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FRACTIONAL ISOPERIMETRIC INEQUALITIES AND SUBGROUP DISTORTION

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For John Stallings on his 60th birthday

Introduction

Isoperimetric inequalities measure the complexity of the word problem in finitely presented groups by giving a bound on the number of relators that one must apply in order to show that a word w in the given generators represents the identity. Such bounds are given in terms of the length of w, and the function describing the optimal bound is known as the *Dehn function* of the group. (Modulo a standard equivalence relation $\simeq$, the Dehn function is an invariant of the group, not just the given finite presentation.)

About seven years ago, as the result of efforts by a number of authors [9], [11], [14], it was established that for every positive integer d one can construct finitely presented groups whose Dehn function is polynomial of degree d. The question of whether or not there exist groups whose Dehn functions are of fractional degree has attracted a good deal of interest (e.g., [18], [25]). According to a theorem of M. Gromov, one does not get fractional exponents less than 2, because if a group satisfies a sub-quadratic isoperimetric inequality, then it actually satisfies a linear isoperimetric inequality (see [13], [10], [19], [22]).

The main purpose of this article is to prove the following:

Theorem A. *There exist infinitely many non-integers $r > 2$ such that the Dehn function of some finitely presented group is $\simeq n^r$.*

Our method for constructing groups whose Dehn functions are of fractional degree is geometric in nature. Of the examples that we shall describe, the simplest are the *abc* groups $\Gamma(a, b, c)$ defined in Section 2. These groups are obtained by taking three torus bundles along the circle (each of a different dimension) and amalgamating their fundamental groups along central cyclic subgroups.

Let M be a closed Riemannian manifold. Given a smooth loop in the universal cover $\widetilde{M}$, one might enquire about the area of the smallest disc bounded by that loop. The isoperimetric function $Fill_0^M$ describes the optimal bound on this area as a function of the length of the curve. One can show that this isoperimetric function for M is $\simeq$ equivalent to the Dehn function of its fundamental group.

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Thus Theorem A can be interpreted as a result about the isoperimetric properties of Riemannian manifolds.

Lurking in the background of our proof of Theorem A is an observation concerning subgroup distortion. Let G be a finitely generated group. Following Gromov [14], we define the distortion of a finitely generated subgroup $H \subseteq G$ to be the $\simeq$ equivalence class of the function $\delta : \mathbb{N} \to \mathbb{N}$, where $\delta(n)$ is the radius of the set of vertices in the Cayley graph of H that are a distance at most n from the identity in G.

Theorem B. *For every rational number $r \geq 1$ there exist pairs of finitely presented groups $H \subseteq G$ such that the distortion of H in G is $\simeq n^r$.*

I presented the results described in this paper at seminars in Princeton and New York during the fall of 1993. I gave a fuller account of them in my lectures at the conferences on geometry and group theory in Champousin, Durham and Lyon in the spring and summer of 1994. I would like to take this opportunity to thank the organisers of these three excellent conferences. I also wish to apologise to colleagues for my tardiness in preparing the final draft of this paper.

During the intervening period, further significant progress has been made on the fundamental question of which functions arise as Dehn functions of finite presentations. In [24], M. V. Sapir, J-C. Birget and E. Rips show that by encoding descriptions of certain machines into finite group presentations one can construct groups for which the Dehn function has the form n^r where the exponent r is any number in $[4, \infty)$ whose decimal expansion can be calculated reasonably quickly (this includes all rational numbers and many irrationals). Conversely, they show that if a number arises as the exponent of a Dehn function, then it is a number that can be computed reasonably quickly. In [7] N. Brady and M. R. Bridson construct a dense set of transcendental exponents $r \in [2, \infty)$ for Dehn functions using geometric constructions. With regard to Theorem B, we note that A. Yu. Ol'shanskii and M. V. Sapir [21] have proved comprehensive results describing the set of functions which arise as the distortion functions of (cyclic) subgroups of finitely presented groups. (See also [20].)

1. Definitions

We begin by recalling the definition of the Dehn function of a finite presentation. We fix a finite presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ for the group Γ. We assume that $\mathcal{R}$ is symmetrized [16]. A reduced word $w \in F(\mathcal{A})$ is *null-homotopic* (i.e., represents the identity element in Γ) if and only if there is an equality of the form $w = \prod_i^N g_i^{-1} r_i g_i$ in the free group $F(\mathcal{A})$, where $r_i \in \mathcal{R}$. Isoperimetric inequalities give bounds on the number N of factors in a minimal such expression; this integer is called $Area(w)$. One seeks to bound $Area(w)$ in terms of $|w|$, the length of w. The function giving the optimal such bound is called the Dehn function of the presentation.

For the purposes of this article, it is best to cast the definition of the Dehn function in the following geometric context.

According to van Kampen's lemma (see [16]) the above factorisation of the null homotopic word w can be portrayed by a finite, oriented, planar, combinatorial 2-complex with basepoint. This complex is called a van Kampen diagram for w. The oriented 1-cells of this complex are labelled by elements of $\mathcal{A}$ and their inverses, the boundary label on each face of the 2-complex is an element of $\mathcal{R}$, and the boundary cycle of the complex (read with positive orientation from the basepoint) is the word w. The number of 2-cells in the complex is equal to N, the number of factors in the given equality for w. Conversely, any van Kampen diagram gives rise to an equality in $F(\mathcal{A})$ showing that the boundary cycle of the diagram represents the identity in Γ. A *minimal area* van Kampen diagram is one that has the least number of 2-cells among all van Kampen diagrams which share its boundary label. Notice that every connected and simply connected subdiagram of a minimal area diagram is itself a minimal area diagram.

Definition 1.1. If $\mathcal{P}$ is a finite presentation for the group Γ, and if w is a word in the generators and their inverses that represents the identity in Γ, then $Area(w)$ is defined to be the number of 2-cells in a minimal area van Kampen diagram for w. The *Dehn function* $f : \mathbb{N} \to \mathbb{N}$ *of* $\mathcal{P}$ is defined by:

$$f(n) = \max_{|w| \leq n} Area(w).$$

If $\mathcal{Q}$ is another finite presentation for Γ with Dehn function g, then there exist positive constants k and k' such that

$$g(n) \leq k\, f(kn + k) + k'n + k', \tag{1.1}$$

for all $n \geq 0$ (see [1]). All of the Dehn functions which we shall consider in this article grow at least linearly (and we shall implicitly assume this henceforth). In this setting one can dispense with the term $(k'n + k')$ in (1.1). This motivates a notion of equivalence of functions which we now recall.

