq 0} \cap G^{u,v} .$$

We call the $G_{>0}^{u,v}$ *totally positive varieties*; they will be one of the main objects of study in this paper. The terminology is justified by the following observation made by Lusztig [18]: in the special case $G = SL_n(\mathbb{C})$ and $u = v = w_o$, the variety $G_{>0}^{u,v}$ is the set of all $n \times n$ -matrices (with determinant 1) which are (strictly) totally positive in the usual sense, i.e., all their minors are positive. We note that the decomposition $G_{\geq 0} = \bigcup G_{>0}^{u,v}$ appeared in [18] (without explicit mentioning of double Bruhat cells). The following theorem can be derived from the results in [18].

Theorem 1.3. *For any $u, v \in W$ and any double reduced word $\mathbf{i} \in R(u, v)$, the map $x_{\mathbf{i}}$ restricts to a bijection $H_{>0} \times \mathbb{R}_{>0}^m \rightarrow G_{>0}^{u,v}$.*

Informally speaking, Theorem 1.3 asserts that an element $x \in G^{u,v}$ is totally nonnegative if and only if for some (equivalently, any) double reduced word $\mathbf{i} \in R(u, v)$, the factorization parameters a, t_1, \dots, t_m appearing in (1.4) are well defined and positive. Thus the solution of the factorization problem will lead to a family of total positivity criteria—one for each double reduced word.

1.4. Generalized minors. The main ingredients of our answer to the factorization problem are similar to those in [4]: a family of regular functions on G generalizing minors of a square matrix, and a biregular “twist” $G^{u,v} \rightarrow G^{u^{-1},v^{-1}}$.

We will denote by $G_0 = N_-HN$ the set of elements $x \in G$ that have Gaussian decomposition; this decomposition will be written as

$$(1.5) \quad x = [x]_- [x]_0 [x]_+ .$$

Following [4], for any fundamental weight ω_i , let Δ^{ω_i} denote the regular function on G whose restriction to the open set G_0 is given by

$$(1.6) \quad \Delta^{\omega_i}(x) = [x]_0^{\omega_i} .$$

For the type A_r , the $\Delta^{\omega_i}(x)$ are the *principal minors* of a matrix x . We will use the same terminology in the general case as well.

To define the analogues of arbitrary minors, we will need two special representatives $\overline{w}, \overline{\overline{w}} \in G$ for any element $w \in W$. For a simple reflection s_i , set

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \overline{\overline{s_i}} = \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

where $\varphi_i : SL_2 \rightarrow G$ is the group homomorphism given by

$$(1.7) \quad \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_i(t) , \quad \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_{\overline{i}}(t) .$$

Alternatively, we could define

$$(1.8) \quad \overline{s_i} = x_i(-1)x_{\overline{i}}(1)x_i(-1) , \quad \overline{\overline{s_i}} = x_i(1)x_{\overline{i}}(-1)x_i(1) .$$

It is known (and easy to check) that the families $\{\overline{s_i}\}$ and $\{\overline{\overline{s_i}}\}$ satisfy the braid relations in W . It follows that the representatives \overline{w} and $\overline{\overline{w}}$ can be uniquely and unambiguously defined for any $w \in W$ by the condition that

$$(1.9) \quad \overline{w'w''} = \overline{w'} \cdot \overline{w''} , \quad \overline{\overline{w'w''}} = \overline{\overline{w'}} \cdot \overline{\overline{w''}}$$

whenever $\ell(w'w'') = \ell(w') + \ell(w'')$.

Definition 1.4. For $u, v \in W$, define a regular function $\Delta_{u\omega_i, v\omega_i}$ on G by setting

$$(1.10) \quad \Delta_{u\omega_i, v\omega_i}(x) = \Delta^{\omega_i} \left(\overline{\overline{u^{-1}xv}} \right) .$$

One has to check that this is well defined, i.e., the right-hand side of (1.10) only depends on the weights $u\omega_i$ and $v\omega_i$, not on the particular choice of u and v . This is done in Section 2.3 (cf. Proposition 2.3).

For the type A_r , the functions $\Delta_{u\omega_i, v\omega_i}(x)$ are the minors of a matrix x . In the general case, we will refer to them as *generalized minors*, or simply as minors if there will be no danger of confusion.

1.5. The twist maps. To define the twist maps, we will need the involutive automorphism $x \mapsto x^\theta$ of the group G which is uniquely determined by

$$(1.11) \quad a^\theta = a^{-1} \quad (a \in H) , \quad x_i(t)^\theta = x_{\overline{i}}(t) , \quad x_{\overline{i}}(t)^\theta = x_i(t) .$$

Notice that the involution θ preserves total nonnegativity. For the type A_r , if x is a matrix with determinant 1, then the matrix x^θ is formed by signless cofactors of x ; in other words, the (i, j) -entry of x^θ is simply the minor of x obtained by deleting the i th row and the j th column.

Definition 1.5. For any $u, v \in W$, the twist map $\zeta^{u, v} : x \mapsto x'$ is defined by

$$(1.12) \quad x' = \left(\overline{\overline{[u^{-1}x]^{-1} u^{-1}}} x v^{-1} \overline{\overline{[xv^{-1}]^{-1}}} \right)^\theta .$$

Theorem 1.6. *The right-hand side of (1.12) is well defined for any $x \in G^{u,v}$, and the twist map $\zeta^{u,v}$ establishes a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1},v^{-1}}$. The inverse isomorphism is $\zeta^{u^{-1},v^{-1}}$.*

The specific choice of representatives for u and v in (1.12) is essential for the following important property.

Theorem 1.7. *The twist map $\zeta^{u,v}$ restricts to a bijection $G_{>0}^{u,v} \rightarrow G_{>0}^{u^{-1},v^{-1}}$.*

Example 1.8. Let $G = SL_2(\mathbb{C})$. Then $W = S_2$ consists of two permutations: e and w_o ; thus G is partitioned into four double Bruhat cells. Table 1 shows the conditions under which a matrix $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with determinant 1 belongs to each of these cells, or to the corresponding totally positive variety $G_{>0}^{u,v}$. The table also shows the formulas defining each twist map $\zeta^{u,v}$.

TABLE 1. Double Bruhat decomposition and the twist maps for SL_2

	$u = e$ $v = e$	$u = e$ $v = w_o$	$u = w_o$ $v = e$	$u = w_o$ $v = w_o$
$G^{u,v}$	$x_{12} = 0$ $x_{21} = 0$	$x_{12} \neq 0$ $x_{21} = 0$	$x_{12} = 0$ $x_{21} \neq 0$	$x_{12} \neq 0$ $x_{21} \neq 0$
$G_{>0}^{u,v}$	$x_{11} > 0$ $x_{12} = 0$ $x_{21} = 0$	$x_{11} > 0$ $x_{12} > 0$ $x_{21} = 0$	$x_{11} > 0$ $x_{12} = 0$ $x_{21} > 0$	$x_{11} > 0$ $x_{12} > 0$ $x_{21} > 0$
$\zeta^{u,v}(x)$	$\begin{pmatrix} x_{11}^{-1} & 0 \\ 0 & x_{11} \end{pmatrix}$	$\begin{pmatrix} x_{12}^{-1} & x_{11}^{-1} \\ 0 & x_{12} \end{pmatrix}$	$\begin{pmatrix} x_{21}^{-1} & 0 \\ x_{11}^{-1} & x_{21} \end{pmatrix}$	$\begin{pmatrix} x_{11}x_{12}^{-1}x_{21}^{-1} & x_{21}^{-1} \\ x_{12}^{-1} & x_{22} \end{pmatrix}$

1.6. Formulas for factorization parameters. To give explicit formulas for factorization parameters, we will need some more notation. First, we will write

$$(1.13) \quad |i| = |\bar{i}| = i, \quad \varepsilon(i) = 1, \quad \varepsilon(\bar{i}) = -1,$$

for any $i \in [1, r]$. Let us fix a pair $(u, v) \in W \times W$ and a double reduced word $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$. Recall that \mathbf{i} is a shuffle of a reduced word for u written in the alphabet $[\bar{1}, \bar{r}]$ and a reduced word for v written in the alphabet $[1, r]$. In particular, the length m of \mathbf{i} is equal to $\ell(u) + \ell(v)$. We will add r additional entries i_{m+1}, \dots, i_{m+r} at the end of \mathbf{i} by setting

$$(1.14) \quad i_{m+j} = \bar{j} \quad (j \in [1, r]).$$

For $k \in [1, m+r]$, let us denote

$$(1.15) \quad u_{\geq k} = \prod_{\substack{l=m, \dots, k \\ i_l \in [\bar{1}, \bar{r}]}} s_{|i_l|}, \quad v_{< k} = \prod_{\substack{l=1, \dots, k-1 \\ i_l \in [1, r]}} s_{|i_l|}.$$

This notation means that in the first (resp. second) product in (1.15), the index l is decreasing (resp. increasing); by convention, we also have $u_{\geq k} = e$ and $v_{< k} = v$ for $k > m$. (For example, if $\mathbf{i} = \bar{2} \ 1 \ \bar{3} \ 3 \ 2 \ \bar{1} \ \bar{2} \ 1 \ \bar{1}$, then, say, $u_{\geq 7} = s_1 s_2$ and $v_{< 7} = s_1 s_3 s_2$.) Let us define a regular function $\Delta_k = \Delta_{k, \mathbf{i}}$ on G by

$$(1.16) \quad \Delta_k = \Delta_{k, \mathbf{i}} = \Delta_{u_{\geq k} \omega_{|i_k|}, v_{< k} \omega_{|i_k|}}.$$

For $k \in [1, m + r]$, we denote

$$k^- = \max\{l : l < k, |i_l| = |i_k|\}$$

unless $|i_l| \neq |i_k|$ for all $l < k$, in which case we set $k^- = 0$. For $k \in [1, m + r]$ and $l \in [0, m + r]$, let

$$(1.17) \quad \chi(k, l) = \begin{cases} 1 & \text{if } k < l \text{ and } \varepsilon(i_k) = \varepsilon(i_l); \\ 1/2 & \text{if } k = l; \\ 0 & \text{if } k > l \text{ or } \varepsilon(i_k) \neq \varepsilon(i_l). \end{cases}$$

With all this notation in mind, we now formulate our first main result: a solution to the factorization problem of Section 1.2.

Theorem 1.9. *Let $\mathbf{i} = (i_1, \dots, i_m)$ be a double reduced word for (u, v) , and suppose an element $x \in G^{u,v}$ can be factored as $x = a x_{i_1}(t_1) \cdots x_{i_m}(t_m)$, with $a \in H$ and all t_k nonzero complex numbers. Then the factorization parameters are determined by the following formulas:*

$$(1.18) \quad t_k = \prod_{l=1}^{m+r} \Delta_{l,\mathbf{i}}(x')^{(\chi(k,l^-) - \chi(k,l))a_{|i_l|,|i_k|}};$$

$$(1.19) \quad a^{\omega_i} = \prod_{\substack{1 \leq k \leq m+r \\ |i_k|=i, \varepsilon(i_k) \neq \varepsilon(i_{k^-})}} \Delta_{k,\mathbf{i}}(x')^{\varepsilon(i_k)},$$

where $x' = \zeta^{u,v}(x)$, and we use the convention $\varepsilon(i_0) = 1$.

Formulas (1.19) can be restated as the following closed expression for the element a :

$$(1.20) \quad a = \prod_{\substack{1 \leq k \leq m+1 \\ \varepsilon(i_k) \neq \varepsilon(i_{k-1})}} \left[\overline{u_{\geq k}^{-1} x' v_{< k}} \right]_0^{\varepsilon(i_k)}.$$

Since (1.18) and (1.19) express the $m + r$ independent parameters t_1, \dots, t_m and $a^{\omega_1}, \dots, a^{\omega_r}$ as Laurent monomials in the $m + r$ minors $\Delta_{1,\mathbf{i}}(x'), \dots, \Delta_{m+r,\mathbf{i}}(x')$, we obtain the following important corollary.

Theorem 1.10. *Under the assumptions of Theorem 1.9, the parameters t_1, \dots, t_m and $a^{\omega_1}, \dots, a^{\omega_r}$ are related to the minors $\Delta_{1,\mathbf{i}}(x'), \dots, \Delta_{m+r,\mathbf{i}}(x')$ by an invertible monomial transformation.*

The inverse of this monomial transformation can be computed explicitly: one can show that it is given by

$$(1.21) \quad \Delta_{k,\mathbf{i}}(x') = a^{-uu_{\geq k}\omega_{|i_k|}} \prod_{\substack{l < k \\ i_l \in [\overline{1}, \overline{r}]}} t_l^{u_{\geq l}^{-1} u_{\geq k} \omega_{|i_k|}(h_{|i_l|})} \prod_{\substack{l \geq k \\ i_l \in [1, r]}} t_l^{v_{< l+1}^{-1} v_{< k} \omega_{|i_k|}(h_{|i_l|})}.$$

Since we will not use this formula, we will not prove it in this paper.

1.7. Total positivity criteria. Theorem 1.9 implies a family of criteria for total positivity that generalize the ones in [3, 4]. Each of these criteria asserts that a point $x \in G^{u,v}$ belongs to the totally positive variety $G_{>0}^{u,v}$ if and only if a particular collection of $\ell(u) + \ell(v) + r$ minors evaluated at x are all positive.

For every double reduced word $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$, let $F(\mathbf{i})$ denote the following collection of $m + r$ minors:

$$(1.22) \quad F(\mathbf{i}) = \{\Delta_{k, \mathbf{i}^*} : k \in [1, m + r]\}$$

(cf. (1.16)), where $\mathbf{i}^* = (i_m, \dots, i_1) \in R(u^{-1}, v^{-1})$ is \mathbf{i} written backwards.

Theorem 1.11. *Let $\mathbf{i} \in R(u, v)$. An element $x \in G^{u,v}$ is totally nonnegative if and only if $\Delta(x) > 0$ for any minor $\Delta \in F(\mathbf{i})$.*

As a consequence, for any two double reduced words $\mathbf{i}, \mathbf{i}' \in R(u, v)$, the positivity of all minors from $F(\mathbf{i})$ at a given $x \in G^{u,v}$ is equivalent to the positivity of all minors from $F(\mathbf{i}')$. This phenomenon has the following algebraic explanation. Let

$$(1.23) \quad F(u, v) = \bigcup_{\mathbf{i} \in R(u, v)} F(\mathbf{i}).$$

This can be restated as

$$(1.24) \quad F(u, v) = \{\Delta_{u'\omega_i, v'\omega_i} : i \in [1, r], u' \preceq u, v' \preceq v^{-1}\},$$

where $u' \preceq u$ stands for $\ell(u) = \ell(u') + \ell(u'^{-1}u)$ (the weak order on W).

Theorem 1.12. *For any $\mathbf{i} \in R(u, v)$, the collection $F(\mathbf{i})$ is a transcendence basis for the field of rational functions $\mathbb{C}(G^{u,v})$. Furthermore, every minor in $F(u, v)$ can be expressed as a ratio of two polynomials in the variables $\Delta \in F(\mathbf{i})$ with nonnegative integer coefficients.*

Combining Theorems 1.2, 1.6, and 1.10 yields the following result.

Theorem 1.13. *For any $\mathbf{i} \in R(u, v)$, every minor in $F(u, v)$ is a Laurent polynomial with integer coefficients in the variables $\Delta \in F(\mathbf{i})$.*

We suggest the following common refinement of Theorems 1.12 and 1.13.

Conjecture 1.14. *For any $\mathbf{i} \in R(u, v)$, every minor in $F(u, v)$ is a Laurent polynomial in the variables $\Delta \in F(\mathbf{i})$ with nonnegative integer coefficients.*

Note that Theorems 1.12 and 1.13 do not automatically imply Conjecture 1.14, since there do exist subtraction-free rational expressions that are Laurent polynomials although *not* with nonnegative coefficients (for example, think of $(p^3 + q^3)/(p + q) = p^2 - pq + q^2$).

It is not hard to derive the following special case of Conjecture 1.14 from [3, Theorem 3.7.4].

Theorem 1.15. *Conjecture 1.14 holds for $G = SL_{r+1}$, when either u or v is the identity element e .*

1.8. Fundamental determinantal identities. The subtraction-free rational expressions in Theorem 1.12 can be computed by an explicitly described algorithm. This algorithm is based on repeated application of the following generalized determinantal identities.

The first group of identities follows from [4, Corollary 6.6]. They correspond to pairs of simple roots that generate a root subsystem of type A_2 or B_2 . There are

similar identities for subsystems of type G_2 (see [4, (4.8)–(4.11)]) although we will not reproduce them here.

Theorem 1.16. *Let $u, v \in W$ and $i, j \in [1, r]$.*

(1) *If $a_{ij} = a_{ji} = -1$ and $\ell(vs_i s_j s_i) = \ell(v) + 3$, then*

$$\Delta_{u\omega_i, vs_i \omega_i} \Delta_{u\omega_j, vs_j \omega_j} = \Delta_{u\omega_i, v\omega_i} \Delta_{u\omega_j, vs_i s_j \omega_j} + \Delta_{u\omega_i, vs_j s_i \omega_i} \Delta_{u\omega_j, v\omega_j} .$$

(2) *If $a_{ij} = -2$, $a_{ji} = -1$, and $\ell(vs_i s_j s_i s_j) = \ell(v) + 4$, then*

$$\begin{aligned} &\Delta_{u\omega_j, vs_i s_j \omega_j} \Delta_{u\omega_i, vs_j s_i \omega_i}^2 \Delta_{u\omega_j, vs_j \omega_j} \\ &= \Delta_{u\omega_j, vs_j s_i s_j \omega_j} \Delta_{u\omega_i, vs_j s_i \omega_i}^2 \Delta_{u\omega_j, v\omega_j} \\ &\quad + (\Delta_{u\omega_i, v\omega_i} \Delta_{u\omega_j, vs_j s_i s_j \omega_j} + \Delta_{u\omega_i, vs_i s_j s_i \omega_i} \Delta_{u\omega_j, vs_j \omega_j})^2 \end{aligned}$$

and

$$\begin{aligned} &\Delta_{u\omega_i, vs_i \omega_i} \Delta_{u\omega_i, vs_j s_i \omega_i} \Delta_{u\omega_j, vs_j \omega_j} \\ &= \Delta_{u\omega_i, vs_j s_i \omega_i}^2 \Delta_{u\omega_j, v\omega_j} \\ &\quad + \Delta_{u\omega_i, v\omega_i} (\Delta_{u\omega_i, v\omega_i} \Delta_{u\omega_j, vs_j s_i s_j \omega_j} + \Delta_{u\omega_i, vs_i s_j s_i \omega_i} \Delta_{u\omega_j, vs_j \omega_j}) . \end{aligned}$$

(3) *Each of the above identities has a companion identity, obtained by “transposing” all participating minors, i.e., by replacing every $\Delta_{\gamma, \delta}$ by $\Delta_{\delta, \gamma}$.*

We also make use of the following new identity.

Theorem 1.17. *Suppose $u, v \in W$ and $i \in [1, r]$ are such that $\ell(us_i) = \ell(u) + 1$ and $\ell(vs_i) = \ell(v) + 1$. Then*

$$(1.25) \quad \Delta_{u\omega_i, v\omega_i} \Delta_{us_i \omega_i, vs_i \omega_i} = \Delta_{us_i \omega_i, v\omega_i} \Delta_{u\omega_i, vs_i \omega_i} + \prod_{j \neq i} \Delta_{u\omega_j, v\omega_j}^{-a_{ji}} .$$

The proof of Theorem 1.17 is given in Section 2.3.