In general, given $f, g : \mathbb{N} \to [0, \infty)$, one writes $g \preceq f$ if there exists a positive constant k so that (1.1) holds with $k' = 0$. And one writes $f \simeq g$ if, in addition, $f \preceq g$. It is easily verified that this is an equivalence relation. Henceforth we shall speak of "the Dehn function of the group Γ" with the understanding that this is only defined up to $\simeq$ equivalence.

Throughout this article we shall be concerned with presentations of the following type. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a disjoint collection of finite sets, and consider the presentation

$$\langle \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n, z_1, \dots, z_{n-1} \mid \mathcal{R}_1, \dots, \mathcal{R}_n \rangle, \tag{1.2}$$

where the relators $\mathcal{R}_i$ involve only the letters $\mathcal{A}_i \cup \{z_i, z_{i-1}\}$, if $i \in \{2, \dots, n-1\}$, the relators $\mathcal{R}_1$ involve only the letters $\mathcal{A}_1 \cup \{z_1\}$, and the relators $\mathcal{R}_n$ involve only the letters $\mathcal{A}_n \cup \{z_{n-1}\}$. Assume that in the group defined by this presentation the elements z_i all have infinite order. Notice that one encounters presentations such as this when considering groups of the following form: $G_1 *_{C_1} \cdots *_{C_{n-1}} G_n$, where the amalgamated subgroups C_i are all infinite cyclic.

Given a van Kampen diagram Δ over such a presentation, we consider the equivalence relation $\sim$ on 2-cells in Δ generated by setting $e \sim e'$ if the boundary cycle of each of e and e' is labelled with a relator from the same $\mathcal{R}_i$ and if, in addition, their boundaries intersect in at least one edge.

Definition 1.2. A *monochromatic region* of Δ is the union of an $\sim$ equivalence class of closed 2-cells. If the boundary labels of the 2-cells in this equivalence class belong to $\mathcal{R}_i$, then the monochromatic region is said to be of type $\mathcal{R}_i$.

For example, in the case $n = 1$, the monochromatic regions are precisely the discs obtained as the closures of the connected components of Δ minus its boundary cycle $\partial\Delta$.

Lemma 1.3. *If Δ is a minimal area diagram, then all of its monochromatic regions are homeomorphic to discs, and every monochromatic region of type $\mathcal{R}_1$ or $\mathcal{R}_n$ meets the boundary of Δ in at least one edge.*

Proof. We proceed by induction on n. As the preceding remark indicates, the case $n = 1$ is clear.

If Δ did not contain a monochromatic region of type $\mathcal{R}_1$, then we would be done by application of our inductive hypothesis to Δ viewed as a van Kampen diagram over the subpresentation obtained by deleting $\mathcal{A}_1$ and $\mathcal{R}_1$ and incorporating z_1 into $\mathcal{A}_2$. We therefore proceed under the assumption that there is a monochromatic region of type $\mathcal{R}_1$.

By definition, the frontier of a region is the intersection of that region with the closure of its complement in the plane. We claim that the frontier of each monochromatic region of type $\mathcal{R}_1$ is homeomorphic to a circle (implying that the region is a disc) and that, moreover, it also contains an edge in the boundary of Δ. Indeed, if either of these two conditions were to fail, then the frontier of our region would contain a simple closed loop all of whose 1-cells were in the interior of Δ. But if a 2-cell of type $\mathcal{R}_1$ meets a 2-cell of another type along an edge, then the second 2-cell must be of type $\mathcal{R}_2$ and the edge must be labelled z_1. Hence this simple closed loop in the interior of Δ would be labelled by a word involving only z_1 and z_1^{-1}. But z_1 is assumed to have infinite order in the group defined by the above presentation, so this labelling word would freely reduce to the trivial word, i.e., have area 0. But then, by replacing the (simply connected) subdiagram of Δ bounded by this loop with a diagram of area 0, we would get a contradiction to the hypothesis that Δ is of minimal area.

Having established the asserted properties for regions of type $\mathcal{R}_1$ (and hence, by symmetry, type $\mathcal{R}_n$) it only remains to observe that, given a disc subdiagram of a van Kampen diagram, if the subdiagram contains an edge of the boundary of the ambient diagram, then the closure of the subdiagram's complement is a disjoint union of van Kampen diagrams. Thus we may remove all monochromatic regions of type $\mathcal{R}_1$ from Δ, and applying our inductive hypothesis to the resulting collection of van Kampen diagrams (considered as diagrams over the subpresentation described in the first paragraph of the proof) we deduce that all monochromatic regions of Δ are homeomorphic to discs. $\square$

Remark 1.4. The above argument remains valid if the cyclic groups $\langle z_i \rangle$ are replaced by free groups of finite rank.

Definition 1.5. Let G be a group with finite generating set $\mathcal{A}$. A word w in the free group on $\mathcal{A}$ is called *injective* if no non-empty subword of w is null-homotopic (i.e., represents $1 \in G$).

A null-homotopic word v in the free group on $\mathcal{A}$ is called *irreducible* if no proper subword of v is null-homotopic. (This implies that every van Kampen diagram for v is a topological disc.)

2. The groups G_c and (Γ, b, c)

The Dehn functions for many groups of the form $\mathbb{Z}^m \rtimes \mathbb{Z}$ were calculated in [11], and this classification was later extended to all abelian-by-free groups (see [8], [4]). We shall be concerned with a particular class of examples from [11], namely the groups $G_c = \mathbb{Z}^c \rtimes_{\phi_c} \mathbb{Z}$, where $\phi_c \in GL(c,\mathbb{Z})$ is the unipotent matrix with ones on the diagonal and super-diagonal and zeros elsewhere. G_c has presentation:

$$\langle x_1, \dots , x_c, t \mid [x_i, x_j] = 1 \text{ for all } i, j,\ [x_c, t] = 1,\ [x_i, t] = x_{i+1} \text{ if } i < c\rangle.$$

Notice that the centre of G_c is the infinite cyclic subgroup generated by x_c. To emphasize this fact, we shall change notation and write z_c in place of x_c.

In what follows we shall write d to denote the word metric on G_c corresponding to the generators given in the above presentation. (Recall that the word metric associated to a finite system of generators for a group G is the left-invariant metric in which the distance of each element $g \in G$ from the identity is the length of the shortest word in the generators and their inverses that represents g.)

We shall need the following facts concerning G_c.

Proposition 2.1. (1) *The Dehn function of G_c is $\simeq n^{c+1}$.*

(2) *There exists a constant α_c such that*

$$Area(wz_c^n) \leq \alpha_c\, |w|^{c+1}$$

for all $n \in \mathbb{Z}$ and all words w such that $wz_c^n = 1$ in G_c.

(3) *There exist constants $K_c > 1$ and $k_c > 0$ such that, for all $n > 0$,*

$$k_c n^{1/c} \leq d(1, z_c^n) \leq K_c n^{1/c}.$$

Proof. Assertion (1) is proved in [11] (see also $5A_2$ of [14] and [9]). Upper bounds are obtained on Dehn functions in [11] by using the combings of G_c constructed in [3]. The combing line to $g \in G_c$ is obtained by first writing g as $t^n x$, according to the semi-direct product decomposition $G_c = \mathbb{Z}^c \rtimes \langle t\rangle$, and then representing $x \in \mathbb{Z}^c$ by a choice of word that stays closest to the straight line from the identity to x in $\mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R}$. A standard argument shows that the area of a null-homotopic word w (with respect to a certain finite presentation) is bounded by the product of the length of the word and the length of the longest combing line to any of the vertices on the loop in the Cayley graph that is based at the identity and is labelled w — see the figure on page 600 of [2]. Indeed the standard argument portrayed by this figure proves something more: if u is a word in the combing under consideration and $vu = 1$ in G_c, then the area of vu is bounded by the product of the length of v and the length of the longest combing line to any of the vertices of v. Part (2) is a special case of this observation, because z^n is a combing line for the (asynchronously bounded) combing of G_c described above.

(3) is an observation of Gromov [14], $5.A_2'$ (see also [23]); we shall explain the main point. The Lie algebra of $\mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ has generators $X_1, \dots , X_c, T$ where $[X_i, X_j] = 0$, $[X_c, T] = 0$, and $[X_i, T] = X_{i+1}$ for $i < c$. (The X_i and T are left-invariant vector fields that correspond to the generators of $G_c \subset \mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ in the obvious way.) Note that T and X_1 together with all of their commutators span the Lie algebra. This means that the left-invariant distribution of 2-planes in $\mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ determined by $\{X_1, T\}$ satisfes Hörmander's condition (cf. [17]), so every pair of points $x, y \in \mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ can be joined by a path that is horizontal (i.e. almost everywhere tangent to this distribution) and we obtain a Carnot-Carathéodory metric d_c by defining $d_c(x, y)$ to be the infimum of the lengths of horizontal paths joining them (see [17]).

Let $t > 0$. The Lie algebra homomorphism h_t defined by $h_t(X_i) = t^i X_i$ and $h_t(T) = tT$ induces a map H_t of $\mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ given by the formula $H_t(\exp(Y)) =$

$\exp(h_t(Y))$. Since h_t stretches the tangent vectors to all horizontal curves by a factor t, we have

$$d_c(H_t(x), H_t(y)) = t\ d_c(x, y)$$

for all $x, y \in \mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$. In particular, if $g = z_c^n = \exp(nX_c)$, then setting $\tau = n^{1/c}$ we have

$$d_c(1, g) = d_c(1, \exp(h_\tau(X_c))) = n^{1/c}\ d_c(1, z_c).$$

The desired inequality for G_c follows immediately from this calculation and the fact that, because the action of G_c by left multiplication on $\mathbb{R}^c \rtimes_{\phi_c} \mathbb{R}$ is free, proper, cocompact and by isometries, the restriction of d_c to G_c is Lipschitz equivalent to any word metric on G_c. □

We shall consider the groups $\Gamma(a, b, c)$ obtained as follows: first we amalgamate G_a with $G_b \times \mathbb{Z}$ by identifying the centre of G_a with that of G_b; then we form the amalgamated free product of the resulting group with G_c, identifying the centre of the latter with the right-hand factor of $G_b \times \mathbb{Z}$. In symbols:

$$\Gamma(a, b, c) = G_a *_{z_a = z_b} (G_b \times \langle\, \zeta\, \rangle) *_{\zeta = z_c} G_c.$$

Combining the natural presentations for the G_l we obtain:

A presentation of $\Gamma(a, b, c)$. Generators:

$$x_1, \dots, x_{a-1}, z(= z_a = z_b), t_a, y_1, \dots, y_{b-1}, t_b, u_1, \dots, u_{c-1}, \zeta(= z_c), t_c.$$

Defining relations:

$$[z, x_i] = [x_i, x_j] = 1\ \forall\, i, j;\ [z, t_a] = 1;\ [x_i, t_a] = x_{i+1} \text{ if } i < a - 1 \text{ and } [x_{a-1}, t_a] = z;$$

and

$$[z, y_i] = [\zeta, y_i] = [y_i, y_j] = 1\ \forall\, i, j;\ [z, t_b] = [\zeta, t_b] = [z, \zeta] = 1;$$

$$[y_i, t_b] = y_{i+1} \text{ if } i < b - 1, \text{ and } [y_{b-1}, t_b] = z;$$

and

$$[\zeta, u_i] = [u_i, u_j] = 1\ \forall\, i, j;\ [\zeta, t_c] = 1;\ [u_i, t] = u_{i+1} \text{ if } i < c - 1 \text{ and } [x_{c-1}, t_c] = \zeta.$$

Let $\mathcal{A}_a$ denote the generators x_i together with t_a, let $\mathcal{A}_b$ denote the generators y_i together with t_b, and let $\mathcal{A}_c$ denote the generators u_i together with t_c. Let $\mathcal{R}_a$ denote the first set of relators, $\mathcal{R}_b$ the second set and $\mathcal{R}_c$ the third set.

Notice that the above presentation $\langle \mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c, z, \zeta \mid \mathcal{R}_a, \mathcal{R}_b, \mathcal{R}_c \rangle$ is precisely the sort of presentation to which (1.2) and (1.3) apply. By exploiting this fact we shall prove:

Theorem 2.2. *For all integers $1 \leq b \leq a < c$, the Dehn function of $\Gamma(a, b, c)$ is $\simeq n^{c + \frac{a}{b}}$.*

3. Lower bounds and subgroup distortion

The purpose of this section is to prove the following:

Proposition 3.1. *For any positive integers a, b and c, the Dehn function $f(n)$ for the above presentation of $\Gamma(a,b,c)$ satisfies*

$$n^{c+\frac{a}{b}} \preceq f(n).$$

The clarity of our exposition will be enhanced if we pursue some general considerations first.