For the type A_r , the identities of Theorems 1.16 and 1.17 become certain 3-term determinantal identities known since the early 19th century. We discuss their attribution in Section 4.6.

2. PRELIMINARIES

In what follows, we retain the notation and terminology introduced in Section 1.

2.1. Involutions. Following [4], we define involutive anti-automorphisms $x \mapsto x^T$ (the “transpose”) and $x \mapsto x^t$ of the group G by setting

$$(2.1) \quad a^T = a \quad (a \in H) , \quad x_i(t)^T = x_{\bar{i}}(t) , \quad x_{\bar{i}}(t)^T = x_i(t)$$

and

$$(2.2) \quad a^t = a^{-1} \quad (a \in H) , \quad x_i(t)^t = x_i(t) , \quad x_{\bar{i}}(t)^t = x_{\bar{i}}(t) .$$

These two involutive anti-automorphisms commute with each other and with the involutive anti-automorphism $x \mapsto x^{-1}$ of G . Hence these three maps generate the group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$; in particular, any composition of them is again an involution. Notice that the involutions $x \mapsto x^T$ and $x \mapsto x^t$ preserve total nonnegativity, while $x \mapsto x^{-1}$ does not. Informally, x^t is a “totally nonnegative version” of x^{-1} .

In the notation just introduced, the involution $x \mapsto x^\theta$ that was defined by (1.11) is given by $x^\theta = (x^t)^T = (x^T)^t$.

The involutions $x \mapsto (x^{-1})^t$ and $x \mapsto x^T$ obviously preserve $G_0 = N_-HN$, and we have

$$(2.3) \quad [(x^{-1})^t]_0 = [x^T]_0 = [x]_0 .$$

All three involutions $x \mapsto x^{-1}$, $x \mapsto x^T$ and $x \mapsto x^t$ act on W by $w \mapsto w^{-1}$. The relations between these involutions and the special representatives introduced in Section 1.4 are summarized in the following proposition.

Proposition 2.1. *We have*

$$(2.4) \quad \overline{w}^{-1} = \overline{w}^T = \overline{w^{-1}} = \overline{w}^t, \quad \overline{w^{-1}} = \overline{w}^t = \overline{w}^{-1} = \overline{w}^T .$$

Proof. Since all three involutions are anti-automorphisms, it is enough to check (2.4) for $w = s_i$, in which case it follows by a calculation in SL_2 . \square

2.2. Commutation relations. For convenience of exposition, we collect here some known commutation relations in G that will be used in our proofs. Recall that $x_i(t)$ and $x_{\bar{i}}(t)$ are defined by (1.1), and $\alpha_1, \dots, \alpha_r$ are the simple roots of \mathfrak{g} .

First of all, for every $a \in H$, we have

$$(2.5) \quad ax_i(t) = x_i(a^{\alpha_i}t)a, \quad ax_{\bar{i}}(t) = x_{\bar{i}}(a^{-\alpha_i}t)a .$$

The following relations between the elements $x_i(t)$ can be found, e.g., in [4, Section 3] (some of them appeared earlier in [19]). If $a_{ij} = a_{ji} = 0$, then

$$(2.6) \quad x_i(t_1)x_j(t_2) = x_j(t_2)x_i(t_1) ,$$

for any t_1 and t_2 . If $a_{ij} = a_{ji} = -1$, then

$$(2.7) \quad x_i(t_1)x_j(t_2)x_i(t_3) = x_j\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_j\left(\frac{t_1t_2}{t_1+t_3}\right)$$

whenever $t_1+t_3 \neq 0$. If $a_{ij} = -2$ and $a_{ji} = -1$, then

$$(2.8) \quad x_i(t_1)x_j(t_2)x_i(t_3)x_j(t_4) = x_j(t_2t_3^2t_4q^{-1})x_i(qp^{-1})x_j(p^2q^{-1})x_i(t_1t_2t_3p^{-1}) ,$$

where

$$p = t_1t_2 + (t_1+t_3)t_4, \quad q = t_1^2t_2 + (t_1+t_3)^2t_4 ;$$

this relation holds whenever $p \neq 0$ and $q \neq 0$. In the case when $a_{ij} = -3$, $a_{ji} = -1$ (i.e., when α_i and α_j generate a root subsystem of type G_2), there is also a relation similar to (2.7) and (2.8). This relation is given in [4, (3.6)–(3.10)]; we will not reproduce it here. Each of the relations (2.6)–(2.8) has a counterpart for the elements $x_{\bar{i}}(t)$; it can be obtained by applying the anti-automorphism $x \mapsto x^T$ (cf. (2.1)).

In conclusion, let us describe the commutation relations between the elements $x_i(t)$ and $x_{\bar{j}}(t')$. If $i \neq j$, then $[e_i, f_j] = 0$ in \mathfrak{g} , hence

$$(2.9) \quad x_i(t)x_{\bar{j}}(t') = x_{\bar{j}}(t')x_i(t) ,$$

for any t and t' . To handle the case $i = j$, we will need the following notation. For a nonzero $t \in \mathbb{C}$ and $i \in [1, r]$, we denote

$$(2.10) \quad t^{h_i} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} ,$$

where $\varphi_i : SL_2 \rightarrow G$ is defined by (1.7); alternatively, t^{h_i} is an element of H uniquely determined by the condition that $(t^{h_i})^\gamma = t^{\gamma(h_i)}$ for any weight $\gamma \in P$. Then we have

$$(2.11) \quad x_i(t)x_{\bar{i}}(t') = x_{\bar{i}}\left(\frac{t'}{1+tt'}\right)(1+tt')^{h_i}x_i\left(\frac{t}{1+tt'}\right)$$

whenever $1+tt' \neq 0$. This relation can be first checked for SL_2 by a simple matrix calculation, and then extended to G by applying the homomorphism φ_i . By the same method, we verify the relations

$$(2.12) \quad x_i(t)x_{\bar{i}}(-t^{-1}) = x_{\bar{i}}(t^{-1})t^{h_i}\overline{s_i}$$

and

$$(2.13) \quad \begin{aligned} x_i(t)\overline{s_i} &= x_{\bar{i}}(t^{-1})t^{h_i}x_i(-t^{-1}), \\ \overline{s_i}x_{\bar{i}}(t) &= x_{\bar{i}}(-t^{-1})t^{h_i}x_i(t^{-1}). \end{aligned}$$

2.3. Generalized determinantal identities. We start with some identities for the ‘‘principal minors’’ Δ^{ω_i} . The definition (1.6) implies that, for any $x \in G$, $x^- \in N_-$, $x^+ \in N$, and $a \in H$, we have

$$(2.14) \quad \begin{aligned} \Delta^{\omega_i}(x^-x) &= \Delta^{\omega_i}(xx^+) = \Delta^{\omega_i}(x), \\ \Delta^{\omega_i}(ax) &= \Delta^{\omega_i}(xa) = a^{\omega_i}\Delta^{\omega_i}(x). \end{aligned}$$

In view of (2.3), we also have

$$(2.15) \quad \Delta^{\omega_i}((x^{-1})^t) = \Delta^{\omega_i}(x^T) = \Delta^{\omega_i}(x).$$

The following property is less obvious.

Proposition 2.2. *For any $x \in G$, $j \neq i$, and $t \in \mathbb{C}$, we have*

$$(2.16) \quad \Delta^{\omega_i}(xx_{\bar{j}}(t)) = \Delta^{\omega_i}(x_j(t)x) = \Delta^{\omega_i}(x).$$

Proof. It is possible to deduce the proposition from the commutation relations given in Section 2.2 but we prefer another proof based on representation theory. The group G acts by right translations in the space $\mathbb{C}[G]$ of regular functions on G . It is well known that every $f \in \mathbb{C}[G]$ generates a finite-dimensional subrepresentation of $\mathbb{C}[G]$. In view of (2.14), the function Δ^{ω_i} is a highest weight vector of weight ω_i in $\mathbb{C}[G]$. Since $\omega_i(h_j) = 0$ for $j \neq i$, it follows that Δ^{ω_i} has weight 0 with respect to the subgroup $\varphi_j(SL_2)$ of G (cf. (1.7)). Therefore, Δ^{ω_i} generates a trivial representation of $\varphi_j(SL_2)$. In particular, $\Delta^{\omega_i}(xx_{\bar{j}}(t)) = \Delta^{\omega_i}(x)$, as desired. The equality $\Delta^{\omega_i}(x_j(t)x) = \Delta^{\omega_i}(x)$ now follows from (2.15). \square

Our next proposition justifies the validity of Definition 1.4.

Proposition 2.3. *For any $x \in G$ and any $j \neq i$, we have*

$$(2.17) \quad \Delta^{\omega_i}(x\overline{s_j}) = \Delta^{\omega_i}(\overline{s_j}x) = \Delta^{\omega_i}(x).$$

Proof. Follows from (2.14), (2.16), and (1.8). \square

The extension of principal minors from the open subset G_0 to the whole of G is given as follows. The Bruhat decomposition theorem implies that every $x \in G$ can be written as

$$(2.18) \quad x = x^- a \overline{w} x^+,$$

for some $x^- \in N_-$, $a \in H$, $w \in W$, and $x^+ \in N$; moreover, the elements $a \in H$ and $w \in W$ are uniquely determined by x .

Proposition 2.4. *If $x \in G$ is expressed in the form (2.18), then*

$$(2.19) \quad \Delta^{\omega_i}(x) = \begin{cases} a^{\omega_i} & \text{if } w\omega_i = \omega_i; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By (2.14), we have

$$\Delta^{\omega_i}(x^- a \bar{w} x^+) = a^{\omega_i} \Delta^{\omega_i}(\bar{w}).$$

Thus, to prove (2.19) we only need to show that

$$(2.20) \quad \Delta^{\omega_i}(\bar{w}) = \begin{cases} 1 & \text{if } w\omega_i = \omega_i; \\ 0, & \text{otherwise.} \end{cases}$$

The formula is obvious for $w = e$, the identity element of W . Hence we can assume that $\ell(w) \geq 1$ and write w as us_j for some $u \in W$ and $j \in [1, r]$ with $\ell(u) = \ell(w) - 1$. Since Δ^{ω_i} is a regular function on G , we have

$$(2.21) \quad \Delta^{\omega_i}(\bar{w}) = \Delta^{\omega_i}(\bar{u} \bar{s}_j) = \lim_{t \rightarrow 0} \Delta^{\omega_i}(\bar{u} x_j(t) \bar{s}_j).$$

Substituting into (2.21) the expression for $x_j(t) \bar{s}_j$ given by (2.13) and using (2.14), we obtain

$$\Delta^{\omega_i}(\bar{w}) = \lim_{t \rightarrow 0} t^{\omega_i(h_j)} \Delta^{\omega_i}(\bar{u} x_{\bar{j}}(t^{-1})).$$

Since $\ell(u) = \ell(w) - 1$, the root $u(\alpha_j)$ is positive, implying that $\bar{u} x_{\bar{j}}(t^{-1}) \bar{u}^{-1} \in N_-$. Again using (2.14), we obtain

$$\Delta^{\omega_i}(\bar{w}) = \lim_{t \rightarrow 0} t^{\delta_{ij}} \Delta^{\omega_i}(\bar{u}).$$

It follows that

$$\Delta^{\omega_i}(\bar{w}) = \begin{cases} \Delta^{\omega_i}(\bar{u}) & \text{if } j \neq i; \\ 0, & \text{otherwise.} \end{cases}$$

This implies (2.20) by induction on $\ell(w)$. □

As a corollary, we obtain the following useful characterization of the set G_0 .

Corollary 2.5. *An element $x \in G$ has Gaussian decomposition if and only if $\Delta^{\omega_i}(x) \neq 0$ for any $i \in [1, r]$.*

In subsequent proofs, we will also make use of the following identities.

Proposition 2.6. *For any $x = [x]_- [x]_0 [x]_+ \in G_0$ and any $w \in W$, we have*

$$(2.22) \quad \Delta_{\omega_i, w\omega_i}([x]_+) = \frac{\Delta_{\omega_i, w\omega_i}(x)}{\Delta^{\omega_i}(x)},$$

$$(2.23) \quad \Delta_{w\omega_i, \omega_i}([x]_-) = \frac{\Delta_{w\omega_i, \omega_i}(x)}{\Delta^{\omega_i}(x)}.$$

Proof. Using (1.10) and (2.14), we obtain:

$$\Delta_{\omega_i, w\omega_i}(x) = \Delta^{\omega_i}(x \bar{w}) = [x]_0^{\omega_i} \Delta^{\omega_i}([x]_+ \bar{w}) = \Delta^{\omega_i}(x) \Delta_{\omega_i, w\omega_i}([x]_+),$$

which proves (2.22). The proof of (2.23) is similar. □

The transformation $(-w_o)$ permutes fundamental weights. We will use the notation $i \mapsto i^*$ for the induced permutation of the index set $[1, r]$, so that

$$(2.24) \quad \omega_{i^*} = -w_o(\omega_i) \quad (i \in [1, r]).$$

Proposition 2.7. *For any $x \in G$ and any $u, v \in W$, we have*

$$(2.25) \quad \Delta_{u\omega_i, v\omega_i}(x) = \Delta_{v\omega_i, u\omega_i}(x^T) = \Delta_{vw_0\omega_i^*, uw_0\omega_i^*}(x^t) .$$

Proof. Using (2.15) and (2.4), we obtain:

$$\Delta_{u\omega_i, v\omega_i}(x) = \Delta^{\omega_i}(\overline{u^{-1}xv}) = \Delta^{\omega_i}((\overline{u^{-1}xv})^T) = \Delta^{\omega_i}(\overline{v^{-1}x^T u}) = \Delta_{v\omega_i, u\omega_i}(x^T) ,$$

which proves the first equality in (2.25).

To prove the second equality, let us introduce the anti-automorphism η of G by $\eta(x) = \overline{w_0}x^t\overline{w_0}$. Since η preserves H and interchanges N_- and N , it follows that η preserves G_0 , and $[\eta(x)]_0 = \eta([x]_0) = \overline{w_0}[x]_0^{-1}\overline{w_0}$ for any $x \in G_0$. Using (1.2) and (2.24), we conclude that $\Delta^{\omega_i}(\eta(x)) = \Delta^{\omega_i^*}(x)$. Hence

$$\begin{aligned} \Delta_{u\omega_i, v\omega_i}(x) &= \Delta^{\omega_i}(\overline{u^{-1}xv}) = \Delta^{\omega_i^*}(\eta(\overline{u^{-1}xv})) = \Delta^{\omega_i^*}(\overline{w_0}\overline{v}^{-1}x^t\overline{u}^{-1}\overline{w_0}) \\ &= \Delta^{\omega_i^*}(\overline{w_0v^{-1}x^t\overline{u}w_0}) = \Delta_{vw_0\omega_i^*, uw_0\omega_i^*}(x^t) , \end{aligned}$$

as claimed. □

Proof of Theorem 1.17. First of all, since $\overline{s_i u^{-1}} = \overline{s_i} \overline{u^{-1}}$ and $\overline{v s_i} = \overline{v} \overline{s_i}$, the definition (1.10) implies that it is enough to prove (1.25) in the case when $u = v = e$, the identity element. Thus, we only need to show that

$$(2.26) \quad \Delta_{\omega_i, \omega_i} \Delta_{s_i \omega_i, s_i \omega_i} - \Delta_{s_i \omega_i, \omega_i} \Delta_{\omega_i, s_i \omega_i} = \prod_{j \neq i} \Delta_{\omega_j, \omega_j}^{-a_{ji}} .$$

As in the case of Proposition 2.2, our proof of (2.26) will rely on representation theory. Consider the representation ρ of the group $G \times G$ in $\mathbb{C}[G]$ given by

$$\rho(x_1, x_2)f(x) = f(x_1^T x x_2) .$$

Let us denote the left- and right-hand sides of (2.26) by f_1 and f_2 , respectively. We first verify that the function $f_2 \in \mathbb{C}[G]$ has the following properties:

- (1) f_2 is a highest weight vector in the representation ρ , i.e., it is invariant under the subgroup $N \times N \subset G \times G$;
- (2) f_2 has weight (γ, γ) , where $\gamma = 2\omega_i - \alpha_i$; that is, $\rho(a_1, a_2)f_2 = (a_1 a_2)^\gamma f_2$ for any $a_1, a_2 \in H$;
- (3) $f_2(e) = 1$ (here e stands for the identity element of G).

Property (3) is trivial, while (1) follows from (2.14). Also by (2.14), f_2 has weight $(-\sum_{j \neq i} a_{ji}\omega_j, -\sum_{j \neq i} a_{ji}\omega_j)$. To prove (2), it is enough to show that $-\sum_{j \neq i} a_{ji}\omega_j = \gamma$; but this follows from the equality

$$(2.27) \quad \sum_{j \in [1, r]} a_{ji}\omega_j = \alpha_i ,$$

which can be taken as a definition of the Cartan matrix.

Properties (1)–(3) uniquely determine the restriction of f_2 to G_0 . Since G_0 is dense in G , and f_2 is regular, these properties uniquely determine f_2 . It remains to show that f_1 satisfies (1)–(3).

The normalization condition (3) follows from Proposition 2.4; indeed, in view of (2.19),

$$\Delta_{\omega_i, \omega_i}(e) = \Delta_{s_i \omega_i, s_i \omega_i}(e) = 1 , \quad \Delta_{s_i \omega_i, \omega_i}(e) = \Delta_{\omega_i, s_i \omega_i}(e) = 0 .$$

To prove that f_1 satisfies (2), notice that for any $u, v \in W$, the function $\Delta_{u\omega_i, v\omega_i}$ has weight $(u\omega_i, v\omega_i)$ (this follows from (1.10) and (1.2)). Hence both summands in f_1 have weight $(\omega_i + s_i\omega_i, \omega_i + s_i\omega_i) = (\gamma, \gamma)$.

To prove that f_1 satisfies (1), we first notice that, in view of (2.25), we have $f_1(x^T) = f_1(x)$ for any $x \in G$, hence $\rho(x_1, x_2)f_1 = \rho(x_2, x_1)f_1$ for any $x_1, x_2 \in G$. Therefore, it suffices to show that f_1 is invariant under the action of N by right translations. Let E_1, \dots, E_r be the infinitesimal right translation operators on $\mathbb{C}[G]$ defined by

$$E_j(f)(x) = \frac{d}{dt}f(xx_j(t))|_{t=0} ;$$

each E_j is a derivation of the ring $\mathbb{C}[G]$. It is enough to show that $E_j f_1 = 0$ for all j . If $j \neq i$, then E_j annihilates all four minors that appear in f_1 (this follows from the fact that $\overline{s_i}x_j(t)\overline{s_i}^{-1} \in N$), hence $E_j f_1 = 0$. It remains to prove that $E_i f_1 = 0$. Clearly, we have

$$(2.28) \quad E_i \Delta_{\omega_i, \omega_i} = E_i \Delta_{s_i \omega_i, \omega_i} = 0 .$$

We claim that

$$(2.29) \quad E_i \Delta_{\omega_i, s_i \omega_i} = \Delta_{\omega_i, \omega_i}, \quad E_i \Delta_{s_i \omega_i, s_i \omega_i} = \Delta_{s_i \omega_i, \omega_i} .$$

Combining (2.28) and (2.29) and using the Leibniz rule, we obtain

$$\begin{aligned} E_i f_1 &= E_i(\Delta_{\omega_i, \omega_i} \Delta_{s_i \omega_i, s_i \omega_i} - \Delta_{s_i \omega_i, \omega_i} \Delta_{\omega_i, s_i \omega_i}) \\ &= \Delta_{\omega_i, \omega_i} \Delta_{s_i \omega_i, \omega_i} - \Delta_{\omega_i, \omega_i} \Delta_{s_i \omega_i, \omega_i} = 0, \end{aligned}$$

as required. We will deduce (2.29) from the following lemma which is a standard fact in the representation theory of SL_2 .