Definition 3.2 (Subgroup Distortion)**.** Let $H \subset G$ be a pair of finitely generated groups, and let d_G and d_H be the word metrics associated to a fixed choice of generators for each. The *distortion* of H in G is the function

$$\delta(n) = \max\{d_H(1,h) \mid h \in H \text{ with } d_G(1,h) \leq n\}.$$

One checks easily that, up to $\simeq$ equivalence, this function is independent of the choice of word metrics d_G and d_H.

Example 3.3. According to 2.1(3), the distortion of the centre $Z(G_c)$ in G_c is $\simeq n^c$ for all $c \in \mathbb{Z}_+$. And by comparing $d_{G_a}(1,z^n)$ with $d_{G_b}(1,z^n)$, we see that if $a > b$, then the distortion of G_b in $G_a *_{\langle z\rangle} G_b$, the group formed by amalgamating G_a and G_b along their centres, is $\succeq n^{\frac{a}{b}}$.

If G is a group with finite presentation $\langle \mathcal{A} \mid \mathcal{R}\rangle$ and H is the subgroup generated by a subset $\mathcal{B} \subset \mathcal{A}$, then the HNN extension of G obtained by adding a stable letter that commutes with H has finite presentation

$$\mathcal{P} = \langle\, \mathcal{A},\, \tau \mid \mathcal{R},\, [\tau, b] = 1\ \forall b \in \mathcal{B}\rangle.$$

We shall now explain how the distortion of H in G provides a lower bound on the Dehn function of this HNN extension.

τ-corridors. Consider a reduced van Kampen diagram Δ over $\mathcal{P}$, and consider an oriented edge of $\partial\Delta$ that is labelled τ. If this edge lies in the boundary of some 2-cell, then the relation labelling this 2-cell must be of the form $\tau b \tau^{-1} b^{-1}$. In particular, the 2-cell has a single oriented edge labelled τ besides the one that we started with. If this second τ-edge is in the interior of Δ, then a second 2-cell with a boundary label of the form $\tau b' \tau^{-1} {b'}^{-1}$ must contain this edge in its boundary. (And because the diagram is reduced, $b' \neq b^{-1}$.) By iterating this argument one obtains a chain of 2-cells crossing Δ, beginning at the τ-edge that we first considered and ending at some other τ-edge in $\partial\Delta$. Such a chain of 2-cells is called a *τ-corridor*.

A rigorous treatment of τ-corridors (in greater generality) can be found in [8]. In the present setting we need only observe that τ-corridors exist, that no 2-cell of Δ lies in more than one τ-corridor, that the "sides" of a τ-corridor are labelled by words in the generators $\mathcal{B}$, and that the area of a τ-corridor is equal to the length of each of its sides. More precisely, each τ-corridor is a subdiagram of Δ whose boundary cycle is labelled $\tau v \tau^{-1} v^{-1}$, where v is a word in the generators $\mathcal{B}$, and the number of 2-cells in the corridor is $|v|$.

Proposition 3.4. *If G is a finitely presented group and H is a finitely generated subgroup with distortion $\delta(n)$, then the Dehn function $f(n)$ of the HNN extension $(G, \tau \mid [\tau, h] = 1\ \forall h \in H)$ satisfies:*

$$n\delta(n) \preceq f(n).$$

More generally, if $G = \langle \mathcal{A} \mid \mathcal{R} \rangle$ and H is the subgroup generated by $\mathcal{B} \subset \mathcal{A}$, and if v is an injective word in the generators $\mathcal{A}$ that represents $h \in H$, then for all $m \in \mathbb{N}$, with respect to the presentation $\langle\, \mathcal{A},\, \tau \mid \mathcal{R},\, [\tau, b] = 1\, \forall b \in \mathcal{B} \rangle$,

$$m\, d_{\mathcal{B}}(1, h) \leq Area(v\tau^m v^{-1}\tau^{-m}).$$

Proof. Because v is injective, $W = v\tau^m v^{-1}\tau^{-m}$ is irreducible. In particular, any van Kampen diagram for W is a disc, so each edge in its boundary lies in the boundary of some closed 2-cell. We fix a reduced diagram Δ for W.

There is a τ-corridor beginning at each edge in the segment of $\partial\Delta$ labelled τ^m. The τ-corridor beginning at the i-th edge of the segment of $\partial\Delta$ labelled τ^m ends at the $(m - i)$-th edge of the segment of $\partial\Delta$ labelled τ^{-m}. The side of each such corridor is labelled by a word in the generators $\mathcal{B}$ and their inverses that represents the group element h. Thus each τ-corridor contains at least $d_{\mathcal{B}}(1, h)$ 2-cells. □

Proof of Proposition 3.1. We work with respect to the presentation of $\Gamma(a, b, c)$ defined in the previous section. Let U_n denote the shortest word in the generators $\mathcal{A}_c$ and their inverses that represents ζ^{n^c}, and let V_n denote the shortest word in the generators $\mathcal{A}_a$ and their inverses that represents z^{n^a}. Because $z = [x_{a-1}, t_a]$, the length of V_n is less than $4\, d_{G_a}(1, z^{n^a})$, which by 2.1(3) is no more than $4K_a n$. Similarly $|U_n| \leq 4K_c n$. Thus the family of words $W_n = U_n V_n U_n^{-1} V_n^{-1}$ has lengths $|W_n| \simeq n$. Because z and ζ commute, W_n represents the identity in $\Gamma(a, b, c)$. We will show that

$$Area(W_n) \succeq n^{c + \frac{a}{b}}.$$

Let Δ be a minimal area van Kampen diagram for W_n. Because no proper subword of W_n represents the identity in $\Gamma(a, b, c)$, Δ is a disc. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ denote the segments of $\partial\Delta$ labelled $U_n, V_n, U_n^{-1}, V_n^{-1}$, respectively. Let p_1, p_2, p_3, p_4 be the initial vertices of these arcs. We shall argue, using the monochromatic regions of Lemma 1.3, that there is an arc labelled ζ^{n^c} in the 1-skeleton of Δ that connects p_1 to p_2.