Lemma 2.8. *Suppose $f \in \mathbb{C}[G]$ is such that $E_i f = 0$ and $f(xt^{h_i}) = t^k f(x)$ for some $k \geq 0$. Let $f' \in \mathbb{C}[G]$ be given by $f'(x) = f(x\overline{s_i})$. Then $E_i^k(f') = k!f$.*

The first equality in (2.29) follows by applying this lemma to $f = \Delta_{\omega_i, \omega_i}$ (in this case, $k = 1$ and $f' = \Delta_{\omega_i, s_i \omega_i}$). Similarly, the second equality in (2.29) follows by applying Lemma 2.8 to $f = \Delta_{s_i \omega_i, \omega_i}$ (in this case, $k = 1$ and $f' = \Delta_{s_i \omega_i, s_i \omega_i}$). This completes the proof of Theorem 1.17. \square

2.4. Affine coordinates in Schubert cells. For every $w \in W$, the corresponding Schubert cell $(BwB)/B \subset G/B$ is the image of the Bruhat cell BwB under the natural projection of G onto the flag variety G/B .

Let the subgroups $N_+(w) \subset N$ and $N_-(w) \subset N_-$ be defined by

$$(2.30) \quad N_+(w) = N \cap \tilde{w}N_-\tilde{w}^{-1}, \quad N_-(w) = N_- \cap \tilde{w}^{-1}N\tilde{w},$$

where \tilde{w} is any representative of w in G ; since H normalizes N and N_- , these subgroups do not depend on the choice of \tilde{w} . The following proposition is essentially well known (cf. [11, Corollary 23.60]).

Proposition 2.9. *An element $x \in G$ lies in the Bruhat cell BwB if and only if, for some (equivalently, any) representative $\tilde{w} \in G$ of w , we have $\tilde{w}^{-1}x \in G_0$ and $[\tilde{w}^{-1}x]_- \in N_-(w)$. Furthermore, the element*

$$(2.31) \quad y_+ = \pi_+(x) = \tilde{w}[\tilde{w}^{-1}x]_- \tilde{w}^{-1} \in N_+(w)$$

does not depend on the choice of \tilde{w} , and the correspondence $\pi_+ : x \mapsto y_+$ induces a biregular isomorphism between the Schubert cell $(BwB)/B$ and $N_+(w)$.

Using the transpose map $x \mapsto x^T$, one obtains a counterpart of Proposition 2.9 for the opposite Bruhat cell B_-wB_- .

Proposition 2.10. *An element $x \in G$ lies in B_-wB_- if and only if, for some (equivalently, any) representative $\tilde{w} \in G$ of w , we have $x\tilde{w}^{-1} \in G_0$ and $[x\tilde{w}^{-1}]_+ \in N_+(w)$. Furthermore, the element*

$$(2.32) \quad y_- = \pi_-(x) = \tilde{w}^{-1}[x\tilde{w}^{-1}]_+\tilde{w} \in N_-(w)$$

does not depend on the choice of \tilde{w} , and the correspondence $\pi_- : x \mapsto y_-$ induces a biregular isomorphism between the “opposite Schubert cell” $B_- \setminus (B_-wB_-)$ and $N_-(w)$.

The group $N_-(w)$ is a unipotent Lie group of dimension $\ell = \ell(w)$, hence it is isomorphic to the affine space \mathbb{C}^ℓ as an algebraic variety. We will associate with any $\mathbf{i} = (i_1, \dots, i_\ell) \in R(w)$ the following system of affine coordinates on $N_-(w)$. For $(p_1, \dots, p_\ell) \in \mathbb{C}^\ell$, we set

$$(2.33) \quad y_{\mathbf{i}}(p_1, \dots, p_\ell) = \overline{w}^{-1} \cdot \overline{s_{i_1}} x_{\overline{i_1}}(p_1) \cdots \overline{s_{i_\ell}} x_{\overline{i_\ell}}(p_\ell) .$$

Also, let us define

$$(2.34) \quad w_k = w_{k,\mathbf{i}} = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_k}$$

for $k \in [1, \ell + 1]$, so that $w_1 = w^{-1}$ and $w_{\ell+1} = e$.

Proposition 2.11. *The map $(p_1, \dots, p_\ell) \mapsto y = y_{\mathbf{i}}(p_1, \dots, p_\ell)$ is a biregular isomorphism between \mathbb{C}^ℓ and $N_-(w)$. The inverse map is given by*

$$(2.35) \quad p_k = \Delta_{w_k \omega_{i_k}, w_{k+1} \omega_{i_k}}(y) .$$

Proof. We can rewrite (2.33) as

$$y_{\mathbf{i}}(p_1, \dots, p_\ell) = \prod_{k=1}^{\ell} \overline{w_{k+1}} x_{\overline{i_k}}(p_k) \overline{w_{k+1}}^{-1} .$$

Each factor $\overline{w_{k+1}} x_{\overline{i_k}}(p_k) \overline{w_{k+1}}^{-1}$ belongs to the root subgroup in G corresponding to the root $-w_{k+1}(\alpha_{i_k})$, and these are all the root subgroups in $N_-(w)$ (cf. [5, VI, 1.6]). This implies the first statement in Proposition 2.11. To prove (2.35), we set $\mathbf{i}' = (i_1, \dots, i_{k-1})$ and $\mathbf{i}'' = (i_{k+1}, \dots, i_\ell)$ so that $\mathbf{i} = (\mathbf{i}', i_k, \mathbf{i}'')$. Let $y' = y_{\mathbf{i}'}(p_1, \dots, p_{k-1})$ and $y'' = y_{\mathbf{i}''}(p_{k+1}, \dots, p_\ell)$. In view of (2.33), we have

$$(2.36) \quad \overline{w_k}^{-1} y \overline{w_{k+1}} = y' (\overline{s_{i_k}} x_{\overline{i_k}}(p_k)) (\overline{w_{k+1}}^{-1} y'' \overline{w_{k+1}}) .$$

In this decomposition, the first factor y' belongs to $N_-(s_{i_1} \cdots s_{i_{k-1}}) \subset N_-$, while the last factor $\overline{w_{k+1}}^{-1} y'' \overline{w_{k+1}}$ belongs to $w_{k+1}^{-1} N_-(w_{k+1}^{-1}) w_{k+1} \subset N$. Using (2.14) and (2.13), we conclude that

$$\begin{aligned} \Delta_{w_k \omega_{i_k}, w_{k+1} \omega_{i_k}}(y) &= \Delta^{\omega_{i_k}} (\overline{w_k}^{-1} y \overline{w_{k+1}}) = \Delta^{\omega_{i_k}} (\overline{s_{i_k}} x_{\overline{i_k}}(p_k)) \\ &= \Delta^{\omega_{i_k}} (x_{\overline{i_k}}(-p_k^{-1}) p_k^{h_{i_k}} x_{i_k}(p_k^{-1})) = p_k , \end{aligned}$$

as claimed. □

Note for future use that a similar argument allows us to prove that, for any $i \in [1, r]$, $k \in [1, \ell + 1]$, and $y \in N_-(w)$, we have

$$(2.37) \quad \Delta_{w_k \omega_i, w_k \omega_i}(y) = 1 .$$

This follows from a decomposition similar to (2.36):

$$\overline{w_k}^{-1}y\overline{w_k} = y'(\overline{w_k}^{-1}y''\overline{w_k}),$$

where $y' \in N_-(s_{i_1} \cdots s_{i_{k-1}}) \subset N_-$, and $\overline{w_k}^{-1}y''\overline{w_k} \in w_k^{-1}N_-(w_k^{-1}w_k) \subset N_-$.

As a corollary of Proposition 2.11, we obtain defining equations for $N_-(w)$ as a subvariety in N_- . Notice that $N_- = N_-(w_o)$. Hence, for every $\mathbf{j} = (j_1, \dots, j_n) \in R(w_o)$, any element $y \in N_-$ can be uniquely written as $y = y_{\mathbf{j}}(p_1, \dots, p_n)$ for some $(p_1, \dots, p_n) \in \mathbb{C}^n$ (here $n = \ell(w_o)$). Let us choose \mathbf{j} so that its first $n - \ell$ indices form a reduced word \mathbf{j}_1 for $w_o w^{-1}$, while the last ℓ indices form a reduced word \mathbf{j}_2 for w . Then write $(w_o)_k = s_{j_n} s_{j_{n-1}} \cdots s_{j_k}$ for $k \in [1, n + 1]$, in agreement with (2.34). Finally, let us denote

$$N'_-(w) = N_- \cap \tilde{w}^{-1}N_- \tilde{w},$$

where \tilde{w} is any representative of w in G (cf. (2.30)). The following proposition is an immediate consequence of Proposition 2.11 and the definition (2.33).

Proposition 2.12. *Every $y \in N_-$ is uniquely written as $y = y_1 y_2$ with $y_1 \in N'_-(w)$ and $y_2 \in N_-(w)$. In the above notation, if $y = y_{\mathbf{j}}(p_1, \dots, p_n)$, then*

$$y_1 = \overline{w}^{-1}y_{\mathbf{j}_1}(p_1, \dots, p_{n-\ell})\overline{w}, \quad y_2 = y_{\mathbf{j}_2}(p_{n-\ell+1}, \dots, p_n).$$

Hence y lies in $N_-(w)$ if and only if

$$(2.38) \quad \Delta_{(w_o)_k \omega_{j_k}, (w_o)_{k+1} \omega_{j_k}}(y) = 0$$

for $k = 1, \dots, n - \ell$.

2.5. y -coordinates in double Bruhat cells. Let us fix a pair $(u, v) \in W \times W$ and consider the open subset $G_0^{u,v} = G^{u,v} \cap G_0$ consisting of the elements x in the double Bruhat cell $G^{u,v}$ that admit Gaussian decomposition (1.5). In view of Propositions 2.9 and 2.10, the restrictions $\pi_+ : G^{u,v} \rightarrow N_+(u)$ and $\pi_- : G^{u,v} \rightarrow N_-(v)$ are well defined. Let us also introduce the map $\pi_0 : G_0^{u,v} \rightarrow H$ by

$$(2.39) \quad y_0 = \pi_0(x) = [x]_0,$$

thus obtaining the map

$$\pi^{u,v} = (\pi_+, \pi_0, \pi_-) : G_0^{u,v} \rightarrow N_+(u) \times H \times N_-(v).$$

For $x \in G_0^{u,v}$, we will write $\pi^{u,v}(x) = (y_+, y_0, y_-)$ and call this triple the y -coordinates of x .

Example 2.13. Let $G = SL_2(\mathbb{C})$, and let $u = v = w_o$ (cf. Example 1.8). A matrix $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with determinant 1 belongs to $G_0^{u,v}$ if and only if $x_{11} \neq 0$, $x_{12} \neq 0$, $x_{21} \neq 0$. Using formulas (2.31), (2.32), and (2.39), we see that the y -coordinates of x are given by

$$(2.40) \quad y_+ = \begin{pmatrix} 1 & x_{11}x_{21}^{-1} \\ 0 & 1 \end{pmatrix}, \quad y_0 = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{11}^{-1} \end{pmatrix}, \quad y_- = \begin{pmatrix} 1 & 0 \\ x_{11}x_{12}^{-1} & 1 \end{pmatrix}.$$

Our use of the term ‘‘coordinates’’ for the triple (y_+, y_0, y_-) is justified by the following statement.

Proposition 2.14. *The map $\pi^{u,v}$ is a biregular isomorphism*

$$G_0^{u,v} \rightarrow (N_+(u) \cap G_0 u^{-1}) \times H \times (N_-(v) \cap v^{-1}G_0) .$$

The inverse isomorphism $(y_+, y_0, y_-) \mapsto x$ is given by

$$(2.41) \quad [x]_- = [y_+ \tilde{u}]_-, [x]_0 = y_0, [x]_+ = [\tilde{v}y_-]_+,$$

where \tilde{u} and \tilde{v} are arbitrary representatives of u and v .

Proof. By Proposition 2.9, any $x \in BuB$ can be written as $x = y_+ \tilde{u}b$, where $y_+ = \pi_+(x)$ and $b \in B$. It follows that $x \in G_0$ if and only if $y_+ \tilde{u} \in G_0$, and if this is the case, then $[x]_- = [y_+ \tilde{u}]_-$. Similarly, an element $x \in B_-vB_-$ lies in G_0 if and only if $\tilde{v}y_- \in G_0$, and then $[x]_+ = [\tilde{v}y_-]_+$. It follows that $\pi^{u,v}$ is an embedding of $G_0^{u,v}$ into $(N_+(u) \cap G_0 u^{-1}) \times H \times (N_-(v) \cap v^{-1}G_0)$, and that the inverse map is given by (2.41). The same argument shows that if the triple (y_+, y_0, y_-) lies in $(N_+(u) \cap G_0 u^{-1}) \times H \times (N_-(v) \cap v^{-1}G_0)$, then the element x given by (2.41) lies in $G_0^{u,v}$, and we are done. \square

The following proposition is immediate from the definitions.

Proposition 2.15. *We have $N_+(u)^T = N_-(u^{-1})$ and $(G_0^{u,v})^T = G_0^{v^{-1},u^{-1}}$. If $x \in G_0^{u,v}$ has y -coordinates (y_+, y_0, y_-) , then x^T has y -coordinates (y_-^T, y_0, y_+^T) .*

This proposition shows that the transpose map “interchanges” the coordinates y_+ and y_- , so that any statement about y_- has a counterpart for y_+ . For instance, Proposition 2.9 is a counterpart of Proposition 2.10 in this sense.

Proposition 2.16. *Suppose $x \in G_0^{u,v}$ has the y -coordinates (y_+, y_0, y_-) . Then*

$$(2.42) \quad [\overline{u^{-1}x}]_0^{-1} = [y_+ \overline{u}]_0 y_0^{-1}, [x \overline{v^{-1}}]_0^{-1} = y_0^{-1} [\overline{v}y_-]_0 .$$

Proof. By Proposition 2.9, $x = y_+ \overline{u}b$ for some $b \in B$. It follows that $y_0 = [y_+ \overline{u}]_0 [b]_0$. On the other hand, $\overline{u^{-1}x} = \overline{u^{-1}}y_+ \overline{u}b \in N_-b$, hence

$$[\overline{u^{-1}x}]_0 = [b]_0 = y_0 [y_+ \overline{u}]_0^{-1} .$$

This proves the first equality in (2.42); the second one follows by Proposition 2.15. \square

It will be of special importance for us to specialize Proposition 2.14 to the case when $(u, v) = (e, w)$, where e is the identity element of W , and $w \in W$ is arbitrary. Then we have $G_0^{e,w} = G^{e,w} = HN^w$ where

$$(2.43) \quad N^w = N \cap B_-wB_- .$$

Specializing Proposition 2.14 to this case, we obtain the following statement.

Proposition 2.17. *For any $w \in W$, the map $\pi_- : B_-wB_- \rightarrow N_-(w)$ restricts to a biregular isomorphism $N^w \rightarrow N_-(w) \cap w^{-1}G_0$. The inverse isomorphism $N_-(w) \cap w^{-1}G_0 \rightarrow N^w$ is given by $y \mapsto [\tilde{w}y]_+$, where \tilde{w} is an arbitrary representative of w .*

Using Proposition 2.15, we see that Proposition 2.14 is equivalent to its special case given by Proposition 2.17 combined with the following decomposition:

$$(2.44) \quad G_0^{u,v} = (N^{u^{-1}})^T HN^v .$$

2.6. Factorization problem in Schubert cells. In this section we recall some results from [4] concerning a version of the factorization problem for the variety $N^w = N \cap B_- w B_-$ (cf. (2.43)). We will need the following analogue of Theorem 1.2 which is essentially due to G. Lusztig [18] (cf. also [4, Proposition 1.1]).

Proposition 2.18. *For any $w \in W$ and any reduced word $\mathbf{i} = (i_1, \dots, i_\ell) \in R(w)$, the map $(t_1, \dots, t_\ell) \mapsto x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell)$ is a biregular isomorphism between $\mathbb{C}_{\neq 0}^\ell$ and a Zariski open subset of N^w .*

We will give explicit formulas for the inverse of the product map in Proposition 2.18.

Theorem 2.19. *Let $\mathbf{i} = (i_1, \dots, i_\ell) \in R(w)$, and let $x = x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell) \in N^w$ with all t_k nonzero complex numbers. Then the t_k are recovered from x by*

$$(2.45) \quad t_k = \frac{1}{\Delta_{w_k \omega_{i_k}, \omega_{i_k}}(y) \Delta_{w_{k+1} \omega_{i_k}, \omega_{i_k}}(y)} \prod_{j \neq i_k} \Delta_{w_{k+1} \omega_j, \omega_j}(y)^{-a_{j, i_k}},$$

where w_k is given by (2.34), and $y = \pi_-(x) \in N_-(w)$ (cf. (2.32)).

This theorem is a reformulation of [4, Theorems 1.4, 6.2]. Here we present a new proof which is in some sense more elementary than the one in [4], and also provides additional information that we will need later on.

Proof. There is nothing to prove if $w = e$, so we will assume that $\ell(w) = \ell \geq 1$. Let $y = \pi_-(x)$ and $z = \overline{w}y$. By Proposition 2.17, $z \in G_0$ and $x = [z]_+$. Let us write $i_1 = i$, and denote $w' = s_i w$, $\mathbf{i}' = (i_2, \dots, i_\ell) \in R(w')$, $x' = x_i(-t_1)x = x_{i_2}(t_2) \cdots x_{i_\ell}(t_\ell) \in N^{w'}$, $y' = \pi_-(x') \in N_-(w')$, and $z' = \overline{w'}y'$. Here is the key lemma.

Lemma 2.20. *In the notation just introduced, let us write $y' = y_{\mathbf{i}'}(p_2, \dots, p_\ell)$, in accordance with Proposition 2.11. Then $y = y_{\mathbf{i}}(p_1, p_2, \dots, p_\ell)$, where p_1 is given by*

$$(2.46) \quad p_1 = \Delta_{s_i \omega_i, \omega_i}(x_{\tilde{i}}([z']_0^{-\alpha_i} t_1^{-1}) [z']_-^{-1}).$$

Furthermore, we have

$$(2.47) \quad t_1 = [z']_0^{\omega_i - \alpha_i} [z]_0^{-\omega_i}.$$

Proof. Let us temporarily denote $\tilde{y} = y_{\mathbf{i}}(p_1, p_2, \dots, p_\ell)$ and $\tilde{z} = \overline{w}\tilde{y}$, where p_1 is given by (2.46); our goal is to show that $\tilde{y} = y$ and $\tilde{z} = z$. By Proposition 2.17, it suffices to show that $[\tilde{z}]_+ = x$, or equivalently that $\tilde{z}x^{-1} \in B_-$.

By Proposition 2.12 (applied to $w = s_i$), formula (2.46) implies that

$$(2.48) \quad x_{\tilde{i}}([z']_0^{-\alpha_i} t_1^{-1}) [z']_-^{-1} = \overline{s_i} y'' \overline{s_i}^{-1} x_{\tilde{i}}(p_1),$$

where $y'' \in N_-$. Using (2.5) and (2.12), we can rewrite the left-hand side of (2.48) as follows:

$$\begin{aligned} x_{\tilde{i}}([z']_0^{-\alpha_i} t_1^{-1}) [z']_-^{-1} &= [z']_0 x_{\tilde{i}}(t_1^{-1}) [z']_0^{-1} [z']_-^{-1} = [z']_0 x_{\tilde{i}}(t_1^{-1}) x'(z')^{-1} \\ &= [z']_0 x_{\tilde{i}}(t_1^{-1}) x_i(-t_1) x(z')^{-1} = [z']_0 \overline{s_i} t_1^{-h_i} x_{\tilde{i}}(-t_1^{-1}) x(z')^{-1}. \end{aligned}$$

Substituting this expression into (2.48) and using the fact that $\tilde{z} = \overline{s_i}^{-1} x_{\tilde{i}}(p_1) z'$, we can rewrite (2.48) as follows:

$$(2.49) \quad \tilde{z}x^{-1} = (y'')^{-1} \overline{s_i}^{-1} [z']_0 \overline{s_i} t_1^{-h_i} x_{\tilde{i}}(-t_1^{-1}).$$

It follows that $\tilde{z}x^{-1} \in B_-$, hence $\tilde{y} = y$ and $\tilde{z} = z$. Since the left-hand side of (2.49) is equal to $[z]_- [z]_0$, it follows that

$$(2.50) \quad [z]_0 = \overline{s_i}^{-1} [z']_0 \overline{s_i} t_1^{-h_i} .$$

Finally, (2.47) follows from (2.50) by applying the character $a \mapsto a^{\omega_i}$ to both sides and using (1.2). □

Note that (2.46) can be simplified as follows:

$$(2.51) \quad p_1 = \Delta_{\omega_i, w'^{-1}\omega_i}(xy'^{-1}) / \Delta_{\omega_i, w^{-1}\omega_i}(x) ;$$

since we will not need this formula, the proof is left to the reader.

Continuing with the proof of Theorem 2.19, let us define, for $k = 1, \dots, \ell$:

$$x^{(k)} = x_{i_k}(t_k) \cdots x_{i_\ell}(t_\ell) \in N^{w_k^{-1}} , \quad y^{(k)} = \pi_-(x^{(k)}) , \quad z^{(k)} = \overline{w_k^{-1}} y^{(k)} .$$

Applying (2.47) with x replaced by $x^{(k)}$ yields

$$(2.52) \quad t_k = [z^{(k+1)}]_0^{\omega_{i_k} - \alpha_{i_k}} [z^{(k)}]_0^{-\omega_{i_k}} .$$

On the other hand, combining the definition (1.10) with (2.27), we can rewrite (2.45) as follows:

$$(2.53) \quad t_k = [\overline{w_{k+1}^{-1}} y]_0^{\omega_{i_k} - \alpha_{i_k}} [\overline{w_k^{-1}} y]_0^{-\omega_{i_k}} .$$

Comparing (2.53) with (2.52), we see that Theorem 2.19 would follow from the equality $[\overline{w_k^{-1}} y]_0 = [z^{(k)}]_0$. The latter is obtained by observing that $\overline{w_k^{-1}} y = \overline{w_k^{-1}} \overline{w}^{-1} z = \tilde{y} z^{(k)}$, where $\tilde{y} = y_{(i_1, \dots, i_{k-1})}(p_1, \dots, p_{k-1}) \in N_-$ (this \tilde{y} was denoted by y' in (2.36)). □

2.7. Totally positive bases for $N_-(w)$. Although most of the results in this section were obtained in [4], we prefer to give independent proofs here; in some cases, this will allow us to refine the statements in [4].

We start with the following general definition.

Definition 2.21. Let F be a finite collection of functions on a set X . A subset $\mathcal{B} \subset F$ is called a *totally positive base* for F if \mathcal{B} is a minimal (with respect to inclusion) subset of F with the property that every $f \in F$ is a *subtraction-free expression* (i.e., a ratio of two polynomials with nonnegative integer coefficients) in the elements of \mathcal{B} .

For every $w \in W$, let us denote

$$(2.54) \quad F(w) = \{ \Delta_{w''\omega_i, w'\omega_i}(y) : i \in [1, r], w' \preceq w'' \preceq w^{-1} \} .$$

(As earlier in (1.23), $w' \preceq w''$ stands for $\ell(w'') = \ell(w') + \ell(w'^{-1}w'')$.) To every reduced word $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$ we associate three collections of regular functions on the group $N_-(w)$:

$$(2.55) \quad \begin{aligned} F_1(\mathbf{i}) &= \{ \Delta_{w_k \omega_{i_k}, \omega_{i_k}} : 1 \leq k \leq m \} , \\ F_2(\mathbf{i}) &= \{ \Delta_{w^{-1} \omega_{i_k}, w_k \omega_{i_k}} : 2 \leq k \leq m + 1 \} , \\ F(\mathbf{i}) &= \{ \Delta_{w_k \omega_i, w_l \omega_i} : i \in [1, r], 1 \leq k \leq l \leq m + 1 \} , \end{aligned}$$

where $w_k = s_{i_m} \cdots s_{i_k}$ (cf. (2.34)).

Theorem 2.22. *For any reduced word $\mathbf{i} = (i_1, \dots, i_m) \in R(w)$, each of the collections $F_1(\mathbf{i})$ and $F_2(\mathbf{i})$ of regular functions on $N_-(w)$ is a transcendence basis for $\mathbb{C}(N_-(w))$ and a totally positive base for $F(w)$.*

Proof. Let us first deal with $F_1(\mathbf{i})$. The most important part of the proof is to show that every minor in $F(w)$ is a subtraction-free expression in the minors from $F_1(\mathbf{i})$. Since we obviously have

$$F(w) = \bigcup_{\mathbf{i} \in R(w)} F(\mathbf{i}) ,$$

this statement will directly follow from Lemmas 2.23 and 2.24 below.

Lemma 2.23. *For any two reduced words $\mathbf{i}, \mathbf{i}' \in R(w)$, every minor in $F_1(\mathbf{i}')$ is a subtraction-free expression in the minors from $F_1(\mathbf{i})$.*

Proof. This is an immediate corollary of [4, Corollary 6.7]. The proof in [4] is based on repeated applications of determinantal identities of Theorem 1.16. \square

Lemma 2.24. *Every minor in $F(\mathbf{i})$ is a subtraction-free expression in the minors from $F_1(\mathbf{i})$.*

Proof. We need to show that every minor $\Delta_{w_k \omega_i, w_l \omega_i}$, for $i \in [1, r]$ and $1 \leq k \leq l \leq m + 1$, is a subtraction-free expression in the minors $\Delta_{w_k \omega_i, \omega_i}$. Recall that, by convention, $w_{m+1} = e$, so the statement is trivial for $l = m + 1$. By (2.37), it also holds for $k = l$, since the corresponding minor equals 1. Thus we may assume that $1 \leq k < l \leq m$; increasing k and l if necessary, we can also assume without loss of generality that $i_k = i_l = i$.

Let us arrange all the pairs (k, l) with $1 \leq k \leq l \leq m + 1$ in the following order: $(k', l') < (k, l)$ if either $l' > l$, or $l' = l, k' > k$. Using induction with respect to this linear order, it is enough to show that for every (k, l) such that $1 \leq k < l \leq m$ and $i_k = i_l = i$, the minor $\Delta_{w_k \omega_i, w_l \omega_i}$ is a subtraction-free expression in the minors $\Delta_{w_k, \omega_j, w_l, \omega_j}$ with $j \in [1, r]$ and $(k', l') < (k, l)$. The latter follows from the identity (1.25) applied to $u = w_{k+1}$ and $v = w_{l+1}$. Indeed, this identity can be rewritten as

$$\Delta_{w_k \omega_i, w_l \omega_i} = \frac{\Delta_{w_k \omega_i, w_{l+1} \omega_i} \Delta_{w_{k+1} \omega_i, w_l \omega_i} + \prod_{j \neq i} \Delta_{w_{k+1} \omega_j, w_{l+1} \omega_j}^{-a_{ji}}}{\Delta_{w_{k+1} \omega_i, w_{l+1} \omega_i}} ,$$

providing a desired subtraction-free expression. \square

Lemma 2.24 implies in particular that each minor $\Delta_{w_k \omega_{i_k}, w_{k+1} \omega_{i_k}}(y)$ is a rational function of the minors from $F_1(\mathbf{i})$. By Proposition 2.11, it follows that $F_1(\mathbf{i})$ is a transcendence basis for $\mathbb{C}(N_-(w))$, hence it is a totally positive base for $F(w)$.

To prove that $F_2(\mathbf{i})$ has the same properties, we will apply the anti-automorphism τ_w of G given by

$$(2.56) \quad \tau_w(y) = \overline{w}(y^{-1})^\theta \overline{w}^{-1} ,$$

where θ was defined in (1.11). In view of (2.4), if $\tau_w(y) = y'$, then

$$(2.57) \quad y = \tau_{w^{-1}}(y') , \quad \tau_w(y^T) = y'^T .$$

A straightforward check shows that

$$(2.58) \quad \tau_{w^{-1}}(N_+(w)) = N_+(w^{-1}) , \quad \tau_w(N_-(w)) = N_-(w^{-1}) .$$

Lemma 2.25. *Let $y \in G$, and let $y' = \tau_w(y)$ for some $w \in W$. For any $w', w'' \in W$ such that $w' \preceq w^{-1}$ and $w'' \preceq w^{-1}$, we have*

$$(2.59) \quad \Delta_{w''\omega_i, w'\omega_i}(y) = \Delta_{ww'\omega_i, ww''\omega_i}(y') .$$

Proof. In view of (1.9) and (2.4), the conditions $w' \preceq w^{-1}$ and $w'' \preceq w^{-1}$ imply that

$$\overline{\overline{(ww')^{-1}}} = \overline{\overline{w'}}^{-1} \overline{\overline{w}}^{-1} , \quad \overline{\overline{ww''}} = \overline{\overline{w}} \overline{\overline{w''}} .$$

Combining this with (2.15), we obtain:

$$(2.60) \quad \begin{aligned} \Delta_{ww'\omega_i, ww''\omega_i}(y') &= \Delta^{\omega_i}(\overline{\overline{(ww')^{-1}}}y'\overline{\overline{ww''}}) = \Delta^{\omega_i}(\overline{\overline{w'}}^{-1} \overline{\overline{w}}^{-1}y' \overline{\overline{w}} \overline{\overline{w''}}) \\ &= \Delta^{\omega_i}(\overline{\overline{w'}}^{-1}(y^{-1})^\theta \overline{\overline{w''}}) = \Delta^{\omega_i}(\overline{\overline{w''}^{-1}}y\overline{\overline{w}}) = \Delta_{w''\omega_i, w'\omega_i}(y) , \end{aligned}$$

as claimed. □

By Lemma 2.25, the anti-automorphism τ_w transforms $F(w)$ into $F(w^{-1})$, and $F_2(\mathbf{i})$ into $F_1(\mathbf{i}^*)$, where $\mathbf{i}^* = (i_m, \dots, i_1) \in R(w^{-1})$ is \mathbf{i} written backwards. Thus the fact that $F_2(\mathbf{i})$ is a transcendence basis for $\mathbb{C}(N_-(w))$ and a totally positive base for $F(w)$, follows from the same properties for $F_1(\mathbf{i})$ that we already proved. This completes the proof of Theorem 2.22. □

2.8. Total positivity in y -coordinates. Let $N_{\geq 0} \subset N$ denote the multiplicative semigroup generated by the elements $x_i(t)$ for $i \in [1, r]$ and $t > 0$. For every $w \in W$, let us denote (cf. [4])

$$(2.61) \quad N_{> 0}^w = N_{\geq 0} \cap N^w = N_{\geq 0} \cap B_- w B_- .$$

The following analogue of Theorem 1.3 is due to G. Lusztig [18] (cf. Proposition 2.18).

Proposition 2.26. *For any $w \in W$ and any reduced word $\mathbf{i} = (i_1, \dots, i_\ell) \in R(w)$, the map $(t_1, \dots, t_\ell) \mapsto x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell)$ restricts to a bijection $\mathbb{R}_{> 0}^\ell \rightarrow N_{> 0}^w$.*

We will use Theorem 2.22 to obtain the following criteria for total positivity.

Theorem 2.27. *Let $x \in N^w$, let $y = \pi_-(x) \in N_-(w)$, and let $\mathbf{i} = (i_1, \dots, i_\ell) \in R(w)$. Then the following conditions are equivalent:*

- (1) $x \in N_{> 0}^w$;
- (2) $\Delta(y) > 0$ for any $\Delta \in F(w)$;
- (3) $\Delta(y) > 0$ for any $\Delta \in F_1(\mathbf{i})$;
- (4) $\Delta(y) > 0$ for any $\Delta \in F_2(\mathbf{i})$.

Proof. The equivalence of (2), (3) and (4) is immediate from Theorem 2.22. Let us show the equivalence of (1) and (3). By Proposition 2.26, every $x \in N_{> 0}^w$ is of the form $x = x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell)$ for some $t_1, \dots, t_\ell > 0$. By Theorem 2.19, each t_k is a monomial in ℓ variables $\{\Delta(y) : \Delta \in F_1(\mathbf{i})\}$. It follows that the monomial transformation from $\{\Delta(y) : \Delta \in F_1(\mathbf{i})\}$ to $\{t_1, \dots, t_\ell\}$ is invertible (an explicit expression for the inverse transformation was given in [4, Theorem 4.3] but we will not need it here). Thus every $\Delta(y)$ with $\Delta \in F_1(\mathbf{i})$ is a Laurent monomial in t_1, \dots, t_ℓ . Hence $\Delta(y) > 0$, and (1) \Rightarrow (3) is proved.

To prove (3) \Rightarrow (1), suppose that $\Delta(y) > 0$ for $\Delta \in F_1(\mathbf{i})$. Let us define t_1, \dots, t_ℓ via (2.45), and let $\tilde{x} = x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell) \in N_{> 0}^w$. Setting $\tilde{y} = \pi_-(\tilde{x})$, we see that

$\Delta(y) = \Delta(\tilde{y})$ for any $\Delta \in F_1(\mathbf{i})$. By Lemma 2.24, we have $\Delta(y) = \Delta(\tilde{y})$ for any $\Delta \in F(\mathbf{i})$. In particular, $\Delta_{w_k \omega_{i_k}, w_{k+1} \omega_{i_k}}(y) = \Delta_{w_k \omega_{i_k}, w_{k+1} \omega_{i_k}}(\tilde{y})$ for $k = 1, \dots, \ell$. Using (2.35), we conclude that $y = \tilde{y}$ and so $x = \tilde{x} \in N_{>0}^w$, proving (3) \Rightarrow (1). \square

Corollary 2.28. *The map $\tau_w : N_-(w) \rightarrow N_-(w^{-1})$ (cf. (2.58)) restricts to a bijection $\pi_-(N_{>0}^w) \rightarrow \pi_-(N_{>0}^{w^{-1}})$.*

Proof. We have already observed that by Lemma 2.25, τ_w transforms $F(w)$ into $F(w^{-1})$. The corollary then follows from Theorem 2.27. \square

We will now show that using y -coordinates (i.e., passing from a double Bruhat cell $G^{u,v}$ to the open subset $G_0^{u,v}$) will not create problems in the study of totally positive varieties. The following proposition is due to G. Lusztig [18]; for the convenience of the reader, we provide a proof.

Proposition 2.29. *We have $N_{\geq 0} = N \cap G_{\geq 0}$, $N_{\geq 0}^T = N_- \cap G_{\geq 0}$ and $G_{\geq 0} = N_{\geq 0}^T H_{>0} N_{\geq 0}$. In particular, $G_{\geq 0} \subset G_0$, i.e., any totally nonnegative element in G admits the Gaussian decomposition. Furthermore, for any $u, v \in W$, the totally positive variety $G_{>0}^{u,v}$ decomposes as*

$$(2.62) \quad G_{>0}^{u,v} = (N_{>0}^{u^{-1}})^T H_{>0} N_{>0}^v,$$

in the notation of (2.61).

Proof. By the definition of $G_{\geq 0}$, every totally nonnegative element $x \in G$ has the form (cf. (1.3)) $x = x_{\mathbf{i}}(a; t_1, \dots, t_m)$, where $\mathbf{i} = (i_1, \dots, i_m)$ is some word in the alphabet $[1, r] \cup [\bar{1}, \bar{r}]$, the t_k are positive real numbers, and $a \in H_{>0}$. We say that \mathbf{i} is *unmixed* if all the indices from $[\bar{1}, \bar{r}]$ precede those from $[1, r]$. By repeated application of the commutation relations (2.5), (2.9) and (2.11), we can transform x to the form $x = x_{\mathbf{i}'}(a'; t'_1, \dots, t'_m)$ for an unmixed word \mathbf{i}' , $a' \in H_{>0}$, and all $t'_k > 0$. This proves the decomposition $G_{\geq 0} = N_{\geq 0}^T H_{>0} N_{\geq 0}$. The equalities $N_{\geq 0} = N \cap G_{\geq 0}$ and $N_{\geq 0}^T = N_- \cap G_{\geq 0}$ follow from this decomposition of $G_{\geq 0}$ and the uniqueness of the Gaussian decomposition. Finally, (2.62) is proved by the same argument combined with (2.44). \square

Combining Propositions 2.29 and 2.15 with Theorem 2.27, we obtain the following description of the totally positive variety $G_{>0}^{u,v}$ in terms of y -coordinates.

Theorem 2.30. *An element $x \in G_0^{u,v}$ lies in $G_{>0}^{u,v}$ if and only if its y -coordinates (y_+, y_0, y_-) satisfy the following properties:*

- $\Delta(y_-) > 0$ for any $\Delta \in F(v)$;
- $\Delta(y_+^T) > 0$ for any $\Delta \in F(u^{-1})$;
- $y_0 \in H_{>0}$.

3. PROOFS OF THE MAIN RESULTS

This section contains proofs of the main results in Section 1.

3.1. Proofs of Theorems 1.1, 1.2, and 1.3.

Proof of Theorem 1.1. We will explicitly construct a desired biregular isomorphism of $G^{u,v}$ with a Zariski open subset of $\mathbb{C}^{r+\ell(u)+\ell(v)}$ with the help of a “twisted”

version of y -coordinates (cf. Section 2.5). We fix a representative \tilde{u} of u , and associate to any $x \in G^{u,v}$ a triple $(y_{(+)}, y_{(0)}, y_{(-)})$ given by

$$(3.1) \quad y_{(+)} = \pi_+(x), \quad y_{(-)} = \pi_-(x^{-1}), \quad y_{(0)} = [\tilde{u}^{-1}x]_0 ;$$

in view of Propositions 2.9 and 2.10, this triple is well defined and belongs to $N_+(u) \times H \times N_-(v^{-1})$. Our statement is a consequence of the following.

Proposition 3.1. *The correspondence $x \mapsto (y_{(+)}, y_{(0)}, y_{(-)})$ given by (3.1) is a biregular isomorphism of $G^{u,v}$ with the Zariski open subset of $N_+(u) \times H \times N_-(v^{-1})$ consisting of triples such that $y_{(-)}y_{(+)} \in vG_0u^{-1}$.*

Proof. The proof is essentially the same as that of Proposition 2.14; the inverse of the correspondence (3.1) is given by

$$x = y_{(+)}\tilde{u}[\tilde{v}^{-1}y_{(-)}y_{(+)}\tilde{u}]_+^{-1}y_{(0)} ,$$

where \tilde{v} is any representative of v . □

Proof of Theorem 1.2. By Theorem 1.1, every double cell $G^{u,v}$ is smooth, and $\dim(G^{u,v}) = \dim(H \times \mathbb{C}^m)$. It follows from the Grothendieck-Zariski factorization theorem [14, Theorem 8.12.6] that if X and Y are irreducible algebraic varieties over \mathbb{C} of the same dimension, and Y is smooth, then any injective morphism from X to Y is an open embedding. Therefore, Theorem 1.2 is implied by the following proposition. (We thank Michel Brion for explaining this implication to us and providing the reference [14].)