Of the two edges in $\partial\Delta$ incident at p_1, one is labelled by a letter from $\mathcal{A}_a$ and the other is labelled by a letter from $\mathcal{A}_c$. The former must lie in the boundary of a 2-cell of type $\mathcal{R}_a$ and the latter must lie in the boundary of a 2-cell of type $\mathcal{R}_c$. It follows that p_1 is the endpoint of an arc in the interior of Δ that bounds a monochromatic region of type $\mathcal{R}_c$. Let $q \in \partial\Delta$ be the other endpoint of this arc. Since the arc is labelled by a power of ζ, the portion of $\partial\Delta$ connecting p_1 to q must be labelled by a word that represents an element of $\langle\, \zeta\, \rangle$. Thus $q \in \sigma_1$. If $q \neq p_2$, then consider the edge of σ_1 that is between q and p_2 and is incident at q. This edge lies in a monochromatic region of type $\mathcal{R}_c$, and emanating from q there is an arc in the interior of Δ that bounds this monochromatic region. The other endpoint of this arc is a point $q' \in \sigma_1$ that is between q and p_2 and distinct from q. Proceeding in this manner we can connect p_1 to p_2 by a concatenation of arcs all of whose edges are in the interior of Δ and are labelled by ζ or ζ^{-1}. Because Δ has minimal area, its 1-skeleton does not contain any closed loops with such a labelling, so this concatenation is actually a simple arc. In the same way we obtain a ζ-labelled simple arc from p_3 to p_4.

The two paths which we have just constructed must be disjoint, because otherwise there would be a ζ-labelled path from p_1 to p_4, contradicting the fact that the label on the arc of $\partial\Delta$ connecting p_4 to p_1 represents a non-trivial element of $\langle\, z\, \rangle$.

Thus Δ contains a disc subdiagram Δ_0 bounded by the two ζ-labelled paths constructed above and the arcs of $\partial\Delta$ labelled V_n and V_n^{-1}. By construction, the faces of Δ_0 incident at the edges of $\partial\Delta_0$ labelled ζ correspond to relations of type $\mathcal{R}_b$. Therefore Δ_0 cannot contain any 2-cells corresponding to relations of type $\mathcal{R}_c$, because this would contradict the fact (Lemma 1.3) that every monochromatic region of type $\mathcal{R}_c$ in Δ meets $\partial\Delta$ in at least one edge. Thus Δ_0 can be viewed as a van Kampen diagram for $\zeta^{n^c}V_n\zeta^{-n^c}V_n^{-1}$ over the subpresentation $\langle \mathcal{A}_a, \mathcal{A}_b, z, \zeta \mid \mathcal{R}_a, \mathcal{R}_b\rangle$.

This subpresentation defines the subgroup $G_a *_z (G_b \times \langle\, \zeta\, \rangle)$ and the only relations involving ζ are $[z,\zeta] = 1$ and $[y_i,\zeta] = 1$ for all $y_i \in \mathcal{A}_b$. Thus we are in the situation of Proposition 3.4, with $\mathcal{A} = \mathcal{A}_a \cup \mathcal{A}_b \cup \{z\}$, with $\mathcal{B} = \mathcal{A}_b \cup \{z\}$, and with $\tau = \zeta$. By combining 2.1(3) and 3.4 we deduce:

$$k_b n^{c+\frac{a}{b}} \leq n^c\, d_{G_b}(1, z^{n^a}) \leq Area\,\Delta_0 \leq Area(W_n). \tag{3.1}$$

□

4. Upper bounds

In this section we establish the upper bound necessary to complete the proof of Theorem 2.2. To this end, we undertake a direct analysis of van Kampen diagrams over the presentation of $\Gamma(a,b,c)$ given in Section 2.

We wish to establish an upper bound of the form kn^r on the area of null-homotopic words. Because this function is super-additive (i.e. $k(s+t)^r \geq ks^r + kt^r$ for all $s,t > 0$), it suffices to consider null-homotopic words w which are irreducible; for if $w = uv$ with u and v null-homotopic, then by forming the one-point join of minimal area diagrams for u and v we see $Area(w) \leq Area(u) + Area(v)$. As we noted in (1.5), the advantage of restricting our attention to irreducible words is that any van Kampen diagram for such a word is a topological disc.

We fix a word w of length n that is null-homotopic and irreducible. And we fix a minimal area van Kampen diagram Δ for w.

We shall replace Δ by a cellulated disc whose combinatorial structure is simpler. In order to do so we focus on the monochromatic regions of type $\mathcal{R}_a$ and $\mathcal{R}_c$ in this diagram. The frontier of a monochromatic region of type $\mathcal{R}_a$ is topologically a circle (Lemma 1.3). This circle contains a number of subarcs of $\partial\Delta$ (called *∂-arcs* of type $\mathcal{R}_a$), and these are connected by arcs in the interior of Δ whose edges are labelled $z^{\pm 1}$; we call these connecting arcs *z-arcs*. The boundary of each monochromatic region of type $\mathcal{R}_c$ admits a similar description, but with ζ-arcs in place of z-arcs. The combinatorial geometry of this system of labelled arcs encodes most (but not all) of the decomposition of Δ into monochromatic regions (see Figure 1).

We now leave Δ behind and begin working with the following labelled cell complex structure Δ' on the unit disc in $\mathbb{R}^2$: the boundary circle is cellulated and labelled as the boundary of Δ; we then focus on the vertices corresponding to the endpoints of ∂-arcs of type $\mathcal{R}_a$ and $\mathcal{R}_c$, and divide the interior of the disc into faces (2-cells) by connecting these vertices with non-intersecting, labelled arcs in the interior of the disc: two vertices on $\partial\Delta'$ are connected by an arc labelled z^r (respectively, ζ^s) if the corresponding points on $\partial\Delta$ were connected by a z-arc (respectively, a ζ-arc) in Δ with that label. (It is convenient to retain the terminology z-arc and ζ-arc for the arcs just constructed in Δ'.)