Proposition 3.2. *For every $u, v \in W$ and $\mathbf{i} = (i_1, \dots, i_m) \in R(u, v)$, the map $x_{\mathbf{i}}$ restricts to an injective regular map $H \times \mathbb{C}_{\neq 0}^m \rightarrow G^{u,v}$.*

Proof. First let us show that $x_{\mathbf{i}}(H \times \mathbb{C}_{\neq 0}^m) \subset G^{u,v}$. We will show that $x_{\mathbf{i}}(H \times \mathbb{C}_{\neq 0}^m) \subset B_-vB_-$; the inclusion $x_{\mathbf{i}}(H \times \mathbb{C}_{\neq 0}^m) \subset BuB$ is proved similarly (or deduced from the previous one with the help of the transpose map). Let

$$\{k_1 < \dots < k_\ell\} = \{k \in [1, m] : i_k \in [1, r]\} ,$$

so that $(i_{k_1}, \dots, i_{k_\ell}) \in R(v)$. Let us use the fact that, for every $i \in [1, r]$ and nonzero $t \in \mathbb{C}$, we have $x_i(t) \in B_-s_iB_-$ and $x_{\bar{i}}(t) \in B_-$ (cf. (2.12)). It follows that if $(a; t_1, \dots, t_m) \in H \times \mathbb{C}_{\neq 0}^m$, then

$$x_{\mathbf{i}}(a; t_1, \dots, t_m) \in B_- \cdot B_-s_{i_{k_1}}B_- \cdot B_- \cdots B_- \cdot B_-s_{i_{k_\ell}}B_- \cdot B_- = B_-vB_- ,$$

as desired; the last equality follows from the well-known fact that

$$B_-w'B_- \cdot B_-w''B_- = B_-w'w''B_-$$

whenever $\ell(w'w'') = \ell(w') + \ell(w'')$ (cf. [5, IV.2.4]).

It remains to show that the map $x_{\mathbf{i}} : H \times \mathbb{C}_{\neq 0}^m \rightarrow G^{u,v}$ is injective. There is nothing to prove if $u = v = e$, so we can assume that $m = \ell(u) + \ell(v) \geq 1$. Suppose that $i_m = i \in [1, r]$ (the case when $i_m \in [\bar{1}, \bar{r}]$ is treated in the same way). Denote $v' = vs_i$ so that $\mathbf{i}' = (i_1, \dots, i_{m-1}) \in R(u, v')$. Now suppose

$$x_{\mathbf{i}}(a; t_1, \dots, t_m) = x_{\mathbf{i}'}(a'; t'_1, \dots, t'_m) ,$$

where $(a; t_1, \dots, t_m)$ and $(a'; t'_1, \dots, t'_m)$ belong to $H \times \mathbb{C}_{\neq 0}^m$. Multiplying both sides of the last equality on the right by $x_i(-t'_m)$, we obtain:

$$(3.2) \quad x_{\mathbf{i}}(a; t_1, \dots, t_{m-1}, t_m - t'_m) = x_{\mathbf{i}'}(a'; t'_1, \dots, t'_{m-1}) .$$

If $t_m \neq t'_m$, then the left-hand side of (3.2) lies in $G^{u,v}$, while the right-hand side lies in $G^{u,v'}$. But this is impossible because the double Bruhat cells are disjoint. Thus $t_m = t'_m$, and the desired injectivity follows by induction on m . This completes the proof of Proposition 3.2 and Theorem 1.2. \square

Proof of Theorem 1.3. If a double reduced word $\mathbf{i} \in R(u, v)$ is unmixed, i.e., all the indices from $[\bar{1}, \bar{r}]$ precede those from $[1, r]$, then our statement follows by combining (2.62) with Proposition 2.26. The statement for an arbitrary $\mathbf{i} \in R(u, v)$ can be reduced to the case of an unmixed \mathbf{i} by the argument used in the proof of Proposition 2.29, i.e., by repeated application of the commutation relations (2.5), (2.9) and (2.11). \square

3.2. Proofs of Theorems 1.6 and 1.7.

Proof of Theorem 1.6. The fact that the right-hand side of (1.12) is well defined for any $x \in G^{u,v}$ follows from Propositions 2.9 and 2.10. Let us show that $x' = \zeta^{u,v}(x) \in G^{u^{-1},v^{-1}}$. Using (2.31), we can rewrite x' as

$$x' = \bar{u}^{-1} \left(y_+^{-1} [xv^{-1}]_- [xv^{-1}]_0 \right)^\theta,$$

where $y_+ = \pi_+(x)$. It follows that $\bar{u}x' \in G_0$, and

$$(3.3) \quad [\bar{u}x']_- = (y_+^{-1})^\theta.$$

Hence $[\bar{u}x']_- \in N_+(u)^\theta = N_-(u^{-1})$, and we conclude from Proposition 2.9 that $x' \in Bu^{-1}B$. The inclusion $x' \in B_-v^{-1}B_-$ is proved in a similar way (or by using the transpose map); the counterpart of (3.3) is given by

$$(3.4) \quad [x'\bar{v}]_+ = (y_-^{-1})^\theta,$$

where $y_- = \pi_-(x)$ (cf. (2.32)).

We have proved that $x' \in G^{u^{-1},v^{-1}}$. To complete the proof of Theorem 1.6, it suffices to show that $\zeta^{u^{-1},v^{-1}}(\zeta^{u,v}(x)) = x$ for any $x \in G^{u,v}$. Notice that (3.3) and (3.4) can be rewritten as

$$(3.5) \quad [\bar{u}x']_- = \bar{u} \left([\bar{u}^{-1}x]_-^{-1} \right)^\theta \bar{u}^{-1}, \quad [x'\bar{v}]_+ = \bar{v}^{-1} \left([xv^{-1}]_+^{-1} \right)^\theta \bar{v}.$$

The desired equality $\zeta^{u^{-1},v^{-1}}(x') = x$ follows by substituting these expressions into the expression for $\zeta^{u^{-1},v^{-1}}(x')$ obtained from (1.12). \square

The following proposition shows that the twist map respects the Gaussian decomposition.

Proposition 3.3. *The twist map $\zeta^{u,v} : x \mapsto x'$ sends the open subset $G_0^{u,v}$ to $G_0^{u^{-1},v^{-1}}$, and we have*

$$(3.6) \quad [x']_0 = [\bar{u}^{-1}x]_0^{-1} [x]_0 [xv^{-1}]_0^{-1}.$$

Proof. To show that x and x' belong to G_0 simultaneously, it suffices to rewrite (1.12) as

$$x' = \left([\bar{u}^{-1}x]_0 [\bar{u}^{-1}x]_+ x^{-1} [xv^{-1}]_- [xv^{-1}]_0 \right)^\theta.$$

In view of (2.3), this also implies (3.6). \square

Let us now describe the twist map in terms of y -coordinates. Recall the definition (2.56) of the anti-automorphism τ_w of the group G .

Proposition 3.4. *Suppose $x \in G_0^{u,v}$ has y -coordinates (y_+, y_0, y_-) . Then the y -coordinates (y'_+, y'_0, y'_-) of $x' = \zeta^{u,v}(x)$ are given by*

$$(3.7) \quad \begin{aligned} y'_+ &= \tau_{u^{-1}}(y_+) , \\ y'_0 &= [y_+ \bar{u}]_0 y_0^{-1} [\bar{v} y_-]_0 , \\ y'_- &= \tau_v(y_-) . \end{aligned}$$

Proof. The desired expressions for y'_+ and y'_- follow from (3.5); the expression for y'_0 follows from (3.6) combined with (2.42). \square

Proof of Theorem 1.7. Let $x \in G_{>0}^{u,v}$, and let $x' = \zeta^{u,v}(x)$. By Proposition 2.29, $x \in G_0^{u,v}$ so x has well-defined y -coordinates (y_+, y_0, y_-) . By Proposition 3.4, the y -coordinates (y'_+, y'_0, y'_-) of x' are given by (3.7). By Theorem 2.30, the triple (y_+, y_0, y_-) satisfies the properties given there, and it suffices to check that (y'_+, y'_0, y'_-) satisfies the same properties with (u, v) replaced by (u^{-1}, v^{-1}) . In view of (3.7) and (2.59), if $\Delta(y_-) > 0$ for any $\Delta \in F(v)$, then $\Delta(y'_-) > 0$ for any $\Delta \in F(v^{-1})$. Similarly, using (2.57) we obtain that if $\Delta(y_+^T) > 0$ for any $\Delta \in F(u^{-1})$, then $\Delta(y'_+{}^T) > 0$ for any $\Delta \in F(u)$.

It remains to show that $y'_0 \in H_{>0}$. Applying the character $a \rightarrow a^{\omega_i}$ to both sides of the equality $y'_0 = [y_+ \bar{u}]_0 y_0^{-1} [\bar{v} y_-]_0$ in (3.7), we obtain

$$(y'_0)^{\omega_i} = \Delta_{u\omega_i, \omega_i}(y_+^T) y_0^{-\omega_i} \Delta_{v^{-1}\omega_i, \omega_i}(y_-) .$$

Since $\Delta_{v^{-1}\omega_i, \omega_i} \in F(v)$ and $\Delta_{u\omega_i, \omega_i} \in F(u^{-1})$, it follows that $(y'_0)^{\omega_i} > 0$ for any $i \in [1, r]$. Therefore, $y'_0 \in H_{>0}$, as desired. \square

3.3. Proof of Theorem 1.9. First notice that the equivalence of (1.19) and (1.20) follows by applying the character $a \mapsto a^{\omega_i}$ to both sides of (1.20) and simplifying the result. In proving (1.18) and (1.20), we will follow the same strategy that was used in the proof of Theorem 1.3: first treat the case when \mathbf{i} is unmixed, and then extend the result to the general case with the help of commutation relations (2.5), (2.9) and (2.11).

Let us first assume that $\mathbf{i} \in R(u, v)$ is unmixed, i.e., all the indices from $[\bar{1}, \bar{r}]$ precede those from $[1, r]$. Repeatedly using (2.5), we conclude that in this case $x = x_{\mathbf{i}}(a; t_1, \dots, t_m) \in G_0^{u,v}$, and the components in the Gaussian decomposition of x are given by

$$(3.8) \quad \begin{aligned} [x]_- &= x_{i_1}(a^{-\alpha_{|i_1|}} t_1) \cdots x_{i_{\ell(u)}}(a^{-\alpha_{|i_{\ell(u)|}} t_{\ell(u)}}) , \\ [x]_0 &= a , \quad [x]_+ = x_{i_{\ell(u)+1}}(t_{\ell(u)+1}) \cdots x_{i_m}(t_m) . \end{aligned}$$

Since by Theorem 1.6, $x = \zeta^{u^{-1}, v^{-1}}(x')$, formula (3.6) implies that

$$(3.9) \quad a = [x]_0 = [\bar{u} x']_0^{-1} [x']_0 [x' \bar{v}]_0^{-1} .$$

This proves (1.20) since a simple inspection shows that the right-hand side of (3.9) is equal to that of (1.20) when \mathbf{i} is unmixed.

Turning to the proof of (1.18), let us first consider the case $\ell(u) < k \leq m$. Let (y_+, y_0, y_-) be the y -coordinates of x , and (y'_+, y'_0, y'_-) be the y -coordinates of x' .

By (2.45), we have

$$(3.10) \quad t_k = \frac{1}{\Delta_{v^{-1}v_{<k}\omega_{i_k}, \omega_{i_k}}(y_-)\Delta_{v^{-1}v_{<k+1}\omega_{i_k}, \omega_{i_k}}(y_-)} \prod_{j \neq i_k} \Delta_{v^{-1}v_{<k+1}\omega_j, \omega_j}(y_-)^{-a_{j, i_k}} .$$

Using (3.7), (2.59), (2.22), (2.41), and (2.42), we can rewrite $\Delta_{v^{-1}v_{<k}\omega_j, \omega_j}(y_-)$ as follows:

$$(3.11) \quad \begin{aligned} \Delta_{v^{-1}v_{<k}\omega_j, \omega_j}(y_-) &= \Delta_{v\omega_j, v_{<k}\omega_j}(y'_-) = \Delta_{\omega_j, v_{<k}\omega_j}(\overline{v^{-1}}y'_-) \\ &= \Delta_{\omega_j, v_{<k}\omega_j}([\overline{v^{-1}}y'_-]_+) [\overline{v^{-1}}y'_-]_0^{\omega_j} = \Delta_{\omega_j, v_{<k}\omega_j}(x') [y'_0]_0^{-\omega_j} [\overline{v^{-1}}y'_-]_0^{\omega_j} \\ &= \Delta_{\omega_j, v_{<k}\omega_j}(x') / \Delta_{\omega_j, v\omega_j}(x') . \end{aligned}$$

Substituting the expressions given by (3.11) into (3.10), we express t_k as a Laurent monomial in the minors $\Delta_l(x')$ given by (1.16). Using the notation from Section 1.6, this monomial can be written as follows:

$$(3.12) \quad t_k = \frac{1}{\Delta_k(x')\Delta_{k^+}(x')} \prod_{l: l^- < k < l} \Delta_l(x')^{-a_{|i_l|, i_k}} \prod_{j \in [1, r]} \Delta_{m+j}(x')^{a_{j, i_k}} ,$$

where the index k^+ is defined by $(k^+)^- = k$. Formula (1.18) now follows by simple inspection which shows that, for \mathbf{i} unmixed, the right-hand side of (1.18) is equal to the one of (3.12).

The proof of (1.18) for $1 \leq k \leq \ell(u)$ is practically the same as above. In this case, the counterpart of (3.12) is given by

$$(3.13) \quad a^{-\alpha_{|i_k|}} t_k = \frac{1}{\Delta_k(x')\Delta_{k^+}(x')} \prod_{l: l^- < k < l} \Delta_l(x')^{-a_{|i_l|, |i_k|}} \prod_{l: l^- = 0} \Delta_l(x')^{a_{|i_l|, |i_k|}} .$$

To deduce (1.18) from (3.13), first notice that in view of (1.19) and (2.27), we have

$$a^{\alpha_{|i_k|}} = \prod_{1 \leq l \leq m+r} \Delta_l(x')^{(\varepsilon(i_l) - \varepsilon(i_l^-)) a_{|i_l|, |i_k|} / 2} .$$

Thus in order to check (1.18), it suffices to show that, for \mathbf{i} unmixed and $1 \leq k \leq \ell(u)$, the right-hand side of (3.13) is equal to

$$\prod_{l=1}^{m+r} \Delta_l(x')^{(\varepsilon(i_l^-) / 2 + \chi(k, l^-) - \varepsilon(i_l) / 2 - \chi(k, l)) a_{|i_l|, |i_k|}} ;$$

this is again checked by direct inspection.

Now let us prove (1.18) and (1.20) for an arbitrary double reduced word for u and v . Every such word can be obtained from an unmixed one by a sequence of *mixed moves* of the form

$$\dots \bar{j} i \dots \rightsquigarrow \dots i \bar{j} \dots .$$

It therefore suffices to prove the following statement.

Lemma 3.5. *Suppose a reduced word $\mathbf{i}' \in R(u, v)$ is obtained from $\mathbf{i} \in R(u, v)$ by a mixed move. If (1.18) and (1.20) hold for \mathbf{i} , then they also hold for \mathbf{i}' .*

Proof. Suppose \mathbf{i}' is obtained from \mathbf{i} by interchanging $i_k = \bar{j}$ and $i_{k+1} = i$. By (2.5), (2.9) and (2.11), the factorization parameters that appear in two factorizations

$$x = x_{\mathbf{i}}(a; t_1, \dots, t_m) = x_{\mathbf{i}'}(a'; t'_1, \dots, t'_m)$$

of an element $x \in G^{u,v}$ are related as follows. If $i \neq j$, then

$$(3.14) \quad a' = a, \quad t'_l = t_l \quad (l \notin \{k, k + 1\}), \quad t'_k = t_{k+1}, \quad t'_{k+1} = t_k.$$

For $i = j$, a straightforward calculation using (2.5) and (2.11) shows that

$$(3.15) \quad (a')^{\omega_p} = a^{\omega_p} (1 + t_k t_{k+1})^{-\delta_{i,p}},$$

$$(3.16) \quad t'_l = t_l (1 + t_k t_{k+1})^{\varepsilon^{(i)} a_{i,|i|}} \quad (l < k),$$

$$(3.17) \quad t'_k = t_{k+1} (1 + t_k t_{k+1}), \quad t'_{k+1} = t_k (1 + t_k t_{k+1})^{-1},$$

$$(3.18) \quad t'_l = t_l \quad (l > k + 1).$$

We need to show the following: if we substitute the parameters t_l and a^{ω_p} given by (1.18) and (1.19) into (3.14)–(3.18), then the resulting t'_l and $(a')^{\omega_p}$ satisfy the same formulas (1.18) and (1.19) with \mathbf{i} replaced by \mathbf{i}' . This is immediate from the definitions when $i \neq j$, so let us assume $i = j$. By the definition (1.16), we have $\Delta_{l,\mathbf{i}} = \Delta_{l,\mathbf{i}'}$ for $l \neq k + 1$, so we will denote this minor simply by Δ_l . The key calculation is now as follows.

Lemma 3.6. *In the above notation, if t_k and t_{k+1} satisfy (1.18), then*

$$(3.19) \quad 1 + t_k t_{k+1} = \frac{\Delta_{k+1,\mathbf{i}}(x') \Delta_{k+1,\mathbf{i}'}(x')}{\Delta_k(x') \Delta_{(k+1)^+}(x')}.$$

Proof. Let us denote $u' = u_{\geq k+2}$ and $v' = v_{< k}$ (this is unambiguous since these expressions are the same for \mathbf{i} and \mathbf{i}' ; cf. (1.15)). In view of (1.16), we have

$$(3.20) \quad \begin{aligned} \Delta_{k+1,\mathbf{i}} &= \Delta_{u' \omega_i, v' \omega_i}, \quad \Delta_{k+1,\mathbf{i}'} = \Delta_{u' s_i \omega_i, v' s_i \omega_i}, \\ \Delta_k &= \Delta_{u' s_i \omega_i, v' \omega_i}, \quad \Delta_{(k+1)^+} = \Delta_{u' \omega_i, v' s_i \omega_i}, \end{aligned}$$

so (3.19) takes the form

$$(3.21) \quad 1 + t_k t_{k+1} = \frac{\Delta_{u' \omega_i, v' \omega_i}(x') \Delta_{u' s_i \omega_i, v' s_i \omega_i}(x')}{\Delta_{u' s_i \omega_i, v' \omega_i}(x') \Delta_{u' \omega_i, v' s_i \omega_i}(x')}.$$

On the other hand, if t_k and t_{k+1} are given by (1.18), then

$$t_k t_{k+1} = \prod_{l=1}^{m+r} \Delta_l(x')^{\chi(k,l^-) + \chi(k+1,l^-) - \chi(k,l) - \chi(k+1,l)} a_{|i|,i}.$$

By (1.17), we have

$$(3.22) \quad \chi(k, l) + \chi(k + 1, l) = \begin{cases} 1 & \text{if } l > k + 1; \\ 1/2 & \text{if } l \in \{k, k + 1\}; \\ 0 & \text{if } l < k. \end{cases}$$

It follows that

$$\begin{aligned} t_k t_{k+1} &= \Delta_k(x')^{-1} \Delta_{(k+1)^+}(x')^{-1} \prod_{l: l^- < k < l} \Delta_l(x')^{-a_{|i|,i}} \\ &= (\Delta_{u' s_i \omega_i, v' \omega_i}(x') \Delta_{u' \omega_i, v' s_i \omega_i}(x'))^{-1} \prod_{p \neq i} \Delta_{u' \omega_p, v' \omega_p}(x')^{-a_{p,i}}. \end{aligned}$$

Therefore, (3.21) becomes a consequence of (1.25), and we are done. □

If we substitute the expressions given by (1.18), (1.19) and (3.19) into the formulas (3.14)–(3.18), then an easy simplification shows that they will be given by (1.18) and (1.19) for \mathbf{i}' . This completes the proofs of Lemma 3.5 and Theorem 1.9.