Δ' is a somewhat simplified version of Δ. Each monochromatic region $D \subseteq \Delta$ of type $\mathcal{R}_a$ or $\mathcal{R}_c$ determines a face in Δ' whose boundary label is the same as that of D; in particular, it makes sense to speak of these faces in Δ' as being of type $\mathcal{R}_a$ or

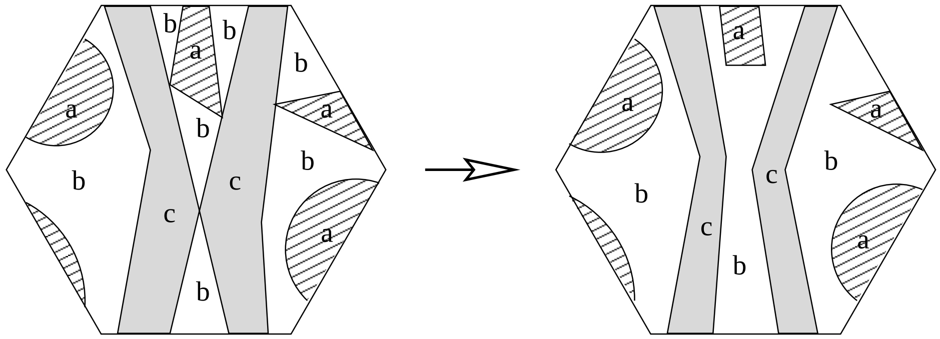

FIGURE 1. Replacing Δ with Δ'

$\mathcal{R}_c$, and henceforth we shall do so. The remaining faces of Δ' are said to be of type $\mathcal{R}_b$, but we should immediately point out that there is not a 1-1 correspondence between monochromatic regions of type $\mathcal{R}_b$ in Δ and faces of type $\mathcal{R}_b$ in Δ': the word labelling the boundary of a face of Δ' that is of type $\mathcal{R}_b$ is a null-homotopic word in the generators of $G_b \times \langle\zeta\rangle$, and this word appears as the label on a circuit in the 1-skeleton of Δ, but this word does not necessarily appear as the label on the boundary cycle of a single monochromatic region of type $\mathcal{R}_b$ in Δ (see Figure 1).

In order to complete the proof we shall show that the sum of the areas of the words labelling the faces of Δ' is bounded by a suitable constant times $n^{c+(a/b)}$. To enhance the clarity of our exposition, we break the proof into a number of steps.

Let the constants $\alpha_a, \alpha_b, \alpha_c, k_a, k_b, k_c, K_a, K_b, K_c$ *be as in Proposition 2.1, and let*

$$K := \max\left\{K_a, K_b, K_c, \frac{1}{k_a}, \frac{1}{k_b}, \frac{1}{k_c}, \alpha_a, \alpha_b, \alpha_c\right\}.$$

In what follows we shall repeatedly use the hypothesis $a \geq b$ to obtain estimates of the form $x^{a/b} + y^{a/b} \leq (x+y)^{a/b}$.

Step 1: Bounding the length of the internal arcs in Δ'.

Lemma 4.1. *If the* ζ*-arcs in the boundary of a face in* Δ' *of type* $\mathcal{R}_c$ *are labelled* $\zeta^{r_1}, \dots, \zeta^{r_s}$*, then* $\sum_i |r_i| \leq Km^c$*, where* m *is the number of occurrences of letters from* $\mathcal{A}_c^{\pm 1} \cup \{\zeta^{\pm 1}\}$ *in* w *(the boundary label of* Δ'*).*

Proof. Killing the generators of $\Gamma(a,b,c)$ other than $\mathcal{A}_c \cup \{\,\zeta\,\}$ gives a retraction of $\Gamma(a,b,c)$ onto G_c. So if an arc of $\partial\Delta'$ is labelled by a word v that represents ζ^{r_i} in $\Gamma(a,b,c)$, then the word obtained from v by deleting the letters other than $\mathcal{A}_c^{\pm 1} \cup \{\zeta^{\pm 1}\}$ also represents ζ^{r_i}. And according to 2.1(3), if this redacted word has length m_i, then $r_i \leq Km_i^c$.

The ζ-arcs in the boundary of the face of type $\mathcal{R}_c$ which we are considering span *disjoint* arcs in $\partial\Delta'$, so by applying the argument of the preceding paragraph to these disjoint arcs we get:

$$\sum_{i=1}^{s} |r_i| \leq \sum_{i=1}^{s} Km_i^c \leq Km^c.$$

$\square$

One obtains a retraction of $\Gamma(a,b,c)$ onto $G_a \times G_b$ by killing the generators $\mathcal{A}_c \cup \{\zeta\}$. Using this retraction in place of the one used in the proof of 4.1, and noting that $a \geq b$, we obtain:

Lemma 4.2. *If the z-arcs in the boundary of a face in Δ' of type $\mathcal{R}_a$ or $\mathcal{R}_b$ are labelled $z^{r_1}, \dots, z^{r_s}$, then $\sum_i |r_i| \leq K\mu^a$, where μ is the number of occurrences of letters from $\mathcal{A}_a^{\pm 1} \cup \mathcal{A}_b^{\pm 1} \cup \{z^{\pm 1}\}$ in w (the boundary label of Δ').*

Step 2: Bounding the sum of the areas of faces of type $\mathcal{R}_a$ and $\mathcal{R}_c$. In this step of the proof we bound the number of 2-cells necessary to fill the faces of Δ' which are of type $\mathcal{R}_a$ or $\mathcal{R}_c$. In fact, a sufficiently sharp bound can be obtained simply by estimating the area of the words labelling their boundaries, viewed as null-homotopic words for G_a or G_c.

Lemma 4.3. *The sum of the areas of the words bounding faces of type $\mathcal{R}_a$ in Δ' is bounded above by:*

$$2Kn^{a+1}.$$

Similarly, the sum of the areas of the words bounding faces of type $\mathcal{R}_c$ is bounded above by:

$$2Kn^{c+1}.$$

Proof. Consider a face E_j of Δ' that corresponds to a monochromatic region of type $\mathcal{R}_a$ in Δ. ∂E_j consists of a number of z-arcs interspersed with subarcs of $\partial\Delta'$, so the word labelling ∂E_j is of the form $W_{E_j} = z^{r_{j,1}} U_{j,1} \dots z^{r_{j,s}} U_{j,s}$. At the cost of applying at most $(\sum_i |U_{j,i}|)(\sum_i |r_{j,i}|)$ relators, we can commute the subwords $U_{j,i}$ past the subwords z^{r_i}, and then freely reducing we get a null-homotopic word of the form $W'_{E_j} = z^{\rho_j} U_j$, with $|\rho_j| \leq \sum_i r_{j,i}$ and $U_j = U_{j,1} \dots U_{j,s}$ in the free group on $\mathcal{A}_a \cup \{\, z \,\}$. But now, by 2.1(2), with respect to the natural presentation for G_a, we have $Area(W'_{E_j}) \leq K|U_j|^{a+1}$. Thus,

$$Area(W_{E_j}) \leq \big(\sum_i |U_{j,i}|\big)\big(\sum_i |r_{j,i}|\big) + K\big(\sum_i |U_{j,i}|\big)^{a+1}.$$

We then use (a weak form of) Lemma 4.2 to replace the second factor in the first summand:

$$Area(W_{E_j}) \leq \big(\sum_i |U_{j,i}|\big) Kn^a + K\big(\sum_i |U_{j,i}|\big)^{a+1}.$$

(Recall that $n = |w| = |\partial\Delta'|$.) By summing over all j and noting that $\sum_{i,j} |U_{i,j}| \leq |w|$, we obtain the desired bound on the areas of faces of type $\mathcal{R}_a$. The argument for faces of type $\mathcal{R}_c$ is entirely similar. □

Step 3 (the hardest): Bounding the area of each face of type $\mathcal{R}_b$. Lemma 1.3 does not yield an easy estimate on the number of monochromatic regions of type $\mathcal{R}_b$ in Δ, and this makes the job of estimating the sum of the areas of the faces of type $\mathcal{R}_b$ in Δ' more difficult than in the case of faces of type $\mathcal{R}_a$ and $\mathcal{R}_c$.

We decompose Δ' as the union of those subdiscs which are the closures of the connected components of Δ' minus the union of its ζ-arcs. Some of these subdiscs are faces of type $\mathcal{R}_c$ — we are not concerned with these. We fix our attention on a subdisc D_j which is not of this type. Notice that no edge in $\partial D_j \cap \partial\Delta'$ is labelled by a letter from $\mathcal{A}_c$, for otherwise the subdiagram of Δ corresponding to D_j would contain a monochromatic region of type $\mathcal{R}_c$ as a proper subdiagram, and hence would have a ζ-arc in its interior.

Let μ_j be the number of edges in ∂D_j that are labelled by letters other than $\zeta^{\pm 1}$. As in 4.2, the sum of the lengths of the z-arcs in the boundary of each face of D_j is at most $K\mu_j^a$. We concentrate on a face $E' \subset D_j$ that is of type $\mathcal{R}_b$: the boundary of E' consists of a number of z-arcs (total length $\leq K\mu_j^a$), a number of ζ-arcs (total length $\leq Kn^c$) and a number of subarcs of $\partial\Delta'$ (total length $\leq \mu_j$).

According to 2.1(3), if a z-arc has length r_l, then one can replace it by a word of length at most $Kr_l^{1/b}$ in the generators $\mathcal{A}_b^{\pm 1} \cup \{z^{\pm 1}\}$, and by 2.1(2) we know that this may be done at the cost of applying at most

$$\alpha_b(Kr_l^{1/b})^{b+1} \;\leq\; K^{2+b} r_l^{1+(1/b)} \tag{4.1}$$

relators. We shall replace each of the z-arcs in $\partial E'$ in this way and count how many relators we must apply in order to do so. For this, as in Step 2, we divide ∂D_j into disjoint subarcs each connecting the endpoints of a z-arc in the boundary of the face $E' \subseteq D_j$. If $\tilde{r}_l$ is the number of occurrences of letters from $\mathcal{A}_a^{\pm 1} \cup \mathcal{A}_b^{\pm 1} \cup \{z^{\pm 1}\}$ in the subarc of ∂D_j corresponding to a z-arc labelled z^{r_l}, then $r_l \leq K\tilde{r}_l^a$ and $\sum_l \tilde{r}_l \leq \mu_j$. These estimates, together with (4.1), show that at the cost of applying at most

$$\begin{aligned} \sum_l K^{2+b} r_l^{1+(1/b)} &\leq K^{2+b} \left(\sum_l r_l \right)^{1+(1/b)} \\ &\leq K^{2+b} \, (K\mu_j^a)^{1+(1/b)} \\ &=: K' \, \mu_j^{a+(a/b)} \end{aligned} \tag{4.2}$$

relators, we can replace all of the labels on z-arcs in $\partial E'$ with equivalent words in the generators $\mathcal{A}_b^{\pm 1} \cup \{z^{\pm 1}\}$. The sum of the lengths of these equivalent words is at most

$$\begin{aligned} \sum_l K \, r_l^{1/b} &\leq K \sum_l (K\tilde{r}_l^a)^{1/b} \\ &= K^{1+(1/b)} \sum_l \tilde{r}_l^{(a/b)} \\ &\leq K' \mu_j^{(a/b)}. \end{aligned} \tag{4.3}$$

At this stage, we have replaced the word labelling $\partial E'$ by a word W which consists of at most $K'\mu_j^{(a/b)}$ letters from $(\mathcal{A}_b \cup \{z\})^{\pm 1}$ and some number of letters $\zeta^{\pm 1}$ coming from ∂D_j. Let us write $Z(E')$ for the number of occurrences of ζ and its inverse. (Later, when we sum over all E' in D_j, we shall use the fact that $\sum Z(E') \leq Kn^c$, by Lemma 4.1.) Since the presentation of $\Gamma(a,b,c)$ with which we are working contains the relations $[\zeta, x] = 1$ for all $x \in \mathcal{A}_b \cup \{z\}$, we may commute all of the other letters in W past all occurrences of $\zeta^{\pm 1}$ at a cost of applying at most

$$K' \mu_j^{(a/b)} \, Z(E') \tag{4.4}$$

relators. Thus (after freely reducing) we obtain a word of the form $U\zeta^p$, where U is a word of length at most $K'\mu_j^{(a/b)}$ in the generators $(\mathcal{A}_b \cup \{z\})^{\pm 1}$ and their inverses.

But $U\zeta^p$ represents the identity in $G_b \times \langle \zeta \rangle$, hence $p = 0$ and, by 2.1(2),

$$Area(U) \le K\,|U|^{b+1} \le K(K')^{b+1} \mu_j^{a+(a/b)}. \tag{4.5}$$

Combining the displayed estimates, we see that for an appropriate constant K'' the following is an upper bound on the area of the word labelling $\partial E'$:

$$K''\left(\mu_j^{a+(a/b)} + \mu_j^{(a/b)}\, Z(E')\right).$$

Thus, summing over all faces of type $\mathcal{R}_b$ in D_j, we obtain a *total* area of at most:

$$K''\left(N_j\, \mu_j^{a+(a/b)} + \mu_j^{(a/b)}\, n^c\right),$$

where N_j is the number of faces of type $\mathcal{R}_b$ in D_j.