3.4. Proofs of Theorems 1.11 and 1.12. We start by recalling a well-known property of reduced words in Coxeter groups (cf. [5, 15]). To state it, we will need the following notion.

We call a *d-move* the transformation of a reduced word that replaces d consecutive entries i, j, i, j, \dots by j, i, j, i, \dots , for some i and j such that d is the order of $s_i s_j$. Note that, for given i and j , the value of d can be determined from the Cartan matrix as follows: if $a_{ij} a_{ji} = 0$ (resp. 1, 2, 3), then $d = 2$ (resp. 3, 4, 6).

Proposition 3.7. *Every two reduced words for the same element of a Coxeter group can be obtained from each other by a sequence of d -moves.*

Applying this proposition to the group $W \times W$, we conclude that every two double reduced words $\mathbf{i}, \mathbf{i}' \in R(u, v)$ can be obtained from each other by a sequence of the following operations: d -moves for each of the alphabets $[1, r]$ and $[\bar{1}, \bar{r}]$, and also mixed moves (cf. Section 3.3) and their inverses.

Proof of Theorem 1.12. Let us first prove that $F(\mathbf{i})$ is a transcendence basis for the field $\mathbb{C}(G^{u,v})$. By Theorem 1.1, $F(\mathbf{i})$ is of cardinality $\dim G^{u,v}$. It is therefore enough to show that $F(\mathbf{i})$ generates $\mathbb{C}(G^{u,v})$. In view of Theorem 1.6, it suffices to show that the collection of “twisted” minors $\Delta_{k,\mathbf{i}}(x')$ (cf. (1.16)) generates $\mathbb{C}(G^{u,v})$. By Theorem 1.2, the field $\mathbb{C}(G^{u,v})$ is generated by the factorization parameters t_k and a^{ω_i} , and the claim follows by Theorem 1.9.

The second statement of the theorem is a consequence of the following lemma.

Lemma 3.8. *Suppose a double reduced word \mathbf{i}' is obtained from \mathbf{i} by a d -move in one of the alphabets $[1, r]$ and $[\bar{1}, \bar{r}]$, or by a mixed move, or by the inverse of a mixed move. Then each element of the set difference $F(\mathbf{i}') \setminus F(\mathbf{i})$ is a subtraction-free expression in the elements of $F(\mathbf{i})$.*

Proof. For d -moves in $[1, r]$ or $[\bar{1}, \bar{r}]$, the desired subtraction-free expressions can be obtained from generalized Plücker relations in Theorem 1.16 (including the omitted relations of type G_2); this part of the argument is essentially borrowed from [4, Proposition 6.10]. For mixed moves and their inverses, the statement follows in the same way from Theorem 1.17 (cf. the proof of Lemma 3.5 above). Lemma 3.8 and Theorem 1.12 are proved. □

Proof of Theorem 1.11. It will suffice to show that the following are equivalent:

- (1) $x \in G_{>0}^{u,v}$;
- (2) $\Delta(x) > 0$ for any $\Delta \in F(u, v)$;
- (3) $\Delta(x) > 0$ for any $\Delta \in F(\mathbf{i})$.

The equivalence of (2) and (3) follows from Theorem 1.12. Let us show that (1) \Rightarrow (3). By Theorem 1.7, if $x \in G_{>0}^{u,v}$, then $x' \in G_{>0}^{u^{-1},v^{-1}}$. The condition (3) now follows by applying Theorems 1.3 and 1.10 to x' and the reduced word $\mathbf{i}^* \in R(u^{-1}, v^{-1})$ opposite to \mathbf{i} .

It remains to show that (2) \Rightarrow (1). First of all, by Corollary 2.5, (2) implies that $x \in G_0^{u,v}$; moreover, $[x]_0 \in H_{>0}$. By Proposition 2.15 and Theorems 2.30 and 2.27, it suffices to show that $y_- = \pi_-(x)$ satisfies $\Delta_{v^{-1}\omega_i, v'\omega_i}(y_-) > 0$, for $i \in [1, r]$ and

any $v' \preceq v^{-1}$. Using (2.41) and (2.22), we obtain (cf. (3.11)):

$$(3.23) \quad \Delta_{v^{-1}\omega_i, v'\omega_i}(y_-) = \frac{\Delta_{\omega_i, v'\omega_i}(x)}{\Delta_{\omega_i, v^{-1}\omega_i}(x)}.$$

By (2), the right-hand side of (3.23) is positive, and the proof is complete. \square

4. GL_n THEORY

Throughout this section, $G = GL_n(\mathbb{C})$ is the group of invertible $n \times n$ matrices with complex entries. In this case, the problems under consideration become quite natural questions in “classical” linear algebra, so we will formulate them here—and state our main results—in an elementary and self-contained way. We will not give any proofs though, since these results can be easily derived from the type A specializations of the corresponding statements in Section 1; pointers to these statements are provided, wherever appropriate.

4.1. Bruhat cells and double Bruhat cells for GL_n . Our first object of interest are the double Bruhat cells. Let us introduce them for the group $G = GL_n(\mathbb{C})$. We will need some notation. Let B (resp. B_-) be the subgroup of upper-triangular (resp. lower-triangular) matrices in G . Let $W = S_n$ be the symmetric group acting on the set $[1, n] = \{1, \dots, n\}$; we will think of W as a subgroup of G by identifying a permutation w with the matrix $w = (\delta_{i, w(j)})$. The double cosets BwB and B_-wB_- are called Bruhat cells (with respect to B and B_- , respectively). The group G has two Bruhat decompositions into a disjoint union of Bruhat cells (see, e.g., [1, Section 2.4]):

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_-vB_-.$$

The *double Bruhat cells* $G^{u,v}$ are defined by

$$G^{u,v} = BuB \cap B_-vB_-;$$

thus G is the disjoint union of all $G^{u,v}$ for $(u, v) \in W \times W$.

As an algebraic variety, a double Bruhat cell $G^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $n + \ell(u) + \ell(v)$, where $\ell(u)$ is the number of inversions of a permutation u ; cf. Theorem 1.1. (In other words, $G^{u,v}$ is isomorphic, as an algebraic variety, to a subset of $\mathbb{C}^{n+\ell(u)+\ell(v)}$ obtained by excluding common zeroes of a finite set of polynomials.)

Each Bruhat cell (hence each double Bruhat cell) can be described explicitly by a set of conditions specifying vanishing and nonvanishing of certain minors. Let us denote by $\Delta_{I,J}$ the minor with the row set I and the column set J ; here I and J are two subsets of the same size in $[1, n]$, and the minor is viewed as a (regular) function on G . (This notation corresponds to that of Definition 1.4, as follows: the function $\Delta_{u\omega_i, v\omega_i}$ in (1.10) becomes the minor $\Delta_{u([1,i]), v([1,i])}$.) The following description of Bruhat cells is probably the most “economical”.

Proposition 4.1. *A matrix $x \in G$ belongs to the Bruhat cell BwB if and only if it satisfies the following conditions:*

- $\Delta_{w([1,i]), [1,i]} \neq 0$, for $i = 1, \dots, n - 1$;
- $\Delta_{w([1, i-1] \cup \{j\}), [1, i]} = 0$, for all (i, j) such that $1 \leq i < j \leq n$ and $w(i) < w(j)$.

This proposition can be proved by specializing Propositions 2.9 and 2.6 and Corollary 2.5. Notice that in our present situation the subgroup $N_-(w)$ (cf. (2.30)) consists of all unipotent lower-triangular matrices y such that $y_{ij} = 0$ whenever $w(i) > w(j)$.

The transpose map $x \mapsto x^T$ transforms a minor $\Delta_{I,J}$ into $\Delta_{J,I}$ and sends a Bruhat cell BwB to $B_w w^{-1} B_w$. Thus Proposition 4.1 implies a similar description of the opposite Bruhat cells $B_w w B_w$. Combining the two sets of conditions yields an explicit description of the double Bruhat cells.

4.2. Factorization problem for GL_n . In the situation under consideration, the maximal torus $H = B \cap B_w$ in G is the subgroup of invertible diagonal matrices. Thus H is naturally identified with $\mathbb{C}_{\neq 0}^n$ by taking the diagonal entries as coordinates. This allows us to state the factorization problem of Section 1.2 in a more symmetric form, as follows.

Let $E_{i,j}$ denote the $n \times n$ matrix whose (i, j) -entry is equal to 1 while all other entries are 0; let $I \in G$ denote the identity matrix. For $i = 1, \dots, n - 1$, let

$$(4.1) \quad x_i(t) = I + tE_{i,i+1} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & t & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$(4.2) \quad x_{\bar{i}}(t) = I + tE_{i+1,i} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & t & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Also, for $i = 1, \dots, n$ and $t \neq 0$, let

$$(4.3) \quad x_{\odot}(t) = I + (t - 1)E_{i,i} = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & t & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The matrices defined in (4.1)–(4.3) are called *elementary Jacobi matrices*. It is easy to see that these matrices generate G as a group.

Consider the alphabet of $3n - 2$ symbols

$$(4.4) \quad \mathcal{A} = \{1, \dots, n - 1, \textcircled{1}, \dots, \textcircled{n}, \bar{1}, \dots, \overline{n - 1}\}.$$

The formulas (4.1)–(4.3) associate a matrix $x_i(t) \in G$ to any symbol $i \in \mathcal{A}$ and any $t \in \mathbb{C}_{\neq 0}$. An analogue of the product map (1.3) is now defined as follows: to any sequence $\mathbf{i} = (i_1, \dots, i_l)$ of symbols in \mathcal{A} , we associate the map $x_{\mathbf{i}} : \mathbb{C}_{\neq 0}^l \rightarrow G$ defined by

$$(4.5) \quad x_{\mathbf{i}}(t_1, \dots, t_l) = x_{i_1}(t_1) \cdots x_{i_l}(t_l).$$

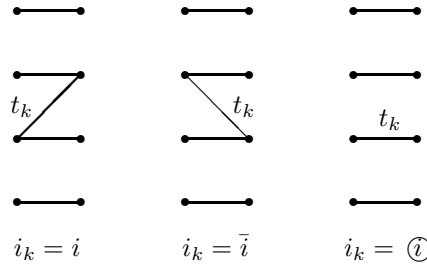
(Thus the difference with (1.3) is that now the factor $a \in H$ is split into elementary factors, which are allowed to be spread along the factorization.)

For instance, the sequence $\mathbf{i} = \textcircled{1} \bar{1} \textcircled{2} 1$ gives rise to the map

$$(4.6) \quad (t_1, t_2, t_3, t_4) \mapsto \begin{pmatrix} t_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t_3 \end{pmatrix} \begin{pmatrix} 1 & t_4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_1 & t_1 t_4 \\ t_2 & t_2 t_4 + t_3 \end{pmatrix}.$$

The matrix $x = x_{\mathbf{i}}(t_1, \dots, t_l)$ has a simple combinatorial description in terms of *planar networks*. This description (cf. [6] and references therein) generalizes the one in [3, Section 2.4], and provides combinatorial formulas for the minors of x as polynomials with nonnegative coefficients in the variables t_1, \dots, t_l .

The planar network $\Gamma(\mathbf{i})$ associated to a sequence \mathbf{i} of symbols from \mathcal{A} (see Figure 1) is constructed as a concatenation of “elementary” networks that correspond to the parameters t_1, \dots, t_l (in this order). Each unbarred, barred, or circled entry i_k of \mathbf{i} corresponds to a fragment of one of the following three kinds, respectively:



(a diagonal edge connects horizontal levels i and $i + 1$; in the examples above, $i = 2$). These fragments are the combinatorial equivalents of the elementary matrices (4.1)–(4.3). Each fragment has a distinguished edge whose weight is t_k ; all other edges have weight 1. All edges are presumed oriented left-to-right.

We number the sources and sinks of the network $\Gamma(\mathbf{i})$ bottom-to-top, and define the weight of a path in $\Gamma(\mathbf{i})$ to be the product of the weights of all edges in the path. One easily checks that the sum of these weights, over all paths that connect a given source i to a given sink j , is nothing but the matrix element x_{ij} of $x = x_{\mathbf{i}}(t_1, \dots, t_l)$.

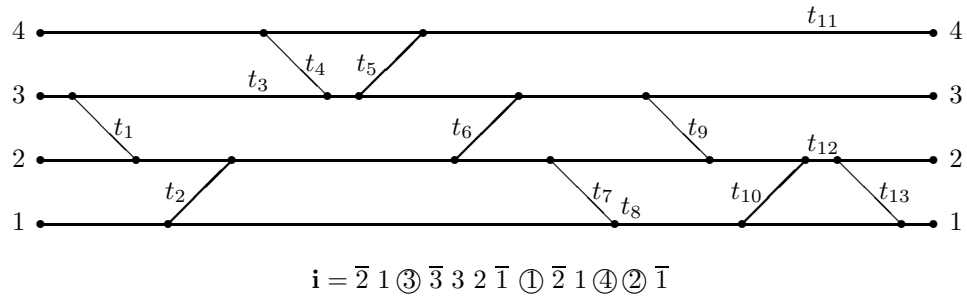


FIGURE 1. Planar network

This observation can be generalized. Let us define the weight of a family of paths in $\Gamma(\mathbf{i})$ to be the product of the weights of all paths in the family. Then the minors of x are computed as follows.

Proposition 4.2. *A minor $\Delta_{I,J}(x)$ equals the sum of weights of all families of vertex-disjoint paths in $\Gamma(\mathbf{i})$ connecting the sources labeled by I with the sinks labeled by J .*

For example, in Figure 1 we have

$$x_{21} = t_7 t_8 + t_{12} t_{13} + t_6 t_9 t_{12} t_{13} \quad \text{and} \quad \Delta_{12,12}(x) = t_8 t_{12} (1 + t_6 t_9).$$

We will be especially interested in a particular class of sequences \mathbf{i} which we call factorization schemes (they are analogues of double reduced words of Section 1.2).

Definition 4.3. Let u and v be two permutations in $W = S_n$. A *factorization scheme* of type (u, v) is a word \mathbf{i} of length $n + \ell(u) + \ell(v)$ in the alphabet \mathcal{A} which is an arbitrary shuffle of three words of the following kind:

- a reduced word for v ;
- a reduced word for u , with all entries barred;
- a permutation of the symbols $\textcircled{1}, \dots, \textcircled{n}$.

These three words will be called, respectively, the *E-part*, the *F-part*, and the *H-part* of a factorization scheme \mathbf{i} .

For example, let

$$(4.7) \quad \begin{aligned} u &= 4312 = s_2 s_3 s_1 s_2 s_1 \in S_4, \\ v &= 4213 = s_1 s_3 s_2 s_1 \in S_4. \end{aligned}$$

Then

$$(4.8) \quad \mathbf{i} = \bar{2} 1 \textcircled{3} \bar{3} 3 2 \bar{1} \textcircled{1} \bar{2} 1 \textcircled{4} \textcircled{2} \bar{1}$$

is a factorization scheme of type (u, v) .

The following result is an analogue of Theorem 1.2.

Theorem 4.4. Let $u, v \in W = S_n$, and let $l = n + \ell(u) + \ell(v)$. For any factorization scheme \mathbf{i} of type (u, v) , the product map $x_{\mathbf{i}}$ given by (4.5) is a biregular isomorphism between $\mathbb{C}_{\neq 0}^l$ and a Zariski open subset of the double Bruhat cell $G^{u,v}$.

The *factorization problem* for GL_n can now be formulated as follows: for a given factorization scheme \mathbf{i} , find explicit formulas for the components t_k in terms of the matrix $x = x_{\mathbf{i}}(t_1, \dots, t_l)$. By Theorem 4.4, each t_k is a rational function in the matrix entries of x . For example, if $\mathbf{i} = \textcircled{1} \bar{1} \textcircled{2} 1$, so that the map $x_{\mathbf{i}}$ is given by (4.6), then the solution to the factorization problem is given by

$$(4.9) \quad t_1 = x_{11}, \quad t_2 = x_{21}, \quad t_3 = \frac{\det(x)}{x_{11}}, \quad t_4 = \frac{x_{12}}{x_{11}}.$$

4.3. The twist maps for GL_n . As in the general case, our solution to the factorization problem for $G = GL_n$ will utilize the “twist maps” $\zeta^{u,v} : x \mapsto x'$, which are defined for any two permutations u and v . The definition (1.12) can be rewritten as

$$(4.10) \quad x' = d_0 [x^T \bar{w}]_+ \bar{w}^T (x^T)^{-1} \overline{v^{-1}} [\overline{v^{-1}^T} x^T]_- d_0^{-1},$$

where the following notation is used. The matrix d_0 is the diagonal $n \times n$ matrix with diagonal entries $1, -1, 1, -1, \dots$. For a matrix $z \in G$, z^T stands for the transpose of z , and $z = [z]_- [z]_0 [z]_+$ denotes the Gaussian decomposition of z (also known as the *LDU* decomposition). Finally, the matrix \bar{w} is obtained from a permutation matrix for w by the following modification: an entry is changed from 1 to -1 whenever it has an odd number of nonzero entries lying below and to the left of it.

By Theorem 1.6, the right-hand side of (4.10) is well defined for any $x \in G^{u,v}$, and the twist map $\zeta^{u,v}$ establishes a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1},v^{-1}}$; the inverse isomorphism is $\zeta^{u^{-1},v^{-1}}$.

We give below a few examples of explicitly computed twist maps.

Example 4.5. Let $G = GL_2(\mathbb{C})$ and $u = v = w_\circ$. Then (cf. Example 1.8)

$$\bar{u} = \overline{v^{-1}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$x' = \begin{pmatrix} x_{11}x_{12}^{-1}x_{21}^{-1} & x_{21}^{-1} \\ x_{12}^{-1} & x_{22} \det(x)^{-1} \end{pmatrix}.$$

Example 4.6. Let $G = GL_3(\mathbb{C})$ and $u = v = w_\circ$. Then

$$x' = \begin{pmatrix} \frac{x_{11}}{x_{31} x_{13}} & \frac{\Delta_{12,13}}{x_{31} \Delta_{12,23}} & \frac{1}{x_{31}} \\ \frac{\Delta_{13,12}}{x_{13} \Delta_{23,12}} & \frac{x_{33}\Delta_{12,12} - \det(x)}{\Delta_{23,12} \Delta_{12,23}} & \frac{x_{32}}{\Delta_{23,12}} \\ \frac{1}{x_{13}} & \frac{x_{23}}{\Delta_{12,23}} & \frac{\Delta_{23,23}}{\det(x)} \end{pmatrix}.$$

Example 4.7. Let $G = GL_4(\mathbb{C})$ and $u = v = w_\circ$. Then x' is equal to

$$\begin{pmatrix} \frac{x_{11}}{x_{14}x_{41}} & \frac{\Delta_{12,14}}{x_{41}\Delta_{12,34}} & \frac{\Delta_{123,134}}{x_{41}\Delta_{123,234}} & \frac{1}{x_{41}} \\ \frac{\Delta_{14,12}}{x_{14}\Delta_{34,12}} & \frac{x_{44}\Delta_{12,12} - \Delta_{124,124}}{\Delta_{34,12}\Delta_{12,34}} & \frac{x_{42}\Delta_{123,134} - x_{41}\Delta_{123,234}}{\Delta_{34,12}\Delta_{123,234}} & \frac{x_{42}}{\Delta_{34,12}} \\ \frac{\Delta_{134,123}}{x_{14}\Delta_{234,123}} & \frac{x_{24}\Delta_{134,123} - x_{14}\Delta_{234,123}}{\Delta_{12,34}\Delta_{234,123}} & \frac{\Delta_{123,123}\Delta_{34,34} - x_{33} \det(x)}{\Delta_{123,234}\Delta_{234,123}} & \frac{\Delta_{34,23}}{\Delta_{234,123}} \\ \frac{1}{x_{14}} & \frac{x_{24}}{\Delta_{12,34}} & \frac{\Delta_{23,34}}{\Delta_{123,234}} & \frac{\Delta_{234,234}}{\det(x)} \end{pmatrix}.$$

Example 4.8. Let $n = 4$, $u = 4312$, and $v = 4213$ (cf. (4.7)). Then

$$\bar{u} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{v^{-1}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$x' = \begin{pmatrix} \frac{x_{11}}{x_{41} x_{13}} & \frac{\Delta_{12,13}}{x_{41} \Delta_{12,23}} & \frac{\Delta_{123,123}}{x_{41} \Delta_{123,234}} & \frac{1}{x_{41}} \\ \frac{\Delta_{14,12}}{\Delta_{34,12} x_{13}} & \frac{x_{43} \Delta_{12,12} - \Delta_{124,123}}{\Delta_{34,12} \Delta_{12,23}} & \frac{\Delta_{123,123} x_{42}}{\Delta_{34,12} \Delta_{123,234}} & \frac{x_{42}}{\Delta_{34,12}} \\ \frac{1}{x_{13}} & \frac{x_{23}}{\Delta_{12,23}} & \frac{\Delta_{23,23}}{\Delta_{123,234}} & \frac{\Delta_{23,23}}{\Delta_{123,123}} \\ 0 & \frac{x_{13}}{\Delta_{12,23}} & \frac{\Delta_{13,23}}{\Delta_{123,234}} & \frac{\Delta_{134,234}}{\det(x)} \end{pmatrix}.$$

Note that, in the course of computing the matrix elements of x' above, one has to take into account the relations

$$(4.11) \quad x_{14} = 0, \quad x_{24} = 0, \quad \Delta_{234,123} = 0$$

satisfied by the matrix elements of $x \in G^{u,v}$. In particular, our computation of x'_{44} used (4.11) in conjunction with Gröbner bases techniques (see, e.g., [7]).

4.4. Double pseudoline arrangements. As an essential new ingredient in our solution to the factorization problem for GL_n , we will represent a factorization scheme \mathbf{i} geometrically by the corresponding *double pseudoline arrangement* (or double wiring diagram). This arrangement is obtained by superimposing two arrangements naturally associated to the E - and F -part of \mathbf{i} (cf. [3]).

To be self-contained, let us recall the definition of a pseudoline arrangement associated to a reduced word. This is best done by an example. Consider $v = 4213 \in S_4$, together with the reduced decomposition $v = s_1 s_3 s_2 s_1$ (cf. (4.7)). The corresponding pseudoline arrangement is given in Figure 2; to each entry i of \mathbf{i} , we associate a crossing at the i th level, counting from the bottom.

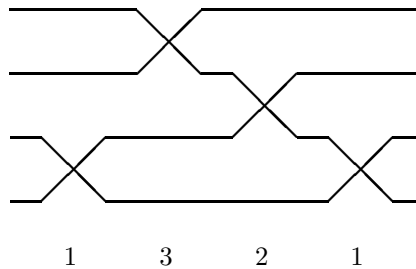


FIGURE 2. Pseudoline arrangement for the reduced word 1321

Let us now consider the factorization scheme \mathbf{i} defined by (4.8). The E -part of \mathbf{i} is 1321, and we already drew the corresponding arrangement. The F -part of \mathbf{i} is 23121. To construct the double pseudoline arrangement for \mathbf{i} , we superimpose the arrangements for 1321 and 23121, aligning them closely in the vertical direction, and placing the intersections so that tracing them left-to-right would produce the

same shuffle of the two reduced words that appear in \mathbf{i} . This results in the double pseudoline arrangement in Figure 3.

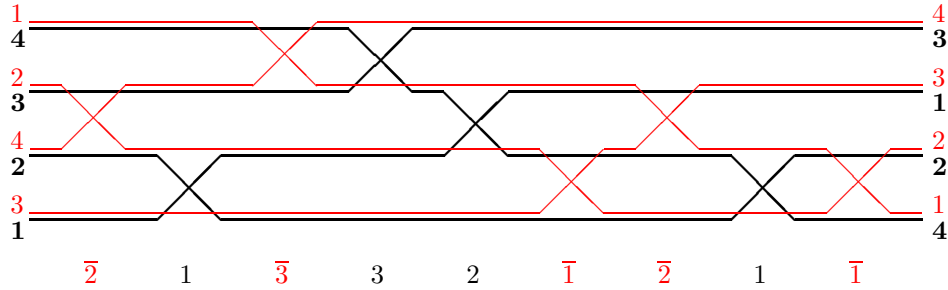


FIGURE 3. Double pseudoline arrangement

The double pseudoline arrangement that corresponds to a factorization scheme \mathbf{i} is denoted by $\text{Arr}(\mathbf{i})$. The two subarrangements of $\text{Arr}(\mathbf{i})$ corresponding to the reduced words for v and u are called the E - and F -part of $\text{Arr}(\mathbf{i})$, and their crossing points are referred to as E - and F -crossings, respectively. These crossings are in an obvious bijection with the noncircled entries of \mathbf{i} .

We next label the pseudolines of $\text{Arr}(\mathbf{i})$ using the following important convention. The pseudolines of the F -part of $\text{Arr}(\mathbf{i})$ are labeled 1 through n bottom-up at the right end of the arrangement (just as in [3]). At the same time, the pseudolines of the E -part are labeled bottom-up at the left end. See Figure 3.

Another numbering that we are going to use is the bottom-to-top numbering of the $n - 1$ horizontal strips containing the crossings of the arrangement. We say that the strip between the j th and $(j + 1)$ st horizontal lines, counting from the bottom, has *level* j , and all the E - and F -crossings contained in this strip are of level j .

Note that arrangement $\text{Arr}(\mathbf{i})$ does not depend on the H -part of the factorization scheme \mathbf{i} . In order to include the H -part into the picture, we associate with each entry \textcircled{j} a bullet \bullet placed on the j th horizontal line. The position of a bullet corresponds to the position of \textcircled{j} in \mathbf{i} , so that when the arrangement is traced left-to-right, the crossings and bullets appear in the same order as the entries of \mathbf{i} that they represent. The resulting “rigged” arrangement is denoted by $\text{Arr}_\bullet(\mathbf{i})$. Figure 4 shows $\text{Arr}_\bullet(\mathbf{i})$ for the factorization scheme (4.8).

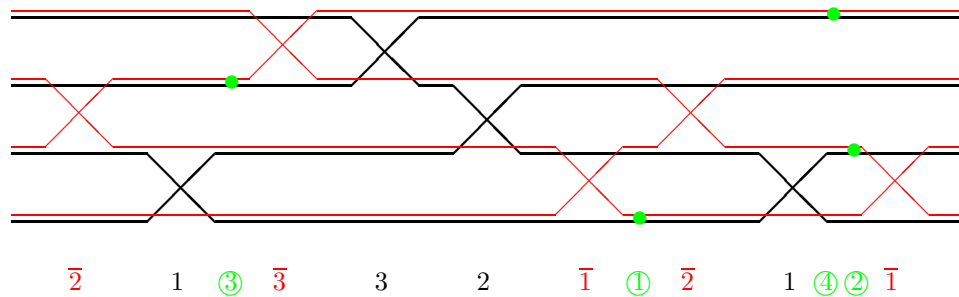


FIGURE 4. Arrangement $\text{Arr}_\bullet(\mathbf{i})$

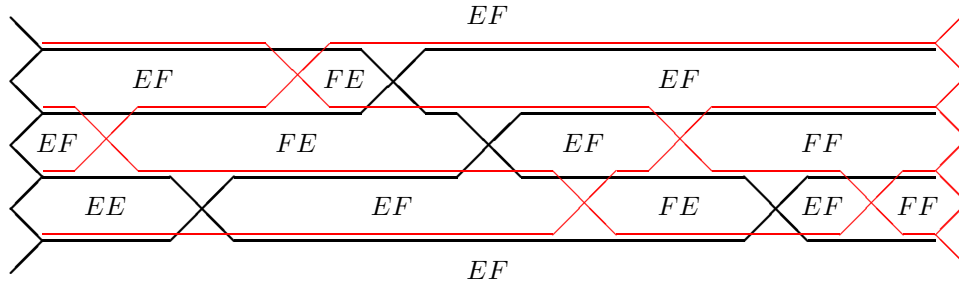


FIGURE 5. Types of chambers

To make our terminology uniform, we will refer to the bullets in $\text{Arr}_\bullet(\mathbf{i})$ as *H-crossings* (despite the fact that they are not crossings geometrically). Thus the total number of crossings in $\text{Arr}_\bullet(\mathbf{i})$ is $l = n + l(u) + l(v)$, and they are associated with the variables t_k in the factorization (4.5). We will occasionally refer to the crossing in $\text{Arr}_\bullet(\mathbf{i})$ associated with a factorization parameter t_k by simply saying “crossing t_k ”. The *H*-crossing lying on the i th horizontal line will also be denoted by d_i .

4.5. Solution to the factorization problem. Let us fix permutations $u, v \in S_n$ and a factorization scheme \mathbf{i} of type (u, v) ; in this section, we present our solution to the corresponding factorization problem. As in [3], the combinatorics needed to formulate the answer involves not only the crossings of the arrangement $\text{Arr}(\mathbf{i})$ but also its *chambers*, which can be defined as horizontal segments between consecutive crossings of the same level. More precisely, each horizontal strip with, say, k crossings breaks down into $k + 1$ chambers (including the ones at the ends of the strip). Two more chambers are located at the bottom and the top of the arrangement. To illustrate, the arrangement in Figure 3 has 14 chambers; in general, there are $l + 1$ of them.

We say that a chamber C is of *type EF* if the left endpoint of C is an *E*-crossing, while its right endpoint is an *F*-crossing. Chambers of types *EE*, *FE* and *FF* are defined in a similar way. Figure 5 shows the types of all 14 chambers of the arrangement in Figure 3. Here and in the sequel, we use the following important convention: on each level, there is a fictitious *E*-crossing at the left border of the arrangement, and a fictitious *F*-crossing at the right border. These fictitious crossings determine the types of the chambers adjacent to the boundary of $\text{Arr}(\mathbf{i})$.

For every chamber C in $\text{Arr}(\mathbf{i})$, let $I(C)$ denote the set of labels of the lines of the *F*-part of the arrangement that pass below C . Analogously, $J(C)$ will consist of the labels of lines of the *F*-part of $\text{Arr}(\mathbf{i})$ that pass below C . The sets $I(C)$ and $J(C)$ are called *chamber sets* for the factorization scheme \mathbf{i} . Figure 6 shows the chamber sets $I(C)$ and $J(C)$ for each chamber of the given arrangement. Note that if C is a chamber of level i , then both $I(C)$ and $J(C)$ have i elements.

Our constructions will also involve the “big” chambers formed by the *E*-part and the *F*-part of a double pseudoline arrangement, taken separately. We will refer to these “big” chambers as *E*-chambers and *F*-chambers, respectively. For example, the arrangement in Figure 3 has 9 *E*-chambers, which are in obvious bijection with the 9 chambers in Figure 2.

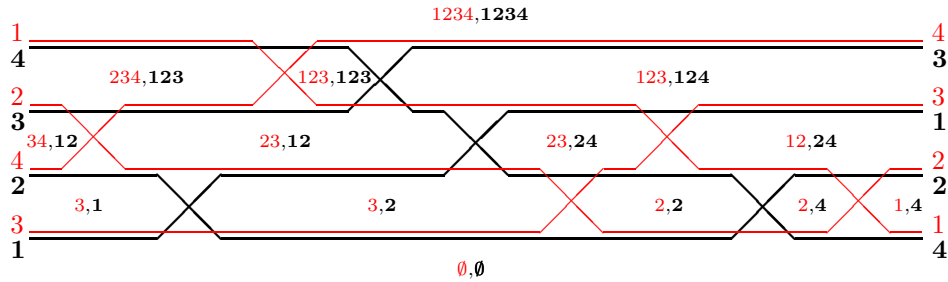


FIGURE 6. Chamber sets

For every chamber C of the arrangement $\text{Arr}(\mathbf{i})$, we denote

$$M_C = \Delta_{I(C), J(C)} ;$$

this minor is considered as a regular function on G (with the convention that $\Delta_{\emptyset, \emptyset} = 1$). For example, if C is the rightmost chamber of level 2 in Figure 6, then $M_C = \Delta_{12, 24}$.

To each $i = 1, \dots, n$ we associate a rational function on $G^{u,v}$ given by

$$(4.12) \quad \Pi_i = \frac{\prod_C M_C}{\prod_{C'} M_{C'}} ,$$

where C runs over all chambers of level i and type FE , while C' runs over all chambers of level i and type EF . For example, in Figure 6 we have

$$\Pi_2 = \frac{\Delta_{23, 12}}{\Delta_{34, 12} \Delta_{23, 24}} .$$

Also, by convention, $\Pi_0 = 1$.

Let \mathcal{C} be a “big” K -chamber of level i , where K is one of the symbols E and F . Let L be the other of these symbols (i.e., $L = F$ if $K = E$, and $L = E$ if $K = F$). We define

$$(4.13) \quad \mathcal{M}_{\mathcal{C}}^{\text{right}} = \frac{\tilde{M} \prod_{C'} M_{C'}}{\prod_{C''} M_{C''}} ,$$

where

- C' runs over all chambers of level i and type LK to the right of \mathcal{C} ;
- C'' runs over all chambers of level i and type KL to the right of \mathcal{C} ;
- $\tilde{M} = M_C$, where C is the (“small”) chamber at the right end of \mathcal{C} (inside \mathcal{C}), unless $K = E$ and C is stuck to the right border, in which case $\tilde{M} = 1$.

Analogously,

$$(4.14) \quad \mathcal{M}_{\mathcal{C}}^{\text{left}} = \frac{\tilde{M} \prod_{C'} M_{C'}}{\prod_{C''} M_{C''}} ,$$

where

- C' runs over all chambers of level i and type KL to the left of \mathcal{C} ;
- C'' runs over all chambers of level i and type LK to the left of \mathcal{C} ;
- $\tilde{M} = M_C$, where C is the (“small”) chamber at the left end of \mathcal{C} (inside \mathcal{C}), unless $K = F$ and C is stuck to the left border, in which case $\tilde{M} = 1$.

We are finally prepared to state our solution to the factorization problem.

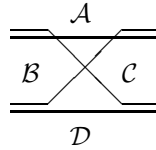
Theorem 4.9. *Let \mathbf{i} be a factorization scheme of type (u, v) , and suppose a matrix $x \in G^{u,v}$ admits the factorization $x = x_{\mathbf{i}}(t_1, \dots, t_l)$ with all t_k nonzero complex numbers. Let $x' = \zeta^{u,v}(x) \in G^{u^{-1}, v^{-1}}$ denote the matrix obtained from x by the “twist” (4.10). Then the factorization parameters t_k are determined as follows.*

- If t_k corresponds to the H -crossing d_i , then

$$(4.15) \quad t_k = \frac{\Pi_i(x')}{\Pi_{i-1}(x')},$$

where Π_i and Π_{i-1} are given by (4.12).

- Let t_k correspond to an E - or F -crossing of level i , and let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be the four “big” chambers surrounding this crossing, as shown:



Then

$$(4.16) \quad t_k = \frac{\mathcal{M}_{\mathcal{A}}^{\text{opp}(d_{i+1})}(x') \mathcal{M}_{\mathcal{D}}^{\text{opp}(d_i)}(x')}{\mathcal{M}_{\mathcal{B}}^{\text{opp}(d_{i+1})}(x') \mathcal{M}_{\mathcal{C}}^{\text{opp}(d_i)}(x')},$$

where we refer to the notation of (4.13)–(4.14) as follows: the superscript $\text{opp}(d_i)$ stands for “left” if the H -crossing d_i is to the right of t_k , and for “right” if d_i is to the left of t_k .

Theorem 4.9 is obtained as a specialization of Theorem 1.9, with the help of the following additional commutation relations:

$$(4.17) \quad \begin{aligned} x_i(a)x_{\textcircled{D}}(b) &= x_{\textcircled{D}}(b)x_i(a), \quad j \notin \{i, i+1\}; \\ x_i(a)x_{\textcircled{B}}(b) &= x_{\textcircled{B}}(b)x_i(a/b); \\ x_i(a)x_{\textcircled{C}}(b) &= x_{\textcircled{C}}(b)x_i(ab), \quad j = i+1; \\ x_{\bar{i}}(a)x_{\textcircled{D}}(b) &= x_{\textcircled{D}}(b)x_{\bar{i}}(a), \quad j \notin \{i, i+1\}; \\ x_{\bar{i}}(a)x_{\textcircled{B}}(b) &= x_{\textcircled{B}}(b)x_{\bar{i}}(ab); \\ x_{\bar{i}}(a)x_{\textcircled{C}}(b) &= x_{\textcircled{C}}(b)x_{\bar{i}}(a/b), \quad j = i+1. \end{aligned}$$

Example 4.10. To illustrate Theorem 4.9, let us compute the factorization parameter t_9 corresponding to the rightmost F -crossing of level 2 in Figure 4. It is given by

$$t_9 = \frac{\mathcal{M}_{\mathcal{A}}^{\text{right}}(x') \mathcal{M}_{\mathcal{D}}^{\text{left}}(x')}{\mathcal{M}_{\mathcal{B}}^{\text{right}}(x') \mathcal{M}_{\mathcal{C}}^{\text{left}}(x')},$$

where

$$\begin{aligned} \mathcal{M}^{\text{right}}(\mathcal{A}) &= \Delta_{123,124}, \\ \mathcal{M}^{\text{right}}(\mathcal{B}) &= \Delta_{23,24}, & \mathcal{M}^{\text{left}}(\mathcal{C}) &= \frac{\Delta_{12,24} \Delta_{23,12}}{\Delta_{23,24} \Delta_{34,12}}, \\ \mathcal{M}^{\text{left}}(\mathcal{D}) &= \frac{\Delta_{2,2}}{\Delta_{3,2}}. \end{aligned}$$

Hence

$$(4.18) \quad t_9 = \frac{\Delta_{2,2} \Delta_{34,12} \Delta_{123,124}}{\Delta_{3,2} \Delta_{12,24} \Delta_{23,12}}(x').$$

Substituting the twisted matrix x' from Example 4.8 into (4.18) and simplifying, we finally obtain

$$(4.19) \quad t_9 = \frac{\Delta_{23,12}(x) (x_{43} \Delta_{12,12}(x) - \Delta_{124,123}(x))}{x_{23} \Delta_{24,12}(x) \Delta_{123,123}(x)}.$$

This answer can be verified directly using the combinatorial interpretation of minors $\Delta_{I,J}$ in terms of planar networks (see Proposition 4.2). From Figure 1 one obtains: $\Delta_{23,12} = t_3 t_7 t_8 t_9 t_{12}$, $x_{43} = t_4$, $\Delta_{12,12} = t_8 t_{12} (1 + t_6 t_9)$, $\Delta_{124,123} = t_4 t_8 t_{12}$, $x_{23} = t_6$, $\Delta_{24,12} = t_4 t_7 t_8 t_9 t_{12}$, $\Delta_{123,123} = t_3 t_8 t_{12}$, implying (4.19).