Finally, summing over all D_j, and using the fact that $\sum_j \mu_j \le n$ (because this is a sum of lengths of disjoint arcs in $\partial\Delta'$), we obtain:

Lemma 4.4. *The sum of the areas of the words bounding faces of type $\mathcal{R}_b$ in Δ' is at most*

$$K''\left(N\ n^{a+(a/b)} + n^{c+(a/b)}\right),$$

where N is the number of faces of type $\mathcal{R}_b$.

Step 4: Bounding the number of faces of type $\mathcal{R}_b$ in Δ'. By combining Lemmas 4.3 and 4.4, and recalling that $1 \le b \le a < c$, we see that Theorem 2.2 is a consequence of the following assertion:

Lemma 4.5. *Δ' contains at most n faces of type $\mathcal{R}_b$.*

Proof. Consider the finite connected graph $\mathcal{G}$ whose vertices are the faces of Δ', and which has an edge connecting two faces if and only if they abut along a z-arc or a ζ-arc. We label the vertices a, b or c, according to the type of face in Δ' to which they correspond. Note that $\mathcal{G}$ is a tree, because the z-arcs and ζ-arcs do not intersect in the interior of Δ'. Every vertex which is adjacent to a vertex of type a or c must be of type b, and vice versa.

We fix a vertex x_0 of valence one in $\mathcal{G}$, and consider the map which sends each vertex x other than x_0 to the first vertex along the unique arc joining x to x_0 in $\mathcal{G}$. This map sends the set of vertices that are of type a or c onto the set of vertices of type b that have valence greater than one. But at most n vertices of $\mathcal{G}$ correspond to faces of Δ' that contain an edge of $\partial\Delta'$, and this collection of vertices includes all those of type a and c (Lemma 1.3) and all those of valence one. Thus $\mathcal{G}$ has at most n vertices of type b, and hence Δ' has at most n faces of type $\mathcal{R}_b$. $\square$

5. Further examples

Thus far we have concentrated on the groups $\Gamma(a, b, c)$. We have done so in order to simplify the exposition as much as possible. However, the methods introduced here apply, more or less directly, to many related classes of groups. For example, a direct translation of the arguments presented above can be used to prove:

Theorem 5.1. *Let a and b be positive integers and let G_a and G_b be as defined in Section 2. Consider the group $J(a, b)$ obtained by amalgamating two copies of G_a and G_b along central cyclic subgroups in the following manner:*

$$J(a, b) := G_a *_{z_a = z_b} (G_b \times G_b') *_{z_a' = z_b'} G_a'.$$

If $a \ge b$ and $(a/b)^2 > a + 1$, then the Dehn function of $J(a, b)$ is $\simeq n^{(a/b)^2}$.

Equally, instead of using the groups G_a as basic building blocks, one could use groups of the form $\mathbb{Z}^a \rtimes F$, where F is a non-abelian free group with a basis all of whose elements act by the unipotent matrix ϕ_a which has ones on the diagonal and super-diagonal and zeros elsewhere.

The interested reader should be able to construct other examples by focusing on the key role played by subgroup distortion; cf. 3.4.

6. Subgroup distortion

Theorem 6.1. *For every rational number $r \geq 1$ there exist pairs of finitely presented groups $H \subseteq G$ such that the distortion of H in G is $\simeq n^r$.*

Proof. We saw in 3.3 that if $a \geq b$, then the distortion $\delta_{a,b}(n)$ of G_b in $G_a *_{\langle z \rangle} G_b$, the group formed by amalgamating G_a and G_b along their centres, is $\succeq n^{\frac{a}{b}}$. On the other hand, the proof of Proposition 3.1 shows that $K\, n^c\, \delta_{a,b}(n)$ is a lower bound for the Dehn function of $\Gamma(a,b,c)$, and we now know that this Dehn function is actually $\simeq n^{c+\frac{a}{b}}$ if $c > a$. Thus $\delta_{a,b}(n) \simeq n^{\frac{a}{b}}$. ☐

7. Isodiametric inequalities

We close with some remarks concerning isodiametric inequalities. These remarks point to the fact that the constructions in earlier sections are relevant to questions concerning the geometry of asymptotic cones for finitely presented groups (cf. [14] and [5]).

Isoperimetric inequalities measure the complexity of the word problem in a group by bounding the number of relators that one must apply in order to show that a null-homotopic word represents the identity. An alternative way of measuring complexity is to bound the length of the elements by which one must conjugate the relators being applied. Geometrically, this corresponds to bounding the diameter of van Kampen diagrams rather than their area. In certain contexts such bounds are the most appropriate measure of complexity (see [15], [12]).

Definition 7.1. Let $\langle \mathcal{A} \mid \mathcal{R} \rangle$ be a finite presentation of the group Γ. Let w be a word that is null-homotopic with respect to this presentation and let Δ be a van Kampen diagram for w. Let p be the basepoint of Δ. Endow the 1-skeleton of Δ with a path metric ρ that gives each edge length 1. The diameter of w is defined by

$$diam(w) := \max\{\rho(p,q) \mid q \text{ a vertex of } \Delta\}.$$

The (unreduced) *isodiametric function* of $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is

$$\Phi(n) := \max_{|w| \leq n} diam(w).$$

The min-area and reduced isodiametric functions are defined similarly except that one quantifies only over van Kampen diagrams that are, respectively, of minimal area (as defined in Section 1) or are reduced. The $\simeq$ equivalence classes of these qualified isodiametric functions depend very much on the chosen presentation of Γ (see [6]), whereas $\Phi(n)$ is $\simeq$ independent of the choice of presentation (see [12]).

Authoritative results concerning the range of possible behaviours of $\simeq$ equivalence classes of (unreduced) isodiametric functions are proved in [24].

Consider the groups $J(a, b)$ defined in Section 5. Direct diagrammatic arguments in the manner of Sections 3 and 4 can be used to prove the following (cf. [5] and [6]). We omit the details.

Proposition 7.2. *If $a \geq b$, then the min-area isodiametric function of $J(a, b)$ is $\simeq n^{a/b}$.*

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