As in Theorem 1.10, formulas (4.15) and (4.16) imply that the factorization parameters t_1, \dots, t_l are related by an invertible monomial transformation to the l minors $\Delta_{I(C),J(C)}(x')$ of the twisted matrix x' that correspond to the chambers of the arrangement $\text{Arr}(\mathbf{i})$, with the bottom chamber excluded. The inverse transformation has the following description which can be deduced from (1.21) (since we left the latter formula without proof, the same is true for our next theorem, although it is not hard to give it a direct proof).

Theorem 4.11. *Formulas (4.15)–(4.16) are equivalent to the following formulas:*

$$(4.20) \quad \Delta_{I(C),J(C)}(x') = \left(\prod t_k \right)^{-1},$$

where the product is over all t_k which correspond to the following types of crossings:

- *E-crossings to the right of C such that C lies between the lines intersecting at t_k ;*
- *F-crossings to the left of C such that C lies between the lines intersecting at t_k ;*
- *H-crossings to the right of C such that C lies above the E-line passing through t_k ;*
- *H-crossings to the left of C such that C lies above the F-line passing through t_k .*

For example, in Figure 6, $\Delta_{3,1}(x') = (t_2 t_6)^{-1}$, $\Delta_{123,124}(x') = (t_1 t_4 t_8 t_{12})^{-1}$, etc.

4.6. Applications to total positivity. In the case $G = GL_n(\mathbb{C})$, the definition of the totally nonnegative variety $G_{\geq 0}$ given in Section 1.3 is modified as follows: $G_{\geq 0}$ is the multiplicative semigroup generated by elementary Jacobi matrices (cf. (4.1)–(4.3)) $x_i(t)$, $x_{\bar{i}}(t)$, and $x_{\odot}(t)$ with $t > 0$. It is known [17, 24] that this definition of total nonnegativity is equivalent to the classical one: an invertible matrix $x \in G$ belongs to $G_{\geq 0}$ if and only if all minors $\Delta_{I,J}(x)$ (in particular, all matrix entries) are nonnegative.

For any two permutations u and v in S_n , the corresponding *totally positive variety* is defined by

$$G_{>0}^{u,v} = G_{\geq 0} \cap G^{u,v} .$$

Each factorization scheme of type (u, v) gives rise to a parametrization of $G_{>0}^{u,v}$, according to the following analogue of Theorem 1.3.

Theorem 4.12. *For any factorization scheme $\mathbf{i} = (i_1, \dots, i_l)$ of type (u, v) , the corresponding product map $x_{\mathbf{i}} : \mathbb{C}_{\neq 0}^l \rightarrow G^{u,v}$ given by (4.5) restricts to a bijection between $\mathbb{R}_{>0}^l$ and $G_{>0}^{u,v}$.*

The twist map $\zeta^{u,v}$ defined by (4.10) respects total positivity: it sends totally nonnegative matrices in $G^{u,v}$ to totally nonnegative matrices in $G^{u^{-1},v^{-1}}$ (cf. Theorem 1.7). Combining this fact with Theorems 4.12 and 4.9 leads to a family of criteria for total positivity, one for each factorization scheme.

For a factorization scheme $\mathbf{i} = (i_1, \dots, i_l)$ of type (u, v) , let $F(\mathbf{i})$ denote the collection of l minors $\Delta_{uI(C),v^{-1}J(C)}$, where C runs over all chambers of the arrangement $\text{Arr}(\mathbf{i})$, excluding the bottom chamber. (This notation agrees with that of (1.22).) We note that the pair $(uI(C), v^{-1}J(C))$ will correspond in the same way as above to a chamber C if we relabel the pseudolines in $\text{Arr}(\mathbf{i})$, numbering the F -pseudolines 1 through n bottom-up at the left end, and the E -pseudolines bottom-up at the right end. See Figure 7.

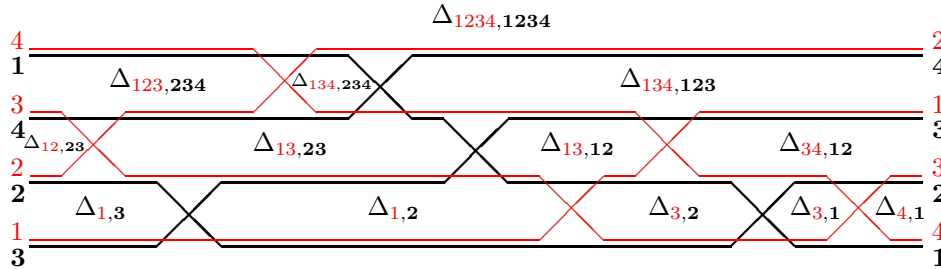


FIGURE 7. Minors $\Delta_{uI(C),v^{-1}J(C)}(x)$

Let $F(u, v)$ denote the union of the collections $F(\mathbf{i})$ for all factorization schemes \mathbf{i} of type (u, v) . The set $F(u, v)$ can be described directly in the following way. A subset $I \subset [1, n]$ is called a w -chamber set if, together with each element j it also contains every i such that $i < j$ and $w(i) < w(j)$. In this terminology (originally introduced in [3, Section 5.3]), $F(u, v)$ consists of all minors $\Delta_{I,J}$ such that I is a u^{-1} -chamber set while J is a v -chamber set.

The following result specializes Theorem 1.11.

Theorem 4.13. *Let x be a matrix in a double Bruhat cell $G^{u,v}$, and let \mathbf{i} be a factorization scheme of type (u, v) . Then the following are equivalent:*

- (1) x is totally nonnegative;
- (2) $\Delta_{I,J}(x) > 0$ for any u^{-1} -chamber set I and any v -chamber set J ;
- (3) $\Delta_{uI(C),v^{-1}J(C)}(x) > 0$ for any chamber C of the arrangement $\text{Arr}(\mathbf{i})$.

For instance, in our running example where u, v , and \mathbf{i} are given by (4.7) and (4.8), a matrix $x \in G^{u,v}$ is totally nonnegative if and only if the 13 minors appearing in Figure 7 are all positive if evaluated at x .

Specializing Theorem 4.13 to the case $u = v = w_o$, we see that the totally positive variety $G_{>0}^{w_o, w_o}$ is the classical variety $G_{>0}$ of the totally positive $n \times n$ matrices, i.e., those matrices whose *all* minors are (strictly) positive. Condition (3) of Theorem 4.13 provides a family of criteria for total positivity, each of which says that a matrix x is totally positive if and only if some collection of n^2 minors are positive at x .

Different factorization schemes \mathbf{i} and \mathbf{i}' of the same type (u, v) can have the same collections of chamber sets, thus leading to the same criteria for total positivity. We will say that \mathbf{i} and \mathbf{i}' (and the corresponding arrangements $\text{Arr}(\mathbf{i})$ and $\text{Arr}(\mathbf{i}')$) are *isotopic* if they can be obtained from each other by a sequence of the following “trivial 2-moves”:

$$\begin{aligned}
 \dots i j \dots &\sim \dots j i \dots, & |i - j| \geq 2, \\
 \dots \bar{i} \bar{j} \dots &\sim \dots \bar{j} \bar{i} \dots, & |i - j| \geq 2, \\
 \dots i \bar{j} \dots &\sim \dots \bar{j} i \dots, & i \neq j, \\
 \dots \textcircled{i} j \dots &\sim \dots j \textcircled{i} \dots, \\
 \dots \textcircled{i} \bar{j} \dots &\sim \dots \bar{j} \textcircled{i} \dots, \\
 \dots \textcircled{i} \textcircled{j} \dots &\sim \dots \textcircled{j} \textcircled{i} \dots.
 \end{aligned}
 \tag{4.21}$$

It is not hard to show that \mathbf{i} and \mathbf{i}' have the same collection of chamber sets $(I(C), J(C))$ if and only if they are isotopic. Thus total positivity criteria in Theorem 4.13 are in a bijection with “isotopy types” of arrangements of type (u, v) .

The set of all isotopy types of arrangements of type (u, v) has a natural structure of a graph defined as follows. We call two isotopy types *adjacent* if the corresponding collections of chamber sets are obtained from each other by exchanging a single pair $(I(C), J(C))$ with another one. The graph obtained this way is always connected, and its study is an interesting combinatorial problem. One can check that the adjacency relation in this graph corresponds to the following *3-moves* and *mixed 2-moves* on double reduced words:

$$\begin{aligned}
 \dots i j i \dots &\rightsquigarrow \dots j i j \dots, & |i - j| = 1, \\
 \dots \bar{i} \bar{j} \bar{i} \dots &\rightsquigarrow \dots \bar{j} \bar{i} \bar{j} \dots, & |i - j| = 1, \\
 \dots i \bar{i} \dots &\rightsquigarrow \dots \bar{i} i \dots
 \end{aligned}
 \tag{4.22}$$

(cf. Sections 3.3 and 3.4); the connectedness property follows from Proposition 3.7.

For $G = GL_2$ and $u = v = w_o$, there are 2 isotopy types. The corresponding collections $F(\mathbf{i})$ are $\{x_{11}, x_{12}, x_{21}, \det(x)\}$ and $\{x_{22}, x_{12}, x_{21}, \det(x)\}$.

In the case of GL_3 and $u = v = w_o$, there are 34 isotopy types, giving rise to 34 different total positivity criteria. Each of these criteria involves 9 minors. Five of them—the minors

$$x_{31}, x_{13}, \Delta_{23,12}, \Delta_{12,23}, \det(x)$$

—are common to all 34 criteria; they correspond to the “unbounded” chambers lying on the periphery of each arrangement. The other four minors that distinguish isotopy types from each other correspond to the bounded chambers. Figure 8 shows a graph with 34 vertices labeled by the quadruples of “bounded” minors that appear in the corresponding total positivity criteria.

For an arbitrary n (and $u = v = w_o$), one obtains various nice (and surprising) total positivity criteria in GL_n by making particular choices of (the isotopy type of) a double pseudoline arrangement in Theorem 4.13. Let us discuss two criteria obtained in this way.

A minor $\Delta_{I,J}$ is called *solid* if both I and J consist of several consecutive indices. A criterion due to Fekete [10] (see also [12, p. 299]) asserts that (strict) total positivity of a matrix is equivalent to the positivity of all its solid minors. Each of the two criteria described below will strengthen this result.

We will consider two factorization schemes of type (w_o, w_o) having the same E - and F -parts (albeit shuffled in a different way). For both of them, the E -part is the lexicographically minimal reduced word for w_o , i.e., the reduced word

$$1, 2, 1, 3, 2, 1, \dots, n-1, n-2, \dots, 1;$$

the F -part is the same but with barred entries. Let \mathbf{i}_1 denote the shuffle of these parts such that all the unbarred entries precede the barred ones. Let \mathbf{i}_2 denote the shuffle of the same parts such that every unbarred entry is immediately followed by the corresponding barred entry (so that \mathbf{i}_2 starts with $1, \bar{1}, 2, \bar{2}, \dots$). A direct check shows that the corresponding collections of minors $F(\mathbf{i}_1)$ and $F(\mathbf{i}_2)$ are given as follows:

- $F(\mathbf{i}_1)$ consists of solid minors $\Delta_{I,J}$ such that $1 \in I \cup J$;
- $F(\mathbf{i}_2)$ consists of solid minors $\Delta_{I,J}$ such that $\min(I) + \max(J) \in \{n, n + 1\}$.

Each of these two collections consists of n^2 minors; and by Theorem 4.13, each of them provides a total positivity criterion that strengthens the one of Fekete’s: a square matrix is totally positive if and only if all the minors in $F(\mathbf{i}_1)$ (respectively, $F(\mathbf{i}_2)$) are positive. It should be mentioned that the first of these criteria was (implicitly) obtained by Cryer [8, Theorems 1.1 and 3.1] using a result of Karlin [16, p. 85]; an explicit statement appears in [13, Theorem 4.1]. The second criterion seems to be new.

The equivalence of conditions (2) and (3) in Theorem 4.13 has the following algebraic explanation (cf. Theorem 1.12).

Theorem 4.14. *For any factorization scheme \mathbf{i} of type (u, v) , the collection of minors $F(\mathbf{i})$ is a totally positive base (cf. Definition 2.21) for the collection $F(u, v)$.*

The most significant part of this theorem is that every minor from $F(u, v)$ can be written as a subtraction-free expression in the minors from $F(\mathbf{i})$. Such an expression can be found in a constructive way. To do this, it will be enough to consider two arrangements $\text{Arr}(\mathbf{i})$ and $\text{Arr}(\mathbf{i}')$ whose isotopy types are adjacent in the graph that we described above; recall that this means that the collection of minors $F(\mathbf{i}')$ is obtained from $F(\mathbf{i})$ by exchanging a single minor Δ with another minor Δ' . It suffices to show that Δ' can be written as a subtraction-free expression in the minors from $F(\mathbf{i})$. This can be done with the help of certain 3-term determinantal identities. These identities are stated in the following proposition, which is a specialization of Theorems 1.16 and 1.17. We use the notation Li, Lij , etc., as a shorthand for $L \cup \{i\}, L \cup \{i, j\}$, etc.

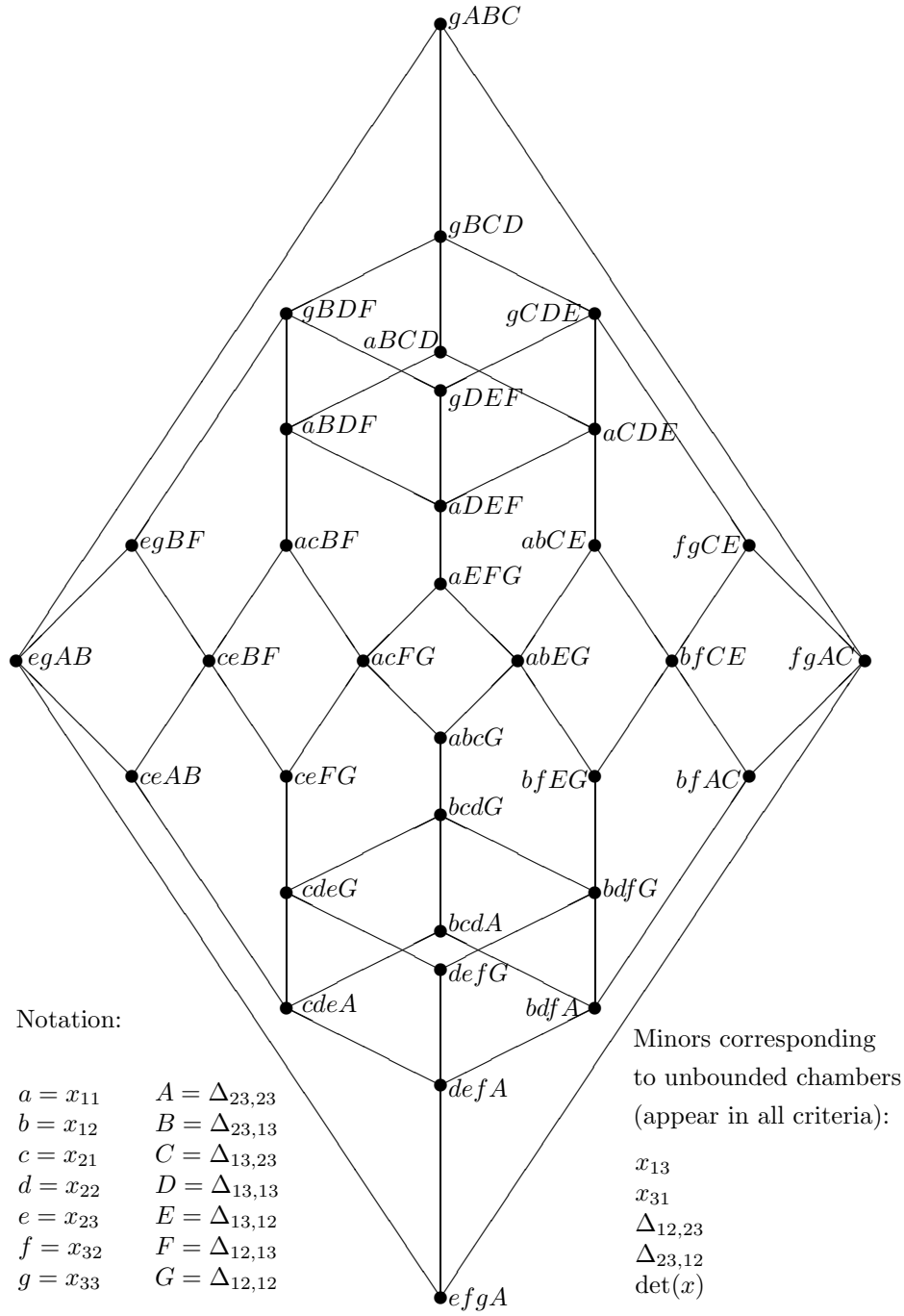


FIGURE 8. Total positivity criteria for GL_3

Proposition 4.15. For any $i, j, k, p \in [1, n]$ and $I, L \subset [1, n]$ such that $i < j < k$, $|I| = |L| + 1$, $L \cap \{i, j, k\} = \emptyset$, $I \cap \{p\} = \emptyset$, we have:

$$(4.23) \quad \begin{aligned} \Delta_{Ip, Lik} \Delta_{I, Lj} &= \Delta_{Ip, Lij} \Delta_{I, Lk} + \Delta_{Ip, Ljk} \Delta_{I, Li} , \\ \Delta_{Lik, Ip} \Delta_{Lj, I} &= \Delta_{Lij, Ip} \Delta_{Lk, I} + \Delta_{Ljk, Ip} \Delta_{Li, I} . \end{aligned}$$

For any $i, i', j, j' \in \{1, \dots, n\}$ and $I, J \subset \{1, \dots, n\}$ such that $i < i'$, $j < j'$, $|I| = |J|$, $I \cap \{i, i'\} = J \cap \{j, j'\} = \emptyset$, we have

$$(4.24) \quad \Delta_{Ii, Jj} \Delta_{Ii', Jj'} = \Delta_{Ii, Jj'} \Delta_{Ii', Jj} + \Delta_{I, J} \Delta_{Ii i', Jj j'} .$$

The identities (4.23)–(4.24) are well known, although their attribution is complicated. As early as in 1819 they were proved by P. Desnanot (see [20, pp. 140–142]). Identities (4.23) are special cases of the (Grassmann-)Plücker relations (see, e.g., [11, (15.53)]), while identity (4.24) plays a crucial role in C. L. Dodgson’s condensation method, and because of that is occasionally associated with the name of Lewis Carroll.

It would be interesting to see which other classical determinantal identities can be generalized to the functions $\Delta_{u\omega_i, v\omega_i}$ on any semisimple group. We conclude the paper by mentioning one challenging problem of this kind: find a generalization of the classical Binet-Cauchy formula for the minors of the product of two matrices:

$$\Delta_{I, J}(xy) = \sum_K \Delta_{I, K}(x) \Delta_{K, J}(y) .$$

At present, we only know such a generalization for the *minuscule* fundamental weights ω_i .

